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ИЗВЕСТИЯ
НАН АРМЕНИИ

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МАТЕМАТИКА

ТОМ 58 № 5 2023

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НАЦИОНАЛЬНАЯ АКАДЕМИЯ НАУК АРМЕНИИ

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A HARDY-LITTLEWOOD TYPE THEOREM FOR HARMONIC BERGMAN-ORLICZ SPACES AND APPLICATIONS

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Abstract. It is well known that a harmonic function is in a Bergman space if and only if it satisfies some Hardy-Littlewood type integral estimates. In this paper, we extend this result to harmonic Bergman-Orlicz spaces. As an application, Lipschitz-type characterizes of harmonic Bergman-Orlicz spaces on the unit ball with respect to pseudo-hyperbolic, hyperbolic and Euclidean metrics are established. In addition, the boundedness of a mapping defined by a difference quotient of harmonic function is discussed.

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1. Introduction and main results

Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$ be two vectors in the n-dimensional real vector space $\mathbb{R}^n$. We write

$$\langle x, y\rangle = x_1y_1 + ... + x_ny_n \quad \text{and} \quad |x| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + ... + x_n^2}.$$

For $a \in \mathbb{R}^n$, let $\mathbb{B}(a, r) = \{x : |x - a| < r\}$, $\mathbb{S}(a, r) = \partial\mathbb{B}(a, r)$ and $\overline{\mathbb{B}(a, r)} = \mathbb{B}(a, r) \cup \mathbb{S}(a, r)$. In particular, we use the notations $\mathbb{B} = \mathbb{B}(0, 1)$, $\mathbb{S} = \partial\mathbb{B}(0, 1)$ and $\overline{\mathbb{B}} = \mathbb{B} \cup \mathbb{S}$ the closure of $\mathbb{B}$. We denote by dv the normalized volume measure on $\mathbb{B}$ and $d\sigma$ the normalized surface measure on $\mathbb{S}$. As usual, the class of all harmonic functions on the unit ball $\mathbb{B}$ will be denoted by $h(\mathbb{B})$.

Given a function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, we say that Φ is a *growth function* if it is continuous and non-decreasing.

For $\alpha > -1$ and a growth function Φ, the *Orlicz space* $L^\Phi_\alpha(\mathbb{B})$ is the set of all functions f such that

$$\int_{\mathbb{B}} \Phi(|f(x)|)dv_\alpha(x) < \infty,$$

where $dv_\alpha(x) = c_\alpha(1 - |x|^2)^\alpha dv(x)$ and c_α is a positive constant so that $v_\alpha(\mathbb{B}) = 1$. Denote by $L^p_\alpha(\mathbb{B})$ the subspace of $L^\Phi_\alpha(\mathbb{B})$ for $\Phi(t) = t^p$ and $0 < p < \infty$.

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The harmonic *Bergman-Orlicz* space $\mathcal{B}_\alpha^\Phi$ is the subspace of $L_\alpha^\Phi(\mathbb{B})$ consisting of all $f \in h(\mathbb{B})$ such that

$$\|f\|_{\alpha,\Phi} = \inf\{\mu > 0 : \int_{\mathbb{B}} \Phi\Big(\frac{|f(x)|}{\mu}\Big) dv_\alpha(x) \leq 1\} < \infty.$$

In particular, if $0 < p < \infty$ and $\Phi(t) = t^p$, then the associated harmonic Bergman-Orlicz space is the weighted harmonic Bergman space $\mathcal{B}_\alpha^p$ (cf. [4, 11]).

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$. For $0 < p < \infty$ and $\alpha > -1$, the standard weighted Bergman space $\mathcal{A}_\alpha^p(\mathbb{D})$ consists of all analytic functions g on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |g(z)|^p dA_\alpha(z) < \infty$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ and $dA(z)$ is the area measure on $\mathbb{C}$ normalized so that $A(\mathbb{D}) = 1$. The famous Hardy-Littlewood theorem for weighted Bergman space $\mathcal{A}_\alpha^p(\mathbb{D})$ asserts that

$$\int_{\mathbb{D}} |g(z)|^p dA_\alpha(z) \approx |g(0)|^p + \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^p dA_\alpha(z). \tag{1.1}$$

It is well-known that integral estimate (1.1) plays an important role in the theory of analytic functions. For the generalizations and applications of (1.1) to the spaces of holomorphic functions, harmonic functions, and solutions to certain PDEs, see [4], [5], [7] - [11, 16] and the references therein.

In [14], B. Sehba considered the analogue of (1.1) in the setting of holomorphic functions spaces on the complex unit ball. By adding some suitable restrictions on the growth function Φ, he generalized the integral estimate (1.1) to the Bergman-Orlicz spaces of holomorphic functions. As applications, characterizations of the Gustavsson-Peetre interpolate and boundedness of Cesàro-type operators on Bergman-Orlicz spaces are discussed.

Motivated by the results in [7, 9, 14], our aim in this paper is to extend (1.1) to the setting of harmonic Bergman-Orlicz space $\mathcal{B}_\alpha^\Phi$. In order to state the our result, we need some more definitions on the growth function Φ.

We say that a growth function Φ is of upper type $q \geq 1$ if there exists $C > 0$ such that, for $s > 0$ and $t \geq 1$,

$$\Phi(st) \leq Ct^q \Phi(s). \tag{1.2}$$

Denote by $\mathcal{U}^q$ the set of growth functions Φ of upper type q, (for some $q \geq 1$), such that the function $t \to \frac{\Phi(t)}{t}$ is non-decreasing.

We say that Φ is of lower type $p > 0$ if there exists $C > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$\Phi(st) \leq Ct^p \Phi(s). \tag{1.3}$$

Denote by $\mathcal{L}_p$ the set of growth functions Φ of lower type p, (for some $p<1$), such that the function $t \to \frac{\Phi(t)}{t}$ is non-increasing.

From the above definitions on Φ, we may always suppose that any $\Phi \in \mathcal{L}_p$ (resp. $\mathscr{U}^q$), is concave (resp. convex) and that Φ is a $\mathcal{C}^1$ function with derivative $\Phi'(t) \approx \frac{\Phi(t)}{t}$ (cf. [14, 15]).

For $f \in h(\mathbb{B})$, recall that the radial derivative $\mathcal{R}$ of f is given by

$$\mathcal{R}f(x) = x \cdot \nabla f(x) = \frac{\partial}{\partial t}(f(tx))_{t=1} = \sum_{m=1}^{\infty} m f_m(x),$$

where ∇ is the usual gradient and the last form is the homogeneous expansion of f. The fundamental theorem of calculus shows that

$$f(x) - f(0) = \int_0^1 (\mathcal{R}f)(tx)\frac{dt}{t}.$$

Theorem 1.1. *Let $\alpha > -1$, $f \in h(\mathbb{B})$ and $\Phi \in \mathscr{U}^q \cup \mathcal{L}_p$. Then the following statements are equivalent.*
(a) $f \in \mathcal{B}_\alpha^\Phi$;
(b) $(1-|x|^2)|\nabla f(x)| \in L_\alpha^\Phi(\mathbb{B})$;
(c) $(1-|x|^2)|\mathcal{R}f(x)| \in L_\alpha^\Phi(\mathbb{B})$.

As applications of Theorem 1.1, we establish several characterizations of harmonic Bergman-Orlicz spaces in terms of Lipschitz-type conditions with respect to pseudo-hyperbolic, hyperbolic and Euclidean metrics, which can be viewed as extensions of [16, Theorem 1.1] into the general setting.

Theorem 1.2. *Let $\alpha > -1$, $\Phi \in \mathscr{U}^q \cup \mathcal{L}_p$ and $f \in h(\mathbb{B})$. Then the following statements are equivalent.*
(a) $f \in \mathcal{B}_\alpha^\Phi$;
(b) There exists a positive continuous function $g \in L_\alpha^\Phi(\mathbb{B})$ such that

$$|f(x) - f(y)| \leq \rho(x,y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$;
(c) There exists a positive continuous function $g \in L_\alpha^\Phi(\mathbb{B})$ such that

$$|f(x) - f(y)| \leq \varrho(x,y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$.

Theorem 1.3. *Let $\alpha > -1$, Φ be a given growth function and $f \in \mathcal{B}_\alpha^\Phi$.*
(1) If $\Phi \in \mathscr{U}^q$, then there exists a positive continuous function $g \in L_{\alpha+k}^\Phi(\mathbb{B})$ ($k \in [q,\infty)$ such that $|f(x) - f(y)| \leq |x-y|\big(g(x) + g(y)\big)$ for all $x, y \in \mathbb{B}$;

(2) *If* $\Phi \in \mathcal{L}_p$, *then there exists a positive continuous function* $g \in L^{\Phi}_{\alpha+k}(\mathbb{B})$ $(k \in [1,\infty)$ *such that* $|f(x)-f(y)| \leq |x-y|\big(g(x)+g(y)\big)$ *for all* $x, y \in \mathbb{B}$.

The organization of this paper is as follows. In Section 2, some necessary terminologies and notations will be introduced. The proof of Theorems 1.1 $\sim$ 1.3 will be presented in Section 3. The Section 4 is devoted to discussing some applications of the main results. Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. For nonnegative quantities X and Y, $X \lesssim Y$ means that X is dominated by Y times some inessential positive constant. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$.

2. Preliminaries

In this section, we introduce notations and collect some preliminary results that we need later.

2.1. Pseudo-hyperbolic metric. For $a \in \mathbb{B}$, let

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{[x,a]^2}, \ x \in \mathbb{B},$$

where $[a,x] = \sqrt{1-2\langle a,x\rangle + |a|^2|x|^2}$. Then φ_a is a Möbius transformation from $\mathbb{B}$ onto $\mathbb{B}$ with $\varphi_a(0) = a$ and $\varphi_a(a) = 0$.

We denote by $\mathcal{M}(\mathbb{B})$ the set of all Möbius transformation on $\mathbb{B}$. It is known that if $\varphi \in \mathcal{M}(\mathbb{B})$, then there exist $a \in \mathbb{B}$ and an orthogonal transformation A such that

$$\varphi(x) = A\varphi_a(x), \quad x \in \mathbb{B}.$$

In terms of φ_a, the *pseudo-hyperbolic metric* ρ and the *hyperbolic metric* ϱ in $\mathbb{B}$ are given by

$$\rho(a,b) = |\varphi_a(b)| = \frac{|a-b|}{[a,b]}, \quad a, b \in \mathbb{B}$$

and

$$\varrho(a,b) = \ln \frac{1+|\varphi_a(b)|}{1-|\varphi_a(b)|},$$

respectively.

Let $a \in \mathbb{B}$ and $r \in (0,1)$, the *pseudo-hyperbolic ball* with center a and radius r is denoted by

$$E(a,r) = \{x \in \mathbb{B} : \rho(a,x) = |\varphi_a(x)| < r\}.$$

A straightforward calculation shows that $E(a,r)$ is actually a Euclidean ball with center c_a and radius r_a given by

$$c_a = \frac{(1-r^2)a}{1-|a|^2r^2} \quad \text{and} \quad r_a = \frac{r(1-|a|^2)}{1-|a|^2r^2}, \tag{2.1}$$

respectively (cf. [1, 14]).

Lemma 2.1. [13] *Let $a \in \mathbb{B}$, $r \in (0,1)$ and $x \in E(a,r)$. Then*

$$1-|a|^2 \approx 1-|x|^2 \approx [a,x] \quad \text{ and } \quad |E(a,r)| \approx (1-|a|^2)^n,$$

where $|E(a,r)|$ denotes the Euclidean volume of $E(a,r)$.

The following standard estimate will be needed in the sequel.

Lemma 2.2. [13] *Let $\alpha > -1$ and $\beta \in \mathbb{R}$. Then for any $x \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^\alpha}{[x,y]^{n+\alpha+\beta}} dv(y) \approx \begin{cases} (1-|x|^2)^{-\beta}, & \beta > 0, \\ \log \dfrac{1}{1-|x|^2}, & \beta = 0, \\ 1, & \beta < 0. \end{cases}$$

2.2. Operators on Orlicz spaces. Let Φ be a growth function. Recall that the lower and the upper indices of Φ are respectively defined by

$$a_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \text{ and } b_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

It is known that when Φ is convex, then $1 \le a_\Phi \le b_\Phi < \infty$ and, if Φ is concave, then $0 \le a_\Phi \le b_\Phi \le 1$ (cf. [6, 14]).

Definition 2.1. *Let Φ be a growth function. A linear operator T defined on $L^\Phi_\alpha(\mathbb{B})$ is said to be of mean strong type $(\Phi,\Phi)_\alpha$ if*

$$\int_{\mathbb{B}} \Phi(|Tf|) dv_\alpha(x) \le C \int_{\mathbb{B}} \Phi(|f|) dv_\alpha(x)$$

for any $f \in L^\Phi_\alpha(\mathbb{B})$, and T is said to be mean weak type $(\Phi,\Phi)_\alpha$ if

$$\sup_{t>0} \Phi(t) v_\alpha(\{x \in \mathbb{B} : |Tf(x)| > t\}) \le C \int_{\mathbb{B}} \Phi(|f|) dv_\alpha(x)$$

for any $f \in L^\Phi_\alpha(\mathbb{B})$, where C is independent of f.

We remark that if $\Phi(t) = t^p$, then the mean strong type $(t^p, t^p)_\alpha$ is the usual strong type (p,p) coincide. The following result comes from [4, Theorem 4.3].

Lemma 2.3. *Let Let Φ_0, Φ_1 and Φ_2 be three convex growth functions. Sup- pose that their upper and lower indices satisfy the following condition*

$$1 \le a_{\Phi_0} \le b_{\Phi_0} < a_{\Phi_2} \le b_{\Phi_2} < a_{\Phi_1} \le b_{\Phi_1} < \infty.$$

If T is of mean weak types $(\Phi_0,\Phi_0)_\alpha$ and $(\Phi_1,\Phi_1)_\alpha$, then it is of mean strong type $(\Phi_2,\Phi_2)_\alpha$.

Let $\beta \in \mathbb{R}$ and consider the operator E_β defined for functions f on $\mathbb{B}$ by

$$E_\beta f(x) = \int_{\mathbb{B}} f(y) \frac{(1-|y|^2)^\beta}{[x,y]^{n+\beta}} dv(y).$$

We refer to [2] and [3] for more details on Bergman type projection E_β. For a proof of the following lemma see, for example [9, Theorem 1.6].

Lemma 2.4. *Let $1 \le p < \infty$ and $\alpha, \beta > -1$. The operator $E_\beta : L^p_\alpha(\mathbb{B}) \to L^p_\alpha(\mathbb{B})$ is bounded if and only if $\alpha + 1 < p(\beta + 1)$.*

Combing Lemmas 2.3 and 2.4, the following result can be easily derived.

Lemma 2.5. *Let $\alpha, \beta > -1$ and Φ be a convex growth function with its lower indice a_Φ. If $1 < p < a_\Phi$ and $\alpha + 1 < p(\beta + 1)$, then E_β is of mean strong type $(\Phi, \Phi)_\alpha$.*

2.3. Harmonic functions. It is well-known that the weighted harmonic Bergman spaces $\mathcal{B}^2_\alpha$ for $\alpha > -1$ is a reproducing kernel Hilbert space with reproducing kernel $R_\alpha(x, y)$:

$$f(x) = \int_{\mathbb{B}} f(y) R_\alpha(x, y) dv_\alpha(y), \; f \in \mathcal{B}^2_\alpha.$$

The reproducing kernels $R_\alpha(x, y)$ can be expressed in terms of zonal harmonics as

$$R_\alpha(x, y) = \sum_{k=0}^{\infty} \frac{(1 + \frac{n}{2} + \alpha)_k}{(\frac{n}{2})_k} Z_k(x, y) = \sum_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x, y),$$

where the series absolutely and uniformly converges on $K \times \mathbb{B}$, for any compact subset K of $\mathbb{B}$. $R_\alpha(x, y)$ is real-valued, symmetric in the variables x and y and harmonic with respect to each variable since the same is true for all $Z_k(x, y)$. For the extension of reproducing kernels $R_\alpha(x, y)$ to all $\alpha \in \mathbb{R}$, see [7, 9].

We recall some useful inequalities concerning harmonic functions which are useful for our investigations.

Lemma 2.6. [4, 13] *Let $0 < p < \infty$, $0 < r < 1$ and $f \in h(\mathbb{B})$. Then there exists some positive constant C such that*

(1) $|f(x)|^p \le C \int_{E(x,r)} |f(y)|^p d\tau(y);$

(2) $|\nabla f(x)|^p \le \frac{C}{(1 - |x|^2)^p} \int_{E(x,r)} |f(y)|^p d\tau(y),$

where $d\tau(x) = (1 - |x|^2)^{-n} dv(x)$ is the invariant measure on $\mathbb{B}$.

The above lemma leads to the following integral inequality (cf. [7, Lemma 5.1]).

Lemma 2.7. *Let $0 < p < 1$ and $\alpha > -1$. Then*

$$\int_{\mathbb{B}} |f(x)g(x)|(1 - |x|^2)^{(n+\alpha)/p - n} dv_\alpha(x) \lesssim \|f(x)g(x)\|_{L^p_\alpha}$$

for all $f, g \in h(\mathbb{B})$.

3. Proofs of main results

The purpose of this section is to prove our main results. Before the proofs, we need the following lemmas.

Lemma 3.1. *Let $\Phi \in \mathcal{L}_p$. Then the growth function Φ_p, defined by $\Phi_p(t) = \Phi(t^{\frac{1}{p}})$ is in $\mathcal{U}^q$ for some $q \geq 1$. Moreover, for $s > 0$ and $t \geq 1$,*

$$\Phi_p(ts) \leq t^{\frac{1}{p}} \Phi_p(s). \tag{3.1}$$

Proof. We only need to prove (3.1) since the assertion $\Phi_p \in \mathcal{U}^q$ for some $q \geq 1$ can be found in [14, Lemma 2.1]. Let $s > 0$ and $t \geq 1$. By the monotonicity of $\frac{\Phi(t)}{t}$, it deduces that

$$\Phi_p(ts) = \Phi(t^{\frac{1}{p}} s^{\frac{1}{p}}) \leq t^{\frac{1}{p}} \Phi(s^{\frac{1}{p}}) = t^{\frac{1}{p}} \Phi_p(s).$$

This gives (3.1).

Lemma 3.2. *Let $\alpha > -1$ and $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$. Then there exists a constant $C > 0$ such that for any $f \in \mathcal{B}_\alpha^\Phi$,*

$$\int_{\mathbb{B}} \Phi((1-|x|^2)|\nabla f(x)|) dv_\alpha(x) \leq C \int_{\mathbb{B}} \Phi(|f(x)|) dv_\alpha(x). \tag{3.2}$$

Proof. Let

$$p_\Phi = \begin{cases} 1, & \text{if } \Phi \in \mathcal{U}^q, \\ p, & \text{if } \Phi \in \mathcal{L}_p. \end{cases} \tag{3.3}$$

By Lemma 2.6, for each $x \in \mathbb{B}$, there exist $C > 0$ such that

$$\big((1-|x|^2)|\nabla f(x)|\big)^{p_\Phi} \leq C \int_{E(x,r)} |f(y)|^{p_\Phi} d\tau(y).$$

Set

$$\Phi_p(t) = \begin{cases} \Phi(t), & \text{if } \Phi \in \mathcal{U}^q, \\ \Phi(t^{\frac{1}{p}}), & \text{if } \Phi \in \mathcal{L}_p. \end{cases} \tag{3.4}$$

It follows from Lemma 3.1 and the convexity of $\Phi_p(t)$ that

$$\Phi((1-|x|^2)|\nabla f(x)|) \leq C \int_{E(x,r)} \Phi(|f(y)|) d\tau(y).$$

Integrating both sides of the above inequality over $\mathbb{B}$ with respect to $dv_\alpha(x)$ and applying Fubini's theorem and Lemma 2.1, (3.2) follows.

Lemma 3.3. *Let $\alpha > -1$, $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$ and $f \in h(\mathbb{B})$. If $(1-|x|^2)|\mathcal{R}f(x)| \in L_\alpha^\Phi(\mathbb{B})$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{B}} \Phi(|f(x) - f(0)|) dv_\alpha(x) \leq C \int_{\mathbb{B}} \Phi((1-|x|^2)|\mathcal{R}f(x)|) dv_\alpha(x). \tag{3.5}$$

Proof. Assume that $f \in h(\mathbb{B})$ and $(1-|x|^2)|\mathcal{R}f(x)| \in L^{\Phi}_{\alpha}(\mathbb{B})$. In view of [9, Theorem 1.4], we see that for large enough s,

$$\mathcal{R}f(x) = \int_{\mathbb{B}} \mathcal{R}f(y) R_s(x,y) dv_s(y).$$

Since $\int_{\mathbb{B}} \mathcal{R}f(y)dv_s(y) = 0$, subtracting this from the previous equation yields

$$\mathcal{R}f(x) = \int_{\mathbb{B}} \mathcal{R}f(y)(R_s(x,y)-1) dv_s(y).$$

Consequently,

$$\begin{aligned} |f(x)-f(0)| &= \Big| \int_0^1 \int_{\mathbb{B}} \mathcal{R}f(y)(R_s(tx,y)-1)dv_s(y)\frac{dt}{t} \Big| \\ &= \Big| \int_{\mathbb{B}} \mathcal{R}f(y) \int_0^1 \frac{R_s(tx,y)-1}{t} dt dv_s(y) \Big|. \end{aligned}$$

Let

$$G(x,y) = \int_0^1 \frac{R_s(tx,y)-1}{t} dt.$$

By an argument similar to the one in the proof of [9, Lemma12.1], we have

$$|G(x,y)| \le \int_0^1 \Big| \frac{R_s(tx,y)-1}{t} \Big| dt \lesssim \int_0^1 \frac{dt}{[tx,y]^{n+s}} \lesssim \frac{1}{[x,y]^{n+s-1}}.$$

Therefore,

$$|f(x)-f(0)| \lesssim \int_{\mathbb{B}} (1-|y|^2)|\mathcal{R}f(y)| \frac{1}{[x,y]^{n+s-1}} dv_{s-1}(y).$$

We first consider the case $\Phi \in \mathcal{U}^q$. Fix p so that $1 < p < a_{\Phi}$. Since $(1-|x|^2)|\mathcal{R}f(x)| \in L^{\Phi}_{\alpha}(\mathbb{B})$, by taking s large enough so that $\alpha + 1 < ps$, we obtain from Lemma 2.5 that

$$\int_{\mathbb{B}} \Phi(|f(x)-f(0)|) dv_{\alpha}(x) \le C \int_{\mathbb{B}} \Phi((1-|x|^2)|\mathcal{R}f(x)|) dv_{\alpha}(x).$$

We now consider the case of $\Phi \in \mathcal{L}_p$. Set $s = (n+\alpha')/p - n$ and $\alpha' > \alpha + p$. By Lemma 2.7, it deduces that

$$\begin{aligned} |f(x)-f(0)|^p &\lesssim \int_{\mathbb{B}} |\mathcal{R}f(y)|^p |G(x,y)|^p dv_{\alpha'}(y) \\ &\lesssim \int_{\mathbb{B}} \frac{|\mathcal{R}f(y)|^p}{[x,y]^{p(n+s-1)}} dv_{\alpha'}(y) \\ &\lesssim \int_{\mathbb{B}} \frac{|(1-|y|^2)\mathcal{R}f(y)|^p}{[x,y]^{n+\alpha'-p}} dv_{\alpha'-p}(y). \end{aligned}$$

As the growth function $t \to \Phi_p(t) = \Phi(t^{\frac{1}{p}})$ is in $\mathcal{U}^q$, proceeding as in the first part of this proof yields that

$$\begin{aligned} &\int_{\mathbb{B}} \Phi(|f(x)-f(0)|) dv_{\alpha}(x) = \int_{\mathbb{B}} \Phi_p(|f(x)-f(0)|^p) dv_{\alpha}(x) \\ \lesssim\ &\int_{\mathbb{B}} \Phi_p((1-|x|^2)|\mathcal{R}f(x)|)^p) dv_{\alpha}(x) \lesssim \int_{\mathbb{B}} \Phi((1-|x|^2)|\mathcal{R}f(x)|) dv_{\alpha}(x). \end{aligned}$$

The proof of this lemma is complete.

Proof of Theorem 1.1. $(a) \Rightarrow (b)$ follows from Lemma 3.2. $(b) \Rightarrow (c)$ is trivial since $|\mathcal{R}f(x) \leq |\nabla f(x)|$ for $x \in \mathbb{B}$. Lemma 3.3 implies $(c) \Rightarrow (a)$.

Proof of Theorem 1.2. We first prove $(b) \Rightarrow (a)$. Assume that (b) holds. Then for each fixed $x \in \mathbb{B}$ and all y sufficiently close to x

$$\frac{|f(x) - f(y)|}{\rho(x,y)} \leq g(x) + g(y), \quad x \neq y.$$

Letting y approach x in the direction of each real coordinate axis, we conclude

$$(1 - |x|^2)|\nabla f(x)| \leq Cg(x).$$

It follows from the assumption $g \in L^\Phi_\alpha(\mathbb{B})$ that

$$\int_{\mathbb{B}} \Phi(|(1 - |x|^2)|\nabla f(x)|)dv_\alpha(x) < \infty.$$

Hence $f \in \mathcal{B}^\Phi_\alpha$ by Theorem 1.1.

$(b) \Rightarrow (a)$. Assume that $f \in \mathcal{B}^\Phi_\alpha$. Fix a small positive r and consider any two points $x, y \in \mathbb{B}$ with $y \in E(x,r)$. Since $E(x,r)$ is a Euclidean ball, by Lemma 2.1, it is given that

$$\begin{aligned} |f(x) - f(y)| &\leq C|x - y| \int_0^1 |\nabla f(sy + (1-s)x)|ds \\ &\leq C\rho(x,y) \sup\{(1 - |x|^2)|\nabla f(\xi)| : \xi \in E(x,r)\} \\ &= \rho(x,y)h(x), \end{aligned}$$

where

$$h(x) = C_r \sup\{(1 - |\xi|^2)|\nabla f(\xi)| : \xi \in E(x,r)\}.$$

If $\rho(x,y) \geq r$, then the triangle inequality implies

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x)| + |f(y)| \\ &\leq \rho(x,y)\Big(\frac{|f(x)|}{r} + \frac{|f(y)|}{r}\Big). \end{aligned}$$

By letting $g(x) = h(x) + \frac{|f(x)|}{r}$, we have

$$|f(x) - f(y)| \leq \rho(x,y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$. It is easy to see that $g(x) = h(x) + \frac{|f(x)|}{r}$ is the desired function provided that $h \in L^\Phi_\alpha(\mathbb{B})$. From (2.1), we can find r' such that $0 < r < r' < 1$ and $E(\xi, r) \subset E(x, r')$ for every $\xi \in E(x,r)$. It follows from Lemma 2.6 and the proof of Lemma 3.2 that

$$|h(x)|^{p_\Phi} \quad \leq \quad C \int_{E(x,r')} |f(y)|^{p_\Phi} d\tau(y)$$

and

$$\Phi(|h(x)|) \quad \leq \quad C \int_{E(x,r')} \Phi(|f(y)|) d\tau(y),$$

where p_Φ is given in (3.3).

Hence by Fubini's theorem and Lemma 2.1,

$$\begin{aligned} \int_{\mathbb{B}} \Phi(|h(x)|) dv_\alpha(x) &\leq C \int_{\mathbb{B}} (1-|x|^2)^\alpha \int_{E(x,r')} \Phi(|f(y)|) d\tau(y) dv(x) \\ &\leq C \int_{\mathbb{B}} \Phi(|f(y)|) d\tau(y) \int_{E(y,r')} (1-|x|^2)^\alpha d\tau(x) \\ &\leq C \int_{\mathbb{B}} \Phi(|f(y)|) dv_\alpha(y). \end{aligned}$$

This proves $(a) \Leftrightarrow (b)$.

As $\rho \leq \varrho$, so the implication $(a) \Rightarrow (c)$ is trivial. For the converse, suppose that there exists a positive continuous function $g \in L^\Phi_\alpha(\mathbb{B})$ such that

$$|f(x) - f(y)| \leq \varrho(x,y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$. By a discussion similar to the proof of $(b) \Rightarrow (a)$, it concludes that

$$(1-|x|^2)|\nabla f(x)| \leq C g(x),$$

which implies that $f \in \mathcal{B}^\Phi_\alpha$. The proof of Theorem 1.2 is finished.

Proof of Theorem 1.3. Since $f \in \mathcal{B}^\Phi_\alpha$, we know that there exists a positive continuous function $g_1 \in L^\Phi_\alpha(\mathbb{B})$ such that

$$|f(x) - f(y)| \leq \rho(x,y)\big(g_1(x) + g_1(y)\big), \quad x, y \in \mathbb{B},$$

from Theorem 1.2. As for $x, y \in \mathbb{B}$,

$$[x,y] \geq 1 - |x|, \quad [x,y] \geq 1 - |y|,$$

we deduce that

$$|f(x) - f(y)| \leq |x-y|\Big(\frac{g_1(x)}{[x,y]} + \frac{g_1(y)}{[x,y]}\Big) \leq |x-y|\big(g(x) + g(y)\big),$$

where

$$g(x) = \frac{g_1(x)}{1-|x|} \leq \frac{2g_1(x)}{1-|x|^2}.$$

This means that $(1 - |\cdot|^2) g(\cdot) \in L^\Phi_\alpha(\mathbb{B})$.

(1) If $\Phi \in \mathcal{U}^q$ and $k \in [q, \infty)$, by (1.2) we have

$$\begin{aligned} \int_{\mathbb{B}} \Phi(|g(x)|) dv_{\alpha+k}(x) &\leq C \int_{\mathbb{B}} \Phi((1-|x|^2)|g(x)|)(1-|x|^2)^{k-q} dv_\alpha(x) \\ &\leq C \int_{\mathbb{B}} \Phi((1-|x|^2)|g(x)|) dv_\alpha(x), \end{aligned}$$

which implies that $g \in L^\Phi_{\alpha+k}(\mathbb{B})$.

(2) If $\Phi \in \mathcal{L}_p$ and $k \in [1,\infty)$, then by Lemma 3.1, $\Phi_p(t) = \Phi(t^{\frac{1}{p}}) \in \mathcal{U}^{1/p}$. It follows from (1.2) again that

$$\begin{aligned}\int_{\mathbb{B}} \Phi(|g(x)|)dv_{\alpha+k}(x) &= \int_{\mathbb{B}} \Phi_p(|g(x)|^p)dv_{\alpha+k}(x)\\ &\leq C\int_{\mathbb{B}} \Phi_p((1-|x|^2)^p|g(x)|^p)(1-|x|^2)^{k-1}dv_\alpha(x)\\ &\leq C\int_{\mathbb{B}} \Phi_p((1-|x|^2)^p|g(x)|^p)dv_\alpha(x)\\ &\leq C\int_{\mathbb{B}} \Phi((1-|x|^2)|g(x)|)dv_\alpha(x),\end{aligned}$$

as desired. The proof is complete.

4. A difference quotient of harmonic function on $\mathbb{B}$

In this section we present an application of our main results.

Let $f \in h(\mathbb{B})$, we define a difference quotient of f by

$$Lf(x,y) = \frac{f(x)-f(y)}{x-y}, \quad x,y \in \mathbb{B}, x \neq y.$$

It is known that in [16], Wulan and Zhu introduced this kind of operator L in the setting of analytic functions spaces and discussed the boundedness of L between standard weighted Bergman space $\mathcal{A}^p_\alpha(\mathbb{D})$. Especially, they obtained the following:

(I) If $\alpha > -1$, and $p \in (0, \alpha+2)$, then L is bounded from $\mathcal{A}^p_\alpha(\mathbb{D})$ into $\mathcal{A}^p_\alpha(\mathbb{D}\times\mathbb{D})$;

(II) If $\alpha > -1$, and $p > \alpha+2$ and $\beta = \frac{p+\alpha-2}{2}$, then L is bounded from $\mathcal{A}^p_\alpha(\mathbb{D})$ into $\mathcal{A}^p_\beta(\mathbb{D}\times\mathbb{D})$.

We now extend this result to the harmonic *Bergman-Orlicz* space $\mathcal{B}^\Phi_\alpha$ as follows.

Theorem 4.1. *Let $\alpha > -1$ and $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$. If $q \in [1, n+\alpha)$, then $L : \mathcal{B}^\Phi_\alpha \to L^\Phi_\alpha(\mathbb{B}\times\mathbb{B})$ is bounded.*

Proof. Let $f \in \mathcal{B}^\Phi_\alpha$. By Theorem 1.2, there exists a positive continuous function $g \in L^\Phi_\alpha(\mathbb{B})$ such that

$$|f(x)-f(y)| \leq \rho(x,y)\big(g(x)+g(y)\big),$$

which in turn gives

$$|Lf(x,y)| = \Big|\frac{f(x)-f(y)}{x-y}\Big| \leq \frac{g(x)+g(y)}{[x,y]}, \quad x \neq y.$$

Set

$$p_\Phi = \begin{cases} 1, & \text{if} \quad \Phi \in \mathcal{U}^q,\\ p, & \text{if} \quad \Phi \in \mathcal{L}_p.\end{cases}$$

Then

$$|Lf(x,y)|^{p_\Phi} \leq \Big(\frac{g(x)}{[x,y]}\Big)^{p_\Phi} + \Big(\frac{g(y)}{[x,y]}\Big)^{p_\Phi}.$$

In what follows, we divide the proof into two cases based on the types of Φ.

Case I. $\Phi \in \mathcal{U}^q$, $q \in [1, n+\alpha)$ and $p_\Phi = 1$.

Applying the convexity of Φ, we have

$$\Phi\Big(\frac{|Lf(x,y)|}{2}\Big) \leq \frac{1}{2}\Phi\Big(\frac{g(x)}{[x,y]}\Big) + \frac{1}{2}\Phi\Big(\frac{g(y)}{[x,y]}\Big).$$

It follows from the definition of $\mathcal{U}^q$ that

$$\begin{aligned}
&\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2}\Big) dv_\alpha(x)dv_\alpha(y)\\
\lesssim\ & \frac{1}{2}\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{g(x)}{[x,y]}\Big) dv_\alpha(x)dv_\alpha(y) + \frac{1}{2}\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{g(y)}{[x,y]}\Big) dv_\alpha(x)dv_\alpha(y)\\
\lesssim\ & \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{g(x)}{[x,y]}\Big) dv_\alpha(x)dv_\alpha(y) \leq \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{2g(x)}{[x,y]}\Big) dv_\alpha(x)dv_\alpha(y)\\
\lesssim\ & \int_{\mathbb{B}} \Phi(g(x))dv_\alpha(x) \int_{\mathbb{B}} \frac{2^q(1-|y|^2)^\alpha}{[x,y]^q} dv(y).
\end{aligned}$$

Since $q \in [1, n+\alpha)$, according to Lemma 2.2, we have

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2}\Big) dv_\alpha(x)dv_\alpha(y) \quad \lesssim \quad \int_{\mathbb{B}} \Phi(g(x))dv_\alpha(x).$$

Case II. $\Phi \in \mathcal{L}_p$ and $p_\Phi = p$.

In view of Lemma 3.1, $\Phi_p(t) = \Phi(t^{\frac{1}{p}}) \in \mathcal{U}^{1/p}$ and by the convexity of Φ_p,

$$\Phi_p\Big(\Big|\frac{Lf(x,y)}{2^{\frac{1}{p}}}\Big|^p\Big) \leq \frac{1}{2}\Phi_p\Big(\Big|\frac{g(x)}{[x,y]}\Big|^p\Big) + \frac{1}{2}\Phi_p\Big(\Big|\frac{g(y)}{[x,y]}\Big|^p\Big).$$

By an argument similar to that in the proof of Case I and Lemma 3.1, we have

$$\begin{aligned}
&\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2^{\frac{1}{p}}}\Big) dv_\alpha(x)dv_\alpha(y) = \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi_p\Big(\Big|\frac{Lf(x,y)}{2^{\frac{1}{p}}}\Big|^p\Big) dv_\alpha(x)dv_\alpha(y)\\
\lesssim\ & \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi_p\Big(\Big|\frac{2g(x)}{[x,y]}\Big|^p\Big) dv_\alpha(x)dv_\alpha(y) \lesssim \int_{\mathbb{B}} \Phi(g(x))dv_\alpha(x) \int_{\mathbb{B}} \frac{2(1-|y|^2)^\alpha}{[x,y]} dv(y)\\
\lesssim\ & \int_{\mathbb{B}} \Phi(g(x))dv_\alpha(x).
\end{aligned}$$

Combing these two cases and using a standard argument based on the closed graph theorem, it concludes that the operator $L : \mathcal{B}_\alpha^\Phi \to L_\alpha^\Phi(\mathbb{B}\times\mathbb{B})$ is bounded.

For the case of $q > n+\alpha$, we can also prove the following result by using an argument similar to the one in the proof of Theorem 4.1.

Theorem 4.2. *Let $\alpha > -1$. Suppose that one of the following two conditions is satisfied:*

(1) $\Phi \in \mathcal{U}^q$ with $q \in (n+\alpha, \infty)$ and $2\beta + n = q + \alpha$;

(2) $\Phi \in \mathcal{L}_p$ and $2\beta + n = pq + \alpha$ for some $pq > n + \alpha$.

Then $L : \mathcal{B}_\alpha^\Phi \to L_\beta^\Phi(\mathbb{B}\times\mathbb{B})$ is bounded.

Proof. Let $f \in \mathcal{B}_\alpha^\Phi$. (1) If $\Phi \in \mathcal{U}^q$ with $q \in (n+\alpha, \infty)$ and $2\beta + n = q + \alpha$, then by the same reasoning as in the proof of the above results, we have

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2}\Big) dv_\beta(x) dv_\beta(y) \lesssim \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{g(x)}{[x,y]}\Big) dv_\beta(x) dv_\beta(y)$$
$$\lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\beta(x) \int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^q} dv(y) \lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\alpha(x),$$

where the last inequality follows from Lemma 2.2. Hence $L : \mathcal{B}_\alpha^\Phi \to L_\beta^\Phi(\mathbb{B}\times\mathbb{B})$ is bounded.

(2) If $\Phi \in \mathcal{L}_p$ and $2\beta + n = pq + \alpha$ for some $pq > n + \alpha$, then by Lemmas 2.2 and 3.1 we have

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2^{\frac{1}{p}}}\Big) dv_\beta(x) dv_\beta(y) \lesssim \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi_p\Big(\Big|\frac{2g(x)}{[x,y]}\Big|^p\Big) dv_\beta(x) dv_\beta(y)$$
$$\lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\beta(x) \int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^{pq}} dv(y) \lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\alpha(x).$$

The result follows.

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EFFECTIVENESS OF THE EVEN-TYPE DELAYED MEAN IN APPROXIMATION OF CONJUGATE FUNCTIONS

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Abstract. In this paper, using the even-type delayed mean of conjugate series, we have obtain the degree of approximation for a conjugate function in the metric of the generalized Höder class with weight. Involving two moduli of continuity, we have shown that the even-type delayed mean are streamlined to guarantee this degree to be of the Jackson order.

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1. Introduction

Looking into mathematical research articles, someone can encounter one class of functions introduced in the past and its generalizations defined in the last decades - the Hölder class. Focusing to our research interest, this provoking class for many researchers has been used to a research problem (question) which specifically is related to the trigonometric Fourier series and "more rarely" to their conjugate series. Factually, the problem consists in determining the degree of approximation of 2π-periodic integrable functions (which belongs to the Hölder class) and for the conjugate functions by various means of their Fourier series and their conjugate series, respectively.

Pertaining to the above problem, a lot of results have been published for dozens of decades. For instance, Prössdorf [25], Leindler [13]–[14], Chandra [1], Mohapatra and Chandra [17], Singh et al. [26]-[27], Das et al. [2]–[3], Mittal and Rhoades [19], Krasniqi [9] and [11], Krasniqi and Szal [10], Lenski at al. [15], Krasniqi et al. [12], Nayak at al. [21]–[22], Singh and Sonker [29], Değer and Kücükaslan [4], Değer [5], Pradhan et al. [24], and Kim [8], are among the researchers who have contributed to the present topic.

In the sequel, we do not recall all results in the mentioned papers, but for our purpose we are going to write some definitions and notations from [21]–[22] and the result in [8], which serve as preliminary materials.

Let $\omega(t)$ be a modulus of continuity, i.e., $\omega(t)$ is a positive nondecreasing continuous function with the properties

$$\omega(0)=0, \quad \omega(t_1+t_2)\le \omega(t_1)+\omega(t_2), \quad \omega(\lambda t)\le(\lambda+1)\omega(t),$$

where $0\le t_1\le t_2\le t_1+t_2\le 2\pi$ and λ is any nonnegative real number.

On one hand, Das, Nath and Ray [3] introduced the space H_p^ω as follows:

$$H_p^\omega := \left\{ f\in L^p[0,2\pi] : \sup_{h\neq 0} \frac{\|f(x+h)-f(x)\|_p}{\omega(|h|)} < \infty \right\}$$

with

$$\|f\|_p^{(\omega)} := \|f\|_p + \sup_{h\neq 0} \frac{\|f(x+h)-f(x)\|_p}{\omega(|h|)},$$

where $L^p[0,2\pi]$ denotes all integrable 2π-periodic functions and

$$\|f\|_p := \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p\,dx\right)^{\frac{1}{p}} & \text{for} \quad 1\le p<\infty \\ \text{ess} \sup\limits_{x\in(0,2\pi)} \{|f(x)|\} & \text{for} \quad p=\infty. \end{cases}$$

It is shown (see [22]) that $\|\cdot\|_p^{(\omega)}$ is a norm in the space H_p^ω and it is a Banach space as well. For an integrable 2π-periodic function $f(x)$, by

$$s_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx + b_k \sin kx$$

we denote the n-th partial sums of Fourier series of f (at the point x)

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos kx + b_k \sin kx. \tag{1.1}$$

We need to recall the delayed mean $\sigma_{n,q}(f;x)$ defined as follows ([33]):

$$\sigma_{n,q}(f;x) := \frac{1}{q}\sum_{i=n}^{n+q-1} s_i(f;x)$$

or

$$\sigma_{n,q}(f;x) := \frac{n+q}{q}\sigma_{n+q-1}(f;x) - \frac{n}{q}\sigma_{n-1}(f;x),$$

where $\sigma_m(f;x)$ denotes the well-known Fejer mean of $s_i(f;x)$.

In his result, Kim [8] used the (even–type) delayed means $\sigma_{n,q}(f;x)$ with $q=rn$, $r=2,4,6,\dots$, which can be expressed by the convolution

$$\sigma_{n,rn}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_{rn}(t)dt,$$

where the kernel $K_{rn}(t)$ is defined by

$$K_{rn}(t) := \frac{2\sin\left(\frac{r}{2}+1\right)nt\sin\left(\frac{rnt}{2}\right)}{rn\left(2\sin\frac{t}{2}\right)^2}.$$

The following theorem already has been proved.

Theorem 1.1 ([8]). *Let $v(t)$ and $\omega(t)$ be moduli of continuity such that $\omega(t)/v(t)$ is nondecreasing. If $f \in H_p^\omega$, $p \geq 1$, then for an even positive integer r,*

$$\|\sigma_{n,rn}(f) - f\|_p^{(v)} = \mathcal{O}\left(\frac{1}{rn}\right) + \mathcal{O}\left(\frac{r}{n}\right) \int_{\frac{\pi}{n}}^{\pi} \frac{\omega(t)dt}{t^3 v(t)}.$$

In addition, if $\omega(t)/(tv(t))$ is non-increasing, then we have

$$\|\sigma_{n,rn}(f) - f\|_p^{(v)} = \mathcal{O}\left(\frac{r}{n}\right) \int_{\frac{\pi}{n}}^{\pi} \frac{\omega(t)dt}{t^3 v(t)}$$

and

$$\|\sigma_{n,rn}(f) - f\|_p^{(v)} = \mathcal{O}\left(r \frac{\omega\left(\frac{\pi}{n}\right)}{v\left(\frac{\pi}{n}\right)}\right).$$

In the other hand, to reveal our intention in this paper, we recall the weighted Lebesgue space $L_\beta^p[0, 2\pi]$. Let f be a 2π-periodic function and $f \in L_\beta^p := L_\beta^p[0, 2\pi]$ for $p \geq 1$, where $L_\beta^p[0, 2\pi]$ denotes all measurable functions and $\|f\|_{p;\beta}$ the weighted norm

$$\|f\|_{p;\beta} := \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \left|\sin\left(\frac{x}{2}\right)\right|^{\beta p} dx\right)^{\frac{1}{p}} & \text{for} \quad 1 \leq p < \infty \\ \operatorname{ess} \sup\limits_{x \in (0, 2\pi)} \left\{|f(x)| \left|\sin\left(\frac{x}{2}\right)\right|^{\beta}\right\} & \text{for} \quad p = \infty \end{cases}$$

with $\beta \geq 0$ a real number (see [30], [32]).

We define the generalized Hölder space with weight by

$$H_{p;\beta}^{(w)} := \left\{ f \in L_\beta^p[0, 2\pi] : \sup_{t \neq 0} \frac{\|f(x+t) - f(x-t)\|_{p;\beta}}{w(|t|)} < \infty \right\}$$

and

$$\|f\|_{p;\beta}^{(w)} := \|f\|_{p;\beta} + \sup_{t \neq 0} \frac{\|f(x+t) - f(x-t)\|_{p;\beta}}{w(|t|)},$$

where $1 \leq p < \infty$, $\beta \geq 0$, and $w(t)$ is a function of modulus continuity type.

Note that $\|\cdot\|_{p;\beta}^{(w)}$ is a norm in $H_{p;\beta}^{(w)}$. The completeness of the space $H_{p;\beta}^{(w)}$ can be debated as long as the completeness of L^p space, and thus the space $H_{p;\beta}^{(w)}$ is a Banach space under the norm $\|\cdot\|_{p;\beta}^{(w)}$.

Assume that the functions $w_1(t)$ and $w_2(t)$ are two moduli of continuity, and $\frac{w_1(t)}{w_2(t)}$ is a non-negative and non-decreasing function. Then,

$$\|f\|_{p;\beta}^{(w_2)} \leq \max\left(1, \frac{w_1(2\pi)}{w_2(2\pi)}\right) \|f\|_{p;\beta}^{(w_1)},$$

which shows that

$$H_{p;\beta}^{(w_1)} \subseteq H_{p;\beta}^{(w_2)} \subseteq L_\beta^p, \quad p \geq 1.$$

The series

$$\sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) \tag{1.2}$$

is the conjugate series of its Fourier series (1.1).

If f is integrable functions in the sens of Lebesgue, then it known that

$$\widetilde{f}(x) := -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt = \lim_{\varepsilon\to 0}\widetilde{f}(x;\varepsilon)$$

exists for almost all x (see [34]), where

$$\widetilde{f}(x;\varepsilon) := -\frac{1}{\pi}\int_{\varepsilon}^{\pi} \psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt$$

with

$$\psi(x;t) := f(x+t) - f(x-t).$$

Regrading to L^p integrability of the function $\widetilde{f}$, conjugate to the function f, we are based on a theorem of M.Riesz which states that ([34]): If $f \in L^p$ for $1 < p < \infty$, then $\widetilde{f} \in L^p$.

We have to mentioned here that, as a whole, the conjugate series of a Fourier series is not necessarily a Fourier series. For example, the series $\sum_{k=2}^{\infty}(\log n)^{-1}\sin kx$ is the conjugate series of the Fourier series $\sum_{k=2}^{\infty}(\log n)^{-1}\cos kx$, however it is not itself a Fourier series (see [34], p. 186). This fact has provoked and motivated us to determine the degree of approximation of the function $\widetilde{f}$, conjugate to the function f, in the metric of the space $H_{p;\beta}^{(w)}$, by using the even–type delayed arithmetic mean $\widetilde{\sigma}_{n,rn}(f;x)$, $(r = 2,4,6,\dots)$ of the series (1.2), which is the aim of the present paper.

More results, in reference to determining the degree of approximation of the function $\widetilde{f}$, conjugate to the function f, in the metrics of the Hölder spaces, the interested reader could find in the work of Chandra [1], Nigam and Hadish [23], Mishra and Khatri [18]-[7], Singh [31], and London et al. [16].

Even though we adopt the same technique (as other authors) for the proof of our result, this last one is new and includes its application for a wide class of functions, which at least is not narrower than classes of functions defined by others.

Our paper is organized as follows. The second section contains some helpful lemmas which play a key role for the proof of the new result, the third section is devoted to the main result, and in the forth section we give a conclusion.

2. Auxiliary lemmas

We need four auxiliary statements. The first one and the third one previously are known, the second one and the fourth one will be proved in sequel (their play a key role in the proof of the main result).

Lemma 2.1 (Generalized Minkowski inequality, [6], [34])**.** *If $z(x,t)$ is a function in two variables defined for $c \le t \le d$, $a \le x \le b$, then*

$$\left\{\int_a^b \left|\int_c^d z(x,t)dt\right|^p dx\right\}^{\frac{1}{p}} \le \int_c^d \left\{\int_a^b |z(x,t)|^p dx\right\}^{\frac{1}{p}} dt, \quad p \ge 1.$$

Lemma 2.2. *Let $w_{rm}(t) := \frac{\sin(r+1)mt - \sin mt}{\left(2\sin\frac{t}{2}\right)^2}$ and $0 \le t \le \frac{1}{n+1}$. Then,*

(i) $w_{rm}(t) = \frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{\left(2\sin\frac{t}{2}\right)^2}$

(ii) $|w_{rm}(t)| \le \frac{rm\pi^2}{4t}$, $(r = 2, 4, 6, \dots)$.

Proof. (i) The equality follows by the formula for converting the difference into product.
(ii) Using the well-known inequalities $|\sin(m\alpha)| \le m|\sin(\alpha)|$ and $\pi\sin(\alpha) \ge 2\alpha$ for $\alpha \in [0, \pi/2]$, we obtain

$$|w_{rm}(t)| = \frac{\left|2\sin\frac{rmt}{2}\right|\left|\cos\frac{(r+2)mt}{2}\right|}{\left(2\sin\frac{t}{2}\right)^2} \le \frac{2rm\left|\sin\frac{t}{2}\right|}{\left(\frac{2t}{\pi}\right)^2} \le \frac{2\pi^2 rm\frac{t}{2}}{4t^2} = \frac{\pi^2 rm}{4t}.$$

The proof is completed. □

Lemma 2.3 ([34])**.** *A function $w(t)$ of modulus of continuity type on the interval $[0, 2\pi]$ satisfies the following condition $\delta_2^{-1}w(\delta_2) \le 2\delta_1^{-1}w(\delta_1)$ for $0 < \delta_1 \le \delta_2$.*

Lemma 2.4. *Let $\frac{w_1(t)}{w_2(t)}$ be a positive and a non-decreasing function, $f \in H_{p;\beta}^{(w_1)}$, $p \ge 1$, and $\beta \ge 0$ a real number. Then,*

(a)
$$\|\psi(x+y;t) - \psi(x-y;t)\|_{p;\beta} = \mathcal{O}\left(w_1(t)\right).$$

(b)
$$\|\psi(x+y;t) - \psi(x-y;t)\|_{p;\beta} = \mathcal{O}\left(w_1(y)\right).$$

(c)
$$\|\psi(x+y;t) - \psi(x-y;t)\|_{p;\beta} = \mathcal{O}\left(\frac{w_1(t)}{w_2(t)}w_2(y)\right).$$

(d)
$$\|\psi(x+y;t) - \psi(x-y;t) - \psi(x+y;t+\pi/m) + \psi(x-y;t+\pi/m)\|_{p;\beta} = \mathcal{O}\left(\frac{w_1(t)}{w_2(t)}w_2(y)\right).$$

Proof. (a) Because of

$$\psi(x+y;t) - \psi(x-y;t) = f(x+y+t) - f(x+y-t) - f(x-y+t) + f(x-y-t),$$

then (applying the Minkowski inequality) we have

$$\begin{aligned}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta} &\leq \|f(x+y+t)-f(x+y-t)\|_{p;\beta}\\ &+\|f(x-y+t)-f(x-y-t)\|_{p;\beta}=\mathcal{O}\left(w_1(t)\right).\end{aligned}$$

(b) Very similarly, rearrangement of the terms into the brackets of the case (a) implies

$$\begin{aligned}\psi(x+y;t)-\psi(x-y;t)=&f(x+t+y)-f(x+t-y)\\ &-f(x-t+y)+f(x-t-y),\end{aligned}$$

and thus, we have

$$\begin{aligned}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta} &\leq \|f(x+t+y)-f(x+t-y)\|_{p;\beta}\\ &+\|f(x-t+y)-f(x-t-y)\|_{p;\beta}=\mathcal{O}\left(w_1(y)\right).\end{aligned}$$

(c) Let $w_2(t)$ be positive and non-decreasing function. Then, for $t \leq y$ and (a), we get

$$\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}=\mathcal{O}\left(w_2(t)\frac{w_1(t)}{w_2(t)}\right)=\mathcal{O}\left(w_2(y)\frac{w_1(t)}{w_2(t)}\right).$$

Now, let $t \geq y$. Since $\frac{w_1(t)}{w_2(t)}$ is positive and non-decreasing function, then based on (b) we obtain

$$\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}=\mathcal{O}\left(w_2(y)\frac{w_1(y)}{w_2(y)}\right)=\mathcal{O}\left(w_2(y)\frac{w_1(t)}{w_2(t)}\right).$$

(d) The proof can be done similarly. □

3. Main results

We managed to prove the following.

Theorem 3.1. *Let $w_1(t)$ and $w_2(t)$ be two moduli of continuity such that $\frac{w_1(t)}{w_2(t)}$ is positive and non-decreasing function. In addition, let f be a 2π-periodic function, Lebesgue integrable on $[0,2\pi]$, belonging to the generalized Hölder class with weight $H_{p;\beta}^{(w_1)}$, $p \geq 1$, and $\beta \geq 0$. Then for the function $\widetilde{f}$, conjugate to the function f, and for an even positive integer r*

$$\begin{aligned}&\|\widetilde{\sigma}_{m,rm}(f)-\widetilde{f}\|_{p;\beta}^{(w_2)}\\ &\quad=\mathcal{O}\left(\frac{1}{r}\left(\frac{1}{m}+\left(1+r+\frac{1}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right)\end{aligned}$$

provided that

$$\int_0^{\eta} t^{-1}w_1(t)dt=\mathcal{O}\left(w_1(\eta)\right) \tag{3.1}$$

and

$$\int_{\eta}^{\pi} t^{-2} w_1(t)dt = \mathcal{O}\left(\eta^{-1} w_1(\eta)\right) \tag{3.2}$$

for $0 < \eta < \pi$.

Proof. It is a well-known fact that the Cesàro mean $\widetilde{\sigma}_m(f;x)$ of the partial sums $\widetilde{s}_j(f;x)$ of the series (1.2) can be expressed as follows (see [34])

$$\widetilde{\sigma}_m(f;x) = -\frac{2}{\pi}\int_0^{\pi}[f(x+t)-f(x-t)]\widetilde{K}_m(t)dt,$$

where

$$\widetilde{K}_m(t) := \frac{(m+1)\sin t - \sin(m+1)t}{(m+1)\left(2\sin\frac{t}{2}\right)^2}.$$

The above equality can be rewritten as follows

$$\widetilde{\sigma}_m(f;x) - \widetilde{f}(x) = \frac{2}{\pi(m+1)}\int_0^{\pi}[f(x+t)-f(x-t)]\frac{\sin(m+1)t}{\left(2\sin\frac{t}{2}\right)^2}dt.$$

Whence, for the even-type delayed arithmetic mean $\widetilde{\sigma}_{m,rm}(f;x)$ of the series (1.2), we can write

$$\begin{aligned}
\widetilde{\tau}_m(x) &:= \widetilde{\sigma}_{m,rm}(f;x) - \widetilde{f}(x)\\
&= \frac{r+1}{r}\left[\sigma_{(r+1)m-1}(f;x) - \widetilde{f}(x)\right] - \frac{1}{r}\left[\sigma_{m-1}(f;x) - \widetilde{f}(x)\right]\\
&= \frac{r+1}{r}\frac{2}{\pi(r+1)m}\int_0^{\pi}[f(x+t)-f(x-t)]\frac{\sin(r+1)mt}{\left(2\sin\frac{t}{2}\right)^2}dt\\
&- \frac{1}{r}\frac{2}{\pi m}\int_0^{\pi}[f(x+t)-f(x-t)]\frac{\sin mt}{\left(2\sin\frac{t}{2}\right)^2}dt\\
&= \frac{2}{rm\pi}\int_0^{\pi}[f(x+t)-f(x-t)]w_{rm}(t)dt, \qquad (3.3)
\end{aligned}$$

where

$$w_{rm}(t) = \frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{\left(2\sin\frac{t}{2}\right)^2}.$$

By definition of the norm $\|\cdot\|_{p;\beta}^{(w_2)}$, we have

$$\|\widetilde{\tau}_m(x)\|_{p;\beta}^{(w_2)} := \|\widetilde{\tau}_m(x)\|_{p;\beta} + \sup_{y\neq 0}\frac{\|\widetilde{\tau}_m(x+y) - \widetilde{\tau}_m(x-y)\|_{p;\beta}}{w_2(y)}. \tag{3.4}$$

Further, we will find the upper bound of $\|\widetilde{\tau}_m(x)\|_{p;\beta}$. First of all, it clear that

$$\begin{aligned}\widetilde{\tau}_m(x) &= \frac{2}{rm\pi}\int_0^{\frac{\pi}{m}}[f(x+t)-f(x-t)]w_{rm}(t)dt\\ &+\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}[f(x+t)-f(x-t)]\\ &\times\left(2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right)\left(\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}\right)dt\\ &+\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}[f(x+t)-f(x-t)]\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{t^2}dt.\end{aligned}$$

Hence, applying Lemma 2.1, we have

$$\begin{aligned}\|\widetilde{\tau}_m(x)\|_{p;\beta} &\le \frac{2}{rm\pi}\int_0^{\frac{\pi}{m}}\|f(x+t)-f(x-t)\|_{p;\beta}\,|w_{rm}(t)|\,dt\\ &+\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|f(x+t)-f(x-t)\|_{p;\beta}\\ &\times\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|\left|\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}\right|dt\\ &+\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|f(x+t)-f(x-t)\|_{p;\beta}\frac{\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|}{t^2}dt\\ &:= P_1+P_2+P_3. \qquad (3.5)\end{aligned}$$

Since $f\in H_{p;\beta}^{(w_1)}$, for $p\ge 1$ and $\beta\ge 0$, then

$$\|f(x+t)-f(x-t)\|_{p;\beta}=\mathcal{O}\left(w_1(t)\right), \qquad (3.6)$$

and consequently by Lemma 2.2 we have

$$\begin{aligned}P_1 &= \frac{\mathcal{O}(1)}{rm}\int_0^{\frac{\pi}{m}}w_1(t)\frac{rm}{t}dt=\mathcal{O}(1)\int_0^{\frac{\pi}{m}}\frac{w_1(t)}{t}dt\\ &=\mathcal{O}(1)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}w_2\left(\frac{\pi}{m}\right)=\mathcal{O}(1)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}w_2\left(\pi\right)=\mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}\right) \qquad (3.7)\end{aligned}$$

taking into account the condition (3.1).

Noticing that the function

$$t\to\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}$$

is bounded for $t\in[\pi/(n+1),\pi]$, and using Lemma 2.3 and (3.6) we can write

$$P_2 = \frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|f(x+t)-f(x-t)\|_{p;\beta}\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|\left|\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}\right|dt$$

(3.8)

$$= \frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\mathcal{O}\left(\frac{tw_1(t)}{t}\right)dt = \frac{4w_2(\pi)}{r\pi^2}\mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}\right) = \mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{rw_2\left(\frac{\pi}{m}\right)}\right).$$

In

$$P_0 := \frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\psi(x;t)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{t^2}dt$$

we substitute t with $t+\pi/m$ and since r is an even natural number, we get

$$\begin{aligned}P_0 &:= -\frac{2}{rm\pi}\int_0^{\pi-\frac{\pi}{m}}\psi(x;t+\pi/m)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt\\ &= -\frac{2}{rm\pi}\left(\int_0^{\frac{\pi}{m}}+\int_{\frac{\pi}{m}}^{\pi}-\int_{\pi-\frac{\pi}{m}}^{\pi}\right)\psi(x;t+\pi/m)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt\end{aligned}$$

Adding these two expressions side by side we obtaining

$$\begin{aligned}P_0 &:= \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\left[\frac{\psi(x;t)}{t^2}-\frac{\psi(x;t+\pi/m)}{(t+\pi/m)^2}\right]2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}dt\\ &+\int_{\pi-\frac{\pi}{m}}^{\pi}\psi(x;t+\pi/m)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt\\ &-\frac{1}{rm\pi}\int_0^{\frac{\pi}{m}}\psi(x;t+\pi/m)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt := P_{00}+P_{01}-P_{02}.\end{aligned} \tag{3.9}$$

The quantity P_{00} can be rewritten as follows

$$\begin{aligned}P_{00} &= \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}(\psi(x;t)-\psi(x;t+\pi/m))\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{t^2}dt\\ &+\frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\psi(x;t+\pi/m)\left(\frac{1}{t^2}-\frac{1}{(t+\pi/m)^2}\right)2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}dt\end{aligned}$$

(3.10)

$$:= P_{000}+P_{001}.$$

Now, applying Lemma 2.1 we have

$$\begin{aligned}\|P_{000}\|_{p;\beta} &= \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|\psi(x;t)-\psi(x;t+\pi/m)\|_{p;\beta}\frac{\left|\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|dt}{t^2}\\ &= \frac{\mathcal{O}(1)}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}(w_1(t)+w_1(t+\pi/m))\frac{dt}{t^2}\\ &= \frac{\mathcal{O}(1)}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^2}dt = \frac{w_2(\pi)}{r\pi^2}\mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}\right) = \mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{rw_2\left(\frac{\pi}{m}\right)}\right)\end{aligned} \tag{3.11}$$

since $\left|\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|\le 1$, $w_1(t+\pi/m)\le 2w_1(t)$ for $t\ge\pi/m$, and condition (3.2).

By the same manner, we obtain

$$\|P_{001}\|_{p;\beta} = \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi} \|\psi(x;t+\pi/m)\|_{p;\beta}\left|\frac{1}{t^2}-\frac{1}{(t+\pi/m)^2}\right|dt$$
$$= \frac{\mathcal{O}(1)}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}(w_1(t+\pi/m))\frac{(t+\pi/m)^2-t^2}{t^2(t+\pi/m)^2}dt$$
$$= \frac{\mathcal{O}(1)}{rm^2}\int_{\frac{\pi}{m}}^{\pi} w_1(t)\frac{2t+\pi/m}{t^2(t+\pi/m)^2}dt$$
$$= \frac{\mathcal{O}(1)}{rm^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}w_2(t)dt = \frac{\mathcal{O}(1)}{rm^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt. \tag{3.12}$$

Thus, from (3.11) and (3.12), we get

$$\|P_{00}\|_{p;\beta} = \mathcal{O}(1)\left(\frac{w_1\left(\frac{\pi}{m}\right)}{rw_2\left(\frac{\pi}{m}\right)}+\frac{1}{rm^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right). \tag{3.13}$$

Since

$$\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right| \le rmt, \quad t\in(0,\pi]$$

then

$$\|P_{02}\|_{p;\beta} = \frac{\mathcal{O}(1)}{rm\pi}\int_0^{\frac{\pi}{m}}\|\psi(x;t+\pi/m)\|_{p;\beta}\frac{rmtdt}{(t+\pi/m)^2}$$
$$= \frac{\mathcal{O}(1)}{\pi}\int_0^{\frac{\pi}{m}}\frac{w_1(t)}{t}dt = \mathcal{O}(1)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}w_2\left(\frac{\pi}{m}\right)$$
$$= \mathcal{O}(1)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}w_2(\pi) = \mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}\right) \tag{3.14}$$

taking into account the condition (3.1).

Moreover, applying Lemma 2.1, we have

$$\|P_{01}\|_{p;\beta} = \frac{1}{rm\pi}\int_{\pi-\frac{\pi}{m}}^{\pi}\|\psi(x;t+\pi/m)\|_{p;\beta}\frac{dt}{(t+\pi/m)^2}$$
$$= \frac{\mathcal{O}(1)}{rm\pi}\int_{\pi-\frac{\pi}{m}}^{\pi}\frac{w_1(t+\pi/m)}{(t+\pi/m)^2w_2(t+\pi/m)}w_2(t+\pi/m)dt$$
$$= \frac{\mathcal{O}(2w_2(\pi))}{rm\pi}\int_{\pi}^{\pi+\frac{\pi}{m}}\frac{w_1(t)}{t^2w_2(t)}dt = \mathcal{O}\left(\frac{1}{rm^2}\right). \tag{3.15}$$

So, from (3.9) we obtain

$$\|P_0\|_{p;\beta} = \mathcal{O}(1)\left(\frac{1}{rm^2}+\left(1+\frac{1}{r}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{rm^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right) \tag{3.16}$$

based on (3.13), (3.14), and (3.15).

Combining (3.5), (3.7), (3.8), and (3.16) we get

$$\|\widetilde{\tau}_m(x)\|_{p;\beta} = \mathcal{O}(1)\left(\frac{1}{rm^2}+\left(1+\frac{1}{r}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{rm^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right). \tag{3.17}$$

Now we are going to estimate the second term in (3.4). For the sake of the interested reader, we will sketch the proof in details and along the same lines. Based on (3.3) we have

$$\widetilde{\tau}_m(x+y)-\widetilde{\tau}_m(x-y)=\frac{2}{rm\pi}\int_0^{\pi}\left[\psi(x+y;t)-\psi(x-y;t)\right]w_{rm}(t)dt. \tag{3.18}$$

We split the integral as follows

$$\begin{aligned}
&\|\widetilde{\tau}_m(x+y)-\widetilde{\tau}_m(x-y)\|_{p;\beta}\\
&\leq \frac{2}{rm\pi}\int_0^{\frac{\pi}{m}}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}\left|w_{rm}(t)\right|dt\\
&\quad+\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}\\
&\quad\times\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|\left|\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}\right|dt\\
&\quad+\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}\frac{\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|}{t^2}dt\\
&:=R_1+R_2+R_3. \qquad (3.19)
\end{aligned}$$

Inasmuch as

$$\left|\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{\left(2\sin\frac{t}{2}\right)^2}\right|=\mathcal{O}\left(rm\right),\quad t\in(0,\pi),$$

then using Lemma 2.4 (c), we have

$$R_1=\mathcal{O}\left(w_2(y)\right)\int_0^{\frac{\pi}{m}}\frac{w_1(t)}{w_2(t)}dt=\mathcal{O}\left(\frac{w_2(y)}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}. \tag{3.20}$$

The function

$$t\to\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}$$

is bounded for $t\in[\pi/(n+1),\pi]$, therefore and using Lemma 2.4 (c) we get

$$R_2=\mathcal{O}\left(\frac{w_2(y)}{rm}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{w_2(t)}dt=\mathcal{O}\left(\frac{w_2(y)}{rm}\right). \tag{3.21}$$

Taking into account that r is an even positive integer number and substituting t with $t+\pi/m$ in

$$R_3^{(0)}:=\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\left[\psi(x+y;t)-\psi(x-y;t)\right]\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{t^2}dt$$

we substitute t with $t+\pi/m$ and since r is an even natural number, we get

$$R_3^{(0)} = -\frac{2}{rm\pi}\int_0^{\pi-\frac{\pi}{m}}[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)] \times \frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt = -\frac{2}{rm\pi}\left(\int_0^{\frac{\pi}{m}}+\int_{\frac{\pi}{m}}^{\pi}-\int_{\pi-\frac{\pi}{m}}^{\pi}\right) \times [\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)]\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt$$

Adding these two latest equalities we have

$$R_3^{(0)} = \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\left[\frac{\psi(x+y;t)-\psi(x-y;t)}{t^2} - \frac{\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)}{(t+\pi/m)^2}\right]2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}dt + \frac{1}{rm\pi}\int_{\pi-\frac{\pi}{m}}^{\pi}[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)] \times \frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt - \frac{1}{rm\pi}\int_0^{\frac{\pi}{m}}[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)] \times \frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt := R_3^{(00)}+R_3^{(01)}-R_3^{(02)}. \tag{3.22}$$

For $R_3^{(00)}$ we can write

$$R_3^{(00)} = \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\frac{\psi(x+y;t)-\psi(x-y;t)-\psi(x+y;t+\pi/m)+\psi(x-y;t+\pi/m)}{t^2} \times 2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}dt + \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)] \times 2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\left(\frac{1}{t^2}-\frac{1}{(t+\pi/m)^2}\right)dt := R_{31}^{(00)}+R_{32}^{(00)}. \tag{3.23}$$

Now, using Lemma 2.1, Lemma 2.4 (d), and condition (3.2), we get

$$\|R_{31}^{(00)}\|_{p;\beta} = \mathcal{O}\left(\frac{w_2(y)}{rm}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^2w_2(t)}dt\right) = \mathcal{O}\left(\frac{w_2(y)}{rmw_2\left(\frac{\pi}{m}\right)}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^2}dt = \mathcal{O}\left(\frac{w_2(y)w_1\left(\frac{\pi}{m}\right)}{rw_2\left(\frac{\pi}{m}\right)}\right). \tag{3.24}$$

Moreover, using Lemma 2.1 and Lemma 2.4 (c), we also get

$$\begin{aligned}\|R_{32}^{(00)}\|_{p;\beta} &= \mathcal{O}\left(\frac{w_2(y)}{rm^2}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{w_1\left(t+\frac{\pi}{m}\right)}{w_2\left(t+\frac{\pi}{m}\right)}\frac{2t+\pi/m}{t^2(t+\pi/m)^2}dt \\ &= \mathcal{O}\left(\frac{w_2(y)}{rm^2}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{2w_1(t)}{w_2(t)}\frac{3t}{t^4}dt = \mathcal{O}\left(\frac{w_2(y)}{rm^2}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt. \qquad (3.25)\end{aligned}$$

Whence, using (3.23), (3.24), and (3.25), we obtain

$$\begin{aligned}\|R_{3}^{(00)}\|_{p;\beta} &= \mathcal{O}(1)\left(\|R_{32}^{(00)}\|_{p;\beta}+\|R_{32}^{(00)}\|_{p;\beta}\right) \\ &= \mathcal{O}\left(\frac{w_2(y)}{r}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right). \qquad (3.26)\end{aligned}$$

By similar reasoning we also have

$$\begin{aligned}\|R_{3}^{(01)}\|_{p;\beta} &= \mathcal{O}\left(\frac{w_2(y)}{rm}\right)\int_{\pi-\frac{\pi}{m}}^{\pi}\frac{w_1\left(t+\frac{\pi}{m}\right)}{w_2\left(t+\frac{\pi}{m}\right)(t+\pi/m)^2}dt \\ &= \mathcal{O}\left(\frac{w_2(y)}{rm}\right)\int_{\pi}^{\pi+\frac{\pi}{m}}\frac{w_1(t)}{t^2w_2(t)}dt = \mathcal{O}\left(\frac{w_2(y)}{rm^2}\right). \qquad (3.27)\end{aligned}$$

Once more, using Lemma 2.1 and Lemma 2.4 (c), we proceed as follows

$$\begin{aligned}\|R_{3}^{(02)}\|_{p;\beta} &= \mathcal{O}\left(\frac{w_2(y)}{rm}\right)\int_{0}^{\frac{\pi}{m}}\frac{w_1\left(t+\frac{\pi}{m}\right)}{w_2\left(t+\frac{\pi}{m}\right)}rm\,dt \\ &= \mathcal{O}\left(w_2(y)\right)\int_{0}^{\frac{\pi}{m}}\frac{2w_1(t)}{w_2(t)}dt = \mathcal{O}\left(\frac{w_2(y)w_1\left(\frac{\pi}{m}\right)}{mw_2\left(\frac{\pi}{m}\right)}\right). \qquad (3.28)\end{aligned}$$

Now, from (3.22), (3.26), (3.27), and (3.28), we find that

$$\begin{aligned}\|R_3\|_{p;\beta} &= \mathcal{O}(1)\left(\|R_{3}^{(00)}\|_{p;\beta}+\|R_{3}^{(01)}\|_{p;\beta}+\|R_{3}^{(02)}\|_{p;\beta}\right) \\ &= \mathcal{O}\left(\frac{w_2(y)}{r}\left(\frac{1}{m^2}+\left(1+\frac{1}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right). \qquad (3.29)\end{aligned}$$

Thus, using (3.19), (3.20), (3.21), and (3.29), we have

$$\begin{aligned}&\frac{\|\widetilde{\tau}_m(x+y)-\widetilde{\tau}_m(x-y)\|_{p;\beta}}{w_2(y)} \\ &\qquad = \mathcal{O}\left(\frac{1}{r}\left(\frac{1}{m}+\left(r+\frac{1}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right). \qquad (3.30)\end{aligned}$$

Finally, inserting (3.17) and (3.30) in (3.4), we obtain

$$\|\widetilde{\tau}_m(x)\|_{p;\beta}^{(w_2)} = \mathcal{O}\left(\frac{1}{r}\left(\frac{1}{m}+\left(1+r+\frac{1}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right).$$

The proof is completed. □

Next, from the main result we are going to extract only one of its particular case. To begin with, for $\gamma \in (0,1]$ and $w(t) = |t|^\gamma$ in $H_{p;\beta}^{(w)}$ class, we obtain the Hölder class with weight

$$H_{p;\beta}^{(\gamma)} := \left\{ f \in L_{\beta}^{p}[0,2\pi] : \sup_{t\neq 0} \frac{\|f(x+t)-f(x-t)\|_{p;\beta}}{|t|^{\gamma}} < \infty \right\}$$

endowed with the norm

$$\|f\|_{p;\beta}^{(\gamma)} := \|f\|_{p;\beta} + \sup_{t\neq 0} \frac{\|f(x+t)-f(x-t)\|_{p;\beta}}{|t|^{\gamma}},$$

where $1 \le p < \infty$ and $\beta \ge 0$.

Remark 3.1. *Note that the functions* $\frac{w_1(t)}{w_2(t)} = |t|^{\gamma_1-\gamma_2}$ *is positive and non-decreasing, and* $\frac{w_1(t)}{tw_2(t)} = |t|^{\gamma_1-\gamma_2-1}$ *is positive and non-increasing for* $0 \le \gamma_2 < \gamma_1 \le 1$, *and* $t \in (0,\pi]$. *Furthermore, the condition (3.1) and (3.2) automatically are satisfied. It worth to mentioned here that is said* $w_1(t)$ *to be of the first kind (see* [20]*) if it satisfies condition (3.2).*

Corollary 3.1. *Let* $w_1(t) = |t|^{\gamma_1}$, $w_2(t) = |t|^{\gamma_2}$ *and* $0 \le \gamma_2 < \gamma_1 \le 1$. *Additionally, let* $f \in H_{p;\beta}^{(\gamma_1)}$ *be a* 2π*-periodic function and Lebesgue integrable on* $[0,2\pi]$, *with* $p \ge 1$ *and* $\beta \ge 0$. *Then for the function* $\widetilde{f}$, *conjugate to the function* f, *and for an even positive integer* r

$$\|\widetilde{\sigma}_{m,rm}(f) - \widetilde{f}\|_{p;\beta}^{(\gamma_2)} = \mathcal{O}\left(\left(3 + \frac{1}{r}\right) \frac{1}{m^{\gamma_1-\gamma_2}} \right).$$

Proof. Since the function $\frac{w_1(t)}{tw_2(t)} = |t|^{\gamma_1-\gamma_2-1}$ is positive and non-increasing for $0 \le \gamma_2 < \gamma_1 \le 1$, then we have

$$\begin{aligned}
\|\widetilde{\tau}_m(x)\|_{p;\beta}^{(w_2)} &= \mathcal{O}\left(\frac{1}{r} \left(\frac{1}{m} + \frac{1}{m^2} + \left(1 + r + \frac{1}{m}\right) \frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)} + \frac{1}{m^2} \int_{\frac{\pi}{m}}^{\pi} \frac{w_1(t)}{t^3 w_2(t)} dt \right) \right) \\
&= \mathcal{O}\left(\frac{1}{r} \left(\frac{2}{m} + (2+r) \frac{\left(\frac{\pi}{m}\right)^{\gamma_1}}{\left(\frac{\pi}{m}\right)^{\gamma_2}} + \frac{1}{m^2} \frac{\left(\frac{\pi}{m}\right)^{\gamma_1}}{\frac{\pi}{m}\left(\frac{\pi}{m}\right)^{\gamma_2}} \int_{\frac{\pi}{m}}^{\pi} \frac{dt}{t^2} \right) \right) \\
&= \mathcal{O}\left(\frac{1}{r} \left(\frac{2}{m} + \frac{2r}{m^{\gamma_1-\gamma_2}} + \frac{1}{m^{1+\gamma_1-\gamma_2}} \left(\frac{m}{\pi} - \frac{1}{\pi} \right) \right) \right) \\
&= \mathcal{O}\left(\frac{1}{r} \left(\frac{3r}{m^{\gamma_1-\gamma_2}} + \frac{1}{m^{\gamma_1-\gamma_2}} \right) \right) = \mathcal{O}\left(\left(3 + \frac{1}{r}\right) \frac{1}{m^{\gamma_1-\gamma_2}} \right).
\end{aligned}$$

The proof has ended. □

4. Conclusion

Using the even-type delayed mean of conjugate series, we have obtained the degree of approximation for a conjugate function in the metric of generalized Höder class with weight. Involving two moduli of continuity and two condition on them, we have shown that this mean are streamlined to guarantee this degree to be of the Jackson order.

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СРАВНЕНИЕ МНОГОЧЛЕНОВ И ГИПЕРБОЛИЧЕСКИЕ С ВЕСОМ ОПЕРАТОРЫ

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Аннотация. В работе на языке кратности нулей подмногочленов найдены достаточные условия, при которых многочлен от двух переменных является гиперболическим с данным весом, когда его главная часть гиперболичен по Гордингу.

MSC2020 number: 12E10; 12D05; 26D05; 35A23.

Ключевые слова: характеристический многогранник (многогранник Ньютона); вес гиперболичности; гиперболические с весом многочлены (операторы).

1. Введение и постановка задачи

Пусть N - множество натуральных чисел, $N_0 := N \cup \{0\}$, N_0^n - n-мерное множество мультииндексов $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_j \in N_0$, $j = 1, \cdots, n$, $R^n (E^n)$ - n-мерное вещественное евклидово пространство точек $\xi = (\xi_1, \cdots, \xi_n)$ $(x = (x_1, \cdots, x_n))$, $\mathbb{C}$ - множество комплексных чисел, $R_+^n := \{\xi \in R^n, \xi_j \geq 0 \ j = 1, \cdots, n\}$ и $R_0^n := \{\xi \in R^n, \xi_1 \cdots \xi_n \neq 0\}$. Для $\xi, \eta \in R^n (E^n)$, $\nu \in R_+^n$, $\alpha \in \mathbb{N}_0^n$ и $t > 0$ обозначим $|\xi| := (\xi_1^2 + \cdots + \xi_n^2)^{\frac{1}{2}}$, $(\xi, \eta) := \xi_1 \cdot \eta_1 + \cdots + \xi_n \cdot \eta_n$, $|\nu| := \nu_1 + \cdots + \nu_n$, $|\xi^\nu| := |\xi_1|^{\nu_1} \cdots |\xi_n|^{\nu_n}$, $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, $t \cdot \xi = (t\xi_1, \cdots, t\xi_n)$ и $D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, где либо $D_j := \frac{\partial}{\partial \xi_j}$ либо $D_j := i^{-1} \frac{\partial}{\partial_j}$ $(i^2 = -1)$ $j = 1, \cdots, n$.

Пусть $P(\xi) = \sum\limits_\alpha a_\alpha \xi^\alpha$ многочлен с постоянными коэффициентами, где сумма распространяется по конечному набору $(P) := \{\alpha \in N_0^n, \ a_\alpha \neq 0\}$. Обозначим $m = m(P) := \max\limits_{\alpha \in (P)} |\alpha|$ и представим многочлен в виде суммы однородных многочленов

$$P(\xi) = \sum_{j=0}^{m} P_j(\xi) := \sum_{j=0}^{m} \left(\sum_{\alpha \in (P), \ |\alpha| = j} a_\alpha \xi^\alpha \right), \quad \xi \in R^n. \tag{1.1}$$

Определение 1.1. *(см. [1] или [2] определение 12.3.3). Многочлен P представленной в виде (1.1) называется гиперболическим по Гордингу относительно вектора $0 \neq \tau \in R^n$, если $P_m(\tau) \neq 0$ и существует число $t_0 > 0$, для которого $P(\xi + it\tau) \neq 0$ при $\xi \in R^n$, $t \in \mathbb{C}$, $|\operatorname{Re} t| \geq t_0$.*

Определение 1.2. *(см [3] или [4]) Функция g, определенная на R^n, называется весом гиперболичности, если 1) $\inf\limits_{\xi \in R^n} g(\xi) > 0$, 2) существуют числа $a \in [0, 1)$ и $c > 0$, для которых $g(\xi + \eta) \leq g(\xi)[1 + |\eta|^a]$, $\xi, \eta \in R^n$.*

Определение 1.3. *(см [3] или [4]). Пусть функция g является весом гиперболичности. Скажем, что многочлен P, представленный в виде (1.1), является g - гиперболическим относительно вектора $0 \neq \tau \in R^n$, если 1) $P_m(\tau) \neq 0$ и 2) существует постоянная $c > 0$, для которого $P(\xi + it\tau) \neq 0$ при $\xi \in R^n$, $t \in \mathbb{C}$, $|\operatorname{Re} t| \geq c \cdot g(\xi)$.*

Если с некоторыми постоянными $\kappa \geq 1$ и $r \in (0, 1)$

$$\kappa^{-1}(1 + |\xi|^r) \leq g(\xi) \leq \kappa(1 + |\xi|^r) \ \xi \in R^n,$$

то g - гиперболический многочлен называется $1/r$ - гиперболическим (см. [5]).

Для однородного многочлена R обозначим $\Sigma(R) := \{\xi \in R^n, \ |\xi| = 1, \ R(\xi) = 0\}$, $l(\eta) = l_R(\eta)$ $(\eta \in R^n)$ - кратность нуля многочлена R в точке η, т.е. $l(\eta) = 0$ при $R(\eta) \neq 0$ и $l(\eta) = r$, при $R(\eta) = 0$, где натуральное число r определяется из условий

$$\sum_{|\alpha| < r} |R^{(\alpha)}(\eta)| := \sum_{|\alpha| < r} |(D^\alpha R)(\eta)| = 0 \text{ и } \sum_{|\alpha| = r} |R^{(\alpha)}(\eta)| \neq 0.$$

Для конечного набора $A := \{\nu^j\} \subset R^n_+$, через $\mathfrak{N}(A)$ обозначим выпуклую оболочку множества $A \cup \{0\}$, называемый характеристическим многогранником (х.м.) или многогранником Ньютона набора A. Многогранник $\mathfrak{N} \subset R^n_+$ называется вполне правильным (в.п.), если 1) $\mathfrak{N}$ имеет вершину в начале координат, 2) отличные от начала координат вершины на каждой оси координат и 3) компоненты внешних (относительно $\mathfrak{N}$) нормалей ($\mathfrak{N}$-нормалей) $(n-1)$ - мерных не координатных граней положительны. Для в.п. многогранника $\mathfrak{N} \subset R^n_+$ и точки $0 \neq \eta \in R^n$ обозначим: $\mathfrak{N}^0$ - множество вершин $\mathfrak{N}$, $\Lambda^{n-1}(\mathfrak{N})$ - множество тех $\mathfrak{N}$ - нормалей $\lambda = (\lambda_1, \cdots, \lambda_n)$ $(n-1)$ - мерных не координатных граней, для которых $\max\limits_{1 \leq j \leq n} \lambda_j = 1$, $d_{\mathfrak{N}}(\lambda) := \max\limits_{\nu \in \mathfrak{N}} (\lambda, \nu), \lambda \in \Lambda^{n-1}(\mathfrak{N})$, $\rho_{\mathfrak{N}} := \max\limits_{\nu \in \mathfrak{N}^0} |\nu|$, $\rho_{\mathfrak{N}}(\eta) := \max\{\nu \in \mathfrak{N}^0, \nu_j = 0$ при $\eta_j = 0, 1 \leq j \leq n\}$ и положим $h_{\mathfrak{N}}(\xi) := \sum\limits_{\nu \in \mathfrak{N}^0} |\xi^\nu|$, $\xi \in R^n$.

Известно, I) (см. [6] или [2] теорему 12.4.6) что если, для многочлена P представленный в виде (1.1), P_m гиперболичен по Гордингу относительно вектора $0 \neq \tau \in R^n$, то для гиперболичности по Гордингу многочлена P относительно τ необходимо и достаточно, чтобы с некоторой постоянной $c > 0$ выполнялись следующие оценки $\widetilde{P_j(\xi)} := \sum_\alpha |P_j^{(\alpha)}(\xi)| \leq c\widetilde{P_m(\xi)},\ \xi \in R^n,\ j = 0, \cdots, m-1$,

II) (см. [1] или [2] теорему 12.5.4) Если оператор $P(D)$ (многочлен P) гиперболичен по Гордингу относительно τ, то задача Коши для оператора $P(D)$ однозначно разрешима в $C^\infty(\Omega(\tau))$, где $\Omega(\tau) := \{x \in E^n, (x, \tau) > 0\}$,

III) (см. [5]) Если оператор $P(D)$ r-гиперболичен, то задача Коши для оператора P поставлена корректно в изотропных пространствах типа Жевре $G^{r_1}(\Omega(\tau))$, при $1 \leq r_1 < r$,

IV) (см. [3] и [4]) Если оператор $P(D)$ $h_{\mathfrak{M}}(\xi_1, \cdots, \xi_{n-1})$-гиперболичен относительно вектора $\tau = (0, \cdots, 0, 1)$, где $\mathfrak{M} \subset R_+^{n-1}$ в.п. многогранник для которого $\max_{\lambda \in \Lambda^{n-1}(\mathfrak{M})} d_{\mathfrak{M}}(\lambda) < 1$, то задача Коши для оператора $P(D)$ поставлена корректно в мультианизотропных пространствах типа Жевре $G^{\mathfrak{N}}(\Omega(\tau))$, где $\mathfrak{N} \in R_+^n$ в.п. многогранник удовлетворяющий некоторым условиям, определяемые многогранником $\mathfrak{M}$.

Наша цель в настоящей заметке на языке кратности нулей подмногочленов P_j $j = 0, 1, \cdots, m-1$ найти условия, при которых многочлен P, представленный в виде (1.1), будет g-гиперболическим относительно вектора τ, когда P_m гиперболичен по Гордингу относительно τ.

2. Сравнение с весом многочленов

Определение 2.1. *(см. [2] определение 10.1.1). Функция $g : R^n \to R_+^1$ называется медленно растущей, если $\inf_{\xi \in R^n} g(\xi) > 0$ и с некоторыми постоянными $c,\ a > 0$ $g(\xi + \eta) \leq c \cdot g(\xi)(1 + |\eta|)^a,\ \xi, \eta \in R^n$.*

Для любого многочлена P и в.п. многогранника $\mathfrak{N} \subset R_+^n$ функции $\tilde{P}(\xi)$ и $h_{\mathfrak{N}}(\xi)$ являются медленно растущими весовыми функциями.

Определение 2.2. *(см. например [7]). Медленно растущая весовая функция g называется медленно меняющийся, если для любого $\kappa > 0$ существует число $c = c(\kappa) > 0$, для которого $g(\xi + \eta) \leq c \cdot g(\xi),\ \xi, \eta \in R^n,\ |\eta| \leq \kappa \cdot g(\xi)$.*

Для любого $r \geq 0$ и $\lambda \in R^n_+$ функции $1+|\xi|^r$, $1+\sum_{j=1}^n |\xi_j|^{\lambda_j}$ являются медленно меняющимися весовыми функциями. Из [8] следует, что функция $h_{\mathfrak{N}}$, где $\mathfrak{N} \in R^n_+$ в.п. многогранник, является медленно меняющийся если $\mathrm{card}\Lambda^{n-1}(\mathfrak{N}) = 1$.

Определение 2.3. *(см. например [9]). Говорят, что многочлен P мощнее многочлена Q и записывают $Q < P$, если с некоторой постоянной $c > 0$*

$$|Q(\xi)| \leq c \cdot (1+|P(\xi)|)\ \xi \in R^n. \tag{2.1}$$

Определение 2.4. *Пусть g медленно растущая весовая функция. Скажем, что многочлен P g-сильнее многочлена Q и запишем $Q \prec^g P$, если с некоторой постоянной $c > 0$*

$$\tilde{Q}(\xi, g(\xi)) \leq c\tilde{P}(\xi, g(\xi)),\ \xi \in R^n, \tag{2.2}$$

где для данного многочлена R

$$\tilde{R}(\xi, t) := \sum_\alpha |R^{(\alpha)}(\xi)| t^{|\alpha|},\ \xi \in R^n,\ t > 0.$$

Предложение 2.1. *Если $Q < P$, то для любой медленно растущей весовой функции g $Q \prec^g P$.*

Доказательство. Непосредственно следует из оценки 10.4.1 работы [2] с применением формулы Тейлора.
На примере покажем, что обратное утверждение не верно.

Пример 2.1 Пусть $n = 2$, $P(\xi) = (\xi_1 - \xi_2)^6$, $Q(\xi) = (\xi_1^2 - \xi_2^2)^2$ и $g(\xi) = (1+|\xi|)^{3/4}$. Так как $P(s+1, s) = 1$, $Q(s+1, s) = (2s+1)^2$, $s = 1, 2, \cdots$, то $Q \not< P$.

С другой стороны имеем, что $\tilde{P}(\xi, g(\xi)) \geq 6! \cdot g^6(\xi) \geq 6!(1+|\xi|)^{4.5}\ \ \xi \in R^n$ и $\tilde{Q}(\xi, g(\xi)) \leq c(1+|\xi|)^4$, $\ \xi \in R^2$ с некоторой постоянной $c > 0$. Следовательно $Q \prec^g P$.

Лемма 2.1. *Пусть g - медленно меняющийся весовая функция а P и Q некоторые многочлены. Для того, чтобы $Q \prec^g P$ необходимо и достаточно, чтобы существовало постоянная $c > 0$, для которого*

$$|Q(\xi)| \leq c\tilde{P}(\xi, g(\xi)),\ \xi \in R^n. \tag{2.3}$$

Доказательство. Выполнение оценки (2.3) при $Q \prec^g P$ очевидно. Докажем обратное. Из оценки (2.3) в силу оценки 10.4.1 работы [2] с некоторыми постоянными c_1, $c_2 > 0$ имеем

$$\tilde{Q}(\xi, g(\xi)) \leq c_1 \sup_{|\eta| \leq g(\xi)} |Q(\xi+\eta)| \leq$$

$$\leq c_2 \sup_{|\eta|\leq g(\xi)} \sum_{\alpha} |P^{(\alpha)}(\xi+\eta)| g^{|\alpha|}(\xi) \quad \xi,\eta \in R^n.$$

Т.к. g - медленно меняющийся весовая функция, то с применением формулы Тейлора с некоторыми постоянными c_3, $c_4 > 0$ отсюда получаем

$$\tilde{Q}(\xi, g(\xi)) \leq c_3 \sup_{|\eta|\leq g(\xi)} \sum_{\alpha} \left[\sum_{\beta} \frac{|P^{(\alpha+\beta)}(\xi)| \cdot |\eta^{\beta}|}{\beta!} \right] g^{|\alpha|}(\xi) \leq$$

$$\leq c_4 \sum_{\gamma} |P^{(\gamma)}(\xi)| g^{|\gamma|}(\xi) \ \xi \in R^n.$$

Лемма доказано. □

На примере покажем, что если g не является медленно меняющийся весовой функцией, то из оценки (2.3) не следует, что $Q \prec^g P$.

Пример 2.2. Пусть $n = 2$, $P(\xi) = \xi_1^8(\xi_1^2 + \xi_2^2)$, $Q(\xi) = \xi_1^2 \cdot (\xi_1^2 + \xi_2^2)^2$ и $g(\xi) = 1 + |\xi_1|^{\frac{2}{7}} + |\xi_2|^{\frac{2}{7}} + |\xi_1|^{\frac{1}{4}} \cdot |\xi_2|^{\frac{1}{4}} =: h_{\mathfrak{N}}(\xi)$, где $\mathfrak{N} \subset R_+^2$ в.п. многогранник с вершинами $(0,0)$, $(2/7,0)$, $(0,2/7)$ и $(1/4,1/4)$ (В силу вышесказанного, т.к. card $\Lambda^1(\mathfrak{N}) = 2 > 1$, функция g не является медленно меняющиейся весовой функцией). Сначала покажем, что $Q \not\prec^g P$. Пусть $\xi^s = (1,s)$ $s = 1,2\cdots$. Легко заметить, что с некоторой постоянной $c_1 > 0$ $\tilde{P}(\xi^s, g(\xi^s)) \leq c_1 \cdot s^2 g^8(\xi^s)$ $s = 1,2,\cdots$. Так как $g(\xi^s) \leq 4 \cdot s^{2/7}$, $s = 1,2,\cdots$, то отсюда получаем, что $\tilde{P}(\xi^s, g(\xi^s)) \leq 4^8 \cdot c_1 \cdot s^{30/7}$, $s = 1,2\cdots$. С другой стороны, учитывая что $g(\xi^s) \geq s^{2/7}$, $s = 1,2,\cdots$, то $\tilde{Q}(\xi^s, g(\xi^s)) \geq |(D_1^2 Q)(\xi^s)| g^2(\xi^s) \geq s^{32/7}$, $s = 1,2,\cdots$. Следовательно $Q \not\prec^g P$. Покажем, что с некоторой постоянной $c_2 > 0$

$$(2.4) \qquad |Q(\xi)| \leq c_2 \cdot \tilde{P}(\xi, g(\xi)) \quad \xi \in R^2.$$

Т.к. $\tilde{P}(\xi, g(\xi)) \geq |P(\xi)| + |(D_1^8 P)(\xi)| q^8(s) \geq \xi_1^8 |\xi|^2 + |\xi|^2 g^8(\xi) \geq \frac{1}{3}|\xi|^2(\xi_1^8 + \xi_1^2\xi_2^2 + |\xi|^{16/7})$ $\xi \in R^2$, то достаточно показать, что с некоторой постоянной $c_3 > 0$

$$(2.4') \qquad |Q(\xi)| \leq c_3 |\xi|^2 (\xi_1^8 + \xi_1^2 \cdot \xi_2^2) \quad \xi \in R^2, \ |\xi| \geq 2.$$

Так как выполнение оценки (2.4′) очевидно при $|\xi_1| \geq |\xi|^{1/3}$ (тогда $|Q(\xi)| \leq |\xi|^2 \cdot \xi_1^8$) с постоянной $c_3 \geq 1$, то докажем его при $|\xi_1| \leq |\xi|^{1/3}$, $|\xi| \geq 2$. Так как тогда $|\xi_2| \geq \frac{1}{4}|\xi|$, то $|\xi|^2 \cdot \xi_1^2 \cdot \xi_2^2 \geq \frac{1}{16}|\xi|^4 \cdot \xi_1^2 = \frac{1}{16} Q(\xi)$. Следовательно оценка (2.4′) выполняется с постоянной $c_3 \geq 16$ и поэтому верна и оценка (2.4).

Лемма 2.2. *Пусть $\mathfrak{N} \subset R_+^n$ в.п. многогранник для которого $d_{\mathfrak{N}} < 1$, P_m и P_k ($k \leq m$) однородные многочлены порядка m и k соответсвенно. Если $P_k \prec^{h_{\mathfrak{N}}} P_m$, то для любого $\eta \in \Sigma(P_m)$ при $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$ $\eta \in \Sigma(P_k)$ и $l_k(\eta) \geq$*

$l_m(\eta)-(m-k)/(1-\rho_{\mathfrak{N}}(\eta))$, где $l_m(\eta)$, $l_k(\eta)$ порядки нулей многочленов P_m, P_k в точке η а число $\rho_{\mathfrak{N}}(\eta)$ по $\mathfrak{N}$ и η определено в предыдущем параграефе.

Доказательство. Пусть $\eta \in \Sigma(P_m)$, для которого $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$. Из определения числа $\rho_{\mathfrak{N}}(\eta)$ следует существование числа $c_1 \geq 1$ для которого при всех $t \geq 1$

$$c_1 t^{\rho_{\mathfrak{N}}(\eta)} \geq h_{\mathfrak{N}}(t\cdot\eta) \geq c_1^{-1} t^{\rho_{\mathfrak{N}}(\eta)}.$$

Используя это, из условия $P_k \prec^{h_{\mathfrak{N}}} P_m$, в силу однородности многочленов P_m, P_k и определения чисел $l_m(\eta)$, $l_k(\eta)$ с некоторыми постоянными c_2, c_3, $c_4 > 0$ при $t \geq 1$ имеем, что

$$\sum_{|\alpha|\geq l_k(\eta)} t^{k-|\alpha|(1-\rho_{\mathfrak{N}}(\eta))}|P_k^{(\alpha)}(\eta)| = \sum_{\alpha} t^{k-|\alpha|(1-\rho_{\mathfrak{N}}(\eta))}|P_k^{(\alpha)}(\eta)| \leq$$

$$\leq c_2 \sum_{\alpha} |P_k^{(\alpha)}(t\cdot\eta)| {h_{\mathfrak{N}}}^{|\alpha|}(t\cdot\eta) \leq c_3 \sum_{\alpha} |P_m^{(\alpha)}(t\cdot\eta)| {h_{\mathfrak{N}}}^{|\alpha|}(t\cdot\eta) \leq$$

$$c_4 \sum_{\alpha} t^{m-|\alpha|(1-\rho_{\mathfrak{N}}(\eta))}|P_m^{(\alpha)}(\eta)| = c_4 \sum_{|\alpha|\geq l_m(\eta)} t^{m-|\alpha|(1-\rho_{\mathfrak{N}}(\eta))}|P_m^{(\alpha)}(\eta)|.$$

Так как $\rho_{\mathfrak{N}}(\eta) \leq \rho_{\mathfrak{N}} < 1$, то отсюда с некоторой постоянной $c_5 > 0$ при $t \geq 1$ получаем

$$\sum_{|\alpha|\geq l_k(\eta)} t^{k-|\alpha|(1-\rho_{\mathfrak{N}}(\eta))}|P_k^{(\alpha)}(\eta)| \leq c_5 t^{m-l_m(\eta)(1-\rho_{\mathfrak{N}}(\eta))}. \tag{2.5}$$

Из оценки (2.5) непосредственно следует, что $P_k^{(\alpha)}(\eta) = 0$ при тех $\alpha \in N_0^n$ для которых $k-|\alpha|(1-\rho_{\mathfrak{N}}(\eta)) > m - l_m(\eta)(1-\rho_{\mathfrak{N}}(\eta))$. Так как при $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$ множество $\{\alpha \in N_0^n,\ |\alpha| < l_m(\eta)-(m-k)/(1-\rho_{\mathfrak{N}}(\eta))\} \neq \emptyset$, то $\eta \in \sum(P_k)$ и $l_k(\eta) \geq l_m(\eta)-(m-k)/(1-\rho_{\mathfrak{N}}(\eta))$. Лемма доказана. □

Следствие 2.1. Пусть $\mathfrak{N} \subset R_+^n$ в.п. многогранник для которого $d_{\mathfrak{N}} < 1$ а P_m и Q_m однородные многочлены порядка m. Если $Q_m \prec^{h_{\mathfrak{N}}} P_m$, то $\Sigma(P_m) \subset \Sigma(Q_m)$ и $l_{P_m}(\eta) \leq l_{Q_m}(\eta)$ $\forall \eta \in \Sigma(P_m)$.

Доказательство. Непосредственно следует из леммы 2.2.

Следствие 2.2. Если при условиях леммы 2.2 $\eta \in R_0^n$, то при $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}})$ $\eta \in \Sigma(P_k)$ и $l_k(\eta) \geq l_m(\eta)-(m-k)/(1-\rho_{\mathfrak{N}})$.

Доказательство. Непосредственно следует из леммы 2.2, т.к. $\rho_{\mathfrak{N}}(\eta) = \rho_{\mathfrak{N}}$ для любого $\eta \in R_0^n$.

Лемма 2.3. *Пусть* $\mathfrak{N} \subset R^n_+$ *в.п. многогранник, для которого* $\rho_{\mathfrak{N}} < 1$, P_m *и* P_k *однородные многочлены порядка* m *и* k $(m \geq k)$ *соответственно, для которых с некоторой постоянной* $c > 0$

$$|P_k(\xi)| \leq c\tilde{P}_m(\xi, h_{\mathfrak{N}}(\xi)) \ \xi \in R^n. \tag{2.6}$$

Тогда для любого $\eta \in \Sigma(P_m)$ *при* $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}})$ $\eta \in \Sigma(P_k)$ *и* $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}})$.

Доказательство. Заметим, что утверждение леммы при card $\Lambda^{(n-1)}(\mathfrak{N}) = 1$ и $\eta \in R^n_0$ непосредственно следует из следствия и леммы 2.1. Докажем утверждение леммы в общем случае. Пусть $\eta \in \Sigma(P_m)$, $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}})$ а точка $\tau \in R^n_0$ выбрано так, чтобы

$$\sum_{|\alpha|=l_k(\eta)} \frac{P_k^{(\alpha)}(\eta)\tau^\alpha}{\alpha!} \neq 0, \sum_{|\alpha|=l_m(\eta)} \frac{P_m^{(\alpha)}(\eta)\tau^\alpha}{\alpha!} \neq 0. \tag{2.7}$$

Существование такой точки $\tau \in R^n_0$ следует из определений чисел $l_k(\eta)$ и $l_m(\eta)$. Рассмотрим поведение $P_k(\xi)$ и $\tilde{P}_m(\xi, h_{\mathfrak{N}}(\xi))$ на последовательности $\xi(s) = s \cdot \eta + s^{\rho_{\mathfrak{N}}} \cdot \tau$, $s = 1, 2, \cdots$ при $s \to \infty$. В силу однородности многочлена P_k, определения числа $l_k(\eta)$ и (2.7) с применением формулы Тейлора, так как $\rho_{\mathfrak{N}} < 1$ при $s \to \infty$ имеем

$$P_k(\xi(s)) = \sum_\alpha \frac{{P_k}^{(\alpha)}(s\eta)(s^{\rho_{\mathfrak{N}}}\tau)^\alpha}{\alpha!} = \sum_\alpha s^{k-|\alpha|(1-\rho_{\mathfrak{N}})} \frac{P_k^{(\alpha)}(\eta)\tau^\alpha}{\alpha!} =$$

$$= \sum_{|\alpha|\geq l_k(\eta)} s^{k-|\alpha|(1-\rho_{\mathfrak{N}})} \frac{P_k^{(\alpha)}(\eta)\tau^\alpha}{\alpha!} =$$

$$= s^{k-l_k(\eta)(1-\rho_{\mathfrak{N}})} \cdot \left(\sum_{|\alpha|=l_k(\eta)} \frac{P_k^{(\alpha)}(\eta)\tau^\alpha}{\alpha!} \right)(1+o(1)). \tag{2.8}$$

Для оценки поведения $\tilde{P}_m(\xi(s), h_{\mathfrak{N}}(\xi(s)))$ при $s \to \infty$ рассмотрим следующие возможные случаи 1) $\eta \in R^n_0$ и 2) $\eta \notin R^n_0$ (в этом случае не умаляя общности за счет перенумерации индексов будем считать, что $\eta_1 \cdots \eta_r \neq 0$ $\eta_{r+1} = \cdots = \eta_n = 0, 1 \leq r < n$).

Рассмотрим случай 1). Так как, в силу определения $\rho_{\mathfrak{N}}$, существует $\nu^0 \in \mathfrak{N}^0$, для которого $|\nu^0| = \rho_{\mathfrak{N}}$ и для любого $\nu \in \mathfrak{N}^0$ при $s \to \infty$

$$|(\xi(s))^\nu| = |(s \cdot \eta)^\nu|(1+o(1)) = s^{|\nu|}|\eta^\nu|(1+o(1)),$$

то существует постоянная $c_1 \geq 1$, для которого при всех $s \geq 1$

$$c_1{}^{-1} s^{\rho_{\mathfrak{N}}} \leq h_{\mathfrak{N}}(\xi(s)) \leq c_1 s^{\rho_{\mathfrak{N}}}.$$

Используя эту оценку, в силу однорододности многочлена P_m и определения числа $l_m(r)$, с некоторыми постоянными c_2, c_3, $c_4 > 0$ при достаточно больших s, с применением формулы Тейлора имеем

$$\tilde{P}_m(\xi(s), h_{\mathfrak{N}}(\xi(s))) =$$

$$= \sum_\alpha |P_m{}^{(\alpha)}(\xi(s))| h_{\mathfrak{N}}{}^{(\alpha)}(\xi(s)) \leq c_2 \sum_\alpha |P_m^{(\alpha)}(\xi(s))| s^{|\alpha|\rho_{\mathfrak{N}}} \leq$$

$$\leq c_2 \sum_\alpha \left[\sum_\beta \left| \frac{P_m{}^{(\alpha+\beta)}(s \cdot \eta)(s^{\rho_{\mathfrak{N}}} \tau)}{\beta!} \right| \right] s^{|\alpha|\rho_{\mathfrak{N}}} \leq$$

$$\leq c_3 \sum_\alpha \sum_\beta s^{m-|\alpha+\beta|(1-\rho_{\mathfrak{N}})} |P_m{}^{(\alpha+\beta)}(\eta)| \leq$$

$$c_4 \sum_\gamma s^{m-|\gamma|(1-\rho_{\mathfrak{N}})} |P_m^{(\gamma)}(\eta)| = c_4 \sum_{|\gamma| \geq l_m(\eta)} s^{m-|\gamma|(1-\rho_{\mathfrak{N}})} |P_m^{(\gamma)}(\eta)|.$$

Так как $\rho_{\mathfrak{N}} < 1$, то отсюда в силу определения числа $l_m(\eta)$, с некоторой постоянной $c_5 > 0$ при достаточно больших s получаем, что

$$\tilde{P}_m(\xi(s), h_{\mathfrak{N}}(\xi(s))) \leq c_5 s^{m-l_m(\eta)(1-\rho_{\mathfrak{N}})}. \tag{2.9}$$

Из оценок (2.8) и (2.9) в силу (2.6) получаем, что $k - l_k(\eta)(1-\rho_{\mathfrak{N}}) \leq m - l_m(\eta)(1-\rho_{\mathfrak{N}})$. Отсюда при $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}})$ следует, что $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}}) > 0$. Следовательно $\eta \in \Sigma(P_k)$ и $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}})$. Этим утверждение леммы в случае 1) доказано.

В случае 2) обозначим $\xi' := (\xi_1, \cdots, \xi_r)$, $\xi'' := (\xi_{r+1}, \cdots, \xi_n)$. Тогда для любого $\nu \in \mathfrak{N}^0$ при $s \to \infty$

$$|(\xi(s))^\nu| = |(\xi'(s))^{\nu'}| \cdot |(\xi''(s))^{\nu''}| = s^{|\nu'|} |\eta^{\nu'}| \cdot s^{|\nu''|\rho_{\mathfrak{N}}} |(\tau'')^{\nu''}| \cdot (1 + o(1)).$$

Так как $\rho_{\mathfrak{N}} < 1$, то из условия в.п. многогранника $\mathfrak{N}$ следует (в случае 2)), что для любого $\nu \in \mathfrak{N}^0$ $|\nu'| + \rho_{\mathfrak{N}} |\nu''| < \rho_{\mathfrak{N}}$. Отсюда получаем, что при $s \to \infty$

$$h_{\mathfrak{N}}(\xi(s)) = o(1) \cdot s^{\rho_{\mathfrak{N}}}. \tag{2.10}$$

Так как $\rho_{\mathfrak{N}} < 1$, то в силу (2.7), определения числа $l_m(\eta)$ и однородности многогранника P_m с применением формулы Тейлора для $P_m(\xi(s))$ при $s \to \infty$ имеем

$$|P_m(\xi(s))| = \left| \sum_\alpha \frac{P_m^{(\alpha)}(s\eta)(s^{\rho_{\mathfrak{N}}} \tau)^\alpha}{\alpha!} \right| =$$

$$= \left| \sum_{\alpha} s^{m-|\alpha|(1-\rho_{\mathfrak{N}})} \frac{P_m^{(\alpha)}(\eta)\tau^\alpha}{\alpha!} \right| = \left| \sum_{|\alpha|\geq l_m(\eta)} s^{m-|\alpha|(1-\rho_{\mathfrak{n}})} \frac{P_m^{(\alpha)}(\eta)\tau^\alpha}{\alpha!} \right| =$$

$$= s^{m-l_m(\eta)(1-\rho_{\mathfrak{N}})} \left| \sum_{|\alpha|=l_m(\eta)} \frac{P_m^{(\alpha)}(\eta)\tau^\alpha}{\alpha!} \right| (1+o(1)). \tag{2.11}$$

Поступая аналогичним образом в силу (2.10) при $s\to\infty$ имеем

$$\sum_{\alpha\neq 0} |P_m^{(\alpha)}(\xi(s))| h_{\mathfrak{N}}^{|\alpha|}(\xi(s)) = o(1) \sum_{\alpha\neq 0} s^{|\alpha|\rho_{\mathfrak{N}}} |P_m^{(\alpha)}(\xi(s))| \leq$$

$$\leq o(1)\cdot \sum_{\alpha\neq 0} s^{|\alpha|\rho_{\mathfrak{N}}} \sum_{\beta} \left| \frac{P_m^{(\alpha+\beta)}(s\eta)\cdot (s^{\rho_{\mathfrak{N}}}\tau)^\beta}{\beta!} \right| \leq$$

$$o(1)\cdot \sum_{\alpha\neq 0}\sum_{\beta} s^{m-|\alpha+\beta|(1-\rho_{\mathfrak{N}})} |P_m^{(\alpha+\beta)}(\eta)| = o(1)\cdot \sum_{\gamma\neq 0} s^{m-|\gamma|(1-\rho_{\mathfrak{N}})} |P_m^{(\gamma)}(\eta)| \leq$$

$$\leq o(1)\cdot \sum_{\gamma\geq l_m(\eta)} s^{m-|\gamma|(1-\rho_{\mathfrak{N}})} |P_m^{(\gamma)}(\eta)| \leq o(1)\cdot s^{m-l_m(\eta)(1-\rho_{\mathfrak{N}})}.$$

Отсюда и из соотношения (2.11) с некоторой постоянной $c_6>0$ при достаточно больших s получаем, что

$$\tilde{P}_m(\xi(s), h_{\mathfrak{N}}(\xi(s))) \geq c_6 s^{m-l_m(\eta)(1-\rho_{\mathfrak{N}})}. \tag{2.12}$$

Из оценок (2.8) и (2.12) в силу (2.6) получаем, что $k - l_k(\eta)(1-\rho_{\mathfrak{N}}) \geq m - l_m(\eta)(1-\rho_{\mathfrak{N}})$. Отсюда при $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}})$ следует, что $\eta\in\Sigma(P_k)$ и $l_k(\eta)\geq l_m(\eta)-(m-k)/(1-\rho_{\mathfrak{N}})$. Этим утверждение леммы вслучае 2), тем самым полностью, доказана. □

Следствие 2.3. *Пусть при условиях леммы 2.3 card* $\Lambda^{n-1}(\mathfrak{N}) = 1$ *и* $\eta\in\Sigma(P_m)$. *Если* $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$, *то* $\eta\in\Sigma(P_k)$ *и* $l_k(\eta)\geq l_m(\eta)-(m-k)/(1-\rho_{\mathfrak{N}}(\eta))$.

Доказательство. Так как при

$$\mathrm{card}\Lambda^{n-1}(\mathfrak{N}) = 1$$

функция $h_{\mathfrak{N}}$ является медленно меняющийся весовой функцией, то всилу леммы 2.1 из оценки (2.3) следует, что $P_k \prec^{h_{\mathfrak{N}}} P_m$. Тогда в силу леммы 2.2 получаем утверждение следствия 2.3.

Лемма 2.4. *Пусть* $k\geq l\geq 0$, $0\leq j\leq l$. *Тогда при всех* x, y, $z\geq 0$ $x^{k-j}y^{l-j}z^j \leq x^k y^l + x^{k-l}z^l$.

Лемма 2.5. *Пусть* $m \geq k \geq 0$, $\delta \in (0,1)$, $m \geq l_0 \geq (m-k)/(1-\delta)$ *и* $k \geq l_1 \geq l_0 - (m-k)/(1-\delta)$. *Тогда для любого* $\kappa \geq 1$ *существует постоянная* $c > 0$, *для которого при всех* $x \geq 1$, $y \in [0,1]$ *и* $z \in [\kappa^{-1}x, \kappa x]$

$$x^k y^{l_1} + x^{k_1 - l_1} z^{l_1} \leq c(x^m y^{l_0} + x^{m-l_0} z^{l_0}). \tag{2.13}$$

Доказательство. Рассмотрим следующие возможные случаи I) $k = m$ и II) $k < m$. Так как в случае I) из условия леммы следует, что $l_1 \geq l_0$, то в случае I) возможны следующие под-случая I.1) $l_1 = l_0$, $k = m$ и I.2) $l_1 > l_0$, $k = m$. В под-случае I.1) выполнение оценки с постоянной $c \geq 1$ очевидно. В случае I.2), при условиях леммы, для любых $x \geq 1$, $y \in [0,1]$ и $z \in [\kappa^{-1}x^\delta, \kappa x]$ имеем

$$x^k y^{l_1} + x^{k-l_1} z^{l_1} = x^m y^{l_1} + x^{m-l_1} z^{l_1} \leq x^m y^{l_0} + (x^{m-l_1} z^{l_1 - l_0}) z^{l_0} \leq$$
$$\leq x^m y^{l_0} + \kappa^{l_1 - l_0} x^{m-l_0} z^{l_0},$$

откуда следует оценка (2.13) с постоянной $c \geq \kappa^{l_1 - l_0}$.

Рассмотрим случай II). Так как

$$x^{m-l_0} z^{l_0} \geq \kappa^{-l_0} x^{m - l_0(1-\delta)} \text{ при } x \geq 1, z \geq \kappa^{-1} x^\delta,$$

то при условиях леммы, с применением неравенства Гелдера при $p = l_0/[l_0 - (m-k)/(1-\delta)]$ (>1), $q = p/(p-1) = (1-\delta)l_0/(m-k)$, для первого слагаемого левого части оценки (2.13) при всех $x \geq 1$, $y \in [0,1]$, $z \in [\kappa^{-1}x^\delta, \kappa x]$ имеем

$$x^k y^{l_1} \leq x^k \cdot y^{l_0 - (m-k)/(1-\delta)} = (x^m y^{l_0})^{[l_0 - (m-k)/(1-\delta)]/l_0} \times$$
$$\times (x^{m-k})^{[m/(1-\delta)l_0 - 1]} \leq \frac{l_0 - (m-k)/(1-\delta)}{l_0} x^m y^{l_0} +$$
$$+ \frac{m-k}{(1-\delta)l_0} x^{m-(1-\delta)l_0} \leq x^m y^{l_0} + \kappa^{l_0} m^{m-l_0} z^{l_0}. \tag{2.14}$$

Для оценки второго слагаемого левой части оценки (2.13) в случае II) рассмотрим следующие возможные под-случаи II.1) $l_0 \leq l_1$ и II.2) $l_0 > l_1$. В под-случае II.1) при условиях леммы при всех $x \geq 1$, $z \in [\kappa^{-1}x^\delta, \kappa x]$ имеем, что

$$x^{k-l_1} z^{l_1} = (x^{k-l_1} z^{l_1 - l_0}) z^{l_0} \leq \kappa^{l_1 - l_0} x^{k - l_0} z^{l_0} \leq$$
$$\leq \kappa^{l_1 - l_0} x^{m-l_0} z^{l_0}. \tag{2.15}$$

В под-случае II.2), с применением неравенства Гелдера когда $p = l_0/l_1$ (>1), $q = p/(p-1) = l_0/(l_0 - l_1)$ при условиях леммы для всех $x \geq 1$, $y \in [0,1]$ и $z \in [\kappa^{-1}x^\delta, \kappa x]$ имеем, что

$$x^{k-l_1} z^{l_1} = (x^{m-l_0} z^{l_0})^{l_1/l_0} x^{k - l_1 m/l_0} \leq \frac{l_1}{l_0} x^{m-l_0} z^{l_0} + \frac{l_0 - l_1}{l_0} x^{(k - l_1 m/l_0) l_0/(l_0 - l_1)}.$$

Так как из условия $l_1 \geq l_0 - (m-k)/(1-\delta) > 0$ леммы следует, что $(kl_0 - l_1 m)/(l_0 - l_1) \leq m - (1-\delta) l_0$, следовательно при $x \geq 1$, $z \in [\kappa^{-1}x, \kappa x]$ $x^{m-l_0(1-\delta)} \leq \kappa^{l_0} x^{m-l_0} z^{l_0}$, то отсюда получаем, что при всех $x \geq 1$, $y \in [0,1]$ и $z \in [\kappa^{-1}x^{\delta}, \kappa x]$ $x^{k-l_1} z^{l_1} \leq 2\kappa^{l_0} x^{m-l_0} z^{l_0}$.

Из оценок (2.14) и (2.15) в под-случае II.1) и оценок (2.14) и (2.7) в под-случае II.2) получаем оценку (2.13) с постоянной $c \geq 3 \cdot \max\{\kappa^{l_1}, \kappa^{l_0}\}$. Этим оценка (2.13) в случае II) и тем самым в общем случае доказана. Лемма 2.6 доказана. □

3. Основные результаты

Предложение 3.1. *Пусть R-однородный многочлен порядка m от двух переменных, $l = l(\eta)$ ($\eta \in R^2, |\eta| = 1$) порядок нуля многочлена R в точке η и $\tau \in R^2$, $|\tau| = 1$, $(\tau, \eta) = 0$. Тогда $D_\tau^l R(\eta) \neq 0$ и при $(t, u) \to (+\infty, 0)$*

$$R(t(\eta + u\tau)) = \frac{1}{l!} \cdot t^m u^l (D_\tau{}^l R(\eta)) \cdot (1 + o(1)),$$

где D_τ производная по направлению вектора τ.

Теорема 3.1. *Пусть $\mathfrak{N} \subset R_+^2$ в.п. многогранник для которого $\rho_{\mathfrak{N}} < 1$, P_m и P_k однородные многочлены от двух переменных порядка m и k ($m \geq k$) соответственно. Если для любого $\eta \in \Sigma(P_m)$ при $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$ $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$, то $P_k \prec^{h_{\mathfrak{N}}} P_m$.*

Доказательство. Предположим противное, что при условиях теоремы существует последовательность $\{\xi^s\} \subset R^2$ для которого при $s \to \infty$

$$\tilde{P}_k(\xi^s, h_{\mathfrak{N}}(\xi^s))/\tilde{P}_m(\xi^s, h_{\mathfrak{N}}(\xi^s)) \to \infty. \tag{3.1}$$

Так как по определению $h_{\mathfrak{N}}$ $h_{\mathfrak{N}}(\xi) \geq 1$ $\forall \xi \in R^2$ ($0 \in \mathfrak{N}^0$), то с некоторой постояной $c_1 > 0$ $\tilde{P}_m(\xi^s, h_{\mathfrak{N}}(\xi^s)) \geq c_1 h_{\mathfrak{N}}^m(\xi^s) \geq c_1$, $s = 1, 2, \cdots$. Следовательно при выполнении соотношения (3.1) $|\xi^s| \to \infty$ при $s \to \infty$. За счет перехода на подпоследовательность последовательности $\{\xi^s\}$ можно считать, что $|\xi^s| \geq 1$, $s = 1, 2, \cdots$ и существует $\eta \in R^2$, $|\eta| = 1$ для которого при $s \to \infty$ $\eta^s := \xi^s/|\xi^s| \to \eta$. Рассмотрим следующие возможные случаи 1) $\eta \notin \Sigma(P_m)$, 2) $\eta \in \Sigma(P_m)$, $l_m(\eta) \leq (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$ и 3) $\eta \in \Sigma(P_m)$, $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$ (в этом случае из условий теоремы имеем, что $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$). В случае 1) в силу однородности многочленов P_m, P_k и условия $\rho_{\mathfrak{N}} < 1$ теоремы с некоторым постоянными c_1, $c_2 > 0$ при достаточно больших s имеем

$$\tilde{P}_m(\xi^s, h_{\mathfrak{N}}(\xi^s)) \geq |P_m(\xi^s)| = |\xi^s|^m |P_m(\eta^s)| \geq \frac{1}{2} |\xi^s|^m |P_m(\eta)|, \tag{3.2}$$

$$\tilde{P}_k(\xi^s, h_{\mathfrak{N}}(\xi^s)) = \sum_\alpha |P_k^{(\alpha)}(\xi^s)| {h_{\mathfrak{N}}}^{|\alpha|}(\xi^s) \le c_1 \sum_\alpha |P_k^{(\alpha)}(\xi^s)| \cdot |\xi^s|^{|\alpha|\rho_{\mathfrak{N}}} =$$

$$= c_1 \sum_\alpha |\xi^s|^{k-|\alpha|(1-\rho_{\mathfrak{N}})} |P_k(\eta^s)| \le c_2 |\xi^s|^k. \tag{3.3}$$

Т.к. $m \ge k$ то оценки (3.2) и (3.3) противоречат соотношению (3.1). Рассмотрим случай 2). Тогда при достаточно больших s, в силу определения числа $l_m(\eta)$, с некоторой постоянной $c_3 > 0$ имеем

$$\tilde{P}_m(\xi^s, h_{\mathfrak{N}}(\xi^s)) \ge \sum_{|\alpha|=l_m(\eta)} |P_m^{(\alpha)}(\xi^s)| {h_{\mathfrak{N}}}^{|\alpha|}(\xi^s) =$$

$$= |\xi^s|^{m-l_m(\eta)} \cdot {h_{\mathfrak{N}}}^{l_m(\eta)}(\xi^s) \cdot \sum_{|\alpha|=l_m(\eta)} |P_m^{(\alpha)}(\eta^s)| \ge$$

$$\ge c_3 |\xi^s|^{m-l_m(\eta)} \cdot {h_{\mathfrak{N}}}^{l_m(\eta)}(\xi^s). \tag{3.4}$$

Так как $\eta^s = \xi^s/|\xi|^s \to \eta$ при $s \to \infty$, то в силу определения числа $\rho_{\mathfrak{N}}(\eta)$ с некоторой постоянной $c_4 > 0$ имеем что $h_{\mathfrak{N}}(\xi^s) \ge c_4 |\xi^s|^{\rho_{\mathfrak{N}}(\eta)}$ $s = 1, 2, \cdots$. Используя это из оценки (3.4), с некоторой постоянной $c_5 > 0$, при достаточно больших s получаем

$$\tilde{P}_m(\xi^s, h_{\mathfrak{N}}(\xi^s)) \ge c_5 |\xi^s|^{m-l_m(\eta)(1-\rho_{\mathfrak{N}}(\eta))}. \tag{3.5}$$

Так как в случае 2) $k \le m - l_m(\eta)(1-\rho_{\mathfrak{N}}(\eta))$ $(l_m(\eta) \le (m-k)/(1-\rho_{\mathfrak{N}}(\eta)))$, то оценки (3.3) и (3.5) противоречат соотношению (3.1).

В случае 3) за счет перехода на подпоследовательность последовательности $\{\xi^s\}$ возможны следующие подслучая 3.1) $\xi^s = |\xi^s|\eta$, $s = 1, 2, \cdots$, 3.2) $\xi^s \ne |\xi^s|\eta$ $s = 1, 2, \cdots$. В подслучае 3.1), в силу однородности многочленов P_m, P_k, определения чисел $l_m(\eta)$, $l_k(\eta)$, так как $\rho_{\mathfrak{N}} < 1$ и с некоторой постоянной $\kappa \ge 1$ (в силу определения $\rho_{\mathfrak{N}}(\eta)$)

$$\kappa |\xi^s|^{\rho_{\mathfrak{N}}(\eta)} \ge h_{\mathfrak{N}}(\xi^s) \ge \kappa^{-1} |\xi^s|^{\rho_{\mathfrak{N}}(\eta)} \; s = 1, 2, \cdots,$$

то существуют постоянные $c_j > 0$, $j = \overline{6,9}$ для которых при достаточно больших s

$$\tilde{P}_m(\xi^s, h_{\mathfrak{N}}(\xi^s)) \ge \sum_{|\alpha|=l_m(\eta)} |P_m^{(\alpha)}(\xi^s)| {h_{\mathfrak{N}}}^{|\alpha|}(\xi^s) \ge c_6 |\xi^s|^{m-l_m(\eta)(1-\rho_{\mathfrak{N}}(\eta))} \times$$

$$\times \sum_{|\alpha|=l_m(\eta)} |P_m^{(\alpha)}(\eta^s)| \ge c_7 |\xi^s|^{m-l_m(\eta)/(1-\rho_{\mathfrak{N}}(\eta))}, \tag{3.6}$$

$$\tilde{P}_k(\xi^s, h_{\mathfrak{N}}(\xi^s)) = \sum_\alpha |P_k^{(\alpha)}(\xi^s)| h_{\mathfrak{N}}^{|\alpha|}(\xi^s) \le c_8 \sum_\alpha |\xi^s|^{k-|\alpha|(1-\rho_{\mathfrak{N}}(\eta))} |P_k^{(\alpha)}(\eta)| =$$

$$(3.7)\qquad = c_8 \sum_{|\alpha| \geq l_k(\eta)} |\xi^s|^{k-|\alpha|(1-\rho_{\mathfrak{N}}(\eta))} |P_k^{(\alpha)}(\eta)| \leq c_9 |\xi^s|^{k-l_k(\eta)(1-\rho_{\mathfrak{N}}(\eta))}.$$

Так как из условия $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}}(\eta)) > 0$ теоремы следует, что $k - l_k(\eta)(1-\rho_{\mathfrak{N}}(\eta)) \leq m - l_m(\eta)(1-\rho_{\mathfrak{N}}(\eta))$, то оценки (3.6) и (3.7) противоречат соотношению (3.1). Рассмотрим подслучай 3.2). Пусть $\tau \in R^2$, $|\tau| = 1$ и $(\tau, \eta) = 0$. Тогда в подслучае 3.2) точки ξ^s $s = 1, 2, \cdots$ можно представить в следующем виде $\xi^s = t_s(\eta + u_s \tau)$ $s = 1, 2, \cdots$, где при $s \to \infty$

$$(3.8)\qquad (t_s, u_s) \to (+\infty, 0) \text{ и } t_s/|\xi^s| \to 1.$$

Поэтому в дальнейшем, не умаляя общности, будем считать, что $t_s \geq 1$, $|u_s| \leq 1$ $s = 1, 2, \cdots$. Так как проядок нуля многочлена $P_k^{(\alpha)}$, $\alpha \in N_0^2$ в точке η при $|\alpha| \leq l_k(\eta)$ не меньше чем $l_k(\eta) - |\alpha|$, то с применением предложения 3.1, в силу однородности многочлена P_k и соотношения (3.8), с некоторой постоянной $c_{10} > 0$ при достаточно больших s имеем.

$$\tilde{P}_k(\xi^s, h_{\mathfrak{N}}(\xi^s)) = \sum_{|\alpha| \leq l_k(\eta)} |P_k^{(\alpha)}(\xi^s)| {h_{\mathfrak{N}}}^{|\alpha|}(\xi^s) + \sum_{|\alpha| > l_k(\eta)} |P_k^{(\alpha)}(\xi^s)| h_{\mathfrak{N}}^{|\alpha|}(\xi^s) \leq$$

$$\leq c_{10} \left[\sum_{|\alpha| \leq l_k(\eta)} |\xi^s|^{k-|\alpha|} |u_s|^{l_k(\eta)-|\alpha|} {h_{\mathfrak{N}}}^{|\alpha|}(\xi^s) + \sum_{|\alpha| > l_k(\eta)} |\xi^s|^{k-|\alpha|} h_{\mathfrak{N}}^{|\alpha|}(\xi^s) \right].$$

Так как $\rho_{\mathfrak{N}} < 1$, следовательно $h_{\mathfrak{N}}(\xi^s)/|\xi^s| \to 0$ при $s \to \infty$, то в силу леммы 2.4, с некоторой постоянной $c_{11} > 0$ при достаточно больших s получаем, что

$$(3.9)\qquad \tilde{P}_k(\xi^s, h_{\mathfrak{N}}(\xi^s)) \leq c_{11} \left[|\xi^s|^k |u_s|^{l_k(\eta)} + |\xi^s|^{k-l_k(\eta)} {h_{\mathfrak{N}}}^{l_k(\eta)}(\xi^s) \right].$$

Для $\tilde{P}_m(\xi^s, h_{\mathfrak{N}}(\xi^s))$, при достаточно больших s, в силу предложения 3.1 и определения числа $l_m(\eta)$, с некоторой постоянной $c_{12} > 0$ имеем

$$\tilde{P}_m(\xi^s, h_{\mathfrak{N}}(\xi^s)) \geq |P_m(\xi^s)| + \sum_{|\alpha| = l_m(\eta)} |P_m^{(\alpha)}(\xi^s)| {h_{\mathfrak{N}}}^{|\alpha|}(\xi^s) \geq$$

$$\frac{1}{2} \left[|\xi^s|^m |u_s|^{l_m(\eta)} |{D_\tau}^{l_m(\eta)} P_m(\eta)| + |\xi^s|^{m-l_m(\eta)} \cdot {h_{\mathfrak{N}}}^{l_m(\eta)}(\xi^s) \cdot \sum_{|\alpha| = l_m(\eta)} |P_m^{(\alpha)}(\eta)| \right] \geq$$

$$(3.10)\qquad \geq c_{12} \left[|\xi^s|^m |u_s|^{l_m(\eta)} + |\xi^s|^{m-l_m(\eta)} h_{\mathfrak{N}}^{l_m(\eta)}(\xi^s) \right].$$

Так как, в силу определения числа $\rho_{\mathfrak{N}}(\eta)$ и условия $\rho_{\mathfrak{N}} < 1$, с некоторой постоянной $\kappa \geq 1$

$$(3.11)\qquad \kappa |\xi^s| \geq h_{\mathfrak{N}}(\xi^s) \geq \kappa^{-1} |\xi^s|^{\rho_{\mathfrak{N}}(\eta)} \quad s = 1, 2, \cdots,$$

числа $k, m, l_0 := l_m(\eta), l_1 := l_k(\eta), \delta := \rho_{\mathfrak{N}}(\eta)$ и тройки $(x_s, y_s, z_s) := (|\xi^s|, |u_s|, h_{\mathfrak{N}}(\xi^s))$ (в силу (3.11)) при достаточно больших s удовлетворяют условиям леммы 2.5, то существует постоянная $c_{13} > 0$ для которого при этих s

$$|\xi^s|^k |u_s|^{l_k(\eta)} + |\xi^s|^{k-l_k(\eta)} h_{\mathfrak{N}}^{l_k(\eta)}(\xi^s) \leq$$

$$\leq c_{13}(|\xi^s|^m |u_s|^{l_m(\eta)} + |\xi^s|^{m-l_m(\eta)} h_{\mathfrak{N}}^{l_m(\eta)}(\xi^s)). \tag{3.12}$$

Из оценок (3.9) и (3.10) в силу (3.12) получаем противоречие с соотношением (3.1). Полученные противоречия доказывают справедливость утверждения теоремы. Теорема 3.1 доказана. □

Следствие 3.1. *Пусть $\mathfrak{N}$, $\mathfrak{N}_1 \subset R^2_+$ в.п. многогранники, для которых $\rho_{\mathfrak{N}} < 1$, $\rho_{\mathfrak{N}_1} < 1$, P_m и P_k однородные многочлены от двух переменных порядка m и k $(m \geq k)$ соответственно. Если для любого $\eta \in \Sigma(P_m)$ $\rho_{\mathfrak{N}}(\eta) = \rho_{\mathfrak{N}_1}(\eta)$, то $P_k \prec^{h_{\mathfrak{N}}} P_m$ тогда и только тогда, когда $P_k \prec^{h_{\mathfrak{N}_1}} P_m$.*

Доказательство. Пусть $P_k \prec^{h_{\mathfrak{N}}} P_m$. Тогда в силу леммы 2.2 $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$ для любого $\eta \in \Sigma(P_m)$ если $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$. Тогда из условий леммы имеем, что $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}_1}(\eta))$ при всех $\eta \in \Sigma(P_m)$, для которых $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}_1}(\eta))$. Отсюда в силу теоремы 3.1 получаем, что $P_k \prec^{h_{\mathfrak{N}_1}} P_m$.

Этим утверждение следствия доказано, так как для доказательства обратного утверждения достаточно поменять местами многогранники $\mathfrak{N}$ и $\mathfrak{N}_1$.

Замечание 3.1. *Легко заметить, что для любого в.п. многогранника $\mathfrak{N} \subset R^2_+$ и $0 \neq \eta \in R^2$ $\rho_{\mathfrak{N}}(\eta) = \max\limits_{\nu \in \mathfrak{N}^0} \nu_1$ при $\eta_2 = 0$, $\rho_{\mathfrak{N}}(\eta) = \max\limits_{\nu \in \mathfrak{N}^0} \nu_2$ при $\eta_1 = 0$ и $\rho_{\mathfrak{N}}(\eta) = \rho_{\mathfrak{N}} = \max\limits_{\nu \in \mathfrak{N}^0} |\nu|$ при $\eta \in R^2_0$.*

Следствие 3.2. *Если при условиях следствия 3.1 $\Sigma(P_m) \subset R^2_0$, то условия $P_k \prec^{h_{\mathfrak{N}}} P_m$, $P_k \prec^{h_{\mathfrak{N}_1}} P_m$ эквивалентны тогда и только тогда, когда $\rho_{\mathfrak{N}} = \rho_{\mathfrak{N}_1}$.*

Доказательство. Непосредственно следует из следствия 3.1 и замечания 3.1. Аналогичным образом (как доказана теорема 3.1) можно доказать следующую:

Теорема 3.2. *Пусть $\mathfrak{N} \subset R^2_+$, P_m и P_k те же, что и в теореме 3.1. Если для любого $\eta \in \Sigma(P_m)$ при $l_m(\eta) > (m-k)/(1-\rho_{\mathfrak{N}})$ следует, что $l_k(\eta) > l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}})$, то существует постоянная $c > 0$ для которого*

$$|P_k(\xi)| \leq c\tilde{P}_m(\xi, h_{\mathfrak{N}}(\xi))| \; \forall \xi \in R^2. \tag{3.13}$$

Следствие 3.3. *Пусть* $\mathfrak{N}$, $\mathfrak{N}_1 \subset R^2$ *в.п. многогранники, для которых* $\rho_{\mathfrak{N}} < 1$, $\rho_{\mathfrak{N}_1} < 1$, P_m *и* P_k *однородные многочлены от двух переменных порядка* m *и* k $m \geq k$ *соответственно. Если* $\rho_{\mathfrak{N}} = \rho_{\mathfrak{N}_1}$, *то для выполнения оценки (3.13) необходимо и достаточно, чтобы с некоторой постоянной* $c_1 > 0$

$$(3.13') \qquad |P_k(\xi)| \leq c\tilde{P}_m(\xi, h_{\mathfrak{N}_1}(s)) \forall \xi \in R^2$$

Доказательство. Проводится аналогично доказательству следствия 3.1 с использованием теоремы 3.2 вместо теоремы 3.1 и леммы 2.3 вместо леммы 2.2.

Теорема 3.3. *Пусть* $\mathfrak{N} \subset R^2_+$ *в.п. многогранник, для которого* $\rho_{\mathfrak{N}} < 1$, *а* P *многочлен от двух переменных представленный в виде (1.1). Если многочлен* P_m *гиперболичен по Гордингу относительно вектора* $0 \neq \tau \in R^2$ *а многочлены* P_k $k = 1, \cdots, m-1$ *удовлетворяют следующему условию: для любых* $\eta \in \Sigma(P_m)$ *и* $k > m - l_m(\eta)/(1-\rho_{\mathfrak{N}}(\eta))$ $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}}(\eta))$ *то многочлен* P $h_{\mathfrak{N}}$*-гиперболичен относительно вектора* τ.

Доказательство. Утверждение теоремы непосредственно следует из теоремы 3.3 работы [9], так как при условиях нашей теоремы, в силу теоремы 3.1 $P_k \prec^{h_{\mathfrak{N}}} P_m$ $k = 0, 1, \cdots, m-1$.

Следствие 3.4. *Пусть* $\mathfrak{N}$, P *и* τ *те же, что и в теореме 3.3 а* $\mathfrak{N}_1 \subset R^2_+$ *некоторый в.п. многогранник. Если* $\rho_{\mathfrak{N}}(\eta) = \rho_{\mathfrak{N}_1}(\eta)$ *при всех* $\eta \in \Sigma(P_m)$, *то многочлен* P $h_{\mathfrak{N}_1}$*-гиперболичен относительно вектора* τ.

Доказательство. Непосредственно следует из теоремы 3.3 и следствия 3.1.

Теорема 3.4. *Пусть* $\mathfrak{N} \subset R^2_+$ *в.п. многогранник для которого* $\rho_{\mathfrak{N}} < 1$ *а* P *- многочлен от двух переменных, представленный в виде (1.1). Если многочлен* P_m *гиперболичен по Гордингу относительно вектора* $0 \neq \tau \in R^2$ *а многочлены* P_k $k = 1, \cdots, m-1$ *удовлетворяют следующему условию: для любых* $\eta \in \Sigma(P_m)$ *и* $k > m - l_m(\eta)/(1-\rho_{\mathfrak{N}})$ $l_k(\eta) \geq l_m(\eta) - (m-k)/(1-\rho_{\mathfrak{N}})$, *то многочлен* P $1/\rho_{\mathfrak{N}}$ *гиперболичен относительно вектора* τ.

Доказательство. Из условий теоремы, в силу теоремы 3.2 с некоторой постоянной $c_1 > 0$ имеем, что

$$|P_k(\xi)| \leq c_1 \tilde{P}_m(\xi, h_{\mathfrak{N}}(\xi)) \ \xi \in R^2, \ k = 0, \cdots, m-1.$$

Так как с некоторой постоянной $\kappa > 0$ в силу условия теормы

$$h_{\mathfrak{N}}(\xi) \leq \kappa(1 + |\xi|^{\rho_{\mathfrak{N}}}) =: g(\xi) \ \xi \in R^2$$

и следовательно с некоторой постоянной $c_2 > 0$ $\tilde{P}(\xi, h_{\mathfrak{R}}(\xi)) \leq c_2 \tilde{P}_m(\xi, g(\xi))$ $\xi \in R^2$, то в силу леммы 2.1 $P_k \prec^g P_m$, так как g является медленно меняющийся весовой функцией. Отсюда в силу теоремы 3.3 работы [9] получаем, что многочлен P g-гиперболичен отнасительно вектора τ или что то же самое, учитывая определение функции g, P $1/\rho_{\mathfrak{R}}$ гиперболичен отнасительно вектора τ. Теорема 3.4. доказана. □

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ON A GENERALIZATION OF AN OPERATOR PRESERVING TURÁN-TYPE INEQUALITY FOR COMPLEX POLYNOMIALS

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Abstract. Let $W(\zeta) = (a_0 + a_1\zeta + ... + a_n\zeta^n)$ be a polynomial of degree n having all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1 + k + k^n$, Govil and Mctume [7] showed that the following inequality holds

$$\max_{\zeta\in\mathbb{T}_1} |D_\alpha W(\zeta)| \geq n\left(\frac{|\alpha| - k}{1 + k^n}\right)\|W\| + n\left(\frac{|\alpha| - (1 + k + k^n)}{1 + k^n}\right)\min_{\zeta\in\mathbb{T}_k} |W(\zeta)|.$$

In this paper, we have obtained a generalization of this inequality involving sequence of operators known as polar derivatives. In addition, the problem for the limiting case is also considered.

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1. Introduction and statement of results

Let $\mathcal{L}$ be the space of all complex polynomials $W(\zeta) = \sum_{v=1}^{n} b_v\zeta^v$ of degree n. For each positive real number k let $\mathbb{T}_k = \{\zeta : |\zeta| = k\}$, $\mathbb{E}_k^- = \{\zeta \in \mathbb{C} : |\zeta| < k\}$ and $\mathbb{E}_k^+ = \{\zeta \in \mathbb{C} : |\zeta| > k\}$ respectively. For any holomorphic function f defined on $\mathbb{T}_1$, we write $\|f\| = \sup_{z\in\mathbb{T}_1} |f(z)|$, the supremum norm of f on $\mathbb{T}_1$. The Bernstein's classical inequality states that

$$\max\{|W'(\zeta)| : \zeta \in \mathbb{T}_1\} \leq n \max\{|W(\zeta)| : \zeta \in \mathbb{T}_1\} \tag{1.1}$$

holds for all polynomials $W \in \mathcal{L}$. This result is best possible and the extremal polynomial for (1.1) is $W(\zeta) = \alpha\zeta^n$, $\alpha \neq 0$. The relationships between the bounds, their refinements and extensions, and the distribution of zeros of W in a certain region of $\mathbb{C}$ have been studied extensively and has deeply influenced the sequel of such type of inequalities throughout the decades. Since the equality in the Bernstein's inequality (1.1) holds for polynomials which have all their zeros at the origin, improvement in (1.1) is not possible if we consider polynomials having all their zeros inside the unit circle. For this reason, in this case, it may be interesting to obtain inequality in the reverse direction, and in this connection, we have the inequality ascribed to Turán[10], which asserts that

$$\max\{|W'(\zeta)| : \zeta \in \mathbb{T}_1\} \geq \frac{n}{2} \max\{|W(\zeta)| : \zeta \in \mathbb{T}_1\} \tag{1.2}$$

holds for all polynomials $W \in \mathcal{L}$ having all zeros in $\mathbb{T}_1 \cup \mathbb{E}_1^-$. This result is best possible and the extremal polynomial for (1.2) is $W(\zeta) = \zeta^n + 1$. One would expect, the refinement of the lower bound estimate in (1.2) under the condition when W is free of zeros on $\mathbb{T}_1$. This assertion was observed in [1] in which the authors proved the following inequality under the same hypothesis as of (1.2)

$$\|W'\| \geq \frac{n}{2}\{\|W\| + \min_{\zeta \in \mathbb{T}_1} |W(\zeta)|\}. \tag{1.3}$$

Inequalities (1.2) and (1.3) are very useful in proving some well known classical polynomial inequalities.

For polynomials of a complex variable, we also have the following more general result, due to Govil [4], which is one of the most known polynomial inequality in this direction and will be useful for our results. More precisely, the inequality

$$\|W'\| \geq \frac{n}{1+k^n}\|W\| \tag{1.4}$$

holds for all polynomials $W \in \mathcal{L}$ having all zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, where $k \geq 1$. As is easy to see that (1.4) becomes equality when $W(\zeta) = \zeta^n + k^n$. Again, excluding the class of polynomials having all zeros on $\mathbb{T}_k$, then one may expect that the bound (1.4) could be amended. In this direction, under the same hypothesis as of (1.4), it was shown by Govil [3] that the following inequality holds good

$$\|W'\| \geq \frac{n}{1+k^n}\{\|W\| + \min_{\zeta \in \mathbb{T}_1} |W(\zeta)|\}. \tag{1.5}$$

The research on mathematical objects associated with Turán type inequalities has been active over a period; there are many research papers published in a variety of journals each year and different approaches have been taken for different targets. The present article is concerned with Turán type inequalities for the polar derivative of a polynomial with restricted zeros. Before moving on to our main results, we will take a moment to introduce the concept of the polar derivative being involved.

Definition 1.1. *Let $W \in \mathcal{L}$, and α is any complex number, then*

$$\begin{aligned} D_\alpha W(\zeta) &= -\left[\frac{W(\zeta)}{(\zeta-\alpha)^n}\right]'(\zeta-\alpha)^{n+1} \\ &= nW(\zeta) + (\alpha - \zeta)W'(\zeta), \end{aligned} \tag{1.6}$$

is called the polar derivative of $W(\zeta)$. Note that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it generalizes the concept of "ordinary derivative" is evident and convincing from the fact that

$$\lim_{\alpha\to\infty} \frac{D_\alpha W(\zeta)}{\alpha} = W'(\zeta) \tag{1.7}$$

uniformly with respect to ζ for $\mathbb{T}_R \cup \mathbb{E}_R^-$, $R > 0$.

In the polar derivative milieu, all the above inequalities have been widely investigated, the research in this field has taken many different directions and resulting in slew of publications see, e.g.,([5], [8], [11], [6]). In this paper our interest is mainly motivated upon the study of various versions of inequalities (1.4) and (1.5), their refinements, strengthening and generalizations in the polar derivative setting by introducing constraints on the zeros of $W \in \mathcal{L}$, the modulus of largest root of W or restrictions on coefficients etc. In this contexture, the inequality

$$\max_{\zeta\in\mathbb{T}_1}|D_\alpha W(\zeta)| \geq \frac{n(|\alpha|-k)}{1+k^n}\|W\| \tag{1.8}$$

holds for all polynomials $W \in \mathcal{L}$ which has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$ and for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$. This result is ascribed to Aziz and Rather [2]. Another result in this direction ascribed to Govil and Mctume [7] acts as a refinement of (1.7) and states that the inequality

$$\max_{\zeta\in\mathbb{T}_1}|D_\alpha W(\zeta)| \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\|W\| + n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\min_{\zeta\in\mathbb{T}_k}|W(\zeta)| \tag{1.9}$$

holds for all polynomials $W \in \mathcal{L}$ which has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$ and for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1+k+k^n$.

Definition 1.2. *Given a polynomial $W \in \mathcal{L}$, we can construct a sequence of polar derivatives or so-called higher order derivatives with respect to finitely many poles as given below*

$$\begin{aligned}
D_{\alpha_1}W(\zeta) &= nW(\zeta) + (\alpha_1-\zeta)W'(\zeta)\\
D_{\alpha_2}D_{\alpha_1}W(\zeta) &= (n-1)D_{\alpha_1}W(\zeta) + (\alpha_2-\zeta)(D_{\alpha_1}W(\zeta))'\\
&\dots\ \dots\ \dots\\
D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta) &= (n-t+1)D_{\alpha_{t-1}}...D_{\alpha_1}W(\zeta) + (\alpha_t-\zeta)(D_{\alpha_{t-1}}...D_{\alpha_1}W(\zeta))',\\
&\textit{for}\ \ 2 \leq t \leq n.
\end{aligned}$$

Like the t^{th} ordinary derivative $W^{(t)}(\zeta)$ of $W(\zeta)$, the t^{th} polar derivative $D_{\alpha_t}...D_{\alpha_2}$ $D_{\alpha_1}W(\zeta)$ of $W(\zeta)$ is a polynomial of degree at most $n-t$. For the sake of simplicity, we use the following notations:

$$A_{\alpha_t}^k = (|\alpha_1|-k)(|\alpha_2|-k)...(|\alpha_t|-k),$$

$$N_t = n(n-1)(n-2)...(n-t+1).$$

In this paper we obtain a generalization of inequalities (1.8) and (1.9), and besides our theorem includes many quality inequalities in this connection as special cases. To be more precise, we prove

Theorem 1.1. *Let $W \in \mathcal{L}$, and $W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then for every real or complex numbers $\alpha_1, \alpha_2, ..., \alpha_t$ with $|\alpha_i| \geq 1 + k + k^n$, $i = 1, 2, ..., t$,*

$$\max_{\zeta \in \mathbb{T}_1} |D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{N_t}{K_t} \left[A_{\alpha_t}^k \|W\| + \{A_{\alpha_t}^k - K_t\} \min_{\zeta \in \mathbb{T}_k} |W(\zeta)| \right], \tag{1.10}$$

where $K_t = (1 + k^n)(1 + k^{n-1}) ... (1 + k^{n-t+1})$.

Remark 1.1. *For $t = 1$, one easily gets inequality (1.9) from Theorem 1.1 and when there is no information about minimum of a polynomial $W(\zeta)$ we get inequality (1.8) as a special case.*

If we choose $\alpha_1 = \alpha_2 = ... = \alpha_t = \alpha$, then by dividing both sides of inequality (1.10) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, therefore taking (1.7) into consideration, we obtain the following result.

Corollary 1.1. *Let $W \in \mathcal{L}$, and $W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then*

$$\max_{\zeta \in \mathbb{T}_1} |W^{(t)}(\zeta)| \geq \frac{N_t}{K_t} \left\{ \|W\| + \min_{\zeta \in \mathbb{T}_k} |W(\zeta)| \right\}, \tag{1.11}$$

where $K_t = (1 + k^n)(1 + k^{n-1}) ... (1 + k^{n-t+1})$.

Inequality (1.5) can easily be obtained from above Corollary 1.1 for $t = 1$.

2. Lemmas

Lemma 2.1. *If all the zeros of an nth degree polynomial W lie in a circular region C and if none of the points $\alpha_t, \alpha_{t-1}, ..., \alpha_1$ lie in the region C, then each of the polar derivatives*

$$D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} W(\zeta), \ t = 1, 2, .., n-1$$

has all of its zeros in C.

This lemma follows by repeated applications of Laguerre's theorem [9].

Lemma 2.2. *Let $W \in \mathcal{L}$, and $W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then for every real or complex numbers α with $|\alpha| \geq k$,*

$$|D_\alpha W(\zeta)| \geq \frac{n(|\alpha| - k)}{1 + k^n} |W(\zeta)|.$$

This lemma is due to Aziz and Rather [2].

Lemma 2.3. *Let $W \in \mathcal{L}$, and $W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then for every real or complex numbers $\alpha_1, \alpha_2, ..., \alpha_t$ with $|\alpha_i| \geq k$, $i = 1, 2, ..., t$,*

$$|D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{N_t}{(1 + k^n)(1 + k^{n-1}) ... (1 + k^{n-t+1})} A_{\alpha_t}^k |W(\zeta)|. \tag{2.1}$$

Proof of Lemma 2.3. Well, Lemma follows trivially if $|\alpha_i| = k$ for at least one i, $1 \leq i \leq t \leq n$. Therefore from now on we will assume that $|\alpha_i| > k$. We will prove this lemma by mathematical induction. Lemma is true for $t = 1$ by Lemma 2.2 i.e., if $|\alpha_1| > k$ then

$$|D_{\alpha_1} W(\zeta)| \geq \frac{n(|\alpha_1| - k)}{1 + k^n} |W(\zeta)|. \tag{2.2}$$

Now for $t = 2$. Except for one value (say a) of α_1, $D_{\alpha_1} W(\zeta)$ will be a polynomial of degree $(n-1)$. Let us take any $\alpha_1 (\alpha_1 \neq a$ if $|a| > k)$ with $|\alpha_1| > k$ and fix it up. Thus $D_{\alpha_1} W(\zeta)$ is a polynomial of degree $(n-1)$ and by Lemma 2.1 it has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$. Therefore on applying ?Lemma 2.2 to $D_{\alpha_1} W(\zeta)$ with $\alpha = \alpha_2$, $|\alpha_2| > k$ we get,

$$|D_{\alpha_2}(D_{\alpha_1} W(\zeta))| \geq \frac{n-1}{1 + k^{n-1}} (|\alpha_2| - k) |D_{\alpha_1} W(\zeta)|.$$

Using (2.2) we have

$$|D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{n(n-1)}{(1 + k^n)(1 + k^{n-1})} (|\alpha_1| - k)(|\alpha_2| - k) |W(\zeta)|.$$

It follows Lemma is true for $t = 2$. We assume that Lemma is true for $t = s$ i.e., for $\zeta \in \mathbb{T}_1$ and $|\alpha_i| > k, \ i = 1, 2, ..., s$

$$|D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{N_s}{(1 + k^n)(1 + k^{n-1}) ... (1 + k^{n-s+1})} A_{\alpha_s}^k |W(\zeta)|, \tag{2.3}$$

and we will prove that Lemma is true for $t = s+1$, $(< n)$. Again except for one value (say α_1') of α_1, $D_{\alpha_1} W(\zeta)$ will be a polynomial of degree $(n-1)$. Let us take any $\alpha_1 (\alpha_1 \neq \alpha_1'$ if $|\alpha_1'| > k)$ with $|\alpha_1| > k$ and fix it up. Thus $D_{\alpha_1} W(\zeta)$ is a polynomial of degree $(n-1)$. Now $D_{\alpha_2} D_{\alpha_1} W(\zeta)$ will be a polynomial of degree $(n-2)$ for every α_2, except for one value α_2', (say), of α_2. Let us take any $\alpha_2 (\alpha_2 \neq \alpha_2'$ if $|\alpha_2'| > k)$ with $|\alpha_2| > k$ and fix it up. Therefore, $D_{\alpha_2} D_{\alpha_1} W(\zeta)$ is a polynomial of degree $(n-2)$. Likewise one can continue and say that $D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)$ will be a polynomial of degree $(n-s)$ for every for every α_s, except for one value α_s', (say), of α_s. Let us take any $\alpha_s (\alpha_s \neq \alpha_s'$ if $|\alpha_s'| > k)$ with $|\alpha_s| > k$ and fix it up. Thus $D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)$ is a polynomial of degree $(n-s)$ with fixed $\alpha_s (|\alpha_s| > k)$, fixed $\alpha_{s-1} (|\alpha_{s-1}| > k)$,...,fixed $\alpha_1 (|\alpha_1| > k)$ and by Lemma 2.1 $D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$. Therefore on applying Lemma 2.2 to $D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)$ with $\alpha = \alpha_{s+1}, \quad |\alpha_{s+1}| > k$ we get,

$$|D_{\alpha_{s+1}}(D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta))| \geq \frac{n-s}{1 + k^{n-s}} (|\alpha_{s+1}| - k) |D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)|,$$

which on being combined with (2.3) gives for $\zeta \in \mathbb{T}_1$ and $|\alpha_{s+1}| > k$

$$|D_{\alpha_{s+1}} D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1 + k^n)(1 + k^{n-1}) ... (1 + k^{n-s})} A_{\alpha_{s+1}}^k |W(\zeta)|. \tag{2.4}$$

Further $\alpha_s(\alpha_s \neq \alpha_s^{'}$ if $|\alpha_s^{'}| > k)$ with $|\alpha_s| > k$ was a fixed element but was chosen arbitrarily. Accordingly (2.4) is true for every $\alpha_{s+1}(|\alpha_{s+1}| > k)$ and every $\alpha_s(\alpha_s \neq \alpha_s^{'}$ if $|\alpha_s^{'}| > k)$ with $|\alpha_s| > k$, i.e.

(2.5)
$$|D_{\alpha_{s+1}}D_{\alpha_s}...D_{\alpha_2}D_{\alpha_1}W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})}A^k_{\alpha_{s+1}}|W(\zeta)|,$$
$$\text{for } \zeta \in \mathbb{T}_1, |\alpha_{s+1}| > k \ \& \ |\alpha_s| > k(\alpha_s \neq \alpha_s^{'} \text{ if } |\alpha_s^{'}| > k).$$

Now if $|\alpha_s^{'}| > k$ then for a fixed $\alpha_{s+1}(|\alpha_{s+1}| > k)$, $D_{\alpha_{s+1}}D_{\alpha_s}...D_{\alpha_2}D_{\alpha_1}$ is continuous function of α_s and hence by continuity we can say (2.5) will be true for a fixed α_{s+1} and $\alpha_s^{'}$ and accordingly (2.5) will be true for every $\alpha_{s+1}(|\alpha_{s+1}| > k)$ and $\alpha_s^{'}$. Thus (2.5) is true for every $\alpha_{s+1}(|\alpha_{s+1}| > k)$ and every $\alpha_s(|\alpha_s| > k)$. That is

(2.6)
$$|D_{\alpha_{s+1}}D_{\alpha_s}...D_{\alpha_2}D_{\alpha_1}W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})}A^k_{\alpha_{s+1}}|W(\zeta)|,$$
$$\text{for } \zeta \in \mathbb{T}_1, |\alpha_{s+1}| > k \ \& \ |\alpha_s| > k.$$

As argued for α_s and $\alpha_s^{'}$, we can argue for α_{s-1} and $\alpha_{s-1}^{'}$ and say that

$$|D_{\alpha_{s+1}}D_{\alpha_s}...D_{\alpha_2}D_{\alpha_1}W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})}A^k_{\alpha_{s+1}}|W(\zeta)|,$$
$$\text{for } \zeta \in \mathbb{T}_1, |\alpha_{s+1}| > k, |\alpha_s| > k \ \& \ |\alpha_{s-1}| > k.$$

One can continue similarly and obtain

$$|D_{\alpha_{s+1}}D_{\alpha_s}...D_{\alpha_2}D_{\alpha_1}W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})}A^k_{\alpha_{s+1}}|W(\zeta)|,$$
$$\text{for } \zeta \in \mathbb{T}_1, |\alpha_{s+1}| > k, |\alpha_s| > k, |\alpha_{s-1}| > k, ..., |\alpha_2| > k, |\alpha_1| > k.$$

Hence Lemma is true for $t = s+1$. This completes the proof of Lemma.

3. Proof of Theorem 1.1

Let $m = \min_{\zeta \in \mathbb{T}_k} |W(\zeta)|$, then $|W(\zeta)| \geq m$ on $\mathbb{T}_k$. Therefore, for every λ with $|\lambda| < 1$, $|W(\zeta)| > |\lambda|$ on $\mathbb{T}_k$. If $W(\zeta)$ has a zero on $\mathbb{T}_k$ then $m = 0$ and the result follows from Lemma 2.3. Therefore from now onwards we will assume that $W(\zeta)$ has all its zeros in $\mathbb{E}_k^-$, where $k \geq 1$. By Rouche's theorem the polynomial

$$F(\zeta) = W(\zeta) + \lambda m$$

also has all its zeros in $\mathbb{E}_k^-$. Thus, on applying Lemma 2.3 to $F(\zeta)$ we obtain for $|\alpha_1| \geq k, |\alpha_2| \geq k, ..., |\alpha_t| \geq k$

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}F(\zeta)| \geq \frac{N_t}{(1+k^n)(1+k^{n-1})...(1+k^{n-t+1})}A^k_{\alpha_t}|F(\zeta)|, \ \zeta \in \mathbb{T}_1,$$

i.e.,

(3.1)
$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta) + m\lambda n(n-1)...(n-t+1)| \geq \frac{N_t}{K_t}A^k_{\alpha_t}|W(\zeta) + \lambda m|, \ \zeta \in \mathbb{T}_1,$$

where $K_t = (1+k^n)(1+k^{n-1})...(1+k^{n-t+1})$.

If we choose the argument of λ such that

$$|W(\zeta)+\lambda m| = |W(\zeta)| + |\lambda| m,$$

then from (3.1), we get

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta) + m\lambda n(n-1)...(n-t+1)| \geq \frac{N_t}{K_t} A^k_{\alpha_t}\{|W(\zeta)| + |\lambda| m\},$$

this gives

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta)| + mN_t|\lambda| \geq \frac{N_t}{K_t} A^k_{\alpha_t}\{|W(\zeta)| + |\lambda| m\}, \ \zeta \in \mathbb{T}_1.$$

Equivalently

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta)| \geq \frac{N_t}{K_t}\left[A^k_{\alpha_t}|W(\zeta)| + |\lambda|\{A^k_{\alpha_t} - K_t\} m\right]. \tag{3.2}$$

Now letting $|\lambda| \to 1$ in (3.2),we get

$$\max_{\zeta\in\mathbb{T}_1}|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta)| \geq \frac{N_t}{K_t}\left[A^k_{\alpha_t}\|W\| + \{A^k_{\alpha_t} - K_t\}\min_{\zeta\in\mathbb{T}_k}|W(\zeta)|\right],$$

which is (1.10) and Theorem 1.1 is thus proved.

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OPERATOR PRESERVING BERNSTEIN-TYPE INEQUALITIES BETWEEN POLYNOMIALS

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Abstract. In this paper, we establish some operator preserving inequalities of Bernstein-type in the uniform-norm between univariate complex coefficient polynomials while taking into account the placement of their zeros. The obtained results produce a variety of interesting results as special cases.

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1. Introduction

Let $P_n := \{P \in \mathbb{C}[z];\ \deg P \leq n\}$ be the space of all univariate complex coefficient polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n and let $P'(z)$ be the derivative of $P(z)$. The famous Bernstein inequality for the uniform-norm on the unit circle states that if $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq nM_1, \tag{1.1}$$

where here and throughout $M_1 = \max_{|z|=1} |P(z)|$ is the uniform-norm of P on the unit circle. On the other hand, concerning the maximum modulus of $P(z)$ on the circle $|z| = R \geq 1$, we have

$$\max_{|z|=R} |P(z)| \leq R^n M_1. \tag{1.2}$$

Inequality (1.1) is due to Bernstein [4], while as inequality (1.2) is a simple deduction from the Maximum Modulus Principle, for reference see ([16], page 346). Equality holds in (1.1) and (1.2) if and only if $P(z)$ is a non-zero multiple of z^n. For $P \in P_n$, it is known (see [2] and [12]) that

$$|P(Rz)| + |Q(Rz)| \leq (R^n + 1)M_1, \tag{1.3}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

For the class of polynomials $P \in P_n$ not vanishing in the interior of the unit circle, the inequalities (1.1) and (1.2) have been respectively replaced by the following inequalities:

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}M_1 \tag{1.4}$$

and

$$\max_{|z|=R}|P(z)| \leq \frac{R^n+1}{2}M_1, \quad R \geq 1. \tag{1.5}$$

Inequalities (1.4) and (1.5) are sharp and equality holds for polynomials having all their zeros on the unit circle. As is well known, inequality (1.4) was conjectured by Erdös and later proved by Lax [8], while inequality (1.5) is due to Ankeny and Rivlin [1]. As an generalization of (1.3), Jain [6] proved the following interesting result:

Theorem A. If $P \in P_n$, then for every β with $|\beta| \leq 1$, $R \geq 1$ and $|z| = 1$, we have

$$\begin{aligned}\left|P(Rz)+\beta\Big(\frac{R+1}{2}\Big)^n P(z)\right| + \left|Q(Rz)+\beta\Big(\frac{R+1}{2}\Big)^n Q(z)\right| \\ \leq \left\{\left|1+\beta\Big(\frac{R+1}{2}\Big)^n\right| + \left|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\right|\right\}M_1.\end{aligned} \tag{1.6}$$

In 2012, Zireh ([17], Lemma 2.6) proved a more general result, which in particular gives the following generalization of (1.6).

Theorem B. If $P \in P_n$, then for every β with $|\beta| \leq 1$, $R > r \geq k,\ k \leq 1$ and $|z| = 1$, we have

$$\begin{aligned}\left|P(Rz)+\beta\Big(\frac{R+k}{r+k}\Big)^n P(rz)\right| + \left|Q(Rz)+\beta\Big(\frac{R+k}{r+k}\Big)^n Q(rz)\right| \\ \leq \left\{\left|1+\beta\Big(\frac{R+k}{r+k}\Big)^n\right| + k^{-n}\left|R^n+\beta\Big(\frac{R+k}{r+k}\Big)^n\right|\right\}M_1,\end{aligned}$$

where $Q(z) = \Big(\frac{z}{k}\Big)^n\overline{P\Big(\frac{k^2}{\bar{z}}\Big)}$.

It is topical in the geometric function theory to study the extremal problems of functions of a complex variable and generalizing the classical polynomial inequalities in various directions. Although the literature on polynomial inequalities is vast and growing and over the years, many authors produced an abundance of various versions and generalizations of the above inequalities by introducing various operators that preserve such type of inequalities between polynomials (for example, see [5], [11] and [12]). It is an interesting problem, as pointed out by Rahman to characterize all such operators, and as part of this characterization Rahman in [12] (see also [9] or Rahman and Schmeisser [[14], pp. 538-551]) introduced a class B_n of operators B that maps $P \in P_n$ to $B[P] \in P_n$.

The class of B_n-operators: For fixed $n \in \mathbb{N}$, Marden ([9], pp. 65) in 1966 defined and studied the differential operator B that to each polynomial $P(z)$ of degree at most n assigns the polynomial

$$B[P](z) := \lambda_0 P(z) + \lambda_1\Big(\frac{nz}{2}\Big)\frac{P'(z)}{1!} + \lambda_2\Big(\frac{nz}{2}\Big)^2\frac{P''(z)}{2!},$$

where $\lambda_0,\ \lambda_1$ and λ_2 are such that all the zeros of

$$\phi(z) = \lambda_0 + \binom{n}{1}\lambda_1 z + \binom{n}{2}\lambda_2 z^2$$

lie in the half plane

$$Re(z) \leq \frac{n}{4}. \tag{1.7}$$

In fact, Marden proved that this operator preserves the zeros of the polynomial in a closed disk, i.e., if all the zeros of $P(z)$ lie in the closed unit disk, then all the zeros of $B[P](z)$ also lie in the same disk. Usually, such operators are called B_n-operators (see [14], page 538) and were also extensilvely studied by Rahman [12]. For more information regarding the B_n-operators (see [10], [13] and [14]). The study of such operators preserving inequalities between polynomials in the geometric function theory is a problem of interest both in mathematics and in the application areas such as physical systems. In addition to having numerous applications, this study has been the inspiration for much more research both from the theoretical point of view, as well as from the practical point of view. Recently, Rather et al. [15] considered the generalized B_n-operator N_v which carries $P \in P_n$ into $N_v[P] \in P_n$ defined by

$$N_v[P](z) := \sum_{v=0}^{m} \lambda_v \left(\frac{nz}{2}\right)^v \frac{P^{(v)}(z)}{v!}, \tag{1.8}$$

where $\lambda_v;\ v = 0, 1, 2, ..., m$, are such that the zeros of the polynomial

$$\phi_v(z) = \sum_{v=0}^{m} \binom{n}{v} \lambda_v z^v,\ m \leq n, \tag{1.9}$$

lie in the half plane (1.7).

It is easy to observe that if we take $\lambda_v = 0$ in (1.8) and (1.9) for $3 \leq v \leq m$, then N_v reduces to the B-operator. They established certain results concerning the upper bound of $|N_v[P]|$ for $|z| \geq 1$. More precisely, they proved the following results:

Theorem C. If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then

$$|N_v[P](z)| \leq |N_v[f](z)| \quad \text{for } |z| \geq 1. \tag{1.10}$$

Equality in (1.10) holds for $P(z) = e^{i\gamma} f(z),\ \gamma \in \mathbb{R}$.

Theorem D. If $P \in P_n$, and $P(z) \neq 0$ in $|z| < 1$, then

$$|N_v[P](z)| \leq \frac{1}{2}\Big\{\big|N_v[\rho_n](z)\big| + |\lambda_0|\Big\} M_1 \quad \text{for } |z| \geq 1, \tag{1.11}$$

where here and throughout $\rho_n = z^n$.
Equality in (1.11) holds for $P(z) = az^n + b,\ |a| = |b| \neq 0$.
Very recently, Mir [11] obtained the following generalizations of the above inequalities by considering a more general problem of investigating the dependence of $|N_v[P(Rz)] - \beta N_v[P(rz)]|$ on the maximum of $|P(z)|$ on $|z| = 1$ for every $|\beta| \leq 1$, $R \geq r \geq 1$, and developed a unified method for arriving at these results. More precisely, Mir proved the following results:

Theorem E. If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and if $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then for any complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$\Big|N_v[P](Rz) - \beta N_v[P](rz)\Big| \leq \Big|N_v[f](Rz) - \beta N_v[f](rz)\Big| \text{ for } |z| \geq 1. \tag{1.12}$$

Equality in (1.12) holds for $P(z) = e^{i\alpha} f(z), \alpha \in \mathbb{R}$.

Theorem F. If If $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$\begin{aligned} &\Big|N_v[P](Rz) - \beta N_v[P](rz)\Big| \\ &\leq \frac{1}{2}\bigg\{\Big(|R^n - \beta r^n||N_v[\rho_n](z)| + |1-\beta||\lambda_0|\Big)M_1 \\ &- \Big(|R^n - \beta r^n||N_v[\rho_n](z)| - |1-\beta||\lambda_0|\Big)m_1\bigg\} \text{ for } |z| \geq 1, \end{aligned} \tag{1.13}$$

where here and throughout $m_1 = \min_{|z|=1} |P(z)|$.
Equality in (1.13) holds for $P(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta| \neq 0$.
The Bernstein-type inequalities are seminal in the field of classical analysis, and over a period, these inequalities have been studied for different operators, in different norms, and for different classes of functions. The present paper is mainly motivated by the desire to establish some new inequalities concerning the N_v-operator in the uniform-norm between polynomials, which in turn yield compact generalizations of inequalities (1.10)-(1.13) and other related results. The essence in the papers of Jain ([6], [7]) and Zireh [17] is the origin of thought for the new inequalities presented in this paper.

2. Main results

In this section, we state our main results. Their proofs are given in the next section. We begin by proving the following inequality giving compact generalizations of Theorems A and B.

Theorem 2.1. *If $P \in P_n$, then for $|\beta| \leq 1$, $R > r \geq k$, $k > 0$, and $|z| \geq 1$ with $Q(z) = z^n\overline{P(1/\overline{z})}$ we have*

$$\begin{aligned}\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \\ + k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right| \\ \leq \left[\frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right| + \left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right||\lambda_0|\right]M_k,\end{aligned} \tag{2.1}$$

where here and throughout $M_k = \max_{|z|=k}|P(z)|$.

Remark 2.1. *One can observe that Theorem 2.1 provides an interesting generalization of Theorem A. For instance, if in (2.1), after substituting the value of $N_v[\rho_n](z)$ and taking $\lambda_v = 0$ for $v = 1, 2, 3, ..., m$, and noting that $N_v[P](z) = \lambda_0 P(z)$, we get Theorem A as a special case when $k = r = 1$.*

Theorem 2.2. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \leq k$, $k > 0$, then for every $|\beta| \leq 1$ and $R > r \geq k$, we have*

$$\begin{aligned}\min_{|z|=1}\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \\ \geq \frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right|m_k,\end{aligned} \tag{2.2}$$

where here and throughout $m_k = \min_{|z|=k}|P(z)|$.

As in remark 2.1, after substituting the value of $N_v[\rho_n](z)$ in (2.2) and taking $\lambda_v = 0$ for $v = 1, 2, 3, ..., m$, and noting that $N_v[P](z) = \lambda_0 P(z)$, we get a result of (Zireh [17], when $\alpha = 0$), see also Dewan and Hans ([5], Theorem 1).

Theorem 2.3. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq k$, $k \leq 1$ then for every $|\beta| \leq 1$ and $R > r \geq k$, we have for $|z| \geq 1$,*

$$\begin{aligned}\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \\ \leq \frac{1}{2}\left[\frac{1}{k^n}\left|R^n + \beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right| + |\lambda_0|\left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right|\right]M_k.\end{aligned} \tag{2.3}$$

Remark 2.2. *By taking $\beta = 0$ and $k = 1$, Theorem 2.3 in particular gives Theorem D and for suitable choices of λ_v; $0 \leq v \leq m$, it yields inequalities (1.4) and (1.5) as well.*

The above inequality (2.3) will be a consequence of a more fundamental inequality presented by the following theorem.

Theorem 2.4. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq k,\ k \leq 1$ then for every $|\beta| \leq 1$ and $R > r \geq k$, we have*

$$\begin{aligned}\Big|N_v[P](Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^n N_v[P](rz)\Big| &\leq \frac{1}{2}\Bigg[\Bigg(\frac{1}{k^n}\Big|R^n + r^n\beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\big|N_v[\rho_n](z)\big| \\ &+ |\lambda_0|\Big|1+\beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\Bigg)M_k - \Bigg(\frac{1}{k^n}\Big|R^n + r^n\beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\big|N_v[\rho_n](z)\big| \\ &- |\lambda_0|\Big|1+\beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\Bigg)m_k\Bigg]. \end{aligned} \tag{2.4}$$

Equality in (2.4) holds for $P(z) = \gamma z^n + \delta$ with $|\gamma| = |\delta| \neq 0$. We shall now discuss some consequences of Theorem 2.4. If in (2.4), after substituting the value of $N_v[\rho_n](z)$, we get for every $|\beta| \leq 1$ and $R > r \geq k$,

$$\begin{aligned}&\Big|N_v[P](Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^n N_v[P](rz)\Big| \\ &\leq \frac{1}{2}\Bigg[\Bigg(\frac{1}{k^n}\Big|R^n + r^n\beta\Big(\frac{R+k}{r+k}\Big)^n\Big||z|^n\Big|\sum_{v=0}^{m}\lambda_v\binom{n}{v}\Big(\frac{n}{2}\Big)^v\Big| \\ &+ |\lambda_0|\Big|1+\beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\Bigg)M_k \\ &- \Bigg(\frac{1}{k^n}\Big|R^n + r^n\beta\Big(\frac{R+k}{r+k}\Big)^n\Big||z|^n\Big|\sum_{v=0}^{m}\lambda_v\binom{n}{v}\Big(\frac{n}{2}\Big)^v\Big| \\ &- |\lambda_0|\Big|1+\beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\Bigg)m_k\Bigg] \quad \text{for } |z| \geq 1, \end{aligned} \tag{2.5}$$

where $\lambda_v;\ 0 \leq v \leq m$ are such that all the zeros of $\phi_v(z)$ defined by (1.9) lie in the half plane (1.7).

Remark 2.3. *Taking $\lambda_v = 0$ for $v = 1, 2, 3, ..., m$ in* (2.5) *and noting that $N_v[P](z) = \lambda_0 P(z)$, we get the following result which is of independent interest, because besides giving generalizations and refinements of* (1.4) *and* (1.5) *it also provides generalizations and refinements of some results of Zireh* [17], *Dewan and Hans* [5] *and Jain (*[6], [7]*).*

Corollary 2.1. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq k,\ k \leq 1$ then for $|\beta| \leq 1$, $R > r \geq k$, we have*

$$\left|P(Rz) + \beta\left(\frac{R+k}{r+k}\right)^n P(rz)\right| \leq \frac{1}{2}\Bigg[\Bigg(\frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right||z|^n + \left|1 + \beta\left(\frac{R+k}{r+k}\right)^n\right|\Bigg)M_k - \Bigg(\frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right||z|^n - \left|1 + \beta\left(\frac{R+k}{r+k}\right)^n\right|\Bigg)m_k\Bigg] \quad \text{for } |z| \geq 1. \tag{2.6}$$

Equality in (2.6) *holds for $P(z) = \gamma z^n + \delta$ with $|\gamma| = |\delta| \neq 0$.*

Self-inversive polynomial: A polynomial $P \in P_n$ is said to be self-inversive if $P(z) = \zeta Q(z),\ |\zeta| = 1$. Finally, we prove the following result for self-inversive polynomials.

Theorem 2.5. *If $P \in P_n$ is self-inversive, then for $|\beta| \leq 1$ and $R > r \geq 1$, we have for $|z| \geq 1$,*

$$\left|N_v[P](Rz) + \beta\left(\frac{R+1}{r+1}\right)^n N_v[P](rz)\right| \leq \frac{1}{2}\Bigg[\left|R^n + r^n\beta\left(\frac{R+1}{r+1}\right)^n\right|\big|N_v[\rho_n](z)\big| + |\lambda_0|\left|1 + \beta\left(\frac{R+1}{r+1}\right)^n\right|\Bigg]M_1. \tag{2.7}$$

Equality in (2.7) *holds for $P(z) = z^n + 1$.*

Remark 2.4. *For $\beta = 0$, the above result in particular reduces to a result of Rather et al. (*[15]*, Theorem 1.4). Taking $\lambda_v = 0$ for $v = 1, 2, 3, ..., m$ in* (2.7) *and noting that $N_v[P](z) = \lambda_0 P(z)$, we get the following result for self-inversive polynomials.*

Corollary 2.2. *If $P \in P_n$ is self-inversive, then for $|\beta| \leq 1$ and $R > r \geq 1$, we have for $|z| \geq 1$,*

$$\left|P(Rz) + \beta\left(\frac{R+1}{r+1}\right)^n P(rz)\right| \leq \frac{1}{2}\Bigg[\left|R^n + r^n\beta\left(\frac{R+1}{r+1}\right)^n\right||z|^n + \left|1 + \beta\left(\frac{R+1}{r+1}\right)^n\right|\Bigg]M_1. \tag{2.8}$$

For $\beta = 0$, the inequality (2.8) *shows that the inequality* (1.5) *also holds for self-inversive polynomials.*

3. Auxiliary results

In order to prove our main results, we need the following lemmas.

Lemma 3.1. *([3]) If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 0$, then for every $R \geq r$ and $rR \geq k^2$,*

$$|P(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |P(rz)| \quad \text{for} \quad |z| = 1.$$

If we take $r = s = 1$ and $\sigma = \frac{n}{2}$ in Theorem 1.1 of Rather et al. [15], we get the following:

Lemma 3.2. *If all the zeros of polynomial $P \in P_n$ lie in $|z| \leq 1$, then all the zeros of $N_v[P(z)]$ defined by (1.8) also lie in $|z| \leq 1$.*

We now prove the following lemma from which we can obtain Theorem C as a special case.

Lemma 3.3. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k > 0$, and $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = k$, then for every $|\beta| \leq 1$, $R > r \geq k$ and for $|z| \geq 1$,*

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq \left|N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)\right|.$$

Proof of Lemma 3.3. By hypothesis $|P(z)| \leq |f(z)|$ for $|z| = k$, therefore any zero of $f(z)$ that lies on $|z| = k$ is also a zero of $P(z)$. On the other hand, for every $\zeta \in \mathbb{C}$ with $|\zeta| > 1$, we have $|P(z)| < |\zeta f(z)|$, for $|z| = k$, when all the zeros of $f(z)$ lie in $|z| < k$, it follows by Rouché's theorem that all the zeros of the polynomial $g(z) = P(z) - \zeta f(z)$ lie in $|z| \leq k$. On applying Lemma 3.1 to the polynomial $g(z)$, we have

$$|g(Rz)| > \left(\frac{R+k}{r+k}\right)^n |g(rz)| \quad \text{for} \quad |z| = k.$$

Since $g(Rz)$ has all its zeros in $|z| \leq \frac{k}{R} \leq 1$. Therefore, if β is any complex number such that $|\beta| \leq 1$, it follows that all the zeros of the polynomial $g(Rz) + \beta\left(\frac{R+k}{r+k}\right)^n g(rz)$ also lie in $|z| \leq 1$. Applying Lemma 3.2 and noting that N_v is a linear operator, we conclude that all the zeros of the polynomial

$$J(z) := N_v[g](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[g](rz)$$

lie in $|z| \leq 1$, for every $|\beta| \leq 1$ and $R > r \geq k$. Replacing $g(z)$ by $P(z) - \zeta f(z)$, we conclude that all the zeros of the polynomial

$$J(z) := N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz) - \zeta\left[N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)\right]$$

lie in $|z| \leq 1$ for all real or complex number β with $|\beta| \leq 1$ and $R > r \geq k$. This implies,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq \left|N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)\right| \quad \text{for } |z| \geq 1. \tag{3.1}$$

If inequality (3.1) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| > \left|N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)\right|.$$

Taking

$$\zeta = \frac{N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)}{N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)},$$

so that $|\zeta| > 1$ and with this choice of ζ, we have $J(z_0) = 0$ for $|z_0| \geq 1$, which is a clear contradiction to the fact that $J(z) \neq 0$ for $|z| \geq 1$. Thus for every complex number β with $|\beta| \leq 1$ and $R > r \geq k$, we have (3.1) holds. This proves Lemma 3.3 completely.

Remark 3.1. *On applying Lemma 3.3 with $f(z) = M_k z^n/k^n$, giving us the following inequality:*

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq \frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right| M_k \quad \textit{for } |z| \geq 1. \tag{3.2}$$

Lemma 3.4. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for every $|\beta| \leq 1$ and $R > r \geq k$, we have*

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right| \quad \textit{for } |z| \geq 1.$$

Proof of Lemma 3.4. Since $P(z) \neq 0$ in $|z| < k$, therefore, all the zeros of polynomial $Q(z/k^2)$ lie in $|z| < k$. Also $|k^n Q(z/k^2)| = |P(z)|$ for $|z| = k$. Applying Lemma 3.3 to $P(z)$ with $f(z)$ replaced by $k^n Q(z/k^2)$, we get for every $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq k^n \left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|.$$

4. Proofs of main results

Proof of Theorem 2.1. Let $M_k = \max_{|z|=k} |P(z)|$, then by Rouché's theorem the polynomial $U(z) = P(z) - \zeta M_k$ has no zeros in $|z| < k$ for every $\zeta \in \mathbb{C}$ with $|\zeta| > 1$. On using Lemma 3.4 to $U(z)$, we have for $|\beta| \leq 1$ and $R > r \geq k$,

$$\left|N_v[U](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[U](rz)\right| \leq k^n \left|N_v[L](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[L](rz/k^2)\right|,$$

where $L(z) = z^n\overline{U(\frac{1}{\overline{z}})} = Q(z) - \overline{\zeta} z^n M_k$. Using $U(z) = P(z) - \zeta M_k$, $L(z) = Q(z) - \overline{\zeta} z^n M_k$, and the fact that N_v is linear and $N_v[1] = \lambda_0$, we get from above inequality for $|\beta| \leq 1$, $|z| \geq 1$ and $R > r \geq k$,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz) - \lambda_0 \zeta M_k\left[1 + \beta\left(\frac{R+k}{r+k}\right)^n\right]\right|$$
$$\leq k^n \left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2) - \frac{\overline{\zeta} M_k}{k^{2n}}\left[R^n + r^n \beta\left(\frac{R+k}{r+k}\right)^n\right] N_v[\rho_n](z)\right|, \tag{4.1}$$

where $Q(z) = z^n\overline{P(\frac{1}{\overline{z}})}$.

Now choosing the argument of ζ suitably on the right hand side of (4.1) such that

$$k^n \left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2) - \frac{\overline{\zeta} M_k}{k^{2n}}\left[R^n + r^n \beta\left(\frac{R+k}{r+k}\right)^n\right] N_v[\rho_n](z)\right|$$
$$= \frac{|\overline{\zeta}| M_k}{k^n}\left|R^n + r^n \beta\left(\frac{R+k}{r+k}\right)^n\right| \left|N_v[\rho_n](z)\right| - k^n \left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|,$$

which is possible by applying inequality (3.2) to the polynomial $Q(z/k^2)$ and using the fact that $\max_{|z|=k}\left|Q(z/k^2)\right| = M_k/k^n$, we get for $|\beta| \le 1$, $R > r \ge k$ and $|z| \ge 1$,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| - |\lambda_0||\zeta|M_k\left|1 + \beta\left(\frac{R+k}{r+k}\right)^n\right|$$
$$\le \frac{|\bar{\zeta}|M_k}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right|$$
$$- k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|.$$

The required result follows on making $|\zeta| \to 1$.

Proof of Theorem 2.2. Let $m_k = \min_{|z|=k}|P(z)|$. In case $m_k = 0$, there is nothing to prove. Assume that $m_k > 0$, so that all the zeros of $P(z)$ lie in $|z| < k$ and we have, $m_k\left|z/k\right|^n \le |P(z)|$ for $|z| = k$. Applying Lemma 3.3 with $f(z)$ replaced by $m_k\left(z/k\right)^n$, we obtain for every $|\beta| \le 1$ and $R > r \ge k$,

$$\min_{|z|=1}\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right|$$
$$\ge \frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right|m_k,$$

which is inequality (2.2). This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. The desired result immediately follows by combining Lemma 3.4 and Theorem 2.1.

Proof of Theorem 2.4. The result follows obviously in case $P(z)$ has a zero on $|z| = k$ (by Theorem 2.3). Therefore, we assume that $P(z)$ has all its zeros in $|z| > k$, so that $m_k = \min_{|z|=k}|P(z)| > 0$. Now for every real or complex number ζ with $|\zeta| < 1$, it follows by Rouché's theorem, that the polynomial $U(z) = P(z) - \zeta m_k$ does not vanish in $|z| < k$. On applying Lemma 3.4 to the polynomial $U(z)$ and noting that N_v is a linear operator with $N_v[1] = \lambda_0$, we get for every $|\beta| \le 1$, $R > r \ge k$ and $|z| \ge 1$,

$$\left|N_v[U](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[U](rz)\right|$$
$$\le k^n\left|N_v[L](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[L](rz/k^2)\right|,$$

where $L(z) = z^n\overline{W(\frac{1}{\overline{z}})}$. Equivalently,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz) - \zeta\lambda_0 m_k\left[1+\beta\left(\frac{R+k}{r+k}\right)^n\right]\right|$$

$$\leq k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right.$$

$$\left. - \frac{\overline{\zeta}m_k}{k^{2n}}\left[R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right]N_v[\rho_n](z)\right| \text{ for } |z| \geq 1, \tag{4.2}$$

where $Q(z) = z^n\overline{P(\frac{1}{\overline{z}})}$.

Now choosing the argument of ζ on the right hand side of (4.2) such that

$$k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right.$$

$$\left. - \frac{\overline{\zeta}m_k}{k^{2n}}\left[R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right]N_v[\rho_n](z)\right|$$

$$= k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|$$

$$- \frac{|\overline{\zeta}|m_k}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\big|N_v[\rho_n](z)\big|,$$

which is possible by Theorem 2.2 applied to $Q(z/k^2)$, we get

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| - |\zeta||\lambda_0|m_k\left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right|$$

$$\leq k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|$$

$$- \frac{|\zeta|}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\big|N_v[\rho_n](z)\big|m_k \quad \text{for } |z| \geq 1.$$

This gives by letting $|\zeta| \to 1$,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq \left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n\right|$$

$$- \left[\frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\big|N_v[\rho_n](z)\big| - |\lambda_0|\left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right|\right]m_k. \tag{4.3}$$

Inequality (4.3) in conjunction with Theorem 2.1 yields (2.4). This completes the proof of Theorem 2.4.

Proof of Theorem 2.5. By hypothesis $P \in P_n$ is self-inversive, therefore $P(z) = \zeta Q(z),\ |\zeta| = 1$. It gives for every $|\beta| \leq 1$, $R > r \geq 1$ and for all z,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+1}{r+1}\right)^n N_v[P](rz)\right| = \left|N_v[Q](Rz) + \beta\left(\frac{R+1}{r+1}\right)^n N_v[Q](rz)\right|.$$

The above equality when combined with Theorem 2.1 (for $k = 1$) yields (2.7). This completes the proof of Theorem 2.5.

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FURTHER RESULTS ON SHARED-VALUE PROPERTIES OF $f'(z) = f(z + c)$

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Abstract. In this paper, we will continue to consider "under what sharing value conditions, does $f'(z) = f(z+c)$ hold?" For example, we prove the following result: Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $f'(z)$ and $f(z + c)$ share ∞ CM and a, b IM, and if $N(r, f) = O(\overline{N}(r, f)), (r \to \infty)$, then $f'(z) = f(z + c)$. The research also includes some improvements of earlier results of such studies.

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1. Introduction

In the study of complex differential equations, Nevanlinna theory has a wide range of applications. In addition, with the difference correspondence of the logarithmic derivative lemma obtained by Chiang-Feng [4], and Halburd-Korhonen [7] respectively, the complex domain differences and the complex difference equations have also been rapidly developed. The related results, readers can refer to [3].

The study of complex differential-difference equations can be traced back to Naftalevich's work in [6, 16, 17], but the results of using Nevanlinna theory to study differential-difference equations are relatively limited, the reader is invited to see [5, 9, 11, 13, 14].

The delay equation $f'(x) = f(x-k)$, $(k > 0)$ have been studied extensively in real analysis. The related results can be found in [1]. Inspired by such results, Liu and Dong [12] discussed the properties of the solutions of complex differential-difference equation $f'(z) = f(z + c)$, $(c(\neq 0) \in \mathbb{C})$ by using Nevanlinna theory.

We have tried to clarify the form of the solutions to the equation $f'(z) = f(z+c)$, but unfortunately, this attempt has not been successful. Then, we investigated this equation from another point of view, namely, "under what sharing value conditions, does $f'(z) = f(z + c)$ hold?" And in [18, Theorem 1.4], we obtained:

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Theorem A. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0, a CM, then $f'(z) = f(z+c)$.*

Afterwards, for entire functions, Qi et al. improved Theorem A to "share 0 CM and a IM " in [19, Theorem 1.2] and "share two distinct constants a, b CM" in [20, Theorem 2.1]. Further, Huang and Fang [10, Theorem 1] improved the value sharing assumption to "share two distinct constants a, b IM". In addition, some authors tried to extend Theorem A to meromorphic functions:

Theorem B [19, Theorem 1.1]**.** *Let $f(z)$ be a non-constant meromorphic function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a CM, and satisfy $f(z+c) = 0 \to f'(z) = 0$, $f(z+c) = \infty \leftarrow f'(z) = \infty$, then $f'(z) = f(z+c)$. Further, $f(z)$ is a transcendental entire function.*

Remark. Let $z_n (n = 1, 2, \ldots)$ be zeros of $f - \alpha$ with multiplicity $\nu(n)$. If z_n are also $\nu(n)$ multiple zeros of $g - \alpha$ at least, then we write $f = \alpha \to g = \alpha$, where $\alpha \in \mathbb{C} \cup \{\infty\}$.

From Theorem 2.1 in [2], we know that:

Theorem C. *Let $f(z)$ be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$. If $f'(z)$ and $f(z+c)$ share $0, \infty$ CM and 1 IM, then $f'(z) = f(z+c)$.*

In this paper, we will continue to consider the above question as $f(z)$ is a meromorphic function. We, for instance, get "Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $f'(z)$ and $f(z+c)$ share ∞ CM and a, b IM, and if $N(r, f) = O(\overline{N}(r, f)), (r \to \infty)$, then $f'(z) = f(z+c)$." The reminder of this paper is organized as follows: In Sections 3 and 4, we will improve Theorem B and Theorem C, respectively. In Section 5, we will give some partially shared values results for $f'(z)$ and $f(z+c)$, which can be seen as the improvements of Theorems B and C as well.

2. Lemmas

Lemma 2.1. [8, Theorem 5.1] *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1. Then,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Throughout the paper, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure.

Lemma 2.2. [22, Lemma 1.2] *Let $f_1(z), f_2(z)$ be two meromorphic functions, then*

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right).$$

Lemma 2.3. [15],[21, Theorem 1.13] *Let $f(z)$ be a non-constant meromorphic function, and $R(f) = \frac{P(f)}{Q(f)}$, where $P(f) = \sum_{i=0}^{p} \alpha_i f^i$ and $Q(f) = \sum_{j=0}^{q} \beta_j f^j$ are two mutually prime polynomials in $f(z)$. If the coefficients $\{\alpha_i(z)\}$, $\{\beta_j(z)\}$ are small functions of $f(z)$ and $\alpha_p(z) \not\equiv 0$, $\beta_q(z) \not\equiv 0$, then*

$$T(r, R(f)) = max\{p, q\} \cdot T(r, f) + S(r, f).$$

Lemma 2.4. *Let $f(z)$ be a non-constant meromorphic function of hyper-order strictly less than 1, and let $a_1, \ldots, a_p \in \mathbb{C}$, $p \geq 2$, be distinct points. Then,*

$$\begin{aligned}(p-1)T(r, f(z+c)) \leq \sum_{k=1}^{p} N\left(r, \frac{1}{f(z+c)-a_k}\right) - N(r, f(z+c)) \\ + N(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f).\end{aligned}$$

Proof. Let

$$P(f) = \prod_{k=1}^{p}(f(z+c) - a_k),$$

then we have

$$\frac{1}{P(f)} = \sum_{k=1}^{p} \frac{b_k}{f(z+c)-a_k}, \tag{2.1}$$

for some constants b_k. From Lemma 2.1 and the lemma of logarithmic derivative, we have

$$m\left(r, \frac{f'}{f(z+c)-a_k}\right) = m\left(r, \frac{f'}{f(z)-a_k}\frac{f(z)-a_k}{f(z+c)-a_k}\right) = S(r, f). \tag{2.2}$$

Hence, by (2.1) and (2.2), it follows that

$$m\left(r, \frac{f'}{P(f)}\right) \leq \sum_{k=1}^{p} m\left(r, \frac{f'}{f(z+c)-a_k}\right) + S(r, f) = S(r, f).$$

From the above equation, we get

$$m\left(r, \frac{1}{P(f)}\right) = m\left(r, \frac{f'}{P(f)}\frac{1}{f'}\right) \leq m\left(r, \frac{1}{f'}\right) + S(r, f). \tag{2.3}$$

From (2.3), we have

$$\begin{aligned}T(r, f') &= m\left(r, \frac{1}{f'}\right) + N\left(r, \frac{1}{f'}\right) + O(1) \\ &\geq m\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{f'}\right) + S(r, f) \\ &= \sum_{k=1}^{p} m\left(r, \frac{1}{f(z+c)-a_k}\right) + N\left(r, \frac{1}{f'}\right) + S(r, f),\end{aligned}$$

which means

$$
\begin{aligned}
&m(r, f(z+c)) + \sum_{k=1}^{p} m\left(r, \frac{1}{f(z+c)-a_k}\right) \\
&\leq m(r, f') + N(r, f') + T(r, f(z+c)) - N(r, f(z+c)) - N\left(r, \frac{1}{f'}\right) + S(r, f) \\
&\leq m\left(r, \frac{f'}{f(z+c)}\right) + m(r, f(z+c)) + N(r, f') + T(r, f(z+c)) \\
&- N(r, f(z+c)) - N\left(r, \frac{1}{f'}\right) + S(r, f) \\
&= 2T(r, f(z+c)) - 2N(r, f(z+c)) + N(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(p-1)T(r, f(z+c)) \leq \sum_{k=1}^{p} N\left(r, \frac{1}{f(z+c)-a_k}\right) - N(r, f(z+c)) \\
+ N(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f).
\end{aligned}
$$

From Lemma 8.3 in [8] and Lemma 2.1, we have the following lemma:

Lemma 2.5. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, then we have*

$$
N(r, f(z+c)) = N(r, f) + S(r, f), \quad \overline{N}(r, f(z+c)) = \overline{N}(r, f) + S(r, f),
$$

$$
N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r, f),
$$

and

$$
T(r, f(z+c)) = T(r, f) + S(r, f).
$$

Lemma 2.6. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c) = \infty \leftarrow f' = \infty$, then $\overline{N}(r, f(z+c)) = \overline{N}(r, f') = \overline{N}(r, f) = S(r, f)$.*

Proof. By the assumption and Lemma 2.5, we have

$$
\begin{aligned}
N(r, f) + \overline{N}(r, f) = N(r, f') &\leq N(r, f(z+c)) + S(r, f) \\
&= N(r, f) + S(r, f),
\end{aligned}
$$

which means that

$$
\overline{N}(r, f) = S(r, f),
$$

and

$$
\overline{N}(r, f(z+c)) = \overline{N}(r, f') = \overline{N}(r, f) = S(r, f).
$$

Lemma 2.7. [21, Lemma 4.3] *Suppose that $f(z)$ is a non-constant meromorphic function and $P(f) = a_p f^p + a_{p-1} f^{p-1} + \cdots + a_0\,(a_p \neq 0)$ is a polynomial in $f(z)$ with degree p and coefficients $a_i\,(i = 0, 1, \ldots, p)$ are constants, suppose furthermore that $b_j\,(j = 1, \ldots, q)\,(q > p)$ are distinct finite values. Then,*

$$m\left(r, \frac{P(f)f'}{(f - b_1)(f - b_2)\cdots(f - b_q)}\right) = S(r, f).$$

Lemma 2.8. *Suppose that $f(z)$ and $g(z)$ are meromorphic functions such that $N(r, f) = N(r, g) = S(r, f)$ and a, b are two distinct finite values. Let*

$$V(z) = \left(\frac{f'}{f - a} - \frac{f'}{f - b}\right) - \left(\frac{g'}{g - a} - \frac{g'}{g - b}\right).$$

If $V(z) \equiv 0$, then either

$$2T(r, f) \leq \overline{N}\left(r, \frac{1}{f - a}\right) + \overline{N}\left(r, \frac{1}{f - b}\right) + S(r, f),$$

or

$$f(z) = g(z).$$

Proof. From $V(z) \equiv 0$, we have

$$\frac{f - a}{f - b} = A\frac{g - a}{g - b}, \tag{2.4}$$

where A is a non-zero constant. If $A = 1$, then we obtain $f(z) = g(z)$. If $A \neq 1$, then it follows from (2.4) that

$$\frac{A - 1}{A}\frac{f - \frac{Ab - a}{A - 1}}{f - b} = \frac{a - b}{g - b}.$$

Since $N(r, f) = N(r, g) = S(r, f)$, we get $N\left(r, \frac{1}{f - \frac{Ab - a}{A - 1}}\right) = S(r, f)$. Clearly, $\frac{Ab - a}{A - 1} \neq a$ and $\frac{Ab - a}{A - 1} \neq b$, and then from the second main theorem, we obtain

$$2T(r, f) \leq \overline{N}\left(r, \frac{1}{f - a}\right) + \overline{N}\left(r, \frac{1}{f - b}\right) + S(r, f).$$

3. The improvement of Theorem B

Proposition 3.1. Let $f(z)$ be a non-constant meromorphic function. If $f'(z)$ and $f(z + c)$ satisfy $f(z + c) = 0 \to f' = 0$ and $f(z + c) = \infty \leftarrow f' = \infty$, then $f(z)$ must be transcendental.

Proof. Suppose $f(z)$ is a non-constant rational function. Then, set

$$f(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are two mutually prime polynomials. Hence,

$$f'(z) = \left(\frac{P}{Q}\right)' = \frac{P'Q - PQ'}{Q^2} = \frac{P_1}{Q_1},$$

and

$$f(z+c) = \frac{P(z+c)}{Q(z+c)},$$

where P_1 and Q_1 are two mutually prime polynomials. If $Q(z)$ is not a constant, then by the assumption that $f(z+c) = \infty \leftarrow f'(z) = \infty$, we have

$$Q_1(z) = 0 \rightarrow Q(z+c) = 0.$$

Let z_1 is a zero of $Q(z)$, then we have $Q_1(z_1) = 0$, and so $Q(z_1 + c) = 0$. From $Q(z_1+c) = 0$, we have $Q_1(z_1+c) = 0$, which implies that $Q(z_1+2c) = 0$. Continuing inductively, we get that $Q(z_1 + nc) = 0$, which is impossible. Hence, $Q(z)$ is a constant. And so, $f(z)$ is a non-constant polynomial. Suppose $\deg f(z) = p \geq 1$, then we know the number of zeros of $f(z+c)$ is p and the number of zeros of f' is $p-1$, which contradicts the assumption $f(z+c) = 0 \rightarrow f' = 0$. (Here, multiple zeros are counted to their multiplicities.) Therefore, $f(z)$ is transcendental.

Remarks. (1). Proposition 3.1 is an improvement of Theorem B and [2, Proposition 1]. Moreover, Proposition 3.1 leads us only to consider the condition that $f(z)$ is a transcendental meromorphic function in this paper.

(2). The main ideas of Proposition 3.1 and Theorem 3.1 come from Theorem B, however, the key way of proof is somewhat different. Hence, for the convenience of the reader, we provide the proof.

Theorem 3.1. *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c) = 0 \rightarrow f' = 0$, $f(z+c) = a \rightarrow f' = a$ and $f(z+c) = \infty \leftarrow f' = \infty$, then $f'(z) = f(z+c)$.*

Proof of Theorem 3.1. Suppose that $f'(z) \not\equiv f(z+c)$. Set

$$F(z) = \frac{f'}{f(z+c)}. \tag{3.1}$$

Then, we see $F(z) \not\equiv 1$. Further, from the assumption $f(z+c) = 0 \rightarrow f' = 0$ and $f(z+c) = \infty \leftarrow f' = \infty$, we know that $F(z)$ is an entire function. Moreover, we have

$$m(r, F) = m\left(r, \frac{f'}{f}\frac{f}{f(z+c)}\right) = S(r, f). \tag{3.2}$$

Hence,

$$T(r, F) = S(r, f). \tag{3.3}$$

By the assumption that $f(z+c) = a \rightarrow f' = a$ and (3.3), it follows that

$$N\left(r, \frac{1}{f(z+c) - a}\right) \leq N\left(r, \frac{1}{F-1}\right) + S(r, f) = S(r, f). \tag{3.4}$$

From Lemmas 2.4-2.5, (3.4), and the sharing values assumption, we obtain that

$$\begin{aligned} T(r,f) &= T(r,f(z+c)) + S(r,f) \\ &\leq \left(N\left(r,\frac{1}{f(z+c)}\right) - N\left(r,\frac{1}{f'}\right)\right) + (N(r,f') - N(r,f(z+c))) \\ &+ N\left(r,\frac{1}{f(z+c)-a}\right) + S(r,f) = S(r,f), \end{aligned}$$

which is a contradiction. Therefore, $f'(z) = f(z+c)$.

4. The improvement of Theorem C

When $f(z)$ is meromorphic, all the previous results were around the condition "$f'(z)$ and $f(z+c)$ share 0, a". What happens if f' and $f(z+c)$ share two arbitrary constants? In this part, we will give some results on the sharing value assumption that "2 IM" for meromorphic functions. As a corollary, we will get an improvement of Theorem C in Theorem 4.2.

Proposition 4.1. Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. Suppose $f'(z)$ and $f(z+c)$ share a, b IM and satisfy $f(z+c) = \infty \leftarrow f' = \infty$. If $f'(z) \not\equiv f(z+c)$ and $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$. Then,

(1).

$$T(r,f') = T(r,f(z+c)) + S(r,f).$$

(2).

$$T(r,f(z+c)) = \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + S(r,f).$$

(3).

$$m\left(r,\frac{f(z+c)-d}{f'-d}\right) = S(r,f), \quad \text{where } d(\neq a,b) \in \mathbb{C}.$$

Proof. (1). Since $f(z+c) = \infty \leftarrow f' = \infty$, we have $\overline{N}(r,f(z+c)) = \overline{N}(r,f') = \overline{N}(r,f) = S(r,f)$, from Lemma 2.6. And $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$ means that

$$N(r,f) \leq k\overline{N}(r,f) + S(r,f) = S(r,f),$$

where k is a positive number. Hence,

$$N(r,f(z+c)) = N(r,f) + S(r,f) = S(r,f), \tag{4.1}$$

and

$$N(r,f') \leq N(r,f) + \overline{N}(r,f) = S(r,f). \tag{4.2}$$

Set

$$H(z) = \frac{f'(z+c)(f(z+c)-f')}{(f(z+c)-a)(f(z+c)-b)}. \tag{4.3}$$

Noting $f'(z)$ and $f(z+c)$ share a, b IM, we obtain the zeros of $f(z+c)-a$ or $f(z+c)-b$ are not poles of $H(z)$, by using an elementary computation. Hence, it follows from (4.1) and (4.2) that

$$N(r,H) \leq 2N(r,f(z+c)) + N(r,f') + S(r,f) = S(r,f). \tag{4.4}$$

Rewrite (4.3) as

$$H(z) = \frac{f'(z+c)f(z+c)}{(f(z+c)-a)(f(z+c)-b)} \frac{f(z+c)-f'}{f(z+c)}.$$

It follows from Lemma 2.5 and Lemma 2.7 that

$$m\left(r, \frac{f'(z+c)f(z+c)}{(f(z+c)-a)(f(z+c)-b)}\right) = S(r,f(z+c)) = S(r,f).$$

Moreover, by Lemma 2.1 and the lemma on the logarithmic derivative, we obtain

$$m\left(r, \frac{f(z+c)-f'}{f(z+c)}\right) \leq m\left(r, \frac{f'}{f(z+c)}\frac{f}{f}\right) + S(r,f) = S(r,f).$$

Therefore,

$$T(r,H) = S(r,f). \tag{4.5}$$

Rewrite (4.3) as

$$H(z)f^2(z+c) - (a+b)H(z)f(z+c) + abH(z) = f'(z+c)f(z+c) - f'(z+c)f'.$$

Note $f'(z) \neq f(z+c)$, we have $H(z) \not\equiv 0$. Further, from (4.1) and (4.2), we get

$$\begin{aligned}
&2T(r,f(z+c)) = T(r, f'(z+c)f(z+c) - f'(z+c)f') + S(r,f)\\
&= m(r, f'(z+c)f(z+c) - f'(z+c)f') + S(r,f)\\
&\leq m\left(r, \frac{f'(z+c)f(z+c) - f'(z+c)f'}{f(z+c)f'}\right) + m(r,f(z+c)) + m(r,f') + S(r,f)\\
&\leq T(r,f(z+c)) + T(r,f') + S(r,f),
\end{aligned}$$

which means that

$$T(r,f(z+c)) \leq T(r,f') + S(r,f). \tag{4.6}$$

On the other hand, from Lemma 2.5 and Lemma 2.6, we conclude that

$$T(r,f') \leq T(r,f) + \overline{N}(r,f) + S(r,f) = T(r,f(z+c)) + S(r,f). \tag{4.7}$$

Combining (4.6) and (4.7), it follows that

$$T(r,f') = T(r,f(z+c)) + S(r,f). \tag{4.8}$$

(2). From (4.1), (4.2), (4.8), the second fundamental theorem and the value sharing condition, we have

$$\begin{aligned}
&T(r,f') = T(r,f(z+c)) + S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{f'-a}\right) + \overline{N}\left(r,\frac{1}{f'-b}\right) + S(r,f) \le \overline{N}\left(r,\frac{1}{f'-f(z+c)}\right) + S(r,f)\\
&\le T(r,f'-f(z+c)) + S(r,f) = m(r,f'-f(z+c)) + S(r,f)\\
&\le m\left(r,\frac{f'-f(z+c)}{f(z+c)}\right) + m(r,f(z+c)) + S(r.f)\\
&\le m\left(r,\frac{f'}{f(z+c)}\frac{f}{f}\right) + T(r,f(z+c)) + S(r,f)\\
&\le T(r,f(z+c)) + S(r,f) = T(r,f') + S(r,f),
\end{aligned}$$

which means that

$$T(r,f(z+c)) = \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + S(r,f), \tag{4.9}$$

and

$$T(r,f') = \overline{N}\left(r,\frac{1}{f'-a}\right) + \overline{N}\left(r,\frac{1}{f'-b}\right) + S(r,f).$$

(3). From the second fundamental theorem, Lemma 2.6 and (4.9), we obtain that

$$\begin{aligned}
&2T(r,f(z+c))\\
&\le \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-d}\right)\\
&+ \overline{N}(r,f(z+c)) + S(r,f)\\
&= T(r,f(z+c)) + \overline{N}\left(r,\frac{1}{f(z+c)-d}\right) + S(r,f) \le 2T(r,f(z+c)) + S(r,f).
\end{aligned}$$

Hence, we have

$$T(r,f(z+c)) = \overline{N}\left(r,\frac{1}{f(z+c)-d}\right) + S(r,f),$$

which means that

$$m\left(r,\frac{1}{f(z+c)-d}\right) = S(r,f). \tag{4.10}$$

Further, we know

$$\begin{aligned}
&m\left(r,\frac{f'-d}{f(z+c)-d}\right)\\
&\le m\left(r,\frac{f'}{f(z+c)-d}\right) + m\left(r,\frac{d}{f(z+c)-d}\right) + S(r,f)\\
&\le m\left(r,\frac{f'}{f-d}\frac{f-d}{f(z+c)-d}\right) + S(r,f) = S(r,f).
\end{aligned} \tag{4.11}$$

Similarly, we have

$$m\left(r, \frac{1}{f'-d}\right) = S(r,f). \tag{4.12}$$

From (4.1), (4.2), (4.8), (4.10), (4.12) and Lemma 2.2, we have

$$\begin{aligned}
& m\left(r, \frac{f(z+c)-d}{f'-d}\right) - m\left(r, \frac{f'-d}{f(z+c)-d}\right) \\
&= T\left(r, \frac{f(z+c)-d}{f'-d}\right) - N\left(r, \frac{f(z+c)-d}{f'-d}\right) \\
&- T\left(r, \frac{f'-d}{f(z+c)-d}\right) + N\left(r, \frac{f'-d}{f(z+c)-d}\right) \\
&= N\left(r, \frac{f'-d}{f(z+c)-d}\right) - N\left(r, \frac{f(z+c)-d}{f'-d}\right) + S(r,f) \\
&= N\left(r, \frac{1}{f(z+c)-d}\right) - N\left(r, \frac{1}{f'-d}\right) + S(r,f) \\
&= T\left(r, \frac{1}{f(z+c)-d}\right) - m\left(r, \frac{1}{f(z+c)-d}\right) \\
&- T\left(r, \frac{1}{f'-d}\right) + m\left(r, \frac{1}{f'-d}\right) + S(r,f) \\
&= T(r, f(z+c)) - T(r, f') + S(r,f) = S(r,f).
\end{aligned}$$

Combining this equation and (4.11), we get

$$m\left(r, \frac{f(z+c)-d}{f'-d}\right) = m\left(r, \frac{f'-d}{f(z+c)-d}\right) + S(r,f) = S(r,f). \tag{4.13}$$

Theorem 4.1. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $f'(z)$ and $f(z+c)$ share a, b IM and satisfy $f(z+c) = \infty \leftarrow f' = \infty$, and if $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$, then $f'(z) = f(z+c)$.*

As a corollary of Theorem 4.1, we are easy to get the following result:

Theorem 4.2. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $f'(z)$ and $f(z+c)$ share ∞ CM and a, b IM, and if $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$, then $f'(z) = f(z+c)$.*

Question. If we omit the condition that ¡°$N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$¡±, would Theorems 4.1 and 4.2 still valid?

Proof of Theorem 4.1. Suppose that $f'(z) \not\equiv f(z+c)$. Set

$$U(z) = \frac{f''(f(z+c)-f')}{(f'-a)(f'-b)}. \tag{4.14}$$

Using the same argument of $H(z)$, we have that $U(z) \not\equiv 0$ and $N(r,U) = S(r,f)$. Further, from the lemma on the logarithmic derivative and the conclusion (3) of

Proposition 4.1, we have

$$\begin{aligned} m(r,U) &= m\left(r, \left(\frac{(a-d)f''}{(a-b)(f'-a)} - \frac{(b-d)f''}{(a-b)(f'-b)}\right)\left(\frac{f(z+c)-d}{f'-d} - 1\right)\right) \\ &\le m\left(r, \frac{f(z+c)-d}{f'-d}\right) + S(r,f) \\ &= S(r,f). \end{aligned}$$

Hence,

$$T(r,U) = S(r,f). \tag{4.15}$$

Define $S_{F\sim G(m,n)}(\alpha)$ for the set of those points $z \in \mathbb{C}$ such that z is an α-point of F with multiplicity m and an α-point of G with multiplicity n. Let $N_{(m,n)}(r, \frac{1}{F-\alpha})$ and $\overline{N}_{(m,n)}(r, \frac{1}{F-\alpha})$ denote the counting function and reduced counting function of F with respect to the set $S_{F\sim G(m,n)}(\alpha)$, respectively.

Let $z_1 \in S_{f'\sim f(z+c)(m,n)}(a)$. Substituting the Taylor expansion of f' and $f(z+c)$ at z_1 into (4.3), (4.14), by calculating carefully, we conclude that $mH(z_1) - nU(z_1) = 0$.

If $mH = nU$ for some m, n, then we have

$$m\left(\frac{f'(z+c)}{f(z+c)-a} - \frac{f'(z+c)}{f(z+c)-b}\right) = n\left(\frac{f''}{f'-a} - \frac{f''}{f'-b}\right). \tag{4.16}$$

Hence,

$$\left(\frac{f(z+c)-a}{f(z+c)-b}\right)^m = A\left(\frac{f'-a}{f'-b}\right)^n,$$

where A is a non-zero constant. Suppose $m \neq n$, then from Lemma 2.3, we get

$$nT(r,f') = mT(r, f(z+c)) + S(r,f),$$

which contradicts the conclusion (1) of Proposition 4.1. Hence, $m = n$. From (4.1), (4.2), (4.16) and Lemma 2.8, it follows that

$$2T(r, f(z+c)) \le \overline{N}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b}\right) + S(r,f),$$

which contradicts the conclusion (2) of Proposition 4.1.

If $mH \not\equiv nU$ for all m, n, then we get

$$\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) \le N\left(r, \frac{1}{mH-nU}\right) = S(r,f).$$

Similarly, we also get

$$\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right) \le N\left(r, \frac{1}{mH-nU}\right) = S(r,f).$$

Hence,

$$\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right) = S(r,f). \tag{4.17}$$

From (4.17) and the conclusions (1)-(2) of Proposition 4.1, we get

$$
\begin{aligned}
&T(r, f(z+c)) \\
&= \overline{N}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b}\right) + S(r,f) \\
&= \sum_{m,n} \left(\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right)\right) + S(r,f) \\
&= \sum_{m+n\geq 5} \left(\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right)\right) + S(r,f) \\
&\leq \frac{1}{5}\sum_{m+n\geq 5} \left(N_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) + N_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right)\right. \\
&\left. + N_{(m,n)}\left(r, \frac{1}{f'-a}\right) + N_{(m,n)}\left(r, \frac{1}{f'-b}\right)\right) + S(r,f) \\
&\leq \frac{4}{5} T(r, f(z+c)) + S(r,f),
\end{aligned}
$$

which is a contradiction. Therefore, $f'(z) = f(z+c)$.

5. Some partially shared values results

In this part, we will give two partially shared values results related to theorems B and C.

Theorem 5.1. *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a IM, and satisfy $f(z+c) = 0 \to f' = 0$, $f(z+c) = \infty \leftarrow f' = \infty$, then $f'(z) = f(z+c)$.*

Proof of Theorem 5.1. Set

$$F(z) = \frac{f'}{f(z+c)}. \tag{5.1}$$

If $F(z) \equiv 1$, then we have $f' = f(z+c)$. In the following, we suppose that $F(z) \not\equiv 1$. Then, by the the same argument of Theorem 3.1, we know $F(z)$ also satisfy

$$T(r, F) = S(r, f). \tag{5.2}$$

In addition, from (5.1)-(5.2), and Lemma 2.5, it follows that

$$T(r, f') = T(r, f(z+c)) + S(r,f) = T(r,f) + S(r,f). \tag{5.3}$$

Hence,

$$S(r, f') = S(r, f(z+c)) = S(r,f). \tag{5.4}$$

Further, since f' and $f(z+c)$ share a IM, we get

$$\begin{aligned}(5.5)\qquad &\overline{N}\left(r,\frac{1}{f'-a}\right)=\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)\leq\overline{N}\left(r,\frac{1}{\frac{f'}{f(z+c)}-1}\right)+S(r,f)\\ &=\overline{N}\left(r,\frac{1}{F-1}\right)+S(r,f)\leq T(r,F)+S(r,f)=S(r,f).\end{aligned}$$

Set

$$G(z)=\frac{f''}{f'-a}-\frac{f'(z+c)}{f(z+c)-a}. \tag{5.6}$$

In the following, we distinguish two cases.

Case 1. If $G(z)\equiv 0$, then we have

$$f'-a=A(f(z+c)-a), \tag{5.7}$$

where A is a non-zero constant.

If $A=1$, then $f'=f(z+c)$. In the following, we suppose that $A\neq 1$, then by (5.7) and $A\neq 1$, we can immediately get $f(z+c)\neq 0$. Hence, 0 is a Picard value of $f(z+c)$, then it follows from (5.4)-(5.5), Lemma 2.6 and the second main theorem that

$$\begin{aligned}T(r,f(z+c))&\leq\overline{N}\left(r,\frac{1}{f(z+c)}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)\\ &+\overline{N}(r,f(z+c))+S(r,f(z+c))=S(r,f(z+c)),\end{aligned}$$

which is a contradiction.

Case 2. If $G(z)\not\equiv 0$, then by (5.4) and the lemma of logarithmic derivative, we obtain

$$m(r,G)=S(r,f).$$

Further, by (5.5) and Lemma 2.6, we get

$$\begin{aligned}N(r,G)&\leq\overline{N}(r,f')+\overline{N}\left(r,\frac{1}{f'-a}\right)\\ &+\overline{N}(r,f(z+c))+\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+S(r,f)=S(r,f).\end{aligned}$$

Therefore,

$$T(r,G)=S(r,f).$$

According to (5.1), we have

$$f''=F'f(z+c)+Ff'(z+c). \tag{5.8}$$

Substituting (5.1) and (5.8) into (5.6), we get

$$G(z)=\frac{F'f(z+c)+Ff'(z+c)}{Ff(z+c)-a}-\frac{f'(z+c)}{f(z+c)-a},$$

which means that

$$(5.9)\quad (FG-F')f^2(z+c)+(aF'-aG(1+F))f(z+c)+a^2G=a(1-F)f'(z+c).$$

If $FG-F'\not\equiv 0$, then by Lemma 2.6 and (5.9), we have

$$\begin{aligned}2T(r,f(z+c)) &= T(r,f'(z+c))+S(r,f(z+c))\\ &\le T(r,f(z+c))+\overline{N}(r,f(z+c))+S(r,f(z+c))\\ &\le T(r,f(z+c))+S(r,f(z+c)),\end{aligned}$$

and so, $T(r,f(z+c))=S(r,f(z+c))$, which is impossible. Hence, $FG-F'\equiv 0$. Namely,

$$\frac{F'}{F}=G=\frac{f''}{f'-a}-\frac{f'(z+c)}{f(z+c)-a},$$

which implies that

$$(5.10)\qquad \frac{f'}{f(z+c)}=B\frac{f'-a}{f(z+c)-a},$$

where B is a non-zero constant. Note that f' and $f(z+c)$ share a IM.

If a is a picard value of f' and $f(z+c)$, then f' and $f(z+c)$ share a CM. From Theorem B, we have $f'=f(z+c)$.

If a is not a picard value of f' and $f(z+c)$, then compare both side of (5.10), we also get f' and $f(z+c)$ share a CM. Otherwise, suppose z_2 is a common zero of $f'-a$ and $f(z+c)-a$, then from (5.10), we have $1=\frac{f'(z_2)}{f(z_2+c)}=0$ or $1=\frac{f'(z_2)}{f(z_2+c)}=\infty$, which is impossible. Hence, from Theorem B, the conclusion holds as well.

Theorem 5.2. *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1, and let a,b be two distinct non-zero constants. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c)=a\to f'=a$, $f(z+c)=b\to f'=b$, $f(z+c)=\infty\leftarrow f'=\infty$ and $\delta(0,f)>0$, then $f'(z)=f(z+c)$.*

Here, we define $\delta(0,f)$ as following

$$\delta(0,f)=1-\limsup_{r\to\infty}\frac{N(r,\frac{1}{f})}{T(r,f)}.$$

Question. If we omit the condition that ¡°$\delta(0,f)>0$¡±, is Theorem 5.2 still valid?

Proof of Theorem 5.2. Suppose that $f'(z)\not\equiv f(z+c)$. Set

$$(5.11)\qquad F(z)=\frac{f'}{f(z+c)}.$$

Similarly as above, we know $F(z)\not\equiv 1$. And the equation (3.2) also holds, namely,

$$m(r,F)=S(r,f).$$

Combining this equation, Lemma 2.5 and the assumption that $f(z+c)=\infty \leftarrow f'=\infty$, it follows that

$$
\begin{aligned}
T(r,F) =& N(r,F)+S(r,f) \le N\left(r,\frac{1}{f(z+c)}\right)+S(r,f) \\
=& N\left(r,\frac{1}{f}\right)+S(r,f).
\end{aligned} \tag{5.12}
$$

Moreover, by the assumption that $f(z+c)=a \rightarrow f'=a$, $f(z+c)=b \rightarrow f'=b$ and (5.12), we get that

(5.13)

$$
\begin{aligned}
&N\left(r,\frac{1}{f(z+c)-a}\right)+N\left(r,\frac{1}{f(z+c)-b}\right) \le N\left(r,\frac{1}{\frac{f'}{f(z+c)}-1}\right)+S(r,f) \\
&= N\left(r,\frac{1}{F-1}\right)+S(r,f) \le T(r,F)+S(r,f) \le N\left(r,\frac{1}{f}\right)+S(r,f).
\end{aligned}
$$

Therefore, from Lemmas 2.4-2.5, the assumption that $f(z+c)=\infty \leftarrow f'=\infty$ and (5.13), we get

$$
\begin{aligned}
T(r,f) = T(r,f(z+c))+S(r,f) &\le (N(r,f')-N(r,f(z+c))) \\
&+N\left(r,\frac{1}{f(z+c)-a}\right)+N\left(r,\frac{1}{f(z+c)-b}\right)+S(r,f) \\
&\le N(r,\frac{1}{f})+S(r,f),
\end{aligned}
$$

which contradicts the assumption that $\delta(0,f)>0$. Therefore, $f'(z)=f(z+c)$.

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О РАВНОМЕРНОЙ СХОДИМОСТИ РЯДОВ ФУРЬЕ ПО ДВОЙНОЙ СИСТЕМЕ УОЛША ПО СФЕРАМ

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Аннотация. В данной работе построена интегрируемая функция U двух переменных, коэффициенты Фурье по двойной системе Уолша которой на спектре положительны и расположены в убывающем порядке по всем направлениям, и для каждой почти везде конечной измеримой функции $f(x,y)$, $(x,y) \in [0,1)^2$ и для любого $\delta > 0$ можно найти ограниченную функцию $g(x,y)$ с

$$|\{(x,y) \in [0,1)^2 : \ g(x,y) \neq f(x,y)\}| \leq \delta,$$

и такую,что $|c_{k,s}(g)| = c_{k,s}(U)$ на спектре функции g и ее сферические частичные суммы ряда Фурье по двойной системе Уолша сходятся равномерно на $[0,1)^2$.

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Ключевые слова: сферическая частичная сумма; двойной ряд Фурье; равномерная сходимость; система Уолша.

1. Введение

Пусть $\{W_k(x)\}_{n=0}^{\infty}$– система Уолша (см. например [1], стр. 12), и пусть $c_k(g) = \int_0^1 g(x)W_k(x)dx, k \geq 0$– коэффициенты Фурье-Уолша функции $g \in L^1[0,1]$. Положим $spec(f) = \{k \in N \cup \{0\} : c_k(f) \neq 0\}$ и $S_n(x,g) = \sum_{k=0}^{n} c_k(g)W_k(x)$, где N множество натуральных чисел.

Система Уолша одна из популярных ортонормированных систем, которая является базисом во всех пространствах $L^p[0,1), p \in (1,\infty)$.

В ряде работ изучалась сходимость рядов Фурье по системе Уолша. Приведем те результаты, которые связаны с настоящей работой.

В [4] М. Г. Григоряном доказанно существование (**универсальной**) функции $U \in L^1[0,1)$, которая относительно системы Уолша обладает универсальным (L^1, L^∞) свойством, а именно, он доказал следующую теорему:

[1]Исследование выполнено при финансовой поддержке Комитета по науке РА в рамках научного проекта № 21AG-1A066

Теорема 1.1. *Существует функция со строго убывающими коэффициентами Фурье-Уолша, обладающая следующим свойством: для каждой почти везде конечной измеримой на* $[0,1)$ *функции* f *и для любого* $\delta > 0$ *можно найти функцию* $g \in L^{\infty}[0,1)$ *с* $|\{x \in [0,1):\ g(x) \neq f(x)\}| \leq \delta$*, такую,что ее ряд Фурье по системе Уолша сходится равномерно на* $[0,1)$ *и такую,что* $|c_k(g)| = c_k(U),\ k \in spec(g)$.

Заметим, что идея исправления функции с целью улучшения ее свойств принадлежит Н. Н. Лузину [6]. Широко известно также и усиленное С-свойство Д. Е. Меньшова [7]. Отметим, что в работах [8] – [11] были получены некоторые результаты связанные с существованием и описанием структуры функций, ряды Фурье которых по системе Уолша и по тригонометрической системе универсальны в том или ином смысле в различных функциональных классах.

В этой статье мы рассмотрим вопрос: можно ли получить результат аналогичный Теореме 1 в двумерном случае.

Пусть $T = [0,1]^2$, и пусть $f \in L^p(T), p \in [1,\infty)$. Коэффициенты Фурье функции f по двойной системе Уолша $\{W_k(x)W_s(y)\}_{k,s=0}^{\infty}$ обозначим через

$$c_{k,s}(f) = \iint\limits_T f(t,\tau)W_k(t)W_s(\tau)dtd\tau \tag{1.1}$$

Положим

$$\Lambda(f) = spec\{c_{k,s}(f)\} = spec(f) = \{(k,s) : c_{k,s}(f) \neq 0;\ k,s \in N \cup \{0\}\}. \tag{1.2}$$

Прямоугольные и сферические частичные суммы двойного ряда Фурье - Уолша определяются соответственно следующим образом:

$$S_{N,M}(x,y,f) = \sum_{k=0}^{N}\sum_{s=0}^{M} c_{k,s}(f)W_k(x)W_s(y), \tag{1.3}$$

$$S_R(x,y,f) = \sum_{k^2+s^2 \leq R^2} c_{k,s}(f)W_k(x)W_s(y). \tag{1.4}$$

Говорят, что двойной ряда Фурье-Уолша функции $f \in L^1[0,1)^2$ сходится в $L^p[0,1)^2, p > 0$ по прямоугольникам (по сферам), если

$$\lim_{N\to\infty, M\to\infty} \iint\limits_T |S_{N,M}(x,y,f) - f(x,y)|^p\, dxdy = 0,$$

(соотвественно, если)

$$\lim_{R\to\infty} \iint\limits_T |S_R(x,y,f) - f(x,y)|^p\, dxdy = 0.$$

Аналогично определяются почти всюду и равномерная сходимость по прямоугольникам и по сферам.

Отметим, что ряд классических результатов (такие теоремы, как теорема Л. Карлесона [15]: ряд Фурье любой функции $f \in L^2[0,2\pi]$ сходится почти всюду на $[0,2\pi]$, теорема М. Рисса [14]: ряд Фурье любой функции $f \in L^p[0,2\pi], p > 1$ сходится по норме $L^p[0,2\pi]$, теорема А. М. Колмогорова [13]: ряд Фурье каждой функции $f \in L[0,2\pi]$ сходится в метрике $L^p[0,2\pi], 0 < p < 1$) невозможно перенести с одномерного случая на двумерный. В этом случае даже разные (сферические, прямоугольные, квадратные частичные суммы резко отличаются друг от друга по своим свойствам в таких вопросах, как сходимости в $L^p[0,2\pi], p \geq 1$ и сходимости почти всюду (см. [25-34]). В работе [16], Фефферманом получены следующие результаты:

Теорема 1.2. *Для любого $p \neq 2$ существует такая функция из класса $L^p(0,2\pi)^2$, что сферические частичные суммы ряда Фурье по тригонометрической системе этой функции не сходятся по норме L^p.*

Теорема 1.3. *Существует непрерывная функция двух переменных, со всюду расходящимися прямоугольными частичными суммами ряда Фурье по тригонометрической системе.*

В работе [17] Д. Г. Харрис доказал, что для любого $p \in [1,2)$ существует такая функция из $L^p(0,1)^2$, что сферические частичные суммы ряда Фурье-Уолша этой функции расходятся почти всюду и по $L^p[0,1)^2$ норме.

В работе [18] М. Г. Григоряном доказано существование функции $f_0 \in L^1(0,2\pi)^2$, двойной ряд Фурье которой по тригонометрической системе по сферам расходится в метриках $L^p(0,2\pi)^2$ для любого $p \in (0,1)$.

В работе [19] Р. Д. Гецадзе доказал, что существует непрерывная функция прямоугольные частичные суммы двойного ряда Фурье-Уолша которой расходятся почти всюду.

Отметим, что до сих по неизвестно сходятся ли почти всюду сферические частичные суммы двойного ряда Фурье-Уолша каждой непрерывной функции?

Определение 1.1. *Будем говорить, что члены в двойной последовательности $\{c_{k,s}(f)\}_{k,s=0}^{\infty}$ на спектре $\Lambda(f)$ расположены в убывающем порядке, если $c_{k_2,s_2}(f) < c_{k_1,s_1}(f)$, когда $k_2 \geq k_1, s_2 \geq s_1, k_2 + s_2 > k_1 + s_1, \ \ (k_2,s_2),(k_1,s_1) \in \Lambda(f)$.*

В этой работе доказывается

Теорема 1.4. *Существует функция* $U \in L^1[0,1)^2$ *такая, что*

а) коэффициенты Фурье функции U *по двойной системе Фурье-Уолша на ее спектре положительны и расположены в убывающем порядке,*

б) для каждой почти везде конечной измеримой на $[0,1)^2$ *функции* f *и для любого* $\delta > 0$ *можно найти функцию* $g \in L^\infty[0,1)^2$ *с*

$$|\{(x,y) \in [0,1)^2 : \ g(x,y) \neq f(x,y)\}| \leq \delta,$$

сферические частичные суммы ряда Фурье которой по двойной системе Уолша сходятся равномерно на $[0,1)^2$,

в) $|c_{k,s}(g)| = c_{k,s}(U), \ (k,s) \in spec(g)$.

2. Доказательство лемм

Для краткости записи условимся употреблять следующие обозначения: $\|f\|_\infty \doteq \sup\limits_{x,y\in[0,1)^2} |f(x,y)|$, $\|f\|_1 \doteq \int_{[0,1)^2} |f(x,y)|dxdy$ (те же самые обозначения будут применяться для $\sup\limits_{x\in[0,1)} |f(x)|$ и $\int_0^1 |f(x)|dx$). Под двоичным прямоугольником мы будем понимать декартово произведение $\Delta_1 \times \Delta_2$, где $\Delta_i, \ i = 1,2$ двоичные полуинтервалы вида $\Delta_s^{(\nu)} = [\frac{\nu-1}{2^s}, \frac{\nu}{2^s}), \ 1 \leq \nu \leq 2^s, \ s \geq 1$.

Мы будем использовать следующую лемму, доказанную в [4].

Лемма 2.1. *Для любых чисел* $k_0 \in N, \gamma_0 \neq 0, \varepsilon_0, \eta_0, \delta_0 \in (0,1), \epsilon_0 \in (0, \frac{|\gamma_0|}{\delta_0})$ *и для любого двоичного полуинтервала* $\Delta_0 = [\frac{\nu-1}{2^s}, \frac{\nu}{2^s})$, $1 \leq \nu \leq 2^s$, $s \geq 1$, *можно найти измеримое множество* $E \subset \Delta_0$, *функцию* $g(x)$, *полиномы* $H(x)$ *и* $Q(x)$ *вида*

$$H(x) = \sum_{l=2^{k_0}}^{2^k-1} b_l W_l(x) \ , \quad Q(x) = \sum_{l=2^{k_0}}^{2^k-1} \varepsilon_l b_l W_l(x),$$

обладающие следующими свойствами:

1) $\varepsilon_l = \pm 1$ *или* $0, \ 0 < b_{l+1} < b_l < \epsilon_0, \ \forall l \in [2^{k_0}, 2^k)$,

2) $\int_0^1 |H(x)|dx < \eta_0$,

3) $|E| > (1-\delta_0)|\Delta_0|$,

4) $g(x) = \begin{cases} \gamma_0, & x \in E \\ 0, & x \notin \Delta_0 \end{cases}$,

5) $\|g - Q\|_\infty < \varepsilon_0$, *б)* $\max\limits_{2^{k_0}\le n<2^k}\left\|\sum\limits_{l=2^{k_0}}^{n}\varepsilon_l b_l W_l(x)\right\|_\infty<\dfrac{3|\gamma_0|}{\delta_0}.$

Лемма 2.2. *Для любых чисел $n_0\in N$, $\gamma\neq 0$, ε, $\delta,\epsilon,\eta\in(0,1)$ и для любого двоичного прямоугольника $\Delta=\Delta_1\times\Delta_2$ существуют измеримое множество $E\subset\Delta$, функция $g(x,y)$, полиномы $H(x,y)$ и $Q(x,y)$ вида*

$$H(x,y)=\sum_{k=2^{n_0}}^{2^{n_1}-1}\sum_{s=2^{m_0}}^{2^{m_1}-1}c_{k,s}W_k(x)W_s(y),$$

$$Q(x,y)=\sum_{k=2^{n_0}}^{2^{n_1}-1}\sum_{s=2^{m_0}}^{2^{m_1}-1}\varepsilon_{k,s}c_{k,s}W_k(x)W_s(y),\quad \varepsilon_{k,s}=\pm1,$$

обладающие следующими свойствами: коэффициенты $c_{k,s}$, $k\in[2^{n_0},2^{n_1}),s\in[2^{m_0},2^{m_1})$ расположены в убывающем порядке и

$$0<c_{k,s}<\varepsilon, \tag{1}$$

$$|E|>(1-\delta)|\Delta|, \tag{2}$$

$$\|H\|_1<\eta, \tag{3}$$

$$g(x,y)=\begin{cases}\gamma, & (x,y)\in E\\ 0, & x\notin\ \Delta\end{cases}, \tag{4}$$

$$\|g\|_\infty\le\frac{9}{\delta^2}|\gamma|, \tag{5}$$

$$\|Q-g\|_\infty<\varepsilon, \tag{6}$$

$$\max_{N_0^2+M_0^2\le R^2\le N_1^2+M_1^2}\left\|\sum_{N_0^2+M_0^2\le k^2+s^2\le R^2}\varepsilon_{k,s}c_{k,s}W_k(x)W_s(y)\right\|_\infty\le\frac{22}{\delta^2}|\gamma|, \tag{7}$$

где $N_0=2^{n_0}$, $M_0=2^{m_0}, N_1=2^{n_1}-1$, $M_1=2^{m_1}-1$.

Доказательство Леммы 2.2. Применяя лемму 2.1, полагая в ее формулировке

$$\gamma_0=\gamma,\ \Delta_0=\Delta_1,\ k_0=n_0,\quad \eta_0=\sqrt{\eta},\ \epsilon_0=\sqrt{\epsilon},\ \delta_0=\frac{\delta}{2},\ \varepsilon_0=\frac{\varepsilon\delta}{16},$$

определим измеримое множество $E_1\subset\Delta_1$, функцию $g_1(x)$, полиномы $H_1(x)$ и $Q_1(x)$ вида

$$H_1(x)=\sum_{k=2^{n_0}}^{2^{n_1}-1}b_k^{(1)}W_k(x)\ =\sum_{k=N_0}^{N_1}b_k^{(1)}W_k(x), \tag{2.1}$$

$$Q_1(x)=\sum_{k=2^{n_0}}^{2^{n_1}-1}\varepsilon_k^{(1)}b_k^{(1)}W_k(x) \ = \sum_{k=N_0}^{N_1}\varepsilon_k^{(1)}b_k^{(1)}W_k(x), \tag{2.2}$$

где

$$\varepsilon_k^{(1)}=0,\pm1 \ \ \forall k\in[2^{n_0},2^{n_1})=[N_0,N_1],$$

удовлетворяющие условиям:

$$0<b_{k+1}^{(1)}<b_k^{(1)}<\sqrt{\epsilon}, \quad \forall k\in[N_0,N_1), \tag{2.3}$$

$$|E_1|>(1-\delta/2)|\Delta_1|, \ \ \int_0^1|H_1(x)|dx<\sqrt{\eta}, \tag{2.4}$$

$$g_1(x)=\begin{cases}\gamma, & x\in \ E_1\\ 0, & x\notin \ \Delta_1\end{cases}, \tag{2.5}$$

$$\max_{N_0\le n\le N_1}\left\|\sum_{k=N_0}^{m}\varepsilon_k^{(1)}b_k^{(1)}W_k(x)\right\|_\infty\le\frac{2}{\delta}|\gamma|, \tag{2.6}$$

$$\|Q_1-g_1\|_\infty<\min\{|\gamma|;\delta\varepsilon/4\}. \tag{2.7}$$

Положим

$$M_0=2^{m_0}>N_1^2+1 \tag{2.8}$$

Снова применим Лемму 2.1, полагая в ее формулировке

$$\gamma_0=1,\Delta_0=\Delta_2, \ \ k_0=m_0, \quad \epsilon_0=\sqrt{\epsilon}\ , \ \delta_0=\frac{\delta}{2}, \ \varepsilon_0=\frac{\varepsilon}{4\|Q_1\|_\infty}.$$

Тогда определяются измеримое множество $E_2\subset[0,1]$, функция $g_2(y)$, полином $H_2(y)$ и $Q_2(y)$ вида

$$H_2(y)=\sum_{k=2^{m_0}}^{2^{m_1}-1}b_k^{(2)}W_k(y) \ = \sum_{k=M_0}^{M_1}b_k^{(2)}W_k(y), \tag{2.9}$$

$$Q_2(y)=\sum_{k=2^{m_0}}^{2^{m_1}-1}\varepsilon_s^{(2)}b_s^{(2)}W_s(y)=\sum_{k=M_0}^{M_1}\varepsilon_s^{(2)}b_s^{(2)}W_s(y), \tag{2.10}$$

$$\varepsilon_s^{(2)}=0,\pm1 \ \ \forall s\in[2^{m_0},2^{m_1})=[M_0,M_1]$$

удовлетворяющие условиям:

$$0<b_{s+1}^{(2)}<b_s^{(2)}<\sqrt{\epsilon}, \ \ s\in[M_0,M_1), \tag{2.11}$$

$$|E_2|>(1-\frac{\delta}{2})|\Delta_2|, \ \ \int_0^1|H_2(y)|dy<\sqrt{\eta}, \tag{2.12}$$

$$g_2(y) = \begin{cases} 1: & y \in E_2 \\ 0: & y \notin \Delta_2 \end{cases}, \tag{2.13}$$

$$\|Q_2(y) - g_2(y)\|_\infty < \frac{\varepsilon}{4(\|Q_1\|_\infty + 1)}, \tag{2.14}$$

$$\max_{M_0 \le m \le M_1} \left\| \sum_{k=M_0}^{m} \varepsilon_k^{(2)} b_k^{(2)} W_k(y) \right\|_\infty \le \frac{2}{\delta}. \tag{2.15}$$

Определим множество E, функцию $g(x,y)$, полиномы $H(x,y)$ и $Q(x,y)$ следующим образом:

$$g(x,y) = g_1(x) g_2(y), \quad E = E_1 \times E_2, \tag{2.16}$$

$$H(x,y) = H_1(x) H_2(y) = \sum_{k,s=N_0,M_0}^{N_1,M_1} c_{k,s} W_k(x) W_s(y), \tag{2.17}$$

$$Q(x,y) = Q_1(x) Q_2(y) = \sum_{k,s=N_0,M_0}^{N_1,M_1} \varepsilon_{k,s} c_{k,s} W_k(x) W_s(y) =$$

$$= \sum_{k=N_0}^{N_1} \varepsilon_k^{(1)} b_k^{(1)} W_k(x) \sum_{s=M_0}^{M_1} \varepsilon_s^{(2)} b_s^{(2)} W_s(y), \tag{2.18}$$

где

$$c_{k,s} = \begin{cases} b_k^{(1)} b_s^{(2)}, & N_0 \le k \le N_1,\ M_0 \le s \le M_1 \\ 0, & k \notin [N_0, N_1], \quad n \notin [M_0, M_1] \end{cases}, \tag{2.19}$$

$$\varepsilon_{k,s} = \begin{cases} \varepsilon_k^{(1)} \varepsilon_s^{(2)}, & N_0 \le k \le N_1,\ M_0 \le s \le M_1 \\ 0, & k \notin [N_0, N_1], \quad n \notin [M_0, M_1] \end{cases}. \tag{2.20}$$

Отсюда и из (2.3), (2.4), (2.11), (2.17) и (2.19) следует, что члены в последовательности $\{c_{k,s},\ N_0 \le k \le N_1,\ \ M_0 \le s \le M_1\}$ положительны, расположены в убывающем порядке и

$$0 < c_{k,s} < \epsilon, \qquad |E| > (1-\delta)|\Delta|,$$

$$\iint_T |H(x,y)| dx dy = \int_0^1 |H_1(x)| dx \int_0^1 |H_1(y)| dy < \eta.$$

Из (2.5)-(2.7), (2.13)-(2.15) и (2.16) следует

$$g(x,y) = \begin{cases} \gamma, & (x,y) \in E \\ 0, & x \notin \Delta \end{cases},$$

$$\|g\|_\infty \le \|g_1\|_\infty \cdot \|g_2\|_\infty \le \frac{9}{\delta^2} |\gamma|.$$

В силу (2.7), (2.14)-(2.16) и (2.18) для всех $(x,y)\in[0,1)^2$ имеем

$$|Q(x,y)-g(x,y)|\leq|Q_2(y)-g_2(y)|\cdot Q_1(x)+$$
$$+|Q_1(x)-g_1(x)|\cdot g_2(y)\leq\varepsilon.$$

Теперь проверим выполнение утверждения 5). Пусть $N_0^2+M_0^2<R^2<N_1^2+M_1^2$, тогда для некоторого p_0 имеем $p_0\leq R<p_0+1$. Из (2.8), (2.18)-(2.20) следует $R^2-N_1^2\geq(p_0-1)^2$ и, следовательно, получим

$$\max_{N_0^2+M_0^2\leq R^2\leq N_1^2+M_1^2}\left\|\sum_{N_0^2+M_0^2\leq k^2+s^2\leq R^2}\varepsilon_{k,s}c_{k,s}W_k(x)W_s(y)\right\|_\infty\leq$$
$$\leq\left\|\sum_{k=N_0}^{N_1}\sum_{s=M_0}^{p_0-1}\varepsilon_{k,s}c_{k,s}W_k(x)W_s(y)\right\|_\infty+\left\|\sum_{k=N_0}^{l}\varepsilon_{k,p_0}c_{k,p_0}W_k(x)W_{p_0}(y)\right\|_\infty=$$
$$=\left\|\sum_{k=N_0}^{N_1}\varepsilon_k^{(1)}b_k^{(1)}W_k(x)\right\|_\infty\cdot\left\|\sum_{s=M_0}^{p_0-1}\varepsilon_s^{(2)}b_s^{(2)}W_s(y)\right\|_\infty+$$
$$+|b_{m_0}^{(2)}|\cdot\max_{N_0\leq m\leq N_1}\left\|\sum_{k=N_0}^{m}\varepsilon_k^{(1)}b_k^{(1)}W_k(x)\right\|_\infty<\frac{12}{\delta^2}|\gamma|.$$

Лемма 2.2 доказана.

Лемма 2.3. *Для любых чисел $\epsilon,\ \varepsilon,\delta,\eta\in(0,1)$, $n_0\in N$ и для любого полинома $f(x,y)\neq0,\ (x,y)\in[0,1)^2$ по двойной системе Уолша можно найти измеримое множество $E\subset[0,1)^2$, функцию $g(x,y)$, полиномы $H(x,y)$ и $Q(x,y)$ вида*

$$H(x,y)=\sum_{k=2^{n_0}}^{2^{n_1}-1}\sum_{s=2^{m_0}}^{2^{m_1}-1}c_{k,s}W_k(x)W_s(y),$$

$$Q(x,y)=\sum_{k=2^{n_0}}^{2^{n_1}-1}\sum_{s=2^{m_0}}^{2^{m_1}-1}\varepsilon_{k,s}c_{k,s}W_k(x)W_s(y),\quad\varepsilon_{k,s}=\pm1,$$

где $N_0=2^{n_0},M_0=2^{m_0}$, обладающие следующими свойствами: коэффициенты $c_{k,s},\ k\in[2^{n_0},2^{n_1}),s\in[2^{m_0},2^{m_1})$ расположены в убывающем порядке и

(1) $$0<c_{k,s}<\epsilon,$$

(2) $$g(x,y)=f(x,y),\quad\forall(x,y)\in E,\quad|E|>1-\delta,$$

(3) $$\|Q-g\|_\infty<\ \varepsilon,$$

(4) $$\|H\|_1<\eta,$$

(5) $$\|g\|_\infty\leq\frac{9}{\delta^2}\|f\|_\infty,$$

$$(6)\qquad \max_{N_0^2+M_0^2\le R^2\le N^2+M^2}\left\|\sum_{N_0^2+M_0^2\le k^2+s^2\le R^2}\varepsilon_{k,s}\,c_{k,s}\,W_k(x)\,W_s(y)\right\|_\infty \le \frac{22}{\delta^2}\|f\|_\infty,$$

где $N_0=2^{n_0},\ M_0=2^{m_0}, N=2^{\bar n}-1,\ M=2^{\bar m}-1.$

Доказательство Леммы 2.3. Ясно, что полином по двойной системе Уолша $f(x,y)$ есть ступенчатая функция вида

$$(2.21)\qquad f(x,y)=\sum_{\nu=1}^{\nu_0}\gamma_\nu\chi_{\Delta_\nu}(x,y)$$

где $\Delta_\nu=[\frac{\nu_1-1}{2^{s_1}},\frac{\nu_1}{2^{s_1}})\times[\frac{\nu_2-1}{2^{s_2}},\frac{\nu_2}{2^{s_2}})$ и $\gamma_\nu\neq 0,\ 1\le\nu<\nu_0$. ($\chi_E(x,y)$- характеристическая функция множества E). Последовательным применением леммы 2.2, для каждого $\nu=1,2,...,\nu_0$ можно определить функции $g_\nu(x,y)$, множества $E_\nu\subset\Delta_\nu$, натуральные числа n_ν,m_ν и полиномы вида

$$(2.22)\qquad H_\nu(x,y)=\sum_{k=N_{\nu-1}}^{N_\nu-1}\sum_{s=M_{\nu-1}}^{M_\nu-1}c_{k,s}^{(\nu)}W_k(x)W_s(y),$$

$$(2.23)\qquad Q_\nu(x,y)=\sum_{k=N_{\nu-1}}^{N_\nu-1}\sum_{s=M_{\nu-1}}^{M_\nu-1}\varepsilon_{k,s}^{(\nu)}c_{k,s}^{(\nu)}W_k(x)W_s(y),$$

где $N_\nu=2^{n_\nu},M_\nu=2^{m_\nu}$, обладающие следующими свойствами: члены в последовательности

$$\{c_{k,s}^{(\nu)},\ N_{\nu-1}\le k<N_\nu,\ M_{\nu-1}\le s<M_\nu\}$$

положительны, расположены в убывающем порядке и для всех $\nu=1,2,\dots$

$$(2.24)\qquad \max_{N_\nu\le k<N_{\nu+1},M_\nu\le s<M_{\nu+1}}c_{k,s}^{(\nu+1)}<\min_{N_{\nu-1}\le k<N_\nu,M_{\nu-1}\le s<M_\nu}c_{k,s}^{(\nu)},$$

$$(2.25)\qquad |E_\nu|>(1-\delta)\cdot|\Delta_\nu|,$$

$$(2.26)\qquad g_\nu(x,y)=\begin{cases}\gamma_\nu,\ (x,y)\in E_\nu\\ 0,\ \ x\notin\ \Delta_\nu\end{cases},$$

$$(2.27)\qquad |g_\nu(x,y)|\le\frac{9}{\delta^2}|\gamma_\nu|,$$

$$(2.28)\qquad \|Q_\nu-g_\nu\|_\infty<\frac{\varepsilon}{\nu_0}\frac{\min_{(x,y)\in([0,1)^2}|f(x,y)|}{\|f\|_\infty},$$

$$(2.29)\qquad \max_{N_{\nu-1}^2+M_{\nu-1}^2\le R^2<N_\nu^2+M_\nu^2}\left|\sum_{N_{\nu-1}^2+M_{\nu-1}^2\le k^2+s^2\le R^2}\varepsilon_{k,s}^{(\nu)}c_{k,s}^{(\nu)}W_k(x)W_s(y)\right|\le\frac{12}{\delta^2}|\gamma_\nu|,$$

$$\|H_\nu\|_1 < \frac{\eta}{\nu_0}. \tag{2.29}$$

Определим множество E, функцию $g(x,y)$, числа $c_{k,s}$, $\varepsilon_{k,s}$ и полиномы $H(x,y)$ и $Q(x,y)$ следующим образом:

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu, \tag{2.30}$$

$$g(x,y) = \sum_{\nu=1}^{\nu_0} g_\nu(x,y), \tag{2.31}$$

$$c_{k,s} = \begin{cases} c_{k,s}^{(\nu)}, \ N_{\nu-1} \le k < N_\nu, \quad M_{\nu-1} \le s < M_\nu, \quad 1 \le \nu \le \nu_0 \\ 0, \quad \text{в остальных случаях} \end{cases}, \tag{2.32}$$

$$\varepsilon_{k,s} = \begin{cases} \varepsilon_{k,s}^{(\nu)}, \ N_{\nu-1} \le k < N_\nu, \quad M_{\nu-1} \le s < M_\nu, \quad 1 \le \nu \le \nu_0 \\ 0, \quad \text{в остальных случаях} \end{cases}, \tag{2.33}$$

$$H(x,y) = \sum_{\nu=1}^{\nu_0} H_\nu(x,y) = \sum_{\nu=1}^{\nu_0} \sum_{k=N_{\nu-1}}^{N_\nu-1} \sum_{s=M_{\nu-1}}^{M_\nu-1} c_{k,s}^{(\nu)} W_k(x) W_s(y) =$$

$$= \sum_{k,s=N_0,M_0}^{N,M} c_{k,s} W_k(x) W_s(y), \tag{2.34}$$

$$Q(x,y) = \sum_{\nu=1}^{\nu_0} Q_\nu(x,y) = \sum_{\nu=1}^{\nu_0} \sum_{k=N_{\nu-1}}^{N_\nu-1} \sum_{s=M_{\nu-1}}^{M_\nu-1} \varepsilon_{k,s}^{(\nu)} c_{k,s}^{(\nu)} W_k(x) W_s(y) =$$

$$= \sum_{k,s=N_0,M_0}^{N,M} \varepsilon_{k,s} c_{k,s} W_k(x) W_s(y), \tag{2.35}$$

где $N = N_{\nu_0} - 1 = 2^{\bar{n}} - 1, M = M_{\nu_0} - 1 = 2^{\bar{m}} - 1$ $(\bar{n} = n_{\nu_0}, \bar{m} = m_{\nu_0})$. В силу (2.24) - (2.26) и (2.30) - (2.35) имеем: все ненулевые члены в последовательности $\{c_{k,s}(H)\ ,\ (k,s) \in spec(H)\}$ положительны, расположены в убывающем порядке и

$$\varepsilon_{k,s} = \pm 1, \quad 0 < c_{k,s} < \epsilon, \quad \forall (k,s) \in [N_0, N) \times [M_0, M), \quad |E| > 1 - \delta,$$

$$g(x,y) = f(x,y), \quad \forall (x,y) \in E, \qquad |Q(x,y) - g(x,y)| < \varepsilon, \ \forall (x,y) \in [0,1)^2$$

$$\iint_T |H(x,y)| dxdy < \sum_{\nu=1}^{\nu_0} \iint_T |H_\nu(x,y)| dxdy < \eta.$$

т.е утверждения 1)-4) выполнены. Теперь проверим выполнение утверждений 5) и 6). Принимая во внимание равенство $g_\nu(x,y) = 0$ при $(x,y) \in ([0,1)^2 \setminus \Delta_\nu)$ $, \forall \nu \in$

$[1,\nu_0]$ (см. (2.26)) в силу (2.21) и (2.27) для всех $(x,y)\in([0,1)^2$ и $\nu\in[1,\nu_0]$ будем иметь

$$\left|\sum_{s=1}^{\nu} g_s(x,y)\right| \le \sum_{s=1}^{\nu} |g_s(x,y)| \le \sum_{s=1}^{\nu_0} \frac{9}{\delta^2}|\gamma_s|\chi_{\Delta_s}(x,y) = \frac{9}{\delta^2}|f(x,y)|. \tag{2.36}$$

Пусть $R\in[N_0^2+M_0^2, N^2+M^2]$, тогда для некоторого $\nu\in[1,\nu_0]$, $N_\nu^2+M_\nu^2\le R^2\le N_{\nu+1}^2+M_{\nu+1}^2$. Ясно, что (см. (2.32), (2.33) и (2.35))

$$\sum_{N_0^2+M_0^2\le k^2+s^2\le R^2} \varepsilon_{k,s}c_{k,s}W_k(x)W_s(y) =$$

$$= \sum_{s=1}^{\nu} Q_s(x,y) + \sum_{N_\nu^2+M_\nu^2\le k^2+s^2\le R^2} \varepsilon_{k,s}^{(\nu)}c_{k,s}^{(\nu)}W_k(x)W_s(y).$$

Отсюда и из соотношений (2.28), (2.29) и (2.36) для всех $(x,y)\in[0,1)^2$ будем иметь

$$\left|\sum_{N_0^2+M_0^2\le k^2+s^2\le R^2} \varepsilon_{k,s}c_{k,s}W_k(x)W_s(y)\right| \le \sum_{s=1}^{\nu} |Q_s(x,y)-g_s(x,y)| +$$

$$+\left|\sum_{s=1}^{\nu} g_s(x,y)\right| + \left|\sum_{N_\nu^2+M_\nu^2\le k^2+s^2\le R^2} \varepsilon_{k,s}^{(\nu)}c_{k,s}^{(\nu)}W_k(x)W_s(y)\right| \le$$

$$\le \frac{\varepsilon}{\nu_0}\nu \min_{(x,y)\in([0,1)^2} |f(x,y)| + \frac{9}{\delta^2}|f(x,y)| + \frac{12|\gamma_\nu|}{\delta^2} \le \frac{22}{\delta^2}|f(x,y)|.$$

Лемма 2.3 доказана.

3. Доказательство Теоремы 2.2.

Пронумеровав все полиномы Уолша с рациональными коэффициентами мы можем представить их в виде последовательности

$$\{f_n(x,y)\}_{n=1}^{\infty}. \tag{3.1}$$

Последовательным применением леммы 2.3 для всех $n\in N$ можно определить функции

$$\{g_n^{(j)}(x,y)\}_{j=1}^{n},$$

множества

$$\{E_n^{(j)}\}_{j=1}^{n},$$

и полиномы

$$\{H_n^{(j)}(x,y)\}_{j=1}^{n},\ \ \{Q_n^{(j)}(x,y)\}_{j=1}^{n},$$

вида

$$H_n^{(j)}(x,y)=\sum_{k,s=N_n^{(j-1)},M_n^{(j-1)}}^{N_n^{(j)}-1,M_n^{(j)}-1} a_{k,s}^{(n,j)}W_k(x)W_s(y),\ 1\le j\le n, \tag{3.2}$$

$$Q_n^{(j)}(x,y)=\sum_{k,s=N_n^{(j-1)},M_n^{(j-1)}}^{N_n^{(j)}-1,M_n^{(j)}-1} \delta_{k,s}^{(n,j)}a_{k,s}^{(n,j)}W_k(x)W_s(y)\ \ (\delta_{k,s}^{(n,j)}=\pm1,0), \tag{3.3}$$

где

$$N_p^{(i_1)}<N_p^{(i_2)},\ \text{если}\ 0\le i_1<i_2\le p\ \text{и}\ N_p^{(p)}<N_{p+1}^{(0)},\ p\ge1,$$

$$M_p^{(i_1)}<M_p^{(i_2)},\ \text{если}\ 0\le i_1<i_2\le p\ \text{и}\ M_p^{(p)}<M_{p+1}^{(0)},\ p\ge1, \tag{3.4}$$

удовлетворяющие условиям: для фиксированных $n\in[1,\infty)$ и $j\in[1,n]$ коэффициенты $a_{k,s}^{(n,j)}$, $(k,s)\in spec(H_n^{(j)})$ положительны, расположены в убывающем порядке и

$$B_n^{(j)}<A_n^{(j-1)},\ 1\le j\le n,\qquad B_n^{(1)}<\frac{A_{n-1}^{(n-1)}}{2^n},n=1,2,...,$$

где

$$A_n^{(j)}=\min_{(\tilde k,\tilde s)\in spec(H_n^{(j)})} a_{\tilde k,\tilde s}^{(n,j)},\quad B_n^{(j)}=\max_{(\tilde k,\tilde s)\in spec(H_n^{(j)})} a_{\tilde k,\tilde s}^{(n,j)}, \tag{3.5}$$

$$g_n^{(j)}(x,y)=f_n(x,y)\ \text{ при }\ (x,y)\in E_n^{(j)},\ \ |E_n^{(j)}|=1-2^{-j-n}, \tag{3.6}$$

$$\|g_n^{(j)}-Q_n^{(j)}\|_\infty<4^{-4n}, \tag{3.7}$$

$$\|g_n^{(j)}\|_\infty<9\cdot4^{j+n}\|f_n\|_\infty, \tag{3.8}$$

$$\max_{(R_n^{(j-1)})^2\le R^2\le(R_n^{(j)})^2}\left\|\sum_{(R_n^{(j-1)})^2\le k^2+s^2\le R^2}\delta_{k,s}^{(n,j)}a_{k,s}^{(n,j)}W_k(x)W_s(y)\right\|<$$

$$<22\cdot4^{j+n}\|f_n\|_\infty,\quad R_n^{(j)}=\sqrt{(N_n^{(j)})^2+(M_n^{(j)})^2}, \tag{3.9}$$

$$\|H_n^{(j)}\|_1<4^{-4n}. \tag{3.10}$$

Определим функцию $U(x,y)$ и числа $a_{k,s}$, $k,s=0,1,2...$ следующим образом:

$$U(x,y)=\sum_{n=1}^\infty\sum_{j=1}^n H_n^{(j)}(x,y)=$$

$$=\sum_{n=1}^\infty\sum_{j=1}^n\left(\sum_{k,s=N_n^{(j-1)},M_n^{(j-1)}}^{N_n^{(j)}-1,M_n^{(j)}-1}a_{k,s}^{(n,j)}W_k(x)W_s(y)\right)=\sum_{k,s=0}^\infty b_{k,s}W_k(x)W_s(y) \tag{3.11}$$

(3.12)
$$b_{k,s}=\begin{cases}a_{k,s}^{(n,j)},\ (k,s)\in[N_n^{(j-1)},N_n^{(j)})\times[M_n^{(j-1)},M_n^{(j)}),\ 1\le j\le n,\quad n\in[1,\infty)\\ 0,\quad \text{в остальных случаях}\end{cases}.$$

В силу (3.10) - (3.12) имеем

$$\iint_T |U(x,y)|dxdy\le\sum_{n=1}^{\infty}\sum_{j=1}^{n}\left(\iint_T |H_n^{(j)}(x,y)|dxdy\right)<\sum_{n=1}^{\infty}\sum_{j=1}^{n}4^{-(n+j)}<1,$$

и

$$\iint_T\left|U(x,y)-\sum_{k,s=0}^{N_n^{(n)},M_n^{(n)}}a_{k,s}W_k(x)W_s(y)\right|dxdy\le$$

$$\le\sum_{n=q}^{\infty}\sum_{j=1}^{n}\left(\iint_T |H_n^{(j)}(x,y)|dxdy\right)\le 2^{-q}\to 0.$$

Отсюда и из (1.1) следует

(3.13) $$b_{k,s}=c_{k,s}(U)\ ,k,s=0,1,2,...$$

Учитывая соотношения (3.5), (3.12) и (3.13), получим, что коэффициенты Фурье-Уолша функции U на спектре $\Lambda(U)$ (см.(1.2)) положительны и расположены в убывающем порядке.

Пусть $f(x,y)-$ любая почти везде конечная измеримая функция определенная на $[0,1)^2$. Принимая во внимание многомерный аналог теоремы Лузина (см [2], стр.323-325) и пункт б) теоремы 2.2 , без ограничения общности можно считать, что $f(x,y)\in C[0,1)^2$. Нетрудно видеть, что из последовательности (3.1) можно выбрать подпоследовательность $\{f_{k_n}(x)\}_{n=1}^{\infty}$ такую, что

(3.14) $$\lim_{N\to\infty}\left\|\sum_{n=1}^{N}f_{k_n}(x,y)-f(x,y)\right\|_{\infty}=0\ ,$$

(3.15) $$\|f_{k_n}(x,y)\|_{\infty}\le 4^{-3n-2},\quad n\ge 2,$$

(3.16) $$k_1>j_0=[\log_{\frac{1}{2}}\delta]+1.$$

где $[a]$- целая часть числа a. Пусть

(3.17) $$Q_1(x,y)=Q_{k_1}^{(j_0+1)},\quad E_1=E_{k_1}^{(j_0+1)},\quad g_1=g_{k_1}^{(j_0+1)}.$$

Предположим, что уже определены числа $k_1=\nu_1<...<\nu_{q-1}$ функции $f_{\nu_1}(x,y),...,f_{\nu_{q-1}}(x,y)$, $g_0(x,y),g_1(x,y).....g_{q-1}(x)$, множества $E_n,\ 1\le n\le q-1$

и полиномы

$$Q_n(x,y)=Q_{\nu_n}^{(n+j_0)}(x,y)=\sum_{k,s=N_{\nu_n}^{(n+j_0-1)},M_{\nu_n}^{(n+j_0-1)}}^{N_{\nu_n}^{(n+j_0)}-1,M_{\nu_n}^{(n+j_0)}-1}\delta_{k,s}^{(\nu_n,n+j_0)}\ a_{k,s}^{(\nu_n,n+j_0)}W_k(x)W_s(y),$$

которые для всех $1\le n\le q-1$ удовлетворяют условиям:

$$\|g_n\|_\infty<2^{-(n-8)},\quad g_n(x,y)=f_{k_n}(x,y),\quad (x,y)\in E_n,\quad |E_n|>1-\delta 2^{-n},$$

$$\left\|\sum_{k=1}^{n}[Q_k(x,y)-g_k(x,y)]\right\|_\infty<4^{-(n-1)},\ 1\le n\le q-1, \tag{3.18}$$

$$\max_{(R_{\nu_n}^{(n+j_0-1)})^2\le R^2\le (R_{\nu_n}^{(n+j_0)})^2}\left\|\sum_{(R_{\nu_n}^{(n+j_0-1)})^2\le k^2+s^2\le R^2}\delta_{k,s}^{(\nu_n,n+j_0)}\ a_{k,s}^{(\nu_n,n+j_0)}W_k(x)W_s(y)\right\|_\infty$$

$$<2^{-n},\quad R_{\nu_n}^{(j)}=\sqrt{(N_{\nu_n}^{(j)})^2+(M_{\nu_n}^{(j)})^2}.$$

Нетрудно видеть, что из последовательности (3.1) можно выбрать функцию $f_{\nu_q}(x,y)$, чтобы

$$\left\|f_{\nu_q}(x,y)-\left(f_{k_q}(x,y)-\sum_{i=1}^{q-1}[Q_i(x,y)-g_i(x,y)]\right)\right\|_\infty<4^{-3q-2}. \tag{3.19}$$

Положим

$$g_q(x,y)=f_{k_q}(x,y)+[g_{\nu_q}^{(q+j_0)}(x,y)-f_{\nu_q}(x,y)], \tag{3.20}$$

(3.21)

$$Q_q(x,y)=Q_{\nu_q}^{(q+j_0)}(x,y)=\sum_{k,s=N_{\nu_q}^{(q+j_0-1)},M_{\nu_q}^{(q+j_0-1)}}^{N_{\nu_q}^{(q+j_0)}-1,M_{\nu_q}^{(q+j_0)}-1}\delta_{k,s}^{(\nu_q,n+j_0)}a_{k,s}^{(\nu_q,q+j_0)}W_k(x)W_s(y),$$

$$E_q=E_{\nu_q}^{(q+j_0)}. \tag{3.22}$$

Из (3.6), (3.16), (3.20) и (3.22) вытекает, что

$$g_q(x,y)=f_{k_q}(x,y),\quad (x,y)\in E_q,\quad |E_q|>1-\delta 2^{-q}. \tag{3.23}$$

В силу (3.7) и (3.18)-(3.21) имеем

$$\left\|\sum_{j=1}^{q}[Q_j(x,y)-g_j(x,y)]\right\|_\infty=\left\|\sum_{j=1}^{q-1}[Q_j(x,y)-g_j(x,y)]+Q_q(x,y)-g_q(x,y)\right\|_\infty\le$$

(3.24)

$$\le\left\|f_{\nu_q}(x,y)-\left(f_{k_q}(x,y)-\sum_{i=1}^{q-1}[Q_i(x,y)-g_i(x,y)]\right)\right\|_\infty+\|g_{\nu_q}^{(q+j_0)}-Q_{\nu_q}^{(q+j_0)}\|_\infty<4^{-3q}.$$

Ясно, что (см. (3.9), (3.21))

$$\max_{M_{\nu_q}^{(q+j_0-1)}\le l<M_{\nu_q}^{(q+j_0)}}\left\|\sum_{k=M_{\nu_q}^{(q+j_0-1)}}^{l}\delta_{k,s}^{(\nu_q,q+j_0)}a_{k,s}^{(\nu_q,q+j_0)}W_k(x)W_s(y)\right\|_\infty<2^{-q}, \tag{3.25}$$

Из (3.8), (3.12), (3.17)-(3.19) следует

$$\|g_q(x,y)\|_\infty\le\left\|f_{\nu_q}(x,y)-\left(f_{k_q}(x,y)-\sum_{i=1}^{q-1}[Q_i(x,y)-g_i(x,y)]\right)\right\|_\infty+$$

$$+\left\|\sum_{j=1}^{q-1}[(Q_j(x,y))-g_j(x,y)]\right\|_\infty+\|g_{\nu_q}^{(q+j_0)}(x,y)\|_\infty<$$

$$<4^{-3q-3}+4^{-3q+3}+4^{q+j_0}\|f_{\nu_q}(x)\|_\infty<2^{-q+8}. \tag{3.26}$$

Ясно, что по индукции можно определить последовательности множеств $\{E_q\}_{q=1}^\infty$, функций $\{g_q(x,y)\}_{q=1}^\infty$ $(g_1(x,y)=f_{k_1}(x,y))$ и полиномов $\{Q_q(x,y\}$, удовлетворяющих условиям (3.23)-(3.26) для всех $q\ge1$. Положим

$$E=\bigcap_{q=1}^\infty E_q\ . \tag{3.27}$$

Из (3.6), (3.22) и (3.27) вытекает

$$|E|>1-\delta.$$

В силу (3.14) имеем

$$\left\|\sum_{q=1}^\infty g_q\right\|_\infty\le\sum_{q=1}^\infty\|g_q\|_\infty<\infty. \tag{3.28}$$

Определим функцию $g(x,y)$, числа $\{\delta_{k,s}\}$ следующим образом:

$$g(x,y)=\sum_{q=1}^\infty g_q(x,y), \tag{3.29}$$

(3.30)

$$\delta_{k,s}=\begin{cases}\delta_{k,s}^{(\nu_q,q+j_0)},\ (k,s)\in[N_{\nu_q}^{(q+j_0-1)},N_{\nu_q}^{(q+j_0)})\times[M_{\nu_q}^{(q+j_0-1)},M_{\nu_q}^{(q+j_0)}),\ q=1,2,\dots,\\ 0,\quad (k,s)\notin\bigcup_{q=1}^\infty[N_{\nu_q}^{(q+j_0-1)},N_{\nu_q}^{(q+j_0)})\times[M_{\nu_q}^{(q+j_0-1)},M_{\nu_q}^{(q+j_0)})\end{cases}.$$

Из (3.14), (3.23), (3.28) и (3.29) имеем

$$g(x,y)\in L^\infty[0,1)^2,\ g(x,y)=f(x,y),\quad (x,y)\in E.$$

Покажем, что сферические частичные суммы двойного ряда

$$\sum_{k,s=0}^\infty\delta_{k,s}c_{k,s}(U)\ W_k(x)W_s(y).$$

сходятся к $g(x,y)$ равномерно на $[0,1)^2$ по сферам. В силу (3.21), (3.26),(3.29) и (3.30) для всех $R \in [R_{\nu_q}^{(q+j_0-1)}, R_{\nu_q}^{(q+j_0)})$, где $R_{\nu_q}^{(j)} = \sqrt{(N_{\nu_q}^{(j)})^2 + (M_{\nu_n}^{(q)})^2}$, будем иметь

$$\left\| \sum_{k^2+s^2 \le R^2} \delta_{k,s} c_{k,s}(U) W_k(x) W_s(y) - g(x,y) \right\| \le$$

$$\le \sum_{j=q}^{\infty} \|g_j(x,y)\| + \left\| \sum_{j=1}^{q-1} [Q_j(x,y) - g_j(x,y)] \right\| +$$

$$+ \max_{(R_{\nu_q}^{(q+j_0-1)})^2 \le R^2 \le (R_{\nu_q}^{(q+j_0)})^2} \left\| \sum_{(R_{\nu_q}^{(q+j_0-1)})^2 \le k^2+s^2 \le R^2} \delta_{k,s}^{(\nu_q,q+j_0)} c_{k,s}^{(\nu_q,q+j_0)} W_k(x) W_s(y) \right\| <$$

$$< 2^{-q-1}.$$

Отсюда следует

$$\delta_{k,s} c_{k,s}(U) = c_{k,s}(g)), .k, s = 0, 1, 2, ..$$

Следовательно, ряд Фурье функции $g(x,y)$ по двойной системе Уолша сходится к ней равномерно на $[0,1)^2$ по сферам. Из (3.3) и (3.30) вытекает $|c_{k,s}(g)| = c_{k,s}(U), (k,s) \in spec(g)$. Теорема 2.2 доказана.

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DISTRIBUTION OF ZEROS AND CRITICAL POINTS OF A POLYNOMIAL, AND SENDOV'S CONJECTURE

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Abstract. According to the Gauss Lucas theorem, the critical points of a Complex polynomial $p(z) := \sum_{j=0}^{n} a_j z^j$ where $a_j \in \mathbb{C}$ always lie in the convex hull of its zeros. In this paper, we prove certain relations between the distribution of zeros of a polynomial and its critical points. Using these relations, we prove the well-known Sendov's conjecture for certain special cases.

MSC2020 numbers: 30A10; 30C15.
Keywords: polynomials; zeros; critical points.

1. Introduction and statement of results

Polynomials are the mathematical expressions of the form $p(z) := \sum_{j=0}^{n} a_j z^j$ where $a_j \in \mathbb{C}$ and have been studied since ancient times with regard to their zeros, the values of the variable z that make $p(z)$ vanish. If we plot the zeros of a polynomial $p(z)$ and the zeros of the derived polynomial $p'(z)$ (the critical points of $p(z)$) in the complex plane, there are interesting geometric relations between the two sets of points. It was shown by Gauss that the zeros of $p'(z)$ are the positions of equilibrium in the force field due to particles of equal mass/charge situated at each zero of $p(z)$, if each particle attracts/repels the other particle with a force equal to the inverse of the distance between them. Concerning the location of critical points of a polynomial , the Gauss-lucas theorem states that the critical points of a polynomial p lie in the convex hull of its zeros. Regarding the distribution of critical points of p within the convex hull of its zeros the well known Sendov's Conjecture asserts:

"If all the zeros of a polynomial p lie in $|z| \leq 1$ and if z_0 is any zero of $p(z)$, then there is a critical point of p in the disk $|z - z_0| \leq 1$."

The conjecture was posed by Bulgarian mathematician Blagovest Sendov in 1958, but is often attributed to Ilieff because of a reference in Hayman's *Research Problems in Function Theory* [5] in 1967. A good number of papers have been published on this conjecture (for details see [6]) but the general conjecture remains open. Rubenstein [11] in 1968 proved the conjecture for all polynomials of degree 3 and 4. In 1969 Schmeisser [12] showed that, if the convex hull containing all zeros of p has its vertices on $|z| = 1$, then p satisfies the conjecture (for the proof see [10, Theorem 7.3.4]). Later Schmeisser [13] also proved the conjecture for the Cauchy class of polynomials. In 1996 Borcea [3] showed that the conjecture holds true for polynomials with atmost six distinct zeros and in 1999 Brown and Xiang [4] proved the conjecture for polynomials of degree upto eight. Dégot [6] proved that for every zero (say) z_0 of a polynomial p there exists lower bound N_0 depending upon the modulus of z_0 such that $|z - z_0| \leq 1$ contains a critical point of p if $deg(p) > N_0$. Chalebgwa [5] gave an explicit formula for such a N_0. More recent important work in this area includes that of Kumar [8] and Sofi, Ahanger and Gardner [14]. As for the latest, Terence Tao [15] following on the work of Degot [6], proved that the Sendov's conjecture holds for polynomials with sufficiently high degree.

In this paper, we prove certain relations between the distribution of zeros of a polynomial p in the complex plane and the distribution of its critical points. Using these relations we prove the Sendov's Conjecture for the case when all the zeros of a polynomial lie on a circle or a line within the closed unit disk.

Theorem 1.1. *Let $p(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial with zeros $z_1, z_2, \cdots, z_n$. Suppose z_1 is a zero of p such that $|z_j - z_1 + re^{i\theta}| \leq r$ for all $2 \leq j \leq n$, $0 < \theta \leq 2\pi$ and $r > 0$, then $|z - z_1| \leq r$ always contains a critical point of p.*

In case z_1 is the largest zero of p in modulus, then $|z_j| \leq |z_1|$ for all $2 \leq j \leq n$, and therefore we have $|z_j - z_1 + |z_1|e^{i\theta}| \leq |z_1|$ where $\theta = arg(z_1)$ for all $2 \leq j \leq n$. Hence by Theorem 1.1, $|z - z_1| \leq |z_1|$ contains a critical point of p. So we have proved the following:

Corollary 1.1. *If $p(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial with all its zeros $z_1, z_2, \cdots, z_n$ and suppose z_1 is the zero of p with largest modulus, that is, $|z_j| \leq |z_1|$ for all $2 \leq j \leq n$, then $|z - z_1| \leq |z_1|$ always contains a critical point of p.*

Theorem 1.2. *Let $p(z) := \sum_{j=0}^{n} a_j z^j$ be a non-constant polynomial with all its zeros $z_1, z_2, \cdots, z_n$ lying inside the closed disk $|z| \leq r$. Suppose all $z_j, 1 \leq j \leq n$ lie on a circle or on a line within the closed disk $|z| \leq r$, then for every $z_i, 1 \leq i \leq n$, there exists a critical point of p in $|z - z_i| \leq r$.*

Taking $r = 1$, Theorem 1.2 shows that the Sendov's Conjecture holds when all the zeros of p lie on a circle or a line within the disk containing all the zeros of p.

Sendov's conjecture is about how far away are critical points of a polynomial p from a given zero of p. In the converse direction we prove the following :

Theorem 1.3. *Let $p(z) := \sum_{j=0}^{n} a_j z^j$ be a non-constant polynomial with its zeros $z_1, z_2, \cdots, z_n$ and critical points $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$. Then for every critical point $\zeta_j, 1 \leq j \leq n-1$ there exists a zero $z_i, 1 \leq i \leq n$ such that*

$$|z_i - \zeta_j|^2 \leq |z_i|^2 - |\zeta_j|^2. \tag{1.1}$$

and

$$\left|\frac{z_i}{2} - \zeta_j\right| \leq \left|\frac{z_i}{2}\right|. \tag{1.2}$$

Remark 1: In the above result if we assume that all the zeros of p lie in the closed unit disk, we get from Theorem 1.3 that for every critical point $\zeta_j, 1 \leq j \leq n-1$ of p there exists a zero $z_i, 1 \leq i \leq n$ of p such that

$$|z_i - \zeta_j| \leq 1. \tag{1.3}$$

and $\left|\frac{z_i}{2} - \zeta_j\right| \leq \frac{1}{2}$, which is the main result by A.Aziz in [1]. Our proof of the result from which it follows is very short and simple.

Remark 2: If a unique zero of p satisfies (1.3) for all $\zeta_j, 1 \leq j \leq n-1$ then by Biernacki's theorem [10, Theorem 4.5.2] Sendov's conjecture is true for p. Therefore, if Sendov's conjecture is false, (1.3) must be true for at least two different zeros of p and hence for any polynomial p with all its zeros in the closed unit disk there will always be at least two different zeros say z_1, z_2 such that $|z - z_i| \leq 1$ for $i = 1, 2$ contains some critical point of p.

2. Proofs of the theorems

Proof of Theorem 1.1 . Let $\zeta_1, \zeta_2 \ldots, \zeta_{n-1}$ be the critical points of p and assume to the contrary $|z_1 - \zeta_j| > r$ for all $1 \leq j \leq n-1$.Then

$$\left|\frac{p''(z_1)}{p'(z_1)}\right| = \left|\sum_{j=1}^{n-1} \frac{1}{z_1 - \zeta_j}\right| < \frac{n-1}{r}.$$

Let $p(z) = a_n(z - z_1)q(z)$, where $q(z) = \prod_{j=2}^{n}(z - z_j)$. Then $p'(z_1) = q(z_1)$ and $p''(z_1) = 2q'(z_1)$, so that

$$\left|\frac{2q'(z_1)}{q(z_1)}\right| = \left|\frac{p''(z_1)}{p'(z_1)}\right| < \frac{n-1}{r} \quad \text{and hence} \quad \left|\frac{q'(z_1)}{q(z_1)}\right| = \left|\sum_{j=2}^{n} \frac{1}{z_1 - z_j}\right| < \frac{n-1}{2r}.$$

Now by our assumption, for all $2 \leq j \leq n$

$$\left|\frac{z_j - z_1 + re^{i\theta}}{re^{i\theta}}\right| \leq 1,$$

we have

$$\mathfrak{Re}\left(\frac{1}{1 - \frac{z_j - z_1 + re^{i\theta}}{re^{i\theta}}}\right) \geq \frac{1}{2}.$$

Equivalently $\mathfrak{Re}\left(\frac{re^{i\theta}}{z_1 - z_j}\right) \geq \frac{1}{2}$. This gives

$$\mathfrak{Re}\left(\sum_{j=2}^{n} \frac{re^{i\theta}}{z_1 - z_j}\right) \geq \frac{n-1}{2}.$$

Therefore

$$\left|\sum_{j=2}^{n} \frac{re^{i\theta}}{z_1 - z_j}\right| \geq \frac{n-1}{2}.$$

Hence

$$\left|\frac{q'(z_1)}{q(z_1)}\right| = \left|\sum_{j=2}^{n} \frac{1}{z_1 - z_j}\right| \geq \frac{n-1}{2r},$$

a contradiction to (2.1). □

Proof of Theorem 1.3. Case I : All the zeros of p lie on the circle. Since the relative distances between zeros and critical points will not change under a translation we may assume without loss of generality that all the zeros of p lie on a circle with centre at the origin and in that case $|z_i| = |z_j|$ for all $1 \leq i, j \leq n$. Therefore by Corollary 1.1 for every $i, 1 \leq i \leq n$, there exists a zero of p' inside $|z - z_i| \leq |z_j| \leq r$.

Case II : All the zeros of p lie on a line.
Again since a rigid motion will not alter the relative distances between zeros and critical points of p, it is sufficient to prove the result in case all the zeors of p lie on the real line. Let $\zeta_1, \zeta_2 \ldots, \zeta_{n-1}$ be the critical points of p and assume to the contrary that there exists a zero of p say z_1 such that $|z_1 - \zeta_j| > r$ for all $1 \leq j \leq n-1$. By the same argument as in the proof of Theorem 1.1 we have

$$\left|\sum_{j=2}^{n} \frac{1}{z_1 - z_j}\right| < \frac{n-1}{2r}. \tag{2.1}$$

We may also assume that $z_1 > 0$. If there exists any zero of p, say z_j such that

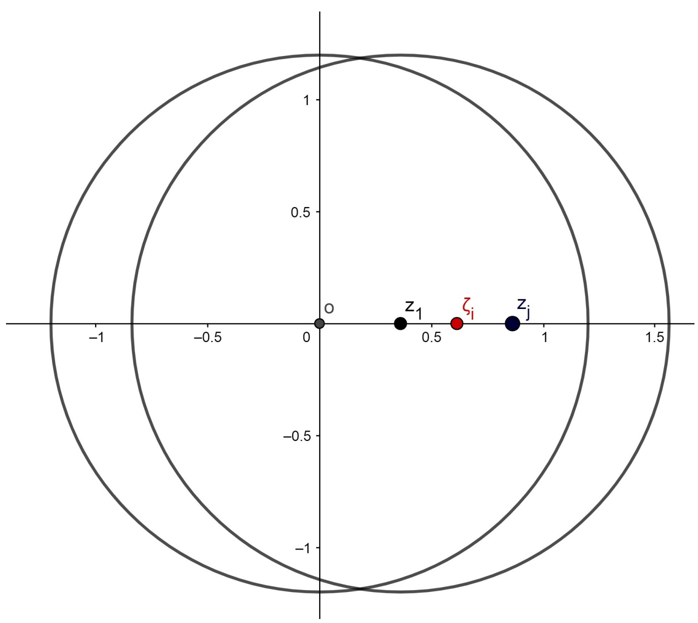

Рис. 1. If $z_j > z_1$ for some $2 \leq j \leq n$ then $|z - z_1| \leq r$ contains a critical point of p.

$z_j > z_1$, then by Rolle's theorem p will have a critical point between z_1 and z_j

which will be within a distance r of z_1 and there is nothing to prove (see Figure 1). So we may assume $z_j \leq z_1$ for all $2 \leq j \leq n$ and hence $0 \leq z_1 - z_j \leq 2r$ for all $2 \leq j \leq n$. This implies that for all $2 \leq j \leq n$ $\frac{1}{z_1 - z_j} \geq \frac{1}{2r}$, which gives

$$\left|\sum_{j=2}^{n} \frac{1}{z_1 - z_j}\right| = \sum_{j=2}^{n} \frac{1}{z_1 - z_j} \geq \frac{n-1}{2r}$$

a contradiction to (2.2) and hence the result. □

Proof of Theorem 1.3. To prove (1.1) assume the contrary. Therefore there exist a critical point say ζ_1 of p such that

$$|z_i - \zeta_1|^2 > |z_i|^2 - |\zeta_1|^2, 1 \leq i \leq n. \tag{2.2}$$

Without loss of generality we may assume that $\zeta_1 \geq 0$. From (2.3) we get

$$|z_i|^2 + \zeta_1^2 - 2\zeta_1 Re(z_i) > |z_i|^2 - \zeta_1^2, 1 \leq i \leq n.$$

This gives $Re(z_i) < \zeta_1$ for all $1 \leq i \leq n$. But this implies that ζ_1, a critical point of p, lies outside the convex hull of $z_1, z_2, \cdots, z_n$, the zeros of p violating the Gauss-Lucas theorem and this contradiction proves (1.1). A similar argument proves (1.2). □

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