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ИЗВЕСТИЯ
НАН АРМЕНИИ

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МАТЕМАТИКА

ТОМ 58 № 6 2023

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SIMPLE PROOF OF THE RISK BOUND FOR DENOISING BY EXPONENTIAL WEIGHTS FOR ASYMMETRIC NOISE DISTRIBUTIONS

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Abstract. In this note, we consider the problem of aggregation of estimators in order to denoise a signal. The main contribution is a short proof of the fact that the exponentially weighted aggregate satisfies a sharp oracle inequality. While this result was already known for a wide class of symmetric noise distributions, the extension to asymmetric distributions presented in this note is new.

MSC2020 numbers: 62J05; 62H12.

Keywords: signal denoising; exponential weights; PAC-Bayesian bounds.

1. Introduction

Let us consider the problem of denoising an n dimensional noisy signal $\boldsymbol{Y}$ using a family of candidates $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$. More precisely, we assume that

$$\boldsymbol{Y} = \boldsymbol{\theta}^* + \boldsymbol{\xi}$$

where $\boldsymbol{\theta}^* \in \mathbb{R}^n$ is the n dimensional true signal and $\boldsymbol{\xi}$ is random noise. Only the noisy vector $\boldsymbol{Y}$ is observed and the goal is to construct an estimator $\widehat{\boldsymbol{\theta}}$ such that the expected error $\mathbf{E}[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2]$ is as small as possible, where $\|\boldsymbol{v}\|$ stands for the Euclidean norm of $\boldsymbol{v} \in \mathbb{R}^n$. We consider the framework in which to achieve the aforementioned goal we are given a set of vectors $\{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m\}$. An estimator $\widehat{\boldsymbol{\theta}}$ is considered a good estimator, if the regret

$$\mathbf{E}[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2] - \min_{j=1,\ldots,m} \|\boldsymbol{\theta}_j - \boldsymbol{\theta}^*\|^2 \tag{1.1}$$

is as small as possible. This problem has been coined model-selection aggregation in (17), where it is also proved that the optimal rate of the difference in (1.1) is $\log m$. The problem of aggregation has been extensively studied in the literature, see for instance (3; 20; 22; 21; 13; 2; 16; 18; 1; 14; 4). In this note, we consider the

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exponentially weighted aggregate (EWA) defined as follows. Let $\pi_0(1), \ldots, \pi_0(m)$ be some nonnegative weights summing to one. Each $\pi_0(j)$ represents our prior confidence in the approximation of $\boldsymbol{\theta}^*$ by $\boldsymbol{\theta}_j$. Based on these prior weights and the observed vector $\boldsymbol{Y}$, we define

$$\widehat{\boldsymbol{\theta}} = \sum_{j=1}^{m} \boldsymbol{\theta}_j \widehat{\pi}(j), \qquad \text{with} \qquad \widehat{\pi}(j) = \frac{\exp\{-\|\boldsymbol{Y} - \boldsymbol{\theta}_j\|^2/\beta\}\pi_0(j)}{\sum_{\ell=1}^{m} \exp\{-\|\boldsymbol{Y} - \boldsymbol{\theta}_\ell\|^2/\beta\}\pi_0(\ell)}.$$

In this expression, $\beta > 0$ is a tuning parameter of the method. As established in the aforementioned references, in different settings one can prove that EWA satisfies the inequality

$$\mathbf{E}[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2] \leqslant \min_{j=1,\ldots,m} \Big(\|\boldsymbol{\theta}_j - \boldsymbol{\theta}^*\|^2 + \beta \log(1/\pi_0(j))\Big). \tag{1.2}$$

In particular, if π_0 is the uniform distribution over $\{1, \ldots, m\}$, one obtains the rate-optimal remainder term $\beta \log m$ for the difference in (1.1).

As pointed out in some papers (8; 9; 6), it is helpful to extend the above-described framework to the case of aggregating a family of estimators which is potentially infinite. This is equivalent to considering a subset $S_0 \subset \mathbb{R}^n$ and aiming at finding an "optimal" way of combining all its elements in order to estimate $\boldsymbol{\theta}^*$. These types of considerations have led to the following extension of the estimator (1.2):

$$\widehat{\boldsymbol{\theta}} = \int_{\mathbb{R}^n} \boldsymbol{\theta}\, \widehat{\pi}(d\boldsymbol{\theta}), \qquad \text{with} \qquad \frac{d\widehat{\pi}}{d\pi_0}(\boldsymbol{\theta}) = \frac{\exp\{-\|\boldsymbol{Y} - \boldsymbol{\theta}\|^2/\beta\}}{\int_{\mathbb{R}^n} \exp\{-\|\boldsymbol{Y} - \boldsymbol{u}\|^2/\beta\}\pi_0(d\boldsymbol{u})}. \tag{1.3}$$

Notice that this estimator is the Bayesian posterior mean in the case where $\boldsymbol{\xi}$ is drawn from the Gaussian distribution with zero mean and covariance matrix $(\beta/2)\mathbf{I}_n$. The goal of this note is to provide an alternative and simple proof of the fact that EWA $\widehat{\boldsymbol{\theta}}$ satisfies (1.2) and its extension to aggregating an infinite set, provided that the distribution of the noise ξ satisfies some suitable conditions. We also slightly extend the existing results by including noise distributions that are not symmetric with respect to the origin. This is particularly suitable for estimating the parameters of Bernoulli or binomial distributions.

Notation. We use boldface letters for vectors, which are always seen as one-column matrices. For any vector $\boldsymbol{v}$, $\|\boldsymbol{v}\|$ and $\|\boldsymbol{v}\|_\infty$ are respectively the Euclidean norm and the sup-norm. By convention, throughout this work, $0 \cdot \infty = 0$. For a probability distribution π on $\mathbb{R}^n$, we denote by $\mathrm{Var}_\pi(\boldsymbol{\theta})$ the variance with respect to π defined by $\int_{\mathbb{R}^n} \|\boldsymbol{\theta}\|^2\, \pi(d\boldsymbol{\theta}) - \|\int_{\mathbb{R}^n} \boldsymbol{\theta}\, \pi(d\boldsymbol{\theta})\|^2$. For two probability distributions μ and ν defined on the same probability space and such that μ is absolutely continuous with respect to ν, the Kullback-Leibler divergence is defined by $D_{\mathrm{KL}}(\mu||\nu) = \int \frac{d\mu}{d\nu}(x) \log \frac{d\mu}{d\nu}(x)\, \nu(dx)$.

2. Main result

This section is devoted to stating and briefly discussing the main result, the proof being postponed to Section 4 below. Prior to stating the result, we recall the Bernstein condition. For some $v > 0$ and $b \geqslant 0$, we say that a random variable η satisfies the (v, b)-Bernstein condition, if

$$\mathbf{E}[e^{t\eta}] \leqslant \exp\Big\{\frac{v^2t^2}{2(1-b|t|)}\Big\}, \qquad \forall t \in (-1/b, 1/b).$$

This condition is clearly on the distribution of the random variable. One can check that if η satisfies the (v, b)-Bernstein condition, then it is sub-exponential with zero mean, and the variance of η is at least equal to v. Many common distributions satisfy this assumption. For instance, any sub-Gaussian distribution with variance proxy τ satisfies the $(\tau, 0)$-Bernstein condition. Any random variable supported by $[-A, A]$ satisfies the Bernstein condition with $(v, b) = (A^2, 0)$ but also with $(v, b) = (\mathrm{Var}(\eta), A/3)$ (19). We will see that the latter is more useful for our purposes than the former.

Similarly, if $\mathcal{F}$ is a sigma-algebra and v and b are two $\mathcal{F}$- measurable random variables, we say that η is (v, b)-Bernstein conditionally to $\mathcal{F}$, if almost surely, the inequality $\mathbf{E}[e^{t\eta}|\mathcal{F}] \leqslant \exp\{v^2t^2/(1 - b|t|)\}$ is satisfied for every $t \in \mathbb{R}$ such that $|t|b < 1$.

Theorem 1. *Let π_0 be a probability distribution supported by $S_0 \subset \mathbb{R}^n$ with a diameter measured in sup-norm bounded by $\mathcal{D}_0$. Assume that the distribution of $\boldsymbol{\xi}$ satisfies the following assumption: for some sigma algebra $\mathcal{F}$ and for some $b : [0, 1] \to [0, \infty)$ and continuously differentiable function $v : [0, 1] \to [0, \infty)$ vanishing at the origin, for every $\alpha \in (0, 1]$, there exists an n-dimensional random vector $\boldsymbol{\zeta}$ such that*

$$\mathbf{E}[\boldsymbol{\zeta}|\mathcal{F}] = 0, \qquad \boldsymbol{\xi} + \boldsymbol{\zeta} \stackrel{\mathscr{D}}{=} (1 + \alpha)\boldsymbol{\xi}.$$

and, conditionally to $\mathcal{F}$, the entries ζ_i are independent and satisfy the $(v(\alpha), b(\alpha))$-Bernstein condition. Then, for every $\beta \geqslant 2b(0)\mathcal{D}_0$, we have

$$\mathbf{E}[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2] \leqslant \inf_\pi \Big\{ \int_{\mathbb{R}^n} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|^2\, \pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0)\Big\}$$
$$+\Big(\frac{2v'(0)}{\beta - 2b(0)\mathcal{D}_0} - 1\Big)\mathbf{E}[\mathrm{Var}_{\widehat{\pi}}(\boldsymbol{\vartheta})],$$

where the inf *is over all the probability distributions. As a consequence, for* $\beta \geqslant 2v'(0) + 2b(0)\mathcal{D}_0$*, we get*

$$\mathbf{E}[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2] \leqslant \inf_{\pi} \left\{ \int_{\mathbb{R}^n} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|^2 \,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0) \right\}. \tag{2.1}$$

Let us briefly comment on this result. First, the link between (2.1) and (1.2) might not be easy to see. It is obtained by considering a prior distribution π_0 supported by the finite set $\{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m\}$ and by upper bounding the infimum in (2.1) by the minimum over all the Dirac measures $\delta_{\boldsymbol{\theta}_j}$. One easily checks that $D_{\mathrm{KL}}(\delta_{\boldsymbol{\theta}_j}||\pi_0) = \log(1/\pi_0(j))$, which allows to infer (1.2) from (2.1).

Second, one may wonder where the form of the upper bound in (2.1) comes from. The presence of the KL-divergence in this bound may seem surprising. The reason is that there is a deep connection between the KL-divergence and the exponential weights. Indeed, according to the Varadhan-Donsker variational formula, the "posterior" distribution $\widehat{\pi}$ defined in (1.3) is solution to following problem:

$$\widehat{\pi} \in \operatorname*{argmin}_{\pi} \left\{ \int_{\mathbb{R}^n} \|\boldsymbol{\theta} - \boldsymbol{Y}\|^2 \,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0) \right\},$$

where the min is over all the probability distributions. This result will be the starting point of the proof.

Finally, one can wonder how restrictive the assumptions of this theorem are. We will show below that they are satisfied for a broad class of noise distributions.

3. Instantiation to some well-known noise distributions

The main theorem stated in the previous section requires a general and a rather abstract condition to be satisfied by the noise distribution. This section shows that many distributions encountered in applications satisfy this assumption with some parameters $v'(0)$ and $b(0)$ which are easy to determine.

3.1. **Centered Bernoulli noise.** Assume that each ξ_i is a centered Bernoulli random variable: it takes the value $1 - \rho_i$ with probability ρ_i and the value $-\rho_i$ with probability $1 - \rho_i$. Here, $\rho_i \in (0, 1)$. Then, one can set

$$\mathbf{P}\big(\zeta_i = \alpha\xi_i \,|\, \xi_i\big) = \frac{1 + \alpha - \alpha|\xi_i|}{\alpha + 1}, \quad \mathbf{P}\big(\zeta_i = -\operatorname{sgn}(\xi_i)(1 + \alpha - \alpha|\xi_i|) \,|\, \xi_i\big) = \frac{\alpha|\xi_i|}{\alpha + 1}.$$

We see that conditionally to ξ_i, the random variable ζ_i is zero mean and takes its values in an interval of length $\alpha(1 - \rho_i) + \alpha\rho_i + 1 = \alpha\rho_i + 1 + \alpha - \alpha\rho_i = 1 + \alpha$. This implies that ζ_i satisfies the $((1 + \alpha)^2/4, 0)$-Bernstein condition, conditionally to ξ_i. In other terms, ζ_i is sub-Gaussian with variance proxy $(1 + \alpha)^2/4$. However, this does not help in applying Theorem 1, since the function $v(\alpha) = (1+\alpha)^2/4$ does not vanish at the origin. On the positive side, since the conditional variance of ζ_i given ξ_i is smaller than $\alpha(1+\alpha)$ and the support is included in $[-(1+\alpha),(1+\alpha)]$, the conditional distribution of ζ_i given ξ_i satisfies the Bernstein condition with $v(\alpha) = \alpha(1+\alpha)$ and $b(\alpha) = (1+\alpha)/3$, see (19, Exercise 2.8.5). This yields the following result.

Corollary 1. Let π_0 be a probability distribution supported by $S_0 \subset \mathbb{R}^n$ such that $\mathcal{D}_0 = \sup_{\boldsymbol{\theta},\boldsymbol{\theta}'\in S_0} \|\boldsymbol{\theta}-\boldsymbol{\theta}'\|_\infty < \infty$. Assume that $\boldsymbol{\xi}$ has independent entries ξ_i satisfying $\mathbf{P}(\xi_i = 1-\rho_i) = 1 - \mathbf{P}(\xi_i = -\rho_i) = \rho_i$ for some $\rho_i \in (0,1)$. Then, for every $\beta \geqslant (2/3)\mathcal{D}_0$, we have

$$\begin{aligned}\mathbf{E}[\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|^2] \leqslant \inf_\pi \bigg\{ \int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\boldsymbol{\theta}^*\|^2\,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi\|\pi_0)\bigg\}\\ +\Big(\frac{6}{3\beta-2\mathcal{D}_0}-1\Big)\mathbf{E}[\mathrm{Var}_{\widehat\pi}(\boldsymbol{\vartheta})].\end{aligned} \tag{3.1}$$

In particular, if $\beta \geqslant 2+(2/3)\mathcal{D}_0$, the last term in (3.1) is nonpositive and, therefore, can be neglected.

This corollary can be used in cases where the observations Y_i are independent Bernoulli random variables with mean θ_i^*. In such a situation, it is natural to choose a prior distribution π_0 that is concentrated on the unit hypercube $[0,1]^n$, the diameter of which in sup-norm is equal to 1. The corollary implies that in such a situation the inequality stated in (2.1) is true provided that $\beta \geqslant 8/3$. We refer the reader to (10) for an application of this result to graphon estimation.

3.2. **Gaussian noise.** In the case of the Gaussian noise $\boldsymbol{\xi}$ with independent entries having 0 mean and variance equal to σ_i^2, one can check that the conditions of Theorem 1 are satisfied with the random vector $\boldsymbol{\zeta}$ which is independent of $\boldsymbol{\xi}$ and has independent entries drawn from the Gaussian distribution $\mathcal{N}(0,(2\alpha+\alpha^2)\sigma_i^2)$. This means that in the Bernstein condition one can choose $\mathcal{F} = \sigma(\boldsymbol{\xi})$, $b = 0$ and $v(\alpha) = (2\alpha+\alpha^2)\max_{1\leqslant i\leqslant n}\sigma_i^2$, which leads to the following result.

Corollary 2. Let π_0 be a probability distribution on $\mathbb{R}^n$. Assume that $\boldsymbol{\xi}$ has independent entries $\xi_i \sim \mathcal{N}(0,\sigma_i^2)$, $i = 1,\ldots,n$. Then, for every $\beta > 0$, we have

$$\mathbf{E}[\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|^2] \leqslant \inf_\pi \bigg\{ \int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\boldsymbol{\theta}^*\|^2\,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi\|\pi_0)\bigg\} + \big(4\sigma^2\beta^{-1}-1\big)\mathbf{E}[\mathrm{Var}_{\widehat\pi}(\boldsymbol{\vartheta})], \tag{3.2}$$

where $\sigma = \max_{1\leqslant i\leqslant n}\sigma_i$. In particular, if $\beta \geqslant 4\sigma^2$, the last term in (3.2) is nonpositive and, therefore, can be neglected.

Some preliminary versions of this result can be traced back to (12; 11). In the form (2.1), and with an extension to aggregation of projection estimators, the result appeared in (15). Further generalisations to various families of linear estimators have been explored in (7). The proof of the oracle inequality in all these papers is very specific to the Gaussian distribution since it is based on Stein's lemma (integration by parts for the Gaussian measure). The alternative proof presented in this work relies on techniques developed in (8; 5; 6).

3.3. **Bounded noise.** For every $a, b > 0$, let $\mathcal{B}(a, b)$ be the distribution of a random variable that takes the values a and $-b$ with probabilities $b/(a+b)$ and $a/(a+b)$, respectively. If the distribution of ξ_i can be written as a mixture of the distributions $\mathcal{B}(a, b)$ with a mixing distribution with bounded support, then our main theorem can be applied. More precisely, assume that the distribution of ξ_i is given by

$$p_{\xi_i}(dx) = \int_0^A \int_0^B \frac{b\delta_a(dx) + a\delta_{-b}(dx)}{a+b}\, \nu_i(da, db),$$

where ν_i is a probability distribution on $[0, A] \times [0, B]$. This means that $\xi_i = \eta_i^{\alpha_i,\beta_i}$ with random variables (α_i, β_i) drawn from ν_i and $\eta_i^{a,b}$ drawn from the binary distribution $\frac{b\delta_a(dx)+a\delta_{-b}(dx)}{a+b}$. Akin to the first subsection of this section, one can choose $\zeta_i^{a,b}$ so that $(1+\alpha)\eta_i^{a,b}$ has the same distribution as $\eta_i^{a,b} + \zeta_i^{a,b}$, for every pair (a, b). Then, clearly, $(1+\alpha)\xi_i$ has the same distribution as $\xi_i + \zeta_i^{\alpha,\beta}$. Let $\mathcal{F}$ be the sigma algebra generated by the random variables $\alpha, \beta, \{\eta_j^{a,b} : (a, b) \in [0, A] \times [0, B], i \in [n]\}$. Conditionally to $\mathcal{F}$, $\zeta_i^{a,b}$ is a binary random variable with zero mean and takes its values in the interval $[-B, A]$, it satisfies the Bernstein condition with $b(\alpha) = (A+B)(1+\alpha)/3$ and $v(\alpha) = (A+B)^2\alpha(1+\alpha)$. Therefore, we get the following result.

Corollary 3. Let π_0 be a probability distribution supported by $S_0 \subset \mathbb{R}^n$ such that $\mathcal{D}_0 = \sup_{\boldsymbol{\theta},\boldsymbol{\theta}'\in S_0} \|\boldsymbol{\theta} - \boldsymbol{\theta}'\|_\infty < \infty$. Assume that $\boldsymbol{\xi}$ has independent entries ξ_i, $i = 1, \ldots, n$, taking values in an interval I_i of length at most L. Then, for every $\beta \geqslant (2/3)L\mathcal{D}_0$, we have

$$\begin{aligned}\mathbf{E}[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2] \leqslant \inf_\pi \bigg\{ \int_{\mathbb{R}^n} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|^2\, \pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0)\bigg\}\\ + \Big(\frac{6L^2}{3\beta - 2L\mathcal{D}_0} - 1\Big)\mathbf{E}[\mathrm{Var}_{\widehat{\pi}}(\boldsymbol{\vartheta})].\end{aligned} \tag{3.3}$$

In particular, if $\beta \geqslant 2L^2 + (2/3)L\mathcal{D}_0$, the last term in (3.3) is nonpositive and, therefore, can be neglected.

This result is well suited for the setting where the components Y_i of the observation $\boldsymbol{Y}$ are bounded. For instance, if we know that $\mathbf{P}(Y_i \in [0, L]) = 1$ for every $i \in \{1, \ldots, n\}$, then it is also natural to choose a prior distribution satisfying $\mathcal{D}_0 = L$. Inequality (2.1) is then satisfied for every $\beta \geqslant (8/3)L^2$. Note that, to the best of our knowledge, this is the first time that such a precise bound is obtained for asymmetric noise distributions. The similar result established in (6, Theorem 2) deals with symmetric distributions only.

3.4. **Centered binomial noise.** Consider the case where ξ_i's are independent and drawn from a centered and scaled binomial distribution $a\mathcal{B}(k, \rho_i) - ak\rho_i$, where $a > 0$ is the scaling factor. This distribution is a particular case of distributions supported by a finite interval considered in the previous subsection. One can therefore apply the last corollary with $L = ak$. However, this leads to a bound which is too crude. Indeed, one can use the fact that ξ_i is equal in distribution to $a(\eta_1 + \ldots + \eta_k)$ where η_j's are iid centered Bernoulli variables. Defining $\bar\zeta_1, \ldots, \bar\zeta_k$ as independent random variables satisfying

$$\mathbf{P}\big(\bar\zeta_j = \alpha\eta_j \,|\, \eta_j\big) = \frac{1 + \alpha - \alpha|\eta_j|}{\alpha + 1}, \quad \mathbf{P}\big(\bar\zeta_j = -\operatorname{sgn}(\eta_j)(1 + \alpha - \alpha|\eta_j|) \,|\, \eta_j\big) = \frac{\alpha|\eta_j|}{\alpha + 1},$$

one easily checks that $\eta_j + \bar\zeta_j$ has the same distribution as $(1 + \alpha)\eta_j$. Therefore, $\xi_i + \zeta_i$, for $\zeta_i = a(\bar\zeta_1 + \ldots + \bar\zeta_k)$, has the same distribution as $(1+\alpha)\xi_i$. Furthermore, conditionally to the sigma-algebra generated by $\{\eta_1, \ldots, \eta_k\}$, ζ_i has zero mean and satisfies the Bernstein condition with $b(\alpha) = a(1 + \alpha)/3$ and $v(\alpha) = a^2 k\alpha(1 + \alpha)$.

Corollary 4. Let π_0 be a probability distribution supported by $S_0 \subset \mathbb{R}^n$ such that $\mathcal{D}_0 = \sup_{\boldsymbol{\theta}, \boldsymbol{\theta}' \in S_0} \|\boldsymbol{\theta} - \boldsymbol{\theta}'\|_\infty < \infty$. Assume that $\boldsymbol{\xi}$ has independent entries ξ_i, $i = 1, \ldots, n$, drawn from the scaled and centered binomial distribution $a(\mathcal{B}(k, \rho_i) - k\rho_i))$. Then, for every $\beta \geqslant (2/3)a\mathcal{D}_0$, we have

$$\begin{aligned} \mathbf{E}[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2] \leqslant \inf_\pi \Big\{ \int_{\mathbb{R}^n} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|^2 \,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi || \pi_0) \Big\} \\ + \Big(\frac{6a^2 k}{3\beta - 2a\mathcal{D}_0} - 1 \Big) \mathbf{E}[\mathrm{Var}_{\widehat\pi}(\boldsymbol{\vartheta})]. \end{aligned} \tag{3.4}$$

In particular, if $\beta \geqslant 2a^2 k + (2/3)a\mathcal{D}_0$, the last term in (3.4) is nonpositive and, therefore, can be neglected.

A typical application of this result concerns the case of observing the average of k Bernoulli variables, that is $Y_i \sim (1/k)\mathcal{B}(k, \theta_i^*)$. In this case, all the θ_i^* belong to $[0, 1]$ and, therefore, it is reasonable to choose a prior distribution π_0 supported by $[0, 1]^n$. This ensures that $\mathcal{D}_0 \leqslant 1$, and, therefore, inequality (2.1) follows from the last corollary provided that $\beta \geqslant 8/(3k)$ (this is obtained by choosing $a = 1/k$).

3.5. **Double exponential noise.** All the previous examples considered in this section are distributions with sub-exponential tails. Let us check that Theorem 1 can also be applied to some distributions that have heavier, say sub-exponential, tails. Let ξ_i be independent drawn from the Laplace distribution[2] with parameters $\mu_i > 0$, $i = 1, \ldots, n$. Then, one can choose $\mathcal{F} = \mu(\boldsymbol{\xi})$ and $\zeta_1, \ldots, \zeta_n$ to be independent, independent of $\boldsymbol{\xi}$, and drawn from the distribution $\frac{1}{(1+\alpha)^2}\delta_0 + \frac{2\alpha+\alpha^2}{(1+\alpha)^2}\mathsf{Lap}((1+\alpha)\mu_i)$. The fact that $\xi_i + \zeta_i$ has the same distribution as $(1+\alpha)\xi_i$ can be checked by computing the characteristic functions of these variables and by verifying that they are equal. As for the Bernstein condition, for every t such that $(1+\alpha)\mu_i|t| \leqslant 1$ we have

$$\begin{aligned}
\mathbf{E}[e^{t\zeta_i}] &= \frac{1}{(1+\alpha)^2} + \frac{2\alpha+\alpha^2}{(1+\alpha)^2} \times \frac{1}{1-(1+\alpha)^2 t^2\mu_i^2} \\
&\big(p := 1-(1+\alpha)^{-2}, z := (1+\alpha)t\mu_i\big) \\
&= 1 - p + \frac{p}{1-z^2} = 1 + \frac{pz^2}{1-z^2} \leqslant 1 + \frac{pz^2}{1-|z|} \\
&\leqslant \exp\Big\{\frac{pz^2}{1-|z|}\Big\} = \exp\Big\{\frac{\alpha(2+\alpha)\mu_i^2 t^2}{1-(1+\alpha)\mu_i|t|}\Big\}
\end{aligned}$$

This means that the (conditional) Bernstein condition is satisfied with $v(\alpha) = \alpha(2+\alpha)\mu^2$ and $b(\alpha) = (1+\alpha)\mu$, where μ is the largest value among μ_i.

Corollary 5. Let π_0 be a probability distribution supported by $S_0 \subset \mathbb{R}^n$ such that $\mathcal{D}_0 = \sup_{\boldsymbol{\theta},\boldsymbol{\theta}'\in S_0} \|\boldsymbol{\theta}-\boldsymbol{\theta}'\|_\infty < \infty$. Assume that $\boldsymbol{\xi}$ has independent entries ξ_i, $i = 1, \ldots, n$, drawn from the Laplace distribution $\mathsf{Lap}(\mu_i)$. Set $\mu = \max_{1\leqslant i\leqslant n}\mu_i$. Then, for every $\beta \geqslant 2\mu\mathcal{D}_0$, we have

$$\begin{aligned}
\mathbf{E}[\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|^2] \leqslant \inf_\pi \bigg\{ \int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\boldsymbol{\theta}^*\|^2\, \pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0)\bigg\} \\
+\Big(\frac{4\mu^2}{\beta-2\mu\mathcal{D}_0} - 1\Big)\mathbf{E}[\mathrm{Var}_{\widehat{\pi}}(\boldsymbol{\vartheta})]. \qquad (3.5)
\end{aligned}$$

In particular, if $\beta \geqslant 4\mu^2 + 2\mu\mathcal{D}_0$, the last term in (3.5) is nonpositive and, therefore, can be neglected.

The last claim improves on (9, Prop. 1), since the latter requires the condition $\beta \geqslant (16\mu^2) \vee (\sqrt{8}\,\mu\mathcal{D}_0)$.

Remark 1. *Let us finally remark that the construction of ζ_i's used in this section can be extended to the case where ξ_i's are scale-mixtures of Laplace distributions with a mixing density supported by a compact set. The only modification in the statement of the final result should be the definition of μ, which should correspond*

[2]This means that the density of ξ_i is equal to $(2\mu_i)^{-1}\exp(-|x|/\mu_i)$.

to the smallest real number such that the mixing density has no mass in (μ, ∞). Similar extension can be carried out in the case of scale-mixtures of Gaussians.

4. Proof of Theorem 1

Since $\widehat{\pi}$ minimizes the criterion $\pi \mapsto \int_{\mathbb{R}^n} \|\boldsymbol{Y}-\boldsymbol{\theta}\|^2\,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0)$, we have

$$\int_{\mathbb{R}^n} \|\boldsymbol{Y}-\boldsymbol{\theta}\|^2\,\widehat{\pi}(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\widehat{\pi}||\pi_0) \leqslant \int_{\mathbb{R}^n} \|\boldsymbol{Y}-\boldsymbol{\theta}\|^2\,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0)$$

for all densities π over $\mathbb{R}^n$. The KL-divergence being always nonnegative, we infer from the last display that

$$\begin{aligned}\|\boldsymbol{Y}-\widehat{\boldsymbol{\theta}}\|^2 &= \int_{\mathbb{R}^n} \|\boldsymbol{Y}-\boldsymbol{\theta}\|^2\,\widehat{\pi}(d\boldsymbol{\theta}) - \int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|^2\,\widehat{\pi}(d\boldsymbol{\theta}) \\ &\leqslant \int_{\mathbb{R}^n} \|\boldsymbol{Y}-\boldsymbol{\theta}\|^2\,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0) - \int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|^2\,\widehat{\pi}(d\boldsymbol{\theta}).\end{aligned} \tag{4.1}$$

Using the decompositions $\|\boldsymbol{Y}-\widehat{\boldsymbol{\theta}}\|^2 = \|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|^2 + 2(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)^\top\boldsymbol{\xi} + \|\boldsymbol{\xi}\|^2$ and $\|\boldsymbol{Y}-\boldsymbol{\theta}\|^2 = \|\boldsymbol{\theta}-\boldsymbol{\theta}^*\|^2 + 2(\boldsymbol{\theta}^*-\boldsymbol{\theta})^\top\boldsymbol{\xi} + \|\boldsymbol{\xi}\|^2$ and taking the expectation of the two sides of (4.1), we get

$$\begin{aligned}\mathbf{E}[\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|^2] + 2\mathbf{E}[(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)^\top\boldsymbol{\xi}] \leqslant \int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\boldsymbol{\theta}^*\|^2\,\pi(d\boldsymbol{\theta}) \\ +\beta D_{\mathrm{KL}}(\pi||\pi_0)\mathbf{E}\bigg[\int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|^2\,\widehat{\pi}(d\boldsymbol{\theta})\bigg]\end{aligned}$$

which can be equivalently written as

$$\begin{aligned}\mathbf{E}[\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|^2] \leqslant \int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\boldsymbol{\theta}^*\|^2\,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi||\pi_0) \\ +2\mathbf{E}[\widehat{\boldsymbol{\theta}}^\top\boldsymbol{\xi}] - \int_{\mathbb{R}^n} \mathbf{E}[\|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|^2\,\widehat{\pi}(\boldsymbol{\theta})]\,d\boldsymbol{\theta}.\end{aligned} \tag{4.2}$$

In addition, we have

$$2\mathbf{E}[\widehat{\boldsymbol{\theta}}^\top\boldsymbol{\xi}] = \frac{\beta}{\alpha}\mathbf{E}\bigg[\int_{\mathbb{R}^n} \log e^{2(\alpha/\beta)\boldsymbol{\theta}^\top\boldsymbol{\xi}}\widehat{\pi}(d\boldsymbol{\theta})\bigg],$$

where $\alpha > 0$ is an arbitrary number. Since the logarithm is concave, the Jensen inequality yields

$$
\begin{aligned}
2\mathbf{E}[\widehat{\boldsymbol{\theta}}^\top \boldsymbol{\xi}] &\leqslant \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} e^{2(\alpha/\beta)\boldsymbol{\theta}^\top\boldsymbol{\xi}}\widehat{\pi}(d\boldsymbol{\theta})\bigg)\bigg]\\
&= \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} e^{2(\alpha/\beta)\boldsymbol{\theta}^\top\boldsymbol{\xi}-\|\boldsymbol{\theta}^*+\boldsymbol{\xi}-\boldsymbol{\theta}\|^2/\beta}\,\pi_0(d\boldsymbol{\theta})\bigg)\\
&\quad - \log\bigg(\int_{\mathbb{R}^n} e^{-\|\boldsymbol{\theta}^*+\boldsymbol{\xi}-\boldsymbol{\theta}\|^2/\beta}\,\pi_0(d\boldsymbol{\theta})\bigg)\bigg]\\
&= \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} e^{(2(1+\alpha)\boldsymbol{\theta}^\top\boldsymbol{\xi}-\|\boldsymbol{\theta}^*-\boldsymbol{\theta}\|^2)/\beta}\,\pi_0(d\boldsymbol{\theta})\bigg)\\
&\quad - \log\bigg(\int_{\mathbb{R}^n} e^{(2\boldsymbol{\theta}^\top\boldsymbol{\xi}-\|\boldsymbol{\theta}^*-\boldsymbol{\theta}\|^2)/\beta}\,\pi_0(d\boldsymbol{\theta})\bigg)\bigg]
\end{aligned}
\tag{4.3}
$$

Let $\boldsymbol{\zeta} = \boldsymbol{\zeta}_\alpha$ be the n dimensional random vector the existence of which is required in the statement of the theorem. Recall that it satisfies

$$
\mathbf{E}[\boldsymbol{\zeta}|\mathcal{F}] = 0, \qquad \boldsymbol{\xi}+\boldsymbol{\zeta} \stackrel{\mathscr{D}}{=} (1+\alpha)\boldsymbol{\xi},
$$

These conditions imply that in the first expectation in (4.3), one can replace $(1+\alpha)\boldsymbol{\xi}$ by $\boldsymbol{\xi}+\boldsymbol{\zeta}$, which yields

$$
\begin{aligned}
2\mathbf{E}[\widehat{\boldsymbol{\theta}}^\top \boldsymbol{\xi}] &\leqslant \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} e^{(2\boldsymbol{\theta}^\top\boldsymbol{\xi}+2\boldsymbol{\theta}^\top\boldsymbol{\zeta}-\|\boldsymbol{\theta}^*-\boldsymbol{\theta}\|^2)/\beta}\,\pi_0(d\boldsymbol{\theta})\bigg)\bigg]\\
&\quad - \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} e^{(2\boldsymbol{\theta}^\top\boldsymbol{\xi}-\|\boldsymbol{\theta}^*-\boldsymbol{\theta}\|^2)/\beta}\,\pi_0(d\boldsymbol{\theta})\bigg)\bigg]\\
&= \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} e^{2\boldsymbol{\theta}^\top\boldsymbol{\zeta}/\beta}\,\widehat{\pi}(d\boldsymbol{\theta})\bigg)\bigg] = \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} e^{2(\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}})^\top\boldsymbol{\zeta}/\beta}\,\widehat{\pi}(d\boldsymbol{\theta})\bigg)\bigg].
\end{aligned}
\tag{4.4}
$$

Since conditionally to $\mathcal{F}$, ζ_i's are independent and each ζ_i satisfies the $(v(\alpha), b(\alpha))$-Bernstein condition, one can use the Jensen inequality to upper bound the expectation in (4.4) as follows

$$
\begin{aligned}
2\mathbf{E}[\widehat{\boldsymbol{\theta}}^\top \boldsymbol{\xi}] &\leqslant \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} \mathbf{E}[e^{2(\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}})^\top\boldsymbol{\zeta}/\beta}|\mathcal{F}]\,\widehat{\pi}(d\boldsymbol{\theta})\bigg)\bigg]\\
&\leqslant \frac{\beta}{\alpha}\mathbf{E}\bigg[\log\bigg(\int_{\mathbb{R}^n} \exp\bigg\{\frac{2\|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|^2 v(\alpha)}{\beta(\beta-2b(\alpha)\|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|_\infty)}\bigg\}\,\widehat{\pi}(d\boldsymbol{\theta})\bigg)\bigg]
\end{aligned}
\tag{4.5}
$$

for every β satisfying $\beta \geqslant 2b(\alpha)\|\boldsymbol{\theta}-\boldsymbol{\theta}'\|_\infty$ for every $\boldsymbol{\theta}, \boldsymbol{\theta}' \in S_0 := \mathrm{supp}(\pi_0)$. Note that for every $\boldsymbol{\theta} \in S_0$, we have $\|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|_\infty \leqslant \mathcal{D}_0$. The inequality in (4.5) being true for any $\alpha > 0$, one can check that

$$2\mathbf{E}[\widehat{\boldsymbol{\theta}}^\top \boldsymbol{\xi}] \leqslant \liminf_{\alpha\to 0} \frac{\beta}{\alpha}\mathbf{E}\biggl[\log\biggl(\int_{\mathbb{R}^n} \exp\Bigl\{\frac{2\|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|^2 v(\alpha)}{\beta(\beta-2b(\alpha)\|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|_\infty)}\Bigr\}\,\widehat{\pi}(d\boldsymbol{\theta})\biggr)\biggr]$$

$$= \mathbf{E}\biggl[\int_{\mathbb{R}^n} \frac{2\|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|^2 v'(0)}{\beta-2b(0)\|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|_\infty}\,\widehat{\pi}(d\boldsymbol{\theta})\biggr]$$

$$\leqslant \frac{2v'(0)}{\beta-2b(0)\mathcal{D}_\infty(S_0)}\mathbf{E}\biggl[\int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\widehat{\boldsymbol{\theta}}\|^2\,\widehat{\pi}(d\boldsymbol{\theta})\biggr]. \tag{4.6}$$

Combining (4.2) and (4.6), we see that

$$\mathbf{E}[\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*\|^2] \leqslant \int_{\mathbb{R}^n} \|\boldsymbol{\theta}-\boldsymbol{\theta}^*\|^2\,\pi(d\boldsymbol{\theta}) + \beta D_{\mathrm{KL}}(\pi\|\pi_0) + \\ +\biggl(\frac{2v'(0)}{\beta-2b(0)\mathcal{D}_\infty(S_0)} - 1\biggr)\mathbf{E}[\mathrm{Var}_{\widehat{\pi}}(\boldsymbol{\vartheta})].$$

This completes the proof.

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ON KHINCHIN'S THEOREM ABOUT THE SPECIAL ROLE OF THE GAUSSIAN DISTRIBUTION

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Abstract. The purpose of this note is to recall one remarkable theorem of Khinchin about the special role of the Gaussian distribution. This theorem allows us to give a new interpretation of the Lindeberg condition: it guarantees the uniform integrability of the squares of normed sums of random variables and, thus, the passage to the limit under the expectation sign. The latter provides a simple proof of the central limit theorem for independent random variables.

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1. Let $\{\xi_{n,j}\} = \{\xi_{n,j}, 1 \le j \le k_n, n \ge 1\}$, $k_n \to \infty$ as $n \to \infty$, be a triangular array (double sequence) of independent in each row random variables on a probability space $(X, \mathscr{B}, P)$. For the sake of simplicity, we always assume that $E\xi_{n,j} = 0$ for all j and n. For any $n \ge 1$, denote $S_n = \sum\limits_{j=1}^{k_n} \xi_{n,j}$, and let DS_n be its variance. The Gaussian (normal) distribution function with parameters a and σ^2, $a, \sigma \in \mathbb{R}$, $\sigma > 0$, is defined by

$$\Phi_{a,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int\limits_{-\infty}^{x} \exp\left\{-\frac{(t-a)^2}{2\sigma^2}\right\} dt, \qquad x \in \mathbb{R}.$$

Khinchin [4] (translation into English can be found in [6]) noted that the Gauss law, as a limiting law for sums of independent random variables, has a very special role that distinguishes it from all infinitely divisible laws. Namely, we arrive at the Gauss law in all cases when the limiting negligibility of the components of the sum of terms under study reaches a sufficiently strong degree; and this happens completely independently of the special properties of the laws of distribution of these terms.

The condition of asymptotic infinitesimality (or, equivalently, limiting negligibility) on the summands $\xi_{n,j}$, in the general case, is formulated as the condition that for any $\varepsilon > 0$, probability of the inequality $|\xi_{n,j}| \ge \varepsilon$ tends to zero uniformly in j as $n \to \infty$:

$$\max_{1 \le j \le k_n} P\left(|\xi_{n,j}| \ge \varepsilon\right) \to 0 \quad \text{as } n \to \infty. \tag{1}$$

Khinchin showed (Theorem 42 in [4]) that if we assume that not only this probability but the probability that all $|\xi_{n,j}|$, $1 \le j \le k_n$, are greater than ε tends to zero as $n \to \infty$, that is,

$$P\left(\max_{1\le j\le k_n} |\xi_{n,j}| \ge \varepsilon\right) \to 0 \quad \text{as } n \to \infty, \tag{2}$$

then the only possible limiting law for normed row sums is the Gauss law.

Theorem 1 (Khinchin)**.** *Let $\{\xi_{n,j}\}$ be a double sequence of independent in each row random variables. If a limiting non-degenerate distribution for the sums S_n exists, then for it to be Gaussian, it is necessary and sufficient that for any $\varepsilon > 0$, random variables $\{\xi_{n,j}\}$ satisfy* (2)*.*

Since condition (2) represents only a somewhat strengthened requirement (1) for the limiting negligibility of summands and does not contain any special assumptions about the nature of the laws of distribution of summands, the above result characterizes the Gauss law as, in a certain sense, a universal limiting law for sums of independent random variables and justifies the exclusive place given to this law in classical studies.

In the bibliographical notes [4], Khinchin mention, that a more general result was obtained by Lévy in [5], however, Khinchin was not able to find the proof of the latter based on the sketch suggested by Lévy. Khinchin's proof is based on the direct investigation of the characteristic functions of summands. Another (shorter) proof, based on the Lévy–Khinchin formula for the decomposition of characteristic functions, was suggested by Gnedenko [2]. The latter can be found in the book [3] by Gnedenko and Kolmogorov (see Theorem 1 on p. 126).

Under conditions of the Khinchin theorem, the limiting distribution (in the case of centered random summands) is Φ_{0,σ^2} with some parameter σ^2. Since the limiting law is non-degenerate, $\sigma^2 > 0$. Note that Khinchin do not impose any restriction on the second moments of the summands, which is a minimal condition on the moments in the central limit theorem (CLT). We say that for a sequence $\{\xi_{n,j}\}$ of (centered) random variables, the CLT holds if

$$\lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{DS_n}} \le x\right) = \Phi_{0,1}(x), \qquad x \in \mathbb{R}.$$

The Khinchin theorem cannot be considered a CLT since the limiting Gaussian distribution is not necessarily the standard one with $\sigma^2 = 1$. However, if we impose the uniform integrability condition on the squares of normed row sums, we will be able to prove CLT based on the Khinchin result.

We will need the following statements (see, for example, Lemma 1 on p. 322 and Theorem 5 on p. 189 in [7]). Under convergence of distribution functions we understand convergence in general, i.e., at each point of continuity of the limiting distribution function.

Proposition 1. *Let $\{F_n\} = \{F_n, n \geq 1\}$ be a sequence of distribution functions. Suppose that any convergent subsequence $\{F_{n'}\}$ of $\{F_n\}$, $\{n'\} \subset \{n\}$, converges to the same distribution function F. Then the sequence $\{F_n\}$ converges to F as well.*

Proof. Let $\mathscr{X}_F$ be the set of continuity points of the distribution function F. Fix some $x \in \mathscr{X}_F$ and assume that $F_n(x)$ does not converge to $F(x)$. Then there exists $\varepsilon > 0$ and an infinite sequence $\{n'\}$ of natural numbers such that

$$|F_{n'}(x) - F(x)| > \varepsilon. \tag{3}$$

By the Helly theorem, from the sequence $\{F_{n'}\}$, one can select a convergent subsequence $\{F_{n''}\}$, and let generalized distribution function G be its limit. By the hypothesis of the proposition, $G = F$, and thus, $F_{n''}(x) \to F(x)$ as $n \to \infty$, which contradicts with (3). This completes the proof. $\square$

We remind that a family of random variables $\{\eta_n, n \geq 1\}$ is uniformly integrable if

$$\sup_n \int\limits_{|\eta_n|>C} |\eta_n| dP \to 0 \qquad \text{as } C \to \infty.$$

Theorem 2. *Let η_n, $n \geq 1$ be a sequence of positive random variables with $E\eta_n < \infty$ such that $\eta_n \to \eta$ as $n \to \infty$. Then $E\eta_n \to E\eta < \infty$ as $n \to \infty$ if and only if the family $\{\eta_n, n \geq 1\}$ is uniformly integrable.*

Now we present the following version of the CLT for independent random variables.

Theorem 3. *Let $\{\xi_{n,j}\}$ be a double sequence of independent in each row random variables such that $E\xi_{n,j}^2 < \infty$, $1 \leq j \leq k_n$, $n \geq 1$. If random variables $\{\xi_{n,j}/\sqrt{DS_n}$, $1 \leq j \leq k_n$, $n \geq 1\}$ satisfy condition (2) and the squares of normed row sums $\{S_n^2/DS_n, n \geq 1\}$ are uniformly integrable, then for the sequence $\{\xi_{n,j}\}$, the CLT holds.*

Proof. Let F_n be the distribution function of $S_n/\sqrt{DS_n}$, $n \geq 1$. Then

$$\int\limits_{\mathbb{R}} x^2 dF_n(x) = E\left(\frac{S_n^2}{DS_n}\right) = 1, \qquad n \geq 1. \tag{4}$$

Further, let $\{F_{n'}\}$, $\{n'\} \subset \{n\}$ be some convergent subsequence of the sequence $\{F_n\}$. Due to the Khincin theorem, $F_{n'}(x) \to \Phi_{0,\sigma^2}(x)$ as $n' \to \infty$ for any $x \in \mathbb{R}$

and some $\sigma > 0$ if random variables $\{\xi_{n,j}/\sqrt{DS_n}\}$ satisfy condition (2). Since $\{S_n^2/DS_n, n \geq 1\}$ are uniformly integrable, due to Theorem 2, we can pass to the limit under the expectation sign, and thus,

$$\lim_{n' \to \infty} \int_{\mathbb{R}} x^2 dF_{n'}(x) = \int_{\mathbb{R}} x^2 \lim_{n' \to \infty} dF_{n'}(x) = \int_{\mathbb{R}} x^2 d\Phi_{0,\sigma^2}(x).$$

Tacking into account (4), we conclude

$$\int_{\mathbb{R}} x^2 d\Phi_{0,\sigma^2}(x) = 1,$$

that is, the parameter σ^2 in the limiting Gaussian distribution is equal to one. Thus, from the uniform integrability of $\{S_n^2/DS_n, n \geq 1\}$, it follows that $F_{n'}(x) \to \Phi_{0,1}(x)$ as $n' \to \infty$ for any $x \in \mathbb{R}$.

Thereby, any convergent subsequence $\{F_{n'}\}$ of the distribution functions of the normed row sums $S_n/\sqrt{DS_n}$ converge to the same limiting distribution $\Phi_{0,1}$. Hence, by Proposition 1, the sequence $\{F_n\}$ converges to $\Phi_{0,1}$ as well. Therefore, for random variables $\{\xi_{n,j}\}$, the CLT holds. □

2. Theorem 3 allows us to give the new probabilistic interpretation of the Lindeberg condition. We will show that from the Lindeberg condition, the uniform integrability of the squares of normed sums of random variables follows, which, in its turn, allows passage to the limit under the expectation sign, and thus, guaranties $\sigma^2 = 1$ in the limiting Gaussian distribution in the Khinchin theorem.

The double sequence $\{\xi_{n,j}\}$ of random variables satisfies the Lindeberg condition if for any $\varepsilon > 0$,

$$\frac{1}{DS_n} \sum_{j=1}^{k_n} \int_{\{|\xi_{n,j}| > \varepsilon\sqrt{DS_n}\}} \xi_{n,j}^2 dP \to 0 \qquad \text{as } n \to \infty. \tag{5}$$

The classical interpretation of the Lindeberg condition is that if a sequence $\{\xi_{n,j}\}$ of random variables satisfies (5), then its elements are asymptotically infinitesimal uniformly in each row, that is, relation (1) holds. Billingsley (see p. 90 in [1]) noted that from the Lindeberg condition, the uniform integrability of the squares of normed sums follows as well.

Proposition 2. *Let $\{\xi_{n,j}\}$ be a double sequence of (centered) independent random variables with finite second moments. If $\{\xi_{n,j}\}$ satisfies the Lindeberg condition* (5), *then squares of normed row sums $\{S_n/\sqrt{DS_n}, n \geq 1\}$, are uniformly integrable.*

Proof. The statement follows from inequality (12.20) in [1], according to which for any $n \geq 1$ and $C > 0$, one has

$$\int\limits_{\{S_n^2 \geq CDS_n\}} \frac{S_n^2}{DS_n} dP \leq K \left(\frac{1}{C} + \frac{1}{DS_n} \sum_{j=1}^{k_n} \int\limits_{\{|\xi_{n,j}| \geq \frac{1}{4} CDS_n\}} \xi_{n,j}^2 dP \right)$$

where K is some universal constant. By (5), for any $C > 0$, there exists $n_0 = n_0(C) > 1$ such that

$$\frac{1}{DS_n} \sum_{j=1}^{k_n} \int\limits_{\{|\xi_{n,j}| \geq \frac{1}{4} CDS_n\}} \leq \frac{1}{C} \qquad \text{for any } n > n_0.$$

Thus,

$$\sup_n \int\limits_{\{S_n^2 \geq CDS_n\}} \frac{S_n^2}{DS_n} dP \leq K \left(\frac{2}{C} + \sup_{1 \leq m \leq n_0(C)} \frac{1}{DS_m} \sum_{j=1}^{k_m} \int\limits_{\{|\xi_{m,j}| \geq \frac{1}{4} CDS_m\}} \xi_{m,j}^2 dP \right),$$

and hence,

$$\sup_n \int\limits_{\{S_n^2 \geq CDS_n\}} \frac{S_n^2}{DS_n} dP \to 0 \qquad \text{as } C \to \infty. \qquad \square$$

The statement above reveals the true essence of the Lindeberg condition. Since uniform integrability condition is the necessary and sufficient condition for taking limit under the expectation sign, we conclude that the Lindeberg condition is one of conditions under which the limiting Gaussian distribution in the Khinchin theorem is the standard one. Tacking into account this fact, we provide the new proof of the well-known Lévy-Lindeberg theorem.

Theorem 4 (Lévy-Lindeberg)**.** *Let $\{\xi_{n,j}\}$ be a double sequence of independent in each row random variables such that $E\xi_{n,j} = 0$, $0 < E\xi_{n,j}^2 < \infty$, $1 \leq j \leq k_n$, $n \geq 1$. If random variables $\{\xi_{n,j}\}$ satisfy Lindeberg condition* (5)*, then the CLT holds.*

Proof. First note that random variables $\{\xi_{n,j}/\sqrt{DS_n}, 1 \leq j \leq k_n, n \geq 1\}$ satisfy condition (2), since for any $\varepsilon > 0$, we have

$$P\left(\max_{1 \leq j \leq k_n} |\xi_{n,j}| \geq \varepsilon \sqrt{DS_n} \right) \leq \sum_{j=1}^{k_n} P(|\xi_{n,j}| \geq \varepsilon \sqrt{DS_n}) \leq$$

$$\leq \frac{1}{\varepsilon^2 DS_n} \sum_{j=1}^{k_n} \int\limits_{\{|\xi_{n,j}| > \varepsilon \sqrt{DS_n}\}} \xi_{n,j}^2 dP \to 0$$

as $n \to \infty$. Further, by Proposition 2, random variables $\{S_n^2/DS_n, n \geq 1\}$, are uniformly integrable. Thus, by Theorem 3, for $\{\xi_{n,j}\}$ the CLT holds. $\square$

3. Let us illustrate the application of Theorem 3 in the case of independent identically distributed (i.i.d.) random variables. Namely, we will use this theorem to prove the following classical result.

Theorem 5 (Lèvy–Khinchin)**.** *Let $\{\eta_n, n \geq 1\}$ be a sequence of i.i.d. random variables such that $E\eta_1 = 0$ and $D\eta_1 = \sigma_0^2 \leq \infty$. Then for $\{\eta_n\}$ the CLT holds.*

Proof. Consider the double array $\{\xi_{n,j}, 1 \leq j \leq n, n \geq 1\}$ of random variables $\xi_{n,j} = \dfrac{\eta_j}{\sigma_0\sqrt{n}}$. It is not difficult to check, that random variables $\{\xi_{n,j}\}$ satisfy condition (2). Further, put $\hat{S}_n = \sum\limits_{j=1}^{n} \xi_{n,j} = \dfrac{1}{\sigma_0\sqrt{n}} \sum\limits_{j=1}^{n} \eta_j$, then $D\hat{S}_n = 1$. With application of inequality (12.19) in [1], for any $C > 0$, we can write

$$
\begin{aligned}
P(\hat{S}_n^2 \geq C^2) \quad &\leq \quad \max_{1\leq j\leq n} P\left(\hat{S}_j^2 \geq C^2\right) \leq P\left(\max_{1\leq j\leq n} |\hat{S}_j| \geq C\right) \leq \\
&\leq \quad K\left(\frac{1}{C^4} + \frac{1}{C^2}\sum_{k=1}^{n}\int\limits_{\{|\xi_{n,j}|>\frac{1}{4}C\}} \xi_{n,j}^2 dP\right),
\end{aligned}
$$

where K is some positive constant. Further, applying equality (3) on p. 223 in [1], we can write

$$
\int\limits_{\hat{S}_n^2 \geq C} \hat{S}_n^2 dP = CP\left(\hat{S}_n^2 \geq C\right) + \int\limits_{C}^{\infty} P\left(\hat{S}_n^2 \geq t\right) dt \leq
$$

$$
\begin{aligned}
&\leq \quad C\left(\frac{K}{C^4} + \frac{K}{C^2\sigma_0^2}\int\limits_{\{|\eta_1|>C\sigma_0/4\}} \eta_1^2 dP\right) + \int\limits_{C}^{\infty}\left(\frac{K}{t^4} + \frac{K}{t^2\sigma_0^2}\int\limits_{\{|\eta_1|>t\sigma_0/4\}} \eta_1^2 dP\right) dt = \\
&= \quad K\left(\frac{1}{C^3} + \int\limits_{C}^{\infty}\frac{dt}{t^4} + \frac{1}{\sigma_0^2}\left(\frac{1}{C} + \int\limits_{C}^{\infty}\frac{dt}{t^2}\right)\int\limits_{\{|\eta_1|>t\sigma_0/4\}} \eta_1^2 dP\right).
\end{aligned}
$$

From here it follows that random variables $\{\hat{S}_n, n \geq 1\}$ are uniformly integrable. Hence, by Theorem 3, for random variables $\{\xi_{n,j}\}$ the CLT holds. It remains to note that $P\left(\hat{S}_n \leq x\right) = P\left(\dfrac{S_n}{\sqrt{DS_n}} \leq x\right)$, $x \in \mathbb{R}^d$. □

We see, that checking the uniform integrability of the squares of normmed sums directly requires some effort. At the same time, the Lindeberg condition for i.i.d. random variables can be checked quite simply. That is why it is preferable in applications.

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ENTIRE FUNCTIONS AND THEIR HIGH ORDER DIFFERENCE OPERATORS

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Abstract. In this paper, we prove that for a transcendental entire function f of finite order such that $\lambda(f-a)<\rho(f)$, where a is an entire function and satisfies $\rho(a)<\rho(f)$, $n\in\mathbb{N}$, if $\Delta_c^n f$ and f share the entire function b satisfying $\rho(b)<\rho(f)$ CM, where $c\in\mathbb{C}$ satisfies $\Delta_c^n f\not\equiv 0$, then $f(z)=a(z)+de^{cz}$, where d,c are two non-zero constants. In particular, if $a=b$, then a reduces to a constant. This result improves and generalizes the recent results of Chen and Chen [3], Liao and Zhang [10] and Lü et al. [11] in a large scale. Also we exhibit some relevant examples to fortify our results.

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1. Introduction and results

In this paper, a meromorphic function f always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with standard notation and main results of Nevanlinna Theory (see, e.g., [7, 12]). By $S(r,f)$ we denote any quantity that satisfies the condition $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside of an exceptional set of finite logarithmic measure. A meromorphic function a is said to be a small function of f if $T(r,a)=S(r,f)$. Moreover, we use notations $\rho(f)$, $\mu(f)$ and $\lambda(f)$ for the order, the lower order and the exponent of convergence of zeros of a meromorphic function f respectively. As usual, the abbreviation CM means "counting multiplicities", while IM means "ignoring multiplicities".

We now introduce some notations. Let $c\in\mathbb{C}\setminus\{0\}$. Then the forward difference $\Delta_c^n f$ for each integer $n\in\mathbb{N}$ is defined in the standard way by

$$\Delta_c^1 f(z)=\Delta_c f(z)=f(z+c)-f(z)$$

$$\Delta_c^n f(z)=\Delta_c\left(\Delta_c^{n-1}f(z)\right)=\Delta_c^{n-1}f(z+c)-\Delta_c^{n-1}f(z),\ \ n\geq 2.$$

Moreover

$$\Delta_c^n f(z)=\sum_{j=0}^{n}(-1)^{n-j}C_n^j f(z+jc),$$

where C_n^j is a combinatorial number.

In 1996, Brück [2] discussed the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [2].

Conjecture A. *Let f be a non-constant entire function such that*

$$\limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} \notin \mathbb{N}\cup\{\infty\}.$$

If f and f' share one finite value a CM, then $f'-a=c(f-a)$, where $c\in\mathbb{C}\setminus\{0\}$.

The conjecture for the special cases (1) $a=0$ and (2) $N\left(r,\frac{1}{f'}\right)=S(r,f)$ had been confirmed by Brück [2]. Though the conjecture is not settled in its full generality, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with its derivatives.

Meromorphic solutions of complex difference equations, and the value distribution and uniqueness of complex differences have become an area of current interest and the study is based on the Nevanlinna value distribution of difference operators established by Halburd and Korhonen [6] and by Chiang and Feng [5] respectively. Recently, many authors (see [3, 4, 10, 11]) have started to consider the sharing values problems of meromorphic functions with their difference operators or shifts. Also it is well known that $\Delta_c f$ can be regarded as the difference counterpart of f'. Now, we recall the following result due to Chen [4], which is difference analogue of the Brück conjecture.

Theorem A. [4] *Let f be a transcendental entire function of finite order which has a finite Borel exceptional value a and let $c(\in\mathbb{C})$ such that $\Delta_c f\not\equiv 0$. If $\Delta_c f(z)$ and $f(z)$ share $b(b\neq a)$ CM then,*

$$\frac{\Delta_c f(z)-b}{f(z)-b}=A,$$

where $A=\frac{b}{b-a}$ is a non-zero constant.

In 2014, Cheng and Cheng [3] further improved Theorem A with the idea of sharing small function and obtained the following result.

Theorem B. [3] *Let f be a transcendental entire function of finite order and a be an entire function such that $\rho(a)<1$ and $\lambda(f-a)<\rho(f)$. Let $n\in\mathbb{N}$ and $c\in\mathbb{C}$ such that $\Delta_c^n f\not\equiv 0$ and b be an entire function such that $b\not\equiv a$ and $\rho(b)<1$. If $\Delta_c^n f$ and f share b CM, then*

$$f(z)=a(z)+de^{cz},$$

where d,c are two non-zero constants.

In 2016, Liao and Zhang [10] improved Theorem B from the case of $\rho(b) < 1$ to the general case of small function such that $\rho(b) < \rho(f)$ and obtained the following result.

Theorem C. [10] *Let f be a transcendental entire function of finite order and a be a small function of f such that $\rho(a) < 1$. Let $n \in \mathbb{N}$ such that $\Delta^n f \not\equiv 0$ and b be an entire function such that $b \not\equiv a$ and $\rho(b) < \rho(f)$. If $\Delta^n f$ and f share b CM, then*
$$\frac{\Delta^n f - b}{f - b} = \frac{b - \Delta^n a}{b - a}.$$
Furthermore f is of the form $f(z) = a(z) + ce^{\beta z}$, where c and β are two non-zero constants such that $\frac{b-\Delta^n a}{b-a} = (e^{\beta} - 1)^n$.

In 2019, Lü et al. [11] asked the following questions.
Question A: Can the condition $\rho(b) < 1$ be weakened in Theorem C.
Question B: Does there exist a joint theorem involve of both cases $a \equiv b$ and $a \not\equiv b$?

In the same paper, Lü et al. [11] gave affirmative answers of Questions A and B by proving the following result.

Theorem D. [11] *Let f be a transcendental entire function of finite order and a be an entire function such that $\lambda(f - a) < \rho(f)$, $\rho(a) < 1$ and $\rho(a) \neq \rho(f)$. Let $n \in \mathbb{N}$ such that $\Delta^n f \not\equiv 0$ and b be an entire function such that $\rho(b) < \max\{1, \rho(f)\}$. If $\Delta^n f$ and f share b CM, then*
$$f(z) = a(z) + ce^{\beta z},$$
where c and β are two non-zero constants. In particular, if $a \equiv b$, then a reduces to a constant.

In the same paper, Lü et al. [11] exhibited the following example to show that the condition $\rho(a) \neq \rho(f)$ is necessary in Theorem D.

Example 1.1. *Let f be a transcendental entire function with $0 < \rho(f) < 1$, $a(z) = f(z) - z$ and $b(z) = 3f(z) - f(z + 1)$. Clearly $\lambda(f - a) = 0 < \rho(f)$, $\rho(b) < 1$ and*
$$\frac{\Delta f - b}{f - b} = 2.$$
Therefore f and Δf share b CM, but f does not satisfies the conclusion of Theorem D.

In the paper, we prove the following main theorem, which extends Theorem D from the case of $\lambda(f - a) < \rho(f)$, $\rho(a) < 1$ and $\rho(a) \neq \rho(f)$ to the general case of entire function such that $\lambda(f - a) < \rho(f)$, $\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$.

Theorem 1.1. *Let f be a transcendental entire function of finite order and a be an entire function such that $\lambda(f-a) < \rho(f)$, $\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $\Delta_c^n f \not\equiv 0$ and b be an entire function such that $\rho(b) < \max\{1, \rho(f)\}$. If $\Delta_c^n f$ and f share b CM, then one of the following cases holds*

(1) *$a = b \in \mathbb{C}$ and $f(z) = a + de^{cz}$, where c and d are two non-zero constants,*
(2) *$a \not\equiv b$ and $f(z) = a(z) + de^{cz}$, where c and d are two non-zero constants.*

Immediately we have the following corollaries.

Corollary 1.1. *Let f be a transcendental entire function such $\rho(f) \geq 1$ and a be an entire function such that $\lambda(f-a) < \rho(f)$ and $\rho(a) < \rho(f)$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $\Delta_c^n f \not\equiv 0$ and b be an entire function such that $\rho(b) < \rho(f)$. If $\Delta_c^n f$ and f share b CM, then one of the following cases holds*

(1) *$a = b \in \mathbb{C}$ and $f(z) = a + de^{cz}$, where c and d are two non-zero constants,*
(2) *$a \not\equiv b$ and $f(z) = a(z) + de^{cz}$, where c and d are two non-zero constants.*

Corollary 1.2. *Let f be a transcendental entire function of finite order and a be an entire function such that $\lambda(f-a) < \rho(f)$, $\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $\Delta_c^n f \not\equiv 0$. If $\Delta_c^n f$ and f share a CM, then a reduces to a constant and $f(z) = a + de^{cz}$, where c and d are two non-zero constants.*

The Corollary 1.2 shows that if a nonzero polynomial a satisfies $\lambda(f-a) < \rho(f)$, then a is not shared CM by $\Delta_c^n f$ and f. For example if we take $f(z) = e^z + z$ and $a(z) = z$, then for any $c \neq 2k\pi i$, $k \in \mathbb{Z}$, we have $\Delta_c f(z) = (e^c - 1)e^z + c$. Hence a is not shared CM by $\Delta_c f$ and f.
This example shows existence of functions which satisfy the conditions of Theorem 1.1.

Example 1.2. *Let $f(z) = e^z$ and $c = \log 2$. Let $a = 0$ and $b \in \mathbb{C} \setminus \{0\}$. Clearly $\lambda(f-a) = 0 < \rho(f)$. Note that*

$$\begin{aligned} \Delta_c^n f(z) &= \sum_{j=0}^{n} (-1)^j C_n^j f(z + (n-j)c) = e^z \sum_{j=0}^{n} (-1)^j C_n^j e^{(n-j)c} \\ &= \left(e^{nc} - C_n^1 e^{(n-1)c} + \ldots + (-1)^n\right) e^z = (e^c - 1)^n e^z = e^z. \end{aligned}$$

Therefore $\Delta_c^n f \equiv f$ and so f and $\Delta_c^n f$ share $b \in \mathbb{C}$ CM.

Following examples show that the condition "$\lambda(f-a) < \rho(f)$" in Theorem 1.1 is sharp.

Example 1.3. *Let* $f(z) = Ae^{z\log(c+1)} - \frac{1-c}{c}$, *where* $c \in \mathbb{R}\setminus\{0\}, c > -1$ *and* A *is an arbitrary constant. Let* $a \in \mathbb{C}\setminus\{0\}$ *such that* $a \neq -\frac{1-c}{c}$ *and* $\frac{1-c}{c} + a = A$. *It is easy to verify that* $\lambda(f-a) = \rho(f)$ *and* $(\Delta_1 f(z) - 1) = c(f(z)-1)$. *Therefore* $\Delta_1 f$ *and* f *share* 1 *CM, but* f *does not satisfy any case of Theorem 1.1.*

Example 1.4. *Let* $f(z) = e^z + 3$, $a = 4$ *and* $c = \pi i$. *Clearly* $\lambda(f-4) = \rho(f) = 1$. *Note that* $\Delta_c f(z) = -2e^z$ *and* $\Delta_c f(z) - 2 = -2(f(z)-2)$. *Therefore* $\Delta_c f$ *and* f *share* 2 *CM, but* f *does not satisfy any case of Theorem 1.1.*

It is easy to see that the conditions "$\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$" in Theorem 1.1 is sharp.

Example 1.5. *Let* $f(z) = e^z$, $a(z) = e^z - 1$ *and* $c = \log 2$. *Note that* $\rho(a) = \rho(f)$ *and* $\Delta_c f(z) = e^z$. *Clearly* $\lambda(f-a) = 0 < \rho(f)$ *and* f *and* $\Delta_c f$ *share* $b(\in \mathbb{C})$ *CM, but but* f *does not satisfy any case of Theorem 1.1.*

It is easy to see that the condition "$\rho(b) < \max\{1, \rho(f)\}$" in Theorem 1.1 is sharp.

Example 1.6. *Let* $f(z) = ze^z$, $a = 0$, $b(z) = (z+c)e^z$ *and* $c = \log 2$. *Note that* $\rho(b) = \rho(f)$ *and* $\Delta_c f(z) = ze^z + 2ce^z$. *Clearly* $\lambda(f) = 0 < \rho(f)$ *and* f *and* $\Delta_c f$ *share* b *CM, but* f *does not satisfy any case of Theorem 1.1.*

Following example shows that the condition "$\lambda(f-a) < \rho(f)$" in Corollary 1.2 is sharp.

Example 1.7. *Let* $f(z) = (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c}z\right)$, *where* $\log$ *denotes the principal branch of the logarithm and* $c = 2\pi i$ *such that* $\log(1+\tau) \neq c$. *Let* $a = 0$. *Note that*

$$\begin{aligned}
\Delta_c f(z) &= (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c}(z+c)\right) - (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c}z\right) \\
&= (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c}z\right)(\exp(\log(1+\tau)) - 1) \\
&= \tau(\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c}z\right) = \tau f(z).
\end{aligned}$$

Clearly f *and* $\Delta_c f$ *share* 0 *CM. On the other hand, we see that* $\rho(f) \leq 1$ *and* $\lambda(f) = \lambda(\exp z - 1) = 1$. *Since* $\lambda(f) \leq \rho(f)$, *it follows that* $\lambda(f) = \rho(f)$. *Also it is clear that* f *does not satisfy any case of Corollary 1.2.*

Following examples show that the condition "$\rho(f) < +\infty$" in Theorem 1.1 and Corollary 1.2 is necessary.

Example 1.8. *Let* $f(z) = e^z\left(e^{s(z)} - 1\right)$*, where* $s(z)$ *is a periodic function with period* $c = \log 2$ *and* $a(z) = -e^z$*. Clearly* $\rho(f) = +\infty$*. Note that* $\Delta_c f = f$ *and so* f *and* $\Delta_c f$ *share* $b(\in \mathbb{C})$ *CM. On the other hand, we see that* $\lambda(f - a) = 0 < \rho(f)$*, but* f *does not satisfy any case of Theorem 1.1.*

Example 1.9. *Let* $f(z) = e^z e^{s(z)}$*, where* $s(z)$ *is a periodic function with period* $c = \log 2$*. Clearly* $\rho(f) = +\infty$*. Note that* $\Delta_c f = f$ *and so* f *and* $\Delta_c f$ *share* 0 *CM. On the other hand, we see that* $\lambda(f) = 0 < \rho(f)$*, but* f *does not satisfy any case of Corollary 1.2.*

Following example assert that Theorem 1.1 does not valid when f is a transcendental meromorphic function.

Example 1.10. *Let* g *be a periodic entire function with period* 1 *such that* $\lambda(g) < \rho(g) = 1$ *and* $g(z)$ *and* $\sin 2\pi z$ *have no common zeros. Let* $a = 0$ *and*

$$f(z) = \frac{g(z)}{\sin 2\pi z} e^{z \log 2}.$$

Clearly $\Delta_1 f$ *and* f *share* 1 *CM, but* f *does not satisfy any case of Theorem 1.1.*

2. Auxiliary lemmas

Lemma 2.1. *[*[12]*, Theorem 1.18] Let* f *and* g *be two non-constant meromorphic functions in the complex plane such that* $\rho(f) < \mu(g)$*. Then* $T(r, f) = o(T(r, g)\ \ (r \to \infty)$*.*

Lemma 2.2. *[*[12]*, Theorem 1.44] Let* g *be a non-constant polynomial and* $f = e^g$*. Then* $\rho(f) = \mu(f) = \deg(g)$*.*

Lemma 2.3. *(*[8]*, Lemma 1.3.1.) Let* $P(z) = \sum_{i=1}^{n} a_i z^i$ *where* $a_n \neq 0$*. Then* $\forall$ $\varepsilon > 0$*, there exists* $r_0 > 0$ *such that* $\forall\ r = |z| > r_0$ *the inequalities* $(1 - \varepsilon)|a_n| r^n \leq |P(z)| \leq (1 + \varepsilon)|a_n| r^n$ *hold.*

Lemma 2.4. [12] *Suppose that* $f_1, f_2, \ldots, f_n$ $(n \geq 2)$ *are meromorphic functions and* $g_1, g_2, \ldots, g_n$ *are entire functions satisfying the following conditions*

(i) $\sum\limits_{j=1}^{n} f_j e^{g_j} = 0$

(ii) $g_i - g_j$ *are non-constants for* $1 \leq i < j \leq n$*;*

(iii) $T(r, f_j) = o\left(T(r, e^{g_h - g_k})\right)$ $(r \to \infty, r \notin E)$ *for* $1 \leq j \leq n$*,* $1 \leq h < k \leq n$*.*

Then $f_j \equiv 0$ *for* $j = 1, 2, \ldots, n$*.*

Lemma 2.5. [5] *Let f be a meromorphic function of finite order ρ and let $c_1,\ c_2 \in \mathbb{C}$ such that $c_1 \neq c_2$. Then for any $\varepsilon > 0$, we have*
$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

Lemma 2.6. [9] *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*
$$U(z,f)P(z,f) = Q(z,f),$$
where $U(z,f), P(z,f), Q(z,f)$ are difference polynomials such that the total degree $\deg(U(z,f)) = n$ in $f(z)$ and its shifts and $\deg(Q(z,f)) \leq n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then for each $\varepsilon > 0$,
$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f)$$
possible outside of an exceptional set of finite logarithmic measure.

Remark 2.1. *From the proof of Lemma 2.6 in* [9]*, we can see that if the coefficients of $U(z,f), P(z,f), Q(z,f)$, namely $a_\lambda(z)$ satisfy $m(r,a_\lambda) = S(r,f)$, then the same conclusion still holds.*

Lemma 2.7. [5] *Let f be a meromorphic function with a finite order ρ, $\eta \in \mathbb{C}\setminus\{0\}$. Let $\varepsilon > 0$ be given. Then there exists a sub set $E \subset (1,\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0,1]$, we have*
$$\exp\left(-r^{\rho-1+\varepsilon}\right) \leq \left|\frac{f(z+c)}{f(z)}\right| \leq \exp\left(r^{\rho-1+\varepsilon}\right).$$

Lemma 2.8. [1] *Let g be a transcendental function of order less than 1 and h be a positive constant. Then there exists an ε set E such that*
$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \quad \frac{g(z+\eta)}{g(z)} \to 1 \ \text{ as } z \to \infty \text{ in } \mathbb{C}\setminus E$$
uniformly in η for $|\eta| \leq h$ Further, the set E may be chosen so that for large $|z| \notin E$, the function g has no zeroes or poles in $|z-\zeta| \leq h$.

Lemma 2.9. *Let f be a transcendental entire function of finite order such that $\rho(f) > 1$ and a be an entire function such that $\lambda(f-a) < \rho(f)$ and $\rho(a) < \rho(f)$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $\Delta_c^n f \not\equiv 0$ and b be an entire function such that $\rho(b) < \rho(f)$. Suppose that f is a solution of the difference equation*
$$\Delta_c^n f - b = (f-b)e^Q,$$
where Q is a polynomial. Then $\deg(Q) = \rho(f) - 1$.

Proof. Proof of the lemma follows directly from the proof of Corollary 2.2. [11].

3. Proof of the theorem

Proof of Theorem 1.1. By the given conditions, we have $\lambda(f-a) < \rho(f)$. Then there exist an entire function $H(\not\equiv 0)$ and a polynomial P such that

$$f = a + He^P, \tag{3.1}$$

where $\lambda(H) = \rho(H) < \rho(f-a)$ and $\deg(P) = \rho(f-a)$.

First we suppose $\rho(f) < 1$. Since $\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$, it follows that $\rho(a) < 1$ and so $\rho(f-a) = \max\{\rho(a), \rho(f)\} < 1$. Consequently

$$\lambda(f-a) < \rho(f) \leq \max\{\rho(a), \rho(f)\} = \rho(f-a).$$

Note that 0 and ∞ are the Borel exceptional values of $f-a$. Then $f-a$ is a function of regular growth and so $\rho(f-a) \in \mathbb{N}$. Therefore we arrive at a contradiction.

Next we suppose $\rho(f) \geq 1$. In this case, the given conditions $\rho(a) < \max\{1, \rho(f)\}$, $\rho(a) \neq \rho(f)$ and $\rho(b) < \max\{1, \rho(f)\}$ reduce to $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$.

Since $\rho(a) < \rho(f)$, it follows that $\rho(H) < \rho(f)$ and $\deg(P) = \rho(f)$. Let

$$P(z) = a_s z^s + a_{s-1} z^{s-1} + \cdots + a_0, \tag{3.2}$$

where $a_s(\neq 0), a_{s-1}, a_{s-2}, \ldots, a_0 \in \mathbb{C}$ and $s \in \mathbb{N}$. Therefore $\rho(f) = \deg(P) = s$. Also from (3.1), we deduce that

$$\Delta_c^n f = \Delta_c^n a + H_n e^P, \tag{3.3}$$

where

$$H_n(z) = \sum_{j=0}^{n} c_j H(z+jc) e^{P(z+jc)-P(z)}, \quad \text{where} \quad c_j = (-1)^{n-j} C_n^j. \tag{3.4}$$

Since $\rho(H) < \rho(f)$, we have $\rho(H(z+ic)) < \rho(f)$ for $i = 0, 1, \ldots, n$. Note that $\deg\left(P(z+ic) - P(z)\right) \leq s-1 = \rho(f)-1$. Then from (3.4), we deduce that $\rho(H_n) < \rho(f)$. Also we see that $\rho(\Delta_c^n a) \leq \rho(a)$.

Since f and $\Delta_c^n f$ share b CM, then there exists a polynomial function Q such that

$$\Delta_c^n f - b = (f-b)e^Q. \tag{3.5}$$

Then from (3.3) and (3.5) we have

$$(\Delta_c^n a - b) - (a-b)e^Q = (He^Q - H_n)e^P. \tag{3.6}$$

Again from (3.5), we deduce that $\deg(Q) = \rho(e^Q) \leq \rho(f)$.

Now we divide the following two cases.

Case 1. Suppose $\rho(f) < 2$. Since $\deg(P) = \rho(f)$, it follows that $\deg(P) < 2$ and so $\deg(P) = 1$. Consequently $\rho(f) = 1$. Therefore by the given conditions, we see that $\lambda(f-a) < 1$, $\rho(a) < 1$, $\rho(a) \neq 1$ and $\rho(b) < 1$.

Now we divide the following two sub-cases.

Sub-case 1.1. Suppose $\deg(Q) = 0$. Let $e^Q = d$. Then from (3.6), we have

$$(\Delta_c^n a - b) - d(a - b) = (dH - H_n)e^P. \tag{3.7}$$

Now from Lemma 2.2, we deduce that $\rho\left((\Delta_c^n a - b) - d(a - b)\right) < \rho(f) = \deg(P) = \rho\left(e^P\right) = \mu\left(e^P\right)$ and $\rho\left(dH - H_n\right) < \rho(f) = \rho\left(e^P\right) = \mu\left(e^P\right)$. Then from Lemma 2.1, we conclude that $T\left(r, (\Delta_c^n a - b) - d(a - b)\right) = S\left(r, e^P\right)$ and $T\left(r, dH - H_n\right) = S\left(r, e^P\right)$. Now from Lemma 2.4 and (3.7), we deduce that

$$\Delta_c^n a - b \equiv d(a - b) \;\text{ and }\; dH \equiv H_n. \tag{3.8}$$

If $a \equiv b$, then from (3.8), we deduce that $\Delta_c^n a \equiv a$.

Now if a is a transcendental entire function with order less than 1, then by Lemma 2.8, we get

$$\begin{aligned} 1 = \frac{\Delta_c^n a(z)}{a(z)} &= \sum_{j=0}^{n} (-1)^{n-j} C_n^j \frac{a(z + jc)}{a(z)} \to \\ \sum_{j=0}^{n} (-1)^{n-j} C_n^j &= (1 - 1)^n = 0 \end{aligned}$$

as $z \to \infty$ possibly outside a ε set E, which is impossible.

If a is a non-constant polynomial, then $\deg(\Delta_c^n a) < \deg(a)$ and so

$$\deg(a) = \deg(\Delta_c^n a - a) = 0,$$

which is also impossible. Hence a is a constant and then $a = \Delta_c^n a = 0$. Therefore if $a \equiv b$, then $a = b = 0$. Now following Sub-case 1 in the proof of Theorem 4.1 in [11], one can easily conclude that

$$f(z) = a(z) + ce^{\beta z},$$

where c and β are two non-zero constants. In particular, if $a \equiv b$, then $a = b = 0$.

Sub-case 1.2. Suppose $\deg(Q) = 1$. In this case, from Sub-case 2 in the proof of Theorem 4.1 in [11], one can easily conclude that $a = b \in \mathbb{C} \setminus \{0\}$ and

$$f(z) = a + ce^{\beta z},$$

where c and β are two non-zero constants.

Case 2. Suppose $\rho(f) \geq 2$.

Then from Lemma 2.9, we deduce that $\deg(Q) = \rho(f) - 1$. Since $\rho(f) \geq 2$, it follows that $\deg(Q) \geq 1$. Now from Lemma 2.5, we have

$$m\left(r, \frac{H(z + jc)}{H(z)}\right) = O\left(r^{\rho(H) - 1 + \varepsilon}\right), \tag{3.9}$$

where $\varepsilon > 0$ is arbitrary. Since $\rho(H) < \rho(e^P)$, we choose $\varepsilon > 0$ such that $\rho(H) - 1 + 2\varepsilon < \rho(e^P) - 1$. Let

$$b_{n-j}(z) = c_j \frac{H(z+jc)}{H(z)} e^{P_j(z)}, \tag{3.10}$$

for $j = 0, 1, 2, \ldots, n$ and

$$F_n(h) = \sum_{j=0}^{n} b_{n-j} h^j. \tag{3.11}$$

We claim that $H_n - He^Q \equiv 0$. If not, suppose $H_n - He^Q \not\equiv 0$. Then we see that the order of the left side of (3.6) is less than $\rho(f)$, but the order of the right side of (3.6) is equal to $\rho(f)$. This is a contradiction. Hence $H_n - He^Q \equiv 0$. Then from (3.4), (3.10) and (3.11), we have

$$F_n(h) = \sum_{j=0}^{n} b_{n-j} h^j = e^Q. \tag{3.12}$$

Let

$$Q(z) = d_{s-1}z^{s-1} + d_{s-2}z^{s-2} + \cdots + d_0. \tag{3.13}$$

Now from (3.4) and (3.12) , we have

$$\sum_{j=1}^{n} c_j \frac{H(z+jc)}{H(z)} e^{R_j(z)} + (-1)^n - e^{Q(z)} = 0, \tag{3.14}$$

where $R_j(z) = P(z+jc) - P(z)$ $(j = 1, \ldots, n)$. Then from (3.2), we may assume that

$$R_j(z) = jsa_s c z^{s-1} + P_{s-2,j}(z), \tag{3.15}$$

where $\deg(P_{s-2,j}) \leq s-2$. Clearly $\deg(R_j) = s-1$ for $j = 1, 2, \ldots, n$.

Now we divide the following two sub-cases.

Sub-case 2.1. Suppose $n = 1$. Then from (3.14), we have

$$c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)} - 1 \equiv e^{Q(z)}. \tag{3.16}$$

Clearly (3.16) shows that $\frac{H(z+c)}{H(z)}$ is entire. Then from (3.9), we deduce that

$$T\left(r, \frac{H(z+c)}{H(z)}\right) = m\left(r, \frac{H(z+c)}{H(z)}\right) = O\left(r^{\rho(H)-1+\varepsilon}\right)$$

and so

$$\rho\left(\frac{H(z+c)}{H(z)}\right) = \rho(H) - 1 < \rho(f) - 1 = s - 1 = \rho(e^{R_1}).$$

Therefore it is easy to conclude that 0 is a Borel exceptional value of the entire function $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)}$. Consequently 1 is not a Borel exceptional of $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)}$ and so $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)} - 1$ must have infinitely many zeros. Therefore we arrive at a contradiction from (3.16).

Sub-case 2.2. Suppose $n \geq 2$. Then from (3.13) and (3.15), we see that

$$R_j(z) - Q(z) = (jsa_sc - d_{s-1})z^{s-1} + \ldots,$$

where $j = 1, 2, \ldots, n$.

Now we divide following two sub-cases:

Sub-case 2.2.1. Suppose there exists $j_0 (1 \leq j_0 \leq n)$ such that $j_0 sa_sc = d_{s-1}$. Therefore $\deg(R_{j_0} - Q) \leq s - 2$. In this case from (3.14), we have

$$\left(\sum_{\substack{1 \leq j \leq n \\ j \neq j_0}} c_j \frac{H(z+cj)}{H(z)} e^{P(z+jc)-P(z+c)} + B_{j_0} e^{P(z+j_0c)-P(z+c)} \right) e^{R_1(z)} = \tag{3.17}$$
$$= (-1)^{n+1},$$

where

$$B_{j_0}(z) = c_{j_0} \frac{H(z+j_0c)}{H(z)} - e^{Q(z)-R_{j_0}(z)}. \tag{3.18}$$

Let $Q_1(z) = e^{R_1(z)}$. Note that

$$Q_1(z+(j-1)c) \ldots Q_1(z+c) = e^{\left(\sum\limits_{i=2}^{j} P(z+ic)-P(z+(i-1)c) \right)} = e^{P(z+jc)-P(z+c)}$$

for $j = 2, 3, \ldots, n$.

Then (3.17) can be written as

$$U(z, Q_1(z))Q_1(z) = (-1)^{n+1}, \tag{3.19}$$

where

$$\begin{aligned} U(z, Q_1(z)) \quad = \quad & \sum_{\substack{1 \leq j \leq n \\ j \neq j_0}} c_j \frac{H(z+jc)}{H(z)} Q_1(z+(j-1)c)Q_1(z+(j-2)c) \cdots Q_1(z+c) \\ & + B_{j_0}(z) Q_1(z+(j_0-1)c)Q_1(z+(j_0-2)c) \cdots Q_1(z+c) \end{aligned}$$

if $j_0 \geq 2$ and

$$\begin{aligned} U(z, Q_1(z)) \quad = \quad & \sum_{2 \leq j \leq n} c_j \frac{H(z+jc)}{H(z)} Q_1(z+(j-1)c)Q_1(z+(j-2)c) \cdots Q_1(z+c) \\ & + B_{j_0}(z) \end{aligned}$$

if $j_0 = 1$.

From (3.19), it is clear that $U(z, Q_1) \not\equiv 0$ and $\deg(U(z, Q_1)) = n - 1 \geq 1$. Now we want to prove that if a_λ is a coefficient of $U(z, Q_1)$, then $m(r, a_\lambda) = S(r, Q_1)$. Note that from Lemma 2.2, we have

$$\mu(e^{R_1}) = \rho(e^{R_1}) = \deg(R_1) = s - 1$$

and

$$\rho(e^{Q-R_{j_0}}) = \deg(Q - R_{j_0}) \leq s - 2 < s - 1 = \mu(e^{R_1}).$$

Then by Lemma 2.1, we deduce that

$$T(r, e^{Q-R_{j_0}}) = S(r, e^{R_1}) = S(r, Q_1). \tag{3.20}$$

Also it is easy to prove from (3.9) that

$$m\left(r, \frac{H(z+jc)}{H(z)}\right) = S(r, e^{R_1}) = S(r, Q_1)\ (j = 1, 2, \ldots, n). \tag{3.21}$$

Now from (3.18), (3.20) and (3.21), we see that

$$m(r, B_{j_0}(z)) \leq m\left(r, \frac{H(z+j_0c)}{H(z)}\right) + m\left(r, e^{Q(z)-R_{j_0}(z)}\right) \leq S(r, Q_1).$$

Then in view of Remark 2.1 and using Lemma 2.6, we conclude that

$$m(r, Q_1) = S(r, Q_1).$$

Therefore $T(r, Q_1) = m(r, Q_1) = S(r, Q_1)$, which is a contradiction.

Sub-case 2.2.2. Suppose $jsa_sc \neq d_{s-1}$ for $1 \leq j \leq n$. In this case (3.14) can be rewrite as

$$e^{Q(z)} = e^{d_{s-1}z^{s-1}} e^{\tilde{P}_{s-2}(z)} = \sum_{j=0}^{n} c_j \frac{H(z+jc)}{H(z)} e^{R_j(z)}, \tag{3.22}$$

where

$$\tilde{P}_{s-2}(z) = Q(z) - d_{s-1}z^{s-1} = d_{s-2}z^{s-2} + d_{s-3}z^{s-3} + \cdots + d_0. \tag{3.23}$$

Again from (3.15) and (3.22), we have

$$e^{Q(z)} = e^{d_{s-1}z^{s-1}} e^{\tilde{P}_{s-2}(z)} = \sum_{j=1}^{n} c_j \frac{H(z+jc)}{H(z)} e^{jsa_scz^{s-1}} e^{P_{s-2,j}(z)} + (-1)^n. \tag{3.24}$$

Note that

$$ns|a_sc| > (n-1)s|ca_s| > \cdots > s|a_sc|$$

and either $|d_{s-1}| \in \{js|a_sc| : j = 1, 2, \ldots, n\}$ or $|d_{s-1}| \notin \{js|a_sc| : j = 1, 2, \ldots, n\}$. Therefore if we compare $|d_{s-1}|$ with $ns|a_sc|$, $(n-1)s|a_sc|$, $\cdots$, $s|a_sc|$, then it is enough to compare $|d_{s-1}|$ with $ns|a_sc|$. Without loss of generality, we suppose that $ns|a_sc| \leq |d_{s-1}|$.

Let $\arg d_{s-1} = \theta_1$ and $\arg(a_sc) = \theta_2$. Take θ_0 such that $\cos((s-1)\theta_0 + \theta_1) = 1$. Then using Lemma 2.7, we see that for any given ε $(0 < \varepsilon < s - \rho(H))$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure such that for all $z = re^{i\theta_0}$ satisfying $|z| = r \notin [0, 1] \cup E$ we have

$$\exp\left(-r^{\rho(H)-1+\varepsilon}\right) \leq \left|\frac{H(z+jc)}{H(z)}\right| \leq \exp\left(r^{\rho(H)-1+\varepsilon}\right)\ (j = 1, 2, \ldots, n). \tag{3.25}$$

Note that

$$
\begin{aligned}
(3.26)\qquad & \left|\exp\left(d_{s-1}z^{s-1}\right)\right| \\
= & \left|\exp\left(|d_{s-1}|r^{s-1}\left(\cos((s-1)\theta_0+\theta_1)\right)+i\sin((s-1)\theta_0+\theta_1)\right)\right| \\
= & \exp\left(|d_{s-1}|r^{s-1}\right).
\end{aligned}
$$

Similarly we can show that

$$
(3.27)\quad \left|\exp\left(jsa_scz^{s-1}\right)\right| = \exp\left(js|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2)\right),\ j=1,2,\ldots,n.
$$

Using Lemma 2.3 (taking $\varepsilon=\frac{1}{2}$), we deduce from (3.23) that $\left|\tilde{P}_{s-2}(z)\right| \geq \frac{|d_{s-2}|}{2}r^{s-2}$ and so

$$
(3.28)\qquad \left|\exp\left(\tilde{P}_{s-2}(z)\right)\right| \geq \exp\left(\frac{|d_{s-2}|}{2}r^{s-2}\right).
$$

Again using Lemma 2.3 (taking $\varepsilon=\frac{1}{2}$), we deduce that $|P_{s-2,j}(z)|=O(r^{s-2})$ and so

$$
(3.29)\qquad \left|\exp\left(P_{s-2,j}(z)\right)\right| = \exp\left(O\left(r^{s-2}\right)\right)\ j=1,2,\ldots,n.
$$

Now from (3.25), (3.27) and (3.29), we get

$$
\begin{aligned}
(3.30)\qquad & \left|\frac{H(z+jc)}{H(z)}e^{jsa_scz^{s-1}}e^{P_{s-2,j}(z)}\right| \\
\leq & \exp\left(js|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2))+r^{\rho(H)-1+\varepsilon}+O\left(r^{s-2}\right)\right) \\
\leq & \exp\left(ns|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2))+r^{\rho(H)-1+\varepsilon}+O\left(r^{s-2}\right)\right)
\end{aligned}
$$

for $j=1,2,\ldots,n$.

Then from (3.24), (3.26), (3.28) and (3.30), we conclude that

$$
\begin{aligned}
& \exp\left(|d_{s-1}|r^{s-1}\right) = \left|\exp(d_{s-1}z^{s-1})\right| = \left|\frac{\exp(Q(z))}{\exp(\tilde{P}_{s-2}(z))}\right| \\
(3.31) \leq\ & \frac{\left|\sum\limits_{j=1}^{n} c_j\frac{H(z+jc)}{H(z)}e^{jsa_scz^{s-1}}e^{P_{s-2,j}(z)}+(-1)^n\right|}{|\exp(\tilde{P}_{s-2}(z))|} \\
\leq\ & \frac{(n+1)n!\exp\left(ns|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2))+r^{\rho(H)-1+\varepsilon}+O\left(r^{s-2}\right)\right)}{\exp\left(\frac{|d_{s-2}|}{2}r^{s-2}\right)}.
\end{aligned}
$$

Since $\rho(H)-1+\varepsilon<s-1$ and $(n+1)n!=\exp(\log(n+1)n!)=o(r^{s-1})$, from (3.31), we deduce that

$$
(3.32)\quad \exp\left(|d_{s-1}|r^{s-1}\right) \leq \exp\left(ns|a_sc|\cos((s-1)\theta_0+\theta_2)r^{s-1}+o(r^{s-1})\right).
$$

By assumption, we have $d_{s-1}\neq nsa_sc$ and $ns|a_sc|\leq|d_{s-1}|$.

First we suppose $ns|a_s c| = |d_{s-1}|$. In that case $\cos((s-1)\theta_0 + \theta_2) \neq 1$ and so $\cos((s-1)\theta_0 + \theta_2) < 1$. Therefore

$$ns|a_s c|\cos((s-1)\theta_0 + \theta_2) < ns|a_s c| = |d_{s-1}|.$$

Next we suppose $ns|a_s c| < |d_{s-1}|$. Then obviously

$$ns|a_s c|\cos((s-1)\theta_0 + \theta_2) \leq ns|a_s c| < |d_{s-1}|.$$

Then in either case we have

$$ns|a_s c|\cos((s-1)\theta_0 + \theta_2) < |d_{s-1}|.$$

Therefore there exists $\varepsilon_1 > 0$ such that

$$ns|a_s c|\cos((s-1)\theta_0 + \theta_2) + 2\varepsilon_1 < |d_{s-1}|$$

and so from (3.32), we have

$$\exp\left(|d_{s-1}|r^{s-1}\right) \leq \exp\left(\left(|d_{s-1}| - \varepsilon_1\right) r^{s-1}\right),$$

which is a contradiction. This completes the proof. □

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ORIENTATION-DEPENDENT CHORD LENGTH DISTRIBUTION IN A CONVEX QUADRILATERAL

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Abstract. This work contributes to the research devoted to the recognition of a convex body by probabilistic characteristics of its lower-dimensional sections. In this paper, for any convex quadrilateral, five orientation-dependent characteristics are introduced and explicitly evaluated per direction. In terms of these characteristics, simple explicit representations of the orientation-dependent chord length distribution function and the covariogram are obtained not only for an arbitrary convex quadrilateral but also for any right prism based on it.

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1. Introduction

Inferring properties of an unknown convex body $\mathbf{D} \subset \mathbb{R}^n$ with a non-empty interior from its chord length measurements is one of the fundamental problems in geometric tomography. Although it is known that the body cannot be characterized by its chord length distribution (see [1]), there are positive results when the distribution function is known for each separate direction. Such a function is known as an **orientation-dependent chord length distribution function (ODCLD)**.

On the other hand, the problem of finding the ODCLD function is equivalent to the problem of finding the function

$$C_{\mathbf{D}}(x) = L_n(\mathbf{D} \cap \{\mathbf{D} + x\}), \;\; x \in \mathbb{R}^n,$$

where $\mathbf{D} + x = \{\mathcal{P} + x \, : \, \mathcal{P} \in \mathbf{D}\}$ and $L_n(\cdot)$ is the n-dimensional Lebesgue measure in $\mathbb{R}^n$. This function is called the **covariogram** of $\mathbf{D}$.

The hypothesis [2] that $\mathbf{D}$ can be determined from its covariogram was rejected when $n \geq 4$ (see [4], [5]) and confirmed when $\mathbf{D}$ is a planar convex domain (see [6]), or a three-dimensional convex polytope (see [7]). Since then, numerous papers have been published with the objective of achieving an explicit form of the ODCLD function or the covariogram for a specific body $\mathbf{D} \subset \mathbb{R}^n$. In particular, when $n = 2$, 3,

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the research includes the articles [8] and [9], where **D** is a triangle or a parallelogram, [10] and [11], where **D** is a regular polygon, an ellipse, or a prism with a triangular or elliptical base. The most recent research in this direction is reflected in [12], [13], and [14], where the ODCLD function and the covariogram are found for some quadrilateral prisms and their rectangular or trapezoidal bases.

This paper focuses on finding an explicit representation of the ODCLD function for an arbitrary convex quadrilateral. The quadrilateral is closed: it contains its interior points and the boundary.

The necessary terminology and characteristics of the quadrilateral to build the ODCLD function are provided in sections 2 and 3. Particularly, we extend there the concept of a φ-diameter for a polygon introduced by David Mount [3], and then define supplementary measures for a standard image (defined in section 2) of a convex quadrilateral. Readers, already familiar with the concept of X-ray (refer to Chapter 1 of [4]), may benefit while contemplating the origins and significance of the newly introduced orientation-dependent characteristics. To determine the ODCLD function, acquiring orientation-dependent X-rays is sufficient (see, for example, [15]). These X-rays, which exhibit convex functions with up to three graph pieces for a convex quadrilateral, can be accurately determined using φ-diameters and supplementary φ-measures as necessary parameters.

The main synthetic results are presented in section 4, where the ODCLD function and the covariogram of a convex quadrilateral are found in terms of the lengths of orientation-dependent diameters and supplementary measures. As an application, in the last section, the analogs of those results are established for quadrilateral prisms. All orientation-dependent computations are processed in section 5.

2. A standard image of a quadrilateral

In a Cartesian plane, for any convex quadrilateral **D** there are points $B(b,0)$, $b>0$, $A \in \{(x,y) : x \geq 0, y > 0\}$, and $C \in \{(x,y) : x > 0, y > 0\}$ such that **D** is congruent to the quadrilateral $OACB$, where O is the origin of coordinates. We will call such a quadrilateral **an image** of **D**. The side OB will be called **the base**, the sides OA and BC will be called **legs**, α and β will stand for the inclination angles (measured anticlockwise from the positive direction of x-axis) of the legs OA and BC, respectively. If $\alpha \leq \beta$ then the quadrilateral $OABC$ will be called **a standard image** of **D**.

Proposition 2.1. *Every convex quadrilateral* $\boldsymbol{D}$ *has a standard image.*

Proof. Let $OACB$ be an image of $\mathbf{D}$. Then let θ_A and θ_C be the internal angles at the vertices A and C, respectively. If $\beta < \alpha$ then $\theta_A + \theta_C < \pi$.

If $\theta_A < \frac{\pi}{2}$, consider the Euclidean transformation $\mathcal{T}$ that rotates the plane clockwise about the origin by α and then reflects it on the x-axis. Then $OA'C'B'$ becomes a standard image of $\mathbf{D}$, where $A' = \mathcal{T}(B)$, $B' = \mathcal{T}(A)$, and $C' = \mathcal{T}(C)$. Indeed, if α' and β' are the corresponding inclination angles of the legs OA' and $B'C'$, then

$$\alpha' = \alpha \leq \frac{\pi}{2} < \pi - \theta_A = \beta'.$$

If $\theta_C < \frac{\pi}{2}$, let $\mathcal{T}$ be the translation by $\overrightarrow{CO}$ followed by the clockwise rotation by $\alpha + \theta_A$ about O. Denoting $A' = \mathcal{T}(B)$, $B' = \mathcal{T}(A)$, and $C' = \mathcal{T}(O)$ we again obtain a standard image of $\mathbf{D}$ since

$$\alpha' = \theta_C < \pi - \theta_A = \beta'. \qquad \square$$

In addition to the length of the base, b and inclination angles of legs, α and β, we introduce two more parameters for $OACB$, a standard image of $\mathbf{D}$. Let α_0 and β_0 be the inclination angles of the diagonals OC and BA, respectively. Obviously,

$$\alpha_0 < \alpha \leq \beta < \beta_0,$$

and any standard image is determined by the five parameters $b, \alpha_0, \alpha, \beta, \beta_0$. We will utilise the notation

$$\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$$

for a standard image. For example, a rectangle with sides of lengths 1 and $\sqrt{3}$ has two standard images, $\mathbf{D}_s^{(1)} = [1, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{3}]$ and $\mathbf{D}_s^{(2)} = [\sqrt{3}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{6}]$.

The values $\alpha_0, \alpha, \beta, \beta_0$ determine another parameter γ, the inclination angle of AC. It is easy to check that

$$\tan \gamma = \frac{\cot \alpha + \cot \beta - \cot \alpha_0 - \cot \beta_0}{\cot \alpha \cot \beta - \cot \alpha_0 \cot \beta_0}.$$

We classify the standard images into two categories based on the value of γ. Due to convexity of $\mathbf{D}$, either $0 \leq \gamma < \alpha_0$, or $\beta_0 < \gamma < \pi$. If the first inequality occurs, we will call the standard image to be of **Type 1**, otherwise - of **Type 2**. For example, a right-angled trapezoid has five standard images, where three of them are of Type 1, and two are of Type 2. Any parallelogram has only standard images of Type 1, whereas any kite with three congruent obtuse internal angles permits only standard images of Type 2.

3. ORIENTATION-DEPENDENT DIAMETERS AND SUPPLEMENTARY MEASURES

Let $\mathbf{D}_s$ be a standard image of a convex quadrilateral $\mathbf{D} \subset \mathbb{R}^2$. Consider the vector

$$\phi = (\cos\varphi, \sin\varphi) \in \mathbb{S}^1,$$

and let l_φ be the subspace of $\mathbb{R}^2$ spanned by ϕ. By $\phi^\perp$ we denote the orthogonal complement of l_φ. For any $y \in \phi^\perp$, let $l_\varphi + y$ be the line which is parallel to ϕ and passes through y. Denote

$$\chi(l_\varphi + y) = L_1\big((l_\varphi + y) \cap \mathbf{D}_s\big).$$

If the line $l_\varphi + y$ intersects $\mathbf{D}_s$, then we will say that it makes a chord in $\mathbf{D}_s$ of length $\chi(l_\varphi + y)$. Denote

$$\Pi^x_{\mathbf{E}}(\varphi) = \{y \in \Pi_{\mathbf{E}}(\varphi) : \chi(l_\varphi + y) \leq x\},$$

where $\Pi_{\mathbf{E}}(\varphi)$ is the orthogonal projection of $\mathbf{E} \subset \mathbb{R}^2$ onto $\phi^\perp$. Assuming that y is uniformly distributed over $\Pi_{\mathbf{D}_s}(\varphi)$, the chord length distribution function in direction ϕ for $\mathbf{D}_s$ is defined by

$$F_{\mathbf{D}_s}(x, \varphi) = \frac{L_1\big(\Pi^x_{\mathbf{D}_s}(\varphi)\big)}{b_{\mathbf{D}_s}(\varphi)}, \tag{3.1}$$

where $b_{\mathbf{D}_s}(\varphi) = L_1(\Pi_{\mathbf{D}_s}(\varphi))$.

Hereinafter, since $l_{\varphi-\pi} = l_\varphi$ we will assume $\varphi \in [0,\, \pi)$.

To determine the distribution function $F_{\mathbf{D}_s}(x, \varphi)$ we need the quantities (introduced in [13])

$$x_0(\varphi) = \min_{y \in \phi_v^\perp} \chi(l_\varphi + y) \;\text{ and }\; x_1(\varphi) = \max_{y \in \phi_v^\perp} \chi(l_\varphi + y),$$

where $\phi_v^\perp$ is the set of vectors $y \in \phi^\perp$ so that the line $l_\varphi + y$ passes through a vertex of $\mathbf{D}_s$ and makes a chord of positive Lebesgue measure there. The quantity $x_1(\varphi)$ coincides with

$$x_{\max}(\varphi) = \max_{y \in \Pi_{\mathbf{D}_s}(\varphi)} \chi(l_\varphi + y),$$

and any chord of length $x_{\max}(\varphi)$ is known as a **φ-diameter** of $\mathbf{D}_s$ (see [3]). In this paper, where convenient, we will call it a **first-order φ-diameter** of $\mathbf{D}_s$. Any chord of length $x_0(\varphi)$ will be called a **second-order φ-diameter** of $\mathbf{D}_s$.

Below, in addition to $x_0(\varphi)$ and $x_1(\varphi)$, we aim to introduce three more orientation-dependent characteristics $\ell_0(\varphi)$, $\ell(\varphi)$, and $\ell_1(\varphi)$ of the standard image $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$. These characteristics will be non-negative continuous functions and will satisfy to $b_{\mathbf{D}_s}(\varphi) = \ell_0(\varphi) + \ell(\varphi) + \ell_1(\varphi)$ for all $\varphi \in [0, \pi)$. We will call them **supplementary φ-measures** of $\mathbf{D}_s$.

Case 1: $\boldsymbol{D}_s$ has no parallel sides. We have $\gamma > 0$ and $\alpha < \beta$. Then for any φ the first and the second-order φ-diameters are unique. Let them be $(l_\varphi + y_1) \cap \mathbf{D}_s$ and $(l_\varphi + y_0) \cap \mathbf{D}_s$, respectively. If $\varphi \neq \alpha_0$ and $\varphi \neq \beta_0$ then $y_0 \neq y_1$. In the case when y_0, $y_1 \in int\big(\Pi_{\mathbf{D}_s}(\varphi)\big)$, they partition $\Pi_{\mathbf{D}_s}(\varphi)$ into three segments: the middle segment, the side-segment adjacent to y_0, and the other side-segment adjacent to y_1. We denote the lengths of those segments by $\ell(\varphi)$, $\ell_0(\varphi)$, and $\ell_1(\varphi)$, respectively. If $y_0 \in \partial\Pi_{\mathbf{D}_s}(\varphi)$ or $y_1 \in \partial\Pi_{\mathbf{D}_s}(\varphi)$, we define correspondingly $\ell_0(\varphi) = 0$ or $\ell_1(\varphi) = 0$.

When $\varphi = \alpha_0$ or $\varphi = \beta_0$, the first and the second-order φ-diameters coincide. We extend the definitions of ℓ, ℓ_0, and ℓ_1 preserving their continuous dependence on φ:

$$\ell(\alpha_0) = \ell(\beta_0) = |y_0 - y_1| = 0,$$

$$\ell_0(\alpha_0) = \lim_{\varphi\to\alpha_0} \ell_0(\varphi),\ \ell_0(\beta_0) = \lim_{\varphi\to\beta_0} \ell_0(\varphi), \ell_1(\alpha_0) = \lim_{\varphi\to\alpha_0} \ell_1(\varphi),\ \ell_1(\beta_0) = \lim_{\varphi\to\beta_0} \ell_1(\varphi).$$

Case 2: $\boldsymbol{D}_s$ has exactly one pair of parallel sides.

Subcase 2.1: Let $\gamma = 0$ and $\alpha < \beta$. Uniqueness of the first and the second-order φ-diameters takes place if and only if $\varphi \in [0, \alpha_0] \cup [\beta_0, \pi)$. If $\varphi \neq \alpha_0$ and $\varphi \neq \beta_0$, we define y_0, y_1, and then $\ell(\varphi)$, $\ell_0(\varphi)$, $\ell_1(\varphi)$ the same way we did it in Case 1. The values at α_0 and β_0 are defined below:

$$\ell(\alpha_0) = \ell(\beta_0) = 0,$$

$$\ell_0(\alpha_0) = \ell_0(\alpha_0-),\ \ell_0(\beta_0) = \ell_0(\beta_0+),\ \ell_1(\alpha_0) = \ell_1(\alpha_0-),\ \ell_1(\beta_0) = \ell_1(\beta_0+).$$

The case $\alpha_0 < \varphi < \beta_0$ yields $x_0(\varphi) = x_1(\varphi)$, so we face infinitely many φ-diameters. Here by $\ell(\varphi)$ we denote the distance between the two farthest φ -diameters, $(l_\varphi + y_0) \cap \mathbf{D}_s$ and $(l_\varphi + y_1) \cap \mathbf{D}_s$. Using these vectors y_0 and y_1, we define $\ell(\varphi)$, $\ell_0(\varphi)$, $\ell_1(\varphi)$ again by the algorithm provided in Case 1.

Subcase 2.2: Now let $\gamma > 0$ and $\alpha = \beta$. The first and the second-order φ-diameters are unique if and only if $\varphi \in [\alpha_0, \beta_0]$. For $\varphi \in (\alpha_0, \beta_0)$ we define y_0, y_1, and then $\ell(\varphi)$, $\ell_0(\varphi)$, $\ell_1(\varphi)$ by the algorithm of Case 1. For the boundary values we define

$$\ell(\alpha_0) = \ell(\beta_0) = 0,$$

$$\ell_0(\alpha_0) = \ell_0(\alpha_0+),\ \ell_0(\beta_0) = \ell_0(\beta_0-),\ \ell_1(\alpha_0) = \ell_1(\alpha_0+),\ \ell_1(\beta_0) = \ell_1(\beta_0-).$$

If $\varphi \notin [\alpha_0, \beta_0]$ then $x_0(\varphi) = x_1(\varphi)$, so $\mathbf{D}_s$ has infinitely many φ-diameters. We define $\ell(\varphi)$, $\ell_0(\varphi)$, $\ell_1(\varphi)$ the same way as we did it in Subcase 2.1 for $\varphi \in (\alpha_0, \beta_0)$.

Case 3: $\boldsymbol{D}_s$ has two pairs of parallel sides. In a parallelogram, $x_0(\varphi) = x_1(\varphi)$ holds for any value of φ. We define

$$\ell(\varphi) = |y_0 - y_1|,$$

and

$$\ell_0(\varphi) = \ell_1(\varphi) = \frac{b_{\mathbf{D}_s}(\varphi) - \ell(\varphi)}{2},$$

where $(l_\varphi + y_0) \cap \mathbf{D}_s$ and $(l_\varphi + y_1) \cap \mathbf{D}_s$ are the two farthest φ - diameters of $\mathbf{D}_s$.

4. Representation of the orientation-dependent chord length distribution function and the covariogram

The following theorem represents the function introduced in 3.1 in terms of the lengths of orientation-dependent diameters and supplementary measures.

Theorem 4.1. *Let $\boldsymbol{D}_s$ be a standard image of a convex quadrilateral $\boldsymbol{D}$ and $0 \le \varphi < \pi$. If x_1, x_0 are the lengths of respectively the first and the second-order φ-diameters, and ℓ_0, ℓ, ℓ_1 are the supplementary φ-measures of $\boldsymbol{D}_s$, then*

$$F_{\boldsymbol{D}_s}(x,\varphi) = \frac{1}{\ell_0 + \ell + \ell_1} \begin{cases} 0, & \textit{if} \ \ x < 0 \\ \left(\dfrac{\ell_0}{x_0} + \dfrac{\ell_1}{x_1}\right)x, & \textit{if} \ \ 0 \le x < x_0(\varphi) \\ \ell_0 + \dfrac{x - x_0}{x_1 - x_0}\ell + \dfrac{x}{x_1}\ell_1, & \textit{if} \ \ x_0(\varphi) \le x < x_1(\varphi) \\ \ell_0 + \ell + \ell_1, & \textit{if} \ \ x \ge x_1(\varphi) \end{cases}. \tag{4.1}$$

Proof. The statement is obvious when $x < 0$ or $x \ge x_1$. Below we assume $0 \le x < x_1$.

Case A: φ is such that $x_0(\varphi) < x_1(\varphi)$. Let $(l_\varphi + y_1) \cap \mathbf{D}_s$ and $(l_\varphi + y_0) \cap \mathbf{D}_s$ be the first and second-order φ - diameters of $\mathbf{D}_s$. If y_0, $y_1 \in int\big(\Pi_{\mathbf{D}_s}(\varphi)\big)$, then the mentioned diameters partition $\mathbf{D}_s$ into two triangles $\mathbf{T}_0(\varphi)$, $\mathbf{T}_1(\varphi)$, and a trapezoid $\mathbf{T}(\varphi)$, where $\mathbf{T}_0$ is based on the second-order diameter and has a height of length ℓ_0, $\mathbf{T}_1$ is based on the first-order diameter and has a height of length ℓ_1, and the trapezoid $\mathbf{T}$ is based on the mentioned diameters and has a height of length ℓ. Then

$$L_1\big(\Pi^x_{\mathbf{D}_s}(\varphi)\big) = \sum_{i=0}^{1} L_1\big(\Pi^x_{\mathbf{T}_i}(\varphi)\big) + L_1\big(\Pi^x_{\mathbf{T}}(\varphi)\big). \tag{4.2}$$

If $0 \le x < x_0$ then $\Pi^x_{\mathbf{T}}(\varphi) = \varnothing$ and

$$L_1\big(\Pi^x_{\mathbf{T}_i}(\varphi)\big) = \frac{x}{x_i} L_1\big(\Pi_{\mathbf{T}_i}(\varphi)\big) = \frac{x}{x_i}\ell_i.$$

If $x_0 \le x < x_1$ then

$$L_1\big(\Pi^x_{\mathbf{T}_0}(\varphi)\big) = L_1\big(\Pi_{\mathbf{T}_0}(\varphi)\big) = \ell_0,$$
$$L_1\big(\Pi^x_{\mathbf{T}_1}(\varphi)\big) = \frac{x}{x_1} L_1\big(\Pi_{\mathbf{T}_1}(\varphi)\big) = \frac{x}{x_1}\ell_1,$$

and

$$L_1\big(\Pi^x_{\mathbf{T}}(\varphi)\big) = \frac{x - x_0}{x_1 - x_0} L_1\big(\Pi_{\mathbf{T}}(\varphi)\big) = \frac{x - x_0}{x_1 - x_0}\ell.$$

Now according to 3.1 and 4.2, we obtain

$$F_{\mathbf{D}_s}(x,\varphi) = \frac{1}{b_{\mathbf{D}_s}(\varphi)}\left(\frac{x}{x_0}\ell_0 + \frac{x}{x_1}\ell_1\right), \text{ for } 0 \le x < x_0, \tag{4.3}$$

and

$$F_{\mathbf{D}_s}(x,\varphi) = \frac{1}{b_{\mathbf{D}_s}(\varphi)}\left(\ell_0 + \frac{x-x_0}{x_1-x_0}\ell + \frac{x}{x_1}\ell_1\right), \text{ for } x_0 \le x < x_1. \tag{4.4}$$

Formula 4.2 works for such values of φ that imply $y_i \notin int\big(\Pi_{\mathbf{D}_s}\big)$ for $i = 0$ or $i = 1$. In this case, $\mathbf{T}_i$ turns into the segment $(l_\varphi + y_i) \cap \mathbf{D}_s$, and yields

$$L_1\big(\Pi^x_{\mathbf{T}_i}(\varphi)\big) = L_1\big(\Pi_{\mathbf{T}_i}(\varphi)\big) = \ell_i(\varphi) = 0.$$

Since $l_i(\varphi)$ has been defined as a continuous function, the formulas 4.3 and 4.4 remain valid.

Case B: φ is such that $x_0(\varphi) = x_1(\varphi)$. Consider $(l_\varphi + y_1) \cap \mathbf{D}_s$ and $(l_\varphi + y_0) \cap \mathbf{D}_s$, the two farthest φ - diameters of $\mathbf{D}_s$. If $y_0 \ne y_1$ and they both belong to $int\big(\Pi_{\mathbf{D}_s}(\varphi)\big)$ then $\mathbf{D}_S$ will be partitioned into the two triangles $\mathbf{T}_0(\varphi)$, $\mathbf{T}_1(\varphi)$, and the trapezoid $\mathbf{T}(\varphi)$ defined in Case A. If $y_0 = y_1$ or $y_i \notin int\big(\Pi_{\mathbf{D}_s}\big)$ for $i = 0$ or $i = 1$, then $\mathbf{T}$, or correspondingly, $\mathbf{T}_i$, turns into the segment $(l_\varphi + y_i) \cap \mathbf{D}_s$. In all these scenarios the formula 4.2 does operate, and since the functions $\ell_i(\varphi)$ are continuous, it implies 4.3. $\square$

Corollary 4.1. *The function $F_{\boldsymbol{D}_s}(\cdot,\varphi)$ is continuous on the real axis if and only if the φ-diameter of $\boldsymbol{D}_s$ is unique. If for some φ, the φ-diameter of $\boldsymbol{D}_s$ is not unique then $F_{\boldsymbol{D}_S}(\cdot,\varphi)$ holds a jump discontinuity at $x_{\max}(\varphi)$. The jump is equal to*

$$\frac{\ell}{\ell_0 + \ell + \ell_1}.$$

Proof. A φ-diameter is unique if and only if $x_0(\varphi) < x_1(\varphi)$, or $x_0(\varphi) = x_1(\varphi)$ but $\ell(\varphi) = 0$. Due to 4.1, this is equivalent to the continuity of $F_{\mathbf{D}_S}(\cdot,\varphi)$.

If a φ-diameter is not unique then $x_0(\varphi) = x_1(\varphi) = x_{max}(\varphi)$ and $\ell(\varphi) > 0$. Hence, $F_{\mathbf{D}_s}(x_{max}(\varphi)+,\varphi) = 1$ whereas $F_{\mathbf{D}_s}(x_{max}(\varphi)-,\varphi) = \frac{\ell_0+\ell_1}{\ell_0+\ell+\ell_1} = 1 - \frac{\ell}{\ell_0+\ell+\ell_1}$. $\square$

From now on the notation $C_E(t,\varphi)$ will be used for the covariogram $C_E(t\phi))$, where $\mathbf{E} \subset \mathbb{R}^2$ and $t \ge 0$. Further in the text, $\|\mathbf{E}\|$ will stand for the area of $\mathbf{E}$.

Theorem 4.2. *Let $\boldsymbol{D}_s$ be a standard image of a convex quadrilateral $\boldsymbol{D}$ and $0 \le \varphi < \pi$. If x_1, x_0 are the lengths of respectively the first and the second-order φ-diameters, and ℓ_0, ℓ, ℓ_1 are the supplementary φ-measures of $\boldsymbol{D}_s$, then $C_{\boldsymbol{D}_s}(t,\varphi) =$*

$$= \begin{cases} \dfrac{x_0\ell_0 + (x_0+x_1)\ell + x_1\ell_1}{2} - (\ell_0 + \ell + \ell_1)t + \dfrac{1}{2}\left(\dfrac{\ell_0}{x_0} + \dfrac{\ell_1}{x_1}\right)t^2, & \textit{if} \quad 0 \le t < x_0 \\ \dfrac{1}{2}\left(\dfrac{\ell_1}{x_1} + \dfrac{\ell}{x_1 - x_0}\right)(x_1 - t)^2, & \textit{if} \quad x_0 \le t < x_1 \\ 0, & \textit{if} \quad t \ge x_1 \end{cases}.$$

Proof. The case $t \geq x_1$ is obvious so below we assume $0 \leq t < x_1$.

Due to the Matheron's formula [2], p. 86, we have

$$\frac{\partial C_{\mathbf{D}_s}(t,\varphi)}{\partial t} = -L_1\left(\{y \in \phi^{\perp} : L_1\left(\mathbf{D}_s \cap (l_\varphi + y)\right) \geq t\}\right),$$

which can be rewritten in terms of the orientation-dependent chord length distribution function as

$$\frac{\partial C_{\mathbf{D}_s}(t,\varphi)}{\partial t} = -b_{\mathbf{D}_s}(\varphi) \cdot [1 - F_{\mathbf{D}_s}(t,\varphi)].$$

Corollary 4.1 Integration of both parts of the last formula yields

$$C_{\mathbf{D}_s}(t,\varphi) = C_{\mathbf{D}_s}(0,\varphi) - b_{\mathbf{D}_s}(\varphi) \cdot t + b_{\mathbf{D}_s}(\varphi) \cdot \int_0^t F_{\mathbf{D}_s}(u,\varphi)du,\ t \geq 0. \tag{4.5}$$

Since

$$C_{\mathbf{D}_s}(0,\varphi) = \|\mathbf{D}_s\| = \frac{x_0\ell_0 + (x_0+x_1)\ell + x_1\ell_1}{2},$$
$$b_{\mathbf{D}_s}(\varphi) = \ell_0(\varphi) + \ell(\varphi) + \ell_1(\varphi),$$

and

$$\int_0^t \left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right) u du = \frac{1}{2}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right) t^2,$$

then the required form of $C_{\mathbf{D}_s}(t,\varphi)$, where $0 \leq t < x_0$, immediately follows from 4.5 and Theorem 4.1.

If $x_0 \leq t < x_1$ then we use the corresponding part of Theorem 4.1 in 4.5:

$$C_{\mathbf{D}_s}(t,\varphi) = C_{\mathbf{D}_s}(0,\varphi) - b_{\mathbf{D}_s}(\varphi) \cdot t + \frac{1}{2}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right) x_0^2 + \int_{x_0}^t \ell_0 + \frac{u - x_0}{x_1 - x_0}\ell + \frac{u}{x_1}\ell_1 du.$$

Computation of the integral followed by the regrouping of similar terms produces

$$C_{\mathbf{D}_s}(t,\varphi) = \frac{x_1^2\ell}{2(x_1 - x_0)} + \frac{x_1\ell_1}{2} - \left(\frac{x_1\ell}{x_1 - x_0} + \ell_1\right) \cdot t + \frac{1}{2}\left(\frac{\ell_1}{x_1} + \frac{\ell}{x_1 - x_0}\right) \cdot t^2 =$$
$$= \frac{1}{2}\left(\frac{\ell_1}{x_1} + \frac{\ell}{x_1 - x_0}\right)(x_1 - t)^2.$$

5. Computation of orientation-dependent diameters and supplementary measures

For a standard image $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$, we denote

$$\Lambda = \{\alpha,\ \beta\},\ \ \Delta = \{\alpha_0,\ \beta_0\},\ \ \Sigma = \{0,\ \alpha,\ \gamma,\ \beta\},$$

which are the sets of the inclination angles of the legs, diagonals, and the sides of $\mathbf{D}_s$, respectively. For any $\varphi \in [0,\ \pi)$, we define the functions $X_\varphi : \Lambda \times \Delta \times \Sigma \setminus \{\varphi\} \longrightarrow \mathbb{R}$ and $L_\varphi : (\Lambda \times \Delta) \cup (\Delta \times \Lambda) \longrightarrow \mathbb{R}$ by

$$X_\varphi(x,\ y,\ z) = \frac{b \sin x \sin(y - z)}{\sin(y - x)\sin(z - \varphi)},$$

$$L_\varphi(x,\ y) = \frac{b \sin(x - \varphi)\sin y}{\sin(x - y)}.$$

Theorem 5.1. *Let* $\boldsymbol{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ *be a standard image of Type 1 of a convex quadrilateral* $\boldsymbol{D}$*. If* x_1, x_0 *are the lengths of respectively the first and the second-order* φ*-diameters of* $\boldsymbol{D}_s$*, then*

i. $x_0(\varphi) = X_\varphi(\alpha,\, \beta_0,\, \beta)$ *and* $x_1(\varphi) = X_\varphi(\beta,\, \alpha_0,\, \beta)$*, for* $0 \le \varphi < \gamma$*;*
ii. $x_0(\varphi) = X_\varphi(\beta,\, \alpha_0,\, \alpha)$ *and* $x_1(\varphi) = X_\varphi(\beta,\, \alpha_0,\, \beta)$*, for* $\gamma \le \varphi < \alpha_0$*;*
iii. $x_0(\varphi) = X_\varphi(\beta,\, \alpha_0,\, \gamma)$ *and* $x_1(\varphi) = X_\varphi(\beta,\, \alpha_0,\, 0)$*, for* $\alpha_0 \le \varphi < \alpha$*;*
iv. $x_0(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, 0)$ *and* $x_1(\varphi) = X_\varphi(\beta,\, \alpha_0,\, 0)$*, for* $\alpha \le \varphi < \beta$*;*
v. $x_0(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, 0)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \gamma)$*, for* $\beta \le \varphi < \beta_0$*;*
vi. $x_0(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \beta)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \alpha)$*, for* $\beta_0 \le \varphi < \pi$*.*

Proof. Let the quadrilateral $\mathbf{D}_s = OACB$ not have any pair of parallel sides. The lengths of the diagonals AB and OC are

$$d_{AB} = \frac{b\sin\alpha}{\sin(\beta_0 - \alpha)} \text{ and } d_{OC} = \frac{b\sin\beta}{\sin(\beta - \alpha_0)}. \tag{5.1}$$

We denote $\Pi_{\{A\}}(\varphi) = y_A$, $\Pi_{\{C\}}(\varphi) = y_C$, and $\Pi_{\{B\}}(\varphi) = y_B$. Then the first and the second-order φ-diameters of $\mathbf{D}_s$ are respectively equal to

$$\begin{cases} l_\varphi \cap \mathbf{D}_s \text{ and } (l_\varphi + y_A) \cap \mathbf{D}_s, & \text{if } 0 \le \varphi < \gamma \\ l_\varphi \cap \mathbf{D}_s \text{ and } (l_\varphi + y_C) \cap \mathbf{D}_s, & \text{if } \gamma \le \varphi < \alpha_0 \\ (l_\varphi + y_C) \cap \mathbf{D}_s \text{ and } l_\varphi \cap \mathbf{D}_s, & \text{if } \alpha_0 \le \varphi < \alpha \\ (l_\varphi + y_C) \cap \mathbf{D}_s \text{ and } (l_\varphi + y_A) \cap \mathbf{D}_s, & \text{if } \alpha \le \varphi < \beta \\ (l_\varphi + y_B) \cap \mathbf{D}_s \text{ and } (l_\varphi + y_A) \cap \mathbf{D}_s, & \text{if } \beta \le \varphi < \pi \end{cases} . \tag{5.2}$$

To compute $x_0(\varphi)$ we initially assume that the chosen direction ϕ is not parallel to any side or a diagonal of $\mathbf{D}_s$, which means $\varphi \notin \Delta \cup \Sigma$. This allows us to determine uniquely the triangle, where one of its sides is the second-order diameter of $\mathbf{D}_s$ and another side is the diagonal that shares an endpoint with the mentioned diameter. In that triangle, the internal angles that occurred in front of the second-order diameter and in front of the corresponding diagonal, are respectively equal to

$$\beta_0 - \beta \text{ and } \beta - \varphi, \text{ if } 0 < \varphi < \gamma; \quad \alpha - \alpha_0 \text{ and } \pi - \alpha + \varphi, \text{ if } \gamma < \varphi < \alpha_0;$$

$$\alpha_0 - \gamma \text{ and } \pi - \varphi + \gamma, \text{ if } \alpha_0 < \varphi < \alpha; \quad \pi - \beta_0 \text{ and } \varphi, \text{ if } \alpha < \varphi < \beta;$$

$$\pi - \beta_0 \text{ and } \varphi, \text{ if } \beta < \varphi < \beta_0; \quad \beta_0 - \beta \text{ and } \pi - \varphi + \beta, \text{ if } \beta_0 < \varphi < \pi.$$

Since $x_0 \in C[0,\pi)$, by 5.1, 5.2 and the Law of sines we conclude

$$
x_0(\varphi) = \begin{cases} d_{AB}\dfrac{\sin(\beta_0-\beta)}{\sin(\beta-\varphi)} = X_\varphi(\alpha,\,\beta_0,\,\beta), & \text{if } \ 0 \le \varphi < \gamma \\ d_{OC}\dfrac{\sin(\alpha-\alpha_0)}{\sin(\alpha-\varphi)} = X_\varphi(\beta,\,\alpha_0,\,\alpha), & \text{if } \ \gamma \le \varphi < \alpha_0 \\ d_{OC}\dfrac{\sin(\alpha_0-\gamma)}{\sin(\varphi-\gamma)} = X_\varphi(\beta,\,\alpha_0,\,\gamma), & \text{if } \ \alpha_0 \le \varphi < \alpha \\ d_{AB}\dfrac{\sin\beta_0}{\sin\varphi} = -X_\varphi(\alpha,\,\beta_0,\,0), & \text{if } \ \alpha \le \varphi < \beta_0 \\ d_{AB}\dfrac{\sin(\beta_0-\beta)}{\sin(\varphi-\beta)} = -X_\varphi(\alpha,\,\beta_0,\,\beta), & \text{if } \ \beta_0 \le \varphi < \pi \end{cases} . \tag{5.3}
$$

To prove the required identities for $x_1(\varphi)$ we assume again that $\varphi \notin \Delta \cup \Sigma$. Consider the triangle, where one of its sides is the first-order diameter of $\mathbf{D}_s$ and another side is the diagonal that shares an endpoint with the mentioned diameter. In this case, the internal angles of the triangle that occurred in front of the first-order diameter and in front of the corresponding diagonal, are respectively equal to

$$\beta-\alpha_0 \ \text{ and } \ \pi-\beta+\varphi, \ \text{if } 0<\varphi<\gamma \text{ or } \gamma<\varphi<\alpha_0;$$

$$\alpha_0 \ \text{ and } \ \pi-\varphi, \ \text{if } \alpha_0<\varphi<\alpha \text{ or } \alpha<\varphi<\beta;$$

$$\pi-\beta_0+\gamma \ \text{ and } \ \varphi-\gamma, \ \text{if } \beta<\varphi<\beta_0; \quad \beta_0-\alpha \ \text{ and } \ \pi-\varphi+\alpha, \ \text{if } \beta_0<\varphi<\pi.$$

As $x_1 \in C[0,\pi)$, we obtain

$$
x_1(\varphi) = \begin{cases} d_{OC}\dfrac{\sin(\beta-\alpha_0)}{\sin(\beta-\varphi)} = X_\varphi(\beta,\,\alpha_0,\,\beta), & \text{if } \ 0 \le \varphi < \alpha_0 \\ d_{OC}\dfrac{\sin\alpha_0}{\sin\varphi} = X_\varphi(\beta,\,\alpha_0,\,0), & \text{if } \ \alpha_0 \le \varphi < \beta \\ d_{AB}\dfrac{\sin(\beta_0-\gamma)}{\sin(\varphi-\gamma)} = -X_\varphi(\alpha,\,\beta_0,\,\gamma), & \text{if } \ \beta \le \varphi < \beta_0 \\ d_{AB}\dfrac{\sin(\beta_0-\alpha)}{\sin(\varphi-\alpha)} = -X_\varphi(\alpha,\,\beta_0,\,\alpha), & \text{if } \ \beta_0 \le \varphi < \pi \end{cases} . \tag{5.4}
$$

It remains to notice that the formulas 5.3 and 5.4 also hold if $\gamma = 0$ or $\alpha = \beta$. $\square$

Theorem 5.2. *Let $\boldsymbol{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ be a standard image of Type 1 of a convex quadrilateral $\boldsymbol{D}$. If ℓ_0, ℓ and ℓ_1 are the supplementary φ-measures of $\boldsymbol{D}_s$, then*

$$
\ell_0(\varphi) = \begin{cases} L_\varphi(\alpha,\,\beta_0) - L_\varphi(\alpha_0,\,\beta), & \textit{if } \ 0 \le \varphi < \gamma \\ L_\varphi(\alpha_0,\,\beta) - L_\varphi(\alpha,\,\beta_0), & \textit{if } \ \gamma \le \varphi < \alpha_0 \ \textit{ or } \beta_0 \le \varphi < \pi \\ -L_\varphi(\alpha,\,\beta_0), & \textit{if } \ \alpha_0 \le \varphi < \alpha \\ L_\varphi(\alpha,\,\beta_0), & \textit{if } \ \alpha \le \varphi < \beta_0 \end{cases} , \tag{5.5}
$$

$$\ell(\varphi)=\begin{cases}-L_\varphi(\alpha,\,\beta_0), & \textit{if}\ \ 0\le\varphi<\gamma\\ -L_\varphi(\alpha_0,\,\beta), & \textit{if}\ \ \gamma\le\varphi<\alpha_0\\ L_\varphi(\alpha_0,\,\beta), & \textit{if}\ \ \alpha_0\le\varphi<\alpha\\ L_\varphi(\alpha_0,\,\beta)-L_\varphi(\alpha,\,\beta_0), & \textit{if}\ \ \alpha\le\varphi<\beta\\ L_\varphi(\beta_0,\,\alpha), & \textit{if}\ \ \beta\le\varphi<\beta_0\\ -L_\varphi(\beta_0,\,\alpha), & \textit{if}\ \ \beta_0\le\varphi<\pi\end{cases}, \tag{5.6}$$

$$\ell_1(\varphi)=\begin{cases}b\sin\varphi, & \textit{if}\ \ 0\le\varphi<\alpha_0\ \textit{or}\ \beta_0\le\varphi<\pi\\ L_\varphi(\beta,\,\alpha_0), & \textit{if}\ \ \alpha_0\le\varphi<\beta\\ -L_\varphi(\beta,\,\alpha_0), & \textit{if}\ \ \beta\le\varphi<\beta_0\end{cases}. \tag{5.7}$$

Proof. First of all we notice that

$$L_1(\Pi_{\mathbf{E}}(\varphi))=L_1(\mathbf{E})\sin|\varepsilon-\varphi|, \tag{5.8}$$

for any line segment $\mathbf{E}\subset\mathbb{R}^2$, $L_1(\mathbf{E})<\infty$ inclined by $\varepsilon\in[0,\,\pi)$. When $\mathbf{E}$ is a diagonal of $\mathbf{D}_s$, then $L_1(\mathbf{E})$ can be read from 5.1. If $\mathbf{E}$ is a leg, we use either of the notations

$$s_{OA}=\frac{b\sin\beta_0}{\sin(\beta_0-\alpha)}\ \text{ and }\ s_{CB}=\frac{b\sin\alpha_0}{\sin(\beta-\alpha_0)} \tag{5.9}$$

for its length.

Let us first prove 5.6. For φ, being in either of the six intervals

$$[0,\gamma),\ [\gamma,\alpha_0),[\alpha_0,\alpha),\ [\alpha,\beta),\ [\beta,\beta_0),\ [\beta_0,\pi),$$

the corresponding six-term sequence of the quantity $\ell(\varphi)$ becomes

$$L_1(\Pi_{OA}(\varphi)),\ L_1(\Pi_{OC}(\varphi)),\ L_1(\Pi_{OC}(\varphi)),\ L_1(\Pi_{OC}(\varphi))-L_1(\Pi_{OA}(\varphi)),$$

$$L_1(\Pi_{AB}(\varphi)),\ L_1(\Pi_{AB}(\varphi)).$$

Since the inclination angles of OA, OC, and AB are respectively α, α_0, and β_0, the formulas 5.8, 5.9, 5.1 yield $\ell(\varphi)=$

$$\begin{cases}s_{OA}\sin|\alpha-\varphi|=\dfrac{b\sin\beta_0}{\sin(\beta_0-\alpha)}\sin(\alpha-\varphi)=-L_\varphi(\alpha,\,\beta_0), & \text{if}\ \ 0\le\varphi<\gamma\\ d_{OC}\sin|\alpha_0-\varphi|=\dfrac{b\sin\beta}{\sin(\beta-\alpha_0)}\sin(\alpha_0-\varphi)=-L_\varphi(\alpha_0,\,\beta), & \text{if}\ \ \gamma\le\varphi<\alpha_0\\ d_{OC}\sin|\alpha_0-\varphi|=\dfrac{b\sin\beta}{\sin(\beta-\alpha_0)}\sin(\varphi-\alpha_0)=L_\varphi(\alpha_0,\,\beta), & \text{if}\ \ \alpha_0\le\varphi<\alpha\\ d_{OC}\sin|\alpha_0-\varphi|-s_{OA}\sin|\alpha-\varphi|=L_\varphi(\alpha_0,\,\beta)-L_\varphi(\alpha,\,\beta_0), & \text{if}\ \ \alpha\le\varphi<\beta\\ d_{AB}\sin|\beta_0-\varphi|=\dfrac{b\sin\alpha}{\sin(\beta_0-\alpha)}\sin(\beta_0-\varphi)=L_\varphi(\beta_0,\,\alpha), & \text{if}\ \ \beta\le\varphi<\beta_0\\ d_{AB}\sin|\beta_0-\varphi|=\dfrac{b\sin\alpha}{\sin(\beta_0-\alpha)}\sin(\varphi-\beta_0)=-L_\varphi(\beta_0,\,\alpha), & \text{if}\ \ \beta_0\le\varphi<\pi\end{cases}.$$

Similarly, $\ell_0(\varphi) =$

$$\begin{cases} d_{OC}\sin|\alpha_0-\varphi| - s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha,\,\beta_0) - L_\varphi(\alpha_0,\,\beta), & \text{if} \quad 0 \le \varphi < \gamma \\ s_{OA}\sin|\alpha-\varphi| - d_{OC}\sin|\alpha_0-\varphi| = L_\varphi(\alpha_0,\,\beta) - L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \gamma \le \varphi < \alpha_0 \\ s_{OA}\sin|\alpha-\varphi| = -L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \alpha_0 \le \varphi < \alpha \\ s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \alpha \le \varphi < \beta \\ s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \beta \le \varphi < \beta_0 \\ d_{OC}\sin|\alpha_0-\varphi| - s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha_0,\,\beta) - L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \beta_0 \le \varphi < \pi \end{cases},$$

and

$$\ell_1(\varphi) = \begin{cases} b\sin|0-\varphi| = b\sin\varphi, & \text{if} \quad 0 \le \varphi < \gamma \\ b\sin|0-\varphi| = b\sin\varphi, & \text{if} \quad \gamma \le \varphi < \alpha_0 \\ s_{CB}\sin|\beta-\varphi| = L_\varphi(\beta,\,\alpha_0), & \text{if} \quad \alpha_0 \le \varphi < \alpha \\ s_{CB}\sin|\beta-\varphi| = L_\varphi(\beta,\,\alpha_0), & \text{if} \quad \alpha \le \varphi < \beta \\ s_{CB}\sin|\beta-\varphi| = -L_\varphi(\beta,\,\alpha_0), & \text{if} \quad \beta \le \varphi < \beta_0 \\ b\sin|0-\varphi| = b\sin\varphi, & \text{if} \quad \beta_0 \le \varphi < \pi \end{cases},$$

which are equivalent to 5.5 and 5.7, respectively. □

Corollary 5.1. *If a standard image $\boldsymbol{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ of a convex quadrilateral is of Type 1 then*

$$b_{\boldsymbol{D}_s}(\varphi) = \begin{cases} L_\varphi(\beta,\,\alpha_0), & \textit{if} \quad 0 \le \varphi < \gamma \\ L_\varphi(\beta_0,\,\alpha), & \textit{if} \quad \gamma \le \varphi < \alpha \\ b\sin\varphi, & \textit{if} \quad \alpha \le \varphi < \beta \\ L_\varphi(\alpha_0,\,\beta), & \textit{if} \quad \beta \le \varphi < \pi \end{cases}. \tag{5.10}$$

Proof. Since $b_{\mathbf{D}_s}(\varphi) = \ell_0(\varphi) + \ell(\varphi) + \ell_1(\varphi)$, we substitute $\ell_0(\varphi)$, $\ell(\varphi)$, and $\ell_1(\varphi)$ by their corresponding expressions from 5.5, 5.6, and 5.7. To reach 5.10, it remains to check the identity $L_\varphi(x, y) + L_\varphi(y, x) = b\sin\varphi$ over the domain of L_φ. □

The proofs of the following results for a standard image of Type 2 are omitted since they are similar to the ones provided for Type 1.

Theorem 5.3. *Let $\boldsymbol{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ be a standard image of Type 2 of a convex quadrilateral $\boldsymbol{D}$. If x_1, x_0 are the lengths of respectively the first and the second-order φ-diameters of $\boldsymbol{D}_s$, then*

i. $x_0(\varphi) = X_\varphi(\beta,\,\alpha_0,\,\alpha)$ *and* $x_1(\varphi) = X_\varphi(\beta,\,\alpha_0,\,\beta)$, *for* $0 \le \varphi < \alpha_0$;

ii. $x_0(\varphi) = X_\varphi(\beta,\,\alpha_0,\,0)$ *and* $x_1(\varphi) = X_\varphi(\beta,\,\alpha_0,\,\gamma)$, *for* $\alpha_0 \le \varphi < \alpha$;

iii. $x_0(\varphi) = X_\varphi(\beta,\,\alpha_0,\,0)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\,\beta_0,\,0)$, *for* $\alpha \le \varphi < \beta$;

iv. $x_0(\varphi) = -X_\varphi(\alpha,\,\beta_0,\,\gamma)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\,\beta_0,\,0)$, *for* $\beta \le \varphi < \beta_0$;

v. $x_0(\varphi) = -X_\varphi(\alpha,\,\beta_0,\,\beta)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\,\beta_0,\,\alpha)$, *for* $\beta_0 \le \varphi < \gamma$;

vi. $x_0(\varphi) = -X_\varphi(\beta,\,\alpha_0,\,\alpha)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\,\beta_0,\,\alpha)$, *for* $\gamma \le \varphi < \pi$.

Theorem 5.4. *Let $\boldsymbol{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ be a standard image of Type 2 of a convex quadrilateral $\boldsymbol{D}$. If ℓ_0, ℓ and ℓ_1 are the supplementary φ-measures of $\boldsymbol{D}_s$, then*

$$\ell_0(\varphi) = \begin{cases} L_\varphi(\beta_0,\, \alpha) - L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad 0 \le \varphi < \alpha_0 \ \textit{or} \ \ \beta_0 \le \varphi < \gamma \\ L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad \alpha_0 \le \varphi < \beta \\ -L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad \beta \le \varphi < \beta_0 \\ L_\varphi(\beta,\, \alpha_0) - L_\varphi(\beta_0,\, \alpha), & \textit{if} \quad \gamma \le \varphi < \pi \end{cases},$$

$$\ell(\varphi) = \begin{cases} -L_\varphi(\alpha_0,\, \beta), & \textit{if} \quad 0 \le \varphi < \alpha_0 \\ L_\varphi(\alpha_0,\, \beta), & \textit{if} \quad \alpha_0 \le \varphi < \alpha \\ L_\varphi(\beta_0,\, \alpha) - L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad \alpha \le \varphi < \beta \\ L_\varphi(\beta_0,\, \alpha), & \textit{if} \quad \beta \le \varphi < \beta_0 \\ -L_\varphi(\beta_0,\, \alpha), & \textit{if} \quad \beta_0 \le \varphi < \gamma \\ -L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad \gamma \le \varphi < \pi \end{cases},$$

$$\ell_1(\varphi) = \begin{cases} b \sin\varphi, & \textit{if} \quad 0 \le \varphi < \alpha_0 \ \textit{or} \ \beta_0 \le \varphi < \pi \\ -L_\varphi(\alpha,\, \beta_0), & \textit{if} \quad \alpha_0 \le \varphi < \alpha \\ L_\varphi(\alpha,\, \beta_0), & \textit{if} \quad \alpha \le \varphi < \beta_0 \end{cases}.$$

Corollary 5.2. *If a standard image $\boldsymbol{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ of a convex quadrilateral is of Type 2 then*

$$b_{\boldsymbol{D}_s}(\varphi) = \begin{cases} L_\varphi(\beta_0,\, \alpha), & \textit{if} \quad 0 \le \varphi < \alpha \\ b \sin\varphi, & \textit{if} \quad \alpha \le \varphi < \beta \\ L_\varphi(\alpha_0,\, \beta), & \textit{if} \quad \beta \le \varphi < \gamma \\ L_\varphi(\alpha,\, \beta_0), & \textit{if} \quad \gamma \le \varphi < \pi \end{cases}.$$

6. Orientation-dependent chord length distribution function and the covariogram of a convex quadrilateral prism

Denote by $\mathbf{D}_s^h$ the right prism $\{(x, y, z) : (x, y) \in \mathbf{D}_s,\, 0 < z \le h\}$, where $\mathbf{D}_s$ is a standard image of a convex quadrilateral. For a vector

$$\omega = (\cos\varphi \cos\theta,\, \sin\varphi \cos\theta,\, \sin\theta) \in \mathbb{S}^2,$$

let $\omega^\perp$ be the orthogonal complement of $\{t\omega \,:\, t \in \mathbb{R}\}$ in $\mathbb{R}^3$, and $\Pi_{\mathbf{D}_s^h}(\varphi, \theta)$ be the orthogonal projection of $\mathbf{D}_s^h$ onto the plane $\omega^\perp$.

We define the chord length distribution function in direction ω for $\mathbf{D}_s^h$ by

$$F_{\mathbf{D}_s^h}(t, \varphi, \theta) = \frac{L_2\{y \in \Pi_{\mathbf{D}_s^h}(\varphi, \theta) : \chi(l_{(\varphi,\theta)} + y) \le t\}}{b_{\mathbf{D}_s^h}(\varphi, \theta)},$$

where $l_{(\varphi,\theta)} + y$ is the line that passes through $y \in \omega^\perp$ and has direction vector ω,

$$\chi(l_{(\varphi,\theta)} + y) = L_1\big((l_{(\varphi,\theta)} + y) \cap \mathbf{D}_s^h\big),$$

and

$$b_{\mathbf{D}_s^h}(\varphi, \theta) = L_2(\Pi_{\mathbf{D}_s^h}(\varphi, \theta)).$$

As $\{z \in \mathbb{R}^3 : \ z = \frac{h}{2}\}$ is a plane of symmetry of $\mathbf{D}_s^h$, we notice that $F_{\mathbf{D}_s^h}(t,\varphi,\theta) = F_{\mathbf{D}_s^h}(t,\varphi-\pi,\theta)$, for $\varphi \in [\pi, 2\pi)$ and $F_{\mathbf{D}_s^h}(t,\varphi,\theta) = F_{\mathbf{D}_s^h}(t,\varphi,-\theta)$. Based on this observation, from now on we will assume that $\varphi \in [0,\,\pi)$ and $\theta \in [0,\,\frac{\pi}{2}]$.

Denote

$$x_{\max}(\varphi,\theta) = \max_{y \in \Pi_{\mathbf{D}_s^h}(\varphi,\theta)} \chi(l_{(\varphi,\theta)} + y).$$

It is easy to check that

$$x_{\max}(\varphi,\theta) = \begin{cases} \frac{x_{\max}(\varphi)}{\cos\theta}, & \text{if} \ \ 0 \le \theta \le \tan^{-1} \frac{h}{x_{\max}(\varphi)} \\ \frac{h}{\sin\theta}, & \text{if} \ \ \tan^{-1} \frac{h}{x_{\max}(\varphi)} < \theta \le \frac{\pi}{2} \end{cases}. \tag{6.1}$$

Theorem 6.1. *For a $\varphi \in [0,\,\pi)$, let x_1 and x_0 be the lengths of the first and the second-order φ-diameters of $\boldsymbol{D}_s$, respectively. Let $\ell_0,\,\ell,\,\ell_1$ be the supplementary φ-measures of $\boldsymbol{D}_s$, and denote $b_{\boldsymbol{D}_s} = \ell_0 + \ell + \ell_1$. Then, for the direction $\omega = (\cos\varphi\cos\theta,\,\sin\varphi\cos\theta,\,\sin\theta)$, $0 \le \theta \le \frac{\pi}{2}$ and the prism $\boldsymbol{D}_s^h$, the following statements take place:*

(a) *If $\tan^{-1}\frac{h}{x_0} < \theta \le \frac{\pi}{2}$ and $0 \le t < x_{\max}(\varphi,\theta)$, or $0 \le \theta \le \tan^{-1}\frac{h}{x_0}$ and $0 \le t < x_0 \sec\theta$, then*

$$F_{\boldsymbol{D}_s^h}(t,\varphi,\theta) = \frac{a_1 t + a_2 t^2}{\|\boldsymbol{D}_s\| \sin\theta + b_{\boldsymbol{D}_s} h \cos\theta}, \tag{6.2}$$

where

$$a_1 = h\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)\cos^2\theta + b_{\boldsymbol{D}_s}\sin 2\theta, \ \ a_2 = -\frac{3}{2}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)\sin\theta\cos^2\theta;$$

(b) *If $0 \le \theta \le \tan^{-1}\frac{h}{x_0}$ and $x_0\sec\theta \le t < x_{\max}(\varphi,\theta)$, then $x_0 < x_1$ and*

$$F_{\boldsymbol{D}_s^h}(t,\varphi,\theta) = \frac{c_0 + c_1 t + c_2 t^2}{\|\boldsymbol{D}_s\| \sin\theta + b_{\boldsymbol{D}_s} h \cos\theta}, \tag{6.3}$$

where

$$c_0 = (h\cos\theta + \frac{x_0}{2}\sin\theta)\left(\ell_0 - \frac{\ell x_0}{x_1 - x_0}\right),$$

$$c_1 = (h\cos^2\theta + x_1\sin 2\theta)\left(\frac{\ell}{x_1 - x_0} + \frac{\ell_1}{x_1}\right), \ \ c_2 = -\frac{3}{2}\sin\theta\cos^2\theta\left(\frac{\ell}{x_1 - x_0} + \frac{\ell_1}{x_1}\right).$$

Proof. Using the formula (see [11]) that establishes a relation between the orientation-dependent chord length distribution functions of a cylinder and its base, for $0 \le t < x_{\max}(\varphi,\theta)$ we obtain

$$\begin{aligned} F_{\mathbf{D}_s^h}(t,\varphi,\theta) &= \frac{b_{\mathbf{D}_s}\cos\theta}{\|\mathbf{D}_s\|\sin\theta + b_{\mathbf{D}_s} h\cos\theta} \times \\ &\times \left[(h - t\sin\theta)F_{\mathbf{D}_s}(t\cos\theta,\varphi) + 2t\sin\theta - \sin\theta\int_0^t F_{\mathbf{D}_s}(u\cos\theta,\varphi)du\right]. \end{aligned} \tag{6.4}$$

(a) By 6.1, the inequality $\tan\theta > \frac{h}{x_0}$ implies $x_{\max}(\varphi,\theta) = \frac{h}{sin\theta}$, and then

$$t\cos\theta < \frac{h}{\tan\theta} < x_0,$$

for any $t \in [0,\, x_{\max}(\varphi,\theta))$.

If $\tan\theta \le \frac{h}{x_0}$ but $0 \le t < x_0 \sec\theta$, the inequality $t\cos\theta < x_0$ still holds. Therefore, by Theorem 4.1, we substitute $F_{\mathbf{D}_s}(t\cos\theta,\varphi)$ and $F_{\mathbf{D}_s}(u\cos\theta,\varphi)$ in 6.4 by

$$\frac{1}{b_{\mathbf{D}_s}}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right) t\cos\theta \text{ and } \frac{1}{b_{\mathbf{D}_s}}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right) u\cos\theta,$$

respectively. Computation of the integral in 6.4 followed by combining the like terms results in 6.2.

(b) Let now $\tan\theta \le \frac{h}{x_0}$ but $x_0\sec\theta \le t < x_{\max}(\varphi,\theta)$. Then $x_0 < x_1$, otherwise it will contradict to 6.1. Theorem 4.1 yields

$$F_{\mathbf{D}_s}(t\cos\theta,\varphi) = \frac{1}{b_{\mathbf{D}_s}}\left(\ell_0 + \frac{t\cos\theta - x_0}{x_1 - x_0}\ell + \frac{t\cos\theta}{x_1}\ell_1\right), \tag{6.5}$$

and

(6.6)

$$\int_0^t F_{\mathbf{D}_s}(u\cos\theta,\varphi)du =$$

$$\int_0^{x_0\sec\theta} F_{\mathbf{D}_s}(u\cos\theta,\varphi)du + \int_{x_0\sec\theta}^t F_{\mathbf{D}_s}(u\cos\theta,\varphi)du = \frac{1}{b_{\mathbf{D}_s}}\times$$

$$\left[\int_0^{x_0\sec\theta}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)u\cos\theta du + \int_{x_0\sec\theta}^t\left(\ell_0 + \frac{u\cos\theta - x_0}{x_1 - x_0}\ell + \frac{u\cos\theta}{x_1}\ell_1\right)du\right].$$

To reach 6.3, it remains to evaluate 6.6, substitute its value along with 6.5 into 6.4, and simplify. □

Corollary 6.1. *Let*

$$\mu(\varphi,\theta) = L_2\Bigg(\{y \in \Pi_{\boldsymbol{D}_s^h}(\varphi,\theta) : \chi(l_{(\varphi,\theta)} + y) = x_{max}(\varphi,\theta)\}\Bigg).$$

The function $F_{\boldsymbol{D}_s^h}(\cdot,\varphi,\theta)$ is continuous on the real axis if and only if $\mu(\varphi,\theta) = 0$. Otherwise, if $\mu(\varphi,\theta) > 0$ for some pair (φ,θ), then $F_{\boldsymbol{D}_s^h}(\cdot,\varphi,\theta)$ has a jump discontinuity at $x_{\max}(\varphi,\theta)$. The jump is equal to

$$\frac{\mu(\varphi,\theta)}{\|\boldsymbol{D}_s\|\sin\theta + b_{\boldsymbol{D}_s}h\cos\theta}.$$

Proof. For any (φ,θ), the continuity of $F_{\mathbf{D}_s^h}(\cdot,\varphi,\theta)$ at $t = 0$ immediately follows from 6.2. The continuity at $t = x_0\sec\theta$ also takes place. Careful calculations show that the expressions in 6.2 and 6.3 coincide when $t = x_0\sec\theta$. Thus, the only discontinuity may occur at $t = x_{max}(\varphi,\theta)$.

Since

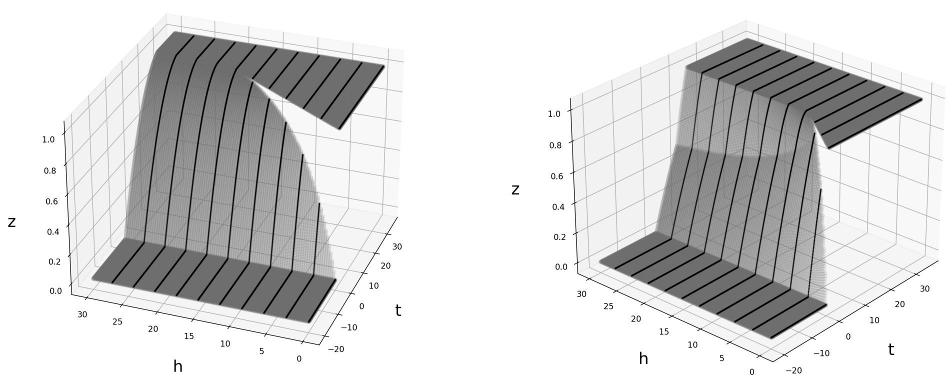
$$F_{\mathbf{D}_s^h}(x_{max}(\varphi,\theta)-,\varphi,\theta) = \frac{L_2\{y \in \Pi_{\mathbf{D}_s^h}(\varphi,\theta) : \chi(l_{(\varphi,\theta)} + y) < x_{max}(\varphi,\theta)\}}{b_{\mathbf{D}_s^h}(\varphi,\theta)} =$$

$$= 1 - \frac{\mu(\varphi,\theta)}{b_{\mathbf{D}_s^h}(\varphi,\theta)},$$

the continuity at $x_{max}(\varphi,\theta)$ holds if and only if $\mu(\varphi,\theta) = 0$. The jump is equal to $\frac{\mu(\varphi,\theta)}{b_{\mathbf{D}_s^h}(\varphi,\theta)} = \frac{\mu(\varphi,\theta)}{\|\mathbf{D}_s\| \sin\theta + b_{\mathbf{D}_s} h \cos\theta}$. $\square$

Remark 6.1. *One can verify that $\mu(\varphi,0) = h \cdot \ell(\varphi)$, so we rediscover Corollary 4.1. For the other extreme, $\mu(\varphi,\frac{\pi}{2}) = \|\boldsymbol{D}_s\|$ holds. The jump in this case is the highest possible,* 1. *We do not aim to compute $\mu(\varphi,\theta)$ for other directions.*

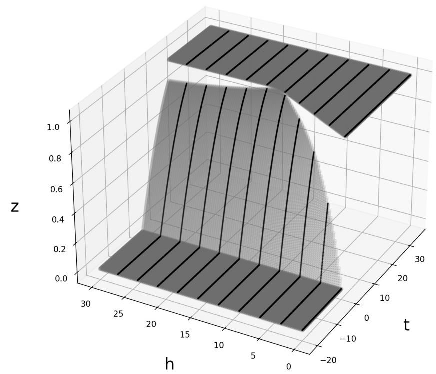
In order to visualize the possible breaks in continuity and smoothness of the ODCLD function, we plot the function $z(t,h) = F_{\mathbf{D}_s^h}(t,\varphi,\theta)$ for a given pair (φ,θ) and different values of the height h. As an example, in Figure 1, this is done for the prism based on the kite $\mathbf{D}_s = [10, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}]$, where $\varphi = \frac{\pi}{6}$, $\theta = \frac{\pi}{3}$, and then $\varphi = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$.



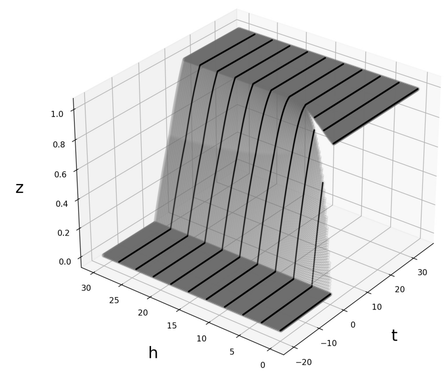
(A) The surface $z(t,h) = F_{\mathbf{D}_s^h}(t, \frac{\pi}{6}, \frac{\pi}{3})$ (B) The surface $z(t,h) = F_{\mathbf{D}_s^h}(t, \frac{\pi}{2}, \frac{\pi}{4})$

Рис. 1. Examples of orientation-dependent chord length distribution functions in right prisms $\mathbf{D}_s^h$ with base $\mathbf{D}_s = [10, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}]$

Each of the highlighted curves on the surface represents the graph of the ODCLD function for the prism of a given height. Figure 2 is created by the same logic for the prisms with a trapezoidal base $\mathbf{D}_s = [10, \frac{\pi}{6}, \frac{\pi}{4}, \frac{2\pi}{3}, \pi - \tan^{-1}\frac{\sqrt{3}}{4-\sqrt{3}}]$.



(A) The surface $z(t,h) = F_{\mathbf{D}_s^h}(t, \frac{\pi}{2}, \frac{2\pi}{5})$



(B) The surface $z(t,h) = F_{\mathbf{D}_s^h}(t, \frac{9\pi}{10}, \frac{2\pi}{7})$

Рис. 2. Examples of orientation-dependent chord length distribution functions in right prisms $\mathbf{D}_s^h$ with base $\mathbf{D}_s = [10, \frac{\pi}{6}, \frac{\pi}{4}, \frac{2\pi}{3}, \pi - \tan^{-1}\frac{\sqrt{3}}{4-\sqrt{3}}]$

Theorem 6.2. *For a $\varphi \in [0, \pi)$, let x_1 and x_0 be the lengths of the first and the second-order φ-diameters of $\boldsymbol{D}_s$, respectively. Let ℓ_0, ℓ, ℓ_1 be the supplementary φ-measures of $\boldsymbol{D}_s$, and denote $b_{\boldsymbol{D}_s} = \ell_0 + \ell + \ell_1$. Then, for the direction $\omega = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$, $0 \le \theta \le \frac{\pi}{2}$, the covariogram $C_{\boldsymbol{D}_s^h}(t\omega) = C_{\boldsymbol{D}_s^h}(t,\varphi,\theta)$ of the prism $\boldsymbol{D}_s^h$ has the following representation:*

(a) *If $\tan^{-1}\frac{h}{x_0} < \theta \le \frac{\pi}{2}$ and $0 \le t < x_{\max}(\varphi,\theta)$, or $0 \le \theta \le \tan^{-1}\frac{h}{x_0}$ and $0 \le t < x_0\sec\theta$, then*

$$C_{\boldsymbol{D}_s^h}(t,\varphi,\theta) = \left(\|\boldsymbol{D}_s\| - b_{\boldsymbol{D}_s}\cos\theta\cdot t + \frac{1}{2}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)\cos^2\theta\cdot t^2\right)(h - \sin\theta\cdot t);$$

(b) *If $0 \le \theta \le \tan^{-1}\frac{h}{x_0}$ and $x_0\sec\theta \le t < x_{\max}(\varphi,\theta)$, then $x_0 < x_1$ and*

$$C_{\boldsymbol{D}_s^h}(t,\varphi,\theta) = \frac{1}{2}\left(\frac{\ell}{x_1 - x_0} + \frac{\ell_1}{x_1}\right)(x_1 - \cos\theta\cdot t)^2(h - \sin\theta\cdot t).$$

Proof. Let $0 \le t < x_{max}(\varphi,\theta)$. Since

$$\mathbf{D}_s^h \cap \left(\mathbf{D}_s^h + t\omega\right) = (\mathbf{D}_s \cap \{\mathbf{D}_s + (t\cos\theta)\phi\}) \times [t\sin\theta, h],$$

we obtain

$$C_{\mathbf{D}_s^h}(t\omega) = L_2(\mathbf{D}_s \cap \{\mathbf{D}_s + (t\cos\theta)\phi\}) \cdot (h - t\sin\theta),$$

and then

$$C_{\mathbf{D}_s^h}(t,\varphi,\theta) = C_{\mathbf{D}_s}(t\cos\theta,\varphi)(h - t\sin\theta). \tag{6.7}$$

The proof now follows from 6.7 and Theorem 4.2. $\square$

Remark 6.2. *Taking $\theta = 0$, it is easy to check that all the results obtained in Section 4 are coherent with the results presented in the current section.*

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CONVERGENCE OF GENERAL FOURIER SERIES OF DIFFERENTIABLE FUNCTIONS

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Abstract. Convergence of classical Fourier series (trigonometric, Haar, Walsh, ... systems) of differentiable functions are trivial problems and they are well known. But general Fourier series, as it is known, even for the function $f(x) = 1$ does not converge. In such a case, if we want differentiable functions with respect to the general orthonormal system (ONS) (φ_n) to have convergent Fourier series, we must find the special conditions on the functions φ_n of system (φ_n). This problem is studied in the present paper. It is established that the resulting conditions are best possible. Subsystems of general orthonormal systems are considered.

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Keywords: Orthonormal system; Fourier coefficients; bounded variation; linner fuctional.

1. Auxiliary notations and results

By V we denote the class of functions with bounded variation on $[0, 1]$ and $V(f)$ is the finite variation of function f. C_V is the set of functions with $f' \in V$. A is the class of absolute continuous functions. A is a Banach space with the norm

$$\|f\|_A = \|f\|_C + \int_0^1 |f'(x)|\, dx,$$

where C is a class of continuous functions.

Let (φ_n) be an ONS on $[0, 1]$, where φ_n are real-valued functions and $f \in \ell_2$, then the numbers

$$C_n(f) = \int_0^1 f(x)\varphi_n(x)\, dx, \quad n = 1, 2, \ldots,$$

are the general Fourier coefficients of function f.

General Fourier series is

$$\sum_{k=1}^{\infty} C_k(f)\varphi_k(x)$$

and its partial sum is

$$S_n(x, f) = \sum_{k=1}^{n} C_k(f)\varphi_k(x).$$

Let $(p = 1, 2, \dots)$

$$B_{np}(x, f) = \sum_{k=n}^{n+p} C_k(f)\varphi_k(x).$$

Lemma 1.1. *Let* (φ_n) *be an ONS on* $[0, 1]$ *and* $f \in C_V$*, then*

$$B_{np}(x, f) = f(1) \int_0^1 \sum_{k=n}^{n+p} \varphi_k(u)\varphi_k(x)\, du$$

$$- \int_0^1 f'(u) \sum_{k=n}^{n+p} \int_0^u \varphi_k(v)\, dv\, du\; \varphi_k(x). \tag{1.1}$$

Proof. Integrating by parts, we get

$$C_k(f) = \int_0^1 f(u)\varphi_k(u)\, du = f(1) \int_0^1 \varphi_k(u)\, du - \int_0^1 f'(u) \int_0^u \varphi_k(v)\, dv\, du.$$

From here we can easily obtain (1.1).

Suppose that

$$H_{np}(u, x) = \sum_{k=n}^{n+p} \varphi_k(u)\varphi_k(x)$$

and

$$A_{np}(u, x) = \int_0^u \sum_{k=n}^{n+p} \varphi_k(v)\, dv \varphi_k(x),$$

then by (1.1) we get

$$B_{np}(x, f) = f(1) \int_0^1 H_{np}(u, x)\, du - \int_0^1 f'(u) A_{np}(u, x)\, du. \tag{1.2}$$

The lemma is proved. □

Lemma 1.2. *Let* (φ_n) *be an ONS on* $[0, 1]$*. Then if* $N = n + p$*,*

$$\lim_{n\to\infty} n^{-\frac{1}{2}} N^{-\frac{3}{2}} \sum_{k=n}^{N} \varphi_k^2(x) = 0 \;\; a.e. \;\; on \;\; [0, 1].$$

Proof. It is obvious that

$$N^{-\frac{3}{2}} \sum_{k=n}^{N} \varphi_k^2(x) \le \sum_{k=n}^{\infty} k^{-\frac{3}{2}} \varphi_k^2(x).$$

Since

$$\sum_{k=n}^{\infty} k^{-\frac{3}{2}} \int_0^1 \varphi_k^2(x)\, dx = \sum_{k=n}^{\infty} k^{-\frac{3}{2}} < +\infty,$$

according to Levy theorem the series

$$\sum_{k=n}^{\infty} k^{-\frac{3}{2}} \varphi_k^2(x)$$

converges a.e. on $[0, 1]$.

So a.e. on $[0,1]$,

$$\lim_{n\to\infty} n^{-\frac{1}{2}} N^{-\frac{3}{2}} \sum_{k=n}^{N} \varphi_k^2(x) = 0.$$

We denote

$$D_N(x) = \max_{1\le i<N} \left| \int_0^{\frac{i}{N}} A_{np}(u,x)\,du \right| = \left| \int_0^{\frac{i_N}{N}} A_{np}(u,x)\,du \right| \ (1 \le i_N < N). \tag{1.3}$$

Lemma 1.3. *Let (φ_n) be an ONS on $[0,1]$ and $i = 1,2,\dots,N$, then if $n+p=N$,*

$$\int_{\frac{i-1}{N}}^{\frac{i}{N}} |A_{np}(u,x)|\,du \le \frac{1}{N}\left(\sum_{k=n}^{n+p} \varphi_k^2(x)\right)^{\frac{1}{2}}. \tag{1.4}$$

Proof. By Bessel inequality

$$\sum_{k=n}^{n+p} \left(\int_0^u \varphi_k(v)\,dv\right)^2 \le 1.$$

Using Cauchy and Hölder inequalities we get

$$\left|\int_{\frac{i-1}{N}}^{\frac{i}{N}} A_{np}(u,x)\,du\right| \le \frac{1}{\sqrt{N}}\left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\sum_{k=n}^{N}\int_0^u \varphi_k(v)\,dv\varphi_k(x)\right)^2 du\right)^{\frac{1}{2}}$$

$$\le \frac{1}{\sqrt{N}}\left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \sum_{k=n}^{N}\left(\int_0^u \varphi_k(v)\,dv\right)^2 du \sum_{k=n}^{N}\varphi_k^2(x)\right)^{\frac{1}{2}} \le \frac{1}{N}\left(\sum_{k=n}^{n+p}\varphi_k^2(x)\right)^{\frac{1}{2}}.$$

Definition 1.1. *By (W,C,x), $x \in G$, we denote the class of any ONS (φ_n) such that for each of them there exists a sequence $(\varepsilon_n(x))$, where $\lim_{n\to\infty}\varepsilon_n(x) = 0$, and*

$$\left|\sum_{k=n}^{n+p} C_k(f)\varphi_k(x)\right| < m_f \varepsilon_n(x)$$

for any $f' \in V$ and p.

Lemma 1.4. *If $\varphi_n(u) = \cos 2\pi n u$, then $(\varphi_n) \in (W,C,x)$ for any $x \in [0,1]$.*

Proof. Let $f' \in V$, then we have

$$\begin{aligned} C_k(f) &= \int_0^1 f(u)\varphi_k(u)\,du = \int_0^1 f(u)\cos 2\pi k u\,du \\ &= f(1)\int_0^1 \cos 2\pi k u\,du - \frac{1}{2\pi k}\int_0^1 f'(u)\sin 2\pi k u\,du \\ &= -\frac{1}{2\pi k}\int_0^1 f'(u)\sin 2\pi k u\,du. \end{aligned}$$

Since

$$\sum_{k=n}^{n+p}\left(\int_0^1 f'(u)\cos 2\pi k u\,du\right)^2 \le \int_0^1 (f'(u))^2\,du,$$

using the Cauchy inequality we get

$$\Big|\sum_{k=n}^{n+p} C_k(f)\varphi_k(x)\Big| = \frac{1}{2\pi}\Big|\sum_{k=n}^{n+p}\int_0^1 f'(u)\sin 2\pi ku\,du\frac{\sin 2\pi kx}{k}\Big|$$
$$\le \Big(\sum_{k=n}^{n+p}\Big(\int_0^1 f'(u)\cos 2\pi ku\,du\Big)^2\Big)^{\frac{1}{2}}\Big(\sum_{k=n}^{n+p}\frac{\sin^2 2\pi kx}{k^2}\Big)^{\frac{1}{2}} \le m_f\,\frac{1}{\sqrt{n}}\,.$$

Lemma 1.5. *If (X_n) is a Haar system, then $(X_n) \in (W, C, x)$.*

Proof. Let $n = 2^m$ and $p \le 2^m$ is any natural number. If $f' \in V$, according to the definition of Haar system (see [20]),

$$|C_{2^m+k}(f)| = O(1)2^{-\frac{3m}{2}} \quad (1 \le k \le 2^m)$$

and $(x \in [0, 1])$

$$\Big|\sum_{k=2^m}^{2^m+p} X_k(x)\Big| \le 2^{\frac{m}{2}}.$$

Then

$$\Big|\sum_{k=2^m}^{2^m+p} C_k(f)X_k(x)\Big| = O(1)2^{-\frac{3m}{2}}2^{\frac{m}{2}} = O(1)2^{-m}.$$

If $n + p = 2^{m+s}$, we get

$$\Big|\sum_{k=2^m}^{2^{m+s}} C_k(f)X_k(x)\Big| = \Big|\sum_{r=m}^{m+s-1}\sum_{k=2^r}^{2^{r+1}} C_k(f)X_k(x)\Big|$$
$$= O(1)\sum_{r=m}^{m+s} 2^{-r} = O(1)2^{-m}.$$

Analogously we can proof that

$$\Big|\sum_{k=m}^{m+p} C_k(f)X_k(x)\Big| = O(1)m^{-1}.$$

Theorem 1.1 (Banach [1])**.** *Let $f \in L_2$ be an arbitrary function ($f \not\simeq 0$). Then there exists an ONS (φ_n) such that*

$$\limsup_{n\to\infty}|S_n(x, f)| = +\infty \ \ \textit{a.e. on } [0, 1],$$

where

$$S_n(x, f) = \sum_{k=1}^{n} C_k(f)\varphi_k(x).$$

Theorem 1.2 (see [7]). *Let $F, f \in L_2$, then*

$$\begin{aligned}\int_0^1 f(u)F(u)\,du &= N\sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(f(u)-f\left(u+\frac{1}{N}\right)\right)du\int_0^{\frac{i}{N}}F(u)\,du\\ &+N\sum_{i=1}^{N}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\int_{\frac{i-1}{N}}^{\frac{i}{N}}(f(u)-f(v))\,dvF(u)\,du\\ &+N\int_{1-\frac{1}{N}}^{1}f(u)\,du\int_0^1F(u)\,du. \qquad (1.5)\end{aligned}$$

2. The main proposition

Problems of convergence of orthogonal series are well studied. There should be noted the results: E. Men'shov [11], H. Rademacher [12], W. Orlich [14], S. Kachmarcz [8], K. Tandori [15], A. Olevsky [13], etc. On the other hand, the convergence problems of Fourier series of functions from some differentiable class are less studied: J. R. McLaughlin [10], S. V. Bochkarev [2], B. S. Kashin [9], L. Gogoladze, V. Tsagareishvili [7], G. Cagareishvili [3]. In the case when the convergence of the Fourier series of differentiable functions is necessary, certain conditions must be imposed on the functions of ONS. This is necessary because, according to Banach Theorem, in the general case the Fourier series does not converge even for the function $f(x) = 1$, $x \in [0, 1]$ (see Theorem 1.1).

In the present paper, we give special conditions which are imposed on functions of ONS (φ_n) under which the Fourier series of the functions of class C_V will be convergent a.e. on $[0, 1]$.

The similar Problems are studied in the papers [2, 9, 7, 3, 5, 6, 16, 17, 18].

3. The main results

We denote $N = n + p$.

Theorem 3.1. *Suppose that (φ_n) is an ONS on $[0, 1]$ and at the point $x \in G$ the series*

$$\sum_{k=1}^{\infty} C_k(l)\varphi_k(x)$$

converges, where $l(u) = 1$, $u \in [0, 1]$. If at the point $x \in G$ (see (1.3))

$$\lim_{n\to\infty} D_N(x) = 0, \qquad (3.1)$$

then the series

$$\sum_{k=1}^{\infty} C_k(f)\varphi_k(x)$$

converges at the point $x \in G$ for any $f \in C_V$.

Proof. Substituting $F(x) = A_{n,p}(u,x)$ and $f = f'$ $(x \in G)$ in (1.5) we get

(3.2)

$$\int_0^1 f'(u)A_{np}(u,x)\,du = N\sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(f'(u) - f'\left(u+\frac{1}{N}\right)\right)du\int_0^{\frac{i}{N}} A_{np}(u,x)\,du$$

$$+ N\sum_{i=1}^{N}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\int_{\frac{i-1}{N}}^{\frac{i}{N}}(f'(u) - f'(v))\,dvA_{n,p}(u,x)\,du$$

$$+ N\int_{1-\frac{1}{N}}^{1} f'(u)\,du\int_0^1 A_{np}(u,x)\,du = a + b + c.$$

Since $f' \in V$ (see (3.1)), we have

$$|a| \le N\frac{1}{N}\sum_{i=1}^{N-1}\sup_{u\in\Delta_{in}}\left|f'(u) - f'\left(u+\frac{1}{N}\right)\right|\max_{1\le i<N}\left|\int_0^{\frac{i}{N}} A_{np}(u,x)\,du\right|$$

$$\le V(f')D_N(x). \tag{3.3}$$

Applying (1.4) and $f' \in V$, we write (see Lemma 1.2 and Lemma 1.3)

$$|b| \le N\,\frac{1}{N}\sum_{i=1}^{N}\max_{u,v\in\Delta_{in}}|f'(u) - f'(v)|\int_{\frac{i-1}{N}}^{\frac{i}{N}}|A_{np}(u,x)|\,du$$

$$\le V(f')\,\frac{1}{N}\left(\sum_{k=n}^{n+p}\varphi_k^2(x)\right)^{\frac{1}{2}}$$

$$= O(1)n^{-\frac{1}{4}}\left(N^{-\frac{3}{2}}\sum_{k=n}^{n+p}\varphi_k^2(x)\right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt[4]{n}}\right). \tag{3.4}$$

Taking into account $f' \in V$ and (1.3), we obtain (see Lemmas 1.2 and 1.3)

$$|c| = O(1)\left|\int_0^1 A_{np}(u,x)\,du\right|$$

$$= O(1)\left(\left|\int_0^{1-\frac{1}{N}} A_{np}(u,x)\,du\right| + \left|\int_{1-\frac{1}{N}}^{1} A_{np}(u,x)\,du\right|\right)$$

$$= O\left(D_N(x) + \frac{1}{\sqrt[4]{n}}\right). \tag{3.5}$$

Thus from (3.2), taking into consideration (3.3), (3.4) and (3.5), we have

$$\left|\int_0^1 f'(u)A_n(u,x)\,dx\right| = O\left(D_N(x) + \frac{1}{\sqrt[4]{n}}\right). \tag{3.6}$$

We consider the function $l(u) = 1$, $u \in [0,1]$. Using the formula (1.1) and bear in mind

$$\int_0^1 \varphi_k(u)\,du = \int_0^1 l(u)\varphi_k(u)\,du = C_k(l),$$

we receive

$$\sum_{k=n}^{n+p} C_k(f)\varphi_k(x) = \sum_{k=n}^{n+p} C_k(l)\varphi_k(x) - \int_0^1 f'(u)A_n(u,x)\,du. \tag{3.7}$$

Finally, by condition of Theorem 1,

$$\lim_{n\to\infty}\left|\sum_{k=n}^{n+p} C_k(l)\varphi_k(x)\right| = 0.$$

So, from (3.6) and (3.7) there holds (see (3.1))

$$\lim_{n\to\infty}\left|\sum_{k=n}^{n+p} C_k(f)\varphi_k(x)\right| = 0$$

for any function $f \in C_V$ at the point $x \in G$. □

Theorem 3.2. *Let* (φ_n) *be an ONS on* $[0,1]$ *and* $x \in G$. *If*

$$\limsup_{n\to\infty} D_N(x) > M > 0, \tag{3.8}$$

then $(\varphi_n) \notin (W, C, x)$.

Proof. Suppose on the contrary that $(\varphi_n) \in (W, C, x)$. This means that for any $f \in C_V$,

$$\left|\sum_{k=n}^{n+p} C_k(f)\varphi_k(x)\right| \le m_f \varepsilon_n(x) \quad \Big(\lim_{n\to\infty} \varepsilon_n(x) = 0\Big).$$

For this propose, if $l(u) = 1$,

$$\left|\sum_{k=n}^{n+p} C_k(l)\varphi_k(x)\right| \le m_l \varepsilon_n(x).$$

Also, if in (1.2) we put $f(u) = q(u) = u$, we obtain

$$B_{n,p}(x,q) = \int_0^1 \sum_{k=n}^{n+p} \varphi_k(u)\,du\varphi_k(x) - \int_0^1 A_n(u,x)\,du.$$

Since

$$\int_0^1 \varphi_k(u)\,du = \int_0^1 l(u)\varphi_k(u)\,du = C_n(l),$$

from the last equality we get

$$B_{n,p}(x,q) = B_{n,p}(x,l) - \int_0^1 A_{np}(u,x)\,du. \tag{3.9}$$

Because of $q, l \in C_V$ we have that

$$|B_{n,p}(x,q)| \le m_q \varepsilon_n(x) \quad \text{and} \quad |B_{n,p}(x,l)| \le m_l \varepsilon_n(x).$$

From here and from (3.9) it obviously follows that

$$\left|\int_0^1 A_{np}(u,x)\,du\right| \le (m_q + m_l)\varepsilon_n(x). \tag{3.10}$$

We consider the increasing sequence (Z_n) such that

$$\lim_{n\to\infty} Zn = +\infty, \quad \lim_{n\to\infty} Z_n\varepsilon_n(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} Z_n \frac{1}{\sqrt[4]{n}} = 0. \tag{3.11}$$

Now we define the sequence of functions (h_N) in such a way:

$$h_N(u) = \begin{cases} 0, & u \in [0, \frac{i_N-1}{N}], \\ 1, & u \in [\frac{i_N}{N}, 1], \\ Nx - i_N + 1, & u \in [\frac{i_N-1}{N}, \frac{i_N}{N}]. \end{cases} \tag{3.12}$$

Substituting $f' = h_N$ we can rewrite (3.2) as

$$\begin{aligned} \int_0^1 h_N(u) A_{np}(u,x)\,dx = N \sum_{i=1}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(h_N(u) - h_N\left(u + \frac{1}{N}\right)\right) du \int_0^{\frac{i}{N}} A_{np}(u,x)\,du \\ + N \sum_{i=1}^{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{i-1}{N}}^{\frac{i}{N}} (h_N(u) - h_N(v))\,dv A_{np}(u,x)\,du \\ + N \int_{1-\frac{1}{N}}^{1} h_N(u)\,du \int_0^1 A_{np}(u,x)\,du = e + f + g. \end{aligned} \tag{3.13}$$

Applying (3.12) we write $|h_N(u) - h_N(v)| \le 1$, $u, v \in [0,1]$. Also, $h_N(u) - h_N(v) = 0$ when $u, v \in [\frac{i-1}{N}, \frac{i}{N}]$, $i = 1, \dots, i_N - 1; i_N + 1, \dots, N$.

For this reason, using Lemma 1.2 and Lemma 1.3, we receive

$$|f| \le N\,\frac{1}{N} \int_{\frac{N-1}{N}}^{\frac{i_N}{N}} |A_{np}(u,x)|\,du \le \frac{1}{N} \left(\sum_{k=n}^{n+p} \varphi_k^2(x) \right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt[4]{n}}\right). \tag{3.14}$$

We estimate the following integrals:

1) $\displaystyle \int_{\frac{i_N-2}{N}}^{\frac{i_N-1}{N}} \left(h_N(u) - h_N\left(u + \frac{1}{N}\right)\right) du = -\int_{\frac{i_N-1}{N}}^{\frac{i_N}{N}} (Nu - i_N + 1)\,du = -\frac{1}{2N};$

2) $\displaystyle \int_{\frac{i_N-1}{N}}^{\frac{i_N}{N}} \left(h_N(u) - h_N\left(u + \frac{1}{N}\right)\right) du = \int_{\frac{i_N-1}{N}}^{\frac{i_N}{N}} (Nu - i_N + 1)\,du - \frac{1}{N} = -\frac{1}{2N}.$

Taking into consideration these equalities we will show that

$$\begin{aligned} |e| &= N \left| \frac{1}{2N} \left(\int_0^{\frac{i_N}{N}} A_{np}(u,x)\,du + \int_0^{\frac{i_N-1}{N}} A_{np}(u,x)\,du \right) \right| \\ &= \frac{1}{2} \left| 2 \int_0^{\frac{i_N}{N}} A_{np}(u,x)\,du - \int_{\frac{i_N-1}{N}}^{\frac{i_N}{N}} A_{np}(u,x)\,du \right|. \end{aligned}$$

Moreover, according to Lemma 1.2 and Lemma 1.3, we conclude that

$$|e| \ge D_N(x) - O(1)\,\frac{1}{\sqrt[4]{n}}. \tag{3.15}$$

Next, by (3.10) and $h_N \in C_V$, we get

$$|g| = O(1)\varepsilon_n(x). \tag{3.16}$$

Finally using (3.13) with (3.14), (3.15) and (3.16) we can write

$$\left| \int_0^1 h_N(u) A_{np}(u,x)\,du \right| \ge D_N(x) - O(1)\varepsilon_n(x) - O(1)\,\frac{1}{\sqrt[4]{n}}. \tag{3.17}$$

We consider the sequence of linear and bounded on A functionals

$$R_n(f) = Z_n \int_0^1 f(u) A_{np}(u,x)\, du.$$

In our case,

$$R_n(h_N) = Z_n \int_0^1 h_N(u) A_{np}(u,x)\, du.$$

According to (3.8), (3.11) and (3.17) ($N = n+p$), we have

$$\lim_{n\to\infty} |R_n(h_N)| \geq \limsup_{n\to\infty} Z_n D_N(x) - O(1) \lim_{n\to\infty} Z_n \varepsilon_n(x) - O(1) \lim_{n\to\infty} Z_n \frac{1}{\sqrt[4]{n}} = +\infty. \tag{3.18}$$

By (3.12),

$$\|h_N\|_A = \|h_N\|_C + \int_0^1 |h'_N(u)|\, du \leq 2.$$

So, according to the Banach–Steinhaus Theorem (see (3.18)), there exists a function $s \in A$ such that

$$\lim_{n\to\infty} |R_n(s)| = \limsup_{n\to\infty} \left| Z_n \int_0^1 s(u) A_{np}(u,x)\, du \right| = +\infty. \tag{3.19}$$

Suppose

$$h(u) = \int_0^u s(v)\, dv.$$

As (see (1.2))

$$\int_0^1 H_{np}(u,x)\, du = \sum_{k=1}^n C_k(l) \varphi_k(x),$$

where $l(u) = 1$, $u \in [0,1]$ and (see p. 7)

$$\left| \sum_{k=1}^n C_k(l) \varphi_k(x) \right| \leq m_l \varepsilon_n(x),$$

then (see (3.11))

$$\lim_n \left| Z_n \int_0^1 H_{np}(u,x)\, du \right| = 0.$$

Using (1.2) when $f = h$, we get

$$B_{np}(x,h) = h(1) \int_0^1 H_{np}(u,x)\, du - \int_0^1 s(u) A_{np}(u,x)\, du.$$

From here

$$Z_n |B_{np}(x,h)| \geq \left| Z_n \int_0^1 s(u) A_{np}(u,x)\, du \right| - \left| h(1) Z_n \int_0^1 H_{np}(u,x)\, du \right|.$$

So, by (3.19), we obtain

$$\limsup_{n\to\infty} Z_n |B_{np}(x,h)| = +\infty. \tag{3.20}$$

On the other hand, as it was assumed $(\varphi_n) \in (W, C, x)$, in view of $h \in C_V$ we have $|B_{np}(x,h)| \leq m_h \varepsilon_n(x)$. From here we get $Z_n|B_{np}(x,h)| \leq Z_n m_h \varepsilon_n(x)$. Thus we have shown (see (3.11)) that

$$\lim_{n\to\infty} Z_n|B_{np}(x,h)| = m_h \lim_{n\to\infty} Z_n\varepsilon_n(x) = 0 \tag{3.21}$$

holds. Thus we obtain that (3.20) contradicts to (3.21), which means that $(\varphi_n) \notin (W, C, x)$. Theorem 3.2 is completely proved. □

Theorem 3.3. *Let (d_n) be a given increasing sequence. Any ONS (φ_n) contains the subsystem (φ_{n_k}) such that the series*

$$\sum_{k=1}^{\infty} d_k |C_{n_k}(f)\varphi_{n_k}(x)|$$

converges a.e. on $[0,1]$ for any $f \in C_V$.

Proof. We suppose that (φ_n) is the complete ONS. Then according to Parseval equality we have

$$\sum_{n=1}^{\infty}\Big(\int_0^u \varphi_n(v)\,dv\Big)^2 = u.$$

Hence there exists a sequence of natural numbers (n_k) such that uniformly with respect to $u \in [0,1]$,

$$\Big|\int_0^u \varphi_{n_k}(v)\,dv\Big| < \frac{k^{-2}}{d_k}, \quad k = 1, 2, \dots . \tag{3.22}$$

Integrating by parts when $f \in C_V$, we obtain

(3.23)

$$C_{n_k}(f) = \int_0^1 f(u)\varphi_{n_k}(u)\,du = f(1)\int_0^1 \varphi_{n_k}(u)\,du - \int_0^1 f'(u)\int_0^u \varphi_{n_k}(v)\,dv\,du.$$

According to (3.22) we conclude that

$$1)\quad \Big|\int_0^1 \varphi_{n_k}(u)\,du\Big| < \frac{k^{-2}}{d_k}, \quad k = 1, 2, \dots ,$$

$$2)\quad \Big|\int_0^1 f'(u)\int_0^u \varphi_{n_k}(v)\,dv\,du\Big| \leq \sup_{u\in[0,1]} |f'(u)|\frac{k^{-2}}{d_k}, \quad k = 1, 2, \dots .$$

From here and (3.23), for any $f \in C_V$ we get

$$|C_{n_k}(f)| = O(1)\,\frac{k^{-2}}{d_k}, \quad k = 1, 2, \dots .$$

Thus

$$\sum_{k=1}^{\infty} d_k|C_{n_k}(f)|\int_0^1 |\varphi_{n_k}(x)|\,dx = O(1)\sum_{k=1}^{\infty} d_k\frac{k^{-2}}{d_k}\Big(\int_0^1 \varphi_{n_k}^2(x)\,dx\Big)^{\frac{1}{2}} < +\infty.$$

As it is known by Levy theorem, a.e. on $[0,1]$,

$$\sum_{k=1}^{\infty} d_k|C_{n_k}(f)\varphi_{n_k}(x)| < +\infty \quad \text{for any} \quad f \in C_V. \qquad \square$$

4. Problems of efficiency

Theorem 4.1. *The system $\varphi_n(u) = \cos 2\pi nu$ on $[0,1]$ satisfies the condition (for any $x \in [0,1]$) $\lim_{n\to\infty} D_N(x) = 0$.*

Proof. We have ($N = n + p$)

$$A_{np}(u,x) = \sum_{k=n}^{n+p} \int_0^u \cos 2\pi kv \; dv \; \cos 2\pi kx = \frac{1}{2\pi} \sum_{k=n}^{n+p} \frac{1}{k} \sin 2\pi ku \cos 2\pi kx.$$

By the Hölder inequality we get ($i = 1, 2, \ldots, N$)

$$\Big| \int_0^{\frac{i}{N}} A_{np}(u,x)\, du \Big| = \frac{1}{2\pi} \Big| \int_0^{\frac{i}{N}} \sum_{k=n}^{n+p} \frac{1}{k} \sin 2\pi k u du \cos 2\pi kx \Big|$$

$$= O(1) \Big(\sum_{k=n}^{n+p} \frac{1}{k^2} \cos^2 2\pi kx \Big)^{\frac{1}{2}} = O\Big(\frac{1}{\sqrt{n}}\Big). \qquad \square$$

Theorem 4.2. *Haar system (X_n) on $[0,1]$ satisfies the condition (see* [20]*)*

$$\lim_{n\to\infty} D_N(x) = 0.$$

Proof. Definition of the Haar system imply that

$$1) \quad \Big| \int_0^u X_{2^s+k}(v)\, dv \Big| \le 2^{-\frac{s}{2}}, \;\text{ when }\; u \in \Big[\frac{k-1}{2^s}, \frac{k}{2^s}\Big]$$

$$\text{and} \quad \int_0^u X_{2^s+k}(v)\, dv = 0, \;\text{ when }\; u \notin \Big[\frac{k-1}{2^s}, \frac{k}{2^s}\Big];$$

$$2) \quad \Big| \int_0^t \int_0^u X_{2^s+k}(v)\, dv\, du \Big| \le 2^{-s}, \;\text{ when }\; t \in \Big[\frac{k-1}{2^s}, \frac{k}{2^s}\Big]$$

$$\text{and} \quad \int_0^t \int_0^u X_{2^s+k}(v)\, dv\, du = 0, \;\text{ when }\; t \notin \Big[\frac{k-1}{2^s}, \frac{k}{2^s}\Big].$$

From here for any $t \in [0,1]$ we get

$$\Big| \int_0^t \sum_{k=1}^{2^s} \int_0^u X_{2^s+k}(v)\, dv\, du X_{2^s+k}(x) \Big| \le 2^{-s} 2^{\frac{s}{2}} = 2^{-\frac{s}{2}}.$$

Hence

$$\Big| \int_0^t \sum_{m=2^r+1}^{2^n} \int_0^u X_m(v)\, dv\, du X_m(x) \Big|$$

$$= \Big| \sum_{s=r}^{n} \int_0^t \sum_{k=1}^{2^s} \int_0^u X_{2^s+k}(v)\, dv\, du\; X_{2^s+k}(x) \Big| \le \sum_{s=r}^{n} 2^{-\frac{s}{2}} = O(1) 2^{-\frac{r}{2}}.$$

Consequently, when $t = \frac{i}{N}$, putting n instead of $2^r + 1$ and $n + p = N$ instead of 2^n, we obtain

$$D_N(x) = \Big| \int_0^{\frac{i_N}{N}} \sum_{m=n}^{n+p} \int_0^u X_m(v)\, dv\, du\; X_m(x) \Big| = O(1) \frac{1}{\sqrt{n}}.$$

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DERIVATIVES OF MEROMORPHIC FUNCTIONS SHARING POLYNOMIALS WITH THEIR DIFFERENCE OPERATORS

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Abstract. In this paper, we investigate the uniqueness of meromorphic functions of finite order $f(z)$ concerning their difference operators $\Delta_c f(z)$ and derivatives $f'(z)$ and prove that if $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, ∞ CM, where $a(z)$ and $b(z)$ are two distinct polynomials, then they assume one of following cases: (1) $f'(z) \equiv \Delta_c f(z)$; (2) $f(z)$ reduces to a polynomial and $f'(z) - A\Delta_c f(z) \equiv (1-A)(c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0)$, where $A(\neq 1)$ is a nonzero constant and $c_n, c_{n-1}, \cdots, c_1, c_0$ are all constants. This generalizes the corresponding results due to Qi et al. and Deng et al.

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1. Introduction and main results

As we know, Nevanlinna theory plays a significant role in the study of the uniqueness theory of meromorphic functions. Recent years, the research about difference analogue of meromorphic functions has become a subject of some interests and there are extensive results on them. For the related results, the readers can refer to[1, 2, 6, 7, 10, 12, 16, 17, 20]. Throughout this paper, c always means a nonzero complex constant. Given a meromorphic function $f(z)$, we recall that a difference operator $\Delta_c f(z)$ is defined by $\Delta_c f(z) = f(z+c) - f(z)$. Suppose that $f(z)$ and $g(z)$ are two meromorphic functions and a is a finite complex constant. If $f(z) - a$ and $g(z) - a$ have the same zeros, then we say that they share a IM(ignoring multiplicities). If $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities, then we say that they share a CM(counting multiplicities). And the above definition also applies when a is a polynomial. Furthermore we use $\rho(f)$ to denote the order of $f(z)$.

In 2013, Chen and Yi[3] studied the unicity of $\Delta_c f(z)$ and $f(z)$ sharing three values CM and proved the following result.

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Theorem 1.1. [3] *Let $f(z)$ be a transcendental meromorphic function such that $\rho(f)$ is not an integer or infinite, and let a and b be two distinct constants. If $\Delta_c f(z)$ and $f(z)$ share a, b, ∞ CM, where $\Delta_c f(z) \not\equiv 0$, then $f(z) \equiv \Delta_c f(z)$.*

Remark 1.1. *In* [3]*, Chen and Yi conjectured that in Theorem 1.1, the condition that "$\rho(f)$ is not an integer" can be omitted.*

In 2014, Zhang et al.[20], Liu et al.[13] respectively confirmed this conjecture and proved the following result.

Theorem 1.2. [20, 13] *Let $f(z)$ be a transcendental entire function of finite order, and let a and b be two distinct constants. If $\Delta_c f(z)$ and $f(z)$ share a, b CM, where $\Delta_c f(z) \not\equiv 0$, then $f(z) \equiv \Delta_c f(z)$.*

Later, Li et al.[11], Cui et al.[5], Lü et al.[14] successively considered a meromorphic function rather than a transcendental meromorphic function in Theorem 1.2 and obtained the following result.

Theorem 1.3. [11, 5, 14] *Let $f(z)$ be a meromorphic function of finite order, and let a and b be two distinct constants. If $\Delta_c f(z)$ and $f(z)$ share a, b, ∞ CM, where $\Delta_c f(z) \not\equiv 0$, then $f(z) \equiv \Delta_c f(z)$.*

In 2019, Li[12] continued the study of the unicity of $\Delta_c f(z)$ and $f(z)$ sharing polynomials CM rather than values CM, which generalizes Theorem 1.3.

Theorem 1.4. [12] *Let $f(z)$ be a transcendental meromorphic function of finite order. If $\Delta_c f(z)$ and $f(z)$ share P_1, P_2, ∞ CM where P_1 and P_2 are two distinct polynomials, then $f(z) \equiv \Delta_c f(z)$.*

During the study of the uniqueness of $\Delta_c f(z)$ and $f(z)$, many researchers may be inspired to think about the following question.

Question 1.1. *Do the theorems above still hold if it is $\Delta_c f(z)$ and $f'(z)$ that share values CM since there are certain similarities between derivatives and difference operators of meromorphic functions?*

In 2018, Qi et al.[15] gave a positive answer to this question and proved the following result.

Theorem 1.5. [15] *Let $f(z)$ be a transcendental meromorphic function such that $\rho(f)$ is not an integer or infinite, and let a and b be two distinct constants. If $\Delta_c f(z)$ and $f'(z)$ share a, b, ∞ CM, then $f'(z) \equiv \Delta_c f(z)$.*

Remark 1.2. *In* [15]*, Qi et al. conjectured that Theorem 1.5 is still valid without the condition that "$\rho(f)$ is not an integer."*

In 2019, Deng et al.[6] not only confirmed this conjecture, but also showed that the condition "$f(z)$ is a transcendental meroporhic function"in Theorem 1.5 can be extended to "$f(z)$ is a meromorphic function."

Theorem 1.6. [6] *Let $f(z)$ be a meromorphic function of finite order, and let a and b be two distinct constants. If $\Delta_c f(z)$ and $f'(z)$ share a, b, ∞ CM, then $f'(z) \equiv \Delta_c f(z)$ or $f(z) = Az + B$, where A, B are all constants and $A \neq a, b$, $Ac \neq a, b$.*

To further generalize and improve Theorem 1.6, a natural problem can be posed as follows.

Question 1.2. *Does Theorem 1.6 still hold if $\Delta_c f(z)$ and $f'(z)$ share polynomials CM?*

In this paper, we study this problem and obtain the following main result.

Theorem 1.7. *Let $f(z)$ be a meromorphic function of finite order, and let $a(z)$ and $b(z)$ be two distinct polynomials. If $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, ∞ CM, then they assume one of following cases.*

(1) $f'(z) \equiv \Delta_c f(z)$;

(2) *$f(z)$ reduces to a polynomial and $f'(z) - A\Delta_c f(z) \equiv (1 - A)(c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0)$, where $A(\neq 1)$ is a nonzero constant and c_n, c_{n-1}, $\cdots$, c_1, c_0 are all constants.*

Remark 1.3. *Theorem 1.6 is a special case of Theorem 1.7, which implies that Theorem 1.7 generalizes the result of Theorem 1.6.*

Example 1.1. *Let $f(z) = 2z + 1$, $c = 2$, $a(z) = 1$, $b(z) = 0$. Then $f'(z) = 2$, $\Delta_c f(z) = 4$. Obviously, $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, ∞ CM and $f'(z) - \frac{1}{2}\Delta_c f(z) = 0$. This example illustrates that the case* (2) in Theorem 1.7 may occur.

Example 1.2. *Let $f(z) = z^2$, $c = 1$, $a(z) = 2z + 3$, $b(z) = 2z + 2$. Then $f'(z) = 2z$, $\Delta_c f(z) = 2z + 1$. Obviously, $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, ∞ CM and $f'(z) - 2\Delta_c f(z) = -(2z + 2)$. This example illustrates that the case* (2) in Theorem 1.7 may occur.

Example 1.3. *Let $f(z) = z^3$, $c = 1$, $a(z) = 3z^2 + \frac{3}{2}z + \frac{1}{2}$, $b(z) = 3z^2 + 6z + 2$. Then $f'(z) = 3z^2$, $\Delta_c f(z) = 3z^2 + 3z + 1$. Obviously, $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, ∞ CM and $f'(z) - 2\Delta_c f(z) = -(3z^2 + 6z + 2)$. This example illustrates that the case* (2) in Theorem 1.7 may occur.

2. Some lemmas

We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna theory, as founded in[9, 18, 19]. Next, we give some lemmas, which play a key role in proving Theorem 1.7.

Lemma 2.1. [4, 8] *Suppose that $f(z)$ is a meromorphic function of finite order, and c is a nonzero complex constant. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.2. [4, 8] *Let $f(z)$ be a meromorphic function of finite order, let c be a nonzero complex constant, and let k be a positive integer. Then*

$$m\left(r, \frac{\Delta_c^k f(z)}{f(z)}\right) = S(r, f).$$

Lemma 2.3. [18, 19] *Suppose that $f_i(z)$ $(i = 1, \cdots, n)$ $(n \geq 2)$ are meromorphic functions and $g_i(z)$ $(i = 1, \cdots, n)$ $(n \geq 2)$ are entire functions satisfying*

(1) $\sum_{i=1}^{n} f_i(z)e^{g_i(z)} \equiv 0$;

(2) *when* $1 \leq k < l \leq n$, $g_k(z) - g_l(z)$ *are not constants;*

(3) *when* $1 \leq i \leq n,\ 1 \leq k < l \leq n$,

$$T(r, f_i) = o\{T\left(r, e^{g_k - g_l}\right)\}, \quad (r \to \infty,\ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or logarithmic measure.

Then $f_i \equiv 0$ for any $i = 1, \cdots, n$.

Lemma 2.4. *Suppose that $a(z)$ is a polynomial satisfying $a(z+c) - a(z) = \frac{a'(z)}{R}$, where R is a nonzero constant. Then*

(1) *when* $c = \frac{1}{R}$, $a(z)$ *is a polynomial of degree one or a constant;*

(2) *when* $c \neq \frac{1}{R}$, $a(z)$ *is a constant.*

Proof. Suppose that $a(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_1 z + a_0$, where $a_n, a_{n-1}, \cdots, a_0$ are all constants. Then

$$\begin{aligned}
a'(z) =& n a_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + (n-2)a_{n-2}z^{n-3} + \cdots + a_1, \\
a(z+c) =& a_n(z+c)^n + a_{n-1}(z+c)^{n-1} + a_{n-2}(z+c)^{n-2} + \cdots + a_1(z+c) + a_0 \\
=& a_n(z^n + C_n^1 z^{n-1}c^1 + C_n^2 z^{n-2}c^2 + \cdots + C_n^{n-1}z^1c^{n-1} + c^n) \\
&+ a_{n-1}(z^{n-1} + C_{n-1}^1 z^{n-2}c^1 + C_{n-1}^2 z^{n-3}c^2 + \cdots + C_{n-1}^{n-2}z^1c^{n-2} + c^{n-1}) + \cdots \\
&+ a_1(z+c) + a_0 \\
=& a_n z^n + (a_n C_n^1 c^1 + a_{n-1})z^{n-1} + (a_n C_n^2 c^2 + a_{n-1}C_{n-1}^1 c^1 + a_{n-2})z^{n-2} + \cdots \\
&+ (a_n C_n^{n-1}c^{n-1} + a_{n-1}C_{n-1}^{n-2}c^{n-2} + \cdots + a_2 C_2^1 c^1 + a_1)z
\end{aligned}$$

$$+ (a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c^1 + a_0).$$

Thus,

$$\begin{aligned} a(z+c) - a(z) =& a_n C_n^1 c^1 z^{n-1} + (a_n C_n^2 c^2 + a_{n-1} C_{n-1}^1 c^1) z^{n-2} \\ &+ (a_n C_n^3 c^3 + a_{n-1} C_{n-1}^2 c^2 + a_{n-2} C_{n-2}^1 c^1) z^{n-3} \\ &+ \cdots + (a_n C_n^{n-1} c^{n-1} + a_{n-1} C_{n-1}^{n-2} c^{n-2} + \cdots + a_2 C_2^1 c^1) z \\ &+ (a_n c^n + a_{n-1} c^{n-1} + \cdots + a_2 c^2 + a_1 c^1). \end{aligned}$$

When $c = \frac{1}{R}$, by $a(z+c) - a(z) = \frac{a'(z)}{R}$, we can get $a_n = a_{n-1} = a_{n-2} = \cdots = a_2 = 0$. Hence, $a(z)$ is a polynomial of degree one or a constant.

When $c \neq \frac{1}{R}$, by $a(z+c) - a(z) = \frac{a'(z)}{R}$, we can get $a_n = a_{n-1} = a_{n-2} = \cdots = a_1 = 0$. Hence, $a(z)$ is a constant.

3. Proof of theorem 1.7

If $\Delta_c f(z) \equiv a(z)$, then by the condition that $\Delta_c f(z)$ and $f'(z)$ share $a(z)$ CM, we can get $f'(z) \equiv a(z)$. Thus $f'(z) \equiv \Delta_c f(z)$. If $\Delta_c f(z) \equiv b(z)$, then we can also get $f'(z) \equiv \Delta_c f(z)$ in the same way. Next, we consider the case of $\Delta_c f(z) \not\equiv a(z), b(z)$.

Note that $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, ∞ CM and $f(z)$ is a meromorphic function of finite order. Then by Lemma 2.1, we have

$$\frac{f'(z) - a(z)}{\Delta_c f(z) - a(z)} = e^{\alpha(z)}, \quad \frac{f'(z) - b(z)}{\Delta_c f(z) - b(z)} = e^{\beta(z)}, \tag{3.1}$$

where $\alpha(z)$ and $\beta(z)$ are two polynomials such that $\max\{\deg \alpha(z), \deg \beta(z)\} \leq \rho(f)$.

It follows from (3.1) that

$$(e^{\alpha(z)} - e^{\beta(z)}) \Delta_c f(z) = a(z) e^{\alpha(z)} - b(z) e^{\beta(z)} - a(z) + b(z). \tag{3.2}$$

If $e^{\alpha(z)} \equiv e^{\beta(z)}$, then from (3.2) we can obtain

$$[a(z) - b(z)](e^{\alpha(z)} - 1) = 0.$$

Since $a(z) \not\equiv b(z)$, we have $e^{\alpha(z)} \equiv 1$. Hence by (3.1), we can get $f'(z) \equiv \Delta_c f(z)$.

Next we consider the case of $e^{\alpha(z)} \not\equiv e^{\beta(z)}$.

It follows from (3.2), (3.1) that

$$\Delta_c f(z) = \frac{a(z) e^{\alpha(z)} - b(z) e^{\beta(z)} - a(z) + b(z)}{e^{\alpha(z)} - e^{\beta(z)}}, \tag{3.3}$$

$$f'(z) = \frac{e^{\alpha(z)} [a(z) e^{\alpha(z)} - b(z) e^{\beta(z)} - a(z) + b(z)]}{e^{\alpha(z)} - e^{\beta(z)}} - a(z) e^{\alpha(z)} + a(z). \tag{3.4}$$

Differentiating (3.3) yields

(3.5)
$$[\Delta_c f(z)]' = \frac{a'(z)e^{2\alpha(z)} + b'(z)e^{2\beta(z)} + [(a(z)-b(z))(\beta'(z)-\alpha'(z)) - a'(z) - b'(z)]e^{\alpha(z)+\beta(z)}}{(e^{\alpha(z)} - e^{\beta(z)})^2} + \frac{[(a(z)-b(z))\alpha'(z) - a'(z) + b'(z)]e^{\alpha(z)} - [(a(z)-b(z))\beta'(z) - a'(z) + b'(z)]e^{\beta(z)}}{(e^{\alpha(z)} - e^{\beta(z)})^2}.$$

It follows from (3.4) that

$$\begin{aligned}\Delta_c f'(z) =& \frac{e^{\alpha(z+c)}[a(z+c)e^{\alpha(z+c)} - b(z+c)e^{\beta(z+c)} - a(z+c) + b(z+c)]}{e^{\alpha(z+c)} - e^{\beta(z+c)}} \\ &- \frac{e^{\alpha(z)}[a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - a(z) + b(z)]}{e^{\alpha(z)} - e^{\beta(z)}} \\ &- a(z+c)e^{\alpha(z+c)} + a(z)e^{\alpha(z)} + a(z+c) - a(z).\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we obtain

$$\begin{aligned}&[a(z+c) - b(z+c)]e^{2\alpha(z)+\alpha(z+c)+\beta(z+c)} + [a(z+c) - b(z+c)]e^{\alpha(z+c)+2\beta(z)+\beta(z+c)} \\ &- 2[a(z+c) - b(z+c)]e^{\alpha(z)+\alpha(z+c)+\beta(z)+\beta(z+c)} - Q_1(z)e^{2\alpha(z)+\alpha(z+c)} \\ &+ [a(z) - b(z)]e^{2\alpha(z)+\beta(z)+\beta(z+c)} + [a(z) - b(z)]e^{\alpha(z)+\alpha(z+c)+2\beta(z)} \\ &- [a(z) - b(z)]e^{\alpha(z)+2\beta(z)+\beta(z+c)} - [a(z) - b(z)]e^{2\alpha(z)+\alpha(z+c)+\beta(z)} \\ &- Q_2(z)e^{\alpha(z+c)+2\beta(z)} - Q_3(z)e^{\alpha(z)+\alpha(z+c)+\beta(z)} + Q_4(z)e^{2\alpha(z)+\beta(z+c)} \\ &+ Q_5(z)e^{2\beta(z)+\beta(z+c)} + Q_6(z)e^{\alpha(z)+\beta(z)+\beta(z+c)} - Q_7(z)e^{\alpha(z)+\alpha(z+c)} \\ &+ Q_7(z)e^{\alpha(z)+\beta(z+c)} + Q_8(z)e^{\alpha(z+c)+\beta(z)} - Q_8(z)e^{\beta(z)+\beta(z+c)} \equiv 0,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}&Q_1(z) = a'(z) + b(z) - b(z+c), \\ &Q_2(z) = b'(z) + a(z) - b(z+c), \\ &Q_3(z) = [a(z) - b(z)][\beta'(z) - \alpha'(z)] - a'(z) - b'(z) - a(z) - b(z) + 2b(z+c), \\ &Q_4(z) = a'(z) + b(z) - a(z+c), \\ &Q_5(z) = b'(z) + a(z) - a(z+c), \\ &Q_6(z) = [a(z) - b(z)][\beta'(z) - \alpha'(z)] - a'(z) - b'(z) - a(z) - b(z) + 2a(z+c), \\ &Q_7(z) = [a(z) - b(z)]\alpha'(z) - a'(z) + b'(z), \\ &Q_8(z) = [a(z) - b(z)]\beta'(z) - a'(z) + b'(z).\end{aligned} \tag{3.8}$$

Next we consider three cases about deg $\alpha(z)$ and deg $\beta(z)$.

Case 1. deg $\alpha(z) >$ deg $\beta(z)$. Then (3.7) can be rewritten as

$$A_3(z)e^{3\alpha(z)} + A_2(z)e^{2\alpha(z)} + A_1(z)e^{\alpha(z)} + A_0(z) \equiv 0, \tag{3.9}$$

where

$$A_3(z) = [a(z+c) - b(z+c)]e^{\Delta_c\alpha(z)+\beta(z+c)} - [a(z) - b(z)]e^{\Delta_c\alpha(z)+\beta(z)} - Q_1(z)e^{\Delta_c\alpha(z)},$$
$$A_2(z) = -2[a(z+c) - b(z+c)]e^{\Delta_c\alpha(z)+\beta(z)+\beta(z+c)} + [a(z) - b(z)]e^{\beta(z)+\beta(z+c)}$$
$$+ [a(z) - b(z)]e^{\Delta_c\alpha(z)+2\beta(z)} - Q_3(z)e^{\Delta_c\alpha(z)+\beta(z)} + Q_4(z)e^{\beta(z+c)} - Q_7(z)e^{\Delta_c\alpha(z)},$$
$$A_1(z) = [a(z+c) - b(z+c)]e^{\Delta_c\alpha(z)+2\beta(z)+\beta(z+c)} - [a(z) - b(z)]e^{2\beta(z)+\beta(z+c)}$$
$$- Q_2(z)e^{\Delta_c\alpha(z)+2\beta(z)} + Q_6(z)e^{\beta(z)+\beta(z+c)} + Q_7(z)e^{\beta(z+c)} + Q_8(z)e^{\Delta_c\alpha(z)+\beta(z)},$$
(3.10)
$$A_0(z) = Q_5(z)e^{2\beta(z)+\beta(z+c)} - Q_8(z)e^{\beta(z)+\beta(z+c)}.$$

Obviously, for any $i = 0, 1, 2, 3$, we have

$$\rho(A_i(z)) < \deg \alpha(z).$$

Hence, it follows from Lemma 2.3 that

$$A_i(z) \equiv 0 \ (i = 0, 1, 2, 3).$$

Next we discuss two subcases as follows.

Subcase 1.1. deg $\beta(z) = 0$. Then $\beta(z)$ is a constant.

It follows from (3.8), (3.10) and $A_3(z) \equiv A_0(z) \equiv 0$ that

$$\Delta_c a(z)e^{\beta} + (1 - e^{\beta})\Delta_c b(z) \equiv a'(z), \tag{3.11}$$

$$\Delta_c a(z)e^{\beta} \equiv a'(z) + b'(z)(e^{\beta} - 1). \tag{3.12}$$

Combining (3.11) with (3.12) yields

$$(e^{\beta} - 1)[b'(z) - \Delta_c b(z)] \equiv 0.$$

If $e^{\beta} = 1$, then by (3.1), we can get $f'(z) \equiv \Delta_c f(z)$ and $e^{\alpha(z)} \equiv e^{\beta(z)} \equiv 1$, which contradicts deg $\alpha(z) >$ deg $\beta(z)$.

If $b'(z) - \Delta_c b(z) \equiv 0$, then it follows from Lemma 2.4 that when $c = 1$, $b(z)$ is a polynomial of degree one or a constant; when $c \neq 1$, $b(z)$ is a constant.

Subcase 1.1.1. $b(z)$ is a constant. We let $b(z) \equiv b$. Then by (3.12), we can get $a'(z) \equiv e^{\beta}\Delta_c a(z)$. It follows from Lemma 2.4 that when $c = e^{-\beta}$, $a(z)$ is a polynomial of degree one or a constant; when $c \neq e^{-\beta}$, $a(z)$ is a constant.

If $a(z)$ is a constant, then we let $a(z) \equiv a$. By (3.8), (3.10) and $A_2(z) \equiv A_1(z) \equiv 0$, we have

$$\begin{cases} \left[e^{2\beta} - e^{\beta} - \alpha'(z)(e^{\beta} - 1)\right] e^{\Delta_c\alpha(z)} + e^{\beta}(1 - e^{\beta}) \equiv 0, \\ e^{\beta}(1 - e^{\beta})e^{\Delta_c\alpha(z)} + e^{2\beta} - e^{\beta} + \alpha'(z)(e^{\beta} - 1) \equiv 0. \end{cases}$$

Thus, we get $e^{\beta}=1$. Similarly, we can get a contradiction.

If $a(z)$ is a polynomial of degree one, then we can get $c=e^{-\beta}$ at once. Next we let $a(z)=Az+B$, where $A(\neq 0), B$ are all constants. By (3.8), (3.10) and $A_2(z)\equiv A_1(z)\equiv 0$, we have

$$\begin{cases} e^{\Delta_c\alpha(z)}\left[\alpha'(z)(Az+B-b)(e^{\beta}-1)-(Az+B-b)e^{2\beta}-(A-Az-B+b)e^{\beta}+A\right] \\ \qquad +(Az+B-b)e^{2\beta}+(A-Az-B+b)e^{\beta}-A\equiv 0, \\ \alpha'(z)(Az+B-b)(e^{\beta}-1)+(Az+B-b)e^{2\beta}+(A-Az-B+b)e^{\beta}-A \\ \qquad -e^{\Delta_c\alpha(z)}\left[(Az+B-b)e^{2\beta}+(A-Az-B+b)e^{\beta}-A\right]\equiv 0. \end{cases}$$

Thus, we get $e^{\beta}=1$. Similarly, we can get a contradiction.

Subcase 1.1.2. $b(z)$ is a polynomial of degree one. Firstly, we can get $c=1$. We let $b(z)=Dz+E$, where $D(\neq 0), E$ are all constants. By (3.12), we can get $a(z)$ is a polynomial of degree one or a constant.

If $a(z)$ is a constant, then by (3.12), we can get $e^{\beta}=1$. Similarly, we can get a contradiction.

If $a(z)$ is a polynomial of degree one, then by (3.12), we can get $e^{\beta}=1$ or $a(z)=Dz+F$, where $F(\neq E)$ is a constant. When $a(z)=Dz+F$, by (3.8), (3.10) and $A_2(z)\equiv A_1(z)\equiv 0$, we have

$$\begin{cases} \left[e^{2\beta}-e^{\beta}-\alpha'(z)(e^{\beta}-1)\right]e^{\Delta_c\alpha(z)}+e^{\beta}(1-e^{\beta})\equiv 0, \\ e^{\beta}(1-e^{\beta})e^{\Delta_c\alpha(z)}+e^{2\beta}-e^{\beta}+\alpha'(z)(e^{\beta}-1)\equiv 0. \end{cases}$$

Thus, we get $e^{\beta}=1$, which implies that we can only get $e^{\beta}=1$ in this case. Similarly, we can get a contradiction.

Subcase 1.2. $\deg\beta(z)\geq 1$. It follows from (3.10), $A_0(z)\equiv 0$ that $Q_8(z)\equiv 0$. By (3.8), we have

$$\beta'(z)\equiv\frac{a'(z)-b'(z)}{a(z)-b(z)}. \tag{3.13}$$

Since $\deg\beta(z)\geq 1$, we can get $\beta'(z)\not\equiv 0$. Then it follows from (3.13) that $a(z)-b(z)\equiv\tilde{A}e^{\beta(z)}$, where $\tilde{A}$ is a nonzero constant. But this is a contradiction.

Case 2. $\deg\beta(z)>\deg\alpha(z)$. Then (3.7) can be rewritten as

$$B_3(z)e^{3\beta(z)}+B_2(z)e^{2\beta(z)}+B_1(z)e^{\beta(z)}+B_0(z)\equiv 0, \tag{3.14}$$

where

$$B_3(z)=[a(z+c)-b(z+c)]e^{\alpha(z+c)+\Delta_c\beta(z)}-[a(z)-b(z)]e^{\alpha(z)+\Delta_c\beta(z)}+Q_5(z)e^{\Delta_c\beta(z)},$$

$$B_2(z)=-2[a(z+c)-b(z+c)]e^{\alpha(z)+\alpha(z+c)+\Delta_c\beta(z)}+[a(z)-b(z)]e^{2\alpha(z)+\Delta_c\beta(z)}$$

$$+[a(z)-b(z)]e^{\alpha(z)+\alpha(z+c)}-Q_2(z)e^{\alpha(z+c)}+Q_6(z)e^{\alpha(z)+\Delta_c\beta(z)}-Q_8(z)e^{\Delta_c\beta(z)},$$

$$B_1(z)=[a(z+c)-b(z+c)]e^{2\alpha(z)+\alpha(z+c)+\Delta_c\beta(z)}-[a(z)-b(z)]e^{2\alpha(z)+\alpha(z+c)}$$

$$- Q_3(z)e^{\alpha(z)+\alpha(z+c)} + Q_4(z)e^{2\alpha(z)+\Delta_c\beta(z)} + Q_7(z)e^{\alpha(z)+\Delta_c\beta(z)} + Q_8(z)e^{\alpha(z+c)},$$

(3.15)

$$B_0(z) = -Q_1(z)e^{2\alpha(z)+\alpha(z+c)} - Q_7(z)e^{\alpha(z)+\alpha(z+c)}.$$

Obviously, for any $i = 0, 1, 2, 3$, we have

$$\rho(B_i(z)) < \deg\ \beta(z).$$

Hence, it follows from Lemma 2.3 that

$$B_i(z) \equiv 0\ (i = 0, 1, 2, 3).$$

Next we discuss two subcases as follows.

Subcase 2.1. $\deg\ \alpha(z) = 0$. Then $\alpha(z)$ is a constant.

It follows from (3.8), (3.15) and $B_3(z) \equiv B_0(z) \equiv 0$ that

$$\Delta_c b(z)e^{\alpha} + (1 - e^{\alpha})\Delta_c a(z) \equiv b'(z), \tag{3.16}$$

$$\Delta_c b(z)e^{\alpha} \equiv b'(z) + a'(z)(e^{\alpha} - 1). \tag{3.17}$$

Combining (3.16) with (3.17) yields

$$(e^{\alpha} - 1)[a'(z) - \Delta_c a(z)] \equiv 0.$$

If $e^{\alpha} = 1$, then by (3.1), we can get $f'(z) \equiv \Delta_c f(z)$ and $e^{\alpha(z)} \equiv e^{\beta(z)} \equiv 1$, which contradicts $\deg\ \beta(z) > \deg\ \alpha(z)$.

If $a'(z) - \Delta_c a(z) \equiv 0$, then it follows from Lemma 2.4 that when $c = 1$, $a(z)$ is a polynomial of degree one or a constant; when $c \neq 1$, $a(z)$ is a constant.

Subcase 2.1.1. $a(z)$ is a constant. We let $a(z) \equiv a$. Then by (3.17), we can get $b'(z) \equiv e^{\alpha}\Delta_c a(z)$. It follows from Lemma 2.4 that when $c = e^{-\alpha}$, $b(z)$ is a polynomial of degree one or a constant; when $c \neq e^{-\alpha}$, $b(z)$ is a constant.

If $b(z)$ is a constant, then we let $b(z) \equiv b$. By (3.8), (3.15) and $B_2(z) \equiv B_1(z) \equiv 0$, we have

$$\begin{cases} \left[e^{2\alpha} - e^{\alpha} - \beta'(z)(e^{\alpha} - 1)\right] e^{\Delta_c\beta(z)} + e^{\alpha}(1 - e^{\alpha}) \equiv 0, \\ e^{\alpha}(1 - e^{\alpha})e^{\Delta_c\beta(z)} + e^{2\alpha} - e^{\alpha} + \beta'(z)(e^{\alpha} - 1) \equiv 0. \end{cases}$$

Thus, we get $e^{\alpha} = 1$. Similarly, we can get a contradiction.

If $b(z)$ is a polynomial of degree one, then we can get $c = e^{-\alpha}$ at once. Next we let $b(z) = Dz + E$, where $D(\neq 0), E$ are all constants. By (3.8), (3.15) and $B_2(z) \equiv B_1(z) \equiv 0$, we have

$$\begin{cases} e^{\Delta_c\beta(z)} \left[\beta'(z)(a - Dz - E)(e^{\alpha} - 1) - (a - Dz - E)e^{2\alpha} + (D + a - Dz - E)e^{\alpha} - D\right] \\ \qquad + (a - Dz - E)e^{2\alpha} - (D + a - Dz - E)e^{\alpha} + D \equiv 0, \\ \beta'(z)(a - Dz - E)(e^{\alpha} - 1) + (a - Dz - E)e^{2\alpha} - (D + a - Dz - E)e^{\alpha} + D \\ \qquad - e^{\Delta_c\beta(z)} \left[(a - Dz - E)e^{2\alpha} - (D + a - Dz - E)e^{\alpha} + D\right] \equiv 0. \end{cases}$$

Thus, we get $e^{\alpha}=1$. Similarly, we can get a contradiction.

Subcase 2.1.2. $a(z)$ is a polynomial of degree one. Firstly, we can get $c=1$. We let $a(z)=Az+B$, where $A(\neq 0), B$ are all constants. By (3.17), we can get $b(z)$ is a polynomial of degree one or a constant.

If $b(z)$ is a constant, then by (3.17), we can get $e^{\alpha}=1$. Similarly, we can get a contradiction.

If $b(z)$ is a polynomial of degree one, then by (3.17), we can get $e^{\alpha}=1$ or $b(z)=Az+F$, where $F(\neq B)$ is a constant. When $b(z)=Az+F$, by (3.8), (3.15) and $B_2(z)\equiv B_1(z)\equiv 0$, we have

$$\begin{cases}\left[e^{2\alpha}-e^{\alpha}-\beta'(z)(e^{\alpha}-1)\right]e^{\Delta_c\beta(z)}+e^{\alpha}(1-e^{\alpha})\equiv 0,\\ e^{\alpha}(1-e^{\alpha})e^{\Delta_c\beta(z)}+e^{2\alpha}-e^{\alpha}+\beta'(z)(e^{\alpha}-1)\equiv 0.\end{cases}$$

Thus, we get $e^{\alpha}=1$, which implies that we can only get $e^{\alpha}=1$ in this case. Similarly, we can get a contradiction.

Subcase 2.2. deg $\alpha(z)\geq 1$. It follows from (3.15), $B_0(z)\equiv 0$ that $Q_7(z)\equiv 0$. By (3.8), we have

$$\alpha'(z)\equiv\frac{a'(z)-b'(z)}{a(z)-b(z)}. \tag{3.18}$$

Since deg $\alpha(z)\geq 1$, we can get $\alpha'(z)\not\equiv 0$. Then it follows from (3.18) that $a(z)-b(z)\equiv\tilde{B}e^{\alpha(z)}$, where $\tilde{B}$ is a nonzero constant. But this is a contradiction.

Case 3. deg $\alpha(z)=$ deg $\beta(z)$.

Subcase 3.1. deg $\alpha(z)=$ deg $\beta(z)=0$. Then, $\alpha(z)$ and $\beta(z)$ are constants, which implies that $e^{\alpha(z)}$ and $e^{\beta(z)}$ are constants, too. It follows from (3.4) that $f'(z)$ can be represented as a linear representation of $a(z)$ and $b(z)$. Thus, $f'(z)$ is a polynomial. Then, $f(z)$ is a polynomial, too. By (3.1) we can deduce that $f'(z)-A\Delta_c f(z)\equiv(1-A)(c_nz^n+c_{n-1}z^{n-1}+\cdots+c_1z+c_0)$, where $A(\neq 0), c_n, c_{n-1},\cdots,c_1,c_0$ are all constants. And when $A=1$, we have $f'(z)\equiv\Delta_c f(z)$. Then from (3.1), we have $e^{\alpha(z)}\equiv e^{\beta(z)}\equiv 1$. But this contradicts the hypothesis $e^{\alpha(z)}\not\equiv e^{\beta(z)}$. Hence, $A\neq 1$.

Subcase 3.2. deg $\alpha(z)=$ deg $\beta(z)\geq 1$. Then (3.7) can be rewritten as

$$\begin{aligned}&C_1(z)e^{3\alpha(z)+\beta(z)}+C_2(z)e^{\alpha(z)+3\beta(z)}+C_3(z)e^{2\alpha(z)+2\beta(z)}+C_4(z)e^{3\alpha(z)}\\&+C_5(z)e^{\alpha(z)+2\beta(z)}+C_6(z)e^{2\alpha(z)+\beta(z)}+C_7(z)e^{3\beta(z)}+C_8(z)e^{2\alpha(z)}\\&+C_9(z)e^{\alpha(z)+\beta(z)}+C_{10}(z)e^{2\beta(z)}\equiv 0,\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}C_1(z)&=[a(z+c)-b(z+c)]e^{\Delta_c\alpha(z)+\Delta_c\beta(z)}-[a(z)-b(z)]e^{\Delta_c\alpha(z)},\\C_2(z)&=[a(z+c)-b(z+c)]e^{\Delta_c\alpha(z)+\Delta_c\beta(z)}-[a(z)-b(z)]e^{\Delta_c\beta(z)},\\C_3(z)&=-2[a(z+c)-b(z+c)]e^{\Delta_c\alpha(z)+\Delta_c\beta(z)}+[a(z)-b(z)]e^{\Delta_c\beta(z)}+\end{aligned}$$

$$+ [a(z) - b(z)]e^{\Delta_c \alpha(z)},$$
$$C_4(z) = -Q_1(z)e^{\Delta_c \alpha(z)}, \quad C_5(z) = -Q_2(z)e^{\Delta_c \alpha(z)} + Q_6(z)e^{\Delta_c \beta(z)},$$
$$C_6(z) = -Q_3(z)e^{\Delta_c \alpha(z)} + Q_4(z)e^{\Delta_c \beta(z)}, \quad C_7(z) = Q_5(z)e^{\Delta_c \beta(z)},$$
$$C_8(z) = -Q_7(z)e^{\Delta_c \alpha(z)}, \quad C_9(z) = Q_7(z)e^{\Delta_c \beta(z)} + Q_8(z)e^{\Delta_c \alpha(z)},$$
$$C_{10}(z) = -Q_8(z)e^{\Delta_c \beta(z)}. \tag{3.20}$$

If $\deg(\alpha(z)-\beta(z)) = \deg(2\alpha(z)-\beta(z)) = \deg(\alpha(z)+\beta(z)) = \deg(3\alpha(z)-\beta(z)) = \deg(3\beta(z)-\alpha(z)) = \deg(\alpha(z)-2\beta(z)) = \deg(3\alpha(z)-2\beta(z)) = \deg(3\beta(z)-2\alpha(z)) = \deg\ \alpha(z) = \deg\ \beta(z)$, then for any $1 \leq i < j \leq 10, 1 \leq n \leq 10$, we can get

$$\rho(C_n(z)) < \rho(e^{g_i(z)-g_j(z)}) = \deg\ \alpha(z).$$

It follows from Lemma 2.3 that $C_n(z) \equiv 0 (n = 1, 2, \cdots, 10)$. Then

$$C_{10}(z) = -Q_8(z)e^{\Delta_c \beta(z)} \equiv 0,$$

which implies that $Q_8(z) \equiv 0$. Thus by (3.8), we have

$$\beta'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 1.2, we can get a contradiction.

Hence, we can only need to discuss the cases that some of $\deg(\alpha(z)-\beta(z))$, $\deg(2\alpha(z)-\beta(z))$, $\deg(\alpha(z)+\beta(z))$, $\deg(3\alpha(z)-\beta(z))$, $\deg(3\beta(z)-\alpha(z))$, $\deg(\alpha(z)-2\beta(z))$, $\deg(3\alpha(z)-2\beta(z))$, $\deg(3\beta(z)-2\alpha(z))$ are less than $\deg\ \alpha(z)$.

Subcase 3.2.1. $\deg(\alpha(z) - \beta(z)) < \deg\ \alpha(z)$. Let $\alpha(z) - \beta(z) = p_1(z)$. Then $\beta(z) = \alpha(z) - p_1(z)$. And (3.19) can be rewritten as

$$D_4(z)e^{4\alpha(z)} + D_3(z)e^{3\alpha(z)} + D_2(z)e^{2\alpha(z)} \equiv 0,$$

where

$$D_4(z) = C_1(z)e^{-p_1(z)} + C_3(z)e^{-2p_1(z)} + C_2(z)e^{-3p_1(z)},$$
$$D_3(z) = C_4(z) + C_6(z)e^{-p_1(z)} + C_5(z)e^{-2p_1(z)} + C_7(z)e^{-3p_1(z)},$$
$$D_2(z) = C_8(z) + C_9(z)e^{-p_1(z)} + C_{10}(z)e^{-2p_1(z)}.$$

Combining this with (3.20), we obtain that for any $i = 2, 3, 4$,

$$\rho(D_i(z)) < \deg\ \alpha(z).$$

It then follows from Lemma 2.3 that

$$D_4(z) = D_3(z) = D_2(z) \equiv 0.$$

It is easy to deduce that $\Delta_c\beta(z) = \Delta_c\alpha(z) - \Delta_c p_1(z)$ since $\beta(z) = \alpha(z) - p_1(z)$. Hence, by (3.20) and $D_2(z) \equiv 0$, we have

$$e^{\Delta_c\alpha(z)}[-Q_7 + (Q_7 e^{-\Delta_c p_1(z)} + Q_8)e^{-p_1(z)} - Q_8 e^{-\Delta_c p_1(z)} e^{-2p_1(z)}] \equiv 0.$$

Equally,

$$-Q_7 + (Q_7 e^{-\Delta_c p_1(z)} + Q_8)e^{-p_1(z)} - Q_8 e^{-\Delta_c p_1(z)} e^{-2p_1(z)} \equiv 0. \tag{3.21}$$

If deg $p_1(z) \geq 1$, then by (3.8) and Lemma 2.3, we can get $Q_7(z) \equiv 0$, and thus

$$\alpha'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 2.2, we can get a contradiction.

Thus deg $p_1(z) = 0$ and so $p_1(z) = \alpha(z) - \beta(z)$ is a constant and

$$\Delta_c p_1(z) \equiv 0, \quad \alpha'(z) \equiv \beta'(z). \tag{3.22}$$

From (3.22) and (3.8), we can get $Q_7(z) \equiv Q_8(z)$. Combining this, (3.21) and (3.22), we have

$$e^{2p_1(z)} - 2e^{p_1(z)} + 1 \equiv 0.$$

Then $e^{p_1(z)} \equiv 1$, which implies that $e^{\alpha(z)} \equiv e^{\beta(z)}$. This contradicts the assumption $e^{\alpha(z)} \not\equiv e^{\beta(z)}$.

Subcase 3.2.2. $\deg(2\alpha(z) - \beta(z)) < \deg \alpha(z)$. Let $2\alpha(z) - \beta(z) = p_2(z)$. Then $\beta(z) = 2\alpha(z) - p_2(z)$. And (3.19) can be rewritten as

$$E_7(z)e^{7\alpha(z)} + E_6(z)e^{6\alpha(z)} + E_5(z)e^{5\alpha(z)} + E_4(z)e^{4\alpha(z)} + E_3(z)e^{3\alpha(z)} + E_2(z)e^{2\alpha(z)} \equiv 0,$$

where

$$E_7(z) = C_2(z)e^{-3p_2(z)}, \quad E_6(z) = C_3(z)e^{-2p_2(z)} + C_7(z)e^{-3p_2(z)},$$
$$E_5(z) = C_1(z)e^{-p_2(z)} + C_5(z)e^{-2p_2(z)}, \quad E_4(z) = C_6(z)e^{-p_2(z)} + C_{10}(z)e^{-2p_2(z)},$$
$$E_3(z) = C_4(z) + C_9(z)e^{-p_2(z)}, \quad E_2(z) = C_8(z).$$

Combining this with (3.20), we obtain that for any $i = 2, ..7$, $\rho(E_i(z)) < \deg \alpha(z)$. It then follows from Lemma 2.3 and (3.20) that $E_2(z) = C_8(z) = -Q_7(z)e^{\Delta_c\alpha(z)} \equiv 0$. Thus $Q_7(z) \equiv 0$. Combining this with (3.8) yields

$$\alpha'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 2.2, we can get a contradiction.

Subcase 3.2.3. $\deg(\alpha(z) + \beta(z)) < \deg \alpha(z)$. Let $\alpha(z) + \beta(z) = p_3(z)$. Then $\beta(z) = -\alpha(z) + p_3(z)$. And (3.19) can be rewritten as

$$F_3(z)e^{3\alpha(z)} + F_2(z)e^{2\alpha(z)} + F_1(z)e^{\alpha(z)} + F_0 + F_{-1}(z)e^{-\alpha(z)} + F_{-2}(z)e^{-2\alpha(z)} + F_{-3}(z)e^{-3\alpha(z)} \equiv 0,$$

where

$$F_3(z) = C_4(z), \quad F_2(z) = C_1(z)e^{p_3(z)} + C_8(z), \quad F_1(z) = C_6(z)e^{p_3(z)},$$
$$F_0(z) = C_3(z)e^{2p_3(z)} + C_9(z)e^{p_3(z)}, \quad F_{-1}(z) = C_5(z)e^{2p_3(z)},$$
$$F_{-2}(z) = C_2(z)e^{3p_3(z)} + C_{10}(z)e^{2p_3(z)}, \quad F_{-3}(z) = C_7(z)e^{3p_3(z)}.$$

Combining this with (3.20), we obtain that for any $i = -3, -2, \cdots, 2, 3$,

$$\rho(F_i(z)) < \deg \alpha(z).$$

It then follows from Lemma 2.3 that

$$F_3(z) = F_2(z) = F_1(z) = F_0(z) = F_{-1}(z) = F_{-2}(z) = F_{-3}(z) \equiv 0.$$

Thus from (3.20) and $F_2(z) \equiv F_{-2}(z) \equiv 0$ we have

$$\left\{[a(z+c) - b(z+c)]e^{\Delta_c \beta(z)} - [a(z) - b(z)]\right\} e^{p_3(z)} - Q_7(z) \equiv 0, \tag{3.23}$$

$$\left\{[a(z+c) - b(z+c)]e^{\Delta_c \alpha(z)} - [a(z) - b(z)]\right\} e^{p_3(z)} - Q_8(z) \equiv 0. \tag{3.24}$$

Obviously, by $\deg(\alpha(z) + \beta(z)) < \deg \alpha(z) = \deg \beta(z)$ and $p_3(z) = \alpha(z) + \beta(z)$, we can get $\deg p_3(z) \le \deg \Delta_c\beta(z) = \deg \beta(z) - 1$.

If $\deg p_3(z) < \deg \Delta_c\beta(z) = \deg \beta(z) - 1$, then by (3.23) we have

$$T(r, e^{\Delta_c\beta(z)}) = T\left(r, \frac{Q_7(z) + [a(z) - b(z)]e^{p_3(z)}}{e^{p_3(z)}[a(z+c) - b(z+c)]}\right) \le S(r, e^{\Delta_c\beta(z)}).$$

Thus $e^{\Delta_c\beta(z)}$ is a constant, which contradicts $0 \le \deg p_3(z) < \deg \Delta_c\beta(z)$.

Hence $\deg p_3(z) = \deg \Delta_c\beta(z)$.

If $\deg p_3(z) = \deg \Delta_c\beta(z) \ge 1$, then (3.23) can be rewritten as

$$[a(z+c) - b(z+c)]e^{\Delta_c\beta(z) + p_3(z)} \equiv [a(z) - b(z)]e^{p_3(z)} + Q_7(z), \tag{3.25}$$

where $Q_7(z) \not\equiv 0$. By the second fundamental theroem and (3.25), we have

$$\begin{aligned} T(r, e^{p_3(z)}) &\le N(r, e^{p_3(z)}) + N\left(r, \frac{1}{e^{p_3(z)}}\right) + N\left(r, \frac{1}{e^{p_3(z)} + \frac{Q_7(z)}{a(z)-b(z)}}\right) + S(r, e^{p_3(z)}) \\ &\le N\left(r, \frac{1}{\frac{a(z+c)-b(z+c)}{a(z)-b(z)} e^{\Delta_c\beta(z)+p_3(z)}}\right) + S(r, e^{p_3(z)}) \le S(r, e^{p_3(z)}). \end{aligned}$$

Thus $e^{p_3(z)}$ is a constant, which contradicts $\deg p_3(z) \ge 1$.

If $\deg p_3(z) = \deg(\alpha(z) + \beta(z)) = \deg \Delta_c\beta(z) = 0$, then $\alpha(z)$ and $\beta(z)$ are polynomials of degree one. We let

$$\alpha(z) = a_1 z + a_0, \quad \beta(z) = -a_1 z + b_0, \tag{3.26}$$

where $a_1 \neq 0$, a_0 and b_0 are all constants. Then it follows from (3.8), (3.23), (3.24) and (3.26) that

(3.27)
$$\left\{[a(z+c)-b(z+c)]e^{-a_1 c}-a(z)+b(z)\right\}e^{p_3(z)}-[a(z)-b(z)]a_1+a'(z)-b'(z)\equiv 0,$$

(3.28)
$$\{[a(z+c)-b(z+c)]e^{a_1 c}-a(z)+b(z)\}e^{p_3(z)}+[a(z)-b(z)]a_1+a'(z)-b'(z)\equiv 0.$$

Combining (3.27) with (3.28) yields

(3.29)
$$\left\{[a(z+c)-b(z+c)](e^{a_1 c}+e^{-a_1 c})-2[a(z)-b(z)]\right\}e^{p_3(z)}+2[a'(z)-b'(z)]\equiv 0.$$

If $a(z)-b(z)$ is a nonzero constant, then $a(z+c)-b(z+c)$ is a nonzero constant, too. In addition, we can get $a(z+c)-b(z+c)=a(z)-b(z)$ and $a'(z)-b'(z)\equiv 0$. From this and (3.29), we have $e^{a_1 c}+e^{-a_1 c}-2=0$. Hence $e^{a_1 c}=1$. Substituting $e^{a_1 c}=1$ into (3.27), we can deduce that $a_1=0$, a contradiction.

If $a(z)-b(z)$ is a nonconstant, then $\deg(a'(z)-b'(z))<\deg(a(z)-b(z))$. Next we let $h(z)=a(z)-b(z)=h_n z^n+\cdots+h_1 z+h_0$, where $h_n(\neq 0)$, h_{n-1}, h_{n-2}, $\cdots$, h_1, h_0 are all constants and $n\geq 1$. Substituting this into (3.29), we have $e^{a_1 c}+e^{-a_1 c}-2=0$. Then $e^{a_1 c}=1$. Substituting $e^{a_1 c}=1$ into (3.27), we can deduce that $a_1=0$, a contradiction.

Subcase 3.2.4. $\deg(3\alpha(z)-\beta(z))<\deg\alpha(z)$. Let $3\alpha(z)-\beta(z)=p_4(z)$. Then $\beta(z)=3\alpha(z)-p_4(z)$. And (3.19) can be rewritten as

$$\begin{aligned}&G_{10}(z)e^{10\alpha(z)}+G_9(z)e^{9\alpha(z)}+G_8(z)e^{8\alpha(z)}+G_7(z)e^{7\alpha(z)}+G_6(z)e^{6\alpha(z)}\\&+G_5(z)e^{5\alpha(z)}+G_4(z)e^{4\alpha(z)}+G_3(z)e^{3\alpha(z)}+G_2(z)e^{2\alpha(z)}\equiv 0,\end{aligned}$$

where

$$\begin{aligned}&G_{10}(z)=C_2(z)e^{-3p_4(z)},\quad G_9(z)=C_7(z)e^{-3p_4(z)},\\&G_8(z)=C_3(z)e^{-2p_4(z)},\quad G_7(z)=C_5(z)e^{-2p_4(z)},\\&G_6(z)=C_1(z)e^{-p_4(z)}+C_{10}(z)e^{-2p_4(z)},\quad G_5(z)=C_6(z)e^{-p_4(z)},\\&G_4(z)=C_9(z)e^{-p_4(z)},\quad G_3(z)=C_4(z),\quad G_2(z)=C_8(z).\end{aligned}$$

Combining this with (3.20), we obtain that for any $i=2,3,\cdots,10$, $\rho(G_i(z))<\deg\,\alpha(z)$. It then follows from Lemma 2.3 and (3.20) that $G_2(z)=C_8(z)=-Q_7(z)e^{\Delta_c\alpha(z)}\equiv 0$. Thus $Q_7(z)\equiv 0$. Combining this with (3.8) yields

$$\alpha'(z)\equiv\frac{a'(z)-b'(z)}{a(z)-b(z)}.$$

Therefore, using the same method as in the proof of Subcase 2.2, we can get a contradiction.

Subcase 3.2.5. $\deg(3\beta(z)-\alpha(z)) < \deg \alpha(z)$. Let $3\beta(z)-\alpha(z) = p_5(z)$. Then $\alpha(z) = 3\beta(z) - p_5(z)$. And (3.19) can be rewritten as

$$\begin{aligned} &J_{10}(z)e^{10\beta(z)} + J_9(z)e^{9\beta(z)} + J_8(z)e^{8\beta(z)} + J_7(z)e^{7\beta(z)} + J_6(z)e^{6\beta(z)} \\ &+ J_5(z)e^{5\beta(z)} + J_4(z)e^{4\beta(z)} + J_3(z)e^{3\beta(z)} + J_2(z)e^{2\beta(z)} \equiv 0, \end{aligned}$$

where

$$\begin{aligned} &J_{10}(z) = C_1(z)e^{-3p_5(z)}, \quad J_9(z) = C_4(z)e^{-3p_5(z)}, \\ &J_8(z) = C_3(z)e^{-2p_5(z)}, \quad J_7(z) = C_6(z)e^{-2p_5(z)}, \\ &J_6(z) = C_2(z)e^{-p_5(z)} + C_8(z)e^{-2p_5(z)}, \quad J_5(z) = C_5(z)e^{-p_5(z)}, \\ &J_4(z) = C_9(z)e^{-p_5(z)}, \quad J_3(z) = C_7(z), \quad J_2(z) = C_{10}(z). \end{aligned}$$

Combining this with (3.20), we obtain that for any $i = 2, 3, \cdots, 10$, $\rho(J_i(z)) < \deg \beta(z)$. It then follows from Lemma 2.3 and (3.20) that $J_2(z) = C_{10}(z) = -Q_8(z)e^{\Delta_c \beta(z)} \equiv 0$. Thus $Q_8(z) \equiv 0$. Combining this with (3.8) yields

$$\beta'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 1.2, we can get a contradiction.

Subcase 3.2.6. $\deg(\alpha(z)-2\beta(z)) < \deg \alpha(z)$. Let $\alpha(z)-2\beta(z) = p_6(z)$. Then $\alpha(z) = 2\beta(z) + p_6(z)$. And (3.19) can be rewritten as

$$K_7(z)e^{7\beta(z)}+K_6(z)e^{6\beta(z)}+K_5(z)e^{5\beta(z)}+K_4(z)e^{4\beta(z)}+K_3(z)e^{3\beta(z)}+K_2(z)e^{2\beta(z)} \equiv 0,$$

where

$$\begin{aligned} &K_7(z) = C_1(z)e^{3p_6(z)}, \quad K_6(z) = C_3(z)e^{2p_6(z)} + C_4(z)e^{3p_6(z)}, \\ &K_5(z) = C_2(z)e^{p_6(z)} + C_6(z)e^{2p_6(z)}, \quad K_4(z) = C_5(z)e^{p_6(z)} + C_8(z)e^{2p_6(z)}, \\ &K_3(z) = C_7(z) + C_9(z)e^{p_6(z)}, \quad K_2(z) = C_{10}(z). \end{aligned}$$

Combining this with (3.20), we obtain that for any $i = 2, 3, \cdots, 7$, $\rho(K_i(z)) < \deg \beta(z)$. It then follows from Lemma 2.3 and (3.20) that $K_2(z) = C_{10}(z) = -Q_8(z)e^{\Delta_c \beta(z)} \equiv 0$. Thus $Q_8(z) \equiv 0$. Combining this with (3.8) yields

$$\beta'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 1.2, we can get a contradiction.

Subcase 3.2.7. $\deg(3\alpha(z)-2\beta(z)) < \deg \alpha(z)$. Let $3\alpha(z)-2\beta(z) = p_7(z)$. Then $\beta(z) = \frac{3}{2}\alpha(z) - \frac{1}{2}p_7(z)$. And (3.19) can be rewritten as

$$L_{\frac{11}{2}}(z)e^{\frac{11}{2}\alpha(z)} + L_5(z)e^{5\alpha(z)} + L_{\frac{9}{2}}(z)e^{\frac{9}{2}\alpha(z)} + L_4(z)e^{4\alpha(z)}$$

$$+ L_{\frac{7}{2}}(z)e^{\frac{7}{2}\alpha(z)} + L_3(z)e^{3\alpha(z)} + L_{\frac{5}{2}}(z)e^{\frac{5}{2}\alpha(z)} + L_2(z)e^{2\alpha(z)} \equiv 0,$$

where

$$\begin{aligned}
&L_{\frac{11}{2}}(z) = C_2(z)e^{-\frac{3}{2}p_7(z)}, \quad L_5(z) = C_3(z)e^{-p_7(z)},\\
&L_{\frac{9}{2}}(z) = C_1(z)e^{-\frac{1}{2}p_7(z)} + C_7(z)e^{-\frac{3}{2}p_7(z)}, \quad L_4(z) = C_5(z)e^{-p_7(z)},\\
&L_{\frac{7}{2}}(z) = C_6(z)e^{-\frac{1}{2}p_7(z)}, \quad L_3(z) = C_4(z) + C_{10}(z)e^{-p_7(z)},\\
&L_{\frac{5}{2}}(z) = C_9(z)e^{-\frac{1}{2}p_7(z)}, \quad L_2(z) = C_8(z).
\end{aligned}$$

Combining this with (3.20), we obtain that for any $i = 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}$,

$$\rho(L_i(z)) < \deg\ \alpha(z).$$

It then follows from Lemma 2.3 and (3.20) that $L_2(z) = C_8(z) = -Q_7(z)e^{\Delta_c\alpha(z)} \equiv 0$. Thus $Q_7(z) \equiv 0$. Combining this with (3.8) yields

$$\alpha'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 2.2, we can get a contradiction.

Subcase 3.2.8. $\deg(3\beta(z) - 2\alpha(z)) < \deg\ \alpha(z)$. Let $3\beta(z) - 2\alpha(z) = p_8(z)$. Then $\alpha(z) = \frac{3}{2}\beta(z) - \frac{1}{2}p_8(z)$. And (3.19) can be rewritten as

$$\begin{aligned}
&M_{\frac{11}{2}}(z)e^{\frac{11}{2}\beta(z)} + M_5(z)e^{5\beta(z)} + M_{\frac{9}{2}}(z)e^{\frac{9}{2}\beta(z)} + M_4(z)e^{4\beta(z)}\\
&+ M_{\frac{7}{2}}(z)e^{\frac{7}{2}\beta(z)} + M_3(z)e^{3\beta(z)} + M_{\frac{5}{2}}(z)e^{\frac{5}{2}\beta(z)} + M_2(z)e^{2\beta(z)} \equiv 0,
\end{aligned}$$

where

$$\begin{aligned}
&M_{\frac{11}{2}}(z) = C_1(z)e^{-\frac{3}{2}p_8(z)}, \quad M_5(z) = C_3(z)e^{-p_8(z)},\\
&M_{\frac{9}{2}}(z) = C_4(z)e^{-\frac{3}{2}p_8(z)} + C_2(z)e^{-\frac{1}{2}p_8(z)}, \quad M_4(z) = C_6(z)e^{-p_8(z)},\\
&M_{\frac{7}{2}}(z) = C_5(z)e^{-\frac{1}{2}p_8(z)}, \quad M_3(z) = C_7(z) + C_8(z)e^{-p_8(z)},\\
&M_{\frac{5}{2}}(z) = C_9(z)e^{-\frac{1}{2}p_8(z)}, \quad M_2(z) = C_{10}(z).
\end{aligned}$$

Combining this with (3.20), we obtain that for any $i = 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}$,

$$\rho(M_i(z)) < \deg\ \beta(z).$$

It then follows from Lemma 2.3 and (3.20) that $M_2(z) = C_{10}(z) = -Q_8(z)e^{\Delta_c\beta(z)} \equiv 0$. Thus $Q_8(z) \equiv 0$. Combining this with (3.8) yields

$$\beta'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 1.2, we can get a contradiction. This completes the proof of Theorem 1.7.

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CROSSING MALMQUIST SYSTEMS WITH CERTAIN TYPES

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Abstract. In this paper, we will present the expression of meromorphic solutions on the crossing differential or difference Malmquist systems of certain types using Nevanlinna theory. For instance, we consider the admissible meromorphic solutions of the crossing differential Malmquist system

$$\begin{cases} f_1'(z) = \dfrac{a_1(z)f_2(z) + a_0(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) = \dfrac{a_2(z)f_1(z) + b_0(z)}{f_1(z) + d_2(z)}, \end{cases}$$

where $a_1(z)d_1(z) \not\equiv a_0(z)$ and $a_2(z)d_2(z) \not\equiv b_0(z)$.

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1. Introduction

The Malmquist theorem, originally published in [6], states that the Malmquist type differential equation

$$f'(z) = R(z, f(z)), \tag{1.1}$$

where $R(z, f(z))$ is a rational function in z and f, admits a transcendental meromorphic solution, then (1.1) reduces to a differential Riccati equation

$$f'(z) = a_0(z) + a_1(z)f(z) + a_2(z)f(z)^2, \tag{1.2}$$

where $a_i(z)(i = 0, 1, 2)$ are rational functions. The original proof in [6] was independent of Nevanlinna theory, however, Nevanlinna theory is an efficient method to prove and generalize the above result, some details can be found in [4, Chapter 10]. We assume that the reader is familiar with the basic notations of Nevanlinna theory, see [3, 4, 5, 12].

To generalize the Riccati or Malmquist equations, as far as we know, Tu and Xiao [7] firstly considered the meromorphic solutions of system of higher-order algebraic differential equations, which will be called the crossing Malmquist systems in the paper. Recently, there are some results for the meromorphic solutions of

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several systems, see [9, 10]. We give the following presentation for our proceeding consideration, which is a corollary of [7, Theorem 2], where the admissible meromorphic solutions imply that the coefficients of the system are rational functions or small functions with respect to $f_1(z)$ and $f_2(z)$.

Theorem A. If the following system

$$\tag{1.3} \begin{cases} f_2'(z) = \dfrac{a_{p_1}(z)f_1(z)^{p_1} + \cdots + a_1(z)f_1(z) + a_0(z)}{b_{q_1}(z)f_1(z)^{q_1} + \cdots + b_1(z)f_1(z) + b_0(z)}, \\ f_1'(z) = \dfrac{c_{p_2}(z)f_2(z)^{p_2} + \cdots + c_1(z)f_2(z) + c_0(z)}{d_{q_2}(z)f_2(z)^{q_2} + \cdots + d_1(z)f_2(z) + d_0(z)} \end{cases}$$

has a paired admissible meromorphic solution (f_1, f_2), then $d_1 d_2 \leq 4$, where $d_i := \max\{p_i, q_i\}, i = 1, 2$.

Obviously, Theorem A can be viewed as the generalization of Malmquist theorem. Moreover, the case $q_1 \geq 1$, $q_2 \geq 1$ can occur. See the example below given by Tu and Xiao [7].

Example 1.1. $(f_1, f_2) = (e^z, e^{-z})$ *is a paired entire solution of the crossing Malmquist system*

$$\tag{1.4} \begin{cases} f_1'(z) = \dfrac{1}{f_2(z)}, \\ f_2'(z) = -\dfrac{1}{f_1(z)}. \end{cases}$$

Actually, all meromorphic solutions of (1.4) can be expressed by $(f_1, f_2) = (e^{\frac{1}{c}z+d_1}, e^{-\frac{1}{c}z+d_2})$, where $e^{d_1+d_2} = c$. From the two equations in (1.4), then $f_1 f_2 = c$ follows immediately, where c is a non-zero constant. Thus, by $\frac{f_1'}{f_1} = \frac{1}{c}$, we get $f_1 = e^{\frac{1}{c}z+d_1}$, then $f_2 = e^{-\frac{1}{c}z+d_2}$, where $e^{d_1+d_2} = c$.

We proceed to consider the admissible meromorphic solutions of the generalization of the system (1.4) as follows

$$\tag{1.5} \begin{cases} f_1'(z) = \dfrac{a_1(z)f_2(z) + a_0(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) = \dfrac{a_2(z)f_1(z) + b_0(z)}{f_1(z) + d_2(z)}, \end{cases}$$

where $a_1(z)d_1(z) \not\equiv a_0(z)$ and $a_2(z)d_2(z) \not\equiv b_0(z)$. We obtain the following theorem.

Theorem 1.1. *The admissible entire solutions (f_1, f_2) of (1.5) satisfy one of the following cases:*

(i) *If $a_1(z) = 0$, $d_2(z) = 0$, then $f_1(z) = \frac{\int (a_0(z)+b_0(z))dz}{f_2(z)+d_1(z)}$, where $d_1'(z) = -a_2(z)$.*

(ii) *If $a_1(z) = 0$, $d_2(z) \neq 0$, then $f_1(z) + d_2 = \frac{\int (a_0(z)+b_0(z)-a_2(z)d_2)dz}{f_2(z)+d_1(z)}$, where $d_2(z) = d_2$ and $d_1'(z) = -a_2(z)$.*

(iii) If $a_1(z) \neq 0$, $d_2(z) \neq 0$, *then*

$$f_1(z) + d_2(z) = \frac{\int (a_0(z) + b_0(z) - a_1(z)d_1(z) - a_2(z)d_2(z))dz}{f_2(z) + d_1(z)},$$

where $d_1'(z) = -a_2(z)$ *and* $d_2'(z) = -a_1(z)$.

The another example below, given by Gao [2], shows that the case $d_1 d_2 = 4$ can occur in Theorem A.

Example 1.2. $(f_1, f_2) = (\frac{1}{e^z+1}, \frac{1}{e^z-1})$ *is a paired meromorphic solution of the crossing Malmquist system*

$$\begin{cases} f_1'(z) = \dfrac{-f_2(z)^2 - f_2(z)}{(2f_2(z)+1)^2}, \\ f_2'(z) = \dfrac{f_1(z)^2 - f_1(z)}{(2f_1(z)-1)^2}. \end{cases} \tag{1.6}$$

The system (1.6) has no any transcendental entire solutions. Otherwise, assume that (f_1, f_2) are transcendental entire functions, using the Valiron-Mohon'ko theorem [4, Theorem 2.2.5] and a basic formula $T(r, f') \leq T(r, f) + S(r, f)$ for an entire function f, then

$$\begin{aligned} 2T(r, f_2) &= T(r, f_2^2) + O(1) = T(r, f_1') + O(1) \leq T(r, f_1) + O(1) \\ &= \frac{1}{2}T(r, f_1^2) + O(1) = \frac{1}{2}T(r, f_2') + O(1) \leq \frac{1}{2}T(r, f_2) + O(1), \end{aligned}$$

thus $T(r, f_2) = O(1)$, which is impossible. We find that $(f_1, f_2) = (\frac{1}{1-e^z}, -\frac{1}{e^z+1})$ is also a paired meromorphic solution of (1.6). However, we have not obtained all meromorphic solutions satisfying the system (1.6). Remark that all the above two solutions (f_1, f_2) of (1.6) are meromorphic functions with no zeros. We obtain the following theorem to describe the partial meromorphic solutions of (1.6).

Theorem 1.2. *If* $f_1(z)$ *and* $f_2(z)$ *are two finite order meromorphic solutions of* (1.6) *with no zeros and simple poles only, then* $f_1(z) = \frac{1}{\alpha e^z+1}$ *and* $f_2(z) = \frac{1}{\alpha e^z-1}$, *where* α *is a non-zero constant.*

Without loss of generalization, we rewrite (1.3) as follows

$$\begin{cases} f_1'(z) = \dfrac{a_2(z)f_2(z)^2 + a_1(z)f_2(z) + a_0(z)}{b_2(z)f_2(z)^2 + b_1(z)f_2(z) + b_0(z)}, \\ f_2'(z) = \dfrac{c_2(z)f_1(z)^2 + c_1(z)f_1(z) + c_0(z)}{d_2(z)f_1(z)^2 + d_1(z)f_1(z) + d_0(z)}, \end{cases} \tag{1.7}$$

where $a_i(z), b_i(z), c_i(z), d_i(z) (i = 0, 1, 2)$ are small functions with respect to $f_1(z)$ and $f_2(z)$. From Theorem A, we see that there are four cases for d_1 and d_2 as follows

(i) (d_1, d_2): (4,1), (1,4);

(ii) (d_1, d_2): (3,1), (1,3);

(*iii*) (d_1, d_2): (2,2), (2,1), (1,2);

(*iv*) (d_1, d_2): (1,1).

Three new examples in the following remark with Example 1.1 and Example 1.2 show that there exist meromorphic solutions for all cases $(i)-(iv)$ indeed.

Remark 1.1. *For the case* $(d_1, d_2) = (4, 1)$*, we see that*

$$(f_1(z), f_2(z)) = (\sec z, \tan \frac{z}{2})$$

solves the following system

$$\begin{cases} f_1'(z) = \dfrac{2f_2(z)(1+f_2(z)^2)}{(1-f_2(z)^2)^2}, \\ f_2'(z) = \dfrac{f_1(z)}{f_1(z)+1}. \end{cases} \tag{1.8}$$

For the case $(d_1, d_2) = (3, 1)$*, we see that* $(f_1(z), f_2(z)) = \left(\frac{e^z}{(e^z-1)^2}, \frac{1}{e^z-1}\right)$ *solves the following system*

$$\begin{cases} f_1'(z) = -f_2(z) - 3f_2(z)^2 - 2f_2(z)^3, \\ f_2'(z) = -f_1(z). \end{cases} \tag{1.9}$$

For the case $(d_1, d_2) = (2, 1)$*, we see that* $(f_1(z), f_2(z)) = (\frac{1}{e^z-1}, e^z)$ *solves the following system*

$$\begin{cases} f_1'(z) = \dfrac{-f_2(z)}{(f_2(z)-1)^2}, \\ f_2'(z) = \dfrac{1+f_1(z)}{f_1(z)}. \end{cases} \tag{1.10}$$

The examples on $(d_1, d_2) = (1, 4), (1, 3), (1, 2)$ *can be constructed easily by the above.*

Gao [1, Theorem 1.2] obtained a difference version of Theorem A as follows.

Theorem B. If the following system

$$\begin{cases} f_2(z+c_1)\cdots f_2(z+c_n) = \dfrac{a_{p_1}(z)f_1(z)^{p_1} + \cdots + a_1(z)f_1(z) + a_0(z)}{b_{q_1}(z)f_1(z)^{q_1} + \cdots + b_1(z)f_1(z) + b_0(z)}, \\ f_1(z+d_1)\cdots f_1(z+d_m) = \dfrac{c_{p_2}(z)f_2(z)^{p_2} + \cdots + c_1(z)f_2(z) + c_0(z)}{d_{q_2}(z)f_2(z)^{q_2} + \cdots + d_1(z)f_2(z) + d_0(z)} \end{cases} \tag{1.11}$$

has a paired admissible meromorphic solution (f_1, f_2), where f_1 and f_2 are all meromorphic functions with hyper-order less than one. Then $d_1 d_2 \le nm$, where $d_i := \max\{p_i, q_i\}$.

Gao [1] also obtained that (e^z, e^{-z}) is a paired transcendental meromorphic solution of the crossing difference Malmquist system

$$\begin{cases} f_1(z+1)f_1(z-1) = \dfrac{1}{f_2(z)^2}, \\ f_2(z+1)f_2(z-1) = \dfrac{1}{f_1(z)^2}. \end{cases} \tag{1.12}$$

Our proceeding theorem shows that all transcendental entire solutions with finite order of (1.12).

Theorem 1.3. *The transcendental entire solutions with finite order of* (1.12) *should satisfy one of the following two cases:*

(*i*) $(f_1(z), f_2(z)) = \left(e^{\alpha z+\beta}, e^{-\alpha z+\nu}\right)$, *where* $\nu + \beta = ki\pi$ *and* k *is an integer;*
(*ii*) $(f_1(z), f_2(z)) = \left(e^{\frac{B}{4}z^2+\frac{A+B}{2}z+D}, e^{-\frac{B}{4}z^2-\frac{A+B}{2}z+H}\right)$, *where* $\frac{B}{2} + 2D + 2H = 2ki\pi$ *and* k *is an integer.*

2. Lemmas

To prove Theorem 1.1, we need the following modification of Hayman inequality which relates to the zeros of f and $f^{(n)} - b$, where b is a non-zero small function with respect to f.

Lemma 2.1. [11] *Let* $f(z)$ *be a transcendental meromorphic function satisfying*

$$N\left(r, \frac{1}{f}\right) = S(r, f).$$

For any small functions $b(z)(\not\equiv 0)$ *of* f, *then*

$$N\left(r, \frac{1}{f^{(n)} - b}\right) \neq S(r, f).$$

In order to prove Theorem 1.2, we need the following lemma, which can be found in [8, Theorem 1.1].

Lemma 2.2. *Let* f *and* g *be transcendental entire functions with finite order, such that* f *and* g' *share* 0 *CM,* g *and* f' *share* 0 *CM. Then* f *and* g *satisfy one of the following three cases:*

(1) $f = \gamma g$, *where* γ *is a non-zero constant;*
(2) $f = \lambda \sin(az + b)$ *and* $g = \gamma \cos(az + b)$, *where* a, b, λ, γ *are constants with* $a\lambda\gamma \neq 0$ *and* $\lambda = i\gamma^2$;
(3) $fg = \beta f'g'$, *where* β *is a non-zero constant.*

3. Proofs of Theorems

Proof of Theorem 1.1. Firstly, rewrite (1.5) into

$$\begin{cases} f_1'(z) - a_1(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) - a_2(z) = \dfrac{b_2(z)}{f_1(z) + d_2(z)}, \end{cases} \tag{3.1}$$

where $b_1(z) = a_0(z) - a_1(z)d_1(z)$ and $b_2(z) = b_0(z) - a_2(z)d_2(z)$.

Using Valiron-Mohon'ko theorem [4, Theorem 2.2.5], we have

$$\begin{aligned} T(r, f_2(z)) + S(r, f_2(z)) &= T(r, f_1'(z)) \leq 2T(r, f_1(z)) + S(r, f_1(z)) \\ &\leq 2T(r, f_2'(z)) \leq 4T(r, f_2(z)) + S(r, f_2(z)). \end{aligned}$$

Hence, we assume that $S(r) := S(r, f_1(z)) = S(r, f_2(z))$. We will discuss four cases for the entire functions $f_1(z)$ and $f_2(z)$ below.

Case 1. If $a_1(z) = 0$, $d_2(z) = 0$, then

$$N\left(r, \frac{1}{f_2(z) + d_1(z)}\right) = N(r, f_1'(z)) + S(r) = S(r), \tag{3.2}$$

$$N\left(r, \frac{1}{f_2'(z) - a_2(z)}\right) = N(r, f_1(z)) + S(r) = S(r),$$

which can be written as

$$N\left(r, \frac{1}{(f_2(z) + d_1(z))' - d_1'(z) - a_2(z)}\right) = S(r). \tag{3.3}$$

By Lemma 2.1, (3.2) and (3.3), for avoiding a contradiction, then $d_1'(z) = -a_2(z)$ holds. In this case, from (3.1), we have

$$\begin{cases} f_1'(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) + d_1'(z) = \dfrac{b_2(z)}{f_1(z)}. \end{cases} \tag{3.4}$$

It follows from (3.4),

$$\begin{cases} f_1'(z)f_2(z) + f_1'(z)d_1(z) = b_1(z), \\ f_2'(z)f_1(z) + d_1'(z)f_1(z) = b_2(z). \end{cases} \tag{3.5}$$

Summing the two equations in (3.5), we get

$$(f_1(z)(f_2(z) + d_1(z)))' = b_1(z) + b_2(z),$$

thus

$$f_1(z) = \frac{\int (b_1(z) + b_2(z))dz}{f_2(z) + d_1(z)} = \frac{\int (a_0(z) + b_0(z))dz}{f_2(z) + d_1(z)}.$$

Case 2. If $a_1(z) = 0$, $d_2(z) \neq 0$, then we affirm that $d_2(z)$ must be a constant. From the second equation of (3.1), we have

$$N\left(r, \frac{1}{f_1(z) + d_2(z)}\right) = S(r).$$

From the first equation of (3.1), we have

$$N\left(r, \frac{1}{f_1'(z)}\right) = N\left(r, \frac{1}{(f_1(z) + d_2(z))' - d_2'(z)}\right) = S(r),$$

for avoiding a contradiction, we have $d_2(z)$ must be a constant d_2. Furthermore, the second equation of (3.1) shows that

$$N\left(r, \frac{1}{f_2'(z) - a_2(z)}\right) = N\left(r, \frac{f_1(z) + d_2(z)}{b_2(z)}\right) = S(r),$$

which implies that

$$N\left(r, \frac{1}{(f_2(z) + d_1(z))' - d_1'(z) - a_2(z)}\right) = S(r). \tag{3.6}$$

The first equation of (3.1) shows also that

$$N\left(r, \frac{1}{f_2(z) + d_1(z)}\right) = S(r). \tag{3.7}$$

By Lemma 2.1, (3.6) and (3.7), $-d_1'(z) - a_2(z) = 0$ holds for avoiding a contradiction, that is $d_1'(z) = -a_2(z)$, so we have

$$\begin{cases} f_1'(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) + d_1'(z) = \dfrac{b_2(z)}{f_1(z) + d_2}. \end{cases} \tag{3.8}$$

It follows from (3.8), we get

$$\Big((f_1(z) + d_2)(f_2(z) + d_1(z))\Big)' = b_1(z) + b_2(z),$$

thus

$$f_1(z) + d_2 = \frac{\int (b_1(z) + b_2(z))dz}{f_2(z) + d_1(z)} = \frac{\int (a_0(z) + b_0(z) - a_2(z)d_2)dz}{f_2(z) + d_1(z)}.$$

Case 3. If $a_1(z) \neq 0$, $d_2(z) = 0$, then (3.1) changes into

$$\begin{cases} f_1'(z) - a_1(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) - a_2(z) = \dfrac{b_2(z)}{f_1(z)}, \end{cases} \tag{3.9}$$

where $b_1(z) = a_0(z) - a_1(z)d_1(z)$ and $b_2(z) = b_0(z)$. The first equation of (3.9) implies that

$$N\left(r, \frac{1}{f_1'(z) - a_1(z)}\right) = S(r),$$

the second equation of (3.9) implies that

$$N\left(r, \frac{1}{f_1(z)}\right) = S(r).$$

By Lemma 2.1 and the above two equations, we get a contradiction.

Case 4. If $a_1(z) \neq 0$, $d_2(z) \neq 0$, then

$$N\left(r, \frac{1}{f_2'(z) - a_2(z)}\right) = N\left(r, \frac{f_1(z) + d_2(z)}{b_2(z)}\right) = S(r),$$
$$N\left(r, \frac{1}{f_2(z) + d_1(z)}\right) = N\left(r, \frac{f_1'(z) - a_1(z)}{b_1(z)}\right) = S(r).$$

Since

$$N\left(r, \frac{1}{(f_2(z) + d_1(z))' - d_1'(z) - a_2(z)}\right) = S(r),$$

by Lemma 2.1, we obtain $-d_1'(z) - a_2(z) = 0$ for avoiding a contradiction, that is $d_1'(z) = -a_2(z)$. In addition,

$$N\left(r, \frac{1}{f_1'(z) - a_1(z)}\right) = S(r),$$
$$N\left(r, \frac{1}{f_1(z) + d_2(z)}\right) = S(r),$$

we can have $d_2'(z) = -a_1(z)$. Thus, we have

$$\begin{cases} f_1'(z) + d_2'(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) + d_1'(z) = \dfrac{b_2(z)}{f_1(z) + d_2(z)}. \end{cases} \tag{3.10}$$

From (3.10), we get

$$\Big((f_1(z) + d_2(z))(f_2(z) + d_1(z))\Big)' = b_1(z) + b_2(z),$$

thus

$$\begin{aligned} f_1(z) + d_2(z) &= \frac{\int (b_1(z) + b_2(z))dz}{f_2(z) + d_1(z)} \\ &= \frac{\int (a_0(z) + b_0(z) - a_1(z)d_1(z) - a_2(z)d_2(z))dz}{f_2(z) + d_1(z)}. \end{aligned}$$

The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. Since $f_1(z)$ and $f_2(z)$ are meromorphic solutions with finite order of (1.6) with no zeros and simple poles only, then we assume that $f_1(z) = \frac{1}{g_1(z)}$ and $f_2(z) = \frac{1}{g_2(z)}$, where $g_1(z)$ and $g_2(z)$ are entire functions with finite order. Thus, the system (1.6) means that

$$\begin{cases} g_1'(z) = \dfrac{1 + g_2(z)}{(2 + g_2(z))^2} g_1^2(z), \\ g_2'(z) = \dfrac{g_1(z) - 1}{(2 - g_1(z))^2} g_2^2(z). \end{cases} \tag{3.11}$$

From the above system and g_1, g_2 are entire functions with simple zeros only, then we see that g_1 and $2+g_2$ have the same zeros and same multiplicities, g_2 and $2-g_1$ have the same zeros and same multiplicities. Hence, we assume

$$\begin{cases} g_1(z) = e^{P(z)}(2+g_2(z)), \\ g_2(z) = e^{Q(z)}(2-g_1(z)), \end{cases} \tag{3.12}$$

and

$$\begin{cases} g_1'(z) = e^{2P(z)}(1+g_2(z)), \\ g_2'(z) = e^{2Q(z)}(g_1(z)-1), \end{cases} \tag{3.13}$$

where $P(z)$ and $Q(z)$ are polynomials. Then, we rewrite (3.13) as

$$\begin{cases} \dfrac{(g_1(z)-1)'}{1+g_2(z)} = e^{2P(z)}, \\ \dfrac{(1+g_2(z))'}{g_1(z)-1} = e^{2Q(z)}. \end{cases} \tag{3.14}$$

From (3.14), we can get that $(g_1-1)'$ and $1+g_2$ share 0 CM, $(1+g_2)'$ and g_1-1 share 0 CM. By Lemma 2.2, then we discuss three cases for g_1 and g_2 below.

Case 1. $g_1 - 1 = \gamma(1+g_2)$, where γ is a non-zero constant, $e^{2P(z)} = \gamma^2 e^{2Q(z)}$. Substitute $g_1 - 1 = \gamma(1+g_2)$ into (3.12), we have

$$\begin{cases} \gamma(1+g_2)+1 = e^{P(z)}(2+g_2), \\ g_2 = e^{Q(z)}[2-\gamma(1+g_2)-1], \end{cases} \tag{3.15}$$

we see that (3.15) is represented by

$$\begin{cases} e^{P(z)} = \dfrac{\gamma(1+g_2)+1}{2+g_2}, \\ e^{Q(z)} = \dfrac{g_2}{1-\gamma(1+g_2)}. \end{cases} \tag{3.16}$$

(i) If $e^{P(z)} = \gamma e^{Q(z)}$, then we have

$$\frac{\gamma(1+g_2)+1}{2+g_2} = \frac{\gamma g_2}{1-\gamma(1+g_2)}. \tag{3.17}$$

Then

$$\gamma g_2^2 + 2\gamma g_2 = -\gamma^2 g_2^2 - 2\gamma^2 g_2 - \gamma^2 + 1. \tag{3.18}$$

So we obtain $\gamma = -1$, then $g_1 = -g_2$ follows. However, in this case, the first equation of (3.11) reduces into

$$g_1'(z) = \frac{1-g_1}{(2-g_1)^2} g_1^2,$$

which has no any transcendental entire solutions by Malmquist theorem.

(ii) If $e^{P(z)} = -\gamma e^{Q(z)}$, then

$$\frac{\gamma(1+g_2)+1}{2+g_2} = \frac{-\gamma g_2}{1-\gamma(1+g_2)}. \tag{3.19}$$

Then

$$-\gamma g_2^2 - 2\gamma g_2 = -\gamma^2 g_2^2 - 2\gamma^2 g_2 - \gamma^2 + 1. \tag{3.20}$$

So we obtain $\gamma = 1$, then $g_1 - g_2 = 2$ follows. From the first equation of (3.12), we see $e^{P(z)} \equiv 1$. Thus, the first equation of (3.14) also implies that

$$\frac{(1+g_2)'}{1+g_2} = 1,$$

so $g_2 = \alpha e^z - 1$, where α is a non-zero constant. Then $g_1 = \alpha e^z + 1$.

Case 2. If $g_1 - 1 = \lambda \sin(az+b)$ and $1 + g_2 = \gamma \cos(az+b)$, where a, b, λ, γ are constants with $a\lambda\gamma \neq 0$ and $\lambda = i\gamma^2$, then $e^{2P(z)} = \gamma^2 e^{2Q(z)}$ follows by (3.13). From (3.12), we have

$$\begin{cases} e^{P(z)} = \dfrac{1 + i\gamma^2 \sin(az+b)}{1 + \gamma\cos(az+b)}, \\ e^{Q(z)} = \dfrac{\gamma\cos(az+b) - 1}{1 - i\gamma^2\sin(az+b)}. \end{cases} \tag{3.21}$$

(i) If $e^{P(z)} = \gamma e^{Q(z)}$, then

$$\frac{1 + i\gamma^2 \sin(az+b)}{1 + \gamma\cos(az+b)} = \frac{\gamma(\gamma\cos(az+b) - 1)}{1 - i\gamma^2\sin(az+b)}.$$

Thus

$$\gamma^4 \sin^2(az+b) + 1 = -\gamma^3 \sin^2(az+b) + \gamma^3 - \gamma,$$

which is impossible for the reason that there is no γ satisfying

$$\begin{cases} \gamma^4 = -\gamma^3, \\ \gamma^3 - \gamma = 1. \end{cases}$$

(ii) If $e^{P(z)} = -\gamma e^{Q(z)}$, we have

$$\frac{1 + i\gamma^2 \sin(az+b)}{1 + \gamma\cos(az+b)} = \frac{-\gamma(\gamma\cos(az+b) - 1)}{1 - i\gamma^2\sin(az+b)}.$$

Then

$$\gamma^4 \sin^2(az+b) + 1 = \gamma^3 \sin^2(az+b) - \gamma^3 + \gamma, \tag{3.22}$$

which is also impossible for the reason that there is no γ satisfying

$$\begin{cases} \gamma^4 = \gamma^3, \\ -\gamma^3 + \gamma = 1. \end{cases} \tag{3.23}$$

Case 3. If $(g_1 - 1)(1 + g_2) = \beta(g_1 - 1)'(1 + g_2)' = \beta g_1' g_2'$, where β is a non-zero constant, we have $e^{2P(z)+2Q(z)} = \frac{1}{\beta} := \tau^2$. From (3.12), we have

$$\begin{cases} g_1 = -e^{P(z)+Q(z)} g_1 + 2e^{P(z)+Q(z)} + 2e^{P(z)}, \\ g_2 = -e^{P(z)+Q(z)} g_2 - 2e^{P(z)+Q(z)} + 2e^{Q(z)}. \end{cases} \tag{3.24}$$

If $\tau \neq -1$, we have

$$
\begin{cases} g_1 = \dfrac{2\tau + 2e^{P(z)}}{1+\tau}, \\ g_2 = \dfrac{-2\tau + 2e^{Q(z)}}{1+\tau}. \end{cases} \tag{3.25}
$$

Substitute (3.25) into the first equation of (3.13), we have

$$
\frac{2P'(z)}{1+\tau} = e^{P(z)} \left(1 + \frac{-2\tau + 2e^{Q(z)}}{1+\tau} \right).
$$

The above equation implies that $1 + \frac{-2\tau}{1+\tau} = 0$, that is $\tau = 1$ and $e^{P(z)+Q(z)} = 1$, thus $P(z) = z + b$, where b is a constant. In the same way, substitute (3.25) into the second equation of (3.13), we have

$$
\frac{2Q'(z)}{1+\tau} = e^{Q(z)} \left(\frac{2\tau + 2e^{P(z)}}{1+\tau} - 1 \right).
$$

The above equation implies that $\frac{2\tau}{1+\tau} - 1 = 0$, that is $\tau = 1$ and $e^{P(z)+Q(z)} = 1$, thus $Q(z) = z + a$, where a is a constant. However, this is in contradiction with $e^{P(z)+Q(z)} = 1$, so this case is omitted.

If $\tau = -1$, from the two equations in (3.24), we have $e^{P(z)+Q(z)} = -1$, $e^{P(z)} = 1$ and $e^{Q(z)} = -1$. From the first equation of (3.12), we have $g_1 = 2 + g_2$. Thus, the first equation of (3.14) also implies that

$$
\frac{(1+g_2)'}{1+g_2} = 1,
$$

so $g_2 = \alpha e^z - 1$, where α is a non-zero constant. Then $g_1 = \alpha e^z + 1$. The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. If $(f_1(z), f_2(z))$ is the paired transcendental entire solutions of the complex difference system (1.12), then we have $f_1(z)$ and $f_2(z)$ must have no zeros, thus we assume that $f_1(z) = e^{h_1(z)}$ and $f_2(z) = e^{h_2(z)}$, where $h_1(z)$ and $h_2(z)$ are non-constant polynomials. So

$$
\begin{cases} e^{h_1(z+1)} e^{h_1(z-1)} = e^{-2h_2(z)}, \\ e^{h_2(z+1)} e^{h_2(z-1)} = e^{-2h_1(z)}, \end{cases} \tag{3.26}
$$

it follows

$$
\begin{cases} h_1(z+1) + h_1(z-1) + 2h_2(z) = 2ki\pi, \\ h_2(z+1) + h_2(z-1) + 2h_1(z) = 2mi\pi, \end{cases} \tag{3.27}
$$

where k, m are integers. Shifting forward and backward on (3.27), we have

$$
\begin{cases} h_1(z+2) + h_1(z) + 2h_2(z+1) = 2ki\pi, \\ h_2(z+2) + h_2(z) + 2h_1(z+1) = 2mi\pi, \end{cases} \tag{3.28}
$$

and

$$\begin{cases} h_1(z) + h_1(z-2) + 2h_2(z-1) = 2ki\pi, \\ h_2(z) + h_2(z-2) + 2h_1(z-1) = 2mi\pi. \end{cases} \tag{3.29}$$

The first equation of (3.28) and the first equation of (3.29) can be rewritten as follows

$$\begin{cases} 2h_2(z+1) = -h_1(z+2) - h_1(z) + 2ki\pi, \\ 2h_2(z-1) = -h_1(z-2) - h_1(z) + 2ki\pi. \end{cases} \tag{3.30}$$

Combining the above system (3.30) and the second equation of (3.27), we have

$$2(2mi\pi - 2h_1(z)) = 4ki\pi - 2h_1(z) - h_1(z+2) - h_1(z-2),$$

thus, we have

$$h_1(z+2) + h_1(z-2) - 2h_1(z) = 4ki\pi - 4mi\pi. \tag{3.31}$$

From (3.31), we also have

$$h_1(z+2) - h_1(z) = h_1(z) - h_1(z-2) + 4ki\pi - 4mi\pi,$$

which implies that

$$F(z+2) = F(z) + 4ki\pi - 4mi\pi$$

by letting $F(z) = h_1(z) - h_1(z-2)$. We discuss two cases below.

Case 1. If $m = k$, then $F(z)$ must be a periodic function with period 2, thus $F(z)$ is a non-zero constant 2α for the reason that $h_1(z)$ is a non-constant polynomial. Thus $h_1(z) - h_1(z-2) = 2\alpha$, it follows $h_1(z) = \alpha z + \beta$.

Case 2. If $m \neq k$, then $F(z)$ must be a non-constant linear polynomial, that is $F(z) = Bz + A$. Thus, $h_1(z) - h_1(z-2) = Bz + A$, $B \neq 0$. In this case, we have $h_1(z)$ is a linear polynomial when $B = 0$ and is a polynomial with degree two when $B \neq 0$, we assume that $h_1(z) = \frac{B}{4}z^2 + \frac{A+B}{2}z + D$, where D is any constant.

Using the similar method as above, we also obtain

$$h_2(z+2) + h_2(z-2) - 2h_2(z) = 4mi\pi - 4ki\pi, \tag{3.32}$$

which implies that

$$h_2(z+2) - h_2(z) = h_2(z) - h_2(z-2) + 4mi\pi - 4ki\pi,$$

it follows

$$G(z+2) = G(z) + 4mi\pi - 4ki\pi$$

by letting $G(z) = h_2(z) - h_2(z-2)$. There are two cases to be discussed as follows.

Case 1. If $m = k$, then $G(z)$ must be a periodic function with period 2, thus $G(z)$ is also a non-zero constant 2μ. Then $h_2(z) - h_2(z-2) = 2\mu$, that is $h_2(z) = \mu z + \nu$.

Case 2. If $m \neq k$, then $G(z)$ must be a non-constant linear polynomial, that is $G(z) = Ez + F$. Thus, $h_2(z) - h_2(z-2) = Ez + F$, $E \neq 0$. In this case, we have $h_2(z)$ is a linear polynomial when $E = 0$ and is a polynomial with degree two when $E \neq 0$, we assume that $h_2(z) = \frac{E}{4}z^2 + \frac{E+F}{2}z + H$, where H is any constant.

We also remark that the degree of $h_1(z)$ and $h_2(z)$ are equal. Substitute $h_1(z) = \alpha z + \beta$ and $h_2(z) = \mu z + \nu$ into the first equation of (3.27), we have $\mu = -\alpha$ and $\nu + \beta = ki\pi$. Substitute $h_1(z) = \frac{B}{4}z^2 + \frac{A+B}{2}z + D$ and $h_2(z) = \frac{E}{4}z^2 + \frac{E+F}{2}z + H$ into the system of (3.27), we have $E = -B$, $F = -A$, $\frac{B}{2} + 2D + 2H = 2ki\pi$, $\frac{E}{2} + 2D + 2H = 2mi\pi$ and $B = 2ki\pi - 2mi\pi$. The proof of Theorem 1.3 is thus completed.

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Содержание

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Vol. 58, No. 6, 2023

Contents

A. S. Dalalyan, Simple proof of the risk bound for denoising by exponential weights for asymmetric noise distributions 3

L. A. Khachatryan, On Khinchin's theorem about the special role of the Gaussian distribution 15

S. Majumder, N. Sarkar and D. Pramanik, Entire functions and their high order difference operators 22

D. M. Martirosyan, Orientation-dependent chord length distribution in a convex quadrilateral.. 36

V. Tsagareishvili, Convergence of general Fourier series of differentiable functions 54

M.-H. Wang, J.-F. Chen, Derivatives of meromorphic functions sharing polynomials with their difference operators 66

F. N. Wang, K. Liu, Crossing Malmquist systems with certain types .... 83 – 95