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ՄԱԹԵՄԱՏԻԿԱ

МАТЕМАТИКА

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ՏԵՂԵԿԱԳԻՐ ИЗВЕСТИЯ

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$(P,Q)-\varepsilon$-PSEUDO CONDITION SPECTRUM FOR 2×2 MATRICES. LINEAR OPERATOR AND APPLICATION

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Abstract. We define a new type of spectrum, called the $(P,Q)-\varepsilon$-pseudo condition spectra

$$\Sigma^{(2)}_{(P,Q)-\varepsilon}(T)=\sigma^{(2)}_{(P,Q)}(T)\bigcup\left\{\lambda\in\mathbb{C}:\|(\lambda-T)^{(2)}_{(P,Q)}\|\|\lambda-T\|>\frac{1}{\varepsilon}\right\}.$$

This $(P,Q)-\varepsilon$-pseudo condition spectrum shares some properties of the usual spectrum such as non emptiness. Our aim in this paper is to show some properties of $(P,Q)-\varepsilon$-pseudo condition spectra of a linear operator T in Banach spaces and reveal the relation between their $(P,Q)-\varepsilon$-pseudo condition spectra. Additionally, we investigate the $(P,Q)-\varepsilon$-pseudo condition spectrum of a block matrix in a Banach space.

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Keywords: Banach spaces; matrix; (P,Q)-outer generalized inverse; $(P,Q)-\varepsilon$-pseudo condition spectrum.

1. Introduction

For the past ten years, there has been in the field of mathematics digital technology has a keen interest in the study of the notion of pseudo-spectrum and pseudo condition spectra. The development of this notion is explained by the fact that in a certain number of mathematical engineering problems were natural non-self-employed operators. This original observation suggests that in some cases, knowledge of the spectrum of an operator alone does not sufficiently understand his action. It is as well as to make up for this apparent lack of information contained in the spectrum, new subsets of the complex plane called pseudo-spectra have been introduced. There are several generalizations of the concept of the spectrum in literature such as Ransford spectrum [8], pseudo spectrum [12], condition spectrum [1, 2, 5, 10], pseudo spectra of multivalued linear operator [3]. Unlike the spectrum, which is a purely algebraic concept, both the pseudo spectrum and condition spectrum depend on the norm. Also, both these sets contain the spectrum as a subset.

Consider two idempotent elements $P,Q\in\mathcal{B}(X)$ i.e. $P^2=P$ and $Q^2=Q$.

Definition 1.1. *Let $T \in \mathcal{B}(X)$. An operator $S \in \mathcal{B}(X)$ satisfying,*

$$STS = S, \quad ST = P \quad and \quad I - TS = Q$$

will be called a (P,Q)-outer generalized inverse of T and it is denoted by $T^{(2)}_{P,Q}$.

The detailed treatment of outer generalized inverses of operators on Banach and Hilbert spaces can be found in [4, 7].

Definition 1.2. *For an element $T \in \mathcal{B}(X)$, the (P,Q)-resolvent set is defined as*

$$\rho^{(2)}_{(P,Q)}(T) := \Big\{\lambda \in \mathbb{C} : (\lambda - T)^{(2)}_{P,Q} \ exist\Big\}.$$

The complement of the set $\rho^{(2)}_{(P,Q)}(T)$ is called (P,Q)-spectrum and it is denoted by $\sigma^{(2)}_{(P,Q)}(T)$.

From now onwards, we consider the idempotent $P \neq 0$ and $P \neq I$ and we fix the operator $Q = I - P$. If $\lambda \in \rho^{(2)}_{(P,Q)}(T)$, then we denote $(\lambda - T)^{(2)}_{P,Q}$ by $R_T(\lambda)$. For given $T \in \mathcal{B}(X)$, if $R_T(\lambda)$ exists for some $\lambda \in \mathbb{C}$, then from Definition 1.1,

$$R_T(\lambda)(\lambda - T) = P \quad \text{and} \quad (\lambda - T)R_T(\lambda) = P \tag{1.1}$$

By (Eq. 1.1), $TP = PT$. Consequently, if $TP \neq PT$ then $\sigma^{(2)}_{(P,Q)}(T) = \mathbb{C}$. Because of this reason, in the rest of the paper we assume $TP = PT$.

In this note, we dedicate to research the $(P,Q) - \varepsilon$-pseudo condition spectra of a linear operator and its properties. The remainder of this paper is organized as follows. In Section 2, we first suggest a characterize for the $(P,Q) - \varepsilon$-pseudo condition spectra of a linear operator. Then, in Section 3, we investigate the $(P,Q)-$ ε-pseudo condition spectra, of a block matrix in a Banach space.

2. $(P,Q) - \varepsilon$-Pseudo condition spectra of linear operator

The $(P,Q) - \varepsilon$-pseudo spectrum were studied in [6, 11]. Let $\varepsilon > 0$ and $T \in \mathcal{B}(X)$. The $(P,Q) - \varepsilon$-pseudo spectrum is defined as

$$\sigma^{(2)}_{(P,Q)-\varepsilon}(T) := \Big\{\lambda \in \mathbb{C} : (\lambda - T)^{(2)}_{P,Q} \text{ does not exist or } \Big\|(\lambda - T)^{(2)}_{P,Q}\Big\| > \varepsilon\Big\}.$$

By convention, we write $\|R_T(\lambda)\| = \infty$ if $R_T(\lambda)$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma^{(2)}_{(P,Q)}(T)$. It is well known that $\rho^{(2)}_{(P,Q)-\varepsilon}(T)$ for any $T \in \mathcal{B}(X)$ is a nonempty open subset, the following remark prove the same for $(P,Q) - \varepsilon$-pseudo resolvent set. In this section, we define the pseudo spectra of linear relation and study some properties.

Definition 2.1. *($(P,Q)-\varepsilon$-pseudo spectra of T)*
Let $T \in \mathcal{B}(X)$ where X is a normed space and $\varepsilon > 0$ we define the $(P,Q)-\varepsilon$-pseudo spectra of T by

$$\sigma^{(2)}_{(P,Q)-\varepsilon}(T) \quad = \quad \sigma^{(2)}_{(P,Q)}(T) \bigcup \left\{ \lambda \in \mathbb{C} : \quad \|R_T(\lambda)\| > \frac{1}{\varepsilon} \right\}.$$

We denote the $(P,Q)-\varepsilon$-pseudo resolvent set of T

$$\rho^{(2)}_{(P,Q)-\varepsilon}(T) = \mathbb{C} \backslash \sigma^{(2)}_{(P,Q)-\varepsilon}(T) = \rho^{(2)}_{(P,Q)}(T) \bigcap \left\{ \lambda \in \mathbb{C} : \quad \|R_T(\lambda)\| \leq \frac{1}{\varepsilon} \right\}.$$

Definition 2.2. *($(P,Q)-\varepsilon$-pseudo condition spectra of T)*
Let $T \in \mathcal{B}(X)$ where X is a normed space and $\varepsilon > 0$ we define the $(P,Q)-\varepsilon$-pseudo condition spectra of T by

$$\Sigma^{(2)}_{(P,Q)-\varepsilon}(T) \quad = \quad \sigma^{(2)}_{(P,Q)}(T) \bigcup \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| > \frac{1}{\varepsilon} \right\}$$

with the convention that $\|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| = \infty$, if $(\lambda - T)^{(2)}_{(P,Q)}$ is not exists. Notice that the uniqueness of $\Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ allows us to consider the $(P,Q)-\varepsilon$-pseudo condition spectrum and $(P,Q)-\varepsilon$-pseudo spectrum.

Theorem 2.1. *Let $T \in \mathcal{B}(X)$ and $0 < \varepsilon < 1$. Then,*

(1) $\sigma^{(2)}_{(P,Q)}(T) = \bigcap\limits_{0<\varepsilon<1} \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$

(2) *If $0 < \varepsilon_1 < \varepsilon_2 < 1$, then*

$$\sigma^{(2)}_{(P,Q)}(T) \subset \Sigma^{(2)}_{(P,Q)-\varepsilon_1}(T) \subset \Sigma^{(2)}_{(P,Q)-\varepsilon_2}(T).$$

(3) *If $\alpha \in \mathbb{C}$, then*

$$\Sigma^{(2)}_{(P,Q)-\varepsilon}(T + \alpha I) = \alpha + \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$$

Proof. (1) It is clear that $\sigma^{(2)}_{(P,Q)}(T) \subset \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ for all $0 < \varepsilon < 1$. Then,

$$\sigma^{(2)}_{(P,Q)}(T) \subset \bigcap_{0<\varepsilon<1} \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$$

Conversely, if $\lambda \in \bigcap\limits_{0<\varepsilon<1} \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then for all $0 < \varepsilon < 1$, we get $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. We will discuss these two cases:

<u>1^{st} case</u> : If $\lambda \in \sigma^{(2)}_{(P,Q)}(T)$, then we get the desired result.

<u>2^{nd} case</u> : If $\lambda \in \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| > \frac{1}{\varepsilon} \right\}$, then taking limits as $\varepsilon \longrightarrow 0^+$, we get

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| = \infty.$$

We deduce that $\lambda \in \sigma^{(2)}_{(P,Q)}(T)$.

(2) Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon_1}(T)$, so

$$\|(\lambda - T)^{(2)}_{(P,Q)}\|\|\lambda - T\| > \frac{1}{\varepsilon_1} > \frac{1}{\varepsilon_2}.$$

We conclude that $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon_2}(T)$. Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T + \alpha I)$, hence

$$\|((\lambda - \alpha) - T)^{(2)}_{(P,Q)}\|\|(\lambda - \alpha) - T\| > \frac{1}{\varepsilon}.$$

Therefore, $\lambda - \alpha \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. This yields to

$$\lambda \in \alpha + \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$$

Lemma 2.1. *Let $T \in \mathcal{B}(X)$, $0 < \varepsilon < 1$ and P is invertible. Then, $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash \sigma^{(2)}_{(P,Q)}(T)$ if and only if there exists x such that*

$$\|P^{-1}(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|.$$

Proof. Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash\sigma^{(2)}_{(P,Q)}(T)$, then

$$\|(\lambda - T)^{(2)}_{(P,Q)}\|\|\lambda - T\| > \frac{1}{\varepsilon}.$$

Thus

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Moreover

$$\sup_{y\neq 0} \frac{\|(\lambda - T)^{(2)}_{(P,Q)}y\|}{\|y\|} > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Then, there exists a nonzero $y \in X$ such that

$$\|(\lambda - T)^{(2)}_{(P,Q)}y\| > \frac{\|y\|}{\varepsilon\|\lambda - T\|}.$$

Putting $x = (\lambda - T)^{(2)}_{(P,Q)}y$, then $(\lambda - T)x = (\lambda - T)(\lambda - T)^{(2)}_{(P,Q)}y = Py$. Hence,

$$\varepsilon\|\lambda - T\|\|x\| > \|P^{-1}(\lambda - T)x\|.$$

Conversely, we assume that there exists $x \in X$ such that

$$\varepsilon\|\lambda - T\|\|x\| > \|P^{-1}(\lambda - T)x\|.$$

Let $\lambda \notin \sigma^{(2)}_{(P,Q)}(T)$ and $x = (\lambda - T)^{(2)}_{(P,Q)}y$, then

$$\|x\| \leq \|(\lambda - T)^{(2)}_{(P,Q)}\|\|y\|.$$

Moreover,

$$\varepsilon\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\|\|y\| > \|P^{-1}(\lambda - T)x\| = \|y\|.$$

It follows that $1 < \varepsilon\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\|$. We conclude that,

$$\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash\sigma^{(2)}_{(P,Q)}(T).$$

Suppose X is a Banach space with the following property: For all generalized invertible operator $T \in \mathcal{B}(X)$ there exist $B \in \mathcal{B}(X)$ such that B is not generalized invertible and

$$\|T - B\| = \frac{1}{\|T^{(2)}_{(P,Q)}\|}.$$

Theorem 2.2. *Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. Then, there exists $D \in \mathcal{B}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \Sigma^{(2)}_{(P,Q)}(T + D)$.*

Proof. Suppose $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. We will discuss these two cases:

<u>1^{st} *case*</u> : If $\lambda \in \sigma^{(2)}_{(P,Q)}(T)$, then it is sufficient to take $D = 0$.

<u>2^{nd} *case*</u> : If $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash\sigma^{(2)}_{(P,Q)}(T)$. Hence, by assumption, there exists an element $B \in \mathcal{B}(X)$ such that

$$\|\lambda - T - B\| = \frac{1}{\|(\lambda - T)^{(2)}_{(P,Q)}\|}.$$

Let $D = \lambda - T - B$. Then

$$\|D\| = \frac{1}{\|(\lambda - T)^{(2)}_{(P,Q)}\|} \leq \varepsilon\|\lambda - T\|.$$

Also $B = \lambda - (T + D)$, is not generalized invertible. So, $\lambda \in \sigma^{(2)}_{(P,Q)}(T + D)$.

Corollary 2.1. *Let X be a Banach space satisfying the hypothesis of Theorem 2.3. Then, $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ if, and only if, there exists $D \in \mathcal{B}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^{(2)}_{(P,Q)}(T + D)$.*

Theorem 2.3. *Let $T \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$, and $0 < \varepsilon < 1$. If there is $D \in \mathcal{B}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^{(2)}_{(P,Q)}(T + D)$. Then, $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.*

Proof. We assume that there exists D such that $\|D\| < \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^{(2)}_{(P,Q)}(T + D)$. Let $\lambda \notin \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then for all $(\lambda - T)^{(2)}_{(P,Q)}$ a generalized inverse of $\lambda - T$ we have

$$\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| \leq \frac{1}{\varepsilon}.$$

Now, we define the operator $S : X \longrightarrow X$ by

$$S := \sum_{n=0}^{\infty} (\lambda - T)^{(2)}_{(P,Q)} \left(D(\lambda - T)^{(2)}_{(P,Q)}\right)^n.$$

Since,

$$\|D(\lambda - T)^{(2)}_{(P,Q)}\| < 1,$$

we can write

$$S = (\lambda - T)^{(2)}_{(P,Q)} \left(I - D(\lambda - T)^{(2)}_{(P,Q)}\right)^{-1}.$$

Then, there exists $y \in X$ such that

$$S\Big(I - D(\lambda - T)^{(2)}_{(P,Q)}\Big)y = (\lambda - T)^{(2)}_{(P,Q)}y.$$

Let $y = P(\lambda - T)x$. Then,

$$S(\lambda - T - D)Px = Px$$

for every $x \in X$. Hence, $\lambda - T - D$ is generalized invertible, so $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.

Theorem 2.4. *Let $T \in \mathcal{B}(X)$, $k = \|T\|\|T^{(2)}_{(P,Q)}\|$ and $0 < \varepsilon < 1$. Then,*

(i)$\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ if, and only if, $\overline{\lambda} \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T^{'})$.

(ii) If $\lambda_n \notin \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)(T)$ such that $\lambda_n \to \lambda$ for all $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)(T)$, then $\|(\lambda - T)^{(2)}_{(P,Q)}\| = \infty$.

Proof. (i) Using the identity

$$\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| = \|\overline{\lambda} - T^{'}\|\|(\overline{\lambda} - T^{'})^{(2)}_{(P,Q)}\|,$$

it is easy to see that the $(P, Q) - \varepsilon$-pseudo condition spectrum of $T^{'}$ is given by the mirror image of $\Sigma_\varepsilon(T)$ with respect to the real axis.

(ii) Suppose $\|(\lambda - T)^{(2)}_{(P,Q)}\| \leq \dfrac{1}{\delta}$ for some $\delta \in \mathbb{R}$ and since $\lambda_n \to \lambda$ for all $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then there exists $n_0 \in \mathbb{N}$ such that

$$|\lambda_n - \lambda| < \delta - 1 < \delta \leq \frac{1}{\|(\lambda - T)^{(2)}_{(P,Q)}\|} \quad \text{for all } n \geq n_0.$$

Hence, $\lambda \notin \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. This is a contradiction.

Theorem 2.5. *Let $T, E \in \mathcal{B}(X)$ such that $\|E\| < \dfrac{\varepsilon}{2}\|\lambda - T\|$ and $0 < \varepsilon < 1$. Then,*

$$\Sigma^{(2)}_{(P,Q)-(\frac{\varepsilon}{2}-\|E\|)}(T) \subseteq \Sigma^{(2)}_{(P,Q)-\varepsilon}(T + E) \subseteq \Sigma^{(2)}_{(P,Q)-\tau_\varepsilon}(T)$$

where, $0 < \tau_\varepsilon = \dfrac{\varepsilon^2}{2} + \varepsilon < 1$ and $0 < \frac{\varepsilon}{2} - \|E\| < 1$.

Proof. Let $\lambda \in \Sigma^{(2)}_{(P,Q)-(\frac{\varepsilon}{2}-\|E\|)}(T)$. Then, by Theorem 2.3, there exists a bounded operator $D \in \mathcal{B}(X)$ with

$$\|D\| < \Big(\frac{\varepsilon}{2} - \|E\|\Big)\|\lambda - T\|$$

such that

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T + D) = \sigma^{(2)}_{(P,Q)}\Big((T + E) + (D - E)\Big).$$

The fact that

$$\|D - E\| \leq \|D\| + \|E\| < \Big(\frac{\varepsilon}{2} - \|E\|\Big)\|\lambda - T\| + \|E\| < \varepsilon\|\lambda - T\|,$$

allows us to deduce that $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T+E)$. Now, let us prove the second inclusion. Suppose $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T+E)$, then there exists $D \in \mathcal{B}(X)$ verifying

$$\|D\| < \varepsilon\|\lambda - T - E\| \leq \varepsilon\|\lambda - T\| + \varepsilon\|E\|$$

and $\lambda \in \sigma^{(2)}_{(P,Q)}(T+E+D)$. The fact that $\|D+E\| \leq \tau_\varepsilon\|\lambda - T\|$ allows us to deduce that $\lambda \in \Sigma^{(2)}_{(P,Q)-\tau_\varepsilon}(T)$.

Theorem 2.6. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then, $\Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ has no isolated points.*

Proof. Suppose $\Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ has an isolated point μ. Then there exists an $\delta > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$ and there exists a generalized invertible $(\lambda - T)^{(2)}_{(P,Q)}$ such that

$$\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| < \frac{1}{\varepsilon}.$$

Let $\mu \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash\sigma^{(2)}_{(P,Q)}(T)$. Then, using the Hahn-Banach Theorem, there exist $x' \in X'$ such that

$$x'\Big((\mu - T)^{(2)}_{(P,Q)}\Big) = \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| \quad \text{with} \quad \|x'\| = 1.$$

Now, we define

$$\begin{cases} \phi : \rho^{(2)}_{(P,Q)}(T) \longrightarrow \mathbb{R}, \\ \qquad\qquad \lambda \longrightarrow \phi(\lambda) = x'\Big((\lambda - T)^{(2)}_{(P,Q)}\Big). \end{cases}$$

Since ϕ is is well-defined and continuous; in $B(\mu, \delta)$ and for all $\lambda \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, we have

$$|\phi(\lambda)| \quad = \quad \Big|x'((\lambda - T)^{(2)}_{(P,Q)})\Big| = \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| < \frac{1}{\varepsilon}.$$

But, $\phi(\mu) = \|\mu - T\|\|(\mu - T)^{(2)}_{(P,Q)}\| \geq \dfrac{1}{\varepsilon}$. This contradicts the maximum modulus principle.

Definition 2.3. *We define $T \in \mathcal{B}(X)$ to be of d-class operator if*

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{1}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \qquad \forall \lambda \in \mathbb{C}\backslash\sigma^{(2)}_{(P,Q)}(T).$$

In fact, we have the following theorem

Theorem 2.7. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. If $T \in \mathcal{B}(X)$ is of d−class operator, then*

$$\Sigma^{(2)}_{(P,Q)-\varepsilon}(T) \subseteq \Big\{\lambda \in \mathbb{C} : d(\lambda, \sigma^{(2)}_{(P,Q)}(T)) \leq \varepsilon\|\lambda - T\|\Big\}.$$

Proof. Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then

$$\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| > \frac{1}{\varepsilon}.$$

Now, if $T \in \mathcal{B}(X)$ is a d-class, we already have

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{1}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \qquad \forall \lambda \in \mathbb{C}\backslash\sigma^{(2)}_{(P,Q)}(T).$$

Hence,

$$\frac{1}{\varepsilon} < \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{\|\lambda - T\|}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \qquad \forall \lambda \in \mathbb{C}\backslash\sigma^{(2)}_{(P,Q)}(T).$$

Therefore,

$$\lambda \in \Big\{\lambda \in \mathbb{C} : d(\lambda, \sigma^{(2)}_{(P,Q)}(T)) \leq \varepsilon\|\lambda - T\|\Big\}.$$

Theorem 2.8. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then*
(i) If $T = \mu I$ for some number μ, then T is of d-class operator and $\sigma^{(2)}_{(P,Q)}(T) = \{\mu\}$.
(ii) If T is of d-class operator, then $\alpha T + \beta$ is also of d-class operator for every number α, β.

Proof. (i) Let $T = \mu$. for some number μ. Then clearly $\sigma^{(2)}_{(P,Q)}(T) = \{\mu\}$. Hence for all $\lambda \in \mathbb{C} \setminus \sigma^{(2)}_{(P,Q)}(T)$, we have $\lambda \neq \mu$. Thus

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{1}{|\lambda - \mu|} = \frac{1}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))}.$$

This shows that a is of d-class operator.
(ii) Next suppose that T is of d-class operator and $B = \alpha T + \beta$ for some complex numbers α, β. We want to prove that B is of d-class operator. If $\alpha = 0$, then it follows from (i). So assume that $\alpha \neq 0$. Let $w \notin \sigma^{(2)}_{(P,Q)}(B) = \{\alpha\lambda + \beta : \lambda \in \sigma^{(2)}_{(P,Q)}(B)\}$. Then, $\lambda := \frac{w-\beta}{\alpha} \notin \sigma^{(2)}_{(P,Q)}(B)$ and since T is of d-class operator,

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{1}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \qquad \forall \lambda \in \mathbb{C}\backslash\sigma^{(2)}_{(P,Q)}(T).$$

Now

$$\|(w - B)^{(2)}_{(P,Q)}\| = \|(\alpha\lambda + \beta - (\alpha T + \beta))^{(2)}_{(P,Q)}\| = \frac{1}{|\alpha|}\|(\lambda - T)^{(2)}_{(P,Q)}\|.$$

Therefore,

$$\begin{aligned}\|(w - B)^{(2)}_{(P,Q)}\| &= \frac{1}{|\alpha| d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \\ &= \frac{1}{d(\lambda\alpha, \sigma^{(2)}_{(P,Q)}(\alpha T))} = \frac{1}{d(w, \sigma^{(2)}_{(P,Q)}(B))}.\end{aligned}$$

This shows that B is of d-class operator.

Remark 2.1. *Under what additional conditions can we conclude that, if T is of d-class operator and $\sigma^{(2)}_{(P,Q)}(T) = \{\mu\}$, then $T = \mu$.*

Theorem 2.9. *Let $T \in \mathcal{B}(X)$ and for every $0 < \varepsilon < 1$ such that $\varepsilon < \|\lambda - T\|$ we have*
(i) $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ *if, and only if,* $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon\|\lambda-T\|}(T)$.
(ii) $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ *if, and only if,* $\lambda \in \Sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{\|\lambda-T\|}}(T)$.

Proof. *(i)* If $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \text{ and } \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| \geq \frac{1}{\varepsilon}.$$

Hence,

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \text{ and } \|(\lambda - T)^{(2)}_{(P,Q)}\| \geq \frac{1}{\varepsilon\|\lambda - T\|},$$

which implies that $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon\|\lambda-T\|)}(T)$. The converse is similar.
(ii) Let $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \text{ and } \|(\lambda - T)^{(2)}_{(P,Q)}\| \geq \frac{1}{\varepsilon}.$$

Hence it follows that

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \text{ and } \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| \geq \frac{\|\lambda - T\|}{\varepsilon}.$$

This proves that

$$\lambda \in \Sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{\|\lambda-T\|}}(T).$$

The converse is similar.

3. Application for matrix 2×2

In this article we will apply the results of the previous section to determine the $(P, Q) - \varepsilon$-pseudo condition spectrum of 2×2 matrix operators by mean of measure of non-strict-singularity. Let X and Y be tow Banach spaces and consider the 2×2 block operator matrix defined on $X \times Y$ by

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where, $T_1, T_2 \in \mathcal{B}(X)$. Defining the norm of the linear operator matrix T as

$$\|T\| = \max\Big\{\|T_1\|, \|T_2\|\Big\}.$$

Now, we state an auxiliary result.

Lemma 3.1. [9, Lemma 3.1] *Let* $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ *and* $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$. *If* $T^{(2)}_{P,Q}$ *exist, then,*

$$T^{(2)}_{(P,Q)} = \begin{pmatrix} (T_1)^{(2)}_{(P_1,Q_1)} & 0 \\ 0 & (T_2)^{(2)}_{(P_2,Q_2)} \end{pmatrix}.$$

Theorem 3.1. *Let* $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ *and* $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$. *If* $T^{(2)}_{P,Q}$ *exist, then,*

$$\Sigma^{(2)}_{(P_1,Q_1)-\varepsilon}(T_1) \bigcup \Sigma^{(2)}_{(P_2,Q_2)-\varepsilon}(T_2) \subset \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$$

Proof. Let $\lambda \in \Sigma^{(2)}_{(P_1,Q_1)-\varepsilon}(T_1) \bigcup \Sigma^{(2)}_{(P_2,Q_2)-\varepsilon}(T_2)$. These imply

$$\lambda \notin \Sigma^{(2)}_{(P_1,Q_1)}(T_1) \text{ or } \|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\|\|(\lambda - T_1)\| > \frac{1}{\varepsilon}$$

or

$$\lambda \notin \Sigma^{(2)}_{(P_2,Q_2)}(T_2) \text{ or } \|(\lambda - T_2)^{(2)}_{(P_2,Q_2)}\|\|(\lambda - T_2)\| > \frac{1}{\varepsilon}.$$

If either $(\lambda - T_1)^{(2)}_{(P_1,Q_1)}$ *or* $(\lambda - T_2)^{(2)}_{(P_2,Q_2)}$ *does not exists*, by Lemma 3.1, it follows:

$$(\lambda - T)^{(2)}_{(P,Q)} = \begin{pmatrix} (\lambda - T_1)^{(2)}_{(P_1,Q_1)} & 0 \\ 0 & (\lambda - T_2)^{(2)}_{(P_2,Q_2)} \end{pmatrix}$$

does not exists, then we have $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.
On the other hand, if $(\lambda - T_1)^{(2)}_{(P_1,Q_1)}$ and $(\lambda - T_2)^{(2)}_{(P_2,Q_2)}$ exists, it holds either

$$\|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\|\|(\lambda - T_1)\| > \frac{1}{\varepsilon} \text{ or } \|(\lambda - T_2)^{(2)}_{(P_2,Q_2)}\|\|(\lambda - T_2)\| > \frac{1}{\varepsilon}.$$

Without loss of generality, assume that $\|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\|\|(\lambda - T_1)\| > \frac{1}{\varepsilon}$ holds. Therefore,

$$\begin{aligned} &\|(\lambda - T)^{(2)}_{(P,Q)}\|\|(\lambda - T)\| = \\ &= \max\left\{\|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\|, \|(\lambda - T_2)^{(2)}_{(P_2,Q_2)}\|\right\} \max\{\|(\lambda - T_1)\|, \|(\lambda - T_2)\|\} \\ &\geqslant \|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\|\|(\lambda - T_1)\| > \frac{1}{\varepsilon}. \end{aligned}$$

This proves that $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.

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К ВОПРОСУ СХОДИМОСТИ И СУММИРУЕМОСТИ ОБЩИХ РЯДОВ ФУРЬЕ

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Аннотация. В работе рассмотрены вопросы сходимости и суммируемости рядов Фурье функций класса Lip 1 относительно общих ортонормированных систем (ОНС). Найдены достаточные условия, которым должны удовлетворять функции ОНС, чтобы ряд Фурье по этой системе каждой функции из класса Lip 1 п.в. сходился, безусловно сходился или суммировался методом (C, α), $\alpha > 0$. Доказана неулучшаемость некоторых полученных результатов.

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Ключевые слова: ортонормированная система; ряд Фурье; сходимость; безусловная сходимость; коэффициенты Фурье.

1. Вспомогательные обозначения и утверждения

Пусть $f \in L_2(I)$ $(I = [0, 1])$ и (φ_n) – ортонормированная на I система (ОНС) функций. Числа

$$C_n(f) = \int_0^1 f(x)\varphi_n(x)dx \tag{1.1}$$

– коэффициенты Фурье функции f. Как известно, Lip 1 является пространством Банаха с нормой

$$||f||_{\mathrm{Lip}\,1} = ||f||_C + \sup_{x,y\in[0,1]} \frac{|f(x) - f(y)|}{|x - y|}. \tag{1.2}$$

Теорема 1.1 (Банах [1])**.** *Пусть $f \in L_2$ $(f \not\sim 0)$ – любая функция. Тогда существует ОНС такая, что*

$$\limsup_{n\to\infty} |S_n(x, f)| = +\infty \ \text{ п.в. на } [0, 1],$$

где

$$S_n(x, f) = \sum_{k=1}^{n} C_k(f)\varphi_k(x).$$

Теорема 1.2 (Меньшова-Радемахера, см. [2, с. 332]). *Пусть* $(\varphi_n(x))$, $x \in [0,1]$, *произвольная ОНС. Тогда для п.в.* $x \in [0,1]$ *сходится всякий ряд*

$$\sum_{n=1}^{\infty} a_n \varphi_n(x),$$

коэффициенты которого удовлетворяют условию

$$\sum_{n=1}^{\infty} a_n^2 \log^2 n < \infty.$$

Теорема 1.3 (Орлич, см. [2, с. 350]). *Если для некоторого* $\varepsilon > 0$

$$\sum_{k=1}^{\infty} a_k^2 \log^2(k+1)(\log\log(k+2))^{1+\varepsilon} < +\infty, \tag{1.3}$$

тогда ряд $\sum_{k=1}^{\infty} a_k \varphi_k(x)$ *п.в. безусловно сходится на* $[0,1]$.

Теорема 1.4 ([3, с. 132]). *Пусть* (φ_n) *ОНС на* $[0,1]$. *Если*

$$\sum_{n=1}^{\infty} a_n^2 (\log\log n)^2 < +\infty,$$

тогда ряд $\sum_{n=1}^{\infty} a_n \varphi_n(x)$ *суммируем методом* (C,α), $\alpha > 0$, *п.в. на* $[0,1]$.

Справедливо равенство (см. [4])

$$\int_0^1 f(x)F(x)dx = \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right) \int_0^{\frac{i}{n}} F(x)dx + \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(f(x) - f\left(\frac{i}{n}\right) \right) F(x)dx + f(1)\int_0^1 F(x)dx, \tag{1.4}$$

где $F, f \in L_2$ и функция $f(x)$ принимает лишь конечные значения в каждой точке отрезка $[0,1]$ (см. (1.1)).

Пусть w_n неубывающая последовательность положительных чисел, удовлетворяющая условию

$$\mathrm{w}_n \le n, \tag{1.5}$$

и

$$P_n(a, \sqrt{\mathrm{w}}, x) = \sum_{k=1}^{n} a_k \sqrt{\mathrm{w}_k} \varphi_k(x).$$

Легко видеть, что если $f \in L_2(I)$ (см. (1.1)),

$$\sum_{k=1}^{n} C_k(f) \sqrt{\mathrm{w}_k} a_k = \int_0^1 f(x) P_n(a, \sqrt{\mathrm{w}}, x)dx. \tag{1.6}$$

Пусть

$$D_n(a,\sqrt{\mathrm{w}}) = \frac{1}{n}\sum_{i=1}^{n-1}\bigg|\int_0^{\frac{i}{n}} P_n(a,\sqrt{\mathrm{w}},x)dx\bigg|. \tag{1.7}$$

Лемма 1.1. *Пусть* $(a_n)\in\ell_2$. *Через* H_n *обозначим множество всех* i $(i = 1,\dots,n-1)$, *для каждого из которых найдется точка* $x\in[\frac{i}{n},\frac{i+1}{n})$ *такая, что*

$$\operatorname{sign}\int_0^x P_n(a,\sqrt{\mathrm{w}},u)du \neq \operatorname{sign}\int_0^{\frac{i+1}{n}} P_n(a,\sqrt{\mathrm{w}},u)du,$$

тогда

$$\frac{1}{n}\sum_{i\in H_n}\bigg|\int_0^{\frac{i}{n}} P_n(a,\sqrt{\mathrm{w}},u)du\bigg| = O(1). \tag{1.8}$$

Доказательство. В силу непрерывности функций $\int_0^x P_n(a,\sqrt{\mathrm{w}},u)du$ найдется точка $x_{in}\in[\frac{i}{n},\frac{i+1}{n})$ такая, что

$$\int_0^{x_{in}} P_n(a,\sqrt{\mathrm{w}},u)du = 0.$$

Отсюда

$$\int_0^{\frac{i}{n}} P_n(a,\sqrt{\mathrm{w}},u)du = \int_{x_{in}}^{\frac{i}{n}} P_n(a,\sqrt{\mathrm{w}},u)du.$$

Следовательно, используя неравенство Гельдера, заключаем

$$\begin{aligned}\sum_{i\in H_n}\bigg|\int_0^{\frac{i}{n}} P_n(a,\sqrt{\mathrm{w}},u)du\bigg| &\le \sum_{i\in H_n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} |P_n(a,\sqrt{\mathrm{w}},u)|du \\ &\le \int_0^1 |P_n(a,\sqrt{\mathrm{w}},u)|du \le \bigg(\int_0^1 P_n^2(a,\sqrt{\mathrm{w}},u)du\bigg)^{\frac12} = \bigg(\sum_{k=1}^n a_k^2 w_k\bigg)^{\frac12} \\ &\le \sqrt{w_n}\bigg(\sum_{k=1}^n a_k^2\bigg)^{\frac12} = O(1)\sqrt{w_n}.\end{aligned}$$

Умножая это неравенство на $\frac{1}{n}$ и используя неравенство (1.5), получаем справедливость леммы 1.1. □

2. Постановка задач

Вопросы ортогональных рядов изучены, например, в монографии [2] и в работах [6]–[10]. Отметим, что из теорем Меньшова [5] и Банаха [1] следует, что сходимость общих ортогональных рядов и сходимость общих рядов Фурье для функций из некоторого дифференциального класса являются разными задачами. В первом случае решающую роль играют коэффициенты ортогонального ряда. Во втором случае принадлежность функции f $(f\not\simeq 0)$ к любому дифференциальному классу не гарантирует сходимость ее ряда Фурье относительно общих ОНС. Стало быть, надо наложить условия на функции ОНС, чтобы ряд Фурье по этой системе имел «хорошие» свойства. Точнее, рассмотрим следующие классы функций:

A_1, непрерывные функции.

A_2, функции ограниченной вариации.

A_3, абсолютно непрерывные функции.

A_4, классы H_ω.

Рассмотрим также свойства рядов Фурье:

B_1, сходимость почти всюду.

B_2, суммируемость методами Чезаро п.в.

B_3, безусловная сходимость.

B_4, абсолютная сходимость и т.д.

Ставятся задачи: каким условиям должны удовлетворять функции ОНС, чтобы ряд Фурье каждой функции из класса A_i, $i = 1, 2, 3, 4, \ldots$, имел свойство B_j, $j = 1, 2, 3, 4, \ldots$. Некоторые из вышеотмеченных задач были рассмотрены в работах [11]–[20].

3. Основные результаты

Теорема 3.1. *Пусть* (φ_n) *ОНС на* $[0,1]$, $h(x) = 1$ *и*

$$\sum_{k=1}^{\infty} C_k^2(h)\mathrm{w}_k < \infty. \tag{3.1}$$

Если для любой последовательности $(a_n) \in \ell_2$ *(см. (1.7))*

$$D_n(a, \sqrt{\mathrm{w}}) = O(1), \tag{3.2}$$

то $\sum_{n=1}^{\infty} C_n^2(f)\mathrm{w}_n < +\infty$ *для любой функции* $f \in \mathrm{Lip}\,1$.

Доказательство. В равенстве (1.4) положим $F(x) = P_n(a, \sqrt{\mathrm{w}}, x)$, тогда

$$\int_0^1 f(x)P_n(a,\sqrt{\mathrm{w}},x)dx = \sum_{i=1}^{n-1}\left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(a,\sqrt{\mathrm{w}},x)dx$$
$$+ \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(f(x) - f\left(\frac{i}{n}\right)\right)P_n(a,\sqrt{\mathrm{w}},x)dx$$
$$+ f(1)\int_0^1 P_n(a,\sqrt{\mathrm{w}},x)\,dx = I_1 + I_2 + I_3. \tag{3.3}$$

Полагая $f \in \mathrm{Lip}\,1$ и учитывая (3.2) имеем

$$(3.4)\quad |I_1| \le \sum_{i=1}^{n-1} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right| \left| \int_0^{\frac{i}{n}} P_n(a, \sqrt{\mathrm{w}}, x) dx \right|$$

$$= O(1)\,\frac{1}{n} \sum_{i=1}^{n-1} \left| \int_0^{\frac{i}{n}} P_n(a, \sqrt{\mathrm{w}}, x) dx \right| = O(1) D_n(a, \sqrt{\mathrm{w}}) = O(1).$$

Пусть $\Delta_{in} = [\frac{i-1}{n}, \frac{i}{n}]$ и $f \in \mathrm{Lip}\,1$. Тогда имеем

$$|I_2| \le \sum_{i=1}^{n} \max_{x \in \Delta_{in}} \left| f(x) - f\left(\frac{i}{n}\right) \right| \int_{\frac{i-1}{n}}^{\frac{i}{n}} |P_n(a, \sqrt{\mathrm{w}}, x)| dx$$

$$= O(1)\,\frac{1}{n} \sum_{k=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |P_n(a, \sqrt{\mathrm{w}}, x)| dx$$

$$= O(1)\,\frac{1}{n} \int_0^1 |P_n(a, \sqrt{\mathrm{w}}, x)|\, dx = O(1)\,\frac{1}{n} \left(\int_0^1 P_n^2(a, \sqrt{\mathrm{w}}, x) dx \right)^{\frac{1}{2}}$$

$$(3.5)\quad = O(1)\,\frac{1}{n} \left(\sum_{k=1}^{n} a_k^2 \mathrm{w}_k \right)^{\frac{1}{2}} = O(1)\,\frac{\sqrt{\mathrm{w}_n}}{n} \left(\sum_{k=1}^{n} a_k^2 \right)^{\frac{1}{2}} = O(1).$$

Далее, используя неравенство Коши и (3.1) получим:

$$|I_3| = \left| f(1) \int_0^1 P_n(a, \sqrt{\mathrm{w}}, x)\, dx \right| = O(1) \left| \int_0^1 \sum_{k=1}^{n} a_k \sqrt{\mathrm{w}_k} \varphi_k(x)\, dx \right|$$

$$= O(1) \left| \sum_{k=1}^{n} a_k \sqrt{\mathrm{w}_k} \int_0^1 \varphi_k(x)\, dx \right| = O(1) \left| \sum_{k=1}^{n} a_k \sqrt{\mathrm{w}_k} C_k(h) \right|$$

$$= O(1) \left(\sum_{k=1}^{n} a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} C_k^2(h) \mathrm{w}_k \right)^{\frac{1}{2}} = O(1).$$

Из последнего неравенства и (3.3)–(3.5) будем иметь

$$\left| \int_0^1 f(x) P_n(a, \sqrt{\mathrm{w}}, x) dx \right| = O(1).$$

Отсюда и из (1.6) для любого $(a_n) \in \ell_2$

$$\sum_{k=1}^{n} a_k \sqrt{\mathrm{w}_k} C_k(f) = O(1).$$

Если теперь в качестве a_k возьмем $|a_k| \operatorname{sign} C_k(f)$, будем иметь

$$\sum_{k=1}^{n} |a_k \sqrt{\mathrm{w}_k} C_k(f)| = O(1).$$

Таким образом, для любого $(a_n) \in \ell_2$ сходится ряд $\sum_{k=1}^{\infty} a_k \sqrt{\mathrm{w}_k} C_k(f)$.

Отсюда в силу известной теоремы $(C_k(f)\sqrt{\mathrm{w}_k}) \in \ell_2$, т.е., для любой функции $f \in \mathrm{Lip}\,1$

$$\sum_{k=1}^{\infty} C_k^2(f) \mathrm{w}_k < +\infty.$$

Теорема 3.1 доказана. □

Теорема 3.2. *Пусть* (φ_n) *ОНС на* $[0,1]$ *и выполняется условие* (3.1). *Если для любой последовательности* $(a_n) \in \ell_2$ *выполняется условие (см.* (1.7)*)*

$$D_n(a, \sqrt{\mathrm{w}}) = O(1) \quad \textit{при} \;\; \mathrm{w}_k = \log^2 k,$$

тогда для любой функции $f \in \mathrm{Lip}\,1$ *ряд* $\sum\limits_{k=1}^{\infty} C_k(f)\varphi_k(x)$ *сходится п.в. на* $[0,1]$.

Доказательство. Действительно, полагая в теореме 3.1 $\mathrm{w}_k = \log^2 k$, получим

$$\sum_{k=1}^{\infty} C_k^2(f) \log^2 k < \infty.$$

Теперь справедливость теоремы 3.2 вытекает из теоремы 1.2. □

Теорема 3.3. *Пусть* (φ_n) *ОНС на* $[0,1]$ *и выполняется условие* (3.1). *Если для любой последовательности* $(a_n) \in \ell_2$ *и для некоторого* $\varepsilon > 0$ *выполняется условие (см.* (1.7)*)*

$$D_n(a, \sqrt{\mathrm{w}(\varepsilon)}) = O(1) \quad \textit{при} \;\; \mathrm{w}_k(\varepsilon) = \log^2(k+1)(\log\log(k+2))^{1+\varepsilon},$$

тогда ряд Фурье любой функции $f \in \mathrm{Lip}\,1$

$$\sum_{n=1}^{\infty} C_n(f)\varphi_n(x)$$

безусловно сходится п.в. на $[0,1]$.

Доказательство. Действительно, полагая в теореме 3.1

$$\mathrm{w}_k(\varepsilon) = \log^2(k+1)(\log\log(k+2))^{1+\varepsilon}$$

получим

$$\sum_{k=1}^{\infty} C_k^2(f)\mathrm{w}_k(\varepsilon) < \infty.$$

Теперь справедливость теоремы 3.3 вытекает из теоремы 1.3. □

Теорема 3.4. *Пусть* (φ_n) *ОНС на* $[0,1]$ *и выполняется условие* (3.1). *Если для любой последовательности* $(a_n) \in \ell_2$ *выполняется условие (см.* (1.7)*)*

$$D_n(a, \sqrt{\mathrm{w}}) = O(1) \quad \textit{при} \;\; \mathrm{w}_n = (\log\log n)^2,$$

тогда ряд Фурье любой функции $f \in \mathrm{Lip}\,1$

$$\sum_{n=1}^{\infty} C_n(f)\varphi_n(x)$$

суммируется методом (C,α), $\alpha > 0$, *п.в. на* $[0,1]$.

Доказательство. Действительно, полагая в теореме 3.1 $\mathrm{w}_n = (\log\log n)^2$, получим

$$\sum_{n=1}^{\infty} C_n^2(f)(\log\log n)^2 < \infty.$$

Теперь справедливость теоремы 3.4 вытекает из теоремы 1.4. □

Теорема 3.5. *Пусть* (φ_n) *заданная на* $[0,1]$ *ОНС и выполняется условие* (3.1). *Если для некоторой последовательности* $(b_n) \in \ell_2$

$$\limsup_{n\to\infty} D_n(b, \sqrt{\mathrm{w}}) = +\infty, \tag{3.6}$$

тогда существует функция $g \in \mathrm{Lip}\,1$ *такая, что*

$$\sum_{n=1}^{\infty} C_n^2(g)\mathrm{w}_n = +\infty.$$

Доказательство. Не ограничивая общности допустим, что

$$\left|\int_0^1 P_n(b, \sqrt{\mathrm{w}}, x)\,dx\right| = O(1). \tag{3.7}$$

Действительно, если

$$\limsup_{n\to\infty} \left|\int_0^1 P_n(b, \sqrt{\mathrm{w}}, x)\,dx\right| = +\infty,$$

то в виду того, что

$$\left|\int_0^1 P_n(b, \sqrt{\mathrm{w}}, x)\,dx\right| = \left|\int_0^1 \sum_{k=1}^{n} b_k\sqrt{\mathrm{w}_k}\varphi_k(x)\,dx\right|$$
$$= \left|\sum_{k=1}^{n} b_k\sqrt{\mathrm{w}_k}\int_0^1 \varphi_k(x)\,dx\right| \le \sum_{k=1}^{n} |b_k|\sqrt{\mathrm{w}_k}|C_k(h)|, \quad h(x) = 1,$$

получим

$$\sum_{k=1}^{\infty} |b_k|\sqrt{\mathrm{w}_k}|C_k(h)| = +\infty.$$

Отсюда, поскольку $b_k \in \ell_2$, вытекает, что

$$\sum_{k=1}^{\infty} |C_k(h)|^2\mathrm{w}_k = +\infty.$$

Так как $h \in \mathrm{Lip}\,1$ $(h(x) = 1)$, получаем доказательство теоремы 3.5. Следовательно, в дальнейшем будем считать, что справедливо (3.7).

Теперь рассмотрим последовательность функций

$$g_n(x) = \int_0^x \mathrm{sign} \int_0^u P_n(b, \sqrt{\mathrm{w}}, v)\,dv\,du. \tag{3.8}$$

В равенстве (1.4) положим $f = g_n$ и $F(x) = P_n(b, \sqrt{\mathrm{w}}, x)$, получим

$$\left|\int_0^1 g_n(x)P_n(b,\sqrt{\mathrm{w}},x)dx\right| = \left|\sum_{i=1}^{n-1}\left(g_n\left(\frac{i}{n}\right)-g_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx + \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(g_n(x)-g_n\left(\frac{i}{n}\right)\right)P_n(b,\sqrt{\mathrm{w}},x)dx + g_n(1)\int_0^1 P_n(b,\sqrt{\mathrm{w}},x)dx\right| \geq |M_1|-|M_2|-|M_3|. \tag{3.9}$$

Из (3.7) и (3.8) легко следует, что

$$|M_3| = \left|g_n(1)\int_0^1 P_n(b,\sqrt{\mathrm{w}},x)dx\right| = O(1). \tag{3.10}$$

Согласно (3.8), $|g_n(x) - g_n(\frac{i}{n})| \leq \frac{1}{n}$ при $x \in [\frac{i-1}{n}, \frac{i}{n}]$, $i = 1, 2, \dots, n$, и поскольку $(b_k) \in \ell_2$,

$$|M_2| \leq \frac{1}{n}\int_0^1 |P_n(b,\sqrt{\mathrm{w}},x)|dx \leq \frac{1}{n}\left(\int_0^1 P_n^2(b,\sqrt{\mathrm{w}},x)dx\right)^{\frac{1}{2}} = \frac{1}{n}\left(\sum_{k=1}^{n} b_k^2\mathrm{w}_k\right)^{\frac{1}{2}} \leq \frac{\sqrt{\mathrm{w}_n}}{n}\left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} = O(1). \tag{3.11}$$

Пусть $E_n = \{1, 2, \dots, n-1\} \setminus H_n$, где H_n – множество, которое было определено в лемме 1.1. Тогда, если $i \in E_n$, то

$$\left(g_n\left(\frac{i}{n}\right)-g_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx = -\frac{1}{n}\left|\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right|.$$

Отсюда

$$\left|\sum_{i\in E_n}\left(g_n\left(\frac{i}{n}\right)-g_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right| = \frac{1}{n}\sum_{i\in E_n}\left|\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right|.$$

Используя последнее равенство, будем иметь

$$|M_1| = \left|\sum_{i=1}^{n-1}\left(g_n\left(\frac{i}{n}\right)-g_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right| = \left|\sum_{i\in E_n}\left(g_n\left(\frac{i}{n}\right)-g_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx + \sum_{i\in H_n}\left(g_n\left(\frac{i}{n}\right)-g_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right| \tag{3.12}$$

$$\geq \frac{1}{n}\sum_{i\in E_n}\left|\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right|$$
$$-\left|\sum_{i\in H_n}\left(g_n\left(\frac{i}{n}\right)-g_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right|.$$

Из (3.8) и (1.8), получаем

$$\left|\sum_{i\in H_n}\left(g_n\left(\frac{i}{n}\right)-g_n\left(\frac{i+1}{n}\right)\right)\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right|$$
$$\leq \frac{1}{n}\sum_{i\in H_n}\left|\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)\right|dx = O(1).$$

Отсюда и из (3.12) имеем

$$|M_1| \geq \frac{1}{n}\sum_{i\in E_n}\left|\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right| - \frac{1}{n}\sum_{i\in H_n}\left|\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right|$$
$$= \frac{1}{n}\sum_{i=1}^{n-1}\left|\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right| - \frac{2}{n}\sum_{i\in H_n}\left|\int_0^{\frac{i}{n}} P_n(b,\sqrt{\mathrm{w}},x)dx\right|$$
$$\geq D_n(b,\sqrt{\mathrm{w}}) - O(1). \tag{3.13}$$

Наконец из (3.9)–(3.11) и (3.13) заключаем

$$\left|\int_0^1 g_n(x)P_n(b,\sqrt{\mathrm{w}},x)dx\right| \geq |M_1|-|M_2|-|M_3| \geq D_n(b,\sqrt{\mathrm{w}}) - O(1). \tag{3.14}$$

Рассмотрим последовательность линейных, ограниченных на Lip 1 функционалов

$$U_n(f) = \int_0^1 f(x)P_n(b,\sqrt{\mathrm{w}},x)dx.$$

Согласно (3.6) и (3.14)

$$\limsup_{n\to\infty} |U_n(g_n)| = +\infty.$$

С другой стороны (см. (1.2)),

$$\|g_n\|_{\mathrm{Lip}\,1} = \|g_n\|_C + \sup_{x,y\in[0,1]}\left|\frac{g_n(x)-g_n(y)}{x-y}\right| \leq 2.$$

Следовательно, в силу теоремы Банаха–Штейнгауза, существует функция $g \in \mathrm{Lip}\,1$ такая, что

$$\limsup_{n\to\infty} |U_n(g)| = +\infty.$$

Таким образом, при $(b_k) \in \ell_2$ расходится ряд

$$\sum_{k=1}^{\infty} b_k C_k(g)\sqrt{\mathrm{w}_k}.$$

Отсюда и заключаем, что

$$\sum_{k=1}^{\infty} C_k^2(g)\mathrm{w}_k = +\infty.$$

Теорема 3.5 полностью доказана. □

Из теоремы 3.5 вытекает неулучшаемость теоремы 3.1 в определенном смысле.

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EXISTENCE AND STABILITY OF INTEGRO DIFFERENTIAL EQUATION WITH GENERALIZED PROPORTIONAL FRACTIONAL DERIVATIVE

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Abstract. In this study, integro-differential equations of arbitrary order are studied. The fractional order is expressed in terms of the ψ-Hilfer type proportional fractional operator. This research exposes the dynamical behaviour of integro-differential equations with fractional order, such as existence, uniqueness, and stability solutions. To prove the results, the initial value problem and nonlocal conditions are used.

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1. Introduction

Fractional differential equations (FDEs) is considered as a branch of mathematical analysis that deals with the investigation and applications of integrals and derivatives of arbitrary order. Therefore, FDE is an extension of the integer-order calculus that considers integrals and derivatives of any real or complex order, see [9, 11, 13]. The topic of FDEs includes the study of analytic and numerical methods, as well as theoretical features such as existence, uniqueness, periodicity, and asymptotic behaviour. One can refer to [4, 5, 10, 16, 20, 21] for recent works on FDEs.

Nowadays, there are noteworthy potentials that have been spent on getting new classes of fractional operators by introducing more general or new kernels. Vanterler Da C. Sousa recently presented a fractional derivative with kernel of function, and the classical features of current fractional derivatives are explored in [15]. The theoretical analysis and current progress of the ψ-Hilfer fractional derivative can be observed in [16, 17]. In This work, we use the generalized fractional calculus for a special example of the proportional derivatives discussed in [6]. The new fractional derivative operator contains two parameters and has features, including maintaining the semi-group property and convergence to the original function as it tends to zero. Additionally, it is fully behaved and has fundamental features over the classical derivatives with the meaning that it generalizes already existing fractional derivatives in the literature. Some recent contributions on fractional differential equations in terms of the generalized proportional derivatives can be located in the papers, see [1] - [3].

The existence and uniqueness of the solution play an essential role in the study of FDEs, see [16, 20]. In this paper, we study the existence and uniqueness of solution for a certain type of nonlinear integro-differential equation(IDE) with initial and nonlocal conditions. Further the stability of solutions is also being discussed.

The paper is constructed as follows: In Section 2, we present the main definitions and interesting results. In Section 3, existence and stability results are established for proposed problems. In Section 4, existence and stability of solutions for nonlocal IDE is discussed.

2. Preliminaries

Some basic definitions and results introduced in this section. Let C be the Banach space of all continuous functions $\mathfrak{h} : J \to R$ with the norm

$$\|\mathfrak{h}\|_C = \sup\left\{|\mathfrak{h}(t)| : t \in J\right\}.$$

We denote the weighted spaces of all continuous functions defined by

$$C_{\nu,\psi}(J,R) = \left\{\mathfrak{g} : J \to R : (\psi(t)-\psi(0))^{\nu}\, \mathfrak{g}(t) \in C\right\},\ 0 \le \nu < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{\nu,\psi}} = \sup_{t\in J}\left|(\psi(t)-\psi(0))^{\nu}\, \mathfrak{g}(t)\right|.$$

Definition 2.1. [6]*If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$. Then the generalized proportional fractional(GPF) integral*

$$\left(\mathscr{I}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \int_0^t \psi^{'}(s) e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \frac{(\psi(t)-\psi(s))^{\alpha-1}}{\vartheta^{\alpha}\Gamma(\alpha)}\mathfrak{h}(s)ds. \tag{2.1}$$

Definition 2.2. [6] *If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a,b]$, where $\psi^{'}(s) > 0$, the GPF derivative of order α of the function $\mathfrak{h}$ with respect to another function isdefined by with $\psi^{'}(t) \neq 0$ is describe as*

$$\left(\mathscr{D}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^{n} \times \int_0^t \psi^{'}(s) e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \frac{(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\mathfrak{h}(s)ds. \tag{2.2}$$

Definition 2.3. [6]*If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a,b]$, where $\psi^{'}(s) > 0$, the GPF derivative in Caputo sence of order α of the function $\mathfrak{h}$ with* *respect to another function is defined by with* $\psi^{'}(t) \neq 0$ *is describe as*

$$\left(\mathscr{D}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \mathscr{I}^{n-\alpha,\vartheta;\psi}\left(\mathscr{D}^{n,\vartheta;\psi}\mathfrak{h}\right)(t). \tag{2.3}$$

Definition 2.4. [6] *The ψ-Hilfer GPF derivative of order α and type β over $\mathfrak{h}$ with respect to another function is defined by*

$$\left(\mathscr{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}\right)(t) = \mathscr{I}^{\beta(1-\alpha),\vartheta;\psi}\left(\mathscr{D}^{1,\vartheta;\psi}\right)\mathscr{I}^{(1-\beta)(1-\alpha),\vartheta;\psi}\mathfrak{h}(t). \tag{2.4}$$

Lemma 2.1. *Let $\alpha, \beta > 0$, Then we have the following semigroup property*

$$(\mathscr{I}^{\alpha,\vartheta;\psi}\mathscr{I}^{\beta,\vartheta;\psi}\mathfrak{g})(t) = (\mathscr{I}^{\alpha+\beta,\vartheta;\psi}\mathfrak{g})(t),$$

and

$$(\mathscr{D}^{\alpha,\vartheta;\psi}\mathscr{I}^{\alpha,\vartheta;\psi}\mathfrak{g})(t) = \mathfrak{g}(t).$$

Lemma 2.2. *Let $n-1 < \alpha < n$ where $n \in N, \vartheta \in (0,1]$, $0 \leq \beta \leq 1$, with $\nu = \alpha + \beta(n-\alpha)$, such that $n-1 < \nu < n$. If $\mathfrak{g} \in C_\nu$ and $\mathfrak{I}^{n-\nu,\vartheta;\psi}\mathfrak{g} \in C_\nu^n$, then*

$$(\mathscr{I}^{\alpha,\vartheta;\psi}\mathscr{I}^{\alpha,\beta,\vartheta;\psi}\mathfrak{g})(t) = \mathfrak{g}(t) - \sum_{k=1}^{n} \frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\nu-k}}{\vartheta^{\nu-k}\Gamma\nu-k+1}\mathscr{I}^{k-\nu,\vartheta;\psi}(a),$$

Lemma 2.3. [15]*(Grönwall's Lemma* [18]*) Let $\alpha > 0$, $a(t) > 0$ is locally integrable function on J and if $\mathfrak{g}(t)$ be a increasing and nonnegative continuous function on J, such that $|\mathfrak{g}(t)| \leq K$ for some constant K. Moreover if $\mathfrak{h}(t)$ be a nonnegative locally integrable function on J with*

$$\mathfrak{h}(t) \leq a(t) + g(t)\int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{h}(s)ds, \quad (t) \in J,$$

with some $\alpha > 0$. Then

$$\mathfrak{h}(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\mathfrak{g}(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{n\alpha-1}\right]a(s)ds, \quad (t) \in J.$$

Theorem 2.1. [8]*(Schauder fixed point theorem) Let B be closed, convex and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has atleast one fixed point in B.*

Theorem 2.2. [8]*(Krasnoselskii's fixed point theorem) Let X be a Banach space, let Ω be a bounded closed convex subset of X and let T_1, T_2 be mapping from Ω into X such that $T_1x + T_2y, \in \Omega$ for every pair $x, y \in \Omega$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1x + T_2x = x$ has a solution on Ω.*

Theorem 2.3. [8]*(Banach Fixed Point Theorem) Suppose Q be a non-empty closed subset of a Banach space E. Then any contraction mapping $\mathfrak{P}$ from Q into itself has a unique fixed point.*

3. Solution of Initial Value Problem

In this section, we consider the Initial value problem(IVP) for fractional IDE of the form

$$\begin{cases} \mathfrak{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t) = \mathfrak{g}\left(t, \mathfrak{h}(t), \int_0^t k(t,s,\mathfrak{h}(s))ds\right), & t \in J := [0,T], \\ \mathfrak{I}^{1-\nu,\vartheta;\psi}\mathfrak{h}(t)|_{t=0} = \mathfrak{h}_0, \end{cases} \tag{3.1}$$

where $\mathfrak{D}^{\alpha,\beta;\psi}$ is ψ-Hilfer GPF of orders $\alpha \in (0,1)$, type $\beta \in [0,1]$ and $\vartheta \in [0,1]$, $\mathfrak{h}$ is the given continuous function, $\mathfrak{I}^{1-\nu;\psi}$ is GPF fractional integral of orders $1-\nu(\nu = \alpha + \beta - \alpha\beta)$. Let R be a Banach space, $\mathfrak{g} : J \times R \times R \to R$ is a given continuous function. For brevity let us take

$$H\mathfrak{h}(t) = \int_0^t k(t,s,\mathfrak{h}(s))ds.$$

We make the following hypotheses to prove our main results, for every $t \in J$. We declare

(H1) There exists a constant $\ell_{\mathfrak{g}} > 0$ such that

$$|\mathfrak{g}(s,\mathfrak{h}_1(\cdot),\mathfrak{h}_2(\cdot)) - \mathfrak{g}(s,\mathfrak{y}_1(\cdot),\mathfrak{y}_2(\cdot))| \le \ell_{\mathfrak{g}}\left(|\mathfrak{h}_1(\cdot) - \mathfrak{y}_1(\cdot)| + |\mathfrak{h}_2(\cdot) - \mathfrak{y}_2(\cdot)|\right),$$

Set $\tilde{\mathfrak{g}} = \mathfrak{g}(s,0,0)$.

(H2) For all $\mathfrak{h}, \mathfrak{y} \in R$, there exists a there exists a constant $\ell_{\mathfrak{h}} > 0$, such that

$$\int_0^t |k(t,s,\mathfrak{h}) - k(t,s,\mathfrak{y})| \le \ell_{\mathfrak{h}}\,|\mathfrak{h}(\cdot) - \mathfrak{y}(\cdot)|\,.$$

Set $\tilde{k} = \int_0^s |k_\omega(s,\tau,0)|\,d\tau$.

(H3) There exists $\lambda_\varphi > 0$, we have $\mathfrak{I}^{\alpha;\psi}\varphi(t) \le \lambda_\varphi\varphi(t)$.

Lemma 3.1. *A function $\mathfrak{h}$ is the solution Eq.* (3.1)*, if and only if $\mathfrak{h}$ satisfies the random integral equation*

$$\begin{aligned} \mathfrak{h}(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t) - (\psi(0))^{\nu-1} \\ &+ \frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds. \end{aligned} \tag{3.2}$$

Theorem 3.1. *Assume that hypotheses* $(H1)$ *and* $(H2)$ *are satisfied. Then, Eq.* (3.1) *has at least one solution.*

Proof. Consider the operator $\mathfrak{P} : C_{1-\nu,\psi} \to C_{1-\nu,\psi}$, where the equivalent integral Eq. (3.2) which can be written in the operator form

$$\begin{aligned} \mathfrak{P}\mathfrak{h}(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t) - (\psi(0))^{\nu-1} \\ &+ \frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds. \end{aligned} \tag{3.3}$$

For any $\mathfrak{h} \in J$, we have

$$\left|\mathfrak{P}(\mathfrak{h}(t))\left(\psi(t)-\psi(0)\right)^{1-\nu}\right| \leq \left|\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}\right|$$
$$+\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))\right| ds$$
$$\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))-\mathfrak{g}(s,0,0)\right.$$
$$\left.+\mathfrak{g}(s,0,0)\right| ds$$
$$\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left(\ell_{\mathfrak{g}}\left|\mathfrak{h}(s)\right|+\ell_{\mathfrak{g}}\left|H\mathfrak{h}(s)\right|+\left|\tilde{\mathfrak{g}}\right|\right) ds$$
$$\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left(\ell_{\mathfrak{g}}\left|\mathfrak{h}(s)\right|\right.$$
$$\left.+\ell_{\mathfrak{g}}\int_0^s \left|k(s,\tau,\mathfrak{h}(\tau))-k(s,\tau,0)\right| d\tau+\ell_{\mathfrak{g}}\int_0^s \left|k(s,\tau,0)\right| d\tau+\left|\tilde{\mathfrak{g}}\right|\right) ds$$
$$\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}\left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})r+\ell_{\mathfrak{g}}\left\|\tilde{k}\right\|_{C_{1-\nu,\psi}}+\left\|\tilde{\mathfrak{g}}\right\|_{C_{1-\nu,\psi}}\right)=r.$$

This proves that $\mathfrak{P}$ transforms the ball $B_r = \left\{\mathfrak{h} \in C_{1-\nu,\psi} : \|\mathfrak{h}\|_{C_{1-\nu,\psi}} \leq r\right\}$ into itself. We shall show that the operator $\mathfrak{P} : B_r \to B_r$ satisfies all the conditions of Theorem 2.1. The proof will be given in several steps.

Step 1: $\mathfrak{P}$ is continuous. Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n \to \mathfrak{h}$ in $C_{1-\nu,\psi}$, we derive

$$\left|\left(\mathfrak{P}\mathfrak{h}_n(t)-\mathfrak{P}\mathfrak{h}(t)\right)\left(\psi(t)-\psi(0)\right)^{1-\nu}\right| \leq \frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\times$$
$$\times \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}_n(s),H\mathfrak{h}_n(s)))-\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s)))\right| ds$$
$$\leq \left(\psi(t)-\psi(0)\right)^{1-\nu}\frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}\left(\psi(t)-\psi(0)\right)^{\alpha+\nu-1}\times$$
$$\times \left\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot),H\mathfrak{h}_n(\cdot)))-\mathfrak{g}(\cdot,\mathfrak{h}(\cdot),H\mathfrak{h}(\cdot)))\right\|_{C_{1-\nu,\psi}}$$
$$\leq \frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}\left(\psi(T)-\psi(0)\right)^{\alpha}\left\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot),H\mathfrak{h}_n(\cdot)))-\mathfrak{g}(\cdot,\mathfrak{h}(\cdot),H\mathfrak{h}(\cdot)))\right\|_{C_{1-\nu,\psi}}.$$

Since $\mathfrak{g}$ is continuous, then we have $\|\mathfrak{P}\mathfrak{h}_n-\mathfrak{P}\mathfrak{h}\|_{C_{1-\nu,\psi}} \to 0$ as $n \to \infty$.

Step 2: $\mathfrak{P}(B_r)$ is uniformly bounded. This is clear since $\mathfrak{P}(B_r) \subset B_r$ is bounded.

Step 3: We show that $\mathfrak{P}(B_r)$ is equicontinuous.
Let $t_1, t_2 \in J, t_1 > t_2$ be a bounded set of $C_{1-\nu,\psi}$ as in Step 2, and $\mathfrak{h} \in B_r$. Then,

$$\begin{aligned}
&\left|(\psi(t_1)-\psi(0))^{1-\nu}\,\mathfrak{P}\mathfrak{h}(t_1)-(\psi(t_2)-\psi(0))^{1-\nu}\,\mathfrak{P}\mathfrak{h}(t_2)\right|\\
&\leq \left|\frac{(\psi(t_1)-\psi(0))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^{t_1}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds\right.\\
&\qquad\left.-\frac{(\psi(t_2)-\psi(0))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^{t_2}\psi^{'}(s)(\psi(t_2)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds\right|\\
&\leq \frac{\|\mathfrak{g}\|_{C_{1-\nu,\psi}}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)\left|(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}\right|.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Steps 1-3 together with the Arzela-Ascoli theorem, we can conclude that $\mathfrak{P}$ is continuous and compact. From an application of Theorem 2.1, we deduce that $\mathfrak{P}$ has a fixed point $\mathfrak{h}$ which is a solution of the problem (3.1). □

Lemma 3.2. *Assume that hypotheses (H1) and (H2) are satisfied. If*

$$\frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\vartheta^{\alpha}\Gamma(\alpha)}\left(\psi(T)-\psi(0)\right)^{\alpha}B(\nu,\alpha)<1.$$

Then, (3.1) *has a unique solution.*

Next, we shall give the definitions Generalized Ulam-Hyers-Rassias(g-UHR) stability stable for the problem (3.1). Let $\epsilon > 0$ be a positive real number and $\varphi : J \to R^{+}$ be a continuous function. We consider the following inequalities

$$\left|\mathfrak{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{y}(t)-\mathfrak{g}(t,\mathfrak{y}(t),H\mathfrak{y}(t))\right|\leq\varphi(t). \tag{3.4}$$

Definition 3.1. *Eq.* (3.1) *is g-UHR stable with respect to* φ *if there exists a real number* $C_{f,\varphi} > 0$ *such that for each solution* $\mathfrak{y} : R \to C_{1-\nu,\psi}$ *of the inequality* (3.4) *there exists a solution* $\mathfrak{h} : R \to C_{1-\nu,\psi}$ *of Eq.* (3.1) *with*

$$|\mathfrak{y}(t)-\mathfrak{h}(t)|\leq C_{f,\varphi}\varphi(t),\quad t\in J.$$

Theorem 3.2. *The hypotheses (H1), (H2) and (H3) hold. Then Eq.*(3.1) *is g-UHR stable.*

Proof. Let $\mathfrak{y}$ be solution of inequality (3.4) and by Lemma 3.2 there exists a unique solution $\mathfrak{h}$ for the Eq. (3.1). Thus we have

$$\begin{aligned}
\mathfrak{h}(t)=&\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t)-(\psi(0))^{\nu-1}\\
&+\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^{t}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds.
\end{aligned}$$

By differentiating inequality (3.4) for each $t \in J$, we have

$$\Bigg|\mathfrak{y}(t) - \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t)-\psi(0))^{\nu-1} - \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))ds\Bigg| \leq \lambda_{\varphi}\varphi(t).$$

Hence, it follows

$$\begin{aligned}
|\mathfrak{y}(t)-\mathfrak{h}(t)| &\leq \Bigg|\mathfrak{y}(t) - \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t)-(\psi(0))^{\nu-1} \\
&\quad + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds.\Bigg| \\
&\leq \Bigg|\mathfrak{y}(t) - \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t)-(\psi(0))^{\nu-1} \\
&\quad - \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))ds\Bigg| \\
&\quad + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))-\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))\right|ds \\
&\leq \lambda_{\varphi}\varphi(t) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{y}(s)-\mathfrak{h}(s)\right|ds \\
&\leq \lambda_{\varphi}\varphi(t) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\lambda_{\varphi}\varphi(s)ds := C_{f,\varphi}\varphi(t).
\end{aligned}$$

Thus, Eq.(3.1) is g-UHR stable. □

4. Solution of fractional nonlocal IVP

In this section, we study the existence, uniqueness and stability of IDE involving ψ-Hilfer derivative given by

$$\begin{cases} \mathfrak{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t),H\mathfrak{h}(t)), \\ \mathfrak{I}^{1-\nu,\vartheta;\psi}\mathfrak{h}(t) = \sum_{i=1}^{m} c_i\mathfrak{h}(\tau_i), \quad \tau_i \in J, \end{cases} \tag{4.1}$$

where $\tau_i, i = 0,1,...,m$ are prefixed points satisfying $0 < \tau_1 \leq ... \leq \tau_m < b$ and c_i is real numbers. Here, nonlocal condition $\mathfrak{h}(0) = \sum_{i=1}^{m} c_i\mathfrak{h}(\tau_i)$ can be applied in physical problems yields better effect than the initial conditions $\mathfrak{h}(0) = \mathfrak{h}_0$. Further, Eq. (3.1) is equivalent to mixed integral type of the form

(4.2)

$$\mathfrak{h}(t) = \begin{cases} \frac{T(\psi(t)-\psi(0))^{\nu-1}}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}\sum_{i=1}^{m} c_i \displaystyle\int_0^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)\times \\ (\psi(\tau_i)-\psi(s))^{\alpha-1}\,\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds \\ +\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds, \end{cases}$$

where

$$T = \frac{1}{\vartheta^{\nu-1}\Gamma(\nu) - \sum_{i=1}^{m} c_i e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(0))}(\psi(\tau_i)-\psi(0))^{\nu-1}}.$$

Theorem 4.1. *Assume that (H1) and (H2) are satisfied. Then, Eq.*(4.1) *has at least one solution.*

Consider the operator $\mathscr{N}: C_{1-\nu,\psi} \to C_{1-\nu,\psi}$, it is well defined and given by

(4.3)

$$\mathscr{N}\mathfrak{h}(t) = \begin{cases} \frac{T(\psi(t)-\psi(0))^{\nu-1}}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} \sum_{i=1}^{m} c_i \int_0^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi^{'}(s) \times \\ (\psi(\tau_i)-\psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s)) ds \\ + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi^{'}(s) (\psi(t)-\psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s)) ds. \end{cases}$$

Set $\widetilde{\mathfrak{g}}(s) = \mathfrak{g}(s, 0, 0)$. Consider the ball $B_r = \left\{ \mathfrak{h} \in C_{1-\nu,\psi} : \|\mathfrak{h}\|_{C_{1-\nu,\psi}} \le r \right\}$.
Now we subdivide the operator $\mathscr{N}$ into two operator $\mathscr{N}_1$ and $\mathscr{N}_2$ on B_r as follows;

$$\mathscr{N}_1\mathfrak{h}(t) = \frac{T(\psi(t)-\psi(0))^{\nu-1}}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} \sum_{i=1}^{m} c_i \int_0^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi^{'}(s) \times$$

$$\times (\psi(\tau_i)-\psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s)) ds,$$

and

$$\mathscr{N}_2\mathfrak{h}(t) = \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi^{'}(s) (\psi(t)-\psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s)) ds.$$

The proof is divided into several steps.
Step.1 $\mathscr{N}_1\mathfrak{h} + \mathscr{N}_2\mathfrak{y} \in B_r$ for every $\mathfrak{h}, \mathfrak{y} \in B_r$.

$$\begin{aligned} &\left| \mathscr{N}_1\mathfrak{h}(t) (\psi(t)-\psi(0))^{1-\nu} \right| \\ &\le \frac{T}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} |\mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s))| ds \\ &\le \frac{T}{\vartheta^{\alpha}\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi^{'}(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} (|\mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s)) - \mathfrak{g}(s,0,0)| + |\mathfrak{g}(s,0,0)|) ds \\ &\le \frac{T}{\vartheta^{\alpha}\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_a^{\tau_i} \psi^{'}(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} (\ell_{\mathfrak{g}} (|\mathfrak{h}(s)| + |H\mathfrak{h}(s))|) + |\widetilde{\mathfrak{g}}(s)|) ds \\ &\le \frac{T}{\vartheta^{\alpha}\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_a^{\tau_i} \psi^{'}(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} \left(\ell_{\mathfrak{g}} \left(|\mathfrak{h}(s)| + \ell_{\mathfrak{h}} |\mathfrak{h}(s))| + \left| \tilde{\mathfrak{k}}(s) \right| \right) + |\tilde{\mathfrak{g}}(s)| \right) ds. \end{aligned}$$

This gives

$$\|\mathscr{N}_1\mathfrak{h}\|_{C_{1-\nu,\psi}} \le \frac{B(\nu,\alpha)T}{\vartheta^{\alpha}\Gamma(\alpha)} \sum_{i=1}^{m} c_i (\psi(\tau_i)-\psi(a))^{\alpha+\nu-1} \times$$

$$\left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}}) \|\mathfrak{h}\|_{C_{1-\nu,\psi}} + \ell_{\mathfrak{g}} \left\| \tilde{\mathfrak{k}} \right\|_{C_{1-\nu,\psi}} + \|\widetilde{\mathfrak{g}}\|_{C_{1-\nu,\psi}} \right). \tag{4.4}$$

For operator $\mathscr{N}_2$

$$
\begin{aligned}
&\left|\mathscr{N}_2\mathfrak{h}(t)\left(\psi(t)-\psi(0)\right)^{1-\nu}\right| \\
&\leq \frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))\right| ds \\
&\leq \frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\left(\psi(t)-\psi(0)\right)^{\alpha+\nu-1}B(\nu,\alpha)\times \\
&\times\left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\left\|\mathfrak{h}\right\|_{C_{1-\nu,\psi}}+\ell_{\mathfrak{g}}\left\|\tilde{\mathfrak{k}}\right\|_{C_{1-\nu,\psi}}+\left\|\widetilde{\mathfrak{g}}\right\|_{C_{1-\nu,\psi}}\right).
\end{aligned}
$$

Thus, we obtain

(4.5)

$$
\left\|\mathscr{N}_2\mathfrak{h}\right\|_{1-\nu}\leq \frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}\left(\psi(t)-\psi(0)\right)^{\alpha}\left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\left\|\mathfrak{h}\right\|_{C_{1-\nu,\psi}}+\ell_{\mathfrak{g}}\left\|\tilde{\mathfrak{k}}\right\|_{C_{1-\nu,\psi}}+\left\|\widetilde{\mathfrak{g}}\right\|_{C_{1-\nu,\psi}}\right).
$$

Linking (4.4) and (4.5), for every $\mathfrak{h},\mathfrak{y}\in B_r$,

$$
\left\|\mathscr{N}_1\mathfrak{h}+\mathscr{N}_2\mathfrak{y}\right\|_{C_{1-\nu,\psi}}\leq\left\|\mathscr{N}_1\mathfrak{h}\right\|_{C_{1-\nu,\psi}}+\left\|\mathscr{N}_2\mathfrak{y}\right\|_{C_{1-\nu,\psi}}\leq r.
$$

Step.2 $\mathscr{N}_1$ is a contration mapping.

For any $\mathfrak{h},\mathfrak{y}\in B_r$

$$
\begin{aligned}
&\left|\left(\mathscr{N}_1\mathfrak{h}(t)-\mathscr{N}_1\mathfrak{y}(t)\right)\left(\psi(t)-\psi(0)\right)^{1-\nu}\right| \\
&\leq \frac{T}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\int_a^{\tau_i}\psi^{'}(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))-\mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))\right| ds \\
&\leq \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\int_a^{\tau_i}\psi^{'}(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{h}(s)-\mathfrak{y}(s)\right| ds \\
&\leq \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\nu-1}B(\nu,\alpha)\left\|\mathfrak{h}-\mathfrak{y})\right\|_{C_{1-\nu,\psi}}.
\end{aligned}
$$

This gives

$$
\left\|(\mathscr{N}_1\mathfrak{h}-\mathscr{N}_1\mathfrak{y})\right\|_{C_{1-\nu,\psi}}\leq\frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\nu-1}B(\nu,\alpha)\left\|\mathfrak{h}-\mathfrak{y})\right\|_{C_{1-\nu,\psi}}.
$$

The operator $\mathscr{N}_1$ is contraction.

Step.3 The operator $\mathscr{N}_2$ is compact and continuous.

According to Step 1, we know that

$$
\left\|\mathscr{N}_2\mathfrak{h}\right\|_{1-\nu,\psi}\leq \frac{B(\nu,\alpha)}{\Gamma(\alpha)}\left(\psi(t)-\psi(0)\right)^{\alpha}\left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\left\|\mathfrak{h}\right\|_{C_{1-\nu,\psi}}+\ell_{\mathfrak{g}}\left\|\tilde{\mathfrak{k}}\right\|_{C_{1-\nu,\psi}}+\left\|\widetilde{\mathfrak{g}}\right\|_{C_{1-\nu,\psi}}\right).
$$

So operator $\mathscr{N}_2$ is uniformly bounded.

Now we prove the compactness of operator $\mathscr{N}_2$.

For $0 < t_1 < t_2 < T$, we have

$$|\mathscr{N}_2\mathfrak{h}(t_1) - \mathscr{N}_2\mathfrak{h}(t_2)| \leq \left| \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^{t_1} \psi^{'}(s) \left(\psi(t_1) - \psi(s)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s))ds \right.$$
$$\left. - \frac{1}{\vartheta^{\alpha} \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\Gamma(\alpha)} \int_0^{t_2} \psi^{'}(s) \left(\psi(t_2) - \psi(a)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s))ds \right|$$
$$\leq \|\mathfrak{g}\|_{C_{1-\nu,\psi}} B(\nu, \alpha) \left| \left(\psi(t_1) - \psi(0)\right)^{\alpha+\nu-1} - \left(\psi(t_2) - \psi(0)\right)^{\alpha+\nu-1} \right|$$

tending to zero as $t_1 \to t_2$. Thus $\mathscr{N}_2$ is equicontinuous. Hence, the operator $\mathscr{N}_2$ is compact on B_r by the Arzela-Ascoli Theorem. It follows from Theorem 2.2 that the problem (4.1) has at least one solution.

Theorem 4.2. *If hypothesis (H1) and the constant*

$$\delta = \frac{\ell_{\mathfrak{g}}(1 + \ell_{\mathfrak{h}})B(\nu, \alpha)}{\vartheta^{\alpha}\Gamma(\alpha)} \left(T \sum_{i=1}^{m} c_i \left(\psi(\tau_i) - \psi(a)\right)^{\alpha+\nu-1} + \left(\psi(b) - \psi(a)\right)^{\alpha} \right) < 1$$

holds. Then, Eq. (4.1) *has a unique solution.*

Next, we shall give the definitions of g-UHR stability for Eq. (4.1)

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{y}(t) - \mathfrak{g}(t, \mathfrak{h}(t), H\mathfrak{h}(t))\right| \leq \varphi(t). \tag{4.6}$$

Definition 4.1. *Eq.* (4.1) *is g-UHR stable with respect to* $\varphi \in C_{1-\nu,\psi}$ *if there exists a real number* $C_{f,\varphi} > 0$ *such that for each solution* $\mathfrak{y} \in C_{1-\nu,\psi}$ *of the inequality* (4.6) *there exists a solution* $\mathfrak{h} \in C_{1-\nu,\psi}$ *of Eq.* (4.1) *with*

$$|\mathfrak{y}(t) - \mathfrak{h}(t)| \leq C_{\mathfrak{g},\varphi}\varphi(t).$$

Theorem 4.3. *Let hypotheses (H1) - (H3) are fullfilled. Then Eq.* (4.1) *is g-UHR stable.*

Let $\mathfrak{y}$ be solution of inequality (4.6) and by Theorem 4.2 there $\mathfrak{h}$ is unique solution of equation

$$\mathfrak{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t) = \mathfrak{g}(t, \mathfrak{h}(t), H\mathfrak{h}(t)),$$
$$\mathfrak{I}^{1-\nu,\vartheta;\psi}\mathfrak{h}(t) = \sum_{i=1}^{m} c_i \mathfrak{h}(\tau_i), \quad \tau_i \in J$$

is given by

$$\mathfrak{h}(t) = A_{\mathfrak{h}} + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s))ds,$$

where

$$A_{\mathfrak{h}} = \frac{T \left(\psi(t) - \psi(0)\right)^{\nu-1}}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} \sum_{i=1}^{m} c_i \int_0^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \times$$
$$\psi^{'}(s) \left(\psi(\tau_i) - \psi(s)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s))ds.$$

Thus $A_{\mathfrak{h}} = A_{\mathfrak{y}}$.

By differentiating inequality (4.6), we have

$$\left|\mathfrak{y}(t) - A_{\mathfrak{y}} - \frac{1}{\vartheta^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s) \left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{y}(s), H\mathfrak{y}(s)) ds\right|$$
$$\leq \lambda_\varphi \varphi(t).$$

Hence, it follows

$$|\mathfrak{y}(t) - \mathfrak{h}(t)|$$
$$\leq \left|\mathfrak{y}(t) - A_{\mathfrak{h}} - \frac{1}{\vartheta^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s) \left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s)) ds\right|$$
$$\leq \left|\mathfrak{y}(t) - A_{\mathfrak{h}} - \frac{1}{\vartheta^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s) \left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{y}(s), H\mathfrak{y}(s)) ds\right|$$
$$+ \frac{1}{\vartheta^\alpha \Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \left|\mathfrak{g}(s, \mathfrak{y}(s), H\mathfrak{y}(s)) - \mathfrak{g}(s, \mathfrak{h}(s), H\mathfrak{h}(s))\right| ds$$
$$\leq \lambda_\varphi \varphi(t) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{g}})}{\vartheta^\alpha \Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \left|\mathfrak{y}(t) - \mathfrak{h}(t)\right| ds.$$

By Lemma 2.3, there exists a constant $M^* > 0$ independent of $\lambda_\varphi \varphi(t)$ such that

$$|\mathfrak{y}(t) - \mathfrak{h}(t)| \leq M^* \lambda_\varphi \varphi(t) := C_{f,\varphi} \varphi(t).$$

Thus, Eq.(4.1) is g-UHR stable.

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OPERATORS ON MIXED-NORM AMALGAM SPACES VIA EXTRAPOLATION

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Abstract. Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty)^n$. We establish versions of the Rubio de Francia extrapolation theorem, and further obtain the bounds for some classical operators and the commutators in harmonic analysis on the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$. As an application, a characterization of the mixed-norm amalgam spaces is given.

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1. Introduction

In 1926, the first appearance of amalgam spaces can be traced to Wiener [36]. But the first systematic study of these spaces was undertaken by Holland [20] in 1975. Feichtinger initially called these spaces Wiener-type spaces in the early 1980's in a series of papers [14, 15, 16], and then, following a suggestion of Benedetto, adopted the name Wiener amalgam spaces. That is, for p, $q \in (0,\infty)$, the amalgam space $(L^p, L^q)(\mathbb{R})$ is defined by

$$(L^p,\ell^q)(\mathbb{R}) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}) :\ \left[\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(x)|^p\ dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} < \infty \right\}.$$

Wiener amalgam spaces are a central object of the time-frequency analysis, another area with links to several mathematical subjects as well as its applications. The mixed amalgam spaces provide for a basic tool for harmonic analysis. And that makes these spaces extremely prominent to us. Very recently, lots of vital work has been done in the study of amalgam spaces. In 2011, Ruzhansky, Sugimoto, Toft and Tomita [29] established various properties of global and local changes of variables as well as properties of canonical transforms on Wiener amalgam spaces. In 2016, Delgado, Ruzhansky and Wang proved the metric approximation property for Wiener amalgam spaces in [8] and [9]. In 2022, Wang [35] obtained global

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regularity estimates for solutions of non-divergence elliptic equations on amalgam spaces spaces if the coefficient matrix is symmetric. For some historical notes and for an introduction about Wiener amalgam spaces on the real line can also be refered to [18].

Recently, to study the weak solutions of boundary value problems for a t-independent elliptic systems in the upper half plane, Auscher and Mourgoglou [2] introduced a particular amalgam space, the slice space $E_t^p(\mathbb{R}^n)$. Moreover, Auscher and Prisuelos-Arribas [3] introduced a more general slice space $(E_r^q)_t(\mathbb{R}^n)$ for $r \in (1, \infty)$, $t \in (0, \infty)$ and $q \in [1, \infty)$, and studied the boundedness of some classical operators on these spaces. For further study and a deeper account of developments on the slice spaces we may consult [19, 40] and the references therein. In 2022, Zhang and Zhou [39] first introduced the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$, as natural generalizations of the amalgam space $(L^p, L^q)_t(\mathbb{R}^n)$.

For $t \in (0, \infty)$, $\vec{p} \in (1, \infty)^n$ and $\vec{q} \in [1, \infty)^n$, the main purpose of this paper is to establish a version of the Rubio de Francia extrapolation theorem on mixed-norm amalgam spaces, and obtain the boundedness of some classical operators and the linear commutators by this theorem over the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$. Moreover, we study characterizations of mixed-norm amalgam spaces via the Littlewood–Paley functions. The bounds for the commutators and the characterizations of the mixed-norm amalgam spaces are new results even for the classical amalgam spaces.

This paper is organized as follows. Main definitions and necessary lemmas will be given in Section 2. In Section 3, we establish versions of the Rubio de Francia extrapolation theorem over the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$. In Section 4, the boundedness of some operators and the commutators are given on mixed-norm amalgam spaces by the results of Section 3. In the final section, characterizations of mixed-norm amalgam spaces via the Littlewood–Paley functions is given.

Finally, we make some convention on notation. Let $\vec{p} = (p_1, ..., p_n)$ be n-tuples and $1 \leq p_1, ..., p_n \leq \infty$. $\vec{p} < \vec{q}$ means that $p_i < q_i$ holds, $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$ means that $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ holds, and $\vec{p}/p_0$ means that p_i/p_0 holds, where $p_0 \in [1, \infty)$, $i = 1, ..., n$. For $\alpha > 0$ and a cube $Q \subset \mathbb{R}^n$. $A \sim B$ means that A is equivalent to B, that is, $A \lesssim B$ $(A \leq CB)$ and $B \lesssim A$ $(B \leq CA)$, where C is a positive constant. Throughout this paper, the letter C will be used for positive constants independent of relevant variables that may change from one occurrence to another.

2. Definitions and main lemmas

To state our main definitions, we begin with the definition of the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ which introduced by Benedek and Panzone [4] in 1961.

Let $\vec{p} \in [1,\infty]^n$. The mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined by the set of all measurable functions f on $\mathbb{R}^n$, such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |f(x_1,...,x_n)|^{p_1}\, dx_1\right)^{\frac{p_2}{p_1}} \cdots dx_n\right)^{\frac{1}{p_n}} < \infty,$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1,...,n\}$.

Definition 2.1. *Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$. The mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ is defined as the space of all measurable functions f on $\mathbb{R}^n$ satisfying*

$$\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} := \left\| \frac{\|f\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} < \infty,$$

with the usual modification for $q_i = \infty$ for each $i = 1,...,n$.

Remark 2.1. *If $p_1 = \cdots = p_n = p$ and $q_1 = \cdots = q_n = q$, then $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ is the slice space $(E_p^q)_t(\mathbb{R}^n)$ and the amalgam space $(L^p, L^q)_t(\mathbb{R}^n)$ (see* [3, 2]*).*

Definition 2.2. *Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$. The weak mixed-norm amalgam space $W(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ is defined as the space of all measurable functions f on $\mathbb{R}^n$ satisfying*

$$\|f\|_{W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda \left\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} < \infty,$$

with the usual modification for $q_i = \infty$, $i = 1,...,n$.

Note that if $p_1 = \cdots = p_n = p$ and $q_1 = \cdots = q_n = p$, then $W(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ is the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$, where

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}} < \infty.$$

We still recall the definition of Muckenhoupt's weights $A_p (1 \leq p \leq \infty)$. These weights introduced in [26] were used to characterize the boundedness of the Hardy–Littlewood maximal operator on weighted Lebesgue spaces. For a locllay integrable function f and for every $x \in \mathbb{R}^n$, the centered Hardy–Littlewood maximal operator is defined by,

$$Mf(x) := \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|\, dy,$$

and the uncentered Hardy–Littlewood maximal operator is defined by,

$$M_u f(x) := \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Definition 2.3. *Let $1 < p < \infty$. A weight w is said to be of class A_p if*

$$[w]_{A_p} := \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{\frac{1}{1-p}}\,dx\right)^{p-1} < \infty.$$

A weight w is said to be of class A_1 if

$$[w]_{A_1} := \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty, \quad \textit{for almost all } x\in\mathbb{R}^n.$$

For $p = \infty$, $A_\infty := \cup_{p\geq 1} A_p$.

Some vital lemmas over the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ will be given in the following.

(b) For any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_n(\cdots(M_1(f)\cdots))(x) = \sup_{I_n\in\mathbb{I}_{x_n}} \left\{\frac{1}{|I_n|}\int_{I_n}\cdots\sup_{I_1\in\mathbb{I}_{x_1}}\left[\frac{1}{|I_1|}\int_{I_1}|f(y_1,...,y_n)|\,dy_1\right]\cdots dy_n\right\},$$

where, for any $k \in \{1,...,n\}$, $\mathbb{I}_{x_k}$ denotes the set of all intervals in $\mathbb{R}$ containing x_k. Then, it is easy to see that, for any $x \in \mathbb{R}^n$,

$$M(f)(x) \leq M_n(\cdots(M_1(f)\cdots))(x).$$

Lemma 2.1. [39] *Let $t \in (0,\infty)$. Given $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$,*

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}, \quad f \in (L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n) \textit{ and } g \in (L^{\vec{p}'}, L^{\vec{q}'})_t(\mathbb{R}^n).$$

where $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = \frac{1}{\vec{q}} + \frac{1}{\vec{q}'} = 1$.

Lemma 2.2. [39] *Let $t \in (0,\infty)$. Given $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$,*

$$[(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)]' = (L^{\vec{p}'}, L^{\vec{q}'})_t(\mathbb{R}^n),$$

where $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = \frac{1}{\vec{q}} + \frac{1}{\vec{q}'} = 1$, and as for dual space of mixed-norm amalgam spaces, then

$$[(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)]' := \left\{f : \|f\|_{[(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)]'} := \sup_{\|g\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\leq 1} \int_{\mathbb{R}^n} f(x)g(x)dx\right\}.$$

Lemma 2.3. [39] *Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$. For any constant $\rho \in [1,\infty)$, we have*

$$C_1 \|f\|_{(L^{\vec{p}},L^{\vec{q}})_{\rho t}(\mathbb{R}^n)} \leq \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq C_2 \|f\|_{(L^{\vec{p}},L^{\vec{q}})_{\rho t}(\mathbb{R}^n)},$$

where the positive constants C_1, C_2 are independent of f and t.

Lemma 2.4. [5, 21]*If $\vec{q} = (q_1,...,q_n)$ satisfies one of the following conditions:*

(a) $1 < q_1,...,q_n < \infty$;

(b) $q_1 = \cdots = q_n = \infty$.

Then for any $f \in L^{\vec{q}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is bounded on $L^{\vec{q}}(\mathbb{R}^n)$.

The boundedness of the Hardy–Littlewood maximal operator on mixed Lebesgue spaces $L^{\vec{q}}(\mathbb{R}^n)$ for the case of (a) is just [21, Lemma 3.5]. And the case of (c) holds by a similar argument to the bounds for M on $L^\infty(\mathbb{R})$ (see [5, p.14]).

Lemma 2.5. *Let $t \in (0,\infty)$ and $\vec{p} \in (1,\infty)^n$. Assume that $\vec{q}$ satisfies the conditions of Lemma 2.4, then the Hardy–Littlewood maximal operator is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.*

Proof. Fix $x \in \mathbb{R}^n$ and $t > 0$, and split the sumprem into $0 < r \le t$ and $t < r$, and then

$$Mf(y) \le \sup_{0<r\le t} \frac{1}{|Q(y,r)|}\int_{Q(y,r)} |f(z)|dz + \sup_{r>t} \frac{1}{|Q(y,\tau)|}\int_{Q(y,r)} |f(z)|dz := I + II.$$

For I, since $y \in Q(x,t)$, $Q(y,r) \subset Q(x,2t)$. Then

$$I \lesssim \sup_{0<r\le t} \frac{1}{|Q(y,r)|}\int_{Q(y,r)} |f(z)|\, \chi_{Q(x,2t)}(z)dz \le M(f\chi_{Q(\cdot,2t)})(y).$$

For II, for any $z, \xi \in \mathbb{R}^n$, $\xi \in Q(z,t)$ is equivalent to $z \in Q(\xi,t)$. If $z \in Q(y,r)$, $\xi \in Q(z,t)$, then $\xi \in Q(y,2r)$. Besides, owing to $x \in Q(y,t)$, then $x \in Q(y,2r)$. Applying the Fubini's theorem and the Hölder inequality, then we get

$$\begin{aligned} II &= \sup_{r>t} \frac{1}{|Q(y,r)|}\int_{Q(y,r)} |f(z)|_{Q(z,t)} d\xi dz \\ &\lesssim \sup_{r>t} \frac{1}{|Q(y,2r)|}\int_{Q(y,2r)} \frac{1}{|Q(\xi,t)|}\int_{Q(\xi,t)} |f(z)|dzd\xi \\ &\le M_u\left(\frac{1}{|Q(\cdot,t)|}\int_{Q(\cdot,t)} |f(z)|dz\right)(x) \le M_u\left(\frac{\|f\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}}}\right)(x). \end{aligned}$$

Therefore, by Lemmas 2.3 and 2.4, we write

$$\begin{aligned} \|Mf\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &\lesssim \left\| \frac{\|M(f\chi_{Q(\cdot,2t)})\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\quad + \left\| \frac{\left\| M_u\left(\frac{\|f\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}}}\right)\chi_{Q(\cdot,t)} \right\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\lesssim \left\| \frac{\|f\chi_{Q(\cdot,2t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\quad + \left\| \frac{\|f\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \sim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}. \end{aligned}$$

This completes the proof of the Lemma 2.5. □

3. Extrapolation

In this section, we establish a new version of extrapolation theorem on mixed-norm amalgam spaces via the algorithm of Rubio de Francia for generating A_1

weights with certain properties (see [17]). To state our results, we begin with some necessary definitions.

A weight w is a positive and locally integrable function on $\mathbb{R}^n$. For $p \in (0, \infty)$, the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right]^{\frac{1}{p}} < \infty.$$

The weak weighted Lebesgue space $L^{p,\infty}_w(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\infty}_w(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{\frac{1}{p}} < \infty.$$

For $p = \infty$,

$$\|f\|_{L^\infty_w(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| < \infty.$$

Theorem 3.1. *Given a family of extrapolation pairs $\mathcal{F}$. Assume that for some $1 \le p_0 \le q_0 < \infty$, and for all $w \in A_1$,*

$$\left[\int_{\mathbb{R}^n} f(x)^{q_0} w(x) dx\right]^{\frac{1}{q_0}} \le C_{w,p_0} \left[\int_{\mathbb{R}^n} g(x)^{p_0} w(x)^{p_0/q_0} dx\right]^{\frac{1}{p_0}}, \quad \forall (f,g) \in \mathcal{F}. \tag{3.1}$$

Let $t \in (0,\infty)$, $\vec{r}$, $\vec{s} \in (p_0,\infty)^n$ and $\vec{p}$, $\vec{q} \in (q_0,\infty)^n$ with $1/r_i - 1/p_i = 1/s_i - 1/q_i = 1/p_0 - 1/q_0 > 0$ for each $i = 1, ..., n$. Then

$$\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \le C\|g\|_{(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)}. \tag{3.2}$$

The positive constant C is independent of f and t.

Proof. Let $m := \|M\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)\to(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)}$. By Lemma 2.5, we conclude that the Hardy–Littlewood maximal operator M is bounded on $(L^{(\vec{p}/q_0)'}$, $L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)$. We begin the proof by using the Rubio de Francia iteration algorithm. The algorithm $\mathcal{R} : L^0(\mathbb{R}^n) \to [0,\infty]$ is defined by

$$\mathcal{R}h(x) := \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k m^k},$$

where for $k \ge 1$, $M^k = M \circ \cdots \circ M$ is k iterations of M, and $M^0 h := |h|$. We show the following properties for all $h \in (L^{(\vec{p}/q_0)'}, L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)$:

(A) $|h| \le \mathcal{R}h$,

(B) $\|\mathcal{R}h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \le 2\|h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)}$,

(C) $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \le 2m$.

Property (A) holds since $\mathcal{R}h \geq M^0(h) = |h|$. Property (B) holds by the fact that $m < \infty$, since

$$\|\mathcal{R}h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \leq \sum_{k=0}^{\infty} \frac{\left\|M^k h\right\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)}}{2^k m^k} \leq 2\|h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)}.$$

Let us then prove (C). We may assume that $h \neq 0$, since the claim is trivial otherwise. It is equivalent to prove that $M(\mathcal{R}h)(x) \leq 2m\mathcal{R}h(x)$. By the definition of $\mathcal{R}$ and the sublinearity of the maximal operator, we obtain

$$M(\mathcal{R}h)(x) = M\left(\sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k m^k}\right) \leq \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^k m^k} = 2m \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^{k+1}m^{k+1}} \leq 2m\mathcal{R}h(x).$$

Fix $(f,g) \in \mathcal{F}$ and define $\mathcal{H} := \{h : \|h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \leq 1\}$. Note that $\|f\|^{q_0}_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} = \|f^{q_0}\|_{(L^{\vec{p}/q_0},L^{\vec{q}/q_0})_t(\mathbb{R}^n)}$. By Lemma 2.2 and (A), we see

$$\|f\|^{q_0}_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} = \sup_{h\in\mathcal{H}} \int_{\mathbb{R}^n} f(x)^{q_0} h(x)dx \leq \sup_{h\in\mathcal{H}} \int_{\mathbb{R}^n} f(x)^{q_0} \mathcal{R}h(x)dx. \tag{3.3}$$

To apply our hypothesis, by our convention on families of extrapolation pairs we need to show that the left-hand side in (3.1) is finite. This follows from Hölder's inequality and property (B): for all $h \in \mathcal{H}$,

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{q_0}\mathcal{R}h(x)dx &\lesssim \|f^{q_0}\|_{(L^{\vec{p}/q_0},L^{\vec{q}/q_0})_t(\mathbb{R}^n)} \|\mathcal{R}h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \\ &\lesssim \|f\|^{q_0}_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \|h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} < \infty.\end{aligned}$$

Given this and (C), we can apply our hypothesis (3.1) in (3.3) to get that

$$\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq \sup_{h\in\mathcal{H}} \left(\int_{\mathbb{R}^n} f(x)^{q_0}\mathcal{R}h(x)dx\right)^{\frac{1}{q_0}} \leq \sup_{h\in\mathcal{H}} \left(\int_{\mathbb{R}^n} g(x)^{p_0}(\mathcal{R}h(x))^{p_0/q_0}dx\right)^{\frac{1}{p_0}}. \tag{3.4}$$

Then for any $h \in \mathcal{H}$, by Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)^{p_0}(\mathcal{R}h(x))^{p_0/q_0}dx &\lesssim \|g^{p_0}\|_{(L^{\vec{r}/p_0},L^{\vec{s}/p_0})_t(\mathbb{R}^n)} \left\|(\mathcal{R}h)^{p_0/q_0}\right\|_{(L^{(\vec{r}/p_0)'},L^{(\vec{s}/p_0)'})_t(\mathbb{R}^n)} \\ &\leq \|g\|^{p_0}_{(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)} \|\mathcal{R}h\|^{p_0/q_0}_{(L^{p_0(\vec{r}/p_0)'/q_0},L^{p_0(\vec{s}/p_0)'/q_0})_t(\mathbb{R}^n)}.\end{aligned}$$

Notice that

$$\frac{p_0}{q_0}\left(\frac{\vec{r}}{p_0}\right)' = \left(\frac{\vec{p}}{q_0}\right)' \quad \text{and} \quad \frac{p_0}{q_0}\left(\frac{\vec{s}}{p_0}\right)' = \left(\frac{\vec{q}}{q_0}\right)'.$$

It follows from the property (B) that

$$\int_{\mathbb{R}^n} g(x)^{p_0}\left(\mathcal{R}h(x)\right)^{p_0/q_0}dx \leq \|g\|^{p_0}_{(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)} \|\mathcal{R}h\|^{p_0/q_0}_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \lesssim \|g\|^{p_0}_{(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)}. \tag{3.5}$$

Combined with (3.4) and (3.5), the desired result is concluded. $\square$

Corollary 3.1. *Given a family of extrapolation pairs $\mathcal{F}$, assume that for some $1 \le p_0 < \infty$, and for all $w \in A_1$,*

$$\left[\int_{\mathbb{R}^n} f(x)^{p_0} w(x)dx\right]^{\frac{1}{p_0}} \le C_{w,p_0} \left[\int_{\mathbb{R}^n} g(x)^{p_0} w(x)dx\right]^{\frac{1}{p_0}}, \quad \forall (f,g) \in \mathcal{F}.$$

Let $t \in (0,\infty)$ and $\vec{p}$, $\vec{q} \in (p_0,\infty)^n$. Then

$$\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \le C\|g\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

The positive constant C is independent of f and t.

For a linear operator $\mathcal{T}$ and a locally integrable function b, the commutators of $\mathcal{T}$ is defined for smooth functions f by

$$[b,\mathcal{T}]f(x) = b(x)\mathcal{T}f(x) - \mathcal{T}(bf)(x).$$

Now, we recall the definition of $BMO(\mathbb{R}^n)$. $BMO(\mathbb{R}^n)$ is the Banach function space modulo constants with the norm $\|\cdot\|_{BMO}$ defined by

$$\|b\|_{BMO} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b(y)-b_Q|dy < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ and b_Q implies the mean value of b over Q; that is, $b_Q := \frac{1}{|Q|}\int_Q b(y)dy$.

Theorem 3.2. *Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $w \in A_1$. Let $\mathcal{T}$ be a sublinear operator.*

(a) For $\vec{q} = (1,...,1)$, suppose that the operator $\mathcal{T}$ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$. Then

$$\|\mathcal{T}f\|_{W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

(b) For $\vec{q} = (1,...,1)$, suppose that the commutators $[b,\mathcal{T}]$ with $b \in BMO(\mathbb{R}^n)$ satisfies

(3.6)

$$w\left(\{y\in\mathbb{R}^n : |[b,\mathcal{T}]f(y)| > \lambda\}\right) \lesssim \|b\|_{BMO}\int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)w(y)dy.$$

Then

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,\mathcal{T}]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Proof. By Lemma 2.2, there exists $g \in (L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)$ such that

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |\mathcal{T}f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n:|\mathcal{T}f(x)|>\lambda\}}(x)g(x)dx.$$

Let $w(x) := \left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^\gamma(x)$ with $\gamma > 1$. Then $w \in A_1$. Since $g(x) \le \left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^\gamma(x)$, by Lemma 2.1, the hypothesis that $\mathcal{T}$ is bounded $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$ and Lemma

2.5, then we can obtain that

$$\begin{aligned}\lambda\left\|\chi_{\{x\in\mathbb{R}^n:\ |\mathcal{T}f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &\le \lambda\int_{\mathbb{R}^n}\chi_{\{x\in\mathbb{R}^n:\ |\mathcal{T}f(x)|>\lambda|\}}(x)\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)dx\\ &\lesssim \int_{\mathbb{R}^n}|f(x)|\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)dx\\ &\lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\left\|\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}\right\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\\ &\lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}.\end{aligned}$$

By taking the supremum over all $\lambda>0$, then we get

$$\|\mathcal{T}f\|_{W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\lesssim\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

For the part of (b). Argue similarly for the weight $w(x):=\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)$ with $\gamma>1$. There exists $g\in(L^{\vec{p}'},L^{\vec{1}'})_t(\mathbb{R}^n)$ such that $g(x)\le w(x)$. Lemma 2.1, the hypothesis of $[b,\mathcal{T}]$ and Lemma 2.5 yield

$$\begin{aligned}&\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,\mathcal{T}]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}\le\int_{\mathbb{R}^n}\chi_{\{x\in\mathbb{R}^n:\ |[b,\mathcal{T}]f(x)|>\lambda|\}}(x)\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)dx\\ &\lesssim\|b\|_{BMO}\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)dx\\ &\lesssim\left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}\left\|\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}\right\|_{(L^{\vec{p}'},L^{\vec{1}'})_t(\mathbb{R}^n)}\\ &\lesssim\left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}\|g\|_{(L^{\vec{p}'},L^{\vec{1}'})_t(\mathbb{R}^n)}.\end{aligned}$$

Hence

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,\mathcal{T}]f|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}\lesssim\left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

This completes the proof of Theorem 3.2. □

4. Some estimates on mixed-norm amalgam spaces

In this section, we apply our extrapolation theorem to prove norm inequalities over mixed-norm amalgam spaces.

We apply the results of Section 3 to the singular integral operators, and establish the mapping properties of these operators and the commutators in this subsection.

Let $\delta>0$. The *Calderón–Zygmund singular integral operator of non-convolution type* is a bounded linear operator $T:\ L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ satisfying that, for all $f\in C_c^\infty(\mathbb{R}^n)$ and $x\notin\operatorname{supp}(f)$,

$$T(f)(x):=\int_{\mathbb{R}^n}K(x,y)f(y)dy,$$

where the distributional kernel coincides with a locally integrable function K defined away from the diagonal on $\mathbb{R}^n\times\mathbb{R}^n$. When K also satisfies that, for $x,\ y\in\mathbb{R}^n$ with

$x \neq y$,

$$|K(x,y)| \leq \frac{C_0}{|x-y|^n}, \tag{4.1}$$

$$|K(x,y) - K(x,y+h)| + |K(x,y) - K(x+h,y)| \leq \frac{C_1|h|^\delta}{|x-y|^{n+\delta}}, \tag{4.2}$$

whenever $|x-y| \geq 2|h|$, and we call K the *standard kernel.*

In [13], it is proved that for the Calderón–Zygmund singular integral operator T with the kernel satisfying (4.1) and (4.2), if $1 < p < \infty$ and $w \in A_p$, then T is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$ and $w \in A_1$, then T is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$. In [31], the commutator $[b,T]$ are bounded in the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ whenever $1 < q < \infty$ and $w \in A_p$, and in [28], if $p = 1$ and $w \in A_1$, then

$$w\left(\{y \in \mathbb{R}^n : |[b,T]f(y)| > \lambda\}\right) \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda}\left(1 + \log^+\left(\frac{|f(y)|}{\lambda}\right)\right) w(y)dy.$$

Thus, by Theorems 3.1 and 3.2, we can easily get the boundedness of the Calderón–Zygmund singular integral operator T with the kernel satisfying (4.1) and (4.2) and the linear commutators $[b,T]$ over the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ in the following.

Corollary 4.1. *Let $0 < t < \infty$, $\vec{p} \in (1,\infty)^n$.*

(a) If $\vec{q} \in (1,\infty)^n$, then the Calderón–Zygmund singular integral operator T with the kernel satisfying (4.1) and (4.2) is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1,...,1)$, then the Calderón–Zygmund singular integral operator T with the kernel satisfying (4.1) and (4.2) is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

Corollary 4.2. *Let $0 < t < \infty$, $\vec{p} \in (1,\infty)^n$ and $b \in BMO(\mathbb{R}^n)$. Let T be the Calderón–Zygmund singular integral operator with the kernel satisfying (4.1) and (4.2),*

(a) If $\vec{q} \in (1,\infty)^n$, then the operator $[b,T]$ is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1,...,1)$, then

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,T]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1 + \log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Let $\mathbb{S}^{n-1} (n \geq 2)$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$, $\Omega(x)$ is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta \leq \infty$ and such that

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{4.3}$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$, the homogeneous singular integral operator T_Ω can be defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)dy,$$

and the Marcinkiewicz integral of higher dimension μ_Ω can be defined by

$$\mu_\Omega f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

The commutators of Marcinkiewicz operator μ_Ω and a locally integrable function b can be defined by

$$[b,\mu_\Omega]f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left[b(x)-b(y)\right] f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

Lemma 4.1. [12] *For $\Omega \in L^\theta(\mathbb{S}^{n-1})$ and $1<\theta<\infty$. If $\theta' \le p < \infty$ and $w \in A_{p/\theta'}$, then T_Ω is bounded on $L^p_w(\mathbb{R}^n)$. If $p=1$ and $w \in A_1$, then T_Ω is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

From Theorems 3.1, 3.2 and Lemma 4.1, we can easily get the results as follows.

Corollary 4.3. *Let $0<t<\infty$, $\vec{p} \in (1,\infty)^n$.*

(a) If $\vec{q} \in (1,\infty)^n$, then T_Ω is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1,...,1)$, then T_Ω is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

Lemma 4.2. [10] *For $\Omega \in L^\theta(\mathbb{S}^{n-1})$ and $1<\theta\le\infty$, if $1<p<\infty$, and $w \in A_p$. Then μ_Ω is bounded on $L^p_w(\mathbb{R}^n)$. If $p=1$, $w\in A_1$, then μ_Ω is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

Applying Theorems 3.1, 3.2 and Lemma 4.3, we have the following results.

Corollary 4.4. *Let $0<t<\infty$, $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1<\theta\le\infty$, and $\vec{p} \in (\theta',\infty)^n$.*

(a) If $\vec{q} \in (\theta',\infty)^n$, then μ_Ω is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1,...,1)$, then μ_Ω is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

Lemma 4.3. [10, 11] *Let $\Omega \in L^\theta(\mathbb{S}^{n-1})$, $1<\theta\le\infty$, $b \in BMO(\mathbb{R}^n)$. If $\theta' < p < \infty$, and $w \in A_{p/\theta'}$, then $[b,\mu_\Omega]$ is bounded on $L^p_w(\mathbb{R}^n)$. If $w \in A_1$, then there exists a constant $C>0$ such that*

$$w\left(\{y \in \mathbb{R}^n : \left|[b,\mu_\Omega]f(y)\right| > \lambda\}\right) \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \log^+\left(\frac{|f(y)|}{\lambda}\right)\right) w(y)dy.$$

Therefore we have

Corollary 4.5. *Let $b \in BMO(\mathbb{R}^n)$, $0 < t < \infty$, $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta \le \infty$, and $\vec{p} \in (\theta', \infty)^n$.*

(a) If $\vec{q} \in (\theta', \infty)^n$, then the operator $[b, \mu_\Omega]$ is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, then

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,\mu_\Omega]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

The Bochner–Riesz operators of order $\delta > 0$ in terms of the Fourier transforms is defined by

$$\left(T_R^\delta f\right)^\wedge(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\delta \hat{f}(\xi),$$

where $\hat{f}$ denote the Fourier transform of f. These operators can be defined by

$$T_R^\delta f(x) = \left(f * \phi_{1/R}\right)(x),$$

where $\phi(x) = [(1 - |\cdot|^2)_+^\delta]^\vee(x)$, and $f^\vee$ is the inverse Fourier transform of f.

The associated maximal operators is defined by

$$T_*^\delta f(x) = \sup_{R>0} |T_R^\delta f(x)|.$$

Lemma 4.4. [32, 33, 34] *Let $n \ge 2$. If $1 < p < \infty$ and $w \in A_p$, then $T_*^{(n-1)/2}$ is bounded on $L_w^p(\mathbb{R}^n)$. For a fixed $R > 0$, if $p = 1$, $w \in A_1$, then $T_R^{(n-1)/2}$ is bounded from $L_w^1(\mathbb{R}^n)$ to $L_w^{1,\infty}(\mathbb{R}^n)$.*

Corollary 4.6. *Let $0 < t < \infty$, and $\vec{p} \in (1, \infty)^n$.*

(a) If $\vec{q} \in (1, \infty)^n$, then $T_*^{(n-1)/2}$ is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, then $T_R^{(n-1)/2}$ is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

Lemma 4.5. [1, 24] *Let $n \ge 2$, and $b \in BMO(\mathbb{R}^n)$. If $1 < p < \infty$, $w \in A_p$, and $\delta \ge \frac{n-1}{2}$, then then $[b, T_R^\delta]$ is bounded on $L_w^p(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, and $\delta > \frac{n-1}{2}$, then*

$$w\left(\left\{y \in \mathbb{R}^n : \left|\left[b, T_*^\delta\right] f(y)\right| > \lambda\right\}\right) \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda}\left(1 + \log^+\left(\frac{|f(y)|}{\lambda}\right)\right) w(y)dy.$$

Corollary 4.7. *Let $b \in BMO(\mathbb{R}^n)$, $0 < t < \infty$, and $\vec{p} \in (1, \infty)^n$.*

(a) If $\vec{q} \in (1, \infty)^n$, and $\delta \ge \frac{n-1}{2}$, then the operator $[b, T_R^\delta]$ is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, and $\delta > \frac{n-1}{2}$, then

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,T_*^\delta]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Let $0 < \alpha < n$, the fractional integral operator I_α is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(\xi)}{|x - \xi|^{n-\alpha}}\, d\xi,$$

And the associated fractional maximal operator M_α is defined by

$$M_\alpha f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy.$$

We note that the fractional maximal functions enjoys the same boundedness as that of the fractional integrals since the pointwise inequality $M_\alpha f(x) \lesssim I_\alpha f(x)$.

We also recall the definition of $A_{p,q}$ weights which are closely related to the weighted boundedness of the fractional integrals in [27].

Definition 4.1. *A weight w is said to be of class $A_{p,q}$, for $1 < p, q < \infty$, if*

$$[w]_{A_{p,q}} := \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty,$$

where p' is the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

And a weight w is said to be of class $A_{1,q}$ with $1 < q < \infty$, if

$$[w]_{A_{1,q}} := \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{\frac{1}{q}} \left(\operatorname*{ess\,sup}_Q \frac{1}{w(x)} \right) < \infty.$$

Lemma 4.6. [27] *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$, and $w \in A_{p,q}$, then there exists a positive constant C such that*

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x) w(x)|^q dx \right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f(x) w(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = 1$, and $w \in A_{1,q}$ with $q = \frac{n}{n-\alpha}$, then for all $\lambda > 0$, then there exists a positive constant C such that

$$w\left(\{x \in \mathbb{R}^n : |I_\alpha(f)(x)| > \lambda\}\right) \le C \left(\frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| w(x)^{\frac{1}{q}} dx \right)^q.$$

The universal positive constant C is independent of f and λ.

Corollary 4.8. *Let $0 < t < \infty$, $0 < \alpha < n$. Suppose that $\vec{p}, \vec{r} \in (1, n/\alpha)^n$ such that $1/r_i - 1/p_i = 1/s_i - 1/q_i = \alpha/n$.*

(a) If $\vec{s} \in (1,\infty)^n$, then I_α is bounded from $(L^{\vec{r}}, L^{\vec{s}})_t(\mathbb{R}^n)$ to $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{s} = (1, ..., 1)$, then I_α is bounded from $(L^{\vec{r}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

Proof. By Theorem 3.1 and Lemma 4.6, the case of (a) holds, we only prove the case of (b).

For $r_i > 1$ and $s_i = 1$, $i = 1, ..., n$, let

$$\frac{1}{p_i} = \frac{1}{r_i} - \frac{\alpha}{n}, \quad \frac{1}{q_i} = 1 - \frac{\alpha}{n}, \quad \text{for each } i = 1, ..., n.$$

Take $\theta = q_i = \frac{n}{n-\alpha}$. Then, for $g \in (L^{(\vec{p}/\theta)'}, L^{\vec{\infty}})_t(\mathbb{R}^n)$, by Lemma 2.2, we write

$$\lambda \left\|\chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} = \lambda \left\|\left|\chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}\right|^\theta\right\|^{\frac{1}{\theta}}_{(L^{\vec{p}/\theta},L^{\vec{q}/\theta})_t(\mathbb{R}^n)}$$
$$= \lambda \left(\int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}(x) g(x)dx\right)^{\frac{1}{\theta}}.$$

And letting $w := [M_\eta(|g|)(x)]^{\frac{1}{\theta}}$ with $0 < \eta < 1$, we have $w^\theta \in A_1$ and hence $w^\theta \in A_{\theta\frac{n-\alpha}{n}}$. Then $w \in A_{1,\theta}$. By Lemma 2.1 and Lemma 4.6, we can obtain that

$$\lambda \left\|\chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} = \lambda \left[\int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}(x) M_\eta g(x)dx\right]^{\frac{1}{\theta}}$$
$$\leq C\int_{\mathbb{R}^n} |f(x)|w(x)dx \leq C\|f\|_{(L^{\vec{r}},L^1)_t(\mathbb{R}^n)} \left\|[M_\eta(g)]^{\frac{1}{\theta}}\right\|_{(L^{\vec{r}'},L^{\vec{\infty}})_t(\mathbb{R}^n)}.$$

From Lemma 2.5 we see

$$\left\|[M_\eta(g)]^{\frac{1}{\theta}}\right\|_{(L^{\vec{r}'},L^{\vec{\infty}})_t(\mathbb{R}^n)} = \left\|[M(|g|^\eta)]^{\frac{1}{\eta\theta}}\right\|_{(L^{\vec{r}'},L^{\vec{\infty}})_t(\mathbb{R}^n)} = \|M(|g|^\eta)\|^{\frac{1}{\eta\theta}}_{(L^{\vec{r}'/\eta\theta},L^{\vec{\infty}})_t(\mathbb{R}^n)}$$
$$\leq C\,\||g|^\eta\|^{\frac{1}{\eta\theta}}_{(L^{\vec{r}'/\eta\theta},L^{\vec{\infty}})_t(\mathbb{R}^n)} = C\|g\|^{\frac{1}{\theta}}_{(L^{\vec{r}'/\theta},L^{\vec{\infty}})_t(\mathbb{R}^n)}.$$

since $\frac{1}{r_i'/\theta} = \left(1-\frac{1}{r_i}\right)\theta = \left(1-\frac{1}{p_i}-\frac{\alpha}{n}\right)\theta = \left(\frac{1}{\theta}-\frac{1}{p_i}\right)\theta = 1-\frac{1}{p_i/\theta} = \frac{1}{(p_i/\theta)'}$ for $i = 1, ..., n$, we see

$$\|I_\alpha f\|_{W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq C\|f\|_{(L^{\vec{r}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Thus, the result holds. □

For the boundedness of the commutator for the Riesz potential, we

Lemma 4.7. [7] *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/p - 1/q = \alpha/n$. Let $b \in BMO(\mathbb{R}^n)$ and $w \in A_{p,q}$, then $[b, I_\alpha]$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^q_w(\mathbb{R}^n)$.*

The estimate of the operator $[b, I_\alpha]$ over the mixed-norm amalgam space is immediate in view of Lemma 4.7 and Theorem 3.1 as follows.

Corollary 4.9. *Let $0 < t < \infty$ and $0 < \alpha < n$. Let $b \in BMO(\mathbb{R}^n)$. Suppose that $\vec{p}$, $\vec{r} \in (1, n/\alpha)^n$ such that $1/r_i - 1/p_i = 1/s_i - 1/q_i = \alpha/n$. If $\vec{q} \in (1,\infty)^n$, then $[b, I_\alpha]$ is bounded from $(L^{\vec{r}}, L^{\vec{s}})_t(\mathbb{R}^n)$ to $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.*

5. The Littlewood–Paley functions

The Littlewood–Paley theory, originated in the 1930s and developed in the late 1950s, is a very effective replacement. It has played a very prominent role in harmonic analysis, Complex analysis and PDE (see [6, 22, 30]). Therefore, it is a very interesting problem to discuss the boundedness of the Littlewood–Paley operators. The main purpose of this section is to study the characterization of the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ via the Littlewood–Paley functions. We first state the associated definitions.

Suppose that $\varphi(x) \in L^1(\mathbb{R}^n)$ satisfies the following conditions:

$$\int_{\mathbb{R}^n} \varphi(x)dx = 0. \tag{5.1}$$

There exist constants C, $\delta > 0$, such that

$$|\varphi(x)| \le \frac{C}{(1+|x|)^{n+\delta}}, \quad \forall x \in \mathbb{R}^n. \tag{5.2}$$

and when $2|y| < |x|$, there exist constants γ, $\delta > 0$, such that

$$|\varphi(x+y) - \varphi(x)|dx \le \frac{C|y|^\delta}{(1+|x-y|)^{n+\delta+\gamma}}. \tag{5.3}$$

For $t > 0$, $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$. For all $x \in \mathbb{R}^n$, the Littlewood–Paley g function g_φ, the square function S_φ and the Littlewood–Paley $g^*_{\lambda,\varphi}$-function are defined by

$$g_\varphi(f)(x) = \left(\int_0^\infty |(\varphi_t * f)(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$S_\varphi(f)(x) = \left(\iint_{\Gamma_\alpha(x)} |(\varphi_t * f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$g^*_{\lambda,\varphi}(f)(x) = \left(\iint_{R^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |(\varphi_t * f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

For a locally integrable function b, the commutators of the Littlewood–Paley function g_φ, S_φ and $g^*_{\lambda,\varphi}$ are defined by

$$g_{\varphi,b}(f)(x) = \left(\int_0^\infty \left|\int_{\mathbb{R}^n} (\varphi_t(x-y)f(y)((b(x)-b(y))dy\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$S_{\varphi,b}(f)(x) = \left(\iint_{\Gamma_\alpha(x)} \left|\int_{\mathbb{R}^n} (\varphi_t(y-z)f(z)((b(y)-b(z))dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$g^*_{\lambda,,\varphi,b}(f)(x) = \left(\iint_{R^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left|\int_{\mathbb{R}^n} (\varphi_t(y-z)f(z)((b(y)-b(z))dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma_\alpha(x) = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : \ |x-y| < \alpha t\right\}$ and $\mathbb{R}^{n+1}_+ = \{(y,t) \in \mathbb{R}^{n+1}_+ : \ y \in \mathbb{R}^n, t > 0\}$.

Lemma 5.1. [25] *Suppose that $\varphi \in L^1(\mathbb{R}^n)$ satisfies (5.1), (5.2) and (5.3). If $1 < p < \infty$, $w \in A_p$, then g_φ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$ and $w \in A_1$, then g_φ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

Lemma 5.2. [23] *Let $b \in BMO(\mathbb{R}^n)$. Suppose that $\varphi \in L^1(\mathbb{R}^n)$ satisfies (5.1), (5.2) and (5.3). If $1 < p < \infty$, $w \in A_p$, then $g_{\varphi,b}$ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$ and $w \in A_1$, then*

$$w\left(\{x \in \mathbb{R}^n : |g_{\varphi,b}f(x)| > \lambda\}\right) \lesssim \|b\|_{BMO} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx.$$

Lemma 5.3. [37] *Let $0 < \alpha \leq 1$, if $p \in (1, \infty)$ and $w \in A_p$, then S_φ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, then S_φ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

Lemma 5.4. [23] *Let $\alpha \in (0, 1]$ and $b \in BMO(\mathbb{R}^n)$. If $p \in (1, \infty)$ and $w \in A_p$, then $S_{\varphi,b}$ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, then there exists a constant $C > 0$ such that*

$$w\left(\{x \in \mathbb{R}^n : |S_{\varphi,b}f(x)| > \lambda\}\right) \lesssim \|b\|_{BMO} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx.$$

Lemma 5.5. [38] *Let $\lambda > 2$ and $0 < \gamma < \min\{n(\gamma - 2)/2, \delta\}$. Let $b \in BMO(\mathbb{R}^n)$. If $p \in (1, \infty)$ and $w \in A_p$, then $g^*_{\lambda,\varphi}$ and $g^*_{\lambda,\varphi,b}$ are bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, then $g^*_{\lambda,\varphi}$ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$, and*

$$w\left(\left\{x \in \mathbb{R}^n : \left|g^*_{\lambda,\varphi,b}(f)(x)\right| > \lambda\right\}\right) \lesssim \|b\|_{BMO} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx.$$

Theorem 5.1. *Let $0 < t < \infty$, $\lambda > 2$ and $0 < \gamma < \min\{n(\gamma - 2)/2, \delta\}$. Let $\vec{p} \in (1, \infty)^n$. If $\vec{q} \in (1, \infty)^n$, then*

(a) $C_1 \|f\|_{(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)} \leq \|g_\varphi(f)\|_{(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)} \leq C_2 \|f\|_{(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)}$.

(b) $C_1 \|f\|_{(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)} \leq \|S_\varphi(f)\|_{(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)} \leq C_2 \|f\|_{(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)}$.

(c) $\left\|g^*_{\lambda,\varphi}(f)\right\|_{(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)}$.

(d) If $\vec{q} = (1, ..., 1)$, then the operators g_φ, S_φ, $g^*_{\lambda,\varphi}$ is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

The positive constants C_1 and C_2 are independent of f and t.

Proof. We only need to prove the left case of (a) and (b), since Lemmas 5.1, 5.3, 5.5 and Theorems 3.1, 3.2.

By Lemma 2.1, the boundedness of g_φ over $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ for $\vec{p}$, $\vec{q} \in (1,\infty)^n$ and Hölder's inequality, we see

$$\begin{aligned}
\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &= \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} f(x)g(x)dx \\
&\le \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n}\int_0^\infty |\varphi_t * f(x)|\cdot|\varphi_t * g(x)|\frac{dt}{t}dx \\
&\le \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} g_\varphi(f)(x)g_\varphi(g)(x)dx \\
&\le \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \|g_\varphi(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\|g_\varphi(g)\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)} \\
&\lesssim \|g_\varphi f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\,,
\end{aligned}$$

For the operator S_φ, using these facts, Lemma 2.1 and $\|S_\varphi f\|_{L^2_H(\mathbb{R}^n)} = A\|f\|_{L^2(\mathbb{R}^n)}$ with $A>0$ and H is a Hilbert space, we conclude that

$$\begin{aligned}
\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &= \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} f(x)g(x)dx \\
&= \frac{1}{A^2}\sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} S_\varphi f(x)S_\varphi g(x)dx \\
&\le \frac{1}{A^2}\sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \|S_\varphi f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\|S_\varphi g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)} \\
&\lesssim \frac{1}{A^2}\|S_\varphi f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\,.
\end{aligned}$$

This completes the proof of Theorem 5.1. □

Using Lemmas 5.1, 5.3, 5.5 and Theorems 3.1, 3.2, we obtain the following estimate for the operator $g_{\varphi,b}$, $S_{\varphi,b}$, $g^*_{\lambda,\varphi,b}$ on mixed-norm amalgam spaces.

Theorem 5.2. *Let $0<t<\infty$, $\lambda>2$ and $0<\gamma<\min\{n(\gamma-2)/2,\delta\}$. Let $\vec{p}\in(1,\infty)^n$ and $b\in BMO(\mathbb{R}^n)$, If $\vec{q}\in(1,\infty)^n$, then $g_{\varphi,b}$, $S_{\varphi,b}$ and $g^*_{\lambda,\varphi,b}$ are bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$. If $\vec{q}=(1,...,1)$, then*

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |g_{\varphi,b}f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)},$$

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |S_{\varphi,b}f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}$$

and

$$\left\|\chi_{\left\{x\in\mathbb{R}^n:\ \left|g^*_{\lambda,\varphi,b}f(x)\right|>\lambda\right\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Remark 5.1. *For the Littlewood–Paley functions with the non-convolution type kernels and their commutators, a similar result also holds.*

5.0.1. *Nononvolution type.* A function $K(x,y)$ defined away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$, is said to be a non-convolution type kernel, if for all $y \in \mathbb{R}^n$, there exists a positive constant C, such that K satisfies the following conditions:

$$\int_{\mathbb{R}^n} K(x,y)dy = 0 \tag{5.4}$$

$$|K(x,y)| \leq \frac{C}{(1+|x-y|)^{n+\delta}} \tag{5.5}$$

$$|K(x+z,y) - K(x,y)| \leq \frac{C|z|^{\gamma}}{(1+|x-y|)^{n+\delta+\gamma}} \tag{5.6}$$

for some δ, $\gamma > 0$, and $2|z| \leq |x-y|$.

For any $f \in \mathscr{S}$, $t > 0$, and $z \ni$ supp f, we denote

$$G_t f(z) = \int_{\mathbb{R}^n} K_t(z,y) f(y)dy,$$

where $K_t(z,y) = \frac{1}{t^n} K(\frac{z}{t}, \frac{y}{t})$. Let b be a locally integrable function. Then the Littlewood-Paley g-function, Lusin's area integral and Littlewood-Paley g_λ^*-function with non-convolution type kernels and their commutators are defined by

$$g(f)(x) = \left(\int_0^\infty |G_t f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$g_b(f)(x) = \left(\int_0^\infty |G_t f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$S(f)(x) = \left(\iint_{\Gamma(x)} |G_t f(z)|^2 \frac{dzdt}{t^{n+1}}\right)^{\frac{1}{2}}$$

and

$$g_\lambda^*(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-z|}\right)^{n\lambda} |G_t f(z)|^2 \frac{dzdt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\lambda > 1$, $\Gamma(x) = \{(z,t) \in \mathbb{R}_+^{n+1} : |z-x| < t\}$ and $\mathbb{R}_+^{n+1} = \{(z,t) \in \mathbb{R}_+^{n+1} : z \in \mathbb{R}^n, t > 0\}$.

Theorem 5.3. *Let $0 < t < \infty$. For $1 < \vec{p} < \infty$. If $1 < \vec{q} < \infty$. Then*

(A) $\|g(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \sim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(B) $\|S(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \sim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(C) for $\lambda > 2$ and $0 < \gamma < \min\{n(\gamma-2)/2, \delta\}$, $\|g_\lambda^*(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \sim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

If $\vec{q} \in [1,\infty)$ and $\min\{q_1,...,q_m\} = 1$. Then the operators g_φ, S_φ, $g_{\lambda,\varphi}^*$ are bounded from $(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)$.

Using Lemmas 5.1,5.3,5.5 and Theorems 3.1,3.2, we obtain the following estimate for the operators $g_{\varphi,b}$, $S_{\varphi,b}$, $g_{\lambda,\varphi,b}^*$ on mixed-norm amalgam spaces.

Theorem 5.4. *Let $0 < t < \infty$, $b \in BMO(\mathbb{R}^n)$. For $1 < \vec{p} < \infty$. If $1 < \vec{q} < \infty$. Then*

(A) $\|g_b(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(B) $\|S_b(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(C) for $\lambda > 2$ *and* $0 < \gamma < \min\{n(\gamma-2)/2, \delta\}$, $\left\|g^*_{\lambda,b}(f)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

If $\vec{q} \in [1,\infty)$ *and* $\min\{q_1,...,q_m\} = 1$. *Then the operators* g_φ, S_φ, $g^*_{\lambda,\varphi}$ *satisfy*

$$\left\|\chi_{\{x\in\mathbb{R}^n:|g_b f|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)},$$

$$\left\|\chi_{\{x\in\mathbb{R}^n:|S_b f|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

and

$$\left\|\chi_{\{x\in\mathbb{R}^n:|g^*_{\lambda,b} f|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

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ON THE UNIQUENESS OF L-FUNCTIONS AND MEROMORPHIC FUNCTIONS SHARING A SET

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Abstract. The paper presents general criterions for the uniqueness of a non-constant meromorphic function having finitely many poles and a non-constant L-function in the Selberg class when they share a set. Our results significantly improve all the existing results in this direction [22, 17, 16, 4] with extent to the most general setting. As a consequence, we have incorporated a large number of examples in the application section showing the far reaching applications of our results.

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1. Introduction and main results

At the outset, we assume that by an L-function we always mean an L-function $\mathcal{L}$ in the Selberg class which includes the Riemann zeta function $\zeta(s) = \sum\limits_{n=1}^{\infty} n^{-s}$ and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an L-function is defined [18, 19] to be a Dirichelet series

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{-s}} \tag{1.1}$$

satisfying the following axioms:

- (i) *Ramanujan hypothesis :* $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$;
- (ii) *Analytic continuation :* There is a non-negative integer m such that $(s-1)^m \mathcal{L}(s)$ is an entire function of finite order;
- (iii) *Functional equation:* $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega \overline{\Lambda_{\mathcal{L}}(1-\overline{s})}, \tag{1.2}$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s) Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \nu_j), \tag{1.3}$$

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with positive real numbers Q, λ_j and complex numbers ν_j, ω with $\mathrm{Re}(\nu_j) \geq 0$ and $|\omega| = 1$;

- (iv) *Euler product hypothesis :* $\log \mathcal{L}(s) = \sum\limits_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Also, throughout the paper by any meromorphic function we always mean a meromorphic function defined in $\mathbb{C}$. We denote $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By $\mathbb{N}$ we mean the set of all natural numbers. Though for standard definitions used in this paper we refer our readers to follow [9], yet for the sake of our convenience we denote the order of f by $\rho(f)$, where

$$\rho(f) = \frac{\log(T(r, f))}{\log r}. \tag{1.4}$$

By $S(r, f)$ we mean any quantity satisfying $S(r, f) = O(\log(rT(r, f)))$ for all r possibly outside a set of finite linear measure. If f is a function of finite order, then $S(r, f) = O(\log r)$ for all r.

The importance of L-functions in number theory is needless to say and an L-function can be analytically continued to a meromorphic function in $\mathbb{C}$. Hence like the value distribution of meromorphic functions, the value distribution of L-functions is a natural consequence. In this respect, during the last few years an extensive study for the distribution of zeros of L-functions have been done by various researchers [11, 14, 22, 10, 18, 19]. In due course of time, the study have been confined to the direction of uniquely determining an L-function via the shared values or sets. Hence let us recall these basic definitions of value and set sharing.

Definition 1.1. [6] *For a non-constant meromorphic function f and $a \in \mathbb{C}$, let $E_f(a) = \{(z, p) \in \mathbb{C} \times \mathbb{N} : f(z) = a$ with multiplicity $p\}$*

$$\left(\overline{E}_f(a) = \{(z, 1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\right),$$

then we say f, g share the value a CM(IM) if $E_f(a) = E_g(a) \left(\overline{E}_f(a) = \overline{E}_g(a)\right)$. For $a = \infty$, we define $E_f(\infty) := E_{1/f}(0)$ $\left(\overline{E}_f(\infty) := \overline{E}_{1/f}(0)\right)$.

Definition 1.2. [6] *For a non-constant meromorphic function f and $S \subset \overline{\mathbb{C}}$, let $E_f(S) = \bigcup_{a \in S}\{(z, p) \in \mathbb{C} \times \mathbb{N} : f(z) = a$ with multiplicity $p\}$*

$$\left(\overline{E}_f(S) = \bigcup_{a \in S}\{(z, 1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\right),$$

then we say f, g share the set S CM(IM) if $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$.

Definition 1.3. [12, 13] *Let k be a non-negative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is* *counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.4. [12] *For $S \subset \overline{\mathbb{C}}$ we define $E_f(S, k) = \cup_{a \in S} E_k(a; f)$, where k is a non-negative integer $a \in S$ or infinity. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$. If $E_f(S, k) = E_g(S, k)$, then we say that f and g share the set S with weight k.*

Obviously Definition 1.3 and Definition 1.4 are the refined notions of Definition 1.1 and Definition 1.2 respectively. However, now we recall the first result in this direction due to Steuding.

Theorem A. [19] *If two L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Since every L-function have meromorphic continuation in $\mathbb{C}$, so natural quest for the uniqueness of a meromorphic function and an L-function enters into the course of uniqueness theory vis-a-vis value distribution theory. Since an L-function can have at most one pole in $\mathbb{C}$, so it is reasonable to study the uniqueness of L-functions with meromorphic functions having finitely many poles. Pertinent to that, in 2010 Li proved the following uniqueness theorem.

Theorem B. [14] *Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a non-constant L-function $\mathcal{L}$ share a CM and b IM, then $\mathcal{L} = f$.*

After that in 2018, taking the famous Gross Problem [8] into account, Yuan, Li and Yi [22] proposed an analogous version of the same for L-functions as follows.

Question 1.1. [22] *What can be said about the relationship between a meromorphic function f and an L-function $\mathcal{L}$ if f and $\mathcal{L}$ share one or two sets?*

Apropos of *Question 1.1*, in the same paper Yuan, Li and Yi provided the following result.

Theorem D. [22] *Let $Q(z) = z^n + az^m + b$, where a, b are non-zero constants with $\gcd(m, n) = 1$ and $n \geq 2m + 5$. Further suppose f is a non-constant meromorphic* *function having finitely many poles and* $\mathcal{L}$ *is a non-constant L-function such that* $E_f(S,\infty) = E_{\mathcal{L}}(S,\infty)$*, where* $S = \{z : Q(z) = 0\}$*. Then* $f = \mathcal{L}$*.*

Later on with the aid of weighted sharing Sahoo-Sarkar [17] improved *Theorem D* as follows.

Theorem E. [17] *Let* S *be defined same as in Theorem D and* $n \geq 2m+5$*. Suppose* f *is a non-constant meromorphic function having finitely many poles in* $\mathbb{C}$ *and* $\mathcal{L}$ *is a non-constant L-function. If* f *and* $\mathcal{L}$ *share* $(S,2)$*, then* $f = \mathcal{L}$*.*

Considering the ignoring multiplicities of the shared set Sahoo-Halder proved the following theorem.

Theorem F. [16] *Let* S *be defined same as in Theorem D and* $n \geq \max\{2m + 5, 4q + 9\}$*, where* $q = n - m \geq 1$*. Let* f *be a non-constant meromorphic function having finitely many poles in* $\mathbb{C}$ *and* $\mathcal{L}$ *be a non-constant L-function. If* f *and* $\mathcal{L}$ *share* $(S,0)$*, then* $f = \mathcal{L}$*.*

Pertinent to *Theorem E* and *Theorem F*, Banerjee-Kundu [4] found out some gaps in these theorems and they provided the following theorem rectifying these gaps.

Theorem G. [4] *Let* S *be defined as in* Theorem D*,* f *be a non-constant meromorphic function having finitely many poles in* $\mathbb{C}$ *and* $\mathcal{L}$ *be a non-constant L-function such that* $E_f(S,t) = E_{\mathcal{L}}(S,t)$*. If*

(i) $t \geq 2$ *and* $n \geq 2m + 5$*, or*
(ii) $t = 1$ *and* $n \geq 2m + 6$*, or*
(iii) $t = 0$ *and* $n \geq 2m + 11$*,*
then $f = \mathcal{L}$*.*

In the same paper Banerjee-Kundu proved another result analogous to *Theorem G* which is as follows.

Theorem H. [4] *Let* $S = \{z : z^n + az^{n-m} + b = 0\}$*, where* a, b *are non-zero constants and* $\gcd(n,m) = 1$*. Let* f *be a non-constant meromorphic function having finitely many poles in* $\mathbb{C}$ *and* $\mathcal{L}$ *be a non-constant L-function such that* $E_f(S,t) = E_{\mathcal{L}}(S,t)$*. If*

(i) $t \geq 2$ *and* $n \geq 2m + 5$*, or*
(ii) $t = 1$ *and* $n \geq 2m + 6$*, or*
(iii) $t = 0$ *and* $n \geq 2m + 11$*,*
then $f = \mathcal{L}$*.*

Note that the set S used in *Theorem D-H* are generated from the zeros of the polynomial

$$P(z) = z^n + az^m + b \;\; or \;\; P(z) = z^n + az^{n-m} + b, \tag{1.5}$$

where a, b are non-zero constants and $\gcd(n, m) = 1$. In [4, see Lemma 4], authors proved that these polynomials are critically injective and they may have multiple zero but that must be one in number. On this occasion let us invoke the definition of critically injective polynomial.

Definition 1.5. *Let $P(z)$ be a polynomial such that $P^{'}(z)$ has mutually r distinct zeros given by $d_1, d_2, \ldots, d_r$ with multiplicities $q_1, q_2, \ldots, q_r$ respectively. Then $P(z)$ is said to be a critically injective polynomial if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \cdots, r\}$.*

Any polynomial which is not critically injective is called a non-critically injective polynomial.

Observe that the following points come out of the above discussions.

(i) All the authors always used one of the polynomials given by (1.5).
(ii) The authors always used the set of zeros of critically injective polynomials to show the uniqueness of f and $\mathcal{L}$.
(iii) In the above theorems authors have improved the previous results by relaxing the nature of sharing of the sets.
(iv) The authors also considered the set of zeros of the polynomials having multiple zeros.

Apropos of observation (i) and (ii), One would naturally raise the following questions.

Question 1.2. *Does there exist any other polynomial except the polynomials given by (1.5) whose set of zeros provide uniqueness of f and $\mathcal{L}$?*

Question 1.3. *Does there exist any non-critically injective polynomial whose set of zeros provide the uniqueness of f and $\mathcal{L}$?*

Pertinent to observation (iii) the following questions become inevitable.

Question 1.4. *Can we have the answer of* Question 1.1 *under more relaxed sharing hypothesis than that obtained in the latest results* Theorem G-H*?*

Question 1.5. *Can we have a set with lesser cardinality than that obtained in the latest results* Theorem G-H *for the uniqueness of f and $\mathcal{L}$?*

Finally with respect to observation (iv), we have the following note.

Note 1.1. *Recently in* [5, see paragraph between Theorem H and Theorem I] *Banerjee-Kundu have clarified the fact that all the results obtained till date in this direction of shared set problems for the uniqueness of f and $\mathcal{L}$; i.e.,* Theorem D-H *(except Theorem F) have an analytical gap while considering multiple zero of the polynomials and the sharing of the sets with some non-zero weight. Thus conclusion of* Theorem D-H *(except Theorem F) become false when the multiple zero of the generating polynomials are taken into account and the sharing of the sets with some non-zero weight. But in the same scenario, the results obtained with IM sharing of the sets are correct; i.e., conclusion (iii) of* Theorem G-H *and* Theorem F*. Though* Theorem F *has a different flaw contradicting their own conclusion of cardinality* $n \geq \max\{2m+5, 4q+9\}$ *which is analysed in* [4, Remark 3]*. Another point is that all these results are true when the polynomial has only simple zeros.*

Hence in this paper, we shall solely concentrate on the polynomials having only simple zeros and answer all the above questions from *Question 1.2-1.5* affirmatively which improve all the existing results from *Theorem D-H*. Moreover, we present general criterions for any general polynomial so that the set of zeros of the same would provide the uniqueness of f and $\mathcal{L}$ when shared by these functions. In a nutshell, our results bring all the existing results under a single umbrella in a more improved version with extent to the most general setting.

In the 4th section of this paper, that is in the **"Application"** section we have proved all our claims to be true by exhibiting a number of examples showing the wide-ranging applications of our results.

Before going to our main results, we make a short discussion on the structure of a general polynomial as this will play an important role throughout the rest of this paper.

Let us consider the following general polynomial $P(z)$ of degree n having only simple zeros.

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \tag{1.6}$$

where $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n, a_0 \neq 0$, a_i being the first non-vanishing coefficient from $a_{n-1}, a_{n-2}, \ldots, a_1$. Let

$$S = \{z : P(z) = 0\}. \tag{1.7}$$

Observe that (1.6) can be written in the form

$$P(z) = a_n \prod_{i=1}^{p} (z - \alpha_i)^{m_i} + a_0, \tag{1.8}$$

where p denotes the number of distinct zeros of $P(z)-a_0$. Let us also denote by s the number of distinct zeros of $P^{'}(z)$. Hence we would have

$$P^{'}(z)=na_n\prod_{i=1}^{s}(z-\eta_i)^{r_i}, \tag{1.9}$$

where r_i denotes the multiplicities of distinct zeros of $P^{'}(z)$.

Set

$$R(z)=-\frac{a_nz^n}{a_{n-1}z^{n-1}+\ldots+a_1z+a_0}=-\frac{a_nz^n}{a_i\prod\limits_{j=1}^{k}(z-\beta_j)^{m_j}}=-\frac{a_nz^n}{\phi(z)}, \tag{1.10}$$

where $a_0,a_1,\ldots,a_n$ are as defined in (1.6) and $\beta_1,\beta_2,\ldots,\beta_k$ are the roots of the equation

$$\phi(z)=a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\ldots+a_1z+a_0=0,$$

with multiplicities $m_1,m_2,\ldots,m_k$. Clearly

$$R(z)-1=-\frac{P(z)}{\phi(z)}, \tag{1.11}$$

where $P(z)$ is defined by (1.6) and obviously $P(z)$ and $\phi(z)$ do not share any common zero. Hence S as defined in (1.7) can be treated as

$$S=\{z:P(z)=0\}=\{z:R(z)-1=0\}. \tag{1.12}$$

Let $R^{'}(z)$ has l distinct zeros say $\delta_1,\delta_2,\ldots,\ldots,\delta_l$ with multiplicities $q_1,q_2,\ldots,q_l$ respectively. Then From (1.10) we would have

$$R'(z) = \frac{\gamma\prod\limits_{j=1}^{l}(z-\delta_j)^{q_j}}{\prod\limits_{j=1}^{k}(z-\beta_j)^{p_j}}, \tag{1.13}$$

where $\gamma\in\mathbb{C}-\{0\}$ and $p_j\in\mathbb{N}$ for all $j\in\{1,2,\ldots,k\}$.

Remark 1.1. *Observe that in the definition (1.6) of the general polynomial $P(z)$, the condition $a_i\neq 0$ for $i=\{1,2,\ldots,n-1\}$ is necessary. Because otherwise we would find a non-constant L-function $\mathcal{L}$ and a non-constant meromorphic function f which share the set $S=\{z:P(z)=0\}$ CM but $f\neq\mathcal{L}$.*

For example, let $a_i=0$ for $i=\{1,2,\ldots,n-1\}$. Then $S=\{z:a_nz^n+a_0=0\}$. Consider a non-constant L-function $\mathcal{L}$ and a non-constant meromorphic function f such that $f=\zeta\mathcal{L}$, where ζ is the nth root of unity. Then clearly,

$$a_nf^n+a_0=a_n\mathcal{L}^n+a_0;$$

$$i.e.,\ \prod_{i=1}^{n}(f-\sigma_i)=\prod_{i=1}^{n}(\mathcal{L}-\sigma_i),$$

where $\sigma_i \in S$ *for* $i = \{1, 2, \ldots, n\}$. *That is* f *and* $\mathcal{L}$ *share* S *CM but* $f \neq \mathcal{L}$.

Now we provide the following two theorems as the main results of this paper.

Theorem 1.1. *Let* $P(z)$ *be given by* (1.8) *with* $p \geq 2$ *and* S, s *be defined by* (1.7), (1.9) *respectively. Suppose* f *is a non-constant meromorphic function having finitely many poles and* $\mathcal{L}$ *is a non-constant L-function sharing* (S, t). *Then for* $n \geq \max\{2p+1, 2s+3\}$, *when* $t \geq 1$; *and for* $n \geq \max\{2p+1, 2s+6\}$, *when* $t = 0$; *the following are equivalent :*

(i) $P(f) = P(\mathcal{L}) \implies f = \mathcal{L}$;
(ii) $E_f(S, t) = E_{\mathcal{L}}(S, t) \implies f = \mathcal{L}$.

Theorem 1.2. *Let* $R(z)$ *be defined by* (1.10) *with* $k \geq 2$ *or* $k = 1$ *with* $n > 2m_1$ *and* S, l *be defined by* (1.12), (1.13) *respectively. Let* f *be a non-constant meromorphic function having finitely many poles and* $\mathcal{L}$ *be a non-constant L-function sharing* (S, t). *Then for* $n \geq \max\{2k+3, 2l+3\}$, *when* $t \geq 1$; *and for* $n \geq \max\{2k+3, 2l+6\}$, *when* $t = 0$; *the following are equivalent :*

(i) $R(f) = R(\mathcal{L}) \implies f = \mathcal{L}$;
(ii) $E_f(S, t) = E_{\mathcal{L}}(S, t) \implies f = \mathcal{L}$.

Remark 1.2. *Obviously* Theorem G-H *are the latest results in this direction for simple zeros of the polynomials given by* (1.5). *In the application section* (Example 4.5 *and* Example 4.2), *we shall show that the conclusions of* Theorem G-H *are true for* $n \geq 7$ *when* $E_f(S, 1) = E_{\mathcal{L}}(S, 1)$, *whereas the same is true in* Theorem G-H *for* $n \geq 8$. *Thus our result directly improves* Theorem G-H *by reducing the cardinality of the set* S *when shared by the functions with weight* 1. *We also find that weight* 2 *in* Theorem G-H *can be relaxed to weight* 1 *keeping the carinality of the set fixed as an application of our result. Hence the answer of* Question 1.4 *is also obtained with improvement. Moreover, in* Theorem G-H *the least cardinality of the sets when shared IM is 13 whereas the same result can be obtained when the cardinalities of the sets are 10, which is a significant improvement of* Theorem G-H. *Thus we obtain a threefold improvement of* Theorem G-H *by the application of our main results and obtain the answer of* Question 1.4-1.5.

We shall also obtain similar results in the application section for other polynomials including critically injective polynomials, non-critically injective polynomials and even those polynomials which are still uncertain to be critically injective or non-critically injective (*see* Example 4.1, Example 4.3 *and* Example 4.4). *These results* *provide us the answers of* Question 1.2-1.3 *with improvements in the nature of sharing of the sets as well as the least cardinalities of the sets.*

Remark 1.3. *The reason behind proving two similar but different theorems are clarified in the first two paragraphs of section 5 named* **"Conclusion and an Open Question"**.

For standard definitions and notations we have already suggested our readers to follow [9]. Furthermore, we explain the following notations which will be used throughout the paper for the proof of the *Theorem 1.1* and *Theorem 1.2*.

Definition 1.6. [21] *Let f and g be two non-constant meromorphic functions such that f and g share $(1,0)$. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those 1-points of f and g where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a\in\overline{\mathbb{C}}$.*

Definition 1.7. [12, 13] *Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 1.8. [13] *For $a\in\mathbb{C}\cup\{\infty\}$we denote by $N(r,a;f\mid=1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f\mid\leq m)(N(r,a;f\mid\geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.*

$\overline{N}(r,a;f\mid\leq m)$ $(\overline{N}(r,a;f\mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid<m)$, $N(r,a;f\mid>m)$, $\overline{N}(r,a;f\mid<m)$ and $\overline{N}(r,a;f\mid>m)$ are defined analogously.

Definition 1.9. [2] *Let $a,b_1,b_2,\ldots,b_q\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid\ g\neq b_1,b_2,\ldots,b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i=1,2,\ldots,q$.*

2. Lemmas

For two non-constant meromorphic functions F and G, set

$$H=\left(\frac{F^{''}}{F^{'}}-\frac{2F^{'}}{F-1}\right)-\left(\frac{G^{''}}{G^{'}}-\frac{2G^{'}}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [21] *Let F, G share $(1,0)$ and $H\not\equiv 0$. Then*

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.2. [2] *Let F, G share $(1,t)$, where $t\in\mathbb{N}\cup\{0\}$. Then*

$$\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_E^{1)}(r,1;F)+\left(t-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\leq\frac{1}{2}[N(r,1;F)+N(r,1;G)].$$

Lemma 2.3. *Let f be a non-constant meromorphic function having finite number of poles and $\mathcal{L}$ be an non-constant L-function sharing a set S IM, where $|S|\geq 3$. Then $\rho(f)=\rho(\mathcal{L})=1$. Furthermore, $\overline{N}(r,\infty;f)=O(\log r)=\overline{N}(r,\infty;\mathcal{L})$ and $S(r,f)=O(\log r)=S(r,\mathcal{L})$.*

Proof. Proceeding in a similar method as done in the proof of Theorem 5, [16, p. 6] we can obtain $\rho(f)=\rho(\mathcal{L})=1$. So we omit it.

Since f has finitely many poles and $\mathcal{L}$ has at most one pole in $\mathbb{C}$, so obviously

$$\overline{N}(r,\infty;f)=O(\log r)=\overline{N}(r,\infty;\mathcal{L}). \tag{2.2}$$

Since $\rho(f)=\rho(\mathcal{L})=1$, so from the definition of $S(r,f)$ we get $S(r,f)=O(\log r)=S(r,\mathcal{L})$. □

Lemma 2.4. *Let $F^*-1=\dfrac{a_n\prod\limits_{i=1}^{n}(f-w_i)}{\psi(f)}$ and $G^*-1=\dfrac{a_n\prod\limits_{i=1}^{n}(\mathcal{L}-w_i)}{\psi(\mathcal{L})}$, where f be a non-constant meromorphic function having finite number of poles, $\mathcal{L}$ be an non-constant L-function, $a_n,w_i\in\mathbb{C}-\{0\}$; $\forall i\in\{1,2,\ldots,n\}$ and $\psi(z)$ be a polynomial of degree less than n with $\psi(w_i)\neq 0$; $\forall i\in\{1,2,\ldots,n\}$. Further suppose that F^* and G^* share $(1,t)$, where $t\in\mathbb{N}\cup\{0\}$. Then*

$$\overline{N}_L(r,1;F^*)\leq\frac{1}{t+1}\left[\overline{N}(r,0;f)-N_1(r,0;f')\right]+O(\log r), \tag{2.3}$$

where $N_1(r,0;f')=N(r,0;f'|f\neq 0,w_1,w_2,...,w_n)$. Similar expression also holds for $\overline{N}_L(r,1;G^*)$.

Proof. Since F^* and G^* share $(1,t)$, so in view of *Lemma 2.3* using the first fundamental theorem we find that

$$\begin{aligned}
\overline{N}_L(r,1;F^*) \quad & \le \overline{N}(r,1;F^*| \ge t+2) \le \frac{1}{t+1}\left[N(r,1;F^*) - \overline{N}(r,1;F^*)\right] \\
& \le \frac{1}{t+1}\left[\sum_{i=1}^{n}\left(N(r,w_i;f) - \overline{N}(r,w_i;f)\right)\right] \\
& \le \frac{1}{t+1}\left[N(r,0;f'|f \ne 0) - N_1(r,0;f')\right] \\
& \le \frac{1}{t+1}\left[N(r,0;\frac{f'}{f}) - N_1(r,0;f')\right] \\
& \le \frac{1}{t+1}\left[N(r,\infty;\frac{f}{f'}) - N_1(r,0;f')\right] + O(\log r) \\
& \le \frac{1}{t+1}\left[N(r,\infty;\frac{f^{'}}{f}) - N_1(r,0;f')\right] + O(\log r) \\
& \le \frac{1}{t+1}\left[\overline{N}(r,\infty;f) + \overline{N}(r,0;f) - N_1(r,0;f')\right] + O(\log r) \\
& \le \frac{1}{t+1}\left[\overline{N}(r,0;f) - N_1(r,0;f')\right] + O(\log r).
\end{aligned}$$

This proves the lemma. □

Lemma 2.5. *Let $P(z)$, S and s as defined by* (1.6), (1.7) *and* (1.9) *respectively. Suppose that f, $\mathcal{L}$ share (S,t), where $t \in \mathbb{N} \cup \{0\}$ and f, $\mathcal{L}$ be a non-constant meromorphic function and an L-function respectively. Further suppose that*

(2.4)

$$\mathcal{F} = \frac{P(f) - a_0}{-a_0} = -\frac{a_n}{a_0}\prod_{i=1}^{p}(f - \alpha_i)^{m_i} \text{ and } \mathcal{G} = \frac{P(\mathcal{L}) - a_0}{-a_0} = -\frac{a_n}{a_0}\prod_{i=1}^{p}(\mathcal{L} - \alpha_i)^{m_i}.$$

Then for $n \ge 2s+3$, when $t \ge 1$ and for $n \ge 2s+6$, when $t = 0$ we get the following.

$$\frac{1}{\mathcal{F}-1} = \frac{A}{\mathcal{G}-1} + B,$$

where $A(\ne 0), B \in \mathbb{C}$.

Proof. According to the assumptions of the lemma we clearly have $\mathcal{F}$, $\mathcal{G}$ share $(1,t)$ and

$$\mathcal{F}^{'} = -\frac{na_n}{a_0}\prod_{i=1}^{s}(f - \eta_i)^{r_i} f^{'}; \quad \mathcal{G}^{'} = -\frac{na_n}{a_0}\prod_{i=1}^{s}(\mathcal{L} - \eta_i)^{r_i}\mathcal{L}^{'},$$

where $\sum_{i=1}^{s} r_i = n-1$. Now consider H as given by (2.1) for $\mathcal{F}$ and $\mathcal{G}$.

Case-I: Suppose $H \not\equiv 0$. Then, it can be easily verified that H has only simple poles and these poles come from the following points.

(i) α_i-points of f and $\mathcal{L}$.

(ii) Poles of f and $\mathcal{L}$.

(iii) 1-points of $\mathcal{F}$ and $\mathcal{G}$ having different multiplicities.

(iv) Those zeros of $f^{'}$ and $\mathcal{L}^{'}$ which are not zeros of $\prod_{i=1}^{s}(f-\eta_i)(\mathcal{F}-1)$ and $\prod_{i=1}^{s}(\mathcal{L}-\eta_i)(\mathcal{G}-1)$ respectively.

Therefore we obtain

$$(2.5)\quad N(r,H) \leq \sum_{i=1}^{s}\left[\overline{N}(r,\eta_i;f)+\overline{N}(r,\eta_i;\mathcal{L})\right]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) +\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;\mathcal{L}^{'}),$$

where $\overline{N}_0(r,0;f^{'})$ and $\overline{N}_0(r,0;\mathcal{L}^{'})$ denotes the reduced counting functions of those zeros of $f^{'}$ and $\mathcal{L}^{'}$ which are not zeros of $\prod_{i=1}^{s}(f-\eta_i)(\mathcal{F}-1)$ and $\prod_{i=1}^{s}(\mathcal{L}-\eta_i)(\mathcal{G}-1)$ respectively. Using the second fundamental theorem we get

(2.6)

$$(n+s-1)T(r,f)\leq \overline{N}(r,1;\mathcal{F})+\sum_{i=1}^{s}\overline{N}(r,\eta_i;f)+\overline{N}(r,\infty;f)-N_0(r,0;f^{'})+S(r,f),$$

(2.7)

$$(n+s-1)T(r,\mathcal{L})\leq \overline{N}(r,1;\mathcal{G})+\sum_{i=1}^{s}\overline{N}(r,\eta_i;\mathcal{L})+\overline{N}(r,\infty;\mathcal{L})-N_0(r,0;\mathcal{L}^{'})+S(r,\mathcal{L}).$$

For the sake of our convenience let us denote by $T(r)=T(r,f)+T(r,\mathcal{L})$. Now combining (2.6) and (2.7) with the help of *Lemma 2.2*, *Lemma 2.1* and then (2.5) we get

$$\begin{aligned}(2.8)\,(n+s-1)T(r) &\leq \overline{N}(r,1;\mathcal{F})+\overline{N}(r,1;\mathcal{G})\\ &+ \sum_{i=1}^{s}\left[\overline{N}(r,\eta_i;f)+\overline{N}(r,\eta_i;\mathcal{L})\right]\\ &\quad+\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\right]-N_0(r,0;f^{'})-N_0(r,0;\mathcal{L}^{'})\\ &\quad+S(r,f)+S(r,\mathcal{L})\\ &\leq \frac{n}{2}T(r)+2\sum_{i=1}^{s}\left[\overline{N}(r,\eta_i;f)+\overline{N}(r,\eta_i;\mathcal{L})\right]\\ &\quad+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\right]+\left(\frac{3}{2}-t\right)\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})\\ &\quad+S(r,f)+S(r,\mathcal{L}).\end{aligned}$$

Hence in view of *Lemma 2.3*, for $t\geq 2$; (2.8) reduces to

$$(\frac{n}{2}-s-1)T(r)\leq O(\log r),$$

which is a contradiction for $n\geq 2s+3$ as $\rho(f)=1=\rho(\mathcal{L})$.

We know that $\overline{N}_*(r,1;\mathcal{F},\mathcal{G}) = \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G})$. Hence for $0 \leq t \leq 1$; using *Lemma 2.4* and *Lemma 2.3* we get from (2.8) that

$$(2.9) \quad (\frac{n}{2} - s - 1)T(r) \leq \frac{(\frac{3}{2} - t)}{t+1}\left[\overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{L})\right] + O(\log r).$$

Now for $t = 1$; from (2.9) we get

$$(\frac{n}{2} - s - \frac{5}{4})T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2s + 3$.

For $t = 0$; from (2.9) we get

$$(\frac{n}{2} - s - \frac{5}{2})T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2s + 6$.

Case-II: Suppose $H \equiv 0$. Hence on integration, we obtain $\frac{1}{\mathcal{F} - 1} = \frac{A}{\mathcal{G} - 1} + B$, where $A(\neq 0), B \in \mathbb{C}$. □

Lemma 2.6. *Let $R(z)$, S and l as defined by* (1.10), (1.12) *and* (1.13) *respectively. Suppose that f, $\mathcal{L}$ share (S,t), where $t \in \mathbb{N} \cup \{0\}$ and f, $\mathcal{L}$ be a non-constant meromorphic function and an L-function respectively. Further suppose that*

$$(2.10) \quad \mathbb{F} = R(f) \ \ and \ \ \mathbb{G} = R(\mathcal{L}).$$

Then for $n \geq 2l+3$, when $t \geq 1$ and for $n \geq 2l+6$, when $t = 0$ we get the following.

$$\frac{1}{\mathbb{F} - 1} = \frac{A}{\mathbb{G} - 1} + B,$$

where $A(\neq 0), B \in \mathbb{C}$.

Proof. Clearly $\mathbb{F}, \mathbb{G}$ share $(1,t)$ and in view of (1.13) we have

$$(2.11) \quad \mathbb{F}^{'} = \frac{\gamma \prod\limits_{j=1}^{l}(f - \delta_j)^{q_j}}{\prod\limits_{j=1}^{k}(f - \beta_j)^{p_j}} f', \qquad \mathbb{G}^{'} = \frac{\gamma \prod\limits_{j=1}^{l}(\mathcal{L} - \delta_j)^{q_j}}{\prod\limits_{j=1}^{k}(\mathcal{L} - \beta_j)^{p_j}} \mathcal{L}'.$$

Now consider H as given by (2.1) for $\mathbb{F}$ and $\mathbb{G}$.

Case-I: Suppose $H \not\equiv 0$. Since H has only simple poles and in this case these poles come from the following points.

(i) δ_j -points of f and $\mathcal{L}$.
(ii) Poles of f and $\mathcal{L}$.
(iii) 1-points of $\mathbb{F}$ and $\mathbb{G}$ having different multiplicities.

(iv) Those zeros of $f^{'}$ and $\mathcal{L}^{'}$ which are not zeros of $\prod_{j=1}^{l}(f-\delta_j)(\mathbb{F}-1)$ and $\prod_{j=1}^{l}(\mathcal{L}-\delta_j)(\mathbb{G}-1)$ respectively.

Therefore we obtain

$$N(r,H)\leq \overline{N}(r,\infty;f)+\sum_{j=1}^{l}\overline{N}(r,\delta_j;f)+\overline{N}_0(r,0;f')+\overline{N}(r,\infty;\mathcal{L}) \tag{2.12}$$

$$+\quad \sum_{j=1}^{l}\overline{N}(r,\delta_j;\mathcal{L})+\overline{N}_0(r,0;\mathcal{L}')+\overline{N}_*(r,1;\mathbb{F},\mathbb{G})+S(r,f)+S(r,\mathcal{L}),$$

where we write $\overline{N}_0(r,0;f')$ for the reduced counting function of the zeros of f' that are not zeros of $(\mathbb{F}-1)\prod_{j=1}^{l}(f-\delta_j)^{q_j}$ and $\overline{N}_0(r,0;\mathcal{L}')$ is similarly defined. By using *Lemma 2.1* , *Lemma 2.2* and (2.12) we observe that

$$\overline{N}(r,1;\mathbb{F})+\overline{N}(r,1;\mathbb{G})\leq N(r,H)+\frac{1}{2}\left[N(r,1;\mathbb{F})+N(r,1;\mathbb{G})\right] \tag{2.13}$$

$$-(t-\frac{1}{2})\overline{N}_*(r,1;\mathbb{F},\mathbb{G})\leq \overline{N}(r,\infty;f)+\sum_{j=1}^{l}\overline{N}(r,\delta_j;f)+\overline{N}(r,\infty;\mathcal{L})+\sum_{j=1}^{l}\overline{N}(r,\delta_j;\mathcal{L})$$

$$+\frac{n}{2}\left\{T(r,f)+T(r,\mathcal{L})\right\}+\left(\frac{3}{2}-t\right)\overline{N}_*(r,1;\mathbb{F},\mathbb{G})+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;\mathcal{L}')+S(r,f)+S(r,\mathcal{L}).$$

Set $T(r,f)+T(r,\mathcal{L})=T(r)$. Hence in view of (2.13), using the second fundamental theorem we have

$$(n+l-1)T(r)\leq \overline{N}(r,\infty;f)+\overline{N}(r,1;\mathbb{F})+\sum_{j=1}^{l}\overline{N}(r,\delta_j;f) \tag{2.14}$$

$$+\overline{N}(r,\infty;\mathcal{L})+\overline{N}(r,1;\mathbb{G})+\sum_{j=1}^{l}\overline{N}(r,\delta_j;\mathcal{L})-N_0(r,0;f')-N_0(r,0;\mathcal{L}')$$

$$+S(r,f)+S(r,\mathcal{L})\leq 2\sum_{j=1}^{l}\overline{N}(r,\delta_j;f)+2\sum_{j=1}^{l}\overline{N}(r,\delta_j;\mathcal{L})$$

$$+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\right]+\frac{n}{2}T(r)+\left(\frac{3}{2}-t\right)\overline{N}_*(r,1;\mathbb{F},\mathbb{G})$$

$$+S(r,f)+S(r,\mathcal{L})\leq (2l+\frac{n}{2})T(r)+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\right]$$

$$+\left(\frac{3}{2}-t\right)\overline{N}_*(r,1;\mathbb{F},\mathbb{G})+S(r,f)+S(r,\mathcal{L}).$$

Hence in view of *Lemma 2.3*, for $t\geq 2$; (2.14) reduces to

$$\left(\frac{n}{2}-l-1\right)T(r)\leq O(\log r),$$

which is a contradiction for $n\geq 2l+3$.

For $0 \leq t \leq 1$; using $\overline{N}_*(r,1;\mathbb{F},\mathbb{G}) = \overline{N}_L(r,1;\mathbb{F}) + \overline{N}_L(r,1;\mathbb{G})$, *Lemma 2.4* and *Lemma 2.3* we get from (2.14) that

$$\left(\frac{n}{2} - l - 1\right) T(r) \quad \leq \quad \frac{\left(\frac{3}{2} - t\right)}{t+1} \left[\overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{L})\right] + O(\log r). \tag{2.15}$$

Now for $t = 1$; from (2.15) we get

$$\left(\frac{n}{2} - l - \frac{5}{4}\right) T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2l + 3$.

For $t = 0$; from (2.15) we get

$$\left(\frac{n}{2} - l - \frac{5}{2}\right) T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2l + 6$.

Case-II: Suppose $H \equiv 0$. Now integrating (2.1), we find that

$$\frac{1}{\mathbb{F} - 1} = \frac{A}{\mathbb{G} - 1} + B, \quad \text{where} \quad A(\neq 0), B \in \mathbb{C}. \tag{2.16}$$

Lemma 2.7. [20] *Let F and G be two non-constant meromorphic functions such that*

$$\frac{1}{F - 1} = \frac{A}{G - 1} + B,$$

where $A(\neq 0), B \in \mathbb{C}$. If

$$\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) < T(r),$$

where $T(r) = \max\{T(r,F), T(r,G)\}$. Then either $FG = 1$ or $F = G$.

Lemma 2.8. *Let $\mathcal{F}$, $\mathcal{G}$ be defined by (2.4) with $p \geq 2$ and they share $(1,t)$ for $t \in \mathbb{N} \cup \{0\}$. Then $\mathcal{F}\mathcal{G} \neq a$, where a is non-zero complex constant.*

Proof. On the contrary, suppose that $\mathcal{F}\mathcal{G} = a$. Then

$$\prod_{i=1}^{p}(f - \alpha_i)^{m_i} \prod_{i=1}^{p}(\mathcal{L} - \alpha_i)^{m_i} = a\left(\frac{a_0}{a_n}\right)^2 = a_1(say). \tag{2.17}$$

It is clear from (2.17) that each α_i-point of f is a pole of $\mathcal{L}$ and vice-versa. Now let us consider the following cases.

Case-1: Let $p \geq 4$. Since an L- function has at most one pole, then in view of (2.17) we can say that f has at least three α_i-points which are picard exeptional values. That is, the meromorphic function f omits at least 3 values, so f must be constant. This contradicts our assumption.

Case-2: Let $p = 3$. Again like the arguments made above we can say that f omits two values say α_1, α_2. Hence using the second fundamental theorem in view of *Lemma 2.3*, we obtain

$$\begin{aligned} T(r,f) &\leq \sum_{i=1}^{2} \overline{N}(r,\alpha_i;f) + \overline{N}(r,\infty;f) + O(\log r) \\ &\leq O(\log r), \end{aligned}$$

which is a contradiction.

Case-3: Let $p = 2$. Note that applying similar argument as made in Case-1 we get f omits at-least one of the α_i's say α_1. On the other hand, f cannot omit both the α_i's. For if, f omits both the α_i's, then we again arrive at a contradiction like Case-2. Hence let us assume α_2 points of f are the poles of $\mathcal{L}$. Again as $z = 1$ is the only pole of $\mathcal{L}$, so let $z = 1$ be α_2 point of f of multiplicity r and the pole of $\mathcal{L}$ of multiplicity s. Then $m_2 r = ns$, which implies $m_2 r \geq n$; i.e., $\frac{1}{r} \leq \frac{m_2}{n}$. Now using the second fundamental theorem in view of *Lemma 2.3* we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\infty;f) + O(\log r) \\ &\leq \frac{m_2}{n} T(r,f) + O(\log r), \end{aligned}$$

which is a contradiction as $n > m_2$. □

Lemma 2.9. *Let $\mathbb{F}, \mathbb{G}$ as defined by* (2.10). *Then for*

(i) $k \geq 2$*; or*

(ii) $k = 1$ *with* $n > 2m_1$*;*

$\mathbb{F}\mathbb{G} \neq a$*, where a is non-zero complex constant.*

Proof. On the contrary suppose that $\mathbb{F}\mathbb{G} \equiv a$. Then

$$\frac{f^n}{\prod_{j=1}^{k}(f-\beta_j)^{m_j}} \cdot \frac{\mathcal{L}^n}{\prod_{j=1}^{k}(\mathcal{L}-\beta_j)^{m_j}} \equiv a\left(\frac{a_i}{a_n}\right)^2 = a^{'}\,(say) \tag{2.18}$$

It is clear from (2.18) that β_j point of f is a zero of $\mathcal{L}$ and vice-versa and

$$T(r,f) = T(r,\mathcal{L}) + O(1). \tag{2.19}$$

Now we deal with the following cases.

Case I: Let $k \geq 2$. If z_0 be a zero of $f - \beta_j$ with multiplicity p, then z_0 is a zero of g with multiplicity q such that $m_j p = nq$ i.e., $p \geq \dfrac{n}{m_j}$. Therefore $\overline{N}(r,\beta_j;f) \leq \dfrac{m_j}{n} N(r,\beta_j;f)$.

So, in view of *Lemma 2.3* using the the second fundamental theorem, we get

$$\begin{aligned}(k-1)T(r,f) &\leq \sum_{j=1}^{k}\overline{N}(r,\beta_j;f)+\overline{N}(r,\infty;f)+O(\log r)\\ &\leq \sum_{j=1}^{k}\frac{m_j}{n}T(r,f)+O(\log r)\\ &\leq (1-\frac{1}{n})T(r,f)+O(\log r), \qquad (2.20)\end{aligned}$$

which contradicts $k \geq 2$.

Case-II: For $k=1$, from (2.18) we have

$$\frac{\mathcal{L}^n}{(\mathcal{L}-\beta_1)^{m_1}} = \frac{a^{'}(f-\beta_1)^{m_1}}{f^n}. \qquad (2.21)$$

From (2.21) we see that $\overline{N}(r,0;f) = \overline{N}(r,\beta_1;\mathcal{L}) + \overline{N}(r,\infty;\mathcal{L}) = \overline{N}(r,\beta_1;\mathcal{L}) + O(\log r)$. Also by similar calculation as in Case-I we have $\overline{N}(r,\beta_1;f) = \frac{m_1}{n}N(r,\beta_1;f)$ and $\overline{N}(r,\beta_1;\mathcal{L}) = \frac{m_1}{n}N(r,\beta_1;\mathcal{L})$. Again using the second fundamental theorem in view of *Lemma 2.3* and (2.19) we have

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,\beta_1;f)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+O(\log r).\\ &\leq \frac{2m_1}{n}T(r,f)+O(\log r), \qquad (2.22)\end{aligned}$$

which is a contadiction as $n > 2m_1$. □

3. Proof Of the theorems

Proof Of the theorem 1.1. We prove the theorem step by step as follows.

(i) $\Longrightarrow$ (ii) : Suppose f is a non-constant meromorphic function and $\mathcal{L}$ is a non-constant L-function such that $E_f(S,t) = E_{\mathcal{L}}(S,t)$, where $t \in \mathbb{N}\cup\{0\}$. Consider $\mathcal{F}$ and $\mathcal{G}$ as defined by (2.4). Then for

(i) $t \geq 1$ and $n \geq 2s+3$, or

(ii) $t = 0$ and $n \geq 2s+6$,

in view of the *Lemma 2.5* we get $\frac{1}{\mathcal{F}-1} = \frac{A}{\mathcal{G}-1} + B$, where $A(\neq 0), B \in \mathbb{C}$. Hence we have

$$T(r,\mathcal{F}) = T(r,\mathcal{G}) + O(1). \qquad (3.1)$$

Since

$$T(r,\mathcal{F}) = nT(r,f)+O(1) \ \ \textit{and} \ \ T(r,\mathcal{G}) = nT(r,\mathcal{L})+O(1). \qquad (3.2)$$

So (3.1) implies that

$$T(r,f) = T(r,\mathcal{L}) + O(1). \qquad (3.3)$$

Now in view of *Lemma 2.3* using (3.2) and (3.3) we get

$$\begin{aligned} & \overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+\overline{N}(r,\infty;\mathcal{G}) \\ \leq\ & pT(r,f)+pT(r,\mathcal{L})+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) \\ =\ & 2pT(r,f)+O(\log r)<\frac{2p+1}{n}T(r,\mathcal{F}) \\ \leq\ & T(r,\mathcal{F}) \quad [\because n\geq 2p+1]. \end{aligned}$$

So in view of *Lemma 2.7*, we have either $\mathcal{F}\mathcal{G}=1$ or $\mathcal{F}=\mathcal{G}$. Since $p\geq 2$, so in view of *Lemma 2.6* we have $\mathcal{F}\mathcal{G}\neq 1$. Hence $\mathcal{F}=\mathcal{G}$. That is, we get

$$P(f)=P(\mathcal{L}), \tag{3.4}$$

which by condition (i) implies $f=\mathcal{L}$.

(ii) $\Longrightarrow$ **(i)** : Let $P(f)=P(\mathcal{L})$. That is,

$$\prod_{i=1}^{p}(f-\alpha_i)^{m_i}=\prod_{i=1}^{p}(\mathcal{L}-\alpha_i)^{m_i},$$

which implies f and $\mathcal{L}$ share (S,∞). Therefore, obviously f and $\mathcal{L}$ share (S,t) for $t\in\mathbb{N}\cup\{0\}$. Hence by condition (ii), we have $f=\mathcal{L}$.

Proof of the theorem 1.2. Let us consider $\mathbb{F}$ and $\mathbb{G}$ as defined by (2.10). Let f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function such that $E_f(S,t)=E_{\mathcal{L}}(S,t)$, where $t\in\mathbb{N}\cup\{0\}$. Then $\mathbb{F}$, $\mathbb{G}$ share $(1,t)$. Now for

(i) $t\geq 1$ and $n\geq 2l+3$, or
(ii) $t=0$ and $n\geq 2l+6$,

in view of *Lemma 2.6* we have

$$\frac{1}{\mathbb{F}-1}=\frac{A}{\mathbb{G}-1}+B, \tag{3.5}$$

where $A(\neq 0),B\in\mathbb{C}$.

From (3.5) we easily obtain

$$T(r,f)=T(r,\mathcal{L})+S(r,f). \tag{3.6}$$

Now in view of *Lemma 2.3*, (3.6) and from the construction of $\mathbb{F}$ and $\mathbb{G}$ we get

$$\begin{aligned} & \overline{N}(r,0;\mathbb{F})+\overline{N}(r,\infty;\mathbb{F})+\overline{N}(r,0;\mathbb{G})+\overline{N}(r,\infty;\mathbb{G}) \\ \leq\ & \overline{N}(r,0;f)+\sum_{j=1}^{k}\overline{N}(r,\beta_j;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;\mathcal{L})+\sum_{j=1}^{k}\overline{N}(r,\beta_j;\mathcal{L})+\overline{N}(r,\infty;\mathcal{L}) \\ \leq\ & (1+k)T(r,f)+(1+k)T(r,\mathcal{L})+O(\log r)=2(1+k)T(r,f)+O(\log r) \\ <\ & \frac{2k+3}{n}T(r,\mathbb{F})\leq T(r,\mathbb{F}) \ \ [\because\ \ n\geq 2k+3]. \end{aligned}$$

So in view of *Lemma 2.7*, we have either $\mathbb{F}\mathbb{G} \equiv 1$ or $\mathbb{F} \equiv \mathbb{G}$. Again in view of *Lemma 2.9* we have $\mathbb{F}\mathbb{G} \not\equiv 1$. Thus $\mathbb{F} \equiv \mathbb{G}$; i.e., $R(f) = R(\mathcal{L})$.

Therefore we find that $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$, whenever $R(f) = R(\mathcal{L}) \implies f = \mathcal{L}$. That is $(i) \implies (ii)$.

To show $(ii) \implies (i)$, suppose that $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$. Let $R(f) = R(\mathcal{L})$, then we have $R(f) - 1 = R(\mathcal{L}) - 1$; i.e., $\dfrac{P(f)}{\phi(f)} = \dfrac{P(\mathcal{L})}{\phi(\mathcal{L})}$. Therefore f and $\mathcal{L}$ share (S,∞) and which implies $E_f(S,t) = E_{\mathcal{L}}(S,t)$, hence $f = \mathcal{L}$.

4. Applications

In this section, we prove that all the existing results can be improved as an application of our results. Moreover, there exist other polynomials providing better results than the existing ones including those polynomials which are still uncertain to be critically injective or non-critically injective. Furthermore, in this section we have also exhibited a similar result for non-critically injective polynomials which is yet not considered in this literature. In a word, by executing the following examples we prove the far reaching applications of *Theorem 1.1* and *Theorem 1.2*.

First of all we exhibit examples of critically injective polynomials as the applications of *Theorem 1.1.*

Example 4.1. *Let us consider the following polynomial.*

$$P(z) = z^n + az^{n-m} + bz^{n-2m} + c, \tag{4.1}$$

where $a, b, c \in \mathbb{C}^*$ be such that $P(z)$ has no multiple root, $gcd(m,n) = 1$ and $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$, $c \neq \frac{\beta_i\beta_j}{\beta_i+\beta_j}$. Here $\beta_i = -(c_i^n + ac_i^{n-m} + bc_i^{n-2m})$, where c_i are the roots of the equation $nz^{2m} + (n-m)az^m + b(n-2m) = 0$, for $i = 1, 2, \ldots, 2m$. Suppose S denotes the set of zeros of (4.1).

Obviously, $P(z)$ has only simple zeroes and it is critically injective [6, see Lemma 2.7]. From (4.1) we have

$$P'(z) = z^{n-2m-1}[nz^{2m} + a(n-m)z^m + b(n-2m)] \tag{4.2}$$

$$= nz^{n-2m-1}\left(z^m + \frac{n(n-m)}{2n}\right)^2. \tag{4.3}$$

From (4.1) and (4.2) we find that

$$p = 2m + 1 \quad \text{and} \quad s = m + 1.$$

In [6, see proof of Theorem 1.1] it is also proved that $P(f) = P(g)$ implies $f = g$ for $n \geq 2m+4$, where $f$and g are non-constant meromorphic functions. Hence for a non-constant meromorphic function f having finitely many poles and an L-function $\mathcal{L}$ we have $P(f) = P(\mathcal{L}) \implies f = \mathcal{L}$ when $n \geq 2m+4$. Thus $P(z)$ satisfies the condition (i) of Theorem 1.1 of the present paper and hence $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ for

(1) $n \geq \max\{4m+3, 2m+5\} \geq 7$ when $t \geq 1$, and
(2) $n \geq \max\{4m+3, 2m+8\} \geq 10$ when $t = 0$.

Remark 4.1. *Note that the polynomial*

(4.4)
$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c, \quad where \ \ n \geq 6, \quad c \neq 0, 1.$$

introduced by Frank-Reinders [7] comes as the special case of (4.1) for $m = 1$, $a = -\frac{2n}{n-1}$, $b = \frac{n}{n-2}$ *and* $c \in \mathbb{C} - \{0, \frac{-1}{(n-1)(n-2)}\}$. *Hence* $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ *as* $n \geq 7$ *when* $t \geq 1$ *and* $n \geq 10$ *when* $t = 0$, *where* S *denotes the set of zeros of* (4.4) *and* f, $\mathcal{L}$ *are non-constant meromorphic function having finitely many poles and a non-constant L-function respectively.*

Example 4.2. *Consider the polynomial*

$$P(z) = z^n + az^{n-m} + b = z^{n-m}(z^m + a) + b, \tag{4.5}$$

where n, m *are relatively prime inegers and* a, b *are non-zero constants such that the polynomial has no multiple zero. Suppose* $S = \{z : P(z) = 0\}$. *Here*

$$p = m + 1 \geq 2$$

and

$$P'(z) = z^{n-m-1}(nz^m + a(n-m)); \tag{4.6}$$

i.e., $s = m + 1$.

Suppose that $P(f) = P(\mathcal{L})$, for any non-constant meromorphic function f having finitely many poles and a non-constant L-function $\mathcal{L}$, then we have

$$f^n - \mathcal{L}^n = -a(f^{n-m} - \mathcal{L}^{n-m}). \tag{4.7}$$

If $f^n \not\equiv \mathcal{L}^n$, then we can rewrite (4.7) as

$$\mathcal{L}^m = -a \frac{(h-v)(h-v^2)...(h-v^{n-m-1})}{(h-u)(h-u^2)...(h-u^{n-1})}, \tag{4.8}$$

where $h = \frac{f}{\mathcal{L}}$, $u = exp(2\pi i/n)$ and $v = exp(2\pi i/(n-m))$. Noting that n and $(n-m)$ are relatively prime positive integers, then the numerator and denominator of (4.8) have no common factors. Since $\mathcal{L}$ has atmost one pole at $z = 1$ in the complex plane, and whenever $n \geq 5$ we can see that there exists at least three distinct roots of $h^n = 1$ such that they are Picard exceptional values of h, and so it follows by (4.8) that h and thus $\mathcal{L}$ are constants, which is impossible.

Therefore, we must have $f^n = \mathcal{L}^n$. Then by (4.7) we also have $f^{n-m} = \mathcal{L}^{n-m}$. Since n and $(n-m)$ are relatively prime positive integers, we deduce that $f = \mathcal{L}$. Thus we see that $P(f) = P(\mathcal{L}) \implies f = \mathcal{L}$, when $n \geq 5$.

Now we apply Theorem 1.1 to find the minimum value of n for which we can say that $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$.

Therefore, $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ for

(1) $n \geq max\{2m+3, 2m+5\} = 2m+5$ when $t \geq 1$, and
(2) $n \geq max\{2m+3, 2m+8\} = 2m+8$ when $t = 0$.

In the next example we explore a non-critically injective polynomial in the direction of Theorem 1.1.

Example 4.3. *Let*

$$P(z) = z^n + 2z^{n-1} + z^{n-2} + c, \tag{4.9}$$

where $n(\geq 5)$ is odd, $c \in \mathbb{C}$ such that $P(z)$ does not have any multiple zero. Also we have

$$P'(z) = z^{n-3}\left(nz^2 + 2(n-1)z + (n-2)\right). \tag{4.10}$$

Here $P(z)$ is a non-critically injective polynomial and we see that

$$p = 2, \quad s = 3.$$

Suppose $S = \{z : P(z) = 0\}$. Let f and $\mathcal{L}$ be two non-constant meromorphic and L-function respectively such that

$$P(f) = P(\mathcal{L}).$$

Since $\mathcal{L}$ has at most one pole in $\mathbb{C}$, hence proceeding in the same line of proof of as done in Example 4.4 *of* [15] *for uniqueness polynomial of entire function we also get here $f = \mathcal{L}$.*

Therefore $P(z)$ satisfies condition (i) of Theorem 1.1. Hence we conclude that $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ when

(1) $n \geq max\{2.2+1, 2.3+3\} = 9$ *for* $t \geq 1$, *and*
(2) $n \geq max\{2.2+1, 2.3+6\} = 13$ *for* $t = 0$.

Now we apply *Theorem 1.2* for rest of the examples where in the first example we have considered a polynomial which is still uncertain to be critically injective or non-critically injective [3, see section 5] and the polynomial used in the second example is critically injective.

Example 4.4. *Consider the polynomial*

$$P(z) = az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2,$$

where $n(\geq 6)$ *is an integer and* a, b *are two non-zero complex numbers satisfying* $ab^{n-2} \neq 1, 2$. *Suppose* $S = \{z : P(z) = 0\}$. *It is obvious that* $n(n-1)z^2 - 2n(n-2)bz + (n-1)(n-2)b^2 = 0$; *has two distinct roots, say* α_1 *and* α_2. *Here*

$$R(z) = \frac{az^n}{n(n-1)(z-\alpha_1)(z-\alpha_2)}. \tag{4.11}$$

Hence $S = \{z : R(z) - 1 = 0\}$. *From (4.11) we have*

$$R'(z) = \frac{(n-2)az^{n-1}(z-b)^2}{n(n-1)(z-\alpha_1)^2(z-\alpha_2)^2}. \tag{4.12}$$

Let f a non-constnat meromorphic function having finitely many poles and $\mathcal{L}$ be a non-constant L-function. Since every L-function is meromorphic in $\mathbb{C}$, so $R(f) = R(\mathcal{L}) \implies f = \mathcal{L}$ for $n \geq 6$ directly follows from [1, see page 67].

We also find that in this case $l = 2$, $k = 2$. Since $P(z)$ satisfies condition (i) of Theorem 1.2. Hence we obtain $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ for

(1) $n \geq max\{2k+3, 2l+3\} = 7$ when $t \geq 1$, and
(2) for $n \geq max\{2k+3, 2l+6\} = 10$ when $t = 0$.

Example 4.5. *Consider the polynomial*

$$P(z) = z^n + az^m + b, \tag{4.13}$$

where m *and* n *are positive integers such that* $n \geq m+4$, a *and* b *are finite non-zero complex numbers with* $\dfrac{b^{n-m}}{a^m} \neq \dfrac{(-1)^n m^m (n-m)^{n-m}}{n^n}$. *Then* $P(z)$ *has only simple zeros. Let* S *denotes the set of zeros of* $P(z)$. *Suppose*

$$R(z) = -\frac{z^n}{az^m + b}. \tag{4.14}$$

Then we find that $S = \{z : R(z) - 1 = 0\}$. *From (4.14) we have*

$$R'(z) = -\frac{z^{n-1}[a(n-m)z^m + bn]}{(az^m + b)^2}. \tag{4.15}$$

Now for a non-constant meromorphic function f and a non-constant L-function $\mathcal{L}$ consider $R(f) = R(\mathcal{L})$. Then we have

$$\frac{f^n}{af^m + b} = \frac{\mathcal{L}^n}{a\mathcal{L}^m + b} \implies a(f^n\mathcal{L}^m - f^m\mathcal{L}^n) - b(\mathcal{L}^n - f^n) = 0. \tag{4.16}$$

Let $h = \dfrac{f}{\mathcal{L}}$. Suppose that h is a non-constant meromorphic function. Then from (4.16) we have

$$(4.17)\quad ah^m\mathcal{L}^{n+m}(h^{n-m}-1)+b\mathcal{L}^n(h^n-1)=0$$
$$\implies \mathcal{L}^m = -\frac{b(h^n-1)}{ah^m(h^{n-m}-1)} = -\frac{b}{a}\frac{(h-u)(h-u^2)...(h-u^{n-1})}{h^m(h-v)(h-v^2)...(h-v^{n-m-1})},$$

where $u = exp(2\pi i/n)$, and $v = exp(2\pi i/(n-m))$. Since n and m are co-prime, so is n and $(n-m)$. Hence the numerator and denominator of (4.17) have no common factors. Further, the function $\mathcal{L}$ has atmost one pole in the complex plane, it follows that h has atleast $(n-m-1)$ picard exceptional values among $\{0, v, v^2, ..., v^{n-m-1}\}$.

Clearly this is a contradiction as $n \geq m+4$. Hence h is constant. Thus from (4.17) we must have $h^n = 1 = h^{n-m}$, which in turn implies $h = 1$; i.e., $f = \mathcal{L}$. Therefore we obtain that $R(z)$ satisfies condition (i) of Theorem 1.2.

Now we count the cardinality of the set S for which $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$. In this case, for $R(z)$ we have

$$l = m+1, \quad k = m.$$

Therefore the condition (ii) of the Theorem 1.2 is satisfied if

(1) $n \geq max\{2m+3, 2m+5\} = 2m+5$ for $t \geq 1$ and
(2) $n \geq max\{2m+3, 2m+8\} = 2m+8$ for $t = 0$.

Remark 4.2. *Observe that* Example 4.5 *and* Example 4.2 *answer* Question 1.4 *and* Question 1.5 *with threefold improvement to* Theorem G-H *as discussed in* Remark 1.2 *which inturn improve* Theorem D-H *by relaxing the nature of sharing of the sets or reducing the least cardinalities of the sets or both.*

Remark 4.3. *Further note that* Example 4.1*,* Example 4.3 *and* Example 4.4 *answer* Question 1.2 *and* Question 1.3 *affirmatively. Moreover,* Example 4.1 *and* Example 4.4 *improves* Theorem D-H *either by relaxing the nature of sharing of the sets or reducing the least cardinalities of the sets or both.*

5. Conclusion and an open question

Observe that if we consider *Example 4.2* in the direction of *Theorem 1.1*, then we would obtain the same conclusion for $n \geq \max\{2(n-m)+3, 2(n-m+1)+3\} = \max\{2n-2m+3, 2n-2m+5\}$; i.e., $m \geq \frac{n+5}{2}$, which is absurd. So, *Theorem 1.2* is not applicable for *Example 4.2*, whereas *Theorem 1.1* is applicable for the same. Similarly we would have problems in counting the cardinality of the set if we apply *Theorem 1.2* in case of *Example 4.1* and *Example 4.3*.

Conversely the conclusion of *Example 4.4* and *Example 4.5* can not be obtained as the application of *Theorem 1.1* but *Theorem 1.2*. That is why, we have have proved two theorems in this paper in the most general setting to justify all the existing results as well as to include all the variants of polynomials for the uniqueness of f and $\mathcal{L}$.

Last but not the least, observing *Theorem 1.1-1.2* and *Example 4.1-4.5* carefully, it is obvious that for any polynomial if one can find $P(f) = P(\mathcal{L})$ or $R(f) = R(\mathcal{L})$ implies $f = \mathcal{L}$, then at instant we would be able to find out the set with least possible cardinality and sharing condition. Hence under this circumstances, the following question become indispensable for the uniqueness of f and $\mathcal{L}$.

Question 5.1. *Can one find general criterion(s) for any general polynomial given by (1.6) so that $P(f) = P(\mathcal{L})$ or $R(f) = R(\mathcal{L})$ implies $f = \mathcal{L}$?*

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GENERALIZATIONS OF SOME DIFFERENTIAL INEQUALITIES FOR POLYNOMIALS

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Abstract. We consider polynomials of the form $P(z) = z^s\big(a_0 + \sum\limits_{v=t}^{n-s} a_v z^v\big), t \geq 1, 0 \leq s \leq n-1$ and prove some results for the estimate of the polar derivative $D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$ and generalize the results due to Aziz and Shah [*Indian J. Pure Appl. Math.,* **29**(1998), 163-173], Govil [*J. Approx. Theory,* **66**(1991), 29-35] and others.

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1. Introduction

For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $P(z) := \sum\limits_{j=0}^{n} a_j z^j$ of degree atmost n over the field $\mathbb{C}$ of complex numbers.
If $P \in \mathcal{P}_n$ and P' be its derivative, then concerning the estimate $|P'(z)|$, in terms of $|P(z)|$ on $|z| = 1$, we have the following famous sharp result due to Bernstein [7].

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Since equality holds in (1.1) if and only if P has all its zeros at the origin, it stands natural to ask what happens to inequality (1.1), if we impose restrictions on the location of zeros of P. In this connection the following inequalities are the earliest belonging to this domain of ideas which have a clear impact on the subsequent work carried forward since then.

If $P \in \mathcal{P}_n$ has all zeros in $|z| \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

and if it has all zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

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Inequality (1.2) was conjectured by Erdös and latter verified by Lax [14], whereas inequality (1.3) is due to Turán [16]. Inequality (1.2) was generalized by Malik [15] to read as:

Theorem A. *If $P(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, where $k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|.$$

Govil [13] also generalized inequality (1.2) in a different way. More precisely he proved the following.

Theorem B. *If $P(z)$ is a polynomial of degree n, such that $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ and $|Q'(z)|$ attain their maxima at the same point on the unit circle, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

It is worth mentioning that the Bernstein inequality has been generalized in different forms by replacing the underlying polynomial with more general class of functions. These inequalities have their own importance in the theory of approximation. The results we prove provide extensions, generalizations and refinements of various differential inequalities for polynomials. Before proceeding for the main results, we first define the polar derivative of a polynomial.

For a polynomial $P(z)$ of degree n, the polar derivative of $P(z)$ denoted by $D_\alpha P(z)$, is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

It is to be observed that

$$\lim_{|\alpha|\to\infty} \left| \frac{D_\alpha P(z)}{\alpha} \right| = P'(z).$$

Aziz [2] extended Theorem A to the polar derivative of a polynomial and proved the following.

Theorem C. *If $P(z)$ is a polynomial of degree n, such that $P(z)$ does not vanish in $|z| < k, k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left(\frac{k + |\alpha|}{1+k} \right) \max_{|z|=1} |P(z)|. \tag{1.4}$$

In this paper we prove.

Theorem 1.1. *If $P(z)$ is a polynomial of degree n, such that all zeros of $P(z)$ lie in $|z| > k, k \geq 1$ with s-fold zero at the origin, $0 \leq s < n$, then for every real or* *complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(|\alpha| + k)}{1+k} + \frac{sk(|\alpha| - 1)}{1+k} \right) \max_{|z|=1} |P(z)|. \tag{1.5}$$

The result is sharp for $s = 0$ and equality holds for the polynomial $P(z) = (z+k)^n$. For $s = 0$, inequality (1.5) reduces to a result due to Aziz [2, Theorem 3] whereas for $s = n-1$, we have the following.

Corollary 1.1. *If $P(z)$ is a polynomial of degree n having all $n-1$ zeros at the origin and one zero in $|z| > k, k \geq 1$, then for every α with $|\alpha| \geq 1$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{1}{1+k} \left\{ \big(n(1+k) - k\big)|\alpha| + k \right\} \max_{|z|=1} |P(z)|.$$

On dividing both sides of above inequality by $|\alpha|$ and letting $|\alpha| \to \infty$, we get

$$\max_{|z|=1} |P'(z)| \leq \left(n - \frac{k}{1+k} \right) \max_{|z|=1} |P(z)|.$$

Remark 1.1. *Divide the two sides of inequality (1.5) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get a result due to Aziz and Shah* [5].

Theorem 1.2. *Let $P(z)$ be a polynomial of degree n, such that all zeros of $P(z)$ lie in $|z| > k, k \leq 1$ with s-fold zeros at the origin, then for every real or complex number α with $|\alpha| \geq 1$*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(|\alpha| + k^{n-s})}{1+k^{n-s}} + \frac{sk^{n-s}(|\alpha| - 1)}{1+k^{n-s}} \right) \max_{|z|=1} |P(z)|, \tag{1.6}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain their maxima at the same point on the unit circle, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

The result is sharp for $s = 0$ and equality holds for the polynomial $P(z) = z^n + k^n$.

On dividing both sides of inequality (1.6) by $|\alpha|$ and letting $|\alpha| \to \infty$, it reduces to a following result.

Corollary 1.2. *Let $P(z)$ be a polynomial of degree n, such that all zeros of $P(z)$ lie in $|z| > k, k \leq 1$ with s-fold zeros at the origin, then for every real or complex number α with $|\alpha| \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n + sk^{n-s}}{1+k^{n-s}} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ and $|nP(z) - zP'(z)|$ attain their maximum at the same points on $|z| = 1$.

Remark 1.2. *For $s = 0$, Theorem 1.2 reduces to a result due to Chanam* [6, Theorem 1].

Remark 1.3. *By taking $s = 0$ and letting $|\alpha| \to \infty$ in (1.6), we get a result due to Govil* [13].

Theorem 1.3. *If $P(z)$ is a polynomial of degree n, such that all zeros of $P(z)$ lie in $|z| < k, k \leq 1$, with s-fold zeros at the origin. Then for every real or complex number α with $|\alpha| \leq 1$ and $|z| = 1$*

(1.7)
$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(k+|\alpha|)}{1+k} + \frac{sk(|\alpha|-1)}{1+k} \right) \max_{|z|=1} |P(z)| - (n-s) \frac{1-|\alpha|}{k^{n-1}(1+k)} \min_{|z|=k} |P(z)|.$$

The result is sharp for $s = 0$ and equality holds for the polynomial $P(z) = (z+k)^n$.

Remark 1.4. *A result of Aziz and Shah* [4, Theorem 3] *follows from Theorem 1.3, if we take $s = 0$.*

Corollary 1.3. *For $\alpha = 0$, we get from (1.7),*

$$|nP(z) - zP'(z)| \leq \frac{nk - sk}{1+k} \max_{|z|=1} |P(z)| - (n-s) \frac{1}{k^{n-1}(1+k)} \min_{|z|=k} |P(z)|.$$

If $\max_{|z|=1} |P(z)| = |P(e^{i\phi})|$, then from above inequality, we get the following improvement of a result due to Aziz and Shah [5]

$$\max_{|z|=1} |P'(z)| \geq \frac{n+sk}{1+k} \max_{|z|=1} |P(z)| + \frac{(n-s)}{k^{n-1}(1+k)} \min_{|z|=k} |P(z)|. \tag{1.8}$$

We also prove the following results concerning the growth of polynomials.

Theorem 1.4. *If $P(z) = a_n z^n + \sum\limits_{v=\eta}^{n} a_{n-v} z^{n-v}, 1 \leq \eta < n$ is a polynomial of degree n, having all zeros on $|z| = k, k \leq 1$, then for every positive integer s*

$$\left\{ M(P,\rho) \right\}^s \leq \frac{k^{n-2\eta+1} + k^{n-\eta+1} + \rho^{ns} - 1}{k^{n-2\eta+1} + k^{n-\eta+1}} \left\{ M(P,1) \right\}^s, \quad \rho \geq 1.$$

Remark 1.5. *For $\eta = 1$, we get a result due to Dewan et .al* [9, Theorem 1].

Also if we take $\eta = s = k = 1$, then Theorem 4 reduces to a result due to Ankeny and Rivlin [1].

Theorem 1.5. *If* $P(z) = a_n z^n + \sum_{v=\eta}^{n} a_{n-v} z^{n-v}, 1 \le \eta < n$ *is a polynomial of degree* n*, having all zeros on* $|z| = k, k \le 1$*, then for every positive integer* s

$$\big\{M(P,\rho)\big\}^s \le \frac{1}{k^{n-\eta+1}\big(\eta|a_{n-\eta}|(1+k^{\eta-1}) + n|a_n|k^{\eta-1}(1+k^{\eta+1})\big)}$$
$$\Bigg\{k^{n-\eta+1}\big(\eta|a_{n-\eta}|(1+k^{\eta-1}) + n|a_n|k^{\eta-1}(1+k^{\eta+1})\big)$$
$$+ (\rho^{ns}-1)\big(n|a_n|k^{2\eta} + \eta|a_{n-\eta}|k^{\eta-1}\Bigg)\Bigg\}\big\{M(P,1)\big\}^s,$$

where $\rho \ge 1$.

Remark 1.6. *For* $\eta = 1$*, we get a result due to Dewan et al* [9, Theorem 2].

2. Lemmas

Lemma 2.1. *If* $P(z) = a_n z^n + \sum_{v=\eta}^{n} a_{n-v} z^{n-v}, 1 \le \eta < n$ *is a polynomial of degree* n*, having all zeros on* $|z| = k, k \le 1$*, then*

$$\max_{|z|=1} |P'(z)| \le \frac{n}{k^{n-2\eta+1} + k^{n-\eta+1}} \max_{|z|=1} |P(z)|.$$

The above Lemma is due to Dewan et al. [11].

Lemma 2.2. *If* $P(z) = a_n z^n + \sum_{v=\eta}^{n} a_{n-v} z^{n-v}, 1 \le \eta < n$ *is a polynomial of degree* n*, having all zeros on* $|z| = k, k \le 1$*, then*

$$\max_{|z|=1} |P'(z)| \le \frac{n}{k^{n-\eta+1}} \left(\frac{n|a_n|k^{2\eta} + \eta|a_{n-\eta}|k^{\eta-1}}{\eta|a_{n-\eta}|(1+k^{\eta-1}) + n|a_n|k^{\eta-1}(1+k^{\eta+1})} \right) \max_{|z|=1} |P(z)|.$$

Lemma 2.2 is due to Dewan and Hans [10].

We also need the following lemma which is a simple consequence of maximum modulus principle.

Lemma 2.3. *If* $P(z)$ *is a polynomial of degree* n*, then for some* $\rho \ge 1$*, we have*

$$M(P,\rho) \le \rho^n M(P,1),$$

where $M(P,\rho) = \max_{|z|=\rho} |P(z)|$.

Lemma 2.4. *(see* [6]*). If* $P(z)$ *is a polynomial of degree* n*, such that* $P(z) \ne 0$ *in* $|z| < k, k \le 1$*, then for every real or complex number* α *with* $|\alpha| \ge 1$

$$\max_{|z|=1} |D_\alpha P(z)| \le n \left(\frac{k^n + |\alpha|}{1+k^n} \right) \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ *and* $|Q'(z)|$ *attain their maxima at the same point on the unit circle, where* $Q(z) = z^n \overline{P\big(\frac{1}{\bar{z}}\big)}$.

Lemma 2.5. *If $P(z)$ is a polynomial of degree n, such that all zeros of $P(z)$ lie in $|z| < k, k \leq 1$, then for every real or complex number α with $|\alpha| \leq 1$*

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left\{ \frac{k + |\alpha|}{1+k} \max_{|z|=1} |P(z)| - \frac{1-|\alpha|}{k^{n-1}(1+k)} \min_{|z|=k} |P(z)| \right\}.$$

The above lemma is due to Aziz and Shah [4].

3. Proofs of theorems

Proof of Theorem 1.1. Since $P(z) = z^s \phi(z)$, where $\phi(z)$ is a polynomial of degree $n-s$, which does not vanish in $|z| < k$. Applying inequality (1.4) to the polynomial $\phi(z)$, we get

$$\max_{|z|=1} |D_\alpha \phi(z)| \leq (n-s) \left(\frac{k+|\alpha|}{1+k} \right) \max_{|z|=1} |\phi(z)|. \tag{3.1}$$

Since

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) = nz^s\phi(z) + (\alpha - z)(sz^{s-1}\phi(z) + z^s\phi'(z))$$
$$= z^s D_\alpha \phi(z) + \alpha s z^{s-1} \phi(z),$$

where $D_\alpha \phi(z) = (n-s)\phi(z) + (\alpha - z)\phi'(z)$.
Therefore,

$$zD_\alpha P(z) = z^{s+1} D_\alpha \phi(z) + \alpha s z^s \phi(z).$$

Hence for $|z| = 1$, we have

$$\max_{|z|=1} |D_\alpha P(z)| \leq \max_{|z|=1} |D_\alpha \phi(z)| + |\alpha| s \max_{|z|=1} |\phi(z)|.$$

Using $\max_{|z|=1} |\phi(z)| = \max_{|z|=1} |P(z)|$, we get

$$\max_{|z|=1} |D_\alpha P(z)| \leq \max_{|z|=1} |D_\alpha \phi(z)| + |\alpha| s \max_{|z|=1} |P(z)|. \tag{3.2}$$

Considering (3.1) in (3.2), we obtain

$$\max_{|z|=1} |D_\alpha P(z)| \leq (n-s) \left(\frac{k+|\alpha|}{1+k} \right) \max_{|z|=1} |P(z)| + |\alpha| s \max_{|z|=1} |P(z)|.$$

Equivalently

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(|\alpha|+k)}{1+k} + \frac{sk(|\alpha|-1)}{1+k} \right) \max_{|z|=1} |P(z)|.$$

This completely proves Theorem 1.1. □

Proof of Theorem 1.2. Since $P(z) = z^s \phi(z)$. On applying Lemma 2.4 to $\phi(z)$, the proof follows similarly as that of Theorem 1.1. □

Proof of Theorem 1.3. Since $P(z)$ is a polynomial with a zero of multiplicity s at origin, therefore, we write it as $P(z) = z^s\phi(z)$, where $\phi(z)$ is a polynomial of degree $n-s$. Applying Lemma 2.5 to $\phi(z)$, we get

$$\max_{|z|=1}|D_\alpha\phi(z)| \le (n-s)\left\{\frac{k+|\alpha|}{1+k}\max_{|z|=1}|\phi(z)| - \frac{1-|\alpha|}{k^{n-s-1}(1+k)}\min_{|z|=k}|\phi(z)|\right\}. \tag{3.3}$$

Using (3.3) in (3.2), we get

$$\max_{|z|=1}|D_\alpha P(z)| \le (n-s)\left\{\frac{k+|\alpha|}{1+k}\max_{|z|=1}|\phi(z)| - \frac{1-|\alpha|}{k^{n-s-1}(1+k)}\min_{|z|=k}|\phi(z)|\right\} + |\alpha| s\max_{|z|=1}|P(z)|. \tag{3.4}$$

Since $\max_{|z|=1}|P(z)| = \max_{|z|=1}|\phi(z)|$ and $\min_{|z|=k}|\phi(z)| = \frac{1}{k^s}\min_{|z|=k}|P(z)|$. we have from (3.4)

$$\max_{|z|=1}|D_\alpha P(z)| \le \left(\frac{n(k+|\alpha|)}{1+k} + \frac{sk(|\alpha|-1)}{1+k}\right)\max_{|z|=1}|P(z)| - (n-s)\frac{1-|\alpha|}{k^{n-1}(1+k)}\min_{|z|=k}|P(z)|.$$

This completely proves Theorem 1.3. □

Proof of Theorem 1.4. If we write $\max_{|z|=1}|P(z)| = M(P,1)$, where $P(z) = a_n z^n + \sum_{v=\eta}^{n} a_{n-v}z^{n-v}, 1 \le \eta < n$ is a polynomial of degree n having all zeros on $|z| = k, k \le 1$, then by Lemma 2.1

$$|P'(z)| \le \frac{n}{k^{n-2\eta+1} + k^{n-\eta+1}} M(P,1), \quad \text{for} \quad |z| = 1. \tag{3.5}$$

Since $P'(z)$ is a polynomial of degree $n-1$, it follows from (3.5) by an application of maximum modulus principle that for $r \ge 1$ and $0 \le \phi < 2\pi$,

$$|P'(re^{i\phi})| \le \frac{nr^{n-1}}{k^{n-2\eta+1} + k^{n-\eta+1}} M(P,1). \tag{3.6}$$

Hence for some $\rho \ge 1$ and for each $\phi, 0 \le \phi < 2\pi$

$$\begin{aligned}\{P(\rho e^{i\phi})\}^s - \{P(e^{i\phi})\}^s &= \int_1^\rho \frac{d}{du}\{P(ue^{i\phi})\}^s du\\ &= \int_1^\rho s\{P(ue^{i\phi})\}^{s-1}P'(ue^{i\phi})e^{i\phi}du.\end{aligned}$$

This implies

$$\left|\{P(\rho e^{i\phi})\}^s - \{P(e^{i\phi})\}^s\right| \le s\int_1^\rho \left|P(ue^{i\phi})\right|^{s-1}\left|P'(ue^{i\phi})\right|du.$$

Using (3.5) and Lemma 2.3, we get

$$\left|\{P(\rho e^{i\phi})\}^s - \{P(e^{i\phi})\}^s\right| \le \frac{ns}{k^{n-2\eta+1} + k^{n-\eta+1}}\{M(P,1)\}^s \int_1^\rho u^{ns-1}du.$$

Equivalently,

$$\left|\left\{P(\rho e^{i\phi})\right\}^s\right| \leq \left|\left\{P(e^{i\phi})\right\}^s\right| + \frac{\rho^{ns}-1}{k^{n-2\eta+1}+k^{n-\eta+1}}\left\{M(P,1)\right\}^s$$
$$\leq \left\{M(P,1)\right\}^s + \frac{\rho^{ns}-1}{k^{n-2\eta+1}+k^{n-\eta+1}}\left\{M(P,1)\right\}^s.$$

This in particular gives,

$$\left\{M(P,\rho)\right\}^s \leq \frac{k^{n-2\eta+1}+k^{n-\eta+1}+\rho^{ns}-1}{k^{n-2\eta+1}+k^{n-\eta+1}}\left\{M(P,1)\right\}^s,$$

where $M(P,\rho)=\max\limits_{|z|=\rho}|P(z)|$. This completely proves Theorem 1.4. □

Proof of Theorem 1.5. The proof of Theorem 1.5 follows on the same lines as that of Theorem 1.4 by using Lemma 2.2 instead of Lemma 2.1. □

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SOME RESULTS ON NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATIONS

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Abstract. We describe transcendental entire solutions of certain nonlinear difference-differential equations of the forms:

$$f(z)^2 + f(z)[af'(z) + bf(z+c)] + q(z)e^{Q(z)}f(z+c) = u(z)e^{v(z)},$$

and

$$f(z)^n + f(z)^{n-1}[af'(z) + bf(z+c)] + q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z},$$

where $q(z), Q(z), u(z), v(z)$ are non-zero polynomials, $a, b, c, p_i, \lambda_i (i = 1, 2)$ are non-zero constants such that $\lambda_1 \neq \lambda_2$. Our results are improvements and complements of Li et al. ([8]). Some examples are given to illustrate our results are accurate.

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Keywords: differential-difference equation; entire solution; order; Nevanlinna theory.

1. Introduction and main results

Considering a meromorphic function f in the complex plane $\mathbb{C}$, we assume that the reader is familiar with the fundamental results and standard notation of Nevanlinna theory, such as the proximity function $m(r, f)$, the counting function $N(r, f)$, and the characteristic function $T(r, f)$, see, e.g., [3, 6, 18]. We denote by $S(r, f)$ any real function of growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic function α is said to be a small function of f, if $T(r, \alpha) = S(r, f)$.

In 1964, Hayman [3] considered the following non-linear differential equation

$$f(z)^n + Q_d(f(z)) = g(z), \tag{1.1}$$

where $Q_d(f)$ is a differential polynomial in f with degree d and obtained the following result.

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Theorem 1.1. *([3]) Suppose that $f(z)$ is a non-constant meromorphic function, $d \leq n-1$, and f, g satisfy $N(r,f) + N(r,\frac{1}{g}) = S(r,f)$ in (1.1). Then we have $g(z) = (f(z)+\gamma(z))^n$, where $\gamma(z)$ is meromorphic and a small function of $f(z)$.*

Nowadays, there has been recent interest in connections between the Nevanlinna theory and the difference operator, as well as meromorphic solutions of difference and functional equations. Yang and Laine[16] then investigated finite order entire solutions f of non-linear differential-difference equations of the form

$$f(z)^n + L(z,f) = h(z),$$

where $L(z,f)$ is a linear differential-difference polynomial in f with meromorphic coefficients of growth $S(r,f)$, $h(z)$ is meromorphic, and $n \geq 2$ is an integer. Many authors have investigated this question by utilizing the Nevanlinna value distribution theory and its difference counterparts, see, e.g., [5, 9, 10, 11, 13, 15].

In 2016, Liu [12] investigated and classified the finite order entire solutions of the equation

$$f(z)^n + q(z)e^{Q(z)} f^{(k)}(z+c) = P(z), \tag{1.2}$$

where $q(z), Q(z), P(z)$ are polynomials, $n \geq 2, k \geq 1$ are integers and $c \in \mathbb{C}\backslash\{0\}$. Later, Chen [2] replaced $P(z)$ in (1.2) by $p_1e^{\lambda_1} + p_2e^{\lambda_2}$, where $p_1, p_2, \lambda_1, \lambda_2$ are non-zero constants, and studied its finite order entire solutions when $n \geq 3$.

By observing all the above equations, it is easy to see that the left side of these equations have only one dominant term f^n. It is nature to ask what can we get if the left side of these equations have two dominant terms. In 2021, Li [8] investigated non-linear differential-difference equations which may have two dominated terms on the left-hand side with the same degree:

$$f(z)^n + \omega f(z)^{n-1} f'(z) + q(z)e^{Q(z)} f(z+c) = P(z), \tag{1.3}$$

they replaced $P(z)$ in (1.3) by $u(z)e^{v(z)}$ or $p_1e^{\lambda_1} + p_2e^{\lambda_2}$ respectively, and obtained the following results.

Theorem 1.2. *([8]) Let $c, \widetilde{\omega} \neq 0$ be constants, q, Q, u, v be polynomials such that Q, v are not constants and $q, u \not\equiv 0$. Suppose that f is a transcendental entire solution with finite order of*

$$f(z)^2 + \widetilde{\omega} f(z) f'(z) + q(z)e^{Q(z)} f(z+c) = u(z)e^{v(z)}, \tag{1.4}$$

satisfying $\lambda(f) < \rho(f)$, then $\deg Q = \deg v$, and one of the following relations holds:

(1) $\rho(f) < \deg Q = \deg v$, and $f = Ce^{\frac{-z}{\widetilde{\omega}}}$

(2) $\rho(f) = \deg Q = \deg v$.

Theorem 1.3. *([8]) Suppose that n is a positive integer, ω is a constant and $c, \lambda_1, \lambda_2, p_1, p_2$ are non-zero constants, q, Q are polynomials such that Q is not a constant and $q \not\equiv 0$. If f is a transcendental entire solution with finite order of*

$$f(z)^n + \omega f(z)^{n-1} f'(z) + q(z)e^{Q(z)} f(z+c) = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}, \tag{1.5}$$

where $\lambda_2 \neq \pm\lambda_1$, then the following conclusions hold:

(1) If $n \geq 4$ for $\omega \neq 0$ and $n \geq 3$ for $\omega = 0$, then every solution f satisfies $\rho(f) = \deg Q = 1$.

(2) If $n \geq 1$ and f is a solution of (1.5) with $\lambda(f) < \rho(f)$, then

$$f(z) = \left(\frac{p_2 n}{n + \omega\lambda_2}\right)^{\frac{1}{n}} e^{\frac{\lambda_2 z}{n}}, \quad Q(z) = \left(\lambda_1 - \frac{\lambda_2}{n}\right) z + b_1,$$

or

$$f(z) = \left(\frac{p_1 n}{n + \omega\lambda_1}\right)^{\frac{1}{n}} e^{\frac{\lambda_1 z}{n}}, \quad Q(z) = \left(\lambda_2 - \frac{\lambda_1}{n}\right) z + b_2,$$

where $b_1, b_2 \in \mathbb{C}$ satisfy $p_1 = q\left(\frac{p_2 n}{n+\omega\lambda_2}\right)^{\frac{1}{n}} e^{\frac{\lambda_2 c}{n} + b_1}$ and $p_2 = q\left(\frac{p_1 n}{n+\omega\lambda_1}\right)^{\frac{1}{n}} e^{\frac{\lambda_1 c}{n} + b_2}$, respectively.

In the following, inspired by the ideas of [8], we will investigate non-linear differential-difference equations which may have three dominated terms on the left-hand side of (1.4) and (1.5) and obtain the following results.

Theorem 1.4. *Let a, b, c be non-zero constants, q, Q, u, v be polynomials such that $q, u \not\equiv 0$. Suppose that f is a transcendental entire solution with finite order of*

$$f(z)^2 + f(z)[af'(z) + bf(z+c)] + q(z)e^{Q(z)} f(z+c) = u(z)e^{v(z)}, \tag{1.6}$$

satisfying $\lambda(f) < \rho(f)$, then one of the following relations holds:

(1) If $\rho(f) > \deg Q$, then $\rho(f) = \deg v = 1$, Q reduces to a constant, and $f(z) = d(z)e^{a_1 z}$, where $d(z) = \frac{C_2 u(z-c)}{C_1 q(z-c)}$, here $C_1 = e^{Q + a_1 c}, C_2 = e^{v_0}$, a_1, v_0 are constants satisfying $1 + aa_1 + be^{a_1 c} = 0$.

(2) If $\rho(f) = \deg Q > \deg v$, then $\rho(f) = \deg Q = 1$, v reduces to a constant, and $f(z) = d(z)e^{a_1 z}$, where $d(z) = \frac{C_4 u(z-c)}{C_3 q(z-c)}$, here $C_3 = e^{b_0 + a_1 c}, C_4 = e^{v}$, a_1, b_0 are constants satisfying $1 + aa_1 + be^{a_1 c} = 0$.

(3) If $\rho(f) < \deg Q$, then $\rho(f) = 1, \deg v = \deg Q$, $f(z) = d(z)e^{a_1 z}$, where $d(z)$ is an entire function with $\rho(d) < 1$, a_1 is a non-zero constant satisfying $1 + aa_1 + be^{a_1 c} = 0$.

We exhibit some examples to show the existence of solutions in Theorem 1.4.

Example 1.1. *$f(z) = ze^z$ is a transcendental entire solution of the following differential-difference equation*

$$f(z)^2 + f(z)\left[f'(z) - 2e^{-\frac{1}{2}} f\left(z + \frac{1}{2}\right)\right] + f\left(z + \frac{1}{2}\right) = \left(z + \frac{1}{2}\right) e^{z+\frac{1}{2}}.$$

Here $a = 1, b = -2e^{-\frac{1}{2}}, v(z) = z + \frac{1}{2}$, and $0 = \lambda(f) < \rho(f) = 1$. Then we have $\deg v = \rho(f) = 1 > \deg Q = 0$, and $a_1 = 1$ satisfy $1 + aa_1 + be^{a_1 c} = 0$.

Example 1.2. *$f(z) = ze^z$ is a transcendental entire solution of the following differential-difference equation*

$$f(z)^2 + f(z)\left[f'(z) - 2e^{-\frac{1}{2}} f\left(z + \frac{1}{2}\right)\right] + e^{-z} f\left(z + \frac{1}{2}\right) = \left(z + \frac{1}{2}\right) e^{\frac{1}{2}}.$$

Here $a = 1, b = -2e^{-\frac{1}{2}}, Q(z) = -z$, and $0 = \lambda(f) < \rho(f) = 1$. Then we have $\deg Q = \rho(f) = 1 > \deg v = 0$, and $a_1 = 1$ satisfy $1 + aa_1 + be^{a_1 c} = 0$.

Example 1.3. *$f(z) = ze^z$ is a transcendental entire solution of the following differential-difference equation*

$$f(z)^2 + f(z)\left[f'(z) - 2e^{-\frac{1}{2}} f\left(z + \frac{1}{2}\right)\right] + e^{z^2} f\left(z + \frac{1}{2}\right) = \left(z + \frac{1}{2}\right) e^{z^2+z+\frac{1}{2}}.$$

Here $a = 1, b = -2e^{-\frac{1}{2}}, Q(z) = z^2, v(z) = z^2 + z + \frac{1}{2}$, and $0 = \lambda(f) < \rho(f) = 1$. Then we have $2 = \deg Q = \deg v > \rho(f) = 1$, and $a_1 = 1$ satisfy $1+aa_1+be^{a_1 c} = 0$.

Theorem 1.5. *Let n is a positive integer, $a, b, c, \lambda_i, p_i (i = 1, 2)$ are non-zero constants, q, Q are polynomials such that Q is not a constant and $q \not\equiv 0$. If f is a transcendental entire solution with finite order of*

$$(1.7)\quad f(z)^n + f(z)^{n-1}\left[af'(z) + bf(z + c)\right] + q(z)e^{Q(z)} f(z + c) = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z},$$

satisfying $\lambda(f) < \rho(f)$, then

$$f(z) = \left(\frac{p_2 n}{n + a\lambda_2 + nbe^{\frac{\lambda_2}{n} c}}\right)^{\frac{1}{n}} e^{\frac{\lambda_2}{n} z}, \quad Q(z) = (\lambda_1 - \frac{\lambda_2}{n})z + B_2,$$

or

$$f(z) = \left(\frac{p_1 n}{n + a\lambda_1 + nbe^{\frac{\lambda_1}{n} c}}\right)^{\frac{1}{n}} e^{\frac{\lambda_1}{n} z}, \quad Q(z) = (\lambda_2 - \frac{\lambda_1}{n})z + B_2,$$

where $B_2 \in \mathbb{C}$ satisfy

$$p_1 = q\left(\frac{p_2 n}{n + a\lambda_2 + nbe^{\frac{\lambda_2}{n} c}}\right)^{\frac{1}{n}} e^{\frac{\lambda_2}{n} c + B_2} \quad or \quad p_2 = \left(\frac{p_1 n}{n + a\lambda_1 + nbe^{\frac{\lambda_1}{n} c}}\right)^{\frac{1}{n}} e^{\frac{\lambda_1}{n} c + B_2},$$

respectively.

2. Some Lemmas

In order to prove results above, we need the following lemmas.

Lemma 2.1 ([18], Theorem 1.51). *Let $f_j(z)(j=1,\cdots,n)(n\geq 2)$ be meromorphic functions, and let $g_j(z)(j=1,\cdots,n)$ be entire functions satisfying*

(i)$\sum_{j=1}^{n} f_j(z)e^{g_j(z)}\equiv 0$*;*

(ii) when $1\leq j<k\leq n$, then $g_j(z)-g_k(z)$ is not a constant;

(iii) when $1\leq j\leq n, 1\leq h<k\leq n$, then

$$T(r,f_j)=o\left\{T\left(r,e^{g_h-g_k}\right)\right\} \quad (r\to\infty, r\notin E),$$

where $E\subset(1,\infty)$ is of finite linear measure or logarithmic measure.

Then, $f_j(z)\equiv 0\,(j=1,\cdots,n)$.

Lemma 2.2 ([18], Theorem 1.62). *Let $f_1,f_2,\cdots,f_n$ be non-constant meromorphic functions, and let $f_{n+1}\not\equiv 0$ be a meromorphic function such that $\sum_{j=1}^{n+1} f_j\equiv 1$. Suppose that there exists a subset $I\in\mathbb{R}^+$ with linear measure $mesI=\infty$, such that:*

$$\sum_{i=1}^{n+1} N\left(r,\frac{1}{f_i}\right)+n\sum_{i=1,i\neq j}^{n+1}\overline{N}(r,f_i)<(\sigma+o(1))T(r,f_j), \quad j=1,2,\cdots,n,$$

as $r\in I$ and $r\to\infty$, where σ is a real number satisfying $0\leq\sigma<1$. Then, $f_{n+1}=1$.

Lemma 2.3 ([7], Theorem 3.1). *Let $f(z)$ be a meromorphic function with the hyper-order less than one, and $c\in\mathbb{C}\backslash\{0\}$. Then we have*

$$T(r,f(z+c))=T(r,f(z))+S(r,f).$$

Lemma 2.4 ([18], Theorem 1.21). *Suppose that $f(z)$ is meromorphic in the complex plane and n is a positive integer. Then $f(z)$ and $f^{(n)}(z)$ have the same order.*

Lemma 2.5 ([1], Lemma 3.3). *Let g be a transcendental meromorphic function of order less than 1, and let h be a positive constant. Then there exists an ε-set E such that as $\mathbb{C}\backslash E\ni z\to\infty$, one has*

$$\frac{g'(z+\eta)}{g(z+\eta)}\to 0, \quad \frac{g(z+\eta)}{g(z)}\to 1$$

uniformly in η for $\mid\eta\mid\leq h$. Further, the ε-set E may be chosen so that for large z not in E, the function g has no zeros or poles in $\mid\zeta-z\mid\leq h$.

Lemma 2.6 ([17], Lemma 1). *Let f_1 and f_2 be two meromorphic functions, and let $a,b_1,\ b_2$ be small functions of f_1 and f_2 satisfying $ab_1b_2\not\equiv 0$ and $b_1f_1+b_2f_2=a$.*

Then one has

$$T(r,f_1)\le \overline{N}(r,f_1)+\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+S(r,f_1).$$

Lemma 2.7 ([18], Theorem 1.57)**.** *Let $f_j(z), j=1,2,3$ be meromorphic functions and $f_1(z)$ is not a constant. If $\sum_{j=1}^{3} f_j(z)\equiv 1$, and*

$$\sum_{j=1}^{3} N\left(r,\frac{1}{f_j}\right)+2\sum_{j=1}^{3}\overline{N}(r,f_j)<(\lambda+o(1))T(r),\quad r\in I,$$

where $\lambda<1$, $T(r)=\max_{1\le j\le 3}\{T(r,f_j)\}$ and I represents a set of $r\in(0,\infty)$ with infinite linear measure. Then $f_2\equiv 1$ or $f_3\equiv 1$.

Lemma 2.8 ([18], Theorem 1.42, Theorem 1.44)**.** *Let $f(z)$ be a non-constant meromorphic function in the complex plane. If $0,\infty$ are Picard exceptional values of $f(z)$, then $f(z)=e^{h(z)}$, where $h(z)$ is a non-constant entire function. Moreover, $f(z)$ is of normal growth, and*

(i) if h is a polynomial of degree p, then $\rho(f)=p$;

(ii) if h is a transcendental entire function, then $\rho(f)=\infty$.

Lemma 2.9 ([18], Theorem 1.22)**.** *Suppose $f(z)$ is a non-constant meromorphic function in the complex plane and k is a positive integer, and let $\Psi(z)=\sum_{i=0}^{k} a_i(z)f^{(i)}(z)$, where $a_1(z),a_2(z),\cdots,a_k(z)$ are small functions of $f(z)$. Then*

$$T(r,\Psi)\le T(r,f)+k\overline{N}(r,f)+S(r,f).$$

3. Proof of Theorem 1.4

Let f be a transcendental entire solution with finite order of equation (1.6) satisfying $\lambda(f)<\rho(f)$. Then, by the Hadamard factorization theorem, we can factorize $f(z)$ as

$$f(z)=d(z)e^{h(z)}, \tag{3.1}$$

where h is a polynomial with $\deg h=\rho(f)$, d is the canonical product formed by zeros of f with $\rho(d)=\lambda(f)<\rho(f)$. Obviously, h is a non-constant polynomial. In fact, if h is a constant, then from (3.1), we will have $\rho(f)=\rho(d)=\lambda(f)$, a contradiction. Thus we have that $\deg h\ge 1$. Let $\deg h=m(\ge 1)$, and $h(z)=a_mz^m+a_{m-1}z^{m-1}+\cdots$, where $a_m\ne 0$.

We rewrite (1.6) as

$$f^2+f(af'+b\overline{f})+qe^{Q}\overline{f}=ue^{v}, \tag{3.2}$$

where $\overline{f}=f(z+c)$, for simplicity.

Obviously, we have $\rho(\overline{f})=\rho(f)=\rho(f')$ by Lemma 2.3 and Lemma 2.4. So from (3.2), by the order property, we get

$$\deg v=\rho(ue^v)\le\max\{\rho(f')=\rho(f)=\rho(\overline{f}),\rho(e^Q),\rho(q)\} \\ =\max\{\deg h,\deg Q\}. \tag{3.3}$$

By substituting (3.1) into (3.2), we get

$$\left[d^2+ad(d'+dh')\right]e^{2h}+bd\overline{d}e^{\overline{h}+h}+q\overline{d}e^{Q+\overline{h}}=ue^v. \tag{3.4}$$

Case 1. $\rho(f)>\deg Q$, then we have $\deg h>\deg Q$, and $\deg v\le\deg h$ from (3.3).

Subcase 1.1. $\deg h>\deg v$. From (3.4) we have

$$\left\{\left[d^2+ad(d'+dh')\right]e^{h_1}+bd\overline{d}e^{h_2}\right\}e^{2a_mz^m}+q\overline{d}e^{h_3}e^{a_mz^m}=ue^v, \tag{3.5}$$

where $h_1=2a_{m-1}z^{m-1}+\cdots$, $h_2=(2a_{m-1}+a_mmc)z^{m-1}+\cdots$ and $h_3=Q+(a_mmc+a_{m-1})z^{m-1}+\cdots$ are all polynomials with degree at most $m-1$. So, combining with $\rho(d')=\rho(d)=\rho(\overline{d})<m$, by using Lemma 2.1 to (3.5), we have $q\overline{d}\equiv0$, which yields a contradiction. Thus $\deg h>\deg v$ can not hold.

Subcase 1.2. $\deg h=\deg v$. Let $v(z)=v_mz^m+v_{m-1}z^{m-1}+\cdots$, where $v_m\ne0$. From (3.4) we have

$$\left\{\left[d^2+ad(d'+dh')\right]e^{h_1}+bd\overline{d}e^{h_2}\right\}e^{2a_mz^m}+q\overline{d}e^{h_3}e^{a_mz^m}=ue^{h_4}e^{v_mz^m}, \tag{3.6}$$

where $h_4=v_{m-1}z^{m-1}+\cdots$ is a polynomial with degree at most $m-1$, h_1, h_2 and h_3 are defined as in Subcase 1.1.

If $v_m\ne2a_m$ and $v_m\ne a_m$, combining with $\rho(d')=\rho(d)=\rho(\overline{d})<m$, by using Lemma 2.1 to (3.6), we get $u\equiv0$, a contradiction.

If $v_m=2a_m$, then (3.6) can be reduced to

$$\left\{\left[d^2+ad(d'+dh')\right]e^{h_1}+bd\overline{d}e^{h_2}-ue^{h_4}\right\}e^{2a_mz^m}+q\overline{d}e^{h_3}e^{a_mz^m}=0,$$

by using Lemma 2.1, we have $q\overline{d}\equiv0$, a contradiction. Thus, we have

$$v_m=a_m. \tag{3.7}$$

Rewriting (3.6) as

$$\left\{\left[d^2+ad(d'+dh')\right]e^{h_1}+bd\overline{d}e^{h_2}\right\}e^{2a_mz^m}+(q\overline{d}e^{h_3}-ue^{h_4})e^{a_mz^m}=0.$$

Similarly as above, by Lemma 2.1, we get

$$\begin{cases}[d^2+ad(d'+dh')]e^{h_1}+bd\overline{d}e^{h_2}\equiv0,\\ q\overline{d}e^{h_3}-ue^{h_4}\equiv0.\end{cases}$$

By observing the expressions of h_1 and h_2, we have $h_2=h_1+h_5$, where $h_5=a_mmcz^{m-1}+\cdots$. Noting that $d\not\equiv0$, then above equations can be rewrote as

$$\begin{cases}d+a(d'+dh')+b\overline{d}e^{h_5}\equiv0,\\ q\overline{d}e^{h_3}-ue^{h_4}\equiv0.\end{cases} \tag{3.8}$$

By the second equation of (3.8), we get

$$\overline{d} = \frac{u}{q} e^{h_4 - h_3}. \tag{3.9}$$

Subcase 1.2.1. $\deg h \geq 2$. Firstly, we claim that $h_4 - h_3$ is a non-constant polynomial. In fact, if $h_4 - h_3$ is a constant, then $\overline{d}$ reduces to a rational function, by the first equation of (3.8) and Lemma 2.1, we have $\overline{d} \equiv 0$, a contradiction.

Secondly, we claim that $\overline{h_4} - \overline{h_3} + \overline{h_5}$ is a non-constant polynomial. Substituting (3.9) into the first equation of (3.8), we get

$$\left\{ \left(1 + a\overline{h'}\right) \frac{u}{q} + a \left[\left(\frac{u}{q}\right)' + \frac{u}{q}(h_4' - h_3') \right] \right\} e^{h_4 - h_3} + b\frac{\overline{u}}{\overline{q}} e^{\overline{h_4} - \overline{h_3} + \overline{h_5}} = 0. \tag{3.10}$$

If $\overline{h_4} - \overline{h_3} + \overline{h_5}$ is a constant, say c_1. Then (3.10) becomes

$$\left\{ (1 + a\overline{h'}) \frac{u}{q} + a \left[\left(\frac{u}{q}\right)' + \frac{u}{q}(h_4' - h_3') \right] \right\} e^{h_4 - h_3} + b\frac{\overline{u}}{\overline{q}} e^{c_1} = 0,$$

that gives $\overline{u} \equiv 0$, a contradiction.

Thus, we get $h_4 - h_3$ and $\overline{h_4} - \overline{h_3} + \overline{h_5}$ are non-constant polynomials. If $\overline{h_4} - \overline{h_3} + \overline{h_5} - (h_4 - h_3)$ is not a constant, by (3.10) and Lemma 2.1, we have $\overline{u} \equiv 0$, a contradiction. If $\overline{h_4} - \overline{h_3} + \overline{h_5} - (h_4 - h_3)$ is a constant, say c_2, then (3.10) reduces to

$$(1 + a\overline{h'}) \frac{u}{q} + a \left[\left(\frac{u}{q}\right)' + \frac{u}{q}(h_4' - h_3') \right] + b\frac{\overline{u}}{\overline{q}} e^{c_2} = 0,$$

that is,

$$(1 + a\overline{h'}) uq\overline{q} + a\overline{q}[u'q - q'u + uq(h_4' - h_3')] + b\overline{u}q^2 e^{c_2} = 0, \tag{3.11}$$

then it can be verified that the term $a\overline{h'}uq\overline{q}$ would have a higher degree of z than all the other terms in (3.11), we obtain $a = 0$, which is impossible.

Subcase 1.2.2. $\deg h = 1$.

Noting that $\deg v = \deg h$, so we have $\deg v = 1$. Suppose that $h(z) = a_1 z + a_0$, $v(z) = v_1 z + v_0$, from (3.4) and (3.7), we have

$$\left\{ [d^2 + ad(d' + da_1)]e^{2a_0} + bd\overline{d}e^{2a_0 + a_1 c} \right\} e^{2a_1 z} + \left\{ q\overline{d}e^{Q + a_1 c + a_0} - ue^{v_0} \right\} e^{a_1 z} = 0. \tag{3.12}$$

Since $1 = \deg h > \deg Q$, we have Q is a non-zero constant, by Lemma 2.1, we get

$$\begin{cases} d + a(d' + da_1) + b\overline{d}e^{a_1 c} \equiv 0, \\ q\overline{d}c_3 - uc_4 \equiv 0, \end{cases} \tag{3.13}$$

where $c_3 = e^{Q + a_1 c + a_0}$, $c_4 = e^{v_0}$, by the second equation of (3.13), we get $\overline{d} = \frac{c_4}{c_3} \frac{u}{q}$ is a rational function, then by the first equation of (3.13), we have $1 + aa_1 + be^{a_1 c} = 0$ as $z \to \infty$.

Thus, we get $f(z)=d(z)e^{a_1z+a_0}$, where $d(z)=\frac{c_4}{c_3}\frac{u(z-c)}{q(z-c)}$, $a_0, a_1(\neq 0)$ are constants satisfying $1+aa_1+be^{a_1c}=0$.

Case 2. $\rho(f)=\deg Q>\deg v$. Suppose that $Q(z)=b_mz^m+b_{m-1}z^{m-1}+\cdots$, where $b_m\neq 0$. From (3.4) we have

$$\left\{[d^2+ad(d'+dh')]e^{h_1}+bd\overline{d}e^{h_2}\right\}e^{2a_mz^m}+q\overline{d}e^{h_6}e^{(a_m+b_m)z^m}=ue^v, \tag{3.14}$$

where $h_6=(a_mmc+a_{m-1}+b_{m-1})z^{m-1}+\cdots$ is a polynomial with degree at most $m-1$, and h_1, h_2 are defined as in Subcase 1.1.

If $b_m\neq\pm a_m$, combining with $\rho(d')=\rho(d)=\rho(\overline{d})<m$, by using Lemma 2.1 to (3.14), we get $u\equiv 0$, which yields a contradiction.

If $b_m=a_m$, then (3.14) can be reduced to

$$\left\{[d^2+ad(d'+dh')]e^{h_1}+bd\overline{d}e^{h_2}+q\overline{d}e^{h_6}\right\}e^{2a_mz^m}=ue^v,$$

by using Lemma 2.1, we have $u\equiv 0$, a contradiction.

Thus, we have

$$b_m=-a_m. \tag{3.15}$$

Rewriting (3.14) as

$$\left\{[d^2+ad(d'+dh')]e^{h_1}+bd\overline{d}e^{h_2}\right\}e^{2a_mz^m}=ue^v-q\overline{d}e^{h_6}. \tag{3.16}$$

Similarly as above, by Lemma 2.1, we get

$$\begin{cases} d^2+ad(d'+dh')+bd\overline{d}e^{h_5}\equiv 0,\\ q\overline{d}e^{h_6}-ue^v\equiv 0.\end{cases} \tag{3.17}$$

By the second equation of (3.17), we get

$$\overline{d}=\frac{u}{q}e^{h_6-v}. \tag{3.18}$$

Subcase 2.1. $\deg h\geq 2$.

Firstly, we claim that h_6-v is a non-constant polynomial. Otherwise, if h_6-v is a constant, then $\overline{d}$ reduces to a rational function, by the first equation of (3.17) and Lemma 2.1, we have $\overline{d}\equiv 0$, a contradiction.

Secondly, we claim that $\overline{h_6}-\overline{v}+\overline{h_5}$ is a non-constant polynomial. Substituting (3.18) into the first equation of (3.17), we get

$$\left\{(1+a\overline{h'})\frac{u}{q}+a\left[\left(\frac{u}{q}\right)'+\frac{u}{q}(h_6'-v')\right]\right\}e^{h_6-v}+b\frac{\overline{u}}{\overline{q}}e^{\overline{h_6}-\overline{v}+\overline{h_5}}=0. \tag{3.19}$$

If $\overline{h_6}-\overline{v}+\overline{h_5}$ is a constant, say c_5. Then (3.19) becomes

$$\left\{(1+a\overline{h'})\frac{u}{q}+a\left[\left(\frac{u}{q}\right)'+\frac{u}{q}(h_6'-v')\right]\right\}e^{h_6-v}+b\frac{\overline{u}}{\overline{q}}e^{c_5}=0,$$

that gives $\overline{u}\equiv 0$, a contradiction.

Thus, we get $h_6 - v$ and $\overline{h_6} - \overline{v} + \overline{h_5}$ are non-constant polynomials. If $\overline{h_6} - \overline{v} + \overline{h_5} - (h_6 - v)$ is not a constant, by (3.19) and Lemma 2.1, we have $\overline{u} \equiv 0$, a contradiction. If $\overline{h_6} - \overline{v} + \overline{h_5} - (h_6 - v)$ is a constant, say c_6, by (3.19), we get

$$(1 + a\overline{h'})\frac{u}{q} + a\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(h_6' - v')\right] + b\frac{\overline{u}}{\overline{q}}e^{c_6} = 0,$$

that is,

$$(1 + a\overline{h'})uq\overline{q} + a\overline{q}\left[u'q - q'u + uq(h_6' - v')\right] + b\overline{u}q^2e^{c_6} = 0, \tag{3.20}$$

then it can be verified that the term $a\overline{h'}uq\overline{q}$ would have a higher degree of z than all the other terms in (3.20), we obtain $a = 0$, which is impossible.

Subcase 2.2. $\deg h = 1$.

Noting that $\deg Q = \rho(f) = \deg h$, so we have $\deg Q = \deg h = 1$. Suppose that $h(z) = a_1 z + a_0$, $Q(z) = b_1 z + b_0$, from (3.4) and (3.15), we have

$$\left\{[d^2 + ad(d' + da_1)]e^{2a_0} + bd\overline{d}e^{2a_0 + a_1 c}\right\} e^{2a_1 z} + q\overline{d}e^{a_1 c + a_0 + b_0} = ue^v. \tag{3.21}$$

Since $1 = \deg h > \deg v$, we have v is a non-zero constant, by Lemma 2.1, we get

$$\begin{cases} d + a(d' + da_1) + b\overline{d}e^{a_1 c} \equiv 0, \\ q\overline{d}c_7 - uc_8 \equiv 0, \end{cases} \tag{3.22}$$

where $c_7 = e^{a_1 c + a_0 + b_0}$, $c_8 = e^v$, by the second equation of (3.22), we get $\overline{d} = \frac{c_8}{c_7}\frac{u}{q}$ is a rational function, then by the first equation of (3.22), we have $1 + aa_1 + be^{a_1 c} = 0$ as $z \to \infty$.

Thus $f(z) = d(z)e^{a_1 z + a_0}$, where $d(z) = \frac{c_8}{c_7}\frac{u(z-c)}{q(z-c)}$, $a_0, a_1 (\neq 0)$ are constants satisfying $1 + aa_1 + be^{a_1 c} = 0$.

Case 3. $\rho(f) < \deg Q$, then we have $T(r, f) = S(r, e^Q)$. Thus we get $T(r, f') = S(r, e^Q)$ from Lemma 2.9 and $T(r, \overline{f}) = S(r, e^Q)$ from Lemma 2.3. Therefore, by (3.2), we have

$$\begin{aligned} T(r, e^Q) + S(r, e^Q) &= T(r, f^2 + f(af' + b\overline{f}) + q\overline{f}e^Q) \\ &= T(r, ue^v) = T(r, e^v) + S(r, e^v). \end{aligned}$$

Therefore, $\deg Q = \deg v$. Differentiating (3.2) yields

$$2ff' + f'(af' + b\overline{f}) + f(af'' + b\overline{f'}) + Ae^Q = (u' + uv')e^v, \tag{3.23}$$

with $A = q'\overline{f} + q\overline{f'} + q\overline{f}Q'$.

Eliminating e^v from (3.2) and (3.23) to get

$$B_1 e^Q + B_2 = 0, \tag{3.24}$$

where

$$B_1 = uA - q\overline{f}(u' + uv'),$$

$$B_2 = 2uff' + uf'(af' + b\overline{f}) + uf(af'' + b\overline{f'}) - (u' + uv')[f^2 + f(af' + b\overline{f})].$$

Noticing that $\rho(\overline{f}) = \rho(f) < \deg Q$, and $\rho(f'') = \rho(f') = \rho(f) < \deg Q$ from Lemma 2.4, thus by Lemma 2.1, we get $B_1 \equiv B_2 \equiv 0$. It follows from $B_1 \equiv 0$ that

$$\frac{q'}{q} + \frac{\overline{f'}}{\overline{f}} + Q' = \frac{u'}{u} + v',$$

by integrating, we have $q\overline{f}e^Q = c_9 ue^v$, where c_9 is a non-zero constant.

Subcase 3.1. $c_9 = 1$. By substituting $q\overline{f}e^Q = ue^v$ into (3.2), we see that

$$f + af' + b\overline{f} = 0. \tag{3.25}$$

Subcase 3.1.1. $\deg h \geq 2$. Then $\deg v = \deg Q > \rho(f) = \deg h \geq 2$. By substituting $\overline{f} = \frac{u}{q}e^{v-Q}$ into (3.25), we can get

$$\left\{\frac{u}{q} + a\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(v' - Q')\right]\right\}e^{v-Q} + b\frac{\overline{u}}{\overline{q}}e^{\overline{v}-\overline{Q}} = 0. \tag{3.26}$$

If $\overline{v} - \overline{Q} - (v - Q)$ is a constant, say c_{10}. Then (3.26) becomes

$$\frac{u}{q} + a\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(v' - Q')\right] + b\frac{\overline{u}}{\overline{q}}e^{c_{10}} = 0,$$

that is,

$$uq\overline{q} + a\overline{q}[u'q - q'u + uq(v' - Q')] + b\overline{u}q^2e^{c_{10}} = 0, \tag{3.27}$$

we claim that $v' - Q'$ is not a constant, otherwise $v - Q$ is linear, then $\deg h = \rho(f) = \rho(\overline{f}) = \rho(e^{v-Q}) = 1$, which contradicts with $\deg h \geq 2$. It can be verified that the term $auq\overline{q}(v' - Q')$ would have a higher degree of z than all the other terms in (3.27), we obtain $a = 0$, which is impossible.

If $\overline{v} - \overline{Q} - (v - Q)$ is not a constant, by (3.26) and Lemma 2.1, we have $\overline{u} \equiv 0$, a contradiction.

Subcase 3.1.2. $\deg h = 1$.

By substituting $f = de^h$ into (3.25), we can get

$$[d + a(d' + dh')]e^h + b\overline{d}e^{\overline{h}} = 0, \tag{3.28}$$

substituting $h(z) = a_1 z + a_0$ into (3.28), we have

$$1 + a\left(\frac{d'}{d} + a_1\right) + b\frac{\overline{d}}{d}e^{a_1 c} = 0.$$

Noting that $\rho(d) < \rho(f) = \deg h = 1$, by Lemma 2.5, we have $1 + aa_1 + be^{a_1 c} = 0$ as $z \to \infty$.

Thus $f(z) = d(z)e^{a_1 z + a_0}$, where $d(z)$ is an entire function with $\rho(d) < 1$, $a_0, a_1 (\neq 0)$ are constants satisfying $1 + aa_1 + be^{a_1 c} = 0$.

Subcase 3.2. $c_9 \neq 1$.

In this case, we have $\overline{f} = c_9 \frac{u}{q} e^{v-Q}$. By substituting it into (3.2), we get

$$\begin{aligned} &\left\{c_9^2 \frac{u^2}{q^2} + ac_9^2 \frac{u}{q}\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(v' - Q')\right]\right\} e^{2(v-Q)} \\ &+bc_9^2 \frac{u}{q}\frac{\overline{u}}{\overline{q}} e^{\overline{v}-\overline{Q}+v-Q} = (1-c_9)\overline{u}e^{\overline{v}}, \end{aligned} \tag{3.29}$$

we can easily get $v - Q$ is not a constant because f is transcendental, and so $\overline{v} - \overline{Q} + v - Q$ is not a constant.

If $\overline{v} - \overline{Q} - (v - Q)$ is a constant, say c_{11}. Then (3.29) becomes

$$c_9^2 \frac{u^2}{q^2} + ac_9^2 \frac{u}{q}\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(v' - Q')\right] + bc_9^2 e^{c_{11}} \frac{u}{q}\frac{\overline{u}}{\overline{q}} = (1-c_9)\overline{u}e^{\overline{v}-2(v-Q)}.$$

Since $\deg Q = \deg v > \rho(f) = \deg(v - Q)$, we can easily deduce a contradiction by the fact that $c_9 \neq 1$ and $\overline{u} \not\equiv 0$.

If $\overline{v}-\overline{Q}-(v-Q)$ is not a constant, note that $\deg Q = \deg v > \rho(f) = \deg(v-Q)$, so $\overline{v} - 2(v-Q)$ and $v - Q - \overline{Q}$ are not constants, by Lemma 2.1, we can also deduce a contradiction by the fact that $c_9 \neq 1$ and $\overline{u} \not\equiv 0$.

The proof of Theorem 1.4 is now completed.

4. Proof of Theorem 1.5

Suppose that f is a transcendental entire solution with finite order of equation (1.7) with $\lambda(f) < \rho(f)$. Then, by the Hadamard factorization theorem, we can factorize $f(z)$ as

$$f(z) = d(z)e^{h(z)}, \tag{4.1}$$

where h is a polynomial with $\deg h = \rho(f)$, d is the canonical products formed by zeros of f with $\rho(d) = \lambda(f) < \rho(f)$. Similarly as in the proof of Theorem 1.4, we have $\rho(f) = \deg h \geq 1$.

We rewrite (1.7) as

$$f^n + f^{n-1}(af' + b\overline{f}) + qe^Q\overline{f} = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z}, \tag{4.2}$$

where $\overline{f} = f(z + c)$, for simplicity.

By substituting (4.1) into (4.2), we get

$$d^{n-1}[d + a(d' + dh')]e^{nh} + b\overline{d}d^{n-1}e^{(n-1)h+\overline{h}} + q\overline{d}e^{Q+\overline{h}} = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z}. \tag{4.3}$$

Case 1. $\deg h \geq 2$.

Subcase 1.1. $\deg(Q + \overline{h}) \leq 1$. Rewriting (4.3) as:

$$d^{n-1}[d + a(d' + dh')]e^{nh} + b\overline{d}d^{n-1}e^{(n-1)h+\overline{h}} = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z} - q\overline{d}e^{Q+\overline{h}}. \tag{4.4}$$

Denote that $\alpha = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z} - q\overline{d}e^{Q+\overline{h}}$, since $\rho(f) = \deg h \geq 2$, we have $T(r,\alpha) = S(r,e^h)$. Next, we claim that $\alpha \not\equiv 0$. Otherwise, (4.4) reduces to

$$d + a(d' + dh') + b\overline{d}e^{\overline{h}-h} \equiv 0,$$

it implies $\overline{d} \equiv 0$ because $\deg h \geq 2$, a contradiction.

From (4.4), we have

$$d^{n-1}[d + a(d' + dh')]e^{nh} + b\overline{d}d^{n-1}e^{(n-1)h+\overline{h}} = \alpha. \tag{4.5}$$

Obviously, $d^{n-1}[d+a(d'+dh')] \not\equiv 0$. Otherwise, if $d^{n-1}[d+a(d'+dh')] \equiv 0$, we will have $d = C_1e^{-\frac{z}{a}-h}$, so $\rho(d) = \deg h = \rho(f)$, which contradicts with the fact that $\rho(d) < \rho(f)$. Then from (4.5) and Lemma 2.6, we get

$$T(r,e^{nh}) \leq \overline{N}(r,e^{nh}) + \overline{N}\left(r,\frac{1}{e^{nh}}\right) + \overline{N}\left(r,\frac{1}{e^{(n-1)h+\overline{h}}}\right) + S(r,e^{nh}) = S(r,e^{nh}),$$

a contradiction.

Subcase 1.2. $\deg(Q+\overline{h}) \geq 2$. Dividing both sides of (4.3) by $p_2e^{\lambda_2 z}$, we obtain

$$\sum_{j=1}^{4} f_j = 1, \tag{4.6}$$

where

$$f_1 = \frac{d^{n-1}[d + a(d' + dh')]}{p_2}e^{nh-\lambda_2 z}, \quad f_2 = \frac{b\overline{d}d^{n-1}}{p_2}e^{(n-1)h+\overline{h}-\lambda_2 z},$$
$$f_3 = \frac{q\overline{d}}{p_2}e^{Q+\overline{h}-\lambda_2 z}, \quad f_4 = -\frac{p_1}{p_2}e^{\lambda_1 z-\lambda_2 z}.$$

Since $\deg(nh - \lambda_2 z) \geq 2$, $\deg((n-1)h + \overline{h} - \lambda_2 z) \geq 2$, $\deg(Q + \overline{h} - \lambda_2 z) \geq 2$, $\lambda_1 \neq \lambda_2$, it is obvious to see $f_j (j = 1,2,3,4)$ are not constants. Thus, we deduce:

$$\begin{aligned}\sum_{j=1}^{4} N\left(r,\frac{1}{f_j}\right) &\leq O\left(N\left(r,\frac{1}{\overline{d}}\right)\right) + O\left(N\left(r,\frac{1}{d}\right)\right) + O(\log r)\\ &\leq O(T(r,d)) + O(\log r) = o(T(r,f_j)), (j = 1,2,3),\end{aligned}$$

and

$$\sum_{j=1}^{4} \overline{N}(r,f_j) \leq O(\log r) = o(T(r,f_j)), (j = 1,2,3),$$

as $r \in I$ and $r \to \infty$.

Thus by (4.6) and Lemma 2.2, we deduce $f_4 = \frac{p_1}{p_2}e^{\lambda_1 z-\lambda_2 z} \equiv 1$, which is impossible.

Case 2. $\deg h = 1$. In this case, we claim that $\deg Q = 1$. Otherwise, suppose that $\deg Q \geq 2$, by (1.7), we obtain $q\overline{f}e^Q = H$, where

$$H = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z} - f^n - f^{n-1}(af' + b\overline{f}).$$

Note that $\rho(\overline{f}) = \rho(f') = \rho(f) = \deg h = 1 < \deg Q$, then by combining with Lemma 2.1, we get $q\overline{f} \equiv H \equiv 0$, a contradiction. So we have $\deg Q = \deg h = 1$.

Set $h(z) = A_1 z + B_1$, $Q(z) = A_2 z + B_2$, where $A_1(\neq 0), A_2(\neq 0)$ and B_1, B_2 are constants. By substituting these into (4.3) and dividing both sides by $p_2 e^{\lambda_2 z}$, we have

$$f_1 + f_2 + f_3 = 1, \tag{4.7}$$

where

$$\begin{aligned} f_1 &= -\frac{p_1}{p_2} e^{(\lambda_1 - \lambda_2)z}, \\ f_2 &= \frac{e^{nB_1} d^{n-1}[d + a(d' + dh') + b\overline{d} e^{A_1 c}]}{p_2} e^{(nA_1 - \lambda_2)z}, \\ f_3 &= \frac{q\overline{d} e^{A_1 c + B_1 + B_2}}{p_2} e^{(A_1 + A_2 - \lambda_2)z}. \end{aligned}$$

Obviously, f_1 is not a constant since $\lambda_1 \neq \lambda_2$. We set

$$T(r) = \max\{T(r, f_1), T(r, f_2), T(r, f_3)\} = T(r, \mathrm{e}^z).$$

Since $\rho(d) < 1$, then we have

$$N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_2}\right) + N\left(r, \frac{1}{f_3}\right) \leq O(T(r, d)) + O(\log r) = o(T(r)),$$

and

$$\overline{N}(r, f_1) + \overline{N}(r, f_2) + \overline{N}(r, f_3) \leq O(\log r) = o(T(r)),$$

as $r \to \infty$. Therefore, by using Lemma 2.7, we can deduce that $f_2 \equiv 1$ or $f_3 \equiv 1$. If $f_2 \equiv 1$, that is

$$e^{nB_1} d^{n-1} \left[d + a(d' + dh') + b\overline{d} e^{A_1 c}\right] e^{(nA_1 - \lambda_2)z} = p_2. \tag{4.8}$$

We assert that $A_1 = \frac{\lambda_2}{n}$. Otherwise, suppose that $A_1 \neq \frac{\lambda_2}{n}$, then from $\rho(d') = \rho(d) < 1 = \deg[(nA_1 - \lambda_2)z]$, by using Lemma 2.1 to (4.8), we get $p_2 = 0$, a contradiction. Thus $h = \frac{\lambda_2}{n} z + B_1$. By substituting it into (4.8), we have

$$d^{n-1} \left[d + a\left(d' + d\frac{\lambda_2}{n}\right) + b\overline{d} e^{\frac{\lambda_2}{n} c}\right] = p_2 e^{-nB_1}. \tag{4.9}$$

Next, we assert that d is a constant. Otherwise, if d is a non-constant entire function, then from (4.9) we get that 0 is a Picard exceptional value of d. Thus by Lemma 2.8, we have $d = e^{\alpha}$, where α is a non-constant polynomial, which contradicts with the assumption that $\rho(d) < 1$. So we have that d is a non-zero constant, and (4.9) reduces to

$$d^n e^{nB_1} \left(1 + a\frac{\lambda_2}{n} + b e^{\frac{\lambda_2}{n} c}\right) = p_2.$$

Therefore,

$$f(z) = de^{h(z)} = de^{B_1}e^{\frac{\lambda_2}{n}z} = \left(\frac{np_2}{n + a\lambda_2 + nbe^{\frac{\lambda_2}{n}c}}\right)^{\frac{1}{n}} e^{\frac{\lambda_2}{n}z}.$$

Moreover, from $f_2 \equiv 1$ and (4.7), we also have $f_1 + f_3 \equiv 0$. That is

$$q\overline{d}e^{A_1c+B_1+B_2}e^{(A_1+A_2)z} = p_1e^{\lambda_1 z},$$

which implies that

$$A_2 = \lambda_1 - A_1 = \lambda_1 - \frac{\lambda_2}{n}, \text{ i.e. } Q(z) = \left(\lambda_1 - \frac{\lambda_2}{n}\right)z + B_2,$$

where B_2 satisfies $p_1 = q\left(\frac{np_2}{n+a\lambda_2+nbe^{\frac{\lambda_2}{n}c}}\right)^{\frac{1}{n}} e^{\frac{\lambda_2}{n}c+B_2}$.

If $f_3 \equiv 1$, by using the similar methods as in the case $f_2 \equiv 1$, we get

$$f(z) = \left(\frac{np_1}{n + a\lambda_1 + nbe^{\frac{\lambda_1}{n}c}}\right)^{\frac{1}{n}} e^{\frac{\lambda_1}{n}z},$$

then from (4.7) we have $f_1 + f_2 = 0$. This gives that

$$Q(z) = \left(\lambda_2 - \frac{\lambda_1}{n}\right)z + B_2,$$

where B_2 satisfies $p_2 = q\left(\frac{np_1}{n+a\lambda_1+nbe^{\frac{\lambda_1}{n}c}}\right)^{\frac{1}{n}} e^{\frac{\lambda_1}{n}c+B_2}$.

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