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НАИЛУЧШИЕ РАВНОМЕРНЫЕ ПРИБЛИЖЕНИЯ В УГЛАХ ЦЕЛЫМИ ФУНКЦИЯМИ

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Аннотация. В данной работе исследуется задача об наилучшеМ равномерном приближении в угле целыми функциями. Полученные новые результаты по равномерному приближению являются уточнением ранее известных результатов. Здесь мы также даем положительный ответ на проблему предложенную Кобером: Пусть функция f голоморфна внутри Δ_α, непрерывна и ограничена на Δ_α для $\alpha \in (0, 2\pi)$ и $\rho = \pi/(2\pi - \alpha)$. Если функция $f\left(z^{1/\rho}\right)$ равномерно непрерывна на лучах $\pm l_{\alpha\rho/2}$, то функция f допускает равномерное приближение на Δ_α целыми функциями порядка ρ и конечного типа.

MSC2020 number: 35J25.

Ключевые слова: Равномерное приближение; угол; целая функция.

1. Введение и вспомогательные леммы

Функции f, подлежащие приближению на E целыми функциями, предполагаются из класса $A(E)$: непрерывные на E и голоморфные на внутренности E^o множества E. Такая функция может иметь произвольный рост на E вблизи бесконечности, причем то же самое верно для возможного роста приближающей целой функции.

В этой ситуации, аналогом полиномиального приближения *наилучших* и *Джексоновских* задач являются следующие задачи: для канонического множества E и функции $f \in A(E)$ построить приближающие целые функции g таким образом, чтобы их *рост* в комплексной плоскости $\mathbb{C}$ был *оптимальным* в некотором смысле, этот рост должен быть выражен через рост функции f и величин, характеризующих структуру f на E, в частности - рост некоторых *производных* функции f на границе ∂E области E. С точки зрения классической теории целых функций, задача представляет особый интерес для описания классов функций на канонических множествах, допускающих равномерное и/или касательное приближение целеыми функциями заданного *конечного* порядка, с точными оценками их типа. Для краткости, будем называть такой тип равномерного приближения *оптимальным.*

Задача оптимального равномерного приближения целыми функциями в углах исследовалось в работах [1]-[8]. В данной статье мы ограничиваемся рассмотрением этой проблемы для случая приближения на замкнутых углах $\Delta_\alpha = \{z \in \mathbb{C} : |\arg z| \leq \alpha/2\}$ (см. работы Х. Кобера [3], М.В. Келдыша [4] и Н. Аракеляна [5]). Как это показано в работе [4] (подробности доказательства см. в [1], гл. 2), функцию $f \in A(\Delta_{\alpha+\delta})$ $(\delta > 0)$ можно равномерно приблизить на Δ_α целыми функциями g, для роста которых получены оценки, зависящие лишь от α, δ и роста функции f на $\Delta_{\alpha+\delta}$; эти оценки точно указывали на возможный оптимальный *порядок* функции g в $\mathbb{C}$, но ничего не говорили об их *типе*. Позже более точные результаты были получены в работе [6], в предположении, что $f \in A(\Delta_\alpha)$ с некоторыми дополнительными свойствами функции f на границе Δ_α. Особо отметим следующий результат: если $f \in A(\Delta_\alpha)$ и функция $z \to f\left(z^{1/\rho}\right)$ с $\rho = \pi/\left(2\pi - \alpha\right)$ *равномерно непрерывна* на $\Delta_{\alpha\rho}$, тогда можно функцию f равномерно приблизить на Δ_α целыми функциями g порядка $\leq \rho$. Это дает частичный ответ на гипотезу Х. Кобера [3]; полный ответ смотрите ниже в разделе 2. В частности, в работе [12] также оценивается тип приближающих целых функций для некоторых специальных классов функций.

Целью данной статьи является окончательное уточнение основных результатов об оптимальном равномерном приближении на углах Δ_α целыми функциями, включая прямые теоремы типа Джексона и обратные результаты типа Бернштейна. Отметим, что такое уточнение для случая приближения на вещественной оси $\mathbb{R}$ было достигнуто в работе [6], а для случая приближения на замкнутых полосах - в работе [7]. В данной работе мы будем использовать некоторые аппроксимационные конструкции, развитые в [5] - [7] и [12].

Работа состоит из двух разделов. Первый содержит введение и вспомогательные леммы; а второй - доказательства результатов об оптимальном равномерном приближении на углах целыми функциями.

1.1. **Некоторые обозначения.** 1^0. *Внутренность, замыкание и границу* множества $E \subset \mathbb{C}$ обозначим соответственно через E^0, $\overline{E}$ и ∂E. *Дополнение* E в $\mathbb{C}$ - через E^c. Для $E \subset \mathbb{C}$ пусть $C\left(E\right)$ будет классом непрерывных функций $f : E \to \mathbb{C}$ с равномерной нормой $\|f\|_E = \sup_{z \in E} |f(z)|$, и положим

$$C_b(E) = \{f \in C(E) : \|f\|_E < +\infty\},$$

так что $C(E) = C_b(E)$ для компактного множества E. *Модуль непрерывности* $\omega_f(\delta)$ для $f \in C_b(E)$ определяется для $\delta > 0$ следующим образом:

$$\omega_f(\delta) = \sup_{z,\zeta \in E} \{|f(z) - f(\zeta)| : |z - \zeta| \leq \delta\}.$$

Для открытого множества Ω в $\mathbb{C}$, пусть $C'(\Omega)$ - класс непрерывно дифференцируемых (в смысле $\mathbb{R}^2$) комплексных функций на Ω.

2^0. Предположим теперь, что G - жордановая область с положительно ориентированной кусочно гладкой границей Γ. Класс $C'(\overline{G})$ определим стандартным путем, как класс функций $\varphi \in C(\overline{G}) \cap C'(G)$, допускающих непрерывное продолжение производных φ'_x и φ'_y из G к $\overline{G}$; это определяет их однозначно на Γ. Следующая формула является комплексным вариантом известной *теоремы о дивергенции* для $\varphi \in C'(\overline{G})$:

$$_\Gamma\varphi\,(z)\,dz = i_G 2\overline{\partial}\varphi d\sigma, \tag{1.1}$$

где σ - лебегова плоская мера на G и оператор $\overline{\partial}$ определяется следующим образом:

$$2\overline{\partial}\varphi = \varphi'_x + i\varphi'_y. \tag{1.2}$$

В *полярных* координатах $z = re^{i\theta}$ частные производные функции φ по r и θ обозначим соответственно через φ'_r и φ'_θ. В этих терминах

$$2\overline{\partial}\varphi = e^{i\theta}[\varphi'_r + (i/r)\varphi'_\theta]. \tag{1.3}$$

Отметим также следующее свойство оператора $\overline{\partial}$ для $\varphi, \psi \in C'(\overline{D})$

$$\overline{\partial}(\varphi\psi) = \psi\overline{\partial}\varphi + \varphi\overline{\partial}\psi. \tag{1.4}$$

$H\,(\Omega)$ обозначает класс голоморфных в Ω функций, таких что условие $f \in H\,(\Omega)$ означает, что $f \in C^1\,(\Omega)$ и $\overline{\partial}f \equiv 0$ в Ω. Для замкнутого множества $E \subset \Omega$ обозначим $A\,(E) = C\,(E) \cap H\,(E^o)$ и $A_b\,(E) = C_b\,(E) \cap H\,(E^o)$. Обозначим через $A'\,(E)$ класс функций $E \to \mathbb{C}$, один раз непрерывно дифференцируемых на E в смысле $\mathbb{C}$.

Предположим теперь, что $\varphi \in A'(\overline{G})$ и $\psi \in C'(\overline{G})$. Тогда по (1.4) имеем

$$\overline{\partial}(\varphi\psi) = \varphi\overline{\partial}\psi \text{ на } \overline{G}. \tag{1.5}$$

Следующая формула является важным приложением формулы (1.1), известной для $\varphi \equiv 1$ как обобщенная формула Коши или иногда как *формула Бореля-Помпейю* для ψ, указывающая, что

$$(1.6) \qquad \frac{1}{2\pi i}\int_{\Gamma}\frac{(\varphi\psi)(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int_{D}\frac{\varphi(\zeta)\overline{\partial\psi}(\zeta)}{\zeta - z}d\sigma = \begin{cases} (\varphi\psi)(z), & \text{для} \quad z \in G, \\ 0, & \text{для} \quad z \in \overline{G}^{c}. \end{cases}$$

Заметим, что при $\psi \equiv 1$ (подразумевая $\overline{\partial}\psi \equiv 0$) формула (1.6) является классической формулой Коши для $\varphi \in H(\Omega)$ с $\overline{G} \subset \Omega$. Заметим также, что формула (1.6) следует из отмеченного выше частного случая $\varphi \equiv 1$, с заменой ψ на $\varphi\psi$ и с учетом (1.5). Обозначим через $C_{\zeta}(z)$ ядро Коши:

$$C_{\zeta}(z) = (\zeta - z)^{-1} \text{ для } z \in \Delta_{\alpha},\ \zeta \in \Delta_{\alpha}^{c}.$$

Ниже мы будем обозначать через Δ_{∞}^{o} область голоморфности функции $z \to \log z$, с точками ветвления 0 и бесконечностью. На римановой поверхности Δ_{∞} можно использовать глобальную полярную параметризацию $z = re^{i\theta}$ с $r > 0$ и $\theta \in \mathbb{R}$. В этих терминах функция $z \to \log z := \log r + i\theta$ удовлетворяет в силу (1.3) условию голоморфности $\overline{\partial}(\log z) = 0$. Приведенное выше понятие $A(E)$ может быть легко расширено для $E \subset \Delta_{\infty} \cup \{0\}$.

3^{0}. Положим также:

$d_E(z) := \inf\{|z - z'| : z' \in E\}$ - расстояние $z \in \mathbb{C}$ от $E \subset \mathbb{C}$;

$E^{d} := \{\zeta \in \mathbb{C} : d_E(\zeta) \leq d\}$ - d - окрестность множества $E \subset \mathbb{C}$;

$D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$ для $a \in \mathbb{C}$, $r > 0$ - открытый круг; $D_r = D_r(0)$;

$l_{\theta} := \{z = te^{i\theta} : t \in [0, +\infty)\}$ для $\theta \in \mathbb{R}$ - луч;

$\Delta_{\alpha,\beta} := \{l_{\theta} : \theta \in [\alpha, \beta] \subset \mathbb{R}\}$ - сектор на $\Delta_{\infty} \cup \{0\}$ (на $\mathbb{C}$, если $\beta - \alpha < 2\pi$);

$\Delta_{\alpha}(\beta) := \Delta_{\beta-\alpha/2,\beta+\alpha/2}$ для $\alpha > 0$, $\beta \in \mathbb{R}$ - угол с биссектрисой l_{β} и отверстием α;

$\Delta_{\alpha} := \Delta_{\alpha}(0)$; $\gamma_{\alpha} := l_{\alpha/2} \cup l_{-\alpha/2}$ - граница Δ_{α};

$x^{+} := \max\{x, 0\}$ для $x \in \mathbb{R}$; $\log^{+} x = (\log x)^{+}$ для $x > 0$ и $\log^{+} 0 = 0$;

$s_z := \begin{cases} z/|z| & \text{для } z \neq 0, \\ 0 & \text{для } z = 0 \end{cases}$ - функция *знак*.

4^{0}. Пусть теперь $f \in C(E)$, где $E \subset \mathbb{C}$ замкнутое и неограниченное множество, так что $E \cap \partial D_r \neq \varnothing$ для $r \geq r_0 \geq 0$. Для $r \geq r_0$

$$M_f(r) = M_f(r, E) := \|f\|_{E \cap \overline{D}_r}.$$

Остальные обозначения и понятия мы введем ниже.

1.2. **Предварительные результаты типа Фрагмен-Линделефа.** 1^{o}. Сначала нам понадобится следующая теорема.

Теорема A. *Пусть* $h \in A(\Delta_\alpha)$ *для* $\alpha \in (0, 2\pi)$, $\rho = \pi/\alpha$ *и*

$$\liminf_{r\to\infty} \int_{-\alpha/2}^{\alpha/2} r^{-\rho} \log^+ \left|h(re^{i\theta})\right| \cos(\rho\theta) d\theta = 0. \tag{1.7}$$

Тогда из условия $\|h\|_{\gamma_\alpha} < +\infty$ следует, что $\|h\|_{\Delta_\alpha} = \|h\|_{\gamma_\alpha} < +\infty$.

Теорема A для $\rho = 1$ является версией братьев Неванлинна теоремы Фрагмена-Линделефа для полуплоскости Δ_π (см. [9] и [10, Теорема ??????????]), и легко следует из этого случая.

Следствие 1.1. *Пусть* $g \in A'(\Delta_\alpha)$ *для* $\alpha \in (0, 2\pi)$ *с* $\|g\|_{\Delta_\alpha} < +\infty$. *Тогда из условия*

$$g'(z) = O(|z|^\mu) \text{ при } z \to \infty,\ z \in \partial\Delta_\alpha \tag{1.8}$$

с постоянной $\mu \in \mathbb{R}^+$ *следует, что*

$$g'(z) = O(|z|^\mu) \text{ при } z \to \infty,\ z \in \Delta_\alpha. \tag{1.9}$$

Доказательство. Определим функцию $h \in A(\Delta_\alpha)$ по формуле

$$h(z) = (z+1)^{-\mu} g'(z),\ z \in \Delta_\alpha, \tag{1.10}$$

удовлетворяющую неравенству $\|h\|_{\gamma_\alpha} < +\infty$ по (1.8). Пусть $d(z) = d_{\gamma_\alpha}(z)$ будет *расстоянием* $z = re^{i\theta} \in \Delta_\alpha$ от γ_α. Поскольку для $r \geq 1$

$$d(z) \geq r\sin(\alpha/2 - |\theta|) \geq c_\alpha(\alpha/2 - |\theta|)$$

с $c_\alpha = (2/\alpha)\sin(\alpha/2)$, из неравенства Коши следует, что

$$|g'(z)| \leq c(\alpha/2 - |\theta|)^{-1},\ \ c = c_\alpha^{-1} \|g\|_{\Delta_\alpha}.$$

Отсюда и из (1.10) для некоторого $r_0 > 1$ и $r \geq r_0$ следует, что

$$\left|h(re^{i\theta})\right| \leq 2cr^{-\mu}(\alpha/2 - |\theta|)^{-1},\ \ |\theta| \leq \alpha/2.$$

Отсюда получаем, что

$$\log^+ \left|h(re^{i\theta})\right| \leq (-\mu)^+ \log r + \log^+(\alpha/2 - |\theta|)^{-1} + \log^+(2c),$$

показывая, что условие (1.7) Теоремы A удовлетворяется также для h, так что $\|h\|_{\Delta_\alpha} = \|h\|_{\gamma_\alpha} < +\infty$, которая завершает доказательство (1.9). □

2. Следующая теорема другая версия принципа Фрагмен-Линделефа (см. [11], Гл. I, Теорема 22).

Теорема B. *Пусть $g \in A(\Delta_\alpha)$ для $\alpha \in (0, 2\pi)$, на $\|g\|_{\gamma_\alpha} < +\infty$ и функция g имеет конечный порядок $\rho = \pi/\alpha$ - и тип σ Δ_α, то есть*

$$\sigma := \limsup_{r\to\infty} \frac{\log^+ M_g(r)}{r^\rho} < +\infty.$$

Тогда для $z = re^{i\theta} \in \Delta_\alpha$ имеем

$$|g(z)| \le \|g\|_{\partial\Delta_\alpha} \exp\{\sigma r^\rho \cos(\rho\theta)\}. \tag{1.11}$$

Следствие 1.2. *Пусть g целая функция, удовлетворяющая условиям:*

i) $\|g\|_{\Delta_\alpha} = m < +\infty$ для некоторого $\alpha \in (0, 2\pi)$;

ii) g имеет конечный порядок $\rho = \pi/(2\pi - \alpha)$ - и тип $(= \sigma)$ на угле $\Delta_{\pi/\rho}(\pi)$.

Тогда

$$g'(z) = O(|z|^{\rho-1}) \text{ при } z \to \infty,\ z \in \Delta_\alpha. \tag{1.12}$$

Доказательство. Положим $g_-(z) = g(-z)$ и $\beta = 2\pi - \alpha$. Пусть $d = d(\zeta)$ *расстояние* точки $\zeta = re^{i\theta} \in \Delta_\beta$ от $\partial\Delta_\beta$. Если, в частности, $|\theta| \ge (\pi - \alpha)^+/2$, то $d = r\sin(\beta/2 - |\theta|)$, так что

$$\cos(\rho\theta) = \sin\left[\rho \arcsin(dr^{-1})\right] \le \rho \arcsin(dr^{-1}) \le 2\rho dr^{-1}.$$

Пусть теперь $z \in \partial\Delta_\beta$, $\delta = |z|^{1-\rho}$ и $\zeta = z + \delta e^{i\theta}$ для $|\theta| \le \pi$, так что $d(\zeta) \le \delta$ и $|z|/r \to 1$ при $|z| \to \infty$. Применяя Теорему B к функции $g_- \in A(\Delta_\beta)$, из условия i) и (1.11) для достаточно больших $|z|$, получим:

$$|g(-\zeta)| \le m \exp(2\sigma\rho dr^{\rho-1}) \le c, \text{ где } c = m\exp(4\sigma\rho).$$

Для функции $g(-\zeta)$ и круга $\partial D_\delta(z)$ применяя неравенство Коши, получим

$$|g'(-z)| \le c\,|z|^{\rho-1} \text{ для } z \in \partial\Delta_\beta \text{ и } |z| \ge r_0 > 0.$$

Отсюда и Следствия 1.1 следует, что $g'(-z) = O(|z|^{\rho-1})$ при $z \to \infty$, $z \in \Delta_\beta$,что эквивалентно (1.12). □

Лемма 1.1. *Пусть функция $f \in A_b(\Delta_{\alpha,\beta})$ и f равномерно непрерывна на $\partial\Delta_{\alpha,\beta}$. Тогда функция f равномерно непрерывна также на $\Delta_{\alpha,\beta}$.*

Доказательство. Отметим, что функция f равномерно непрерывна на каждом луче l_θ для $\theta \in (\alpha, \beta)$: поскольку функция $f \in A_b(\Delta_\delta(\beta))$ для некоторого $\delta > 0$, тогда по неравенству Коши, функция f' будет ограниченным на $l_\theta \cap \overline{D}_{r_0}$ для любого фиксированного $r_0 > 0$. Разделяя интервал $[\alpha, \beta]$ на небольшие промежутки длины $< \pi$, достаточно доказать лемму для соответствующих углов, т.е. для случая выпуклого сектора $\beta - \alpha < \pi$, который сводится к сектору Δ_α с $\alpha \in (0, \pi)$.

Пусть Δ_α^ν - параллельный перенос Δ_α с вершиной $\nu \in \Delta_\alpha$. Тогда (по неравенству Коши) функция f' ограничена на угле $\Delta_\alpha^{r_\alpha} \subset \Delta_\alpha$ с $r_\alpha = 2\cos(\alpha/2)$, так что функция f равномерно непрерывна на $\Delta_\alpha^{r_\alpha}$. Обозначим через $\omega_1(\delta)$ $(\omega_2(\delta))$ модуль непрерывности функции f на $\partial\Delta_\alpha$ (на $\Delta_\alpha \cap \overline{D}_{r_\alpha}$), и положим

$$\omega(\delta) = \max\{\omega_1(\delta), \omega_2(\delta)\} \text{ для } \delta > 0.$$

Таким образом, функция f равномерно непрерывна на *границах* замкнутых выпуклых областей

$$\Lambda^\pm = \Delta_\alpha \diagdown (\Delta_\alpha^{\nu_\mp})^o \text{ с } \nu_\pm = \exp(\pm i\alpha/2).$$

Это значит, что функция $\lambda_\delta^\pm \in A_b(\Lambda_\pm)$, определенная по формуле

$$\lambda_\delta^\pm(z) = f(z + \delta\nu_\pm) - f(z) \text{ для } z \in \Lambda^\pm,\ \delta \in (0, 1], \tag{1.13}$$

будет удовлетворять (по принципу максимума модуля) условию

$$\sup_{z\in\Lambda^\pm} \left|\lambda_\delta^\pm(z)\right| = \sup_{\zeta\in\partial\Lambda^\pm} \left|\lambda_\delta^\pm(\zeta)\right| \le \omega(\delta) \to 0 \text{ при } \delta \to 0. \tag{1.14}$$

Теперь, по (1.13), (1.14), формула

$$\varphi_\delta^\pm(z) = (\delta\nu_\pm)^{-1} \int_z^{z+\delta\nu_\pm} \lambda_\delta^\pm(t)dt \text{ для } z \in \Lambda^\pm$$

определяет функцию $\varphi_\delta^\pm \in A'(\Lambda^\pm)$, для $z \in \Lambda^\pm$ удовлетворяющую условиям

$$\left|\varphi_\delta^\pm(z) - f(z)\right| \le \omega(\delta),\ \ \left|(\varphi_\delta^\pm)'(z)\right| \le \omega(\delta)/\delta.$$

Ограниченность функции $(\varphi_\delta^\pm)'$ на выпуклом множестве $\Lambda^\pm$ доказывает равномерную непрерывность функции $\varphi_\delta^\pm$ на $\Lambda^\pm$. Таким образом, функция f является равномерно непрерывной функцией на областях $\Lambda^\pm$ как равномерный предел равномерно непрерывных функций $\varphi_\delta^\pm$, при $\delta \to 0$. Это доказывает, что функция f равномерно непрерывна на $\Delta_\alpha = \Lambda^- \cup \Delta_\alpha^{r_\alpha} \cup \Lambda^+$, являющейся такой функцией на каждом выпуклом множестве. □

1.3. **Гладкое продолжение гладких функций.** Нам нужен некоторый специфический результат о C' - продолжении для функции $f \in A'(\gamma_\sigma)$, $\gamma_\sigma = \partial\Delta_\sigma$, $\sigma \in (0, 2\pi)$. Нашей целью является построение функции $f_* \in C'(\Delta_\sigma)$, удовлетворяющей условиям $f_* = f$ и $\overline{\partial} f_* = 0$ на γ_σ. Также мы ожидаем хороших оценок для функция $\overline{\partial} f_*$ на Δ_σ в терминах роста функции f'.

Лемма 1.2. *Пусть $f \in A'(\Delta_\alpha)$ и $\sigma = 2\pi - \alpha$ и $\partial\Delta_\sigma(\pi) = \gamma_\alpha$. Тогда существует функция $f_* \in C'(\Delta_\sigma(\pi))$, такая что*

(i) $f_* = f$ и $\overline{\partial} f_* = 0$ на γ_α;

(ii) рост функций f_* и $\overline{\partial} f_*$ для $z \in \Delta_\sigma(\pi)$ ограничен неравенствами

$$|f_*(z)| \leq 3M_f(|z| + 2d, \gamma_\alpha) \tag{1.15}$$

$$\left|\overline{\partial} f_*(z)\right| \leq k_\alpha M_{f'}(|z| + 2d, \gamma_\alpha), \tag{1.16}$$

где $d = d_\alpha(z)$ расстояния z от γ_α и $k_\alpha > 0$ зависит лишь от α.

Доказательство. Для $|\theta| \leq \sigma/2$ положим

$$\psi(\theta) = \sin^2(\tau\theta), \ e(\theta) = \exp(is_\theta\sigma/2) \tag{1.17}$$

с $\tau = \pi/\sigma$, и рассмотрим две функции $u, v \in C'(\Delta_\sigma)$, определенные для $z = re^{i\theta} \in \Delta_\sigma$ по формуле

$$u = r\psi(\theta), \ v = (r/\tau)\cos(\tau\theta). \tag{1.18}$$

Определим финкции, ассоцированные с u и v

$$\zeta(z) = u(z)e(\theta) \ \text{ и } \ w(z) = v(z)e(\theta) \text{ для } z \in \Delta_\sigma. \tag{1.19}$$

Очевидно, что $\zeta \in C'(\Delta_\sigma)$ (несмотря на разрыв $e(\theta)$ при $\theta = 0$), поскольку u, u'_r, и u'_θ равны нулю на биссектрисе l_0 угла Δ_σ. Другая функция w даже разрывна на l_0; но отметим, что она имеет очевидные C'-расширения отдельно на углах $\Delta_\sigma^+ := \Delta_{0,\alpha/2}$ и $\Delta_\sigma^- := \Delta_{-\alpha/2,0}$.

Отметим, что поскольку $u, v \geq 0$, то следует, что $\zeta, w \in l_{\sigma/2}$ для $z \in \Delta_\sigma^+$ и $\zeta, w \in l_{-\sigma/2}$ для $z \in \Delta_\sigma^-$. Отсюда слвгует, что $\zeta + w \in l_{\pm\sigma/2}$ для $z \in \Delta_\sigma^\pm$. Отметим также, что согласно (1.7) - (1.9), $\zeta(z) = z$ и $w(z) = 0$ для $z \in \Delta_\sigma$, тогда и только тогда, если $z \in \gamma_\sigma$.

Используя (1.17) - (1.19), легко проверяются следующие соотношения:

$$r\zeta'_r = \zeta, \ rw'_r = w, \ \zeta'_\theta = 2\tau^2 w\sin(\tau\theta), \ \psi(\theta)w'_\theta = -\zeta\sin(\tau\theta). \tag{1.20}$$

Пусть $d = d_\sigma(z)$ будет *расстоянием* $z = re^{i\theta} \in \Delta_\sigma$ от γ_σ. Оценим $|w|$ сверху через d. Положим для этого $\nu = \sigma/2 - |\theta| \in [0, \sigma/2]$ и отметим, что в терминах ν,

$$|w| = v = (r/\tau)\sin(\tau\nu). \tag{1.21}$$

Теперь если $\sigma > \pi$ и $0 \leq \nu < (\sigma - \pi)/2$, то $d = r$, и по (1.21) следует, что $|w| \leq r/\tau \leq 2d$. Если $\sigma \leq \pi$ и $0 \leq \nu \leq \sigma/2$, то $d = r\sin\nu$. Отмечая, что

$$\tau^{-1}\sin(\tau\nu) \leq \nu \leq (\pi/2)\sin\nu, \tag{1.22}$$

по (1.21) снова получаем, что $|w| \leq 2d$. Суммируя, получаем, что

$$|w| \leq 2d_\sigma(z) \text{ для } z \in \Delta_\sigma. \tag{1.23}$$

Теперь определим искомую функцию $f_* \in C'(\Delta_\sigma)$, положив $f_*(z) = f(0)$ для $z = r \in l_0$ и

$$f_*(z) = f(\zeta) - is_\theta\psi(\theta)[f(\zeta + w) - f(\zeta)] \tag{1.24}$$

для $z = re^{i\theta} \in \Delta_\sigma \backslash l_0$, с $\psi(\theta)$ определенной в (1.17).

а) Убедимся сначала, что функция $f_* \in C(\Delta_\sigma)$, несмотря на разрыв функций w и s_θ on l_0.

Упомянем для этого, что $f \circ \zeta \in C'(\Delta_\sigma)$, поскольку $f \in A'(\gamma_\sigma)$ и $\zeta \in C'(\Delta_\sigma)$. Кроме того, $f_*(z) \to f(0)$, при $z \to z_0 = r_0 \in l_0$, так как тогда $\theta \to 0$ и $\zeta \to 0$, что подразумевает $f(\zeta) \to f(0)$ и $\psi(\theta) \to \psi(0) = 0$.

Отметим также, что $f_*(z) = f(z)$ for $z \in \gamma_\sigma$, в этом случае $z = \zeta$ и $w = 0$, что подразумевает $\varphi(z) = 0$. С учетом (1.23), и то, что $|\zeta| \leq r = |z|$ с $|\psi(\theta)| \leq 1$, в силу (1.24) получаем оценку

$$|f_*(z)| \leq 3M_f(|z| + 2d, \gamma_\sigma) \text{ для } z \in \Delta_\sigma, \tag{1.25}$$

где $d = d_\sigma(z)$. Согласно (1.25), рост функции f_* на Δ_σ зависит лишь от роста функции f на γ_σ; в частности, если функция f ограничена на γ_σ, то функция f_* ограничена на Δ_σ.

б) Чтобы убедиться, что $f_* \in C'(\Delta_\sigma)$, достаточно проверить, что $(\varphi\psi)'_r \to 0$ и $(\varphi\psi)'_\theta \to 0$ при $z = re^{i\theta} \to r_0 \in l_0$, т.е. $r \to r_0$ и $\theta \to 0$; это очевидно следует из того факта, что $\varphi \in C'(\Delta_\sigma^-) \cup C'(\Delta_\sigma^+)$ и $\psi(0) = \psi'(0) = 0$.

Для расчета $\overline{\partial} f_*$ по (1.2), (1.3) отметим, что,

$$\overline{\partial} f(\zeta) = f'(\zeta)\overline{\partial}\zeta, \ \overline{\partial} f(\zeta + w) = f'(\zeta + w)\overline{\partial}(\zeta + w).$$

Теперь учитывая (1.4) и (1.17)-(1.19), для $z = re^{i\theta} \in \Delta_\sigma$, получим

$$\begin{aligned} 2\overline{z}\overline{\partial} f_*(z) &= [f'(\zeta) - f'(\zeta + w)\sin(\tau|\theta|)]\zeta + \\ &\quad s_\theta\psi(\theta)[f'(\zeta + w) - f'(\zeta)](\zeta'_\theta - i\zeta) \\ &\quad + i[f'(\zeta)\zeta'_\theta - s_\theta\psi(\theta)f'(\zeta + w)w] \\ &\quad + s_\theta\psi'(\theta)[f(\zeta + w) - f(\zeta)]. \end{aligned} \tag{1.26}$$

Правая часть (1.24) равна нулю при $z \in \gamma_\sigma$, поскольку тогда $|\theta| = \sigma/2$ и $w = \zeta'_\theta = 0$ (см. (1.18) – (1.20)); это означает, что выражения в (1.26) в квадратных скобках равны нулю. Таким образом, мы получаем, что $\overline{\partial} f_*(z) = 0$ для $z \in \gamma_\sigma$.

Для оценки правой части (1.26), сначала отметим, что

$$|f(\zeta + w) - f(\zeta)| \leq \int_\zeta^{\zeta + w} |f'(t)|\,|dt| \leq M_{f'}(|z| + |w|, \gamma_\sigma)\,|w|, \tag{1.27}$$

так как $|\zeta| \le r$. Кроме того $|\psi(\theta)| \le 1$, $|\psi'(\theta)| \le \tau$ для $|\theta| \le \sigma/2$. Учитывая также (1.23), мы из (1.26) и (1.27) получаем, что

$$\left|\overline{\partial} f_*(z)\right| \le k_\sigma M_{f'}(|z| + 2d, \gamma_\sigma) \text{ для } z \in \Delta_\sigma, \tag{1.28}$$

где $d = d_\sigma(z)$ и $k_\sigma > 0$ зависят лишь от σ.

Чтобы определить функцию f_* на $\Delta_\sigma(\pi)$, отметим, что из $z \in \gamma_\alpha$ следует, что $-z \in \gamma_\sigma$. Таким образом, построение функции f_* на Δ_σ по формуле (1.24) для функции $f_- \in A'(\gamma_\sigma)$ вместо f, где $f_-(z) = f(-z)$, мы получаем искомую функцию, просто заменяя $f_*(z)$ на $f_*(-z)$ с $z \in \Delta_\sigma(\pi)$. Тогда неравенства (1.15) и (1.16) следуют соответственно из (1.25) и (1.28). □

Замечание 1.1. *Условие (i) гарантирует, что функция f_* коплексно дифференцируема на γ_α, так что частные производные функции f и f_* будут совпадать на γ_α.*

1.4. **Приближения на Δ_α функциями из класса $A(\Delta^1_\alpha)$.** Следующая лемма одна из основных результатов этой работы.

Лемма 1.3. *Пусть $f \in A'(\Delta_\alpha)$ для $\alpha \in (0, 2\pi)$ и $\varepsilon > 0$. Тогда существует функция $F \in A(\Delta^1_\alpha)$ такая, что*

$$|f(z) - F(z)| < \varepsilon \text{ для } z \in \Delta_\alpha \tag{1.29}$$

и

$$M_F(r) < 6M_f\,(2r) + c\varepsilon \exp(2 + c\varepsilon^{-1} M_{f'}(2r + 5)), \tag{1.30}$$

где $c = c(\alpha) > 0$.

Доказательство. Доказательство леммы реализуем в 2 шага. Сначала приблизим функцию f на Δ_α функциями h принадлежащими классу $A'(\Lambda)$, где область Λ мы выбераем внизу, потом на втором шаге функция $h \in A'(\Lambda)$ приближается функциями $F \in A(\Delta^1_\alpha)$ на Δ_α. Заменив f на $\varepsilon^{-1} f$ и F на $\varepsilon^{-1} F$, можно свести доказательство леммы к случаю $\varepsilon = 1$.

Шаг 1: Продолжим функцию f на $\mathbb{C}$ взяв в качестве C^1-расширение функции f_*, удовлетворяющее условиям Леммы 1.2.

Пусть $\zeta \to n = n\,(|\zeta|) \in \mathbb{N}$ для $\zeta \in \gamma_\alpha$ будет кусочно-постоянной функцией; фиксируем $n\,(|\zeta|)$ однозначно по условию

$$0 < n\,(|\zeta|) - \{M_{f'}(|\zeta|\,, \gamma_\alpha) + 1\} \le 1. \tag{1.31}$$

Определим новую функцию $Q(\zeta, z)$ такую, что $Q(\zeta, \zeta) = 1$ для $\zeta \in \Lambda$:

$$Q(\zeta, z) = \left(\frac{\zeta - \zeta_0}{z - \zeta_0}\right)^n, \tag{1.32}$$

где

$$dist(\zeta, \gamma_\alpha) = (\ln^+ M_{f'}(|\zeta|, \gamma_\alpha) + 1)^{-1} \text{ для } \zeta \in \partial\Lambda;$$

так область будет определен; и ζ_0 определено так, что $dist(\zeta_0, \gamma_\alpha) = 2dist(\zeta, \gamma_\alpha)$ для $\zeta \in \partial\Lambda$. Очевидно, что $Q(\zeta, \cdot) \in H(\Lambda)$ для фиксированного $\zeta \in \Lambda \backslash \Delta_\alpha \equiv U$.

Определим функции h_r на Λ по формулей

$$h_r(z) = f_*(z) + I_r(z) \text{ для } r > 0 \text{ и } z \in \Lambda_r, \tag{1.33}$$

где

$$I_r(z) = \pi^{-1} \int_{\Lambda_r} G_\zeta(z)\, d\sigma_\zeta \text{ для } r > 0 \text{ и } z \in \Lambda_r, \tag{1.34}$$

с подынтегральной функцией $G_\zeta(z) = \left(\overline{\partial} f_*\right)(\zeta)\, Q(\zeta, z)\, C_\zeta(z)$ и $\Lambda_r = \Lambda \cap \overline{D}_r$.

Теперь докажем, что $I_r(z)$ локально равномерно сходится на Λ, при $r \to \infty$, к соответствующему несобственному интегралу

$$I_\infty(z) = \frac{1}{\pi} \int_U G_\zeta(z)\, d\sigma_\zeta \text{ для } z \in \Lambda. \tag{1.35}$$

Пусть K компактное множество и $K \subset \Lambda$. Тогда существует $r_0 > 1$, так, что $K \subset \overline{D}_{r_0}$ и $r'' > r' > 3r_0$. Отсюда следует, что $|\zeta - \zeta_0| < 2|z - \zeta_0|$ и по (1.31), (1.32) для $z \in \Delta_\alpha$ и $\zeta \in U$ имеем

$$|G_\zeta(z)| \le \frac{n}{2}\left(\frac{2}{\ln n}\right)^n \frac{1}{|z - \zeta_0|^2} < c_1 \frac{1}{|z - \zeta_0|^2}, \tag{1.36}$$

здесь и внизу обозначим через $c_j > 0$, $j = 1, 2, \cdots$ постоянные зависящие лишь от α, таким образом по (1.36) получим

$$|I_{r''}(z) - I_{r'}(z)| \le c_2 \int_{r'-r_0}^{r''-r_0} \frac{1}{u^2} du < c_2 \left(\frac{1}{r' - r_0} - \frac{1}{r'' - r_0}\right) \to 0,$$

равномерно для $z \in K$, при $r'', r' \to \infty$. Это доказывает аблолютную и локально равномерную сходимость функции $I_\infty(z)$ для $z \in \Lambda$ и, что $I_\infty \in C(\Lambda)$. Затем по (1.33), (1.35) и (1.36) искомую функцию $h \in A(\Lambda)$ можем определит по формуле

$$h(z) = f_*(z) + I_\infty(z) \text{ для } z \in \Lambda. \tag{1.37}$$

По формуле (1.6) *Бореля-Помпейю* мы можем представить функцию h в виде:

$$h(z) = \frac{1}{2\pi i} \int_{\partial\Lambda_r} \frac{f_*(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi} \int\int_{\Lambda \backslash \Lambda_r} (\overline{\partial} f)(\zeta) \frac{Q(\zeta, z)}{\zeta - z} d\sigma_\zeta +$$

$$+\frac{1}{\pi}\int\int_{\Lambda_r}(\overline{\partial}f)(\zeta)\frac{Q(\zeta,z)-1}{\zeta-z}d\sigma_\zeta. \tag{1.38}$$

Здесь все три интеграла голоморфные функции на Λ_r^o и не зависят от r. Таким образом, получаем, что $h\in A(\Lambda)$.

Из определения (1.37) следует

$$|f(z)-h(z)|=|I_\infty(z)| \text{ для } z\in\Delta_\alpha \tag{1.39}$$

и

$$|h(z)|\leq|f_*(z)|+|I_\infty(z)| \text{ для } z\in\Lambda, \tag{1.40}$$

так что приближения функции f na Δ_α функцией $h\in A(\Lambda)$ и оценка роста функции h на Λ сводятся в этой схеме к оценке I_∞.

Для оценки роста $|f(z)-h(z)|$ для Δ_α, нам нужно оценить $|I_\infty(z)|$ для $z\in\Delta_\alpha$. Представим $I_\infty(z)$ в виде суммы трех интегралов:

$$I_\infty(z)=\frac{1}{\pi}\int_{U_1(z)}G_\zeta(z)\,d\sigma_\zeta+\frac{1}{\pi}\int_{U_2(z)}G_\zeta(z)\,d\sigma_\zeta+\frac{1}{\pi}\int_{U_3(z)}G_\zeta(z)\,d\sigma_\zeta, \tag{1.41}$$

где $U_1(z)=\{\zeta\in\mathbb{C}:\zeta\in U \text{ и } |\zeta-z|\geq1\}$, $U_2(z):=\{\zeta\in\mathbb{C}:\zeta\in U \text{ и } 1/\ln n\leq|\zeta-z|\leq1\}$ и $U_3(z)=\{\zeta\in\mathbb{C}:\zeta\in U \text{ и } |\zeta-z|\leq1/\ln n\}$. В силу (1.32) и (1.36) получаем

$$\frac{1}{\pi}\int_{U_1(z)}|G_\zeta(z)|\,d\sigma_\zeta<\frac{n}{\pi}\left(\frac{2}{\ln n}\right)^n\int_{U_1(z)}\frac{1}{|z-\zeta|^2}<c_{3.} \tag{1.42}$$

По (1.32) и (1.35) имеем

$$\frac{1}{\pi}\int_{U_2(z)}|G_\zeta(z)|\,d\sigma_\zeta<\frac{n\ln n}{\pi}\int_{-1}^{1}du\int_0^{1/\ln n}\left(1-\frac{v\ln n}{2}\right)^n dv<$$

$$<\frac{4n}{\pi(n+1)}<c_4 \tag{1.43}$$

Представив $\zeta-z=re^{i\theta}$ для $r\leq1/\ln n$ и $\theta\in(-\pi/2,\pi/2)$, получаем:

$$\frac{1}{\pi}\int_{U_3(z)}|G_\zeta(z)|\,d\sigma_\zeta<\int_0^{1/\ln n}\int_{-\pi/2}^{\pi/2}\frac{n}{\ln n}\left(1-\frac{r\theta}{2}\right)^n d\theta dr<\int_0^{1/\ln n}c_6. \tag{1.44}$$

Таким образом суммируя (1.42)-(1.44) по (1.41) мы получаем, что $|I_\infty(z)|<c_7$ для $z\in\Delta_\alpha$.

Пусть теперь $z\in\Lambda$. Представим интеграл $I_\infty(z)$ как сумму двух интегралов:

$$I_\infty(z)=\int_{D(z)}G_\zeta(z)\,d\sigma_\zeta+\int_{U\setminus D(z)}G_\zeta(z)\,d\sigma_\zeta=I_4(z)+I_5(z), \tag{1.45}$$

где $D(z)=\{\zeta\in\mathbb{C}:\zeta\in E \text{ и } |\zeta-z|\leq3|z|\}$. Первый интеграл уже оценили в (1.42).

Учитывая, что $n2^n < e^{2n}$ и

$$\int_{D(z)} \frac{1}{|\zeta - z|} d\sigma_\zeta < c_4 \sqrt{mesD\,(z)},$$

по (1.25) и $|\zeta| > |z|$, получаем

$$|I_5\,(z)| < \exp\{2n\,(3l\,|z|)\}. \tag{1.46}$$

Суммируя (1.45) и (1.46), получаем

$$|I_\infty\,(z)| < \exp\left\{2n\,(3l\,|z|) + c_5\right\} \text{ для } z \in \Lambda. \tag{1.47}$$

Таким образом, получаем

$$|h\,(z)| < M_f\,(|z|) + \exp\left\{2n\,(3l\,|z|) + c_5\right\} \text{ для } z \in \Lambda. \tag{1.48}$$

Шаг 2: Теперь приблизим функцию $h \in A'(\Lambda)$ на Δ_α функциями F из класса $A(\Delta^1_\alpha)$. Это будет реализовано аналогично приближению функции f на Δ_α. Для этого мы во-первых должны оценить рост функции $\overline{\partial} h(\zeta)$ для $\zeta \in \Delta^1_\alpha \backslash \Lambda \equiv W$. Из (1.31) и (1.35) следует, что

$$\left|\overline{\partial} I_\infty(\zeta)\right| \le M_f\,(|\zeta| + 1) + 2M_{f'}\,(|\zeta| + 1)\,, \tag{1.49}$$

затем из определения функции h и представления (1.38) имеем

$$\left|\overline{\partial} h(\zeta)\right| \le M_f\,(|\zeta| + 1) + 3M_{f'}\,(|\zeta| + 1)\,. \tag{1.50}$$

Определим функции F_r на Δ^1_α по формуле

$$F_r\,(z) = h_*\,(z) + J_r\,(z) \text{ для } r > 0 \text{ и } z \in \Delta^1_{\alpha,r}, \tag{1.51}$$

где

$$J_r\,(z) = \pi^{-1} \int_{W_r} P_\zeta\,(z)\, d\sigma_\zeta \text{ для } r > 0 \text{ и } z \in W_r, \tag{1.52}$$

с подынтегральной функцией $P_\zeta\,(z) = \left(\overline{\partial} h_*\right)(\zeta)\, Q_1\,(\zeta, z)\, C_\zeta\,(z)$ и $W_r = W \cap \overline{D}_r$, где

$$Q_1\,(\zeta, z) = \left(\frac{\zeta - \zeta'}{z - \zeta'}\right)^m,$$

где $dist\,(\zeta', \gamma_\alpha) = 2dist\,(\zeta, \gamma_\alpha)$ для $\zeta \in \partial\Delta^1_\alpha$ и $m = m\,(|\zeta|) = \max\{n, \ln^+ M_f\,(|\zeta|)\}$. Очевидно, что $F_r\,(z) \in A\left(\Delta^1_\alpha\right)$. Как и выше, мы имеем, что функция $J_r\,(z)$ абсолютно и локально равномерно сходится к функции $J_\infty\,(z)$ для $z \in \Delta^1_\alpha$, при $r \to \infty$.

По (1.51), искомую приближающие функции $F \in A\left(\Delta^1_\alpha\right)$ можем определить по формуле

$$h\,(z) := f_*\,(z) + I_\infty\,(z) \text{ для } z \in \Lambda. \tag{1.53}$$

Из определения (1.51) следует, что

$$|f(z) - h(z)| = |I_\infty(z)| \text{ для } z \in \Delta_\alpha \tag{1.54}$$

и

$$|F(z)| \le |f_*(z)| + |I_\infty(z)| \text{ для } z \in \Lambda, \tag{1.55}$$

поэтому, как и выше, мы должны оценить рост $J_\infty(z)$ на Δ_α и на Δ_α^1.

Тогда повторяя шаги доказательства первого шага и учитывая (1.49), (1.50), получаем

$$|h(z) - F(z)| < \varepsilon \text{ для } z \in \Delta_\alpha \tag{1.56}$$

и

$$M_F(r) < 3M_f(2r) + c\varepsilon \exp(2 + c\varepsilon^{-1} M_{f'}(2r+5)). \tag{1.57}$$

По (1.39) и (1.54) мы получаем оценку (1.29). По (1.47) и (1.57) мы получаем оценку (1.30). □

Следующая лемма доказана в [13].

Лемма 1.4. *Пусть $f \in A'(\Delta_\alpha)$, $\alpha < \beta < \min\{\alpha + \pi/2, \pi + \alpha/2\}$ и $\varepsilon > 0$. Тогда существует функция $F \in A(\Delta_\beta)$ такая, что*

$$|f(z) - F(z)| < \varepsilon \text{ для } z \in \Delta_\alpha \tag{1.58}$$

и рост фунции F удовлетворяет этому неравенству на Δ_β

$$M_F(r) < 3M_f(lr) + c\varepsilon \exp\left\{1 + c\varepsilon^{-1}\lambda(3lr, f)\right\}, \tag{1.59}$$

для $r > 0$, где

$$\lambda(r, f) = \max_{|\zeta| \le r} \left\{(|\zeta| + 1) |f'_\partial(\zeta)|\right\}, \tag{1.60}$$

$l = 1 + \tan((\beta - \alpha)/2) > 1$ и $c = c(\alpha, \beta) > 0$ постоянная, зависящая только от α и β.

2. Приближение целыми функциями

Процесс оптимального равномерного приближения на угле Δ_α целыми функциями будет реализован в два шага. Сначала приблизим функцию $f \in A'(\Delta_\alpha)$ на Δ_α функциями F, голоморфными в большей области Ω, $F \in A(\Omega)$, с оценкой роста F на Ω - Лемма 1.3 и Лемма 1.4, тогда функция F будет равномерно приближаться на Δ_α целыми функциями. Наша основная задача состоит в том, чтобы сопровождать реализацию двух шагов: каждый шаг возможным оптимальным ростом приближающих функций.

2.1. **Приближение ядра Коши.** Реализация второго шага основано на построении соответствующих приближающих целых функций *ядра Коши* (см. Лемма 1 в [5]).

Пусть $\alpha \in (0, 2\pi)$ и $d = d_\alpha(\zeta)$ *расстояние точки* $\zeta \in \Delta^c_\alpha$ *от* γ_α. Тогда для $b > 0$ *существует функция* $Q_b(\zeta, z)$, *непрерывная по* b и $(\zeta, z) \in \Delta^c_\alpha \times \mathbb{C}$ и *удовлетворяющая условиям*:

(i) $Q_b(\zeta, z)$ *целая функция по* z *для любого* $\zeta \in \Delta^c_\alpha$.

(ii) *Для* $\zeta \in \Delta^c_\alpha$ *и* $z \in \overline{D}_{|\zeta|/2} \cup \Delta_\alpha$ *имеем*

$$|Q_a(\zeta, z) - C_\zeta(z)| < (44/d)e^{-b}. \tag{2.1}$$

(iii) *Рост функции* Q_b *на* $\Delta^c_\alpha \times \mathbb{C}$ *ограничивается неравенством*

$$|Q_a(\zeta, z)| < d \exp\left\{\mu(b/d)\, |\zeta|^{1-\rho} (|z| + 1)^\rho\right\}, \tag{2.2}$$

где $\rho = \pi/(2\pi - \alpha)$ *и* $\mu = \mu(\alpha) > 0$ *зависят лишь от* α.

Для случая $\alpha \geq \pi$ понадобится следующая лемма, доказанная в [5]:

Лемма 2.1. *Пусть* $\alpha \in (0, 2\pi)$, *тогда существует функция* $\Omega(\zeta, z)$ *целая по* z *и по* ζ, *удовлетворяющяя неравенствам:*

$$\Omega(\zeta, z) \equiv 1 \text{ для } \zeta = z, \tag{2.3}$$

$$|\Omega(\zeta, z)| \leq c(\alpha) \left(1 + |\zeta - z|^2\right)^{-1} \tag{2.4}$$

если $\zeta \in \Delta^1_\alpha \backslash \Delta^{-1}_\alpha$ *и* $z \in \Delta^1_\alpha$.

В случае $\alpha \geq \pi$

$$|\Omega(\zeta, z)| \leq c(\alpha) \left(1 + |\zeta - z|^2\right)^{-1} \exp\{c(\alpha)(|z| + 1)^\rho\} \tag{2.5}$$

если $\zeta \in \Delta^1_\alpha$, *и* $|\zeta| \leq 2|z|$.

Эти 2 леммы нам понадобяться ниже.

2.2. **Оптимальное равномерное целое приближение на Δ_α.** Следующие теоремы охватывают наиболее простую ситуацию, когда приближаемая функция предварительно голоморфна в большей угловой области, чем угол приближения.

Теорема 2.1. *Пусть $F \in A(\Delta_\beta)$ для $\alpha \in (0, \pi)$, $\alpha < \beta < 2\pi$ и $\varepsilon > 0$. Тогда существует целая функция G такая, что*

$$|F(z) - G(z)| < \varepsilon \text{ для } z \in \Delta_\alpha \tag{2.6}$$

и рост функции G удовлетворяет неравенству

$$\log \frac{|G(z)|}{\varepsilon} < c(1 + |z|^\rho) \left\{ 2 + \log^+ \frac{M_F(2r)}{\varepsilon} \right\} \text{ для } z \in \mathbb{C}, \tag{2.7}$$

где $c = c(\alpha, \beta) > 0$

Доказательство. Как в Лемме 1.3, можем привести доказательство к случаю $\varepsilon = 1$. Докажем теорему, используя метод, развитый в [7].

Для функции Q, взятой из Леммы 2.1 с $d = 1$, положим

$$\Phi(\zeta, z) = Q_{b_{|\zeta|}}(\zeta, z) \text{ для } (\zeta, z) \in \partial\Delta_\beta \times \mathbb{C},$$

где

$$b_t = 1 + \log^+ M_F(t) + 2\log(|t| + 1).$$

Для $r > 0$ введем теперь несобственные интегралы

$$I_r(z) = (2\pi i)^{-1} \int_{\partial\Delta_\beta} R(\zeta, z)\, d\zeta, \text{ для } z \in \mathbb{C} \backslash \partial\Delta_\beta, \tag{2.8}$$

где

$$R(\zeta, z) = F(\zeta) \left[\Phi(\zeta, z) - (\zeta - z)^{-1} \right]. \tag{2.9}$$

Искомую функцию G определим по формуле

$$G(z) = I_0(z) + F(z) \text{ для } z \in \Delta_\beta I_0(z) \text{ и } z \in \mathbb{C} \backslash \overline{\Delta}_\beta. \tag{2.10}$$

Так как в [7] очевидно, что $G \in H(\mathbb{C})$.

Тогда, используя конструкцию доказательства Теоремы 1 работы [7], получаем доказательство этой теоремы. □

Следствие 2.1. *По теореме 2.1 и (2.6), (2.7), функцию $F \in A(\Delta_\beta)$ для $\alpha \in (0, \pi)$, $\alpha < \beta < 2\pi < \infty$ порядка $\rho_F < +\infty$ можно равномерно приблизить на Δ_α целыми функциями G порядка $\rho_G \leq \rho_F + \rho$; если в частности $\sigma_F < +\infty$, то либо $\rho_G < \rho_F + \rho$, либо $\rho_G = \rho_F + \rho$ и $\sigma_G < k\sigma_F$, где постоянная $k > 0$ зависит лишь от α.*

Теорема 2.2. *Пусть $F \in A(\Delta_\alpha^1)$ для $\alpha \in [\pi, 2\pi)$ и $\varepsilon > 0$. Тогда существует целая функция G такая, что*

$$|F(z) - G(z)| < \varepsilon \text{ для } z \in \Delta_\alpha \tag{2.11}$$

и рост функции G удовлетворяет неравенству

$$\log \frac{|G(z)|}{\varepsilon} < c(1+|z|^{\rho})\left\{2+\log^{+}\frac{M_F(2r)}{\varepsilon}\right\} \text{ для } z\in\mathbb{C}, \tag{2.12}$$

где $c = c(\alpha) > 0$.

Доказательство. Как и в лемме 1.3, доказательство можно привести к случаю $\varepsilon = 1$. Докажем теорему как и выше, используя метод, развитый в [7].

Для функции Q, взятой из Леммы 2.1 при $d = 1$, положим

$$\Phi(\zeta, z) := Q_{b_{|\zeta|}}(\zeta, z) \text{ для } (\zeta, z) \in \partial\Delta_\alpha^1 \times \mathbb{C}, \tag{2.13}$$

где $b_t = 1 + \log^+ M_F(t)$. Очевидно, что $\Phi \in C(\partial\Delta_\alpha^1 \times \mathbb{C})$ и целая функция по z для любого фиксированного $\zeta \in \partial\Delta_\alpha^1$.

Для $r > 0$ введем теперь несобственные интегралы

$$I_r(z) = (2\pi i)^{-1}\int_{\partial\Delta_\alpha^1} R(\zeta, z)\, d\zeta, \text{ для } z \in \mathbb{C}\backslash\partial\Delta_\alpha^1, \tag{2.14}$$

где

$$R(\zeta, z) = F(\zeta)\left[\Phi(\zeta, z) - (\zeta - z)^{-1}\right]\Omega(\zeta, z), \tag{2.15}$$

функция $\Omega(\zeta, z)$ взята из Леммы 2.2.

Искомую функцию G определим по формуле

$$G(z) = I_0(z) + F(z) \text{ для } z \in \Delta_\alpha^1 I_0(z) \text{ и } z \in \mathbb{C}\backslash\overline{\Delta}_\alpha^1. \tag{2.16}$$

Так как и в [7], очевидно, что $G \in H(\mathbb{C})$.

После чего, используя конструкцию доказательства Теоремы 1 работы [7], получаем доказательство этой теоремы. □

Следствие 2.2. *По Теореме 1.2 и (2.13), (2.14), функцию $F \in A(\Delta_\alpha^1)$ для $\alpha \in [\pi, 2\pi)$ порядка $\rho_F < +\infty$ можно равномерно приблизить на Δ_α целыми функциями G порядка $\rho_G \le \rho_F + \rho$; если, в частности, $\sigma_F < +\infty$, то либо $\rho_G < \rho_F + \rho$, либо $\rho_G = \rho_F + \rho$ и $\sigma_G < k\sigma_F$, где постоянная $k > 0$ зависит лишь от α.*

Следующая теорема является основным результатом работы.

Теорема 2.3. *Пусть $f \in A'(\Delta_\alpha)$ для $\alpha \in (0, 2\pi)$ и $\varepsilon > 0$. Тогда существует целая функция g, такая что*

$$|f(z) - g(z)| < \varepsilon \tag{2.17}$$

и рост функции g удовлетворяет этому неравенству для $\alpha \geq \pi$

$$\log \frac{|g(z)|}{\varepsilon} < c(2+|z|^{\rho})\{2+\log^{+}\frac{M_f(r)}{\varepsilon}+\varepsilon^{-1}M_{f'}(r,\gamma_\alpha)\} \text{ для } z \in \mathbb{C}, \tag{2.18}$$

и для $\alpha < \pi$

$$\log \frac{|g(z)|}{\varepsilon} < c(2+|z|^{\rho})\{2+\log^{+}\frac{M_f(r)}{\varepsilon}+\varepsilon^{-1}\mu_f(r,\gamma_\alpha)\} \text{ для } z \in \mathbb{C} \tag{2.19}$$

где

$$\mu_f(r,\gamma_\alpha) = \max_{|z|\leq r,\ z\in\gamma_\alpha} \left| z^{1-\rho} f'(z) \right| \tag{2.20}$$

и $r = 2|z| + 3$ и $c = c(\alpha) > 0$.

Доказательство. Непосредственно следует из Леммы 1.3 и Теоремы 1.1 для случая $\alpha < \pi$ с использованием метода, развитого в Теореме 2 в работе [13]; и из Леммы 1.4 и Теоремы 1.2 для случая $\alpha \geq \pi$. □

Следующая теорема следует из Теоремы 2.4 и дает положительный ответ на проблему, предложенную Кобером в [3].

Теорема 2.4. *Пусть $f \in A_b(\Delta_\alpha)$ для $\alpha \in (0, 2\pi)$ и $\rho = \pi/(2\pi - \alpha)$. Если $f\left(z^{1/\rho}\right)$ равномерно непрерывна на лучах $\pm l_{\alpha\rho/2}$, тогда функция f допускает равномерное приближение на Δ_α целыми функциями порядка ρ и конечного типа.*

Доказательство. Для случая $\alpha = \pi$ теорема уже доказана в [3] Г. Кобером.

Пусть $\omega(\delta)$ - модуль непрерывности функции $f\left(z^{1/\rho}\right)$ и возьмем

$$\varphi(z) = \frac{\rho}{\delta}\int_z^{(z^\rho+\delta)^{1/\rho}} f(\zeta)\,\zeta^{1-\rho}d\zeta.$$

Очевидно, что $\varphi \in A'(\Delta_\alpha)$ и

$$|\varphi(z) - f(z)| \leq \omega(\delta) \quad \text{и} \quad |\varphi'(z)| \leq \rho\frac{\omega(\delta)}{\delta}|z|^{\rho-1}.$$

Тогда, применяя к φ Теорему 2.1, завершаем доказательство Теоремы 2.4. □

2.3. **Введение классов B_α.** Можно вывести некоторые результаты о целом приближении также для функций $A(\Delta_\alpha)$. Нам понадобятся следующие определения.

1) Для $\alpha > 0$ через B_α обозначим класс целых функций g порядка ρ, таких, что $\|g\|_{\Delta_\alpha} < +\infty$.

2) Для числа $\sigma > 0$ через $B_{\alpha,\sigma}$ обозначим подкласс функций $g \in B_\alpha$, где $g(z) \leq \exp\{\sigma|z|^\rho\}$.

Теорема 2.5. *Функция* $f \in A_b(\Delta_\alpha)$ *допускает равномерное приближение на* Δ_α *целыми функциями из класса* B_α *тогда и только тогда, когда* $f\left(z^{1/\rho}\right)$ *равномерно непрерывна на лучах* $\pm l_{\alpha\rho/2}$.

Доказательство. Достаточной частью этой теоремы является Теорема 2.2. □

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ON THE GASCA-MAEZTU CONJECTURE FOR $n = 6$

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Abstract. A two-dimensional n-correct set is a set of nodes admitting unique bivariate interpolation with polynomials of total degree at most n. We are interested in correct sets with the property that all fundamental polynomials are products of linear factors. In 1982, M. Gasca and J. I. Maeztu conjectured that any such set necessarily contains $n+1$ collinear nodes. So far, this had only been confirmed for $n \leq 5$. In this paper, we take a step for proving the case $n = 6$.

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Keywords: Gasca-Maeztu conjecture; fundamental polynomial; algebraic curve; maximal line; maximal curve; n-correct set; n-independent set.

1. Introduction

Denote by Π_n the space of bivariate polynomials of total degree $\leq n$, for which

$$N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).$$

Let $\mathcal{X} := \mathcal{X}_s = \{(x_1, y_1), \ldots, (x_s, y_s)\}$ be a set of s distinct nodes in the plane.

The problem of finding a polynomial $p \in \Pi_n$ satisfying the conditions

$$p(x_i, y_i) = c_i, \qquad i = 1, 2, \ldots s, \tag{1.1}$$

for a data $\bar{c} := \{c_1, \ldots, c_s\}$ is called *interpolation problem.*

Definition 1.1. A set of nodes $\mathcal{X}_s$ is called *n-correct* if for any data $\bar{c}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions (1.1).

A necessary condition of n-correctness is: $\#\mathcal{X}_s = s = N$.

Denote by $p|_{\mathcal{X}}$ the restriction of p on $\mathcal{X}$.

Proposition 1.1. *A set of nodes $\mathcal{X}$ with $\#\mathcal{X} = N$ is n-correct if and only if*

$$p \in \Pi_n, \ p|_{\mathcal{X}} = 0 \implies p = 0.$$

A polynomial $p \in \Pi_n$ is called an *n-fundamental polynomial* for $A \in \mathcal{X}$ if

$$p|_{\mathcal{X} \setminus \{A\}} = 0 \text{ and } p(A) = 1.$$

We denote an n-fundamental polynomial of $A \in \mathcal{X}$ by $p^\star_A = p^\star_{A,\mathcal{X}}$.

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Definition 1.2. A set of nodes $\mathcal{X}$ is called *n-independent* if each node has n-fundamental polynomial. Otherwise, it is *n-dependent.* A set $\mathcal{X}$ is called *essentially n-dependent* if none of its nodes has n-fundamental polynomial.

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence is $\#\mathcal{X}_s = s \leq N$.

One can readily verify that a node set $\mathcal{X}_s$ is n-independent if and only if the interpolation problem (1.1) is *solvable*, i.e., for any data $\{c_1, \ldots, c_s\}$ there is a (possibly not unique) polynomial $p \in \Pi_n$ satisfying (1.1).

A *plane algebraic curve* is the zero set of some bivariate polynomial of degree ≥ 1. To simplify notation, we shall use the same letter, say p, to denote the polynomial p and the curve given by the equation $p(x, y) = 0$. In particular, by ℓ (or α) we denote a linear polynomial from Π_1 and the line defined by the equation $\ell(x, y) = 0$.

Definition 1.3. Let $\mathcal{X}$ be a set of nodes. We say, that a line ℓ is a *k-node line* if it passes through exactly k nodes of $\mathcal{X}$.

The following proposition is well-known (see e.g. [8] Prop. 1.3):

Proposition 1.2. *Suppose that a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ. Then we have that $p = \ell q$, where $q \in \Pi_{n-1}$.*

This implies that at most $n+1$ nodes of an n-independent set can be collinear. An $(n+1)$-node line ℓ is called a *maximal line* (C. de Boor, [1]).

Set

$$d(n, k) := N_n - N_{n-k} = (1/2)k(2n + 3 - k).$$

The following is a generalization of Proposition 1.2.

Proposition 1.3 ([14], Prop. 3.1)**.** *Let q be an algebraic curve of degree $k \leq n$ with no multiple components. Then the following hold:*

(i) any subset of q containing more than $d(n, k)$ nodes is n-dependent;

(ii) any subset $\mathcal{X}$ of q containing exactly $d(n, k)$ nodes is n-independent if and only if

$$p \in \Pi_n \text{ and } p|_{\mathcal{X}} = 0 \implies p = qr, \text{ where } r \in \Pi_{n-k}.$$

Thus at most $d(n, k)$ n-independent nodes lie in a curve q of degree $k \leq n$.

Definition 1.4. Let $\mathcal{X}$ be an n-independent set of nodes with $\#\mathcal{X} \geq d(n, k)$. A curve of degree $k \leq n$ passing through $d(n, k)$ points of $\mathcal{X}$ is called maximal.

The following is a characterization of the maximal curves:

Proposition 1.4 ([14], Prop. 3.3)**.** *Let $\mathcal{X}$ be an n-independent set of nodes with with $\#\mathcal{X} \geq d(n,k)$. Then a curve μ of degree k, $k \leq n$, is a maximal curve if and only if*

$$p \in \Pi_n, \ p|_{\mathcal{X}\cap\mu} = 0 \implies p = \mu s, \ s \in \Pi_{n-k}.$$

One readily gets from here that for a GC_n set $\mathcal{X}$ and $\mu \in \Pi_k$:

$$(1.2) \qquad \mu \text{ is a maximal curve } \iff \mathcal{X} \setminus \mu \text{ is a } GC_{n-k} \text{ set.}$$

In the sequel we will need the following results:

Theorem 1.1 (case i=1: [13], Thm. 4.2; case i=2: [10], Thm. 3)**.** *Let $i = 1$ or 2. Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-i)+i$ nodes with $1+i \leq k \leq n-1$. Then at most $2i$ different curves of degree $\leq k$ pass through all the nodes of $\mathcal{X}$.*

Moreover, there are such $2i$ curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but i lie in a maximal curve of degree $k - i$.

Theorem 1.2 ([11], Thm. 2.5, [7], Thm. 3.2)**.** *Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-2) + 3$ nodes, $3 \leq k \leq n-1$. Then at most 3 linearly independent curves of degree $\leq k$ pass through all the nodes of $\mathcal{X}$.*

Moreover, there are such three curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ lie in a curve of degree $k-1$, or all the nodes of $\mathcal{X}$ but three lie in a (maximal) curve of degree $k-2$.

Below we bring a characterization of n-dependent sets $\mathcal{X}$ with $\#\mathcal{X} \leq 3n$.

Theorem 1.3 ([12], Thm. 5.1)**.** *A set $\mathcal{X}$ consisting of at most $3n$ nodes is n-dependent if and only if one of the following conditions holds.*

(i) *$n+2$ nodes are collinear,*

(ii) *$2n+2$ nodes belong to a (possibly reducible) conic,*

(iii) *$\#\mathcal{X} = 3n$, and there exist $\gamma \in \Pi_3$ and $\sigma \in \Pi_n$ such that $\mathcal{X} = \gamma \cap \sigma$.*

Corollary 1.1. *A set $\mathcal{X}$ consisting of at most $3n-1$ nodes is n-dependent if and only if either $n+2$ nodes are collinear, or $2n+2$ nodes belong to a (possibly reducible) conic.*

Consider special n-correct sets: GC_n sets, defined by Chung and Yao:

Definition 1.5 ([5])**.** An n-correct set $\mathcal{X}$ is called GC_n *set*, if the n-fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.

Now we are in a position to present the Gasca-Maeztu, or briefly GM [[6], 1982] Any GC_n set contains $n+1$ collinear nodes.

So far, the GM conjecture has been confirmed to be true only for $n \leq 5$. The case $n = 2$ is trivial. The case $n = 3$ was established by M. Gasca and J. I. Maeztu in [6]. The case $n = 4$ was proved by J. R. Busch [2]. Other proofs of this case have been published since then (see e.g. [3], [8]). The case $n = 5$ was proved by H. Hakopian, K. Jetter and G. Zimmermann [9]. Recently G. Vardanyan provided a simpler and shorter proof for this case [16].

In this paper we make a step in proving the Gasca-Maeztu conjecture for $n = 6$ (see Proposition 3.8). The analogue of this step was crucial in the proof of the case $n = 5$ (see [9], Prop. 3.12; [16], Prop. 2.8).

Definition 1.6. Let $\mathcal{X}$ be an n-correct set. We say, that a node $A \in \mathcal{X}$ *uses a line* ℓ, if $p_A^\star = \ell q, \quad q \in \Pi_{n-1}$.

Since the fundamental polynomial in an n-correct set is unique we get

Lemma 1.1. *Suppose $\mathcal{X}$ is an n-correct set and a node $A \in \mathcal{X}$ uses a line ℓ. Then ℓ passes through at least two nodes from $\mathcal{X}$, at which q from the above definition does not vanish.*

Definition 1.7. For a given set of lines $\ell_1, \ldots, \ell_k$, we define $\mathcal{N}_{\ell_1,\ldots,\ell_k}$ to be the set of those nodes in $\mathcal{X}$ which do not lie in any of the lines ℓ_i, and for which at least one of the lines is not used.

In the case of one line ℓ we have

$$\mathcal{N}_\ell = \{A \in \mathcal{X} : A \notin \ell, \text{ and } A \text{ is not using } \ell\}.$$

Proposition 1.5 ([8], Thm. 3.2)**.** *Assume that $\mathcal{X}$ is a GC_n set, and $\ell_1, \ldots, \ell_k$ are lines. Then the following hold for $\mathcal{N} = \mathcal{N}_{\ell_1,\ldots,\ell_k}$.*

(i) *If $\mathcal{N}$ is nonempty, then it is essentially $(n-k)$-dependent.*

(ii) *$\mathcal{N} = \emptyset$ if and only if the product $\ell_1 \cdots \ell_k$ is a maximal curve.*

For $k = 1$ this result has been proved by Carnicer and Gasca [3].

Assume that $\mathcal{X}_i$ is a set of k_i collinear points:

$$\mathcal{X}_i \subset \ell_i, \quad \#\mathcal{X} = k_i,\ i = 1, 2, 3, \quad \ell_i \text{ is a line.}$$

Assume also that non of the points is an intersection point of the lines.

Consider the set $\mathcal{L}_{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3}$ of lines containing one point from each of $\mathcal{X}_i$ $i = 1, 2, 3$, and denote by M_{k_1,k_2,k_3} the maximal possible number of such lines.

We shall need the following estimate (see [8], [9])

$$M_{3,3,2} = 5. \tag{1.3}$$

1.1. **The m-distribution sequence of a node.** In this section we bring a number of concepts from [9], Section 2.

Suppose that $\mathcal{X}$ is a GC_n set. Consider a node $A \in \mathcal{X}$ together with the set of n used lines denoted by $\mathcal{L}_A$. The $N-1$ nodes of $\mathcal{X} \setminus \{A\}$ belong to the lines of $\mathcal{L}_A$. Let us order the lines of $\mathcal{L}_A$ in the following way:

The line ℓ_1 is a line in $\mathcal{L}_A$ that passes through maximal number of nodes of $\mathcal{X}$, denoted by $k_1 : \mathcal{X} \cap \ell_1 = k_1$. The line ℓ_2 is a line in $\mathcal{L}_A$ that passes through maximal number of nodes of $\mathcal{X} \setminus \ell_1$, denoted by $k_2 : (\mathcal{X} \setminus \ell_1) \cap \ell_2 = k_2$.

In the general case the line $\ell_s,\ s = 1, \ldots, n$, is a line in $\mathcal{L}_A$ that passes through maximal number of nodes of the set $\mathcal{X} \setminus \cup_{i=1}^{s-1} \ell_i$, denoted by $k_s : (\mathcal{X} \setminus \cup_{i=1}^{s-1} \ell_i) \cap \ell_s = k_s$. A correspondingly ordered line sequence

$$\mathcal{S} = (\ell_1, \ldots, \ell_n)$$

is called a *maximal line sequence* or briefly an *m-line sequence* if the respective sequence $(k_1, \ldots, k_n)$ is the maximal in the lexicographic order [9]. Then the latter sequence is called a *maximal distribution sequence* or briefly an *m-d sequence.* Evidently, for the m-d sequence we have that

$$k_1 \geq k_2 \geq \cdots \geq k_n \text{ and } k_1 + \cdots + k_n = N - 1. \tag{1.4}$$

Though the m-distribution sequence for a node A is unique, it may correspond to several m-line sequences.

An intersection point of several lines of $\mathcal{L}_A$ is counted for the line containing it which appears in $\mathcal{S}$ first. A node in $\mathcal{X}$ is called *primary* for the line it is counted for, and *secondary* for the other lines containing it.
According to Lemma 1.1, a used line contains at least two primary nodes:

$$k_i \geq 2 \quad \text{for } i = 1, \ldots, n\,. \tag{1.5}$$

Let $(\ell_1, \ldots, \ell_k)$ be a line sequence.

Definition 1.8. We say that a polynomial has $(s_1, \ldots, s_k)$ *primary zeroes* in the lines $(\ell_1, \ldots, \ell_k)$ if the counted zeroes are primary nodes in the respective lines.

From Proposition 1.2 we get

Corollary 1.2. *If $p \in \Pi_{m-1}$ has $(m, m-1, \ldots, m-k+1)$ primary zeroes in the lines $(\ell_1, \ldots, \ell_k)$ then we have that $p = \ell_1 \cdots \ell_k r$, where $r \in \Pi_{m-k-1}$.*

In some cases a particular line $\widetilde{\ell}$ used by a node is fixed and then the properties of the other factors of the fundamental polynomial are studied.

In this case in the corresponding m-line sequence, called $\widetilde{\ell}$-*m-line sequence*, one takes as the first line ℓ_1 the line $\widetilde{\ell}$, no matter through how many nodes it passes. Then the second and subsequent lines are chosen, as in the case of the m-line sequence. Thus the line ℓ_2 is a line in $\mathcal{L}_A \setminus \{\widetilde{\ell}_1\}$ that passes through maximal number of nodes of $\mathcal{X} \setminus \widetilde{\ell}_1$, and so on.

Correspondingly the $\widetilde{\ell}$-*m-distribution sequence* is defined.

2. The Gasca-Maeztu conjecture for $n = 6$

Now let us formulate the Gasca-Maeztu conjecture for $n = 6$ as:

Theorem 2.1. *Any GC_6 set contains seven collinear nodes.*

To make a step for the proof assume by way of contradiction:

Assumption. *The set $\mathcal{X}$ is a GC_6 set without a maximal line.*

In view of (1.4) and (1.5) the only possible m-distribution sequences for any node $A \in \mathcal{X}$ in the case $n = 6$ with $N = 28$ are

$$\begin{array}{llllll} \text{(i)} & (6,6,6,4,3,2) & \text{(ii)} & (6,6,5,5,3,2) & \text{(iii)} & (6,6,5,4,4,2) \\ \text{(iv)} & (6,6,5,4,3,3) & \text{(v)} & (6,6,4,4,4,3) & \text{(vi)} & (6,5,5,5,4,2) \\ \text{(vii)} & (6,5,5,5,3,3) & \text{(viii)} & (6,5,5,4,4,3) & \text{(ix)} & (6,5,4,4,4,4) \\ \text{(x)} & (5,5,5,5,5,2) & \text{(xi)} & (5,5,5,5,4,3) & \text{(xii)} & (5,5,5,4,4,4). \end{array}$$

Here we omitted the distribution sequences $(6,6,6,5,2,2)$ and $(6,6,6,3,3,3)$. The reason is that $\ell_1\ell_2\ell_3$ is a maximal cubic with 18 $(= 6+6+6)$ nodes and, in view of (1.2), three 6 must be followed by $4,3,2$, as in above (i).

3. Lines used several times

A 2-node line shared. Consider a 2-node line $\widetilde{\ell}$. For the $\widetilde{\ell}$-m-distribution sequence of a node $A \notin \widetilde{\ell}$ there are only the following five possibilities:

$$\begin{array}{llllll} \text{(i)} & (\widetilde{2},6,6,6,4,3) & \text{(ii)} & (\widetilde{2},6,6,5,5,3) & \text{(iii)} & (\widetilde{2},6,6,5,4,4) \\ \text{(vi)} & (\widetilde{2},6,5,5,5,4) & \text{(x)} & (\widetilde{2},5,5,5,5,5). & & \end{array} \tag{3.1}$$

Note that in $\widetilde{\ell}$-m-d sequences, we use the tilde to indicate the place of $\widetilde{\ell}$.

It was proved in [4], Prop. 4.2, that any 2-node line in a GC_n set $\mathcal{X}$ can be used at most by one node from $\mathcal{X}$. This yields the following

Proposition 3.1. *Assume that $\mathcal{X}$ is a GC_6-set, and suppose that $\widetilde{\ell}$ is a 2-node line. Then $\widetilde{\ell}$ can be used by at most one node $A \in \mathcal{X}$. The m-d sequence of A has to be one of (i), (ii), (iii), (vi), and (x), presented in* (3.1).

A 3*-node line shared.* Then, consider a 3-node line $\widetilde{\ell}$. For the $\widetilde{\ell}$-m-d sequence of a node $A \notin \widetilde{\ell}$ there are only the following possibilities:

(i)	$(\widetilde{3},6,6,6,4,2)$	(ii)	$(\widetilde{3},6,6,5,5,2)$	(iv)	$(\widetilde{3},6,6,5,4,3)$
(v)	$(\widetilde{3},6,6,4,4,4)$	(vii)	$(\widetilde{3},6,5,5,5,3)$	(viii)	$(\widetilde{3},6,5,5,4,4)$
(xi)	$(\widetilde{3},5,5,5,5,4)$.				

Here, and in all subsequent cases, denote a respective $\widetilde{\ell}$-m-line sequence by $(\widetilde{\ell}, \ell_2, \ldots, \ell_6)$. Denote also by ℓ_{AB} the line through the nodes A and B.

Suppose that the line $\widetilde{\ell}$ is used by two nodes $A,\ B \in \mathcal{X}$:

$$p_A^\star = \widetilde{\ell}\, q_1 \quad \text{and} \quad p_B^\star = \widetilde{\ell}\, q_2\,, \qquad q_i \in \Pi_5\,.$$

Then we have that the curves: $q_1, q_2 \in \Pi_5$, pass through 6-independent nodes of the set $\mathcal{Y} := \mathcal{X} \setminus (\widetilde{\ell} \cup \{A, B\}),\ \#\mathcal{Y} = 28 - (3+2) = 23$.

Note that $23 = d(6, 5-1) + 1 = d(6,4) + 1 = 7 + 6 + 5 + 4 + 1$.

Therefore, in view of Theorem 1.1, case i=1, we get that all the nodes of $\mathcal{Y}$ but one, denoted by C, belong to a maximal curve μ of degree 4. Note that $p_C^\star = \widetilde{\ell}\mu_4\ell_{AB}$, meaning that the node C uses $\widetilde{\ell}$ too.

Since $\mathcal{X}$ is a GC set we conclude that μ has 4 line-components coinciding with $\ell_2, \ldots, \ell_5$. It is easily seen that these four lines have 6, 6, 6, 4 or 6, 6, 5, 5, nodes, respectively. For $D = A, B, C$, we have that

$$p_D^\star = \widetilde{\ell}\,\ell_2 \cdots \ell_6, \tag{3.2}$$

where ℓ_6 is a line depending on D with two primary nodes.

Thus the $\widetilde{\ell}$-m-d sequence indicated in (3.2) may correspond only to the m-d sequences (i) $(6,6,6,4,3,2)$ and (ii) $(6,6,5,5,3,2)$.

Note that all the 6 nodes in $\mathcal{X} \setminus \mu$, included C, share the 4 line-components of μ. As it is proved in [13], Corollary 6.1, no node in μ uses the line $\widetilde{\ell}$.

Thus we have shown the following:

Proposition 3.2. *Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and suppose that a* 3*-node line $\widetilde{\ell}$ is used by two nodes $A,\ B \in \mathcal{X}$. Then there exists a third node C using $\widetilde{\ell}$ and $\widetilde{\ell}$ is used by exactly three nodes of $\mathcal{X}$.*

Moreover, A, B, and C, share four other lines with either $6,6,6,4$, *or* $6,6,5,5$, *primary nodes, respectively. Furthermore, the m-d sequence of these three nodes is either* $(6,6,6,4,\widetilde{3},2)$, *or* $(6,6,5,5,\widetilde{3},2)$, *respectively.*

A 4*-node line shared.* Now, consider a 4-node line $\widetilde{\ell}$. $\mathcal{X}$. The $\widetilde{\ell}$-m-d sequence of $A \notin \widetilde{\ell}$ has to be one of the following:

(i)	$(\widetilde{4}, 6, 6, 6, 3, 2)$	(iii)	$(\widetilde{4}, 6, 6, 5, 4, 2)$	(iv)	$(\widetilde{4}, 6, 6, 5, 3, 3)$
(v)	$(\widetilde{4}, 6, 6, 4, 4, 3)$	(vi)	$(\widetilde{4}, 6, 5, 5, 5, 2)$	(viii)	$(\widetilde{4}, 6, 5, 5, 4, 3)$
(ix)	$(\widetilde{4}, 6, 5, 4, 4, 4)$	(xi)	$(\widetilde{4}, 5, 5, 5, 5, 3)$	(xii)	$(\widetilde{4}, 5, 5, 5, 4, 4)$.

Suppose that the line $\widetilde{\ell}$ is used by the nodes $A, B, C \in \mathcal{X}$: Then, as in the previous case, we get three curves of degree 5 passing through $21 = 28-(4+3)$ 6-independent nodes of the set $\mathcal{Y} := \mathcal{X} \setminus (\widetilde{\ell} \cup \{A, B, C\})$.

Note that $21 = d(6, 5-2) + 3 = d(6, 3) + 3 = 7 + 6 + 5 + 3$.

This, in view of Theorem 1.2, implies that either

(a) all the nodes of $\mathcal{Y}$ but three, i.e., 18 nodes, belong to a maximal curve μ of degree 3, or

(b) all the nodes of $\mathcal{Y}$, i.e., 21 nodes, belong to a curve q of degree 4.

Since any node outside of μ uses it we get that μ has 3 line-components, passing through $6+6+6$ nodes, respectively.

Concerning (b) note that $\widetilde{\ell}q$ is a maximal curve of degree 4 and any node $D = A, B, C$, uses q :

$$p_D^\star = \widetilde{\ell} q \ell_6,$$

where ℓ_6 is a line depending on D with two primary nodes.

Hence q has 4 line-components. It is easily seen that these four lines have either $6+6+6+3$, $6+6+5+4$, or $6+5+5+5$ nodes, correspondingly. We readily get also that these lines coincide with the lines $\ell_2, \ldots, \ell_5$, of the corresponding $\widetilde{\ell}$-m-distribution (3.2). Hence, these three cases may correspond only to the above cases (i) and (iii) and (vi).

Now suppose that except of A, B, C, another node $D \in \mathcal{X}$ uses $\widetilde{\ell}$. Then we have four curves of degree 5 passing through $20 = 28-(4+4)$ 6-independent nodes. We have that $20 = d(6, 5-2) + 2 = d(6, 3) + 2 = 7 + 6 + 5 + 2$.

Therefore, in view of Theorem 1.1, case i=2, we obtain that all the nodes of $\mathcal{X} \setminus \{A, B, C, D\}$ but two, i.e., 18 nodes belong to a maximal curve μ of degree 3. As was stated above this maximal curve has 3 line-components with $6+6+6$ nodes, correspondingly. We readily get also that these lines coincide with the lines ℓ_2, ℓ_3, ℓ_4. Consequently, this case may correspond only to the above case (i). As it is proved in [10], Corollary, no node in μ uses the line $\widetilde{\ell}$.

By summarizing we obtain the following

Proposition 3.3. *Assume that $\mathcal{X}$ is a GC_6-set without a maximal line, and suppose that a* 4*-node line $\widetilde{\ell}$ is used by three nodes A, B, $C \in \mathcal{X}$. Then, A, B, and C,* *besides* $\widetilde{\ell}$*, share four lines with either* $6,6,6,3;\ \ 6,6,5,4;$ *or* $6,5,5,5,$ *primary nodes, respectively.*

Moreover the m-d sequence for $A,\ B,\ C,$ *is* $(6,6,6,4,\widetilde{3},2),\ (6,6,5,5,\widetilde{3},2),$ $(6,6,5,4,\widetilde{4},2),$ *or* $(6,5,5,5,\widetilde{4},2).$

Proposition 3.4. *Assume that* $\mathcal{X}$ *is a* GC_6*-set without a maximal line, and suppose that some* 4*-node line* $\widetilde{\ell}$ *is used by four nodes* $A,\ B,\ C,\ D \in \mathcal{X}$*. Then,* $\widetilde{\ell}$ *is used by exactly* 6 *nodes.*

Moreover, besides $\widetilde{\ell}$*, these six nodes share also three other lines each passing through* 6 *primary nodes. Furthermore the m-d sequence for all six nodes is* $(6,6,6,\widetilde{4},3,2)$.

A 5*-node line shared.* Now suppose that $\widetilde{\ell}$ is a 5-node line. The $\widetilde{\ell}$-m-d sequence of $A \notin \widetilde{\ell}$ has to be one of the following:

(ii) $(\widetilde{5},6,6,5,3,2)$ (iii) $(\widetilde{5},6,6,4,4,2)$ (iv) $(\widetilde{5},6,6,4,3,3)$

(vi) $(\widetilde{5},6,5,5,4,2)$ (vii) $(\widetilde{5},6,5,5,3,3)$ (viii) $(\widetilde{5},6,5,4,4,3)$

(ix) $(\widetilde{5},6,4,4,4,4)$ (x) $(\widetilde{5},5,5,5,5,2)$ (xi) $(\widetilde{5},5,5,5,4,3)$

(xii) $(\widetilde{5},5,5,4,4,4)$.

Let us start with a well-known

Lemma 3.1. *Given* m *linearly independent polynomials. Then for any point* A *there are* $m-1$ *linearly independent polynomials, in their linear span, vanishing at* A.

Proposition 3.5. *Assume that* $\mathcal{X}$ *is a* GC_6*-set without a maximal line, and* $\widetilde{\ell}$ *is a* 5*-node line used by five nodes of* $\mathcal{X}$*. Then it is used by exactly six nodes.*

Moreover, besides $\widetilde{\ell}$*, these six nodes share also three other lines passing through* $6,6,5$ *primary nodes, respectively. Furthermore the m-d sequence for each of the six nodes is* $(6,6,6,4,3,2),$ *or* $(6,6,5,5,3,2)$.

Proof. Assume that the nodes of the set $\mathcal{A}_5 := \{A_1,\ldots,A_5\} \subset \mathcal{X}$ use the line $\widetilde{\ell}$. Assume that

$$p^{\star}_{A_1} = \widetilde{\ell}\ell_2\cdots\ell_6.$$

Evidently, the nodes $A_2,\ldots,A_5$ belong to the lines $\ell_2,\ldots,\ell_6$.

In view of Lemma 3.1 for any points $T_i,\ i=1,2,3$, there is a polynomial

$$p_0 \in \mathcal{P}_4 := linearspan\{p^{\star}_{A_2},\ldots,p^{\star}_{A_5}\}, \quad p_0 \neq 0,$$

such that $p_0(T_i)=0,\ i=1,\ldots,3$. On the other hand we have that

$$p_0 = \widetilde{\ell}q_0, \quad q_0 \in \Pi_5.$$

Assume that the three points are not intersection points of the six lines. They also are taken outside of $\widetilde{\ell}$, whence $q_0(T_i) = 0,\ i = 1, 2, 3.$

Consider the set of nodes

$$\mathcal{C} := \mathcal{X} \setminus \left(\widetilde{\ell} \cup \mathcal{A}_5\right), \quad |\mathcal{C}| = 28 - 5 - 5 = 18.$$

The following cases of distribution of these 18 nodes in the lines $\ell_2, \ldots, \ell_6$ in some order are possible:

(1) $(6,6,6,0,0)$; (2) $(6,6,5,1,0)$; (3) $(6,6,4,2,0)$; (4) $(6,6,4,1,1)$;
(5) $(6,6,3,3,0)$; (6) $(6,6,3,2,1)$; (7) $(6,6,2,2,2)$; (8) $(6,5,5,2,0)$;
(9) $(6,5,5,1,1)$; (10) $(6,5,4,3,0)$; (11) $(6,5,4,2,1)$; (12) $(6,5,3,3,1)$;
(13) $(6,5,3,2,2)$; (14) $(6,4,4,4,0)$; (15) $(6,4,4,3,1)$; (16) $(6,4,4,2,2)$;
(17) $(6,4,3,3,2)$; (18) $(6,3,3,3,3)$; (19) $(5,5,5,3,0)$; (20) $(5,5,5,2,1)$;
(21) $(5,5,4,4,0)$; (22) $(5,5,4,3,1)$; (23) $(5,5,4,2,2)$; (24) $(5,5,3,3,2)$;
(25) $(5,4,4,4,1)$; (26) $(5,4,4,3,2)$; (27) $(5,4,3,3,3)$; (28) $(4,4,4,4,2)$;

(29) $(4,4,4,3,3)$.

We assume for the convenience that the lines are in the increasing order.

We may assume also that in each above distribution the listed zeros are primary in the respective lines. Indeed, by reordering the lines and making the zeros primary we will get another distribution listed above.

Now one can verify readily that the cases (3)-(29) are not possible, since by adding three arbitrary points $T_i,\ i = 1, 2, 3$, we make the polynomial q_0 to have at least $(6, 5, 4, 3, 2)$ primary zeroes in the lines $\ell_2, \ldots, \ell_6$.

For example, for several particular cases below, we add the three points to the lines $\ell_2, \ldots, \ell_6$, according to the following distributions:
(3) $(0,0,0,1,2)$; (14) $(0,1,0,0,2)$; (25) $(1,1,0,0,1)$; (29) $(2,1,0,0,0)$.

This implies that $q_0 = \ell_2 \cdots \ell_6$ hence $p_0 = \widetilde{\ell}\ell_2 \cdots \ell_6 = p^\star_{A_1}$. Therefore we get $p^\star_{A_1} \in \mathcal{P}_4$, which is a contradiction.

Then note that also the case (1) is not possible since the curve $\widetilde{\ell}\ell_2\ell_3\ell_4 \in \Pi_4$ contains $23 = 5+6+6+6$ nodes, while a maximal quartic contains $22 = 7+6+5+4$ nodes. Thus the only possible case is the distribution (2).

Evidently, the curve $\mu_4 := \widetilde{\ell}\ell_2 \cdots \ell_4$ here is a maximal curve. Hence the node in the line ℓ_5 together with the five nodes of $\mathcal{A}_5$, use the lines $\ell_2, \ldots, \ell_4$. Thus the six nodes besides $\widetilde{\ell}$, share also the three lines ℓ_2, ℓ_3, ℓ_4, passing through $6, 6,$ and 5 primary nodes.

Thus the distribution (2) may correspond only to the following m-d sequences: $(6, 6, 6, \widetilde{4}, 3, 2)$ and $(6, 6, \widetilde{5}, 5, 3, 2)$. □

A 6*-node line shared.* Finally suppose that $\widetilde{\ell}$ is a 6-node line. For the $\widetilde{\ell}$-m-d sequence of a node $A \notin \widetilde{\ell}$ there are only the following possibilities:

$$\begin{array}{llllll} \text{(i)} & (\widetilde{6},6,6,4,3,2) & \text{(ii)} & (\widetilde{6},6,5,5,3,2) & \text{(iii)} & (\widetilde{6},6,5,4,4,2) \\ \text{(iv)} & (\widetilde{6},6,5,4,3,3) & \text{(v)} & (\widetilde{6},6,4,4,4,3) & \text{(vi)} & (\widetilde{6},5,5,5,4,2) \\ \text{(vii)} & (\widetilde{6},5,5,5,3,3) & \text{(viii)} & (\widetilde{6},5,5,4,4,3) & \text{(ix)} & (\widetilde{6},5,4,4,4,4). \end{array}$$

Proposition 3.6. *Assume that $\mathcal{X}$ is a GC_6 set without a maximal line, and $\widetilde{\ell}$ is a* 6*-node line. Assume also that $\widetilde{\ell}$ is used by eight nodes of $\mathcal{X}$. Then it is used by exactly ten nodes of $\mathcal{X}$.*

Moreover, these ten nodes form a GC_3 set and share two more lines with six primary nodes each. Furthermore, each of these ten nodes has the m-d sequence $(6,6,6,4,3,2)$.

Proof. Since $\widetilde{\ell}$ is used by at least eight nodes, we have that $\#\mathcal{N}_{\widetilde{\ell}} \le 28-(6+8) = 14$. By Proposition 1.5 the set $\mathcal{N}_{\widetilde{\ell}}$ is 5-dependent. Since $14 = 3 \times 5 - 1$, one may apply Corollary 1.1 to conclude that either $\mathcal{N}_{\widetilde{\ell}}$ contains $5+2=7$ collinear nodes, which contradicts the hypothesis, or 12 $(= 2 \cdot 5 + 2)$ nodes there are in a conic β. Thus the latter case takes place and $\#\mathcal{N}_{\widetilde{\ell}} \ge 12$.

Now note that $\mathcal{N}_{\widetilde{\ell}} \subset \beta$. Indeed, we may have one or two nodes in $\mathcal{N}_{\widetilde{\ell}}$ outside of β. But in this case those nodes evidently have fundamental polynomial of degree 3, for the set $\mathcal{N}_{\widetilde{\ell}}$, contradicting Proposition 1.5, (i).

Then let us show that $\#\mathcal{N}_{\widetilde{\ell}} = 12$. Assume by way of contradiction that there are ≥ 13 nodes in $\mathcal{N}_{\widetilde{\ell}}$. Then there are at most 9 nodes outside of $\beta \cup \widetilde{\ell}$ and therefore they are contained in a cubic γ. Then we readily get that $\mathcal{X} \subset \widetilde{\ell}\beta\gamma \in \Pi_6$, which contradicts Proposition 1.1.

Finally note that $\widetilde{\ell}\beta$ contains 18 nodes, i.e., is a maximal cubic. Therefore, by Proposition 1.4, it is used by all the 10 nodes in $\mathcal{X} \setminus (\widetilde{\ell} \cup \beta)$, and hence β has to be the product of two 6-node lines. □

Proposition 3.7. *Assume that $\mathcal{X}$ is a GC_6 set without a maximal line, and $\widetilde{\ell}_i,\ i = 1,2$, are two disjoint* 6*-node lines. Assume also that six nodes of $\mathcal{X}$ are using $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$. Then, the six nodes besides $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ share either one more line with* 6 *primary nodes or two more lines each with* 5 *primary nodes. In the first case the lines $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ are used by exactly ten nodes of $\mathcal{X}$ and in the second case they are used by exactly six nodes of $\mathcal{X}$.*

Moreover, in the first and second cases the ten and six nodes form a GC_3 and GC_2 sets, respectively. Furthermore, each of the ten nodes and each of the six nodes has the m-d sequence $(6,6,6,4,3,2)$, *and* $(6,6,5,5,3,2)$, *respectively.*

Proof. We have that $\#\mathcal{N}_{\widetilde{\ell}_1,\widetilde{\ell}_2} \le 28-(6+6+6)=10$. By Proposition 1.5, the set $\mathcal{N}_{\widetilde{\ell}_1,\widetilde{\ell}_2}$ is 4-dependent. Since $10 = 3\times 4-2 = 2\times 4+2$, we can apply Corollary 1.1 and conclude that either $\mathcal{N}_{\widetilde{\ell}_1,\widetilde{\ell}_2}$ contains $4+2=6$ nodes lying in a line $\widetilde{\ell}_3$, or all the ten nodes are lying in a conic β.

In the first case we readily conclude that $\widetilde{\ell}_1\widetilde{\ell}_2\widetilde{\ell}_3$ is a maximal cubic with 18 nodes and hence the remaining ten nodes of $\mathcal{X}$ are using it.

In the second case we readily conclude that $\beta\widetilde{\ell}_1\widetilde{\ell}_2$ is a maximal quartic with 22 nodes and hence the remaining six nodes of $\mathcal{X}$ are using it. Hence the conic β reduces to two lines with 5 primary nodes.

It remains to mention that if a seventh node uses the lines $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ then we get $\#\mathcal{N}_{\widetilde{\ell}_1,\widetilde{\ell}_2} \le 28-(6+6+7)=9=2\times 4+1$ which readily reduces to the first case. □

The following table is an analog of one in [9]. It is obtained from Propositions 3.1 - 3.6, and shows how many times at most a line $\widetilde{\ell}$, under certain restrictions, can be used, provided that the GC_6-set has no maximal line.

(3.3)

$---$total # of nodes on $\widetilde{\ell}$	maximal # of nodes using $\widetilde{\ell}$ in general	no node uses $(6,6,6,4,3,2)$ constellation	no node uses $(6,6,6,4,3,2)$, $(6,6,5,5,3,2)$
6	10	7	7
5	6	6	4
4	6	3	3
3	3	3	1
2	1	1	1

3.1. **The main result.** In this paper we will prove the following

Proposition 3.8. *Assume that $\mathcal{X}$ is a GC_6 set with no maximal line. Then for no node in $\mathcal{X}$ the m-d sequence is* $(6,6,6,4,3,2)$.

Assume by way of contradiction that for a node in $\mathcal{X}$ the m-d sequence is $(6,6,6,4,3,2)$. Let $(\alpha_1,\ldots,\alpha_6)$ be a respective m-line sequence.

Set $\mathcal{X}=\mathcal{A}\cup\mathcal{B}$ (see Fig. 3.1) with

$$\mathcal{A}=\mathcal{X}\cap\{\alpha_1\cup\alpha_2\cup\alpha_3\},\quad \#\mathcal{A}=18,\quad \text{and}\quad \mathcal{B}=\mathcal{X}\setminus\mathcal{A},\quad \#\mathcal{B}=10.$$

Denote $\mathcal{L}_3 := \{\alpha_1,\alpha_2,\alpha_3\}$. Note that no intersection point of the three lines of $\mathcal{L}_3$ belongs to $\mathcal{X}$. The following is the analogue of [9], Lemma 3.2.

Lemma 3.2.

(i) *The set $\mathcal{B}$ is a GC_3 set, and each node $B\in\mathcal{B}$ uses the three lines of $\mathcal{L}_3$ and the three lines it uses within $\mathcal{B}$, i.e.,*

$$p^\star_{B,\mathcal{X}} = \alpha_1\alpha_2\alpha_3 p^\star_{B,\mathcal{B}}\,. \tag{3.4}$$

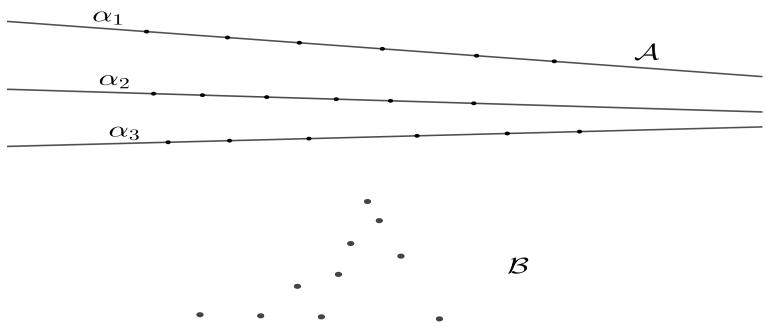

Рис. 3.1. The case $(6, 6, 6, 4, 3, 2)$ with $\mathcal{X} = \mathcal{A} \cup \mathcal{B}$.

(ii) *No node in* $\mathcal{A}$ *uses any of the lines of* $\mathcal{L}_3$.

Proof. (i) Suppose by way of contradiction that the set $\mathcal{B}$ is not 3-correct, i.e., it is a subset of a cubic γ_0. Then $\mathcal{X}$ is a subset of the zero set of the polynomial $\alpha_1\alpha_2\alpha_3\gamma_0 \in \Pi_6$, which contradicts Proposition 1.1.

Now, we readily obtain the formula (3.4).

(ii) Without loss of generality assume that $A \in \alpha_1$ uses the line α_2. Then $p_A^\star = \alpha_2\, q$, where $q \in \Pi_5$. It is easily seen that q has (6,5) primary zeros in the lines (α_3, α_1). Therefore, in view of Corollary 1.2, we obtain that $p_A^\star = \alpha_1\alpha_2\alpha_3 r,\ r \in \Pi_3$, which is a contradiction. □

Lemma 3.3. *No node from* $\mathcal{A}$ *can have the m-d sequence* $(6, 6, 6, 4, 3, 2)$.

Proof. Assume conversely that $A \in \mathcal{A}$ has the m-d sequence $(6, 6, 6, 4, 3, 2)$. Denote a respective m-line sequence by $(\alpha'_1, \ldots, \alpha'_6)$. The lines here, according to Lemma 3.2, (ii), are different from $\alpha_1, \alpha_2, \alpha_3$.

Denote $\mathcal{A}' = \mathcal{X} \cap \{\alpha'_1 \cup \alpha'_2 \cup \alpha'_3\}$. The three lines $\alpha'_1, \alpha'_2, \alpha'_3$ contain at least $9 = 3 + 3 + 3$ nodes outside of $\gamma := \alpha_1 \cup \alpha_2 \cup \alpha_3$. The fourth line ℓ'_4 contains at least $1 = 4 - 3$ node outside of γ denoted by C. Since $\#\mathcal{B} = 10$ we conclude that these four lines have exactly 10 nodes in $\mathcal{B}$ and $12 = 4 \times 3$ nodes in $\mathcal{A}$. Therefore we obtain that $\mathcal{B} \subset \alpha'_1 \cup \cdots \cup \alpha'_4$, and $C \in \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{A}')$.

This, in view of Lemma 3.2, (i), implies that $p_C^\star = \alpha_1\alpha_2\alpha_3\alpha'_1\alpha'_2\alpha'_3$.

From here we readily conclude that the node C uses six lines none of which is a maximal line within $\mathcal{B}$. Indeed, we have that $\alpha_i \cap \mathcal{B} = \emptyset,\ i = 1, 2, 3$, and $|\alpha'_i \cap \mathcal{B}| = 3,\ i = 1, 2, 3$. This contradicts Lemma 3.2, (i). □

Definition 3.1. We say, that a line ℓ is a $k_{\mathcal{A}}$-*node* line if it passes through exactly k nodes of $\mathcal{A}$, $k = 0, 1, 2, 3$.

Lemma 3.4. *(i) Assume that a line $\widetilde{\ell} \notin \mathcal{L}_3$ does not intersect a line $\alpha \in \mathcal{L}_3$ at a node in $\mathcal{X}$. Then the line $\widetilde{\ell}$ can be used by atmost 1 node from $\mathcal{A}$. Moreover, this latter node can belong only to α.*

(ii) If ℓ is $0_{\mathcal{A}}$ or $1_{\mathcal{A}}$-node line then no node from $\mathcal{A}$ uses it.

(iii) If ℓ is $2_{\mathcal{A}}$-node line then it can be used by atmost one node from $\mathcal{A}$.

Proof. (i) Without loss of generality assume that $\alpha = \alpha_1$ and $A \in \alpha_2$ uses $\widetilde{\ell}$: $p_A^\star = \widetilde{\ell} q,\ q \in \Pi_4$. It is easily seen that q has $(6,5,4)$ primary zeros in the lines $(\alpha_1, \alpha_3, \alpha_2)$. Therefore, in view of Corollary 1.2, we conclude that $p_A^\star = \widetilde{\ell}\, \alpha_1\, \alpha_2\, \alpha_3\, r,\ r \in \Pi_1$, which is a contradiction.

Now assume conversely that $A, B \in \alpha_1 \cap \mathcal{X}$ use the line $\widetilde{\ell}$. Choose a point $C \in \alpha_2 \setminus (\widetilde{\ell} \cup \mathcal{X})$. Then, in view of Lemma 3.1, choose numbers a and b, with $|a| + |b| \neq 0$, such that $p(C) = 0$, where $p := ap_A^\star + bp_B^\star$. It is easily seen that $p = \widetilde{\ell} q,\ q \in \Pi_5$ and the polynomial q has $(6,5,4)$ primary zeros in the lines $(\alpha_2, \alpha_3, \alpha_1)$. Therefore $p = \widetilde{\ell} \alpha_1\, \alpha_2\, \alpha_3\, q$, where $q \in \Pi_2$. Thus $p(A) = p(B) = 0$, implying that $a = b = 0$, which is a contradiction.

The items (ii) and (iii) readily follow from (i). □

A node is called an *i_m-node*, $i \leq 2$, if it lies in exactly i maximal lines.

Lemma 3.5. *Let $\widetilde{\ell}$ be a $3_{\mathcal{A}}$ type line passing through a 2_m-node $B \in \mathcal{B}$. Assume also that the node set $\mathcal{B} \setminus \{\widetilde{\ell}\}$ contains 4 collinear nodes. Then the line $\widetilde{\ell}$ can be used by at most three nodes from $\mathcal{A}$.*

Proof. Assume by way of contradiction that the line $\widetilde{\ell}$ is used by four nodes from a set $\mathcal{A}_4 := \{A_1, \ldots, A_4\} \subset \mathcal{A}$. For any chosen points $T_i,\ i = 1,2,3$, (see the proof of Proposition 3.5 for the details) there is a polynomial

$$p_0 \in \mathcal{X}_4 := linearspan\{p_{A_1}^\star, \ldots, p_{A_4}^\star\}, \quad p_0 \neq 0, \tag{3.5}$$

such that $p_0(T_i) = 0,\ i = 1,2,3$. On the other hand we have that

$$p_0 = \widetilde{\ell} q_0, \quad q_0 \in \Pi_5, \text{ and } q_0(T_i) = 0,\ i = 1,2,3.$$

Now consider the set of nodes

$$\mathcal{C} := \mathcal{A} \setminus \left(\widetilde{\ell} \cup \mathcal{A}_4\right), \quad |\mathcal{C}| = 18 - 3 - 4 = 11.$$

Denote by ℓ^* the line passing through the four collinear nodes of $\mathcal{B} \setminus \{\widetilde{\ell}\}$.

The following cases of distribution of above 11 nodes in the three lines of $\mathcal{L}$ are possible:

(1) $(5,5,1)$; (2) $(5,4,2)$; (3) $(5,3,3)$; (4) $(4,4,3)$.

One can verify that the cases (1)-(4) are not possible in the following way. By locating conveniently the three points $T_i,\ i = 1,2,3$, we make the polynomial q_0

to have at least $(6,5,4,3)$ primary zeroes in the lines $\alpha_1,\alpha_2,\alpha_3,\ell^*$, in some order. Thus we get $q_0=\alpha_1\alpha_2\alpha_3\beta,\ \beta\in\Pi_2$, implying $p_0=\alpha_1\alpha_2\alpha_3\gamma,\ \gamma\in\Pi_3$. Hence, in view of Lemma 3.2, (i), we readily get that $p_0\in linearspan\{p^\star_{B,\mathcal{X}}:B\in\mathcal{B}\}$, which contradicts (3.5).

To implement the above described verification in details suppose that

$$|\ell^*\cap\mathcal{C}|=k,\ \ k\le 2.$$

Case 1: $k=0$. In the case of distribution (1), $(5,5,1)$, we add the three points in the form $(1,0,2)$, meaning that we add a point to the line α_1 and the remaining two points to the line α_3. In the case of distributions (2)-(4) we add the three points in the form $(1,1,1),\ (1,2,0),\ (2,1,0)$, respectively.

Then note that the polynomial q_0 has at least $(6,5,4,3)$ primary zeroes in the lines $\alpha_1,\alpha_2,\ell^*,\alpha_3$, in the indicated order.

Case 2: $k=1$. In this case a node denoted by A^* in $\mathcal{C}$ belongs to the line ℓ^*. The following are the cases of distribution of remaining 10 nodes of $\mathcal{C}$ in the lines of $\mathcal{L}$:

$(1')\ (5,5,0)$; $(2')\ (5,4,1)$; $(3')\ (5,3,2)$; $(4')\ (4,4,2)$; $(5')\ (4,3,3)$.

Consider the distribution sequence $(1'),\ (5,5,0)$. In this case we have that $\mathcal{A}_4\cup\{A^*\}\subset\alpha_3$. Note that this is the only case when instead of ℓ^* we use the two maximal lines passing through $B\in\mathcal{B}$, denoted by ℓ_1^{**} and ℓ_2^{**}. Each of these lines passes through 3 nodes in $\mathcal{B}\setminus\widetilde{\ell}$. Note that these lines do not pass through A^* since they intersect ℓ^* at $B\in\mathcal{B}$.

Thus in case $(1')$ we add a point to the line ℓ_1^{**}. Then we add the remaining two points to the lines of $\mathcal{L}$ in the form $(1,0,1)$. Now note that the polynomial q_0 has at least $(6,5,4,3,2)$ zeroes in the following ordered lines: $\alpha_1,\alpha_2,\ell_1^{**},\ell_2^{**},\alpha_3$, counting also $A^*\in\alpha_3$.

In the remaining cases $(2')-(5')$ we add a point to the line ℓ^* to have there 6 zeroes and use it as the first line in the ordered line sequence. Then we add the remaining two points to the lines of $\mathcal{L}$ in the form $(0,0,2),(0,1,1),\ (1,0,1)$ $(1,1,0)$, respectively.

Finally notice that the polynomial q_0 has at least $(6,5,4,3)$ zeroes in the ordered lines: $\ell^*,\alpha_1,\alpha_2,\alpha_3$.

Case 3: $k=2$. In this case two nodes of $\mathcal{A}$ belong to the line ℓ^*. The following are the cases of distribution of the remaining 9 nodes of $\mathcal{C}$ in the three lines of $\mathcal{L}$: $(1'')(5,4,0)$; $(2'')(5,3,1)$; $(3'')(5,2,2)$; $(4'')(4,4,1)$; $(5'')(4,3,2)$; $(6'')(3,3,3)$.

In these cases the line ℓ^* has 6 zeroes and is the first line in the ordered sequence of lines. For the distributions $(1'') - (6'')$ we add the three points in the form $(0,0,3),(0,1,2),\ (0,2,1)\ (1,0,2),\ (2,1,0)$, respectively.

Then note that as above the polynomial q_0 has at least $(6,5,4,3)$ zeroes in the following ordered lines: $\ell^*, \alpha_1, \alpha_2, \alpha_3$. □

3.2. **The proof of the main result.** Consider all the lines passing through a node $B \in \mathcal{B}$ and at least one more node of $\mathcal{X}$. Denote the set of these lines by $\mathcal{L}(B)$. Let $m_k := m_k(B),\ k = 1,2,3$, be the number of $k_{\mathcal{A}}$-node lines from $\mathcal{L}(B)$. Then the following holds:

$$1m_1(B) + 2m_2(B) + 3m_3(B) = \#\mathcal{A} = 18. \tag{3.6}$$

Lemma 3.6. *We have that $m_3(B) \le 5$.*

Proof. The relation (3.6) implies that $m_3(B) \le 6$. Assume by way of contradiction that six lines pass through B and three nodes in $\mathcal{A}$. Therefore these six lines intersect the three lines $\alpha_1, \alpha_2, \alpha_3$, at all the 18 nodes of $\mathcal{A}$.
Note that $\alpha_1\alpha_2\alpha_3$ is a maximal cubic. Hence, by Proposition 1.4, the six lines contain as components the lines $\alpha_1, \alpha_2, \alpha_3$, which is a contradiction. □ *The proof of Proposition 3.8.* We will prove Proposition in three steps. Recall that the set $\mathcal{B}$ is a GC_3 set.

Step 1. The set $\mathcal{B}$ is a Chung-Yao set (with 5 maximal lines, Fig. 3.1).

Let us fix as a node $B \in \mathcal{B}$. Note that all nodes in this case are 2_m-nodes. According to Lemma 3.5 any $3_{\mathcal{A}}$ type line $\widetilde{\ell}$ here is used by at most 3 nodes of $\mathcal{A}$. Indeed, $\widetilde{\ell}$ passes through at most two nodes of $\mathcal{B}$. Thus it intersects at most $4 = 2 \times 2$ maximal lines of $\mathcal{B}$ and the four nodes of the fifth maximal line of $\mathcal{B}$ are outside of $\mathcal{B} \setminus \widetilde{\ell}$ (see Fig. 3.1).

Therefore, in view of Lemma 3.4, the number of usages of the lines $\widetilde{\ell}$ through B with the nodes from $\mathcal{A}$ equals at most:

$$m_2(B) + 3m_3(B) \ge 18 = m_1(B) + 2m_2(B) + 3m_3(B).$$

Hence $m_1 = m_2 = 0$ and $m_3 = 6$, which contradicts Lemma 3.6.

Step 2. $\mathcal{B}$ is a Carnicer-Gasca set (with 4 maximal lines, Fig. 3.2).

We have that there are at most four 3-node lines in $\mathcal{B}$ ([15], Prop. 5). Moreover, any 3-node line passes through $1_m,\ 1_m,\ 1_m$, or through $2_m,\ 1_m,\ 1_m$, nodes [15]. There are exactly six 2_m-nodes in $\mathcal{B}$. Therefore we have at least two nodes in $\mathcal{B}$, denoted by B_0 and B_1, through which no 3-node line passes.

Denote the line passing through the nodes B_0 and B_1, by ℓ_{01}.

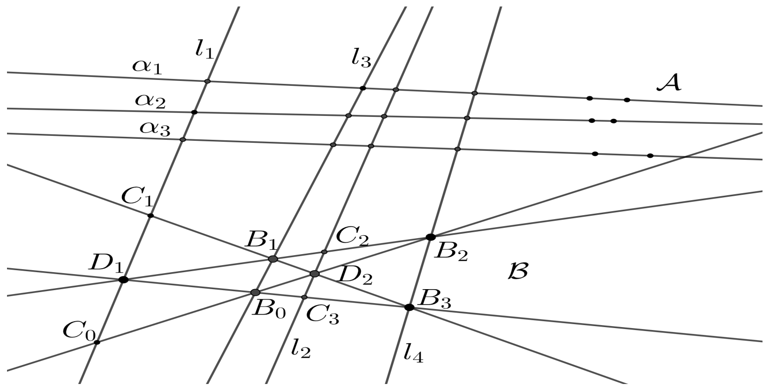

Рис. 3.2. The set $\mathcal{B}$ is a Carnicer-Gasca set

Lemma 3.7. *The line ℓ_{01} is a $3_{\mathcal{A}}$ type 5-node line and is used by exactly six nodes from $\mathcal{A}$.*

Proof. Assume by way of contradiction that ℓ_{01} is a $\leq$ 4-node line. Then, in view of Lemma 3.4, it is used ≤ 1 times from $\mathcal{A}$. On the other hand if ℓ_{01} is a 5-node line but is not used six times from $\mathcal{A}$ then, according to Propostion 3.5, it is used ≤ 4 times from $\mathcal{A}$.

Note that there is no 6-node line through $B := B_0$, since there is no 3-node line through it in $\mathcal{B}$. Next, by Lemma 3.5, any 5-node line through B, except of ℓ_{01}, is used by ≤ 3 nodes from $\mathcal{A}$ (see Fig. 3.2).

Thus, in view of Lemma 3.4, we have that the number of usages of the lines through B with the nodes from $\mathcal{A}$ equals at most:

$$m_2(B) + 4 + 3(m_3(B) - 1) \geq 18 = m_1(B) + 2m_2(B) + 3m_3(B).$$

Hence $m_1 + m_2 \leq 1$, and $m_3 \geq 6$, which contradicts Lemma 3.6. □

Denote by $\mathcal{A}_6 \subset \mathcal{A}$ the set of six nodes that are using the line ℓ_{01}.

We get from Proposition 3.5 that the six nodes of $\mathcal{A}_6$ besides ℓ_{01}, share also three other lines passing through $6, 6, 5$ primary nodes, respectively. Furthermore the m-d sequence for the six nodes is $(6, 6, 6, 4, 3, 2)$, or $(6, 6, 5, 5, 3, 2)$.

Lemma 3.3 implies that the first case cannot take place. Thus the m-d sequence for all nodes of $\mathcal{A}_6$ is $(6, 6, 5, 5, 3, 2)$. Note that, in view of (1.2), $\mathcal{A}_6$ is a GC_3 set, since $\ell_1 \cdots \ell_4$ is a maximal quartic.

Let $\ell_1, \ldots, \ell_6$ be a respective m-line sequence, where $\ell_3 := \ell_{01}$. Note that $\ell_1, \ldots, \ell_4$ are invariable lines, and ℓ_5, ℓ_6 are variable lines, for the nodes of $\mathcal{A}_6$. The lines $\ell_1, \ldots, \ell_4$ have at least $10 = 3 + 3 + 2 + 2$ distinct nodes in $\mathcal{B}$. Since $\#\mathcal{B} = 10$ we conclude that these four lines have exactly 10 nodes in $\mathcal{B}$ and $12 = 4 \times 3$ nodes in $\mathcal{A}$ (see Fig. 3.2).

Recall that we have four maximal lines in $\mathcal{B}$ which are 4-node lines and intersect in primary nodes of $\mathcal{X}$ each of the lines $\ell_1, \ldots, \ell_4$. The nodes B_0 and B_1 are in the line ℓ_3. Denote by B_2, B_3 the two primary nodes in $\ell_4 \cap \mathcal{B}$.

We readily conclude that $B_i,\ i = 0, \ldots, 3$, are 2_m-nodes of $\mathcal{B}$ and the 4 maximal lines are the lines $\ell_{02}, \ell_{03}, \ell_{12}, \ell_{13}$, where ℓ_{ij} is the line passing through the nodes B_i and B_j (see Fig. 3.2).

The remaining two 2_m-nodes of $\mathcal{B}$ are the remaining two intersection points of the maximal lines denoted by $D_1 := \ell_{03} \cap \ell_{12}$ and $D_2 := \ell_{02} \cap \ell_{13}$.
The nodes D_1 and D_2 one by one lie in the lines ℓ_1 and ℓ_2, respectively, since the latters are 3-node lines within $\mathcal{B}$ and each contains at most one 2_m-node.

Lemma 3.8. *The following is true for at least one of* $B \in \{B_2, B_3\}$:
No 3*-node line within* $\mathcal{B}$ *passes through the node* B.

Proof. Consider the node B_2. We have that $\mathcal{X} \setminus (\ell_{02} \cup \ell_{12}) = \{B_3, C_1, C_3\}$. Since ℓ_{23} is a 2-node line in $\mathcal{B}$ we get that the only candidate for 3-node line through B_2 is the line passing through the nodes B_2, C_1, C_3, provided that the latter triple of nodes is collinear (see Fig. 3.2).

Similarly we get that the only candidate for a 3-node line through B_3 is the line passing through the nodes B_3, C_0, C_2, provided they are collinear.

What we need to show is that at least one of the two triples of nodes is not collinear. Assume by way of contradiction that the both triples are collinear, lying in some two lines ℓ_0 and ℓ_0', respectively.

Consider the following collinear sets

$$\mathcal{X}_1 := \{C_0, D_1, C_1\},\ \mathcal{X}_2 := \{C_2, D_2, C_3\},\ \mathcal{X}_3 := \{B_2, B_3\}.$$

It is easily seeen that the four maximal lines of $\mathcal{B}$ pass through one point from each of $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$. Note that the above two lines ℓ_0 and ℓ_0', have the same property. Therefore we get $\#\mathcal{L}_{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3} \geq 6$, which contradicts (1.3). □ Thus from now on one can assume, without loss of generality, that no 3-node line passes through the node B_2.

Lemma 3.9. *The set* $\mathcal{B}$, *except of the lines* ℓ_1 *and* ℓ_2, *may have just one more* 3*-node line, which passes through the nodes* B_3, C_0, C_2, *provided that the latter nodes are collinear.*

Moreover, (ℓ_1, ℓ_2) *is the only disjoint pair of* 3*-node lines in* $\mathcal{B}$.

Proof. Indeed, any 3-node line must pass through a node of $\mathcal{B}$ outside of the lines ℓ_1 and ℓ_2, i.e., through one of $B_0, \ldots, B_3$. Thus it must pass through B_3 and therefore also through C_0 and C_2, provided that the latter triple of nodes is collinear (see Fig. 3.2). It remains to note that the third 3-node line intersects both of the lines ℓ_1 and ℓ_2 at nodes of $\mathcal{B}$. □

Lemma 3.10. *There is a type $3_{\mathcal{A}}$ 4-node line through each of the nodes $B_0, B_1,$ and B_2. Moreover, these lines are used by exactly 3 nodes from $\mathcal{A}$.*

Proof. Denote by B any of the nodes B_0, B_1, B_2. Note that there are no 6-node lines through B. Then we get, from Lemma 3.5, that any type $3_{\mathcal{A}}$ 5-node line through B, except of ℓ_3 or ℓ_4 is used by ≤ 3 nodes from $\mathcal{A}$.

Thus, the number of usages of the lines through B with the nodes from $\mathcal{A}$ equals at most:

$$m_2(B) + 6 + 3(m_3(B) - 1) \geq 18 = m_1(B) + 2m_2(B) + 3m_3(B). \tag{3.7}$$

Hence we obtain that $m_1 + m_2 \leq 3$.

Let us denote by $max_{use} := max_{use}(B)$ the maximum possible usage of lines through B we obtained, i.e., $max_{use} = m_2 + 3m_3 + 3$.

Then, as it follows from the equality in (3.7), the quantity $m_1 + 2m_2$ is divisible by 3. Thus the following four cases are possible here.

1) $m_1 = m_2 = 0$, then $m_3 = 6$, which contradicts Lemma 3.6.

2) $m_1 = m_2 = 1$, then $m_3 = 5,\ max_{use} = 19$,

3) $m_1 = 3, m_2 = 0$, then $m_3 = 5,\ max_{use} = 18$,

4) $m_1 = 0, m_2 = 3$, then $m_3 = 4,\ max_{use} = 18$.

Note that in each of the above cases 2), 3), 4) there are at least 4,5,4 lines of type $3_{\mathcal{A}}$, respectively. Since in the case 2) $max_{use} = 19 = 18 + 1$ one of the type $3_{\mathcal{A}}$ lines may be used less than 3 times, or more precisely 2 times.

It remains to take into account that from these four lines only three may be 5-node lines. For example, for $B = B_0$ these three lines are $\ell_{B_0B_1}, \ell_{B_0C_1}$ and $\ell_{B_0C_2}$ (see Fig. (3.2)). The remaining one certainly is a 4-node line. □

Consider a 4-node line $\widetilde{\ell}$ through B_0, B_1, B_2, mentioned in Lemma 3.10, used by exactly 3 nodes of $\mathcal{A}$. According to Proposition 3.3 the $\widetilde{\ell}$-m-d sequence of each mentioned triple of nodes is either

(a) $(\widetilde{4}, 6, 6, 5, 4, 2)$ or

(b) $(\widetilde{4}, 6, 5, 5, 5, 2)$.

Note that the first five lines in the respective $\widetilde{\ell}$-m-line sequences are invariable for the triples of nodes.

Now assume that the case (a) holds for a triple of nodes. Denote a respective $\widetilde{\ell}$-m-line sequence for the nodes of the triple by $\widetilde{\ell}, \ell_2', \ldots, \ell_6'$. Consider the m-d sequence for the nodes of $\mathcal{A}_6 : (6,6,5,5,3,2)$, and the respective m-line sequence $\ell_1, \ldots, \ell_6$. Note that $\mathcal{A}_6$ is a GC_3 set.

We get from Lemma 3.9 that the pair of the lines ℓ_2', ℓ_3' coincides with ℓ_1, ℓ_2. Then note that in the set $\mathcal{B} \setminus (\ell_1 \cup \ell_2)$ the only $3_{\mathcal{A}}$ lines with two primary nodes are the lines ℓ_3 and ℓ_4. Thus ℓ_4' coincides with one of them.

Now, in view of Corollary 1.2, we readily obtain that any node E of $\mathcal{A}_6$, besides the lines $\ell_2', \ell_3', \ell_4'$, uses also the lines $\widetilde{\ell}$ and ℓ_5'. Indeed, in view of Proposition 1.2, the node E uses one of the lines $\widetilde{\ell}, \ell_5'$ to which it does not belong. Then we get that E uses also the other line. This is a contradiction, since outside of the curve $\widetilde{\ell}\ell_2' \cdots \ell_5' \in \Pi_5$ there are only 3 nodes.

It remains to consider the case when all the three $\widetilde{\ell}$-m-d sequences equal (b). Denote a respective $\widetilde{\ell}$-m-line sequence by $\widetilde{\ell}, \ell_2'', \ldots, \ell_6''$.

Lemma 3.11. *(i) The above three triples are disjoint in this case.*
(ii) Suppose the line ℓ_2'' with 6 primary nodes for a triple coincides with one of the lines ℓ_1 or ℓ_2. Then the triple is not a subset of the set $\mathcal{A}_6$.

Proof. (i) Consider a pair of triples. Note that for them a line among the 4 invariable lines $\ell_2'', \ldots, \ell_5''$ is different. Indeed, assume conversely that all threse lines coincide with each other. Then as above we readily get that also the 4-node lines $\widetilde{\ell}$ coincide, which is a contradiction. Thus the invariable lines for each pair of triples differ at least with two lines. Therefore for any variable line the line sequences are different. Hence the triples are disjoint.

(ii) Assume that the line ℓ_2'' coincides, say, with ℓ_1. Then the three invariable lines $\ell_3'', \ell_4'', \ell_5''$ cannot coincide with ℓ_2, ℓ_3, ℓ_4. Indeed, otherwise any node in $\mathcal{A}$, together with $\ell_2'', \ldots, \ell_5''$ uses also the 4-node line $\widetilde{\ell}$, which is a contradiction since outside of $\widetilde{\ell}, \ell_2'', \ldots, \ell_5''$ there are only 3 nodes.

Thus, by taking into account $\widetilde{\ell}$, we have two invariable lines in the m-line sequence of the triple that are not present in $\{\ell_1, \ldots, \ell_4\}$. Next let us fix the variable line ℓ_5 such that it differs from the two mentioned lines. Indeed, we may choose as ℓ_5 any maximal line of the GC_3 set $\mathcal{A}_6$. Thus the considered triple is disjoint with the three nodes of $\mathcal{A}_6 \setminus \ell_5$ that use the lines $\ell_1, \ldots, \ell_5$. It remains to note that the triple does not coincide with the three nodes of $\mathcal{A}_6 \cap \ell_5$, since the latter three nodes are collinear. □

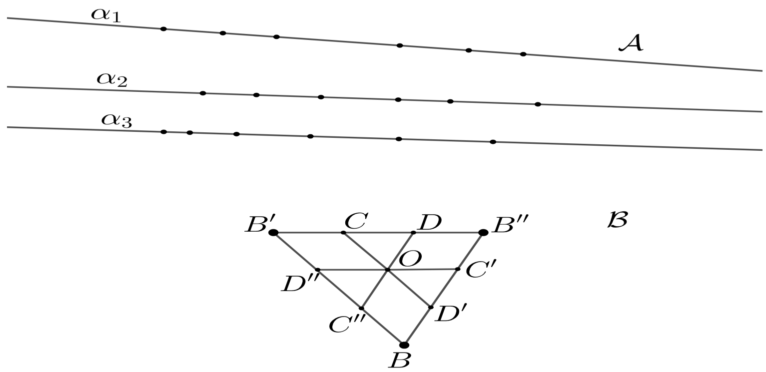

Рис. 3.3. The set $\mathcal{B}$ is a principal lattice

Now suppose that for all three triples the the line ℓ_2'' with 6 primary nodes is the third possible 3-node line of $\mathcal{B}$, different from ℓ_1 and ℓ_2 (Lemma 3.9). Then, in view of Lemma 3.11, (i), this line is used by $9 = 3+3+3$ nodes, which is a contradiction.

Finally suppose that for one of the triples the line ℓ_2'' coincides with one of the two disjoint 6-node lines, say with ℓ_1. Then, in view of Lemma 3.11, (ii), the line ℓ_1 is used by at least $7 = 6 + 1$ nodes from $\mathcal{A}$. Observe that ℓ_1 is used by a node from $\mathcal{B}$ too. Thus in all the 6-node line ℓ_1 is used by at least 8 nodes. This, in view of Proposition 3.6 and Lemma 3.3, is a contradiction.

Step 3. The set $\mathcal{B}$ is a principal lattice (with three maximal lines).

Concider a 2_m-node $B \in \mathcal{B}$. Note that there is no 3-node line through B within $\mathcal{B}$ (see Fig. 3.3).

Assume that the line $\widetilde{\ell} = \ell_{BC}$ (or ℓ_{BD}) is a 5-node line and is used ≥ 5 times from $\mathcal{A}$. Then, according to Proposition 3.5, it is used by exactly 6 nodes from $\mathcal{A}$ and the 6 nodes besides $\widetilde{\ell}$ share also 3 lines with $6, 6, 5$ primary nodes. The two 6-node lines are 3-node disjoint lines within $\mathcal{B}$. Thus they pass through the 1_m nodes C, C', C'', and D, D', D'', respectively. This is a contradiction since the node C belongs to the line $\widetilde{\ell}$ and is not primary.

Thus the lines ℓ_{BC} and ℓ_{BD} are used ≤ 4 times from $\mathcal{A}$. Note that the line ℓ_{BO}, as well as any 4-node line through B, according to Lemma 3.5, is used ≤ 3 times from $\mathcal{A}$.

Hence the maximal possible number of usages of the lines through B with the nodes from $\mathcal{A}$ equals:

$$m_2(B) + 4 + 4 + 3(m_3(B) - 2) \geq 18 = m_1(B) + 2m_2(B) + 3m_3(B).$$

Hence $m_1+m_2 \le 2$. Then m_1+2m_2 is divisible by 3 and $max_{use}(B) = m_2+3m_3+2$. Thus the following two cases are possible:

1) $m_1 = m_2 = 0, m_3 = 6$, which contradicts 3.6,

2) $m_1 = m_2 = 1, m_3 = 5,\ max_{use} = 18$.

Now we readily conclude that:

$B1$) The lines ℓ_{BC} and ℓ_{BD} are 5-node lines used exactly 4 times from $\mathcal{A}$.

$B2$) No node of $\mathcal{A}$ may use two lines through B.

$B3$) There are at least two 4-node lines through B that are of type $3_{\mathcal{A}}$ and are used by exactly 3 nodes from $\mathcal{A}$.

$B3'$) Note that if the line ℓ_{BO} is not of type $3_{\mathcal{A}}$ then there are three above mentioned 4-node lines through B.

Then let us consider the usages of lines passing through the 0_m node $O \in \mathcal{B}$ by the nodes of $\mathcal{A}$. Consider the three lines through $O:\ OC, OC', OC''$, which can be 6-node lines. These lines are used by 3 nodes of $\mathcal{B}$ and therefore, by Proposition 3.6, they can be used by at most 4 nodes of $\mathcal{A}$.

Similarly the lines OB, OB', OB'' can be 5-node lines, in which case, according to Lemma 3.5, they can be used by at most 3 nodes from $\mathcal{A}$.

Thus for the maximal possible number of usages of the lines through O with the nodes from $\mathcal{A}$ we have:

$$m_2(O) + 4 + 4 + 4 + 3(m_3(O) - 3) \ge 18 = m_1(O) + 2m_2(O) + 3m_3(O).$$

Hence $m_1+m_2 \le 3$. Then m_1+2m_2 is divisible by 3 and $max_{use}(O) = m_2+3m_3+3$. Thus the following cases are possible

1) $m_1 = m_2 = 0, m_3 = 6$, which contradicts Lemma 3.6.

2) $m_1 = m_2 = 1, m_3 = 5,\ max_{use} = 19$,

3) $m_1 = 3, m_2 = 0, m_3 = 5,\ max_{use} = 18$,

4) $m_1 = 0, m_2 = 3, m_3 = 4,\ max_{use} = 18$.

Now we readily conclude that

$O1$) At least two of the three lines through $O:\ OC, OC', OC''$, are 6-node lines and are used exactly 4 times from $\mathcal{A}$.

$O2$) At most one node of $\mathcal{A}$ may use two lines through O all others may use only one line through O.

$O3$) From the six lines through $O:\ OC, OC', OC'', OB, OB', OB''$, in view of Lemma 3.6, at most five are of type $3_{\mathcal{A}}$ and possibly are used by ≥ 3 nodes from $\mathcal{A}$.

Thus, by the remarks $B3$), and $B3'$), there are at least six, possibly seven, type $3_{\mathcal{A}}$ 4-node lines, through the 2_m-nodes of $\mathcal{B}$, that are used by exactly 3 nodes from

$\mathcal{A}$. In view of Proposition 3.3 again for these 4-node lines we have one of the two $\widetilde{\ell}$-m-d sequences: (a) $(\widetilde{4}, 6, 6, 5, 4, 2)$ or (b) $(\widetilde{4}, 6, 5, 5, 5, 2)$.

Lemma 3.12. *The second* 4 *in the* $\widetilde{\ell}$*-m-d sequence (a), in a respective* $\widetilde{\ell}$*-m line sequence, corresponds to a* $3_{\mathcal{A}}$ *4-node line passing through* B, B' *or* B''.

Proof. Suppose that $\ell_1, \ldots, \ell_6$ is a respective $\widetilde{\ell}$-m-line sequence. Then ℓ_2 and ℓ_3 pass through the six 1_m-nodes of $\mathcal{B}$. Thus the line ℓ_4 coincides with one of the lines $B'O$, or $B''O$, say with $B'O$ (see Fig. 3.3).

Now ℓ_5 passes through B'' as the only remaining primary node in $\mathcal{B}$. Let us show that it does not pass through any other node of $\mathcal{B}$.

Note that ℓ_5 cannot pass through O since then each of the three nodes will use two lines passing through O, which contradicts the remark $O2$).

Then assume conversely that ℓ_5 passes through one of 1_m nodes, say C'. As we know from the remark $B1$), the line $\ell_{B'C'}$ is used by the fourth node of $\mathcal{A}$ denoted by F. In view of Proposition 1.2 the node F uses one of ℓ_2, ℓ_3 to which it does not belong and then the other. Next we readily get that F uses also the lines ℓ_4 and ℓ_1. Thus the 4-node line ℓ_1 is used by 4 nodes which, in view of Proposition 3.4 and Lemma 3.3, is a contradiction. □

Lemma 3.13. *Any two triples of nodes corresponding to two distributions of type (a) or (b) are disjoint.*

Proof. Note that the variable lines with 2 primary nodes in respective $\widetilde{\ell}$-m line sequences cannot be equal to any of the 4-node line. Indeed the first five lines pass through all the nodes of the set $\mathcal{B}$. Now if a sixth line becomes 4-node line through a 2_m-node of $\mathcal{B}$ then we have two lines passing through the 2_m-node, which contradicts the remark $B2$).

Then since in each case of distributions (a) and (b) we have different 4-node lines therefore the corresponding triples are disjoint. □

Now assume that for at least two pairs of 4-node lines we have the $\widetilde{\ell}$-m-d sequence (a). Then the two disjoint lines ℓ_2 and ℓ_3 in the respective $\widetilde{\ell}$-m line sequence are used by two triples of nodes, i.e., by six nodes.

In view of Proposition 3.7 we get that each of the six nodes has either m-d sequence $(6, 6, 6, 4, 3, 2)$ or $(6, 6, 5, 5, 3, 2)$. The first case contradicts Lemma 3.3. While the second sequence clearly differs from the sequence (a), since there we have two invariable 4-node lines.

Then assume that for one pair of 4-node lines the $\widetilde{\ell}$-m-d sequence (a) takes place and for other 4-node lines the sequence (b) takes place.

In view of the remarks $B3'), O1)$, and $O3)$, let us consider the following

Case 1) There are three 6-node lines through O used by three nodes and therefore there are at least seven 4-node lines through 2_m-nodes, and

Case 2) There are exactly two 6-node lines through O used by three nodes and therefore there are at least six 4-node lines through 2_m-nodes.

Now recall that the two disjoint 6-node lines are ℓ_2 and ℓ_3. Denote the 6-node lines passing through O by $\ell_0, \ell_0', \ell_0''$. Finally denote the above seven possible 4-node lines through the 2_m-nodes of $\mathcal{B}$ by $\alpha_i, i = 1, \ldots, 7$.

In Case 1) we have six different triples. Let us consider only the 4-node and 6-node lines in the respective line sequences:

$(\alpha_1, \alpha_2, \ell_2, \ell_3)$; (α_3, ℓ_2); (α_4, ℓ_3); (α_5, ℓ_0); (α_6, ℓ_0'); (α_7, ℓ_0'').

In Case 2) we have 5 different triples corresponding to:

$(\alpha_1, \alpha_2, \ell_2, \ell_3)$; (α_3, ℓ_2); (α_4, ℓ_3); (α_5, ℓ_0); (α_6, ℓ_0').

Note that in both cases all the possible 6-node lines are used by six nodes, counted also the triple usage of each of lines $\ell_0, \ell_0', \ell_0''$, from $\mathcal{B}$. Therefore, in view of Proposition 3.6 and Lemma 3.3, no place for another 6-node line in the line sequences, and consequently the additional triple usage.

Thus all the lines with 5 primary nodes in the line sequences, actually have to be exact 5-node lines.

Let us show that these 5-node lines are different. Indeed, assume conversely that a 5-node line $\widetilde{\ell}$ is in two m-line sequences used by different triples. Then $\widetilde{\ell}$ is used by 6 nodes. As we know, by Proposition 3.7, these nodes must have the m-d sequence $(6, 6, 5, 5, 3, 2)$, which clearly differs from (a) and (b).

Now in Case 1) we need for 16 $(= 5 \times 3 + 1)$ and in Case 2) we need for 13 $(= 4 \times 3 + 1)$ different 5-node lines.

Below we show that actually there are not that many $3_{\mathcal{A}}$ 5-node lines, which finishes the proof in this case.

For this end let us count the number of 2-node lines in $\mathcal{B}$. There are 9 $(= 3 \times 3)$ such lines through the three 2_m-nodes (see Fig. 3.3). Then there are 3 such lines through the 1_m nodes. Here we take into account that $C, C', C'' \in \ell_2$, and $D, D', D'' \in \ell_3$. Hence in all we may have atmost 12 lines.

Finally, let us consider the case when for all 4-node lines the $\widetilde{\ell}$-m-d sequence (b) takes place. Then in Case 1) we have seven disjoint triples whose union is $\mathcal{A}$, which is a contradiction since $\#\mathcal{A} = 18$.

In Case 2) we have 6 different triples corresponding to:

(α_1, ℓ_2); (α_2, ℓ_2); (α_3, ℓ_3); (α_4, ℓ_3); (α_5, ℓ_0); (α_6, ℓ_0').

In this case no place for another 6-node line too. Thus again all the lines with 5 primary nodes actually are exact 5-node lines. Here we need for 18 ($= 6 \times 3$) 5-node lines. As we showed above there can be atmost 12 such lines.

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PERIODIC ORTHONORMAL SPLINE SYSTEMS WITH ARBITRARY KNOTS AS BASES IN $H^1(\mathbb{T})$

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Abstract. We give a simple geometric characterization of sequences of knots for which the corresponding periodic orthonormal spline system of order k is a basis in the atomic Hardy space on the torus $\mathbb{T}$.

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1. Introduction

This paper belongs to a series of papers studying properties of periodic and non-periodic orthonormal spline systems with arbitrary knots. The detailed study of non-periodic orthonormal spline systems started in 1960's with Z. Ciesielski's papers [4, 5] on properties of the Franklin system, which is an orthonormal system consisting of continuous piecewise linear functions with dyadic knots. Next, the results by J. Domsta (1972), cf. [9], made it possible to extend such study to orthonormal spline systems of higher order with dyadic knots. These systems occurred to be bases or unconditional bases in several function spaces like $L^p[0,1]$, $1 \le p < \infty$, $C[0,1]$, $H^p[0,1]$, $0 < p \le 1$, Sobolev spaces $W^{p,k}[0,1]$, they give characterizations of BMO and VMO spaces, and various spaces of smooth functions.

The extension of these results to orthonormal spline systems with arbitrary knots has begun with the case of piecewise linear systems, i.e. general Franklin systems, or orthonormal spline systems of order 2. This was possible due to precise estimates of the inverse to the Gram matrix of piecewise linear B-spline bases with arbitrary knots, as presented in [14]. We would like to mention here two results by G.G. Gevorkyan and A. Kamont. First, each general Franklin system is an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$, cf. [10]. Second, there is a simple geometric characterization of knot sequences for which the corresponding general Franklin system is a basis or an unconditional basis in $H^1[0,1]$, cf. [12]. We note that in

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both of these results, an essential tool for their proof is the association of a so called characteristic interval to each general Franklin function f_n.

The case of splines of higher order is much more difficult. Let us mention that the basic result – the existence of a uniform bound for L^∞-norms of orthogonal projections on spline spaces of order k with arbitrary order (i.e. a bound depending on the order k, but independent of the sequence of knots) – was a long-standing problem known as C. de Boor's conjecture (1973), cf. [2]. The case of $k = 2$ was settled even earlier by Z. Ciesielski [4], the cases $k = 3, 4$ were solved by C. de Boor himself (1968, 1981), cf. [1, 3], but the positive answer in the general case was given by A. Yu. Shadrin [21] in 2001. A much simplified and shorter proof of this theorem was recently obtained by M. v. Golitschek (2014), cf. [22]. An immediate consequence of A.Yu. Shadrin's result is that if a sequence of knots is dense in $[0, 1]$, then the corresponding orthonormal spline system of order k is a basis in $L^p[0, 1]$, $1 \le p < \infty$ and $C[0, 1]$. Moreover, Z. Ciesielski [6] obtained several consequences of Shadrin's result, one of them being some estimate for the inverse to the B-spline Gram matrix. Using this estimate, G.G. Gevorkyan and A. Kamont [12] extended a part of their result from [11] to orthonormal spline systems of arbitrary order and obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order k is a basis in $H^1[0, 1]$. Further extension required more precise estimates for the inverse of B-spline Gram matrices, of the type known for the piecewise linear case. Such estimates were obtained recently by M. Passenbrunner and A.Yu. Shadrin [19]. Using these estimates, M. Passenbrunner [17] proved that for each sequence of knots, the corresponding orthonormal spline system of order k is an unconditional basis in $L^p[0, 1]$, $1 < p < \infty$. With the help of this result it was obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order k is an unconditional basis in $H^1[0, 1]$ (see [13]).

Another extension of the previous results can be done for *periodic* orthonormal spline systems with arbitrary knots. In the periodic case K. Keryan [15] proved that for any admissible point sequence the corresponding periodic Franklin system (i.e. periodic piecewise linear system) forms an unconditional basis in $L^p[0, 1]$, $1 < p < \infty$. K. Keryan and M. Passenbrunner [16] obtained an essential estimate for general periodic orthonormal spline functions. Combining the estimate with the methods developed in [10] they proved the unconditionality of periodic orthonormal spline systems in $L^p(\mathbb{T})$, $1 < p < \infty$. A result concerning the basis property of periodic orthonormal spline systems of order 2 in Hardy's atomic space on the torus was carried out by M. Poghosyan and K. Keryan. In the paper [20] they gave a simple geometric characterization of knot sequences for which the corresponding general periodic Franklin system is a basis or unconditional basis in $H^1(\mathbb{T})$.

The main result of the present paper is to give a characterization of those knot sequences for which the corresponding periodic orthonormal spline system of fixed order of smoothness is a basis in $H^1(\mathbb{T})$.

The paper is organized as follows. In Section 2 we give necessary definitions and we formulate the main result of this paper – Theorem 2.1. The proof of the main result is presented in Section 3: in Subsection 3.1 some properties of periodic orthonormal spline systems are provided, then in Subsection 3.2 a lower bound for $H^1(\mathbb{T})$ norm of a function is given, and finally in Subsections 3.3 and Sufficiency it is proved the necessity and sufficiency of k-regularity in Theorem 2.1 correspondingly.

2. Definitions, notation and the main result

We begin with some preliminary notations. The parameter $k \geq 2$ will always be used for the order of the underlying polynomials or splines. We use the notation $A(t) \sim B(t)$ to indicate the existence of two constants $c_1, c_2 > 0$, such that $c_1 B(t) \leq A(t) \leq c_2 B(t)$ for all t, where t denotes all implicit and explicit dependencies that the expressions A and B might have. If the constants c_1, c_2 depend on an additional parameter p, we write this as $A(t) \sim_p B(t)$. Correspondingly, we use the symbols $\lesssim, \gtrsim, \lesssim_p, \gtrsim_p$. For a subset E of the real line, we denote by $|E|$ the Lebesgue measure of E.

Now let $k \geq 2$ be an integer and $\mathcal{T} := (s_n)_{n=1}^{\infty}$ be a point sequence from the torus $\mathbb{T}$ such that each point occurs at most k times. Such point sequences are called k admissible.

For $n \geq k$, we define $\hat{\mathcal{S}}_n$ to be the space of polynomial splines of order k with grid points $(s_j)_{j=1}^{n} \subset \mathbb{T}$. For each $n \geq k+1$, the space $\hat{\mathcal{S}}_{n-1}$ has codimension 1 in $\hat{\mathcal{S}}_n$ and, therefore, there exists a function $\hat{f}_n \in \hat{\mathcal{S}}_n$ with $\|\hat{f}_n\|_{L^2(\mathbb{T})} = 1$ that is orthogonal to the space $\hat{\mathcal{S}}_{n-1}$. Observe that this function $\hat{f}_n$ is unique up to sign. In addition, let $(\hat{f}_n)_{n=1}^{k}$ be an orthonormal basis for $\hat{\mathcal{S}}_k$. The system of functions $(\hat{f}_n)_{n=1}^{\infty}$ is called *periodic* orthonormal spline system of order k corresponding to the sequence $(s_n)_{n=1}^{\infty}$.

Now we define the atomic Hardy space on $\mathbb{T}$.

Definition 2.1. *A function $a : \mathbb{T} \to \mathbb{R}$ is called a* periodic atom*, if either $a \equiv 1$ or $\exists\, \Gamma \subset \mathbb{T}$ interval such that all these conditions are satisfied:*

(i) $\operatorname{supp} a \subset \Gamma$,

(ii) $\|a\|_{L^\infty(\mathbb{T})} \le |\Gamma|^{-1}$,
(iii) $\int_{\mathbb{T}} a(x)\,\mathrm{d}x = \int_\Gamma a(x)\,\mathrm{d}x = 0$.

Definition 2.2. *$H^1(\mathbb{T})$ is the family of all the functions f that has representation*

$$f = \sum_{n=1}^{\infty} c_n a_n$$

for some periodic atoms $(a_n)_{n=1}^{\infty}$ and real scalars $(c_n)_{n=1}^{\infty} \in \ell^1$.

The space $H^1(\mathbb{T})$ becomes a Banach space under the norm

$$\|f\|_{H^1(\mathbb{T})} := \inf \sum_{n=1}^{\infty} |c_n|$$

where inf is taken over all (periodic) atomic representations $\sum c_n a_n$ of f. Now, we introduce regularity conditions in the torus $\mathbb{T}$ for sequence $(s_n)_{n=1}^{\infty}$.

Assume that $n \ge k+1$. Let $(\sigma_j)_{j=0}^{n-1}$ be the ordered sequence of knot points consisting of $(s_j)_{j=1}^{n}$ in $\mathbb{T}$ canonically identified with $[0,1)$:

$$\hat{\mathcal{T}} := \hat{\mathcal{T}}_n = (0 \le \sigma_{n,0} \le \sigma_{n,1} \le \cdots \le \sigma_{n,n-2} \le \sigma_{n,n-1} < 1). \tag{2.1}$$

For the integers $\ell \le k$ and $i \in \mathbb{N}_0$, we define $T_{n,i}^{(\ell)} := [\sigma_{n,i}, \sigma_{n,i+\ell}] \subset \mathbb{T}$ interval. Here we observe index i periodically, i.e. we use the notation of periodic extension of the sequence $(\sigma_j)_{j=0}^{n-1}$, i.e. $\sigma_{rn+j} = r + \sigma_j$ for $j \in \{0, \ldots, n-1\}$ and $r \in \mathbb{Z}$ and in the subindices of the B-spline functions, we take the indices modulo n.

Definition 2.3. *Let $\ell \le k$ and $(s_n)_{n=1}^{\infty}$ be an ℓ-admissible point sequence the in the torus $\mathbb{T}$. Then, this sequence is called* ℓ-regular in torus $\mathbb{T}$ with parameter $\gamma \ge 1$ *if*

$$\frac{|T_{n,i}^{(\ell)}|}{\gamma} \le |T_{n,i+1}^{(\ell)}| \le \gamma |T_{n,i}^{(\ell)}|, \qquad n \ge \ell+1,\ i \in \mathbb{N}_0.$$

Let $\hat{P}_n^{(k)}$ be the orthogonal projection operator onto $\hat{\mathcal{S}}_n$ with respect to the canonical inner product in $L^2(\mathbb{T})$ and $\hat{D}_n^{(k)}$ be its Dirichlet kernel.

The following is the main result of this paper.

Theorem 2.1. *Let $k \ge 1$ and let (s_n) be a k-admissible sequence of knots in $\mathbb{T}$ with the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ of the order k. Then, $(\hat{f}_n^{(k)})$ is a basis in $H^1(\mathbb{T})$ if and only if (s_n) is k-regular in the torus with some parameter $\gamma \ge 1$*

3. Proof of Theorem 2.1

Since the sequence of knots $(s_n)_{n=1}^{\infty}$ is dense in the torus $\mathbb{T}$, the linear span of the functions $\{\hat{f}_n^{(k)}, n \ge 1\}$ is linearly dense in $C(\mathbb{T})$, which implies its linear density in $H^1(\mathbb{T})$. Therefore, $\{\hat{f}_n^{(k)}, n \ge 1\}$ is a basis in $H^1(\mathbb{T})$ if and only if the partial sum operators $\hat{P}_n^{(k)}$ are uniformly bounded in $H^1(\mathbb{T})$, i.e. there is a constant $C = C(\mathcal{T})$, that only depends on the knot sequence $(s_n)_{n=1}^{\infty}$, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} = \|\hat{P}_n^{(k)} : H^1(\mathbb{T}) \to H^1(\mathbb{T})\| \le C(\mathcal{T}). \tag{3.1}$$

We show that (3.1) is equivalent to k-regularity of $\mathcal{T}$. This is an immediate consequence of the Propositions 3.1 and 3.2, which contain estimates of norms $\hat{P}_n^{(k)}$ from below and from above, respectively.

Proposition 3.1. *Let $\hat{\mathcal{T}}_n = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$ be a sequence of knots in the torus $\mathbb{T}$ of multiplicities at most k. Let*

$$M = M_n^{(k)} := \max\left\{ \frac{|T_{n,i}^{(k)}|}{|T_{n,i+1}^{(k)}|}, \frac{|T_{n,i+1}^{(k)}|}{|T_{n,i}^{(k)}|} : \quad 0 \le i \le n-1 \right\}.$$

Then there is a constant $C_k > 0$, depending only on k, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \ge C_k \log M_n^{(k)}.$$

Proposition 3.2. *Let $\hat{\mathcal{T}}_n = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$ be a sequence of knots in the torus $\mathbb{T}$ of multiplicities at most k. Let γ be such that*

$$\frac{|T_{n,i}^{(k)}|}{\gamma} \le |T_{n,i+1}^{(k)}| \le \gamma |T_{n,i}^{(k)}|, \qquad n \ge k+1, \ i \in \mathbb{N}_0.$$

Then there is a constant $C_{k,\gamma} > 0$ depending only on k and γ, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \le C_{k,\gamma}.$$

Before we begin to prove the Propositions 3.1 and 3.2, we recall some properties of splines and orthogonal projections $\hat{P}_n^{(k)}$.

3.1. **Properties of periodic orthonormal spline systems.** The key result which let us work with periodic orthonormal spline systems of the order k is the periodic version of A. Yu. Shadrin's [21] theorem, i.e. uniform boundedness of L^∞-norms of projections $\hat{P}_n^{(k)}$. The result was obtained by M. Passenbrunner in [18].

Theorem 3.1 ([18])**.** *There exists a constant C_k depending only on the spline order k such that for any sequence $\hat{\mathcal{T}}$ of knots of multiplicity at most k*

$$\|\hat{P}_{\hat{\mathcal{T}}}^{(k)}\|_\infty = \|\hat{P}_{\hat{\mathcal{T}}}^{(k)} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\| \le C_k.$$

Clearly, this means that

$$\|\hat{P}_{\hat{\mathcal{T}}}^{(k)}\|_\infty = \sup_{t\in\mathbb{T}} \int_{\mathbb{T}} |\hat{D}_{\hat{\mathcal{T}}}^{(k)}(t,s)| ds \le C_k. \tag{3.2}$$

Now, as before, let $\hat{\mathcal{T}}_n = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$ be a sequence of knots in the torus $\mathbb{T}$ of multiplicities at most k. By $\hat{N}_{n,i}^{(k)}$, $i = 0, \dots, n-1$ we denote the L^∞-normalized periodic B-spline basis of $\hat{\mathcal{S}}_n^{(k)}$. These functions are nonnegative, linearly independent and form a partition of unity, i.e. $\sum_{i=0}^{n-1} \hat{N}_{n,i}^{(k)}(t) = 1$ for each $t \in \mathbb{T}$. Moreover, supp $\hat{N}_{n,i}^{(k)} = [\sigma_i, \sigma_{i+k}]$ and $\|\hat{N}_{n,i}^{(k)}\|_{L^1(\mathbb{T})} = \frac{|T_{n,i}^{(k)}|}{k}$. Corresponding to this basis, there exists a biorthogonal basis of $\hat{\mathcal{S}}_n^{(k)}$, which is denoted by $(\hat{N}_{n,i}^{(k)*})_{i=0}^{n-1}$.

Let $\hat{G}_n^{(k)} = [(\hat{N}_{n,i}^{(k)}, \hat{N}_{n,j}^{(k)}), 0 \le i, j \le n-1]$ be the Gram matrix for the system $\{\hat{N}_{n,i}^{(k)}, i = 0, \dots, n-1\}$, and let $A_n^{(k)} = [\hat{a}_{i,j} = (\hat{N}_{n,i}^{(k)*}, \hat{N}_{n,j}^{(k)*}), 0 \le i, j \le n-1]$. the following is a equivalent version of Theorem 3.1 (See [18], Section 3, Remark 3.2):

Theorem 3.2 ([18]). *Let $n \ge 2k$. Then, there exists a constant $q \in (0,1)$ depending only on the spline order k such that*

$$|\hat{a}_{ij}| \lesssim_k \frac{q^{\hat{d}(i,j)}}{\max(|\operatorname{supp} \hat{N}_{n,i}^{(k)}|, |\operatorname{supp} \hat{N}_{n,j}^{(k)}|)}, \qquad 0 \le i, j \le n-1,$$

where $\hat{d}$ is the periodic distance function on $\{0, \dots, n-1\}$.

In particular, since $\hat{D}_n^{(k)}(t,s) = \sum_{i,j=0}^{n-1} \hat{a}_{i,j} \hat{N}_{n,i}^{(k)}(t) \hat{N}_{n,j}^{(k)}(s)$, Theorem 3.2 implies that

$$|\hat{D}_n^{(k)}(t,s)| \lesssim_k \sum_{i,j=0}^{n-1} \frac{q^{\hat{d}(i,j)}}{\max(|\operatorname{supp} \hat{N}_{n,i}^{(k)}|, |\operatorname{supp} \hat{N}_{n,j}^{(k)}|)} \hat{N}_{n,i}^{(k)}(t) \hat{N}_{n,j}^{(k)}(s). \tag{3.3}$$

If the setting of the parameters k and n is clear from the context, we will omit k and n and write $\hat{N}_i$ instead of $\hat{N}_{n,i}^{(k)}$.

3.2. **A lower bound for $H^1(\mathbb{T})$ norm of a function.** In order to prove Proposition 3.1 we will need the periodic version of the claim used in [12] (cf. page 7, estimate (3.4)).

Proposition 3.3 ([12]). *Define $\Phi(x) := \max(0, 1/2 - |x/4|)$ and $\Phi_\epsilon(x) = \frac{1}{\epsilon}\Phi(\frac{x}{\epsilon})$, for $x \in [0,1]$. Then, there is a constant $C > 0$ such that*

$$\|f\|_{H^1[0,1]} \ge C \|f^*\|_{L^1[0,1]}, \quad \text{where} \quad f^*(x) = \sup_{\epsilon > 0} \Big| \int_0^1 \Phi_\epsilon(x-t) f(t) dt \Big|.$$

Using this proposition we prove the following.

Lemma 3.1. *Define the 1-periodic functions $\hat{\Phi}(x) := \max(0, 1/2 - |x/4|)$ and $\hat{\Phi}_\epsilon(x) = \frac{1}{\epsilon}\hat{\Phi}(\frac{x}{\epsilon})$, for $x \in \mathbb{T}$. Then, for some constant $c > 0$ the following holds,*

$$\|f\|_{H^1(\mathbb{T})} \ge c \|f^{**}\|_{L^1(\mathbb{T})}, \quad \text{where} \quad f^{**}(x) = \sup_{\epsilon > 0} \Big| \int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t) f(t) dt \Big|.$$

Proof. Let f be a function from $H^1(\mathbb{T})$. Then, there exists a sequence of periodic atoms $(\hat{a}_i)_{i=1}^\infty$ and coefficients $(\lambda_i)_{i=1}^\infty$ such that,

$$f = \sum_{i=1}^\infty \lambda_i \hat{a}_i, \quad \text{and} \quad \sum_{i=1}^\infty |\lambda_i| \le 2\|f\|_{H^1(\mathbb{T})}. \tag{3.4}$$

Now by (3.4), we get

$$\begin{aligned}\|f^{**}\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} \sup_{\epsilon>0} \Big|\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t)\sum_{i=1}^\infty \lambda_i \hat{a}_i(t)dt\Big|dx \\ &\leq \sum_{i=1}^\infty |\lambda_i| \int_{\mathbb{T}} \sup_{\epsilon>0} \Big|\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t)\hat{a}_i(t)dt\Big|dx \\ &\leq \sum_{i=1}^\infty |\lambda_i| \int_{\mathbb{T}} \sup_{0<\epsilon<1/16} \Big|\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t)\hat{a}_i(t)dt\Big|dx \\ &+ \sum_{i=1}^\infty |\lambda_i| \int_{\mathbb{T}} \sup_{1/16\leq\epsilon} \Big|\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t)\hat{a}_i(t)dt\Big|dx =: \Sigma_1+\Sigma_2\end{aligned}$$

First we estimate Σ_2. We have that

$$\hat{\Phi}_\epsilon(x) \leq \frac{1}{2\epsilon}, \quad x \in \mathbb{T},$$

and $\|\hat{a}_i\|_{L^1(\mathbb{T})} \leq 1$ so we get the following,

$$\Sigma_2 \leq \sum_{i=1}^\infty 8|\lambda_i| \lesssim \|f\|_{H^1(\mathbb{T})}.$$

Now, let $\Gamma_j \subset \mathbb{T}$ be the interval that contains the support of the periodic atom $\hat{a}_j$. Define $J := \{i : |\Gamma_i^c| \geq 1/4\}$ and split Σ_1 into 2 sums. The first sums over all the indices from the set J and the second one sums over indices of J^c. Let's denote the sums by $\Sigma_{1,J}$ and Σ_{1,J^c}, respectively.

Observe $\Sigma_{1,J}$. Fix an arbitrary $i \in J$ and identify tours $\mathbb{T}$ with $[0,1)$ in such a way that 0 coincides with the the center of Γ_i^c. We have that $0 < \epsilon < \frac{1}{16}$ and $|\Gamma_i^c| \geq \frac{1}{4}$. Hence, we get that $\hat{\Phi}_\epsilon(\cdot) = \Phi_\epsilon(\cdot)$. Consequently, by Proposition 3.3 we get

$$\begin{aligned}\Sigma_{1,J} &= \sum_{i\in J} |\lambda_i| \int_{\mathbb{T}} \sup_{0<\epsilon<1/16} \Big|\int_{\mathbb{T}} \Phi_\epsilon(x-t)\hat{a}_i(t)dt\Big|dx \\ &\leq \sum_{i\in J} |\lambda_i| \|\hat{a}_i^*\|_{L^1[0,1]} \lesssim \sum_{i\in J} |\lambda_i| \|\hat{a}_i\|_{H^1[0,1]} \leq \|f\|_{H^1(\mathbb{T})}.\end{aligned}$$

The last inequality comes from $\|\hat{a}_i\|_{H^1[0,1]} \leq 1$. This is true because by the right identification of $\mathbb{T}$ with $[0,1)$, i.e. the starting point 0 is not Γ_i, we made sure that $\hat{a}_i$ is an atom on $[0,1)$.

Consider Σ_{1,J^c}. For all $i \in J^c$ we have $|\Gamma_i|^{-1} \leq 4/3$ and $\|\hat{a}_i\|_{L^\infty(\mathbb{T})} \leq |\Gamma_i|^{-1}$. Thus,

$$\begin{aligned}\Sigma_{1,J^c} &\leq \sum_{i\in J^c} \frac{|\lambda_i|}{|\Gamma_i|} \int_{\mathbb{T}} \sup_{0<\epsilon<1/16} \int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t)dtdx \\ &\leq \frac{4}{3}\sum_{i\in J^c} |\lambda_i| \lesssim \|f\|_{H^1(\mathbb{T})}\end{aligned}$$

Combining all above we get the desired result, i.e.

$$\|f^{**}\|_{L^1(\mathbb{T})} \lesssim \|f\|_{H^1(\mathbb{T})}.$$

3.3. **Necessity of k-regularity: proof of Proposition 3.1.** Since $\hat{P}_n^{(k)}$ is a projection onto $\hat{\mathcal{S}}_n^{(k)}$, it follows that $\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \geq 1$. Therefore we can assume that $M \geq M(k)$, where $M(k) \geq 2$ will be specified later. Let u be such that $M = \frac{|T_{n,u}^{(k)}|}{|T_{n,u+1}^{(k)}|}$ (clearly, the case of $M = \frac{|T_{n,u+1}^{(k)}|}{|T_{n,u}^{(k)}|}$ is analogous). As $M \geq 2$, it follows that

$$|T_{n,u}^{(1)}| \geq (M-1)|T_{n,u+1}^{(k)}| \geq \frac{M}{2}|T_{n,u+1}^{(k)}|.$$

Here we identified the torus $\mathbb{T}$ with $[0,1)$ in such a way that the starting point 0 is not in the intervals $T_{n,u}^{(1)}$ and $T_{n,u+1}^{(k)}$. Now let $\phi_1(\cdot) = \frac{k}{|T_{n,u+1}^{(k)}|}\hat{N}_{u+1}(\cdot)$ and $\phi_2(\cdot) = \phi_1(\cdot + |T_{n,u+1}^{(k)}|)$. Then supp $\phi_1 = [\sigma_{u+1}, \sigma_{u+k+1}]$, supp $\phi_2 = [\sigma_{u+1} - |T_{n,u+1}^{(k)}|, \sigma_{u+1}] \subset [\sigma_u, \sigma_{u+1}]$ and $\|\phi_1\|_1 = \|\phi_2\|_1 = 1$. Put $\phi = \phi_1 - \phi_2$. Then $\int_{\mathbb{T}} \phi(x)dx = 0$, supp$\phi = \Gamma = [\sigma_{u+1} - |T_{n,u+1}^{(k)}|, \sigma_{u+k+1}]$ and $\|\phi\|_\infty \leq \frac{2k}{|\Gamma|}$, so $\|\phi\|_{H^1(\mathbb{T})} \leq 2k$. We need to estimate from below $\|\hat{P}_n^{(k)}\phi\|_{H^1(\mathbb{T})}$. For this, we use Lemma 3.1.

At first, consider $\hat{P}_n^{(k)}\phi_2$. We observe $\hat{D}_n^{(k)}$: the kernel of the projection $\hat{P}_n^{(k)}$. Note that $\hat{D}_n^{(k)}$ is a polynomial of degree at most $k-1$ on $T_{n,u}^{(1)}$, and by comparison of different norms of polynomials of fixed degree (cf. e.g. Theorem 2.6 of Chapter 4 in [8])

$$\int_{T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)|dt \sim_k |T_{n,u}^{(1)}| \max_{t \in T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)|,$$

and the constants in the above equivalences depend only on k. As supp $\phi_2 \subset T_{n,u}^{(1)}$, we find

$$\begin{aligned} |\hat{P}_n^{(k)}\phi_2(x)| &= |\int_{\mathbb{T}} \hat{D}_n^{(k)}(t,x)\phi_2(t)dt| \leq \int_{T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)||\phi_2(t)|dt \\ &\leq \max_{t \in T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)|\|\phi_2\|_{L^1(\mathbb{T})} \leq \frac{C_k}{|T_{n,u}^{(1)}|}\int_{T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)|dt. \end{aligned}$$

Combining 3.2 and the last sequence of inequalities we get $|\hat{P}_n^{(k)}\phi_2(x)| \leq \frac{C_k}{|T_{n,u}^{(1)}|}$, and consequently

$$(\hat{P}_n^{(k)}\phi_2)^{**} \leq \frac{C_k}{|T_{n,u}^{(1)}|}. \tag{3.5}$$

Now, we estimate $(\hat{P}_n^{(k)}\phi_1)^{**}$ from below . Clearly, since $\hat{P}_n^{(k)}$ is a projection onto $\hat{\mathcal{S}}_n^{(k)}$, we have $\hat{P}_n^{(k)}\phi_1 = \phi_1$. Take $x \in T_{n,u}^{(1)}$, $x \leq \sigma_{u+1} - |T_{n,u+1}^{(k)}|$ and let $\epsilon(x) = \sigma_{u+k+1} - x$. Then $\hat{\Phi}_{\epsilon(x)}(x-t) = \hat{\Phi}_{\epsilon(x)}(t-x) \geq \frac{1}{4\epsilon(x)}$ for $t \in$ supp ϕ_1, and consequently

$$\phi_1^{**}(x) \geq \int_0^1 \hat{\Phi}_{\epsilon(x)}(t-x)\phi_1(t)dt \geq \frac{1}{4\epsilon(x)}.$$

Combining this with (3.5) we find that

$$(\hat{P}_n^{(k)}\phi)^{**}(x) \geq \phi_1^{**}(x) - (\hat{P}_n^{(k)}\phi_2)^{**}(x) \geq \frac{1}{4\epsilon(x)} - \frac{C_k}{|T_{n,u}^{(1)}|} \quad \text{for} \quad x \in [\sigma_u, \sigma_{u+1} - |T_{n,u+1}^{(k)}|].$$

Then, as $|T_{n,u}^{(1)}| \geq \frac{M}{2}|T_{n,u+1}^{(k)}|$, we have for $M \geq 32C_k$

$$(\hat{P}_n^{(k)}\phi)^{**}(x) \geq \frac{1}{8\epsilon(x)} \quad \text{for} \quad x \in [\sigma_{u+1} - \frac{|T_{n,u}^{(1)}|}{16C_k}, \sigma_{u+1} - |T_{n,u+1}^{(k)}|]. \tag{3.6}$$

Using again $|T_{n,u}^{(1)}| \geq \frac{M}{2}|T_{n,u+1}^{(k)}|$, we get from the last inequality

$$\begin{aligned} \|(\hat{P}_n^{(k)}\phi)^{**}\|_{L^1(\mathbb{T})} &\geq \int_{\sigma_{u+1} - \frac{|T_{n,u}^{(1)}|}{16C_k}}^{\sigma_{u+1} - |T_{n,u+1}^{(k)}|} (\hat{P}_n^{(k)}\phi)^{**}(x)dx \\ &\geq \int_{2|T_{n,u+1}^{(k)}|}^{\frac{|T_{n,u}^{(1)}|}{16C_k} + |T_{n,u+1}^{(k)}|} \frac{1}{8u}du \geq \frac{1}{8}\log M - C_{k,1}. \end{aligned}$$

Fix $M(k) \geq 32C_k$ and such that $\frac{1}{16}\log M(k) \geq C_{k,1}$; then for $M \geq M(k)$ we have $\frac{1}{8}\log M - C_{k,1} \geq \frac{1}{16}\log M$. As $\|\phi\|_{H^1(\mathbb{T})} \leq 2k$, by Lemma 3.1 we get $\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \geq C_k \log M$. •

3.4. **Sufficiency of k-regularity: proof of Proposition 3.2.** The idea of the proof is analogous to the idea of the proof of Proposition 3.2 in [12]. We recall the mesh (2.1) obtained by the canonical identification of $\mathbb{T}$ with $[0,1)$

$$\hat{\mathcal{T}}_n = (0 \leq \sigma_{n,0} \leq \sigma_{n,1} \leq \cdots \leq \sigma_{n,n-2} \leq \sigma_{n,n-1} < 1).$$

Let $n \geq 2k$ and η be a periodic atom. It is enough to show that

$$\|\hat{P}_n^{(k)}\eta\|_{H^1(\mathbb{T})} \lesssim_{k,\gamma} 1.$$

For this, we find a suitable atomic decomposition of $\hat{P}_n^{(k)}\eta$.

If $\eta \equiv 1$, then also $\hat{P}_n^{(k)}\eta \equiv 1$ and it is a periodic atom.

Now, let $\int_{\mathbb{T}} \eta(t)dt = 0$, and let $\Gamma \subset \mathbb{T}$ be an interval such that supp $\eta \subset \Gamma$, $\|\eta\|_{L^\infty(\mathbb{T})} \leq \frac{1}{|\Gamma|}$. Let

$$\mathcal{G} = \{0 \leq i \leq n-1 : \operatorname{supp} \hat{N}_i \cap \Gamma = \emptyset\}.$$

Put

$$\psi_i = \hat{N}_i \cdot \hat{P}_n^{(k)}\eta \quad \text{for} \quad i \in \mathcal{G}, \quad \text{and} \quad \psi = \hat{P}_n^{(k)}\eta - \sum_{i \in \mathcal{G}} \psi_i. \tag{3.7}$$

We check that the collection $\{\psi, \psi_i, i \in \mathcal{G}\}$ gives a desired (periodic) atomic decomposition of $\hat{P}_n^{(k)}$. Clearly, $\hat{P}_n^{(k)} = \psi + \sum_{i\in\mathcal{G}} \psi_i$. For $i \in \mathcal{G}$ the supports of $\hat{N}_i$ and η are disjoint. Since $\hat{P}_n^{(k)}$ is an orthogonal projection onto $\hat{\mathcal{S}}_n^{(k)}$ we have for $i \in \mathcal{G}$

$$\int_{\mathbb{T}} \psi_i(t)dt = \int_{\mathbb{T}} \hat{N}_i(t) \cdot \hat{P}_n^{(k)}\eta(t)dt = \int_{\mathbb{T}} \hat{N}_i(t) \cdot \eta(t)dt = 0.$$

Since $\int_{\mathbb{T}} \eta(t) = 0$, we have also $\int_{\mathbb{T}} \hat{P}_n^{(k)} \eta(t)dt = 0$, which implies $\int_{\mathbb{T}} \psi(t)dt = 0$.

Now, we estimate $\|\psi_i\|_\infty$ for $i \in \mathcal{G}$. Let $0 \le m \le n-1$ be the unique index such that σ_m is not in Γ, but σ_{m+1} is in Γ. Next, we let $0 \le l \le n-1$ be the unique index such that σ_{l-1} is in Γ, but σ_l is not in Γ, Then $\mathcal{G} = \{0 \le i \le n-1 : i \le m-k\} \cup \{0 \le i \le n-1 : i \ge l\} =: \mathcal{G}_1 \cup \mathcal{G}_2$.

Consider the case $i \in \mathcal{G}_1$. Note that supp $\psi_i \subset$ supp $\hat{N}_i = [\sigma_i, \sigma_{i+k}]$. Recall that (cf. formulae in the Section 3.1)

$$\hat{P}_n^{(k)} \eta(t) = \sum_{j_1, j_2 = 0}^{n-1} \hat{a}_{j_1, j_2} \int_{\mathbb{T}} \hat{N}_{j_1}(u) \eta(u) du \, \hat{N}_{j_2}(t). \tag{3.8}$$

By Theorem 3.2, we have the estimate

$$|\hat{a}_{j_1,j_2}| \lesssim_k \frac{q^{\hat{d}(j_1,j_2)}}{\max(|\operatorname{supp} \hat{N}_{j_1}|, |\operatorname{supp} \hat{N}_{j_2}|)},$$

where $0 < q < 1$, depends only on the order k. Note that if $t \in$ supp ψ_i and j_2 is such that $\hat{N}_{j_2}(t) \ne 0$, then for those indices j_2 we have

$$|T_{n,j_2}^{(k)}| \sim_{k,\gamma} |T_{n,i}^{(k)}|,$$

by the k-regularity. Therefore, for j_2 such that $\hat{N}_{j_2}(t) \ne 0$

$$|\hat{a}_{j_1,j_2}| \lesssim_{k,\gamma} \frac{q^{\hat{d}(j_1,j_2)}}{|T_{n,i}^{(k)}|}.$$

Moreover, the number of the indices j_2 such that $N_{j_2}(t) \ne 0$ doesn't exceed $2k-1$. Thus, (3.8) gives

$$|\psi_i(t)| \le |\hat{P}_n^{(k)} \eta(t)| \lesssim_{k,\gamma} \frac{1}{|T_{n,i}^{(k)}|} \sum_{j_1=0}^{n-1} q^{\hat{d}(i,j_1)} \Big| \int_{\mathbb{T}} \hat{N}_{j_1}(u) \eta(u) du \Big|.$$

Next, note that if j_1 is such that supp $\hat{N}_{j_1} \cap \operatorname{supp}\eta \ne \emptyset$ then $m-k+1 \le j_1 \le l-1$. Moreover, we have $\int_{\mathbb{T}} |\hat{N}_{j_1}(u)\eta(u)| du \le 1$. Therefore the above inequality implies

$$|\psi_i(t)| \lesssim_{k,\gamma} \frac{1}{|T_{n,i}^{(k)}|} \sum_{j_1=m-k+1}^{l-1} q^{\hat{d}(i,j_1)} \lesssim_k \frac{q^{\min\{\hat{d}(i,l), \hat{d}(i,m)\}}}{|T_{n,i}^{(k)}|}.$$

Now, put $\alpha_i = \|\psi_i\|_{L^\infty(\mathbb{T})} |T_{n,i}^{(k)}|$ and $\psi_i = \alpha_i \tilde{\psi}_i$. Clearly, supp $\tilde{\psi}_i =$ supp $\psi_i \subset [\sigma_i, \sigma_{i+k}]$ and $\|\tilde{\psi}_i\|_{L^\infty(\mathbb{T})} \le \frac{1}{|T_{n,i}^{(k)}|}$, so $\tilde{\psi}_i$ is a periodic atom. Since

$$0 \le \alpha_i \lesssim_{k,\gamma} q^{\min\{\hat{d}(i,l), \hat{d}(i,m)\}},$$

we finally get

$$\sum_{0 \le i \le m-k} \psi_i = \sum_{0 \le i \le m-k} \alpha_i \tilde{\psi}_i \quad \text{with} \quad \sum_{0 \le i \le m-k} \alpha_i \lesssim_{k,\gamma} 1. \tag{3.9}$$

Analogously, for $i \geq l$ we get $\psi_i = \alpha_i \tilde{\psi}_i$ with $\tilde{\psi}_i$ a periodic atom and $0 \leq \alpha_i \lesssim_{k,\gamma} q^{\min\{\hat{d}(i,l),\hat{d}(i,m)\}}$, and consequently

$$\sum_{l \leq i \leq n-1} \psi_i = \sum_{l \leq i \leq n-1} \alpha_i \tilde{\psi}_i \quad \text{with} \quad \sum_{l \leq i \leq n-1} \alpha_i \lesssim_{k,\gamma} 1. \tag{3.10}$$

It remains to consider ψ. Since the functions $\hat{N}_j$, $0 \leq j \leq n-1$, are a partition of unity, we have $\psi = \hat{P}_n^{(k)} \eta \cdot \sum_{j=m-k+1}^{l-1} \hat{N}_j$. Let $\tilde{\Gamma} = [\sigma_{m-k+1}, \sigma_{l+k-1}]$. Note that $\operatorname{supp} \psi \subset \tilde{\Gamma}$. We will show that $\|\psi\|_{L^\infty(\mathbb{T})} \lesssim_{k,\gamma} \frac{1}{|\tilde{\Gamma}|}$.

At first, consider the case when Γ contains at least one support of $\hat{N}_j$. Then by the k-regularity $|\Gamma| \sim_{k,\gamma} |\tilde{\Gamma}|$. Using $\int_{\mathbb{T}} |\hat{D}_n^{(k)} \eta(t,s)| ds \lesssim_k 1$ (cf. 3.1) we get

$$|\psi(t)| \leq |\hat{P}_n^{(k)} \eta(t)| \leq \int_{\mathbb{T}} |\hat{D}_n^{(k)} \eta(t,s)||\eta(s)| ds \leq \frac{1}{|\Gamma|} \int_{\mathbb{T}} |\hat{D}_n^{(k)}(t,s)| ds \lesssim_{k,\gamma} \frac{1}{|\tilde{\Gamma}|}.$$

In the other case, i.e. when Γ does not contain any B-spline support, it follows by the k-regularity that $|T_{n,m}^{(k)}| \sim_{k,\gamma} |\tilde{\Gamma}|$. We again use formula (3.8) to estimate $\|\psi\|_\infty$. If j_1 is such that $(\hat{N}_{j_1}, \eta) \neq 0$, then $m-k+1 \leq j_1 \leq l-1$, so by the k-regularity $|T_{n,m}^{(k)}| \sim_{k,\gamma} |T_{n,j_1}^{(k)}|$. This and Theorem 3.2 imply that $|\hat{a}_{j_1,j_2}^{(k)}| \lesssim_{k,\gamma} \frac{1}{|T_{n,m}^{(k)}|}$. As $|(\hat{N}_{j_1}, \eta)| \leq 1$, we get $\sum_{j_1=0}^{n-1} |\hat{a}_{j_1,j_2}||(\hat{N}_{j_1}, \eta)| = \sum_{j_1=m-k+1}^{m+k} |\hat{a}_{j_1,j_2}||(\hat{N}_{j_1}, \eta)| \lesssim_{k,\gamma} \frac{1}{|T_{n,m}^{(k)}|}$. As the functions $\hat{N}_{j_2}$, $0 \leq j_2 \leq n-1$ are a partition of unity, we get for $t \in \tilde{\Gamma}$

$$|\psi(t)| \leq |\hat{P}_n^{(k)} \eta(t)| \lesssim_{k,\gamma} \frac{1}{|T_{n,m}^{(k)}|} \lesssim_{k,\gamma} \frac{1}{|\tilde{\Gamma}|}.$$

It follows from these considerations that $\psi = \alpha \tilde{\psi}$, where $\tilde{\psi}$ is a periodic atom and $0 \leq \alpha \lesssim_{k,\gamma} 1$. Putting together this fact, (3.7), (3.9) and (3.10) we get for periodic atoms $\tilde{\psi}, \tilde{\psi}_i$

$$\hat{P}_n^{(k)} \eta = \sum_{0 \leq i \leq m-k} \alpha_i \tilde{\psi}_i + \alpha \tilde{\psi} + \sum_{l \leq i \leq n-1} \alpha_i \tilde{\psi}_i,$$

where $\alpha, \alpha_i \geq 0$ and $\sum_{0 \leq i \leq m-k} \alpha_i + \alpha + \sum_{l \leq i \leq n-1} \alpha_i \lesssim_{k,\gamma} 1$. This is the desired atomic decomposition of $\hat{P}_n^{(k)} \eta$.

Next we consider the case when $n < 2k$. Let $f \in H^1(\mathbb{T})$, then

$$\begin{aligned} \|\sum_{m=1}^{n} (f, \hat{f}_m) \hat{f}_m\|_{H^1(\mathbb{T})} &\leq \sum_{m=1}^{n} \|f\|_{L^1(\mathbb{T})} \|\hat{f}_m\|_{L^\infty(\mathbb{T})} \|\hat{f}_m\|_{H^1(\mathbb{T})} \\ &\leq \|f\|_{H^1(\mathbb{T})} \sum_{m=1}^{2k} \|\hat{f}_m\|_{L^\infty(\mathbb{T})} \|\hat{f}_m\|_{H^1(\mathbb{T})} \leq C_k \|f\|_{H^1(\mathbb{T})}. \end{aligned}$$

This concludes the proof of the proposition.

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UNIQUENESS OF MEROMORPHIC FUNCTIONS WHEN ITS SHIFT AND FIRST DERIVATIVE SHARE THREE VALUES

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Abstract. This paper brings out some improvements as well as generalization results of a paper of X. Qi and L. Yang [17] [Comput. Methods Funct. Theory, 20, 159-178 (2020)], which deals with the uniqueness results of $f'(z)$ and $f(z+c)$. To be more realistic about the obtained results, we exhibit some examples.

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1. Introduction, definitions and results

We assume that the reader is familiar with meromorphic function, standard notations and main results of Nevanlinna's value distribution theory [9, 20]. As usual, the abbreviation "CM" means "counting multiplicities while "IM" stands for "ignoring multiplicities".

Let f and g be two meromorphic functions in the complex plane $\mathbb{C}$. In particular, let z_n, $n = 1, 2, \ldots$, be the zeros of $f - a$ with multiplicity $h(n)$. If z_n are also $h(n)$ multiple zeros of $g - a$ at least, then we write $f = a \to g = a$, where $a \in \mathbb{C} \cup \{\infty\}$.

The order of f is denoted by $\sigma(f)$ and is defined by

$$\sigma(f) = \limsup_{r \longrightarrow \infty} \frac{\log^+ T(r, f)}{\log r}.$$

Rubel and Yang [18] first investigated the uniqueness of an entire function concerning its derivative, and proved the following result.

Theorem A. *Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share two distinct finite values a, b CM, then $f(z) \equiv f'(z)$.*

Mues and Steinmetz [[14], Satz 1] showed the sharing assumption in Theorem A can be replaced by 2 IM. Afterwards, Mues and Steinmetz [15], Gundersen [[6], Thm. 1] improved Theorem A to a non-constant meromorphic function.

Theorem B. *Let $f(z)$ be a non-constant meromorphic function. If $f(z)$ and $f'(z)$ share two distinct finite values a, b CM, then $f(z) \equiv f'(z)$.*

Gundersen [6] has given a counterexample to show the sharing assumption in Theorem B cannot be improved to 1 CM + 1 IM. Further, 2 CM can be replaced by 3 IM, see [5, 14]. Moreover, the results stated above are still true if $f'(z)$ is changed to $f^{(k)}(z)$, where k is a positive integer. For some of related results, the reader is invited to see [[20],Ch. 8].
As a difference analogue, Heittokangas et al. [10, 11] started to consider meromorphic functions sharing values with their shifts. The background for these considerations lies in the parallel difference version to the usual Nevanlinna theory which starts with the papers [4, 7, 8].

Remark A. *We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. We define $S(f)$ is the family of all meromorphic functions $a(z)$ such that $T(r,a) = S(r,f)$ as $r \to \infty$. Here we call $a(z)$ as a small function with respect to f.*

A key results in [10] reads as follows.

Theorem C. *Let $f(z)$ be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) \equiv f(z+c)$.*

Later on, many authors consider the uniqueness of meromorphic functions of finite order concerning their shifts or differences. Some attempts towards relaxing the sharing assumptions can be found in [1, 2, 3, 13, 21].

In real analysis, the time-delay differential equation $f'(x) = f(x-k),\ k > 0$, has been extensively studied. As for a complex variable counterpart, Liu and Dong studied the complex differential-difference equation $f'(z) = f(z+c)$, where c is a non-zero constant, see [12].

In [16], Qi and Yang looked at this complex differential-difference equation from different perspective. That is, "under what sharing values conditions, does $f'(z) = f(z+c)$ hold?" And they investigated the value sharing problem related to $f'(z)$ and $f(z+c)$ as follows.

Theorem D. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0, a CM, then $f'(z) \equiv f(z+c)$.*

Then, Qi and Yang [17] posed a list of questions related to Theorem D in the following.

Question A. *Can the value sharing condition be improved in Theorem D?*

Question B. *Can the condition "$f(z)$ is transcendental" be deleted in Theorem D?*

Corresponding to the questions above, they obtained the following results.

Theorem E. *Let $f(z)$ be a non-constant meromorphic function of finite order, let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a CM, and satisfy $f(z+c) = 0 \to f'(z) = 0$, $f(z+c) = \infty \leftarrow f'(z) = \infty$. Then $f'(z) \equiv f(z+c)$. Further, $f(z)$ is a transcendental entire function.*

Theorem F. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0 CM and a IM, then $f'(z) \equiv f(z+c)$.*

Theorem G. *Let $f(z)$ be a transcendental entire function of finite order, and let a, b be two distinct finite values. If $f'(z)$ and $f(z+c)$ share a, b IM, then*

$$T(r, f(z+c)) = O\left(T(r, f')\right), \quad T(r, f') = O\left(T(r, f(z+c))\right),$$

as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure.

In the following, we now pose a list of questions relevant to Theorems E-G such that the conclusions of the theorems are intact.

Question 1.1. *Can the value sharing condition "$f'(z)$ and $f(z+c)$ share a CM" be further improved by "$f'(z)$ and $f(z+c)$ share a IM" in Theorem E, where $a(\neq 0) \in \mathbb{C}$?*

Question 1.2. *Can one further weaker the condition "$f'(z)$ and $f(z+c)$ share 0 CM" by "$f(z+c) = 0 \to f'(z) = 0$" in Theorem F?*

Question 1.3. *What happen if the condition "$f'(z)$ and $f(z+c)$ share a, b IM for a transcendental entire function $f(z)$" be replaced by "$f(z+c) = 0 \to f'(z) = 0$ and $f(z+c) = \infty \leftarrow f'(z) = \infty$ for transcendental meromorphic function $f(z)$" in Theorem G?*

Corresponding to the questions above, we get the following main results of this paper.

Theorem 1.1. *Let $f(z)$ be a non-constant meromorphic function of finite order, let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a IM and satisfy $f(z+c) = 0 \to f'(z) = 0$, $f(z+c) = \infty \leftarrow f'(z) = \infty$, then $f'(z) \equiv f(z+c)$ and $f(z)$ is a transcendental entire function of finite order.*

Remark 1.1. *Clearly we see that Theorem E holds for the condition "$f'(z)$ and $f(z+c)$ share a CM" whenever Theorem 1.1 holds for the condition "$f'(z)$ and $f(z+$*

c) share a IM". We know that, in a particular case 'IM' sharing condition becomes 'CM' sharing condition. So Theorem 1.1 is an improvement result of Theorem E.

From Theorem 1.1, it is sufficient to consider the condition that $f(z)$ is an entire function. And then we obtain the following result.

Corollary 1.1. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a IM and satisfy $f(z+c) = 0 \to f'(z) = 0$. Then $f'(z) \equiv f(z+c)$.*

Remark 1.2. *Clearly if we consider "Let $f(z)$ be a transcendental entire function" in Theorem 1.1, then the condition "$f(z+c) = \infty \leftarrow f'(z) = \infty$" doesn't arise. Then this theorem becomes the same as Corollary 1.1. So the proof of Corollary 1.1 follows from the proof of Theorem 1.1.*

Again we observe that Theorem F holds for the condition "$f'(z)$ and $f(z+c)$ share 0 CM" whenever Corollary 1.1 holds for the condition "$f(z+c) = 0 \to f'(z) = 0$". By definition, in a particular situation the condition "$f(z+c) = 0 \to f'(z) = 0$" becomes the condition "$f'(z)$ and $f(z+c)$ share 0 CM". In this sense, Corollary 1.1 is an improvement result of Theorem F.

By Theorem 1.1, we can consider the condition that $f(z)$ is a transcendental meromorphic function and obtain the result as follows.

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of finite order. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c) = 0 \to f'(z) = 0$ and $f(z+c) = \infty \leftarrow f'(z) = \infty$, then*

$$T(r, f') = T(r, f(z+c)) + S(r, f).$$

Remark 1.3. *We observe that if we consider "Let $f(z)$ be a transcendental entire function" in Theorem 1.2, then the condition "$f(z+c) = \infty \leftarrow f'(z) = \infty$" doesn't arise. Then Theorem 1.2 holds for one finite shared-value 0 whenever Theorem G holds for two finite shared-value a and b. In this sense, Theorem 1.2 is an improvement result of Theorem G. Also we observe that Theorem G holds for an entire function whenever Theorem 1.2 holds for a transcendental meromorphic function. In this sense, Theorem 1.2 is a generalization result of Theorem G.*

Remark 1.4. *To be more realistic about Theorem 1.1 and validity of the conditions in theorem, the following examples are relevant.*

Example 1.1. *Let $f(z) = e^{-z}$, $a \in \mathbb{C} \setminus \{0\}$ and $c = \pi i$. Then $f(z+c) = -e^{-z}$ and $f'(z) = -e^{-z}$. So $f'(z)$ and $f(z+c)$ share $0, \infty$ CM and a CM. Thus $f(z)$ satisfy the conditions of Theorem 1.1. Consequently, $f'(z) \equiv f(z+c)$ follows.*

Example 1.2. *Let* $f(z) = \sin z + \cos z$, $b \in \mathbb{C} \setminus \{0\}$ *and* $c = \frac{\pi}{2}$. *Then* $f(z+c) = \cos z - \sin z$ *and* $f'(z) = \cos z - \sin z$. *So* $f'(z)$ *and* $f(z+c)$ *share* $0, \infty$ *CM and a CM. Thus* $f(z)$ *satisfy the conditions of Theorem 1.1. Consequently,* $f'(z) \equiv f(z+c)$ *follows.*

Example 1.3. [12] *We consider the following functions:*

(i) $f(z) = (b_1 z + b_0)e^{ez+B}$, *where* $b_1(\neq 0), b_0, B \in \mathbb{C}$ *and* $c = \frac{1}{e}$;

(ii) $f(z) = b_0 e^{Az+B}$, *where* $c = \frac{\ln|A| + i(argA + 2k\pi)}{A}$, $k \in \mathbb{Z}$ *and* $A(\neq 0) \in \mathbb{C}$;

(iii) $f(z) = g(z)e^{Az+B}$, *where* $g(z)$ *is a transcendental entire function and satisfies* $g'(z) = A\left(g(z+c) - g(z)\right)$ *and* $\sigma(g) < 1$, *where* $A(\neq 0) \in \mathbb{C}$ *and* $c = \frac{\ln|A| + i(argA + 2k\pi)}{A}$.

We observe that the above functions satisfy the conditions of Theorem 1.1. Consequently, $f'(z) \equiv f(z+c)$ *follows.*

Remark 1.5. *The condition "*$f(z+c) = \infty \leftarrow f'(z) = \infty$*" in Theorem 1.1 is sharp and it follows by the following example.*

Example 1.4. [12] *Let* $f(z) = \frac{2}{1-e^{-2z}}$ *and* $c = \pi i$. *Then even* $f'(z)$ *and* $f(z+c)$ *share* 1 *IM and satisfies* $f(z+c) = 0 \to f'(z) = 0$, $f'(z) \not\equiv f(z+c)$ *follows, since* $f(z+c) = \infty \not\leftarrow f'(z) = \infty$.

Remark 1.6. *The condition "*$f(z+c) = 0 \to f'(z) = 0$*" in Theorem 1.1 is sharp and it follows by the following example.*

Example 1.5. *Let* $f(z) = k_1 e^{k_2 z} - 1$ *and* $e^{k_2 c} = 2k_2$, *where* $k_1, k_2 \in \mathbb{C} \setminus \{0\}$. *Then even* $f'(z)$ *and* $f(z+c)$ *share* 1 *IM and satisfies* $f(z+c) = \infty \leftarrow f'(z) = \infty$, $f'(z) \not\equiv f(z+c)$ *follows, since* $f(z+c) = 0 \not\to f'(z) = 0$.

Remark 1.7. *The condition "*$f'(z)$ *and* $f(z+c)$ *share a IM,* $a \in \mathbb{C} \setminus \{0\}$*" in Theorem 1.1 is sharp and it follows by the following example.*

Example 1.6. *Let* $f(z) = \frac{1}{2}e^{4z - \frac{1}{2}}$ *and* $e^{4c} = 5$. *Then even* $f'(z)$ *and* $f(z+c)$ *satisfy* $f(z+c) = 0 \to f'(z) = 0$, $f(z+c) = \infty \leftarrow f'(z) = \infty$, $f'(z) \not\equiv f(z+c)$ *follows, since* $f'(z)$ *and* $f(z+c)$ *does not share a IM.*

Remark 1.8. *Clearly, the function* $f(z)$ *is of finite order in Theorem 1.1 is sharp by the following example.*

Example 1.7. *Let* $f(z)$ *be a transcendental meromorphic function of infinite order such that* $f(z+c) = \frac{e^z - 1}{z + e^z - e^{e^z}}$ *and* $f'(z) = \frac{\left(e^{e^z} - z\right)(e^z - 1)}{z + e^z - e^{e^z}}$. *Clearly,* $f'(z) - 1 = e^z\left(f(z+c) - 1\right)$ *and* $f'(z) = \left(e^{e^z} - z\right) f(z+c)$. *So* $f'(z)$ *and* $f(z+c)$ *share* 1 *CM* *and satisfy* $f(z+c)=0 \to f'(z)=0$, $f(z+c)=\infty \leftarrow f'(z)=\infty$. *Clearly, the conclusion of Theorem 1.1 doesn't hold in this situation, i.e.,* $f'(z) \not\equiv f(z+c)$.

2. Some Lemmas

The following are relevant lemmas of this paper and are used in the sequel.

Lemma 2.1. [19] *Let* $f(z)$ *be a non-constant meromorphic function, and let* $a_i(z)$ *be meromorphic functions such that* $T(r,a_i)=S(r,f)$ *for* $i=0,1,2,\ldots,n$, *where* $a_n(z)\not\equiv 0$. *Then we have*

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0\right) = nT(r,f) + S(r,f).$$

Lemma 2.2. [4, 7] *Let* $f(z)$ *be a non-constant meromorphic function of finite order* σ, *and let* $c \in \mathbb{C}\setminus\{0\}$ *be fixed. Then for each* $\varepsilon > 0$, *we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.3. [10] *Let* $f(z)$ *be a non-constant meromorphic function of finite order, and let* $c \in \mathbb{C}\setminus\{0\}$ *be fixed. Then we have*

$$N(r,0;f(z+c)) \leq N(r,0;f(z)) + S(r,f), \tag{2.1}$$

$$N(r,\infty;f(z+c)) \leq N(r,\infty;f) + S(r,f),$$

$$\overline{N}(r,0;f(z+c)) \leq \overline{N}(r,0;f(z)) + S(r,f) \tag{2.2}$$

$$\text{and} \quad \overline{N}(r,\infty;f(z+c)) \leq \overline{N}(r,\infty;f) + S(r,f). \tag{2.3}$$

Lemma 2.4. *[*[4]*, Lem. 5.1] Let* $f(z)$ *be a non-constant meromorphic function of finite order* σ, *and let* $c \in \mathbb{C}\setminus\{0\}$ *be fixed. Then for each* $\varepsilon > 0$, *we have*

$$T\left(r, f(z+c)\right) = T(r,f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).$$

Furthermore, if $f(z)$ *is a transcendental meromorphic function with finite order, then we have*

$$T(r, f(z+c)) = T(r,f) + S(r,f).$$

Remark 2.1. *By Lemma 2.4, we conclude for a non-constant meromorphic function* $f(z)$ *of finite order that,* $S(r,f) = S(r, f(z+c))$.

Remark 2.2. *By Lemmas 2.2 and 2.4, we can see that, if* $f(z)$ *is a transcendental meromorphic function with finite order, then*

$$N(r,\infty;f) = N(r,\infty;f(z+c)) + S(r,f) \tag{2.4}$$

$$\text{and} \quad N(r,0;f) = N(r,0;f(z+c)) + S(r,f). \tag{2.5}$$

Lemma 2.5. *[[9], p. 55] Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Then we have*

$$T\left(r, f^{(k)}\right) \leq (1+o(1))(k+1)T(r,f),$$

as $r \to \infty$ possibly outside some exceptional set of finite linear measure.

Remark 2.3. *By Remark 2.1 and Lemma 2.5, we conclude for a non-constant meromorphic function $f(z)$ of finite order that $S(r,f') \leq S(r,f) = S(r, f(z+c))$.*

We now introduce some relevant notations for this paper. Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}_0\left(r, a; f'(z) \mid f(z+c) = a\right)$ the reduced counting function of common a-points of $f'(z)$ and $f(z+c)$ of different multiplicities, whereas

$$N_E\left(r, a; f'(z) \mid f(z+c) = a\right)$$

denotes the counting function of common a-points of $f'(z)$ and $f(z+c)$ of equal multiplicities.

Again we denote by $N\left(r, a; f'(z) \mid f(z+c) \neq a\right)$ the counting function of a-points of $f'(z)$ which are not the a-points of $f(z+c)$.

In the following, we define

$$H(z) = \frac{f'(z)}{f(z+c)}. \tag{2.6}$$

Lemma 2.6. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let $H(z)$ be defined in (2.6). If $f'(z)$ and $f(z+c)$ satisfy $f(z+c) = 0 \to f'(z) = 0$ and $f(z+c) = \infty \leftarrow f'(z) = \infty$, then the following conclusions occur:*

(i) *$H(z)$ is an entire function and $T(r,H) = S(r,f)$,*
(ii) *$N_0\left(r, 0; f'(z) \mid f(z+c) = 0\right) + N\left(r, 0; f'(z) \mid f(z+c) \neq 0\right) = S(r,f)$,*
(iii) *$N_0\left(r, \infty; f(z+c) \mid f'(z) = \infty\right) + N\left(r, \infty; f(z+c) \mid f'(z) \neq \infty\right) = S(r,f)$,*
(iv) *$T(r,f') = T(r, f(z+c)) + S(r,f)$,*
(v) *$S(r,f) = S(r, f(z+c)) = S(r,f')$ and*
(vi) *$\overline{N}(r, \infty; f(z+c)) = \overline{N}(r, \infty; f') = S(r,f)$.*

Proof. Clearly from $f(z+c) = 0 \to f'(z) = 0$ and $f(z+c) = \infty \leftarrow f'(z) = \infty$ it follows that H has zeros only and so H is an entire function. But the zeros of $f'(z)$ and $f(z+c)$ of equal multiplicities are not zeros of $H(z)$. Since $f(z)$ is a non-constant meromorphic function of finite order, in view of Logarithmic Derivative Lemma and by Lemma 2.2, we get

$$m(r,H) \leq m\left(r, \frac{f'(z)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f).$$

Then we see that

$$T\,(r,H)=N\,(r,\infty;H)+m\,(r,H)=S(r,f).$$

This completes the proof (i). Note that

$$\begin{aligned}N_0\,(r,0;f'(z)\mid f(z+c)=0)+N\,(r,0;f'(z)\mid f(z+c)\neq 0)\leq\\ \leq N(r,0;H)\leq T(r,H)=S(r,f).\end{aligned}$$

Similarly, we get

$$N_0\,(r,\infty;f(z+c)\mid f'(z)=\infty)+N\,(r,\infty;f(z+c)\mid f'(z)\neq\infty)=S(r,f).$$

This completes the proofs (ii) and (iii). Now using (2.6), the conclusion (i) and the first fundamental theorem of Nevanlinna, we get

$$\begin{aligned}&T\,(r,f')\leq T(r,H)+T(r,f(z+c))=T(r,f(z+c))+S(r,f)\\ \text{and}\quad &T\,(r,f(z+c))\leq T\left(r,\frac{1}{H}\right)+T(r,f')\\ \leq\quad &T(r,H)+T(r,f')+S(r,f)=T(r,f')+S(r,f).\end{aligned}$$

Hence the conclusion (iv) follows. So we use this result whenever needed in the following. By Remark 2.3 and the conclusion (iv), we get $S(r,f)=S(r,f(z+c))=S(r,f')$. Hence the conclusion (v) follows. From the assumption, we have

$$N(r,\infty;f')\leq N(r,\infty;f(z+c))+S(r,f). \tag{2.7}$$

Note that

$$N(r,\infty;f')=N(r,\infty;f)+\overline{N}(r,\infty;f) \tag{2.8}$$

$$\text{i.e.,}\qquad N(r,\infty;f')\geq N(r,\infty;f) \tag{2.9}$$

By (2.4) and (2.9), we get

$$N(r,\infty;f')\geq N(r,\infty;f(z+c))+S(r,f). \tag{2.10}$$

Then from (2.4), (2.8) and (2.7), (2.10), we get respectively

$$\begin{aligned}&N(r,\infty;f')=N(r,\infty;f(z+c))+\overline{N}(r,\infty;f)+S(r,f)\\ \text{and}\qquad &N(r,\infty;f')=N(r,\infty;f(z+c))+S(r,f).\end{aligned}$$

Consequently we get

$$\overline{N}(r,\infty;f')=\overline{N}(r,\infty;f)=S(r,f). \tag{2.11}$$

Note that

$$\overline{N}_E\,(r,\infty;f(z+c)\mid f'(z)=\infty)\leq\overline{N}(r,\infty;f')=S(r,f).$$

Then using the conclusion (iii), we get

$$\begin{aligned}\overline{N}(r,\infty;f(z+c)) = \quad & \overline{N}_0\,(r,\infty;f(z+c) \mid f'(z)=\infty) + \overline{N}_E\,(r,\infty;f(z+c) \mid f'(z)=\infty) \\ (2.12) \qquad & +\overline{N}\,(r,\infty;f(z+c) \mid f'(z)\neq\infty) = S(r,f).\end{aligned}$$

So from (2.11) and (2.12), we get

$$\overline{N}(r,\infty;f') = \overline{N}(r,\infty;f(z+c)) = S(r,f).$$

Hence the result (vi) follows. □

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of finite order, let $H(z)$ be defined in (2.6) and $a \in \mathbb{C}\setminus\{0\}$. If $f'(z)$ and $f(z+c)$ share a IM and satisfy $f(z+c)=0 \to f'(z)=0$ and $f(z+c)=\infty \leftarrow f'(z)=\infty$. Then $f'(z)\equiv f(z+c)$ and further $f(z)$ is a transcendental entire function of finite order.*

Proof. From (2.6), we get

$$H(z)-1 = \frac{f'(z)-f(z+c)}{f(z+c)}. \tag{2.13}$$

Now the two possibilities may arise, i.e., either $H(z)\not\equiv 1$ or $H(z)\equiv 1$. So we consider two cases separately in the following.

Case 1. Suppose $H(z)\equiv 1$. Consequently from (2.13), we get

$$f'(z)\equiv f(z+c). \tag{2.14}$$

Then by the same argument of proof used in Theorem E, we get that $f(z)$ is a transcendental entire function.

Case 2. Suppose $H(z)\not\equiv 1$. Then $H(z)-1$ has zeros. Since $f'(z)$ and $f(z+c)$ share a IM, from (2.13) we get

$$\overline{N}(r,a;f'(z)) = \overline{N}(r,a;f(z+c)) \le N(r,1;H) \le T(r,H) = S(r,f). \tag{2.15}$$

We claim that $\overline{N}_E(r,0;f'(z)\mid f(z+c)=0)\neq S(r,f)$. If not, suppose $\overline{N}_E(r,0;f'(z)\mid f(z+c)=0)=S(r,f)$. Then from the conclusion (ii) of Lemma 2.6, we deduce that

$$\begin{aligned}\overline{N}(r,0;f'(z)) \quad = \quad & \overline{N}_0\,(r,0;f'(z)\mid f(z+c)=0) + \overline{N}_E(r,0;f'(z)\mid f(z+c)=0) \\ & +\overline{N}\,(r,0;f'(z)\mid f(z+c)\neq 0) = S(r,f).\end{aligned}$$

So by the second fundamental theorem of Nevanlinna, we get

$$T(r,f') \le \overline{N}(r,\infty;f') + \overline{N}(r,0;f') + \overline{N}(r,a;f') + S(r,f') = S(r,f'),$$

which is impossible. Hence $\overline{N}_E(r,0;f'(z)\mid f(z+c)=0)\neq S(r,f)$. We now consider

$$H_1(z) = \frac{f''(z)}{f'(z)-a} - \frac{f'(z+c)}{f(z+c)-a}. \tag{2.16}$$

Then the two possibilities may arise, i.e., either $H_1(z) \not\equiv 0$ or $H_1(z) \equiv 0$. First we suppose $H_1(z) \equiv 0$. Then on integration from (2.16), we get

$$f'(z) - a \equiv d\left(f(z+c) - a\right),$$

where $d \in \mathbb{C} \setminus \{0\}$. Let z_0 be a common zero of $f'(z)$ and $f(z+c)$. Then $f'(z_0) = 0$ and $f(z_0 + c) = 0$. So $d = 1$ and then $f'(z) \equiv f(z+c)$. Thus again the conclusions follow.

Next we suppose $H_1(z) \not\equiv 0$. Now by the conclusion (vi) of Lemma 2.6 and (2.15), we get

$$N(r, \infty; H_1) = \overline{N}(r, \infty; f') + \overline{N}(r, a; f') + \overline{N}(r, \infty; f(z+c)) + \overline{N}(r, a; f(z+c)) = S(r, f).$$

Also $m(r, H_1) = S(r, f)$. Therefore $T(r, H_1) = S(r, f)$. Now from (2.6) by differentiation, we get

$$f''(z) = Hf'(z+c) + H'f(z+c).$$

Putting the above value in (2.16), we get

$$\begin{aligned} H_1 &= \frac{Hf'(z+c) + H'f(z+c)}{Hf(z+c) - a} - \frac{f'(z+c)}{f(z+c) - a} \\ &= \frac{H'f^2(z+c) - aHf'(z+c) - aH'f(z+c) + af'(z+c)}{Hf^2(z+c) - af(z+c) - aHf(z+c) + a^2}, \end{aligned}$$

i.e.,
$$\begin{aligned} &HH_1f^2(z+c) - aH_1f(z+c) - aHH_1f(z+c) + a^2H_1 \\ &= H'f^2(z+c) - aHf'(z+c) - aH'f(z+c) + af'(z+c) \end{aligned}$$

i.e.,
$$\begin{aligned} &a(H-1)f'(z+c) = (H' - HH_1)f^2(z+c) + (aH_1 + aHH_1 - aH')f(z+c) - a^2H_1 \\ &Af'(z+c) = Bf^2(z+c) + Cf(z+c) + D, \end{aligned}$$

where $A = a(H-1)$, $B = H' - HH_1$, $C = aH_1 + aHH_1 - aH'$ and $D = -a^2H_1$. Now suppose $B \not\equiv 0$. Note that A, B, C and D are small functions of $f(z+c)$. Then using Lemmas 2.1 and 2.6, we get

$$\begin{aligned} 2T(r, f(z+c)) &\leq T(r, f'(z+c)) + S(r, f(z+c)) \\ &\leq T\left(r, \frac{f'(z+c)}{f(z+c)}\right) + T(r, f(z+c)) + S(r, f(z+c)) \\ &= N\left(r, \infty; \frac{f'(z+c)}{f(z+c)}\right) + T(r, f(z+c)) + S(r, f(z+c)) \\ &= \overline{N}(r, \infty; f(z+c)) + T(r, f(z+c)) + S(r, f(z+c)) \\ &= T(r, f(z+c)) + S(r, f(z+c)), \end{aligned}$$

i.e., $T(r, f(z+c)) \leq S(r, f(z+c))$, which is impossible. So $B \equiv 0$ and then by (2.16), we get

$$\frac{H'}{H} \equiv H_1 = \frac{f''(z)}{f'(z) - a} - \frac{f'(z+c)}{f(z+c) - a}.$$

On integration, we have

$$H \equiv k\frac{f'(z)-a}{f(z+c)-a},$$

where $k \in \mathbb{C}\setminus\{0\}$. So by (2.6), we have

$$H(z) = \frac{f'(z)}{f(z+c)} \equiv k\frac{f'(z)-a}{f(z+c)-a}.$$

Now let z_0 be an a-point of $f(z+c)$ of multiplicity p, where $p \in \mathbb{N}$. Since $f'(z)$ and $f(z+c)$ share a IM, we conclude that z_0 be also an a-point of $f'(z)$ of multiplicity q, where $q \in \mathbb{N}$ and $f'(z_0) = a$. Since H is an entire function, we get that $p \leq q$. If $p < q$, then z_0 must be a zero of $f'(z)$. Consequently, we get $f'(z_0) = 0$. Since $a \in \mathbb{C}\setminus\{0\}$, simultaneously $f'(z_0) = 0$ and $f'(z_0) = a$ are impossible. So the only possibility is $p = q$. This shows that $f'(z)$ and $f(z+c)$ share a CM. Now it is clear that $f'(z)$ and $f(z+c)$ satisfy all conditions of Theorem E and consequently the claimed conclusions arise. □

3. Proofs of the main theorems

Proof of Theorem 1.1. Here $f(z)$ is a non-constant meromorphic function of finite order. So the following two cases separately occur.

Case 1. Let $f(z)$ be a non-constant rational function. Then $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)(\not\equiv 0)$ are two mutually prime polynomials. Following the same argument of proof used in Theorem E, we get $Q(z) \equiv$ constant $= k$, say, where $k \in \mathbb{C}\setminus\{0\}$.

As $f(z)$ is a non-constant rational function, $P(z)$ is a non-constant polynomial. Then $f(z) = \frac{1}{k}P(z)$. Furthermore, we have $f'(z) = \frac{1}{k}P'(z)$ and $f(z+c) = \frac{1}{k}P(z+c)$. As $f(z+c) = 0 \to f'(z) = 0$, we see that $P(z+c) = 0 \to P'(z) = 0$. Therefore $P'(z) \equiv P(z+c)p(z)$, where $p(z)(\not\equiv 0)$ is a polynomial of $\deg(p(z)) \geq 0$. But this contadicts the fact that $\deg(P'(z)) < \deg(P(z+c)p(z))$. So $f(z)$ is not a non-constant rational function.

Case 2. Let $f(z)$ be a transcendental meromorphic function of finite order. The next part of the theorem follows from the Lemma 2.7. □

Proof of Theorem 1.2. The proof of the theorem follows from the conclusion (iv) of Lemma 2.6. □

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UNIQUENESS OF A MEROMORPHIC FUNCTION PARTIALLY AND NORMALLY SHARING SMALL FUNCTIONS WITH ITS DIFFERENT VARIANTS OF GENERALIZED OPERATOR

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Abstract. First of all, in continuation of our previous result related to "2 CM+1 IM" small functions sharing of a meromorphic function of restricted hyper order and its linear shift delay differential operator, in some extend we have been able to answer a question paused by us in [Rendiconti del Circolo Mat. di Palermo, 2021 (Published online)]. As another attempt we improve and extend a result of [Comput. Methods Funct. Theory, 22(2), 197 – 205 (2022)]. Most importantly, we have pointed out a gap in the proof of a recent theorem [Results Math., 76, Article number: 147 (2021)] and citing a proper example we have shown that the result is true only for a particular case. Finally we present the compact version of the same result as an improvement.

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1. Introduction and some useful notations

At the outset we will assume that the readers are familiar with the standard notations and expressions like $m(r,f)$, $N(r,f)$ $(N(r,\infty;f))$, $N(r,\frac{1}{f-a})$ $(N(r,a;f))$, $T(r,f)$ in Nevanlinna theory for meromorphic functions defined on whole complex plane $\mathbb{C}$ (see [10], [19]). In addition, by $S(r,f)$ we mean a quantity satisfies $S(r,f)=o(T(r,f))$ as $r\longrightarrow\infty$ outside of a possible exceptional set E of finite logarithmic measure. We say that $a(z)(\not\equiv\infty)$ is a small function compared to $f(z)$ or slowly moving with respect to $f(z)$ if $T(r,a)=S(r,f)$. We denote by $S(f)$ the set of all small functions compared to $f(z)$ and $\hat{S}(f)$ by $S(f)\cup\{\infty\}$.

Some important terms namely order, hyper-order and ramification index of f will be defined respectively as follows:

$$\rho(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}\ ,\quad \rho_2(f)=\limsup_{r\longrightarrow\infty}\frac{\log\log T(r,f)}{\log r}$$

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$$\text{and} \quad \Theta(a;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where $a \in \mathbb{C} \cup \{\infty\}$.

The following definitions and notations are required in the sequel.

Definition 1.1. *For some $a \in \mathbb{C}$, we denote by $E(a;f)$, the collection of the zeros of $f-a$, where a zero is counted according to its multiplicity. In addition to this, when $a=\infty$, the above notation implies that we are considering the poles. In the same manner, by $\overline{E}(a;f)$, we denote the collection of the distinct zeros or poles of $f-a$ according as $a \in \mathbb{C}$ or $a=\infty$ respectively.*

If $E(a;f) = E(a;g)$ we say that f and g share the value a CM (counting multiplicities) and if $\overline{E}(a;f) = \overline{E}(a;g)$, then we say that f and g share the value a IM (ignoring multiplicities).

Especially, for $a(z) \in S(f)$, if $f-a(z)$ and $g-a(z)$ share 0 CM (IM), then we will say that f and g share $a(z)$ CM (IM). Let z_0 be a zero of $f-a(z)$ and $g-a(z)$ of multiplicity $p(\geq 0)$ and $q(\geq 0)$ respectively. We denote by $\overline{N}_{\otimes}(r,0,f-a(z);g-a(z))$, the reduced counting function of common zeros of $f-a(z)$ and $g-a(z)$ with different multiplicities that is $p \neq q$. On the other hand, for $a(z) \in S(f) \cup \{\infty\}$, if $E(0,f-a(z)) \subseteq E(0,g-a(z))$ $(E(0,g-a(z)) \subseteq E(0,f-a(z)))$, then we say that $f(g)$ and $g(f)$ share the small function $a(z)$ CM partially from $f(g)$ to $g(f)$.

Also we denote $N_{=1}(r,f)$ by the counting function of simple poles of f.

For $c \in \mathbb{C} \setminus \{0\}$, we define the shift of $f(z)$ by $f(z+c)$ or f_c and the difference operators of $f(z)$ by

$$\Delta_c f = f(z+c) - f(z), \qquad \Delta_c^k f = \Delta_c(\Delta_c^{k-1} f) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(z+ic),$$

where $k(\geq 2)$ is an integer. Generalization of shifts and derivatives operators, were recently done in [1]. We have defined the operators namely linear shift, shift-differential and differential operator, linear shift delay differential operator as follows:

$$L_1(f(z)) = a_0(z)f(z) + \sum_{i=1}^{k} a_i(z)f(z+c_i), \quad L_2(f(z)) = \sum_{i=1}^{s} b_i(z)f^{(i)}(z+c_i),$$

$$L_3(f(z)) = \sum_{i=1}^{t} d_i(z)f^{(i)}(z), \quad L(f(z)) = L_1(f(z)) + L_2(f(z)) + L_3(f(z)),$$

where $a_i(z)$ $(i=0,1,\ldots,k)$; $b_i(z)$ $(i=1,\ldots,s)$; $d_i(z)$ $(i=1,\ldots,t) \in S(f)$ and all $c_i's$ are non-zero complex constants. Also by delay-differential operator denoted by $\tilde{L}(f(z))$ and defined by $L_2(f(z)) + L_3(f(z))$. Choosing $c_i = ic$ for $i = 0,1,\ldots,k$, where c is non-zero complex constant, we denote $L_1(f(z))$ as

$$L_cf\left(=\sum_{j=0}^{k}a_j(z)f(z+jc)(\not\equiv 0)\right)$$

with $a_k(z)\neq 0$ $(k\geq 1)$ and it is called as linear c-shift operator. If we impose the restriction $\sum_{j=0}^{k}a_j(z)=s$ on the coefficients of L_cf, then we denote it by L_c^sf. By virtue of the definition, all the operators functioning in the following section $\not\equiv 0, f(z)$.

2. Background and main results

In 2014, Liu et al. [14] were the first to investigate the uniqueness theorem for a finite order entire function sharing two small functions with its linear shift operator as follows:

Theorem A. [14] *Let f be a non-constant entire function of finite order and $a(z)$, $b(z)$ be two distinct small functions related to $f(z)$, let $L_1(f(z))$ be linear shift operator of $f(z)$ with constant coefficients. If $f(z)$ and $L_1(f(z))$ share $a(z)$, $b(z)$ CM, then $f(z)\equiv L_1(f(z))$.*

After that, in 2017, concerning entire function of finite order, Li et al. [13] tackle the "1 CM+ 1 IM" value sharing problem as follows:

Theorem B. [13] *Let $b\in\mathbb{C}\setminus\{0\}$ and let $f(z)$ be a non constant entire function of finite order. If $f(z)$ and $\Delta_c^kf(z)$ share 0 CM and b IM, then $f(z)\equiv\Delta_c^kf(z)$.*

Theorem C. [13] *Let $f(z)$ be a non constant entire function of finite order. If $f(z)$ and $\Delta_c^kf(z)$ share two distinct complex constants a CM and b IM and if*

$$N\left(r,\frac{1}{f(z)-a}\right)=T(r,f)+S(r,f), \tag{2.1}$$

then $f(z)\equiv\Delta_c^kf(z)$.

Recently, adopting the same procedure of [13], Kaish-Rahaman [12] again proved *Theorem C* but they did not mention it. Also Qi-Yang [17] extended *Theorem B* from finite order entire function to entire function of $\rho_2(f)<1$ and asked the following question:

Question 2.1. [17] *If the sharing condition in Theorem B is changed into sharing "a CM+ b IM", where a, b are two distinct constants such that $ab\neq 0$, is the result still valid?*

In aspect of the uniqueness result of a meromorphic function f sharing "3 CM values" with its difference operators, Lu-Lu [15], Cui-Chen [6] contributed remarkably. Again we would like to mention that Kaish-Rahaman [12] proved uniqueness result of a meromorphic function f sharing "2 CM values" with its difference operators with the support of the assumption $N(r,f) = S(r,f)$. So the previous results on "3 CM value" sharing are far better than "2 CM value" sharing result in [12]. Unfortunately, again Kaish-Rahaman [12] did not provide any information about [15] and [6]. So considering these facts the paper of Kaish-Rahaman [12] has hardly any value.

After that, Deng et al. [7], Gau et al. [8] investigated the "3 CM small functions" sharing problem for the difference operator or even k-th order difference operator.

In connection with the *Question 2.1*, Qi-Yang [17] also asked the following question:

Question 2.2. [17] *Can the value sharing condition "3 CM" for a meromorphic function with its difference operators be reduced up to "2 CM + 1 IM"?*

It is to be noted that for finite order entire function, *Question 2.1* has already been answered in *Theorem C*. Recently, by the following results, we have answered of *Questions 2.1* and *2.2* in a compact form for a larger class of operators in view of small functions sharing.

Theorem D. (see Theorem 2.1 & Corollary 2.1, [1]) Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$ and let $a(z)$, $b(z)$ be two distinct small functions. If $L(f(z))$ and $f(z)$ share $a(z)$, ∞ CM and $b(z)$ IM with $\Theta(0; f-a(z))+\Theta(\infty; f) > 0$ and one of the following cases is satisfied:
(i) $a(z) \equiv 0$,
(ii) $a(z) \not\equiv 0$ with $N\left(r, \frac{1}{f-a(z)}\right) = T(r,f) + S(r,f)$,
then $L(f(z)) \equiv f(z)$.

Theorem E. (see Corollaries 2.2 & 2.3, [1]) Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$ and let $b(z)$ $(\not\equiv 0) \in S(f)$. If $L(f(z))$ and $f(z)$ share $a(z)$ CM and $b(z)$ IM and one of the following cases is satisfied:
(i) $a(z) \equiv 0$,
(ii) $a(z) \not\equiv 0$ with $N\left(r, \frac{1}{f-a(z)}\right) = T(r,f) + S(r,f)$,
then $L(f(z)) \equiv f(z)$.

And also in the same paper [1], we asked the following question:

Question 2.3. [1] *Is it possible to remove the condition on ramification index in Theorem D?*

When in *Theorem D*, $a(z) = 0$, $b(z) = 1$ and specifically $L(f(z)) = \Delta_c f(z)$, the condition $\Theta(0; f) + \Theta(\infty; f) > 0$ is no longer required. Recently, by the following theorem, Chen-Xu [3] have been able to prove it.

Theorem F. [3] *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$. If $\Delta_c f(z)$ and $f(z)$ share 0, ∞ CM and 1 IM, then $\Delta_c f(z) \equiv f(z)$.*

In view of partially sharing values, in 2018, Chen [2] investigated the following uniqueness result.

Theorem G. [2] *Let f be a non constant meromorphic function of hyper order $\rho_2(f) < 1$. If $\Delta_c f$ and $f(z)$ share the value 1 CM and satisfy*

$$E(0, f) \subseteq E(0, \Delta_c f) \quad \textit{and} \quad E(\infty, \Delta_c f) \subseteq E(\infty, f),$$

then $\Delta_c f \equiv f$.

In this paper we not only resolve the *Questions 2.3* partially as well as we are able the relax the sharing conditions of $a(z)$ and ∞ in view of partially sharing as follows:

Theorem 2.1. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$ and let $a(z)$, $b(z)$ be two distinct small functions of f. If $L(f(z))$, $f(z)$ share $b(z)$ IM and satisfy*

$$E(0, f - a(z)) \subseteq E(0, L(f(z)) - a(z)), \quad E(\infty, L(f(z))) \subseteq E(\infty, f) \quad \textit{with}$$
$$N_{=1}(r, L(f(z))) = S(r, f),$$

and one of the following cases is satisfied:
(i) $a(z) \equiv 0$,
(ii) $a(z) \not\equiv 0$ with $N\left(r, \frac{1}{f - a(z)}\right) = T(r, f) + S(r, f)$,
then $L(f(z)) \equiv f(z)$.

Consequently we have the following corollary, which is more relaxed with respect to *Theorem E*.

Corollary 2.1. *Under the same situation as in* Theorem 2.1 *if $f(z)$ be a transcendental entire function then $L(f(z)) \equiv f(z)$.*

It is to be noted that *Theorem C* provides the answer of *Question 2.1* with one extra supposition on counting function. Continuous efforts are being put in by researchers to remove the condition, but nobody succeeded. Recently, Huang [11] proved the following result which gives the better answer of the *Question 2.1.*

Theorem H. [11] *Let $f(z)$ be a transcendental entire function of finite order. If $f(z)$ and $(\Delta_c^k f(z))^{(n)}$, $n \geq 0$ share two distinct complex constants a CM and b IM then $f(z) \equiv (\Delta_c^k f(z))^{(n)}$.*

Remark 2.1. *Inspecting closely the proof of Theorem H, we can see that there was a fatal error in the proof of Lemma 2.6 (see p. 6, l. 4 from top,* [11]*).*

For the sake of argument, let us think that the *Lemma 2.6* in [11] is correct and consequently that means *Theorem H* is also true for $ab \neq 0$. Then, from the following example we can exhibit an evidence of lacuna in the proof of *Theorem H.*

Example 2.1. *Let $f(z) = e^{2\lambda z} - 2e^{\lambda z} + 2$, where λ is a complex constant. Choose c, k and n (≥ 1) satisfying $e^{\lambda c} = -1$ and $\lambda^n = \frac{1}{(-2)^{k+1}}$. Now,*

$$\begin{aligned}\Delta_c^k f(z) &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(z+ic) = e^{2\lambda z}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}e^{2\lambda ic}\right)\\ &- 2e^{\lambda z}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}e^{\lambda ic}\right) + 2\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\right)\\ &= e^{2\lambda z}\left(e^{2\lambda c}-1\right)^k - 2e^{\lambda z}\left(e^{\lambda c}-1\right)^k.\end{aligned}$$

Putting $e^{\lambda c} = -1$, we have $\Delta_c^k f(z) = (-2)^{k+1}e^{\lambda z}$. So, $(\Delta_c^k f(z))^{(n)} = (-2)^{k+1}\lambda^n e^{\lambda z}$ $= e^{\lambda z}$. Here $f(z)$ and $(\Delta_c^k f(z))^{(n)}$ share 2 CM and 1 IM but $(\Delta_c^k f(z))^{(n)} \not\equiv f(z)$.

In the above example, $N\left(r, \frac{1}{f(z)-2}\right) = N\left(r, \frac{1}{e^{\lambda z}-2}\right) = T(r, e^{\lambda z}) \neq T(r, f) = 2T(r, e^{\lambda z})$ and this does not conform (2.1). Since *Lemma 2.6* is used to deal "$ab \neq 0$", under subcase 2.3 *(*see p.12, [11]*)*, so the existence of *Theorem H* for the case "$ab \neq 0$" is under question. Thus for $ab \neq 0$, without the aid of supposition (2.1), the existence of *Theorem H* seems to be impossible.

As a result, till now, for the case $ab \neq 0$, Corollary 2.1 is the best possible answer to Question 2.1. We see that it automatically covers the case "$a = 0$" of *Theorem H.* However, in Theorem H the case "$b = 0$" has been resolved conveniently. Hence *Theorem H* is true only when $ab = 0$.

From the above theorem, we see that the only option left to improve *Theorem H* is to manipulate the case $b = 0$. Now we are going to present the next theorem which will significantly extend *Theorem H* for $b = 0$.

Theorem 2.2. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$ and let $a(z)$ be a non zero periodic small function of f with period c. If $(L_c^0 f(z))^{(n)}$ $(n \geq 0)$, $f(z)$ share $a(z)$ CM, 0 IM and $N(r, f) = S(r, f)$, then $(L_c^0 f(z))^{(n)} \equiv f(z)$.*

Very recently, concerning shift and k-th derivative of a meromorphic function, Chen-Xu [4] proved the following result.

Theorem I. [4] *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$. If $f^{(k)}(z)$ and f_c share $0, \infty$ CM and 1 IM, then $f^{(k)}(z) \equiv f_c$.*

In view of partially sharing Qi-Yang [16] proved the following result:

Theorem J. [16] *Let $f(z)$ be a non constant meromorphic function of finite order and let $b \neq 0 \in \mathbb{C}$. If f', f_c share b CM and satisfy*

$$E(0, f_c) \subseteq E(0, f') \ \ \text{and} \ \ E(\infty, f') \subseteq E(\infty, f_c),$$

then $f' \equiv f_c$. Further, $f(z)$ is a transcendental entire function.

Remark 2.2. *In Theorem J, the authors showed that when $f' \equiv f_c$ the meromorphic function is ultimately reduces to an entire function. As it is not possible to get such a meromorphic function satisfying $f' \equiv f_c$ so we wonder that why the result carried forward in meromorphic function. Although we are considering meromorphic functions to continue their research and improve their results, still we believe that, in the next theorem, it would have been better to consider the function as an entire function.*

Related to *Theorem I*, we can have the following theorem which will relax and extend the conditions of the shared values of the same theorem from "CM" to "partially CM small functions sharing". The theorem improves *Theorem J* as well.

Theorem 2.3. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$ and let $a(z)$, $b(z)$ be two distinct small functions of f. If $\tilde{L}(f(z))$, f_c share $b(z)$ IM and satisfy*

$$E(0, f_c - a(z)) \subseteq E(0, \tilde{L}(f(z)) - a(z)) \ \ \text{and} \ \ E(\infty, \tilde{L}(f(z))) \subseteq E(\infty, f_c)$$

and one of the following cases is satisfied:
(i) $a(z) \equiv 0$,
(ii) $a(z) \not\equiv 0$ with $N\left(r, \frac{1}{f_c - a(z)}\right) = T(r, f) + S(r, f)$,
then $\tilde{L}(f(z)) \equiv f_c$.

3. Lemmas

In this section, we present some lemmas, which will be needed to proceed further.

Lemma 3.1. [9] *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f_c}{f}\right) + m\left(r, \frac{f}{f_c}\right) = S(r, f).$$

Lemma 3.2. [9] *Let $T : [0,+\infty) \to [0,+\infty)$ be a non-decreasing continuous function, and let $s \in (0,+\infty)$. If the hyper-order of T is strictly less than 1, i.e.,*

$$\limsup_{r\longrightarrow\infty} \frac{\log\log T(r)}{\log r} = \rho_2 < 1,$$

and $\delta \in (0, 1-\rho_2)$, then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r runs to infinity outside of a set of finite logarithmic measures.

Using this lemma by a simple alteration of the result for finite order meromorphic functions in [5], one can have the following lemma.

Lemma 3.3. *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$, then we have*

$$N(r, f_c) = N(r, f) + S(r, f) \quad \text{and} \quad T(r, f_c) = T(r, f) + S(r, f).$$

Lemma 3.4. [18] *Let f be a non-constant meromorphic function, $a_j \in \hat{S}(f)$, $j = 1, 2, \ldots, q$, $(q \geq 3)$. Then for any positive real number ϵ, we have*

$$(q-2-\epsilon)T(r,f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right), \; r \notin E,$$

where $E \subset [0,\infty)$ and satisfies $\int_E d\log\log r < \infty$.

Lemma 3.5. [1] *Let $f(z)$ be a non constant meromorphic function of $\rho_2(f) < 1$ in $\mathbb{C}$, $p \in \mathbb{C}$. Then for a small function $b(z)$ of f,*

$$m\left(r, \frac{L(f(z)) + pL(b(z))}{f(z) + p\, b(z)}\right) = S(r, f).$$

Lemma 3.6. *Let $f(z)$ be a non constant meromorphic function of $\rho_2(f) < 1$ and $g = L(f(z))$. If for $c \in \mathbb{C}$, $E(0, f_c - a(z)) \subseteq E(0, g - a(z))$ and $E(\infty, g) \subseteq E(\infty, f_c)$ or $N(r, f) = S(r, f)$, then $S(r, g) = S(r, f)$ and $\rho_2(g) = \rho_2(f_c) < 1$.*

Proof. When $E(\infty, g) \subseteq E(\infty, f_c)$, then by *Lemma 3.3* $N(r, g) \leq N(r, f_c) = N(r, f) + S(r, f)$. So, in view of *Lemma 3.5* we obtain that

$$\begin{aligned} T(r,g) &= m(r,g) + N(r,g) \\ &\leq m(r,f) + m\left(r, \frac{g}{f}\right) + N(r,f) + S(r,f) \\ &\leq T(r,f) + S(r,f). \end{aligned} \tag{3.1}$$

When $N(r,f)=S(r,f)$, we also can establish (3.1). Now by the First fundamental Theorem, *Lemma 3.1* and *Lemma 3.5* we get,

$$\begin{aligned} T(r,f) &\leq T\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f)\\ \leq\ & m\left(r,\frac{g-L(a(z-c))}{f-a(z-c)}\right)+m\left(r,\frac{f-a(z-c)}{f_c-a(z)}\right)+m\left(r,\frac{1}{g-L(a(z-c))}\right)\\ &+N\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f)\\ \leq\ & m\left(r,\frac{1}{g-L(a(z-c))}\right)+N\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f). \end{aligned}$$

Since $E(0,f_c-a(z))\subseteq E(0,g-a(z))$, thereby

$$T(r,f)\leq 2T(r,g)+S(r,f). \tag{3.2}$$

Combining (3.1) and (3.2), it follows that $S(r,f)=S(r,g)$ and $\rho_2(g)=\rho_2(f_c)<1$. □

Throughout the paper we use the notation of $P(h)$ and its use, which is given in the following lemma.

Lemma 3.7. *For some meromorphic function h, we define*

$$\begin{aligned} P(h) &= \begin{vmatrix} h(z)-a(z) & a(z)-b(z)\\ h'(z)-a'(z) & a'(z)-b'(z)\end{vmatrix} = \begin{vmatrix} h(z)-b(z) & a(z)-b(z)\\ h'(z)-b'(z) & a'(z)-b'(z)\end{vmatrix}\\ &= \begin{vmatrix} h(z)-a(z) & h'(z)-a'(z)\\ h(z)-b(z) & h'(z)-b'(z)\end{vmatrix}\\ &= [a'(z)-b'(z)]h(z)-[a(z)-b(z)]h'(z)+[a(z)b'(z)-a'(z)b(z)], \end{aligned}$$

$a(z)$, $b(z)\in S(f)\cap S(g)$, where f, g are defined in Lemma 3.6. Then $P(f)$, $P(g)$ $\not\equiv 0$ and

$$m\left(r,\frac{P(h)}{h-a(z)}\right)=S(r,h)+S(r,f)=m\left(r,\frac{P(h)}{h-b(z)}\right).$$

Proof. *On the contrary, if $P(f)\equiv 0$, then by a simple integration we have $f(z)$ is small function which shows $T(r,f)=S(r,f)$, that is not possible. Similarly $P(g)\equiv 0$ gives $T(r,g)=S(r,g)$, which also makes a contradiction. So $P(f)$, $P(g)\not\equiv 0$. Now from the construction of $P(h)$, we can easily deduce that*

$$\begin{aligned} m\left(r,\frac{P(h)}{h-a(z)}\right) &= m\left(r,\frac{(h-a(z))(a'(z)-b'(z))-(a(z)-b(z))(h'(z)-a'(z))}{h-a(z)}\right)\\ &= S(r,h)+S(r,f). \end{aligned}$$

Similarly,

$$m\left(r,\frac{P(h)}{h-b(z)}\right)=S(r,h)+S(r,f).$$

□

Lemma 3.8. *Let f and g be two non constant meromorphic functions as defined in Lemma 3.6. Set*

$$H_f^g = \frac{P(g)}{(g-a(z))(g-b(z))} - \frac{P(f)}{(f-a(z))(f-b(z))},$$

$a(z)$, $b(z) \in S(f) \cap S(g)$. Then the following occurs:
(i) $m(r, H_f^g) = S(r,f)$.
(ii)

$$T(r, H_f^g) \leq \overline{N}_{\otimes}(r, 0, f-a(z); g-a(z)) + \overline{N}_{\otimes}(r, 0, f-b(z); g-b(z)) + S(r,f).$$

(iii) Let $N(r,f) = N(r,g) = S(r,g)$. If $H_f^g \equiv 0$, then either $g \equiv f$ or

$$2T(r,f) \leq \overline{N}\left(r, \frac{1}{f-a(z)}\right) + \overline{N}\left(r, \frac{1}{f-b(z)}\right) + S(r,f).$$

Proof. (i)

$$m(r, H_f^g) = m\left(r, \frac{1}{a(z)-b(z)}\left(\left(\frac{P(g)}{g-a(z)} - \frac{P(g)}{g-b(z)}\right) - \left(\frac{P(f)}{f-a(z)} - \frac{P(f)}{f-b(z)}\right)\right)\right).$$

Now we apply *Lemma 3.7* to obtain

$$m(r, H_f^g) = S(r,f).$$

(ii) Rewriting H_f^g we have

$$H_f^g = \left(\frac{g'-b'(z)}{g-b(z)} - \frac{g'-a'(z)}{g-a(z)}\right) - \left(\frac{f'-b'(z)}{f-b(z)} - \frac{f'-a'(z)}{f-a(z)}\right).$$

Clearly

$$N(r, H_f^g) \leq \overline{N}_{\otimes}(r, 0, f-a(z); g-a(z)) + \overline{N}_{\otimes}(r, 0, f-b(z); g-b(z)) + S(r,f)$$

and so in view of (i), (ii) holds.
(iii) Now $H_f^g \equiv 0$ implies

$$\frac{\begin{vmatrix} f(z)-a(z) & f'(z)-a'(z) \\ f(z)-b(z) & f'(z)-b'(z) \end{vmatrix}}{(f-a(z))(f-b(z))} = \frac{\begin{vmatrix} g(z)-a(z) & g'(z)-a'(z) \\ g(z)-b(z) & g'(z)-b'(z) \end{vmatrix}}{(g-a(z))(g-b(z))}.$$

Integrating we have

$$\frac{f-a(z)}{f-b(z)} = A\frac{g-a(z)}{g-b(z)},$$

where A is a non zero constant. If $A = 1$ then $g \equiv f$. So let $A \neq 1$. Proceeding in a similar way as in page 15 of [1] we have

$$f - \frac{Ab(z)-a(z)}{A-1} = \frac{(a(z)-b(z))A}{A-1} . \frac{f-b(z)}{g-b(z)}.$$

Let $d(z) = \frac{Ab(z)-a(z)}{A-1}$. As $A \neq 0$, 1 and $a(z) \neq b(z)$, so $d(z) \neq a(z)$, $b(z)$. From the above equation it is obvious that any zero of $f - d(z)$ must be a zero of at least one of $a(z)-b(z)$ or $d(z)-b(z)$. Therefore $\overline{N}\left(r,\frac{1}{f-d(z)}\right)=S(r,f)$. So, by *Lemma 3.4* we obtain

$$\begin{aligned}2T(r,f) &\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a(z)}\right)+\overline{N}\left(r,\frac{1}{f-b(z)}\right)+\overline{N}\left(r,\frac{1}{f-d(z)}\right)\\ &+ S(r,f)\overline{N}\left(r,\frac{1}{f-a(z)}\right)+\overline{N}\left(r,\frac{1}{f-b(z)}\right)+S(r,f).\end{aligned}$$

□

4. Proofs of the theorems

The following proof of the **Theorem 2.1** is based on some ideas from Chen-Xu [4].

Proof of Theorem 2.1. Set $g=L(f(z))$. Since $E(0,f-a(z))\subseteq E(0,g-a(z))$, $E(\infty,g)\subseteq E(\infty,f)$, so

$$\frac{g-a(z)}{f-a(z)}=\gamma(z), \tag{4.1}$$

where $\gamma(z)$ is a meromorphic function such that $N(r,\gamma(z))=S(r,f)$.

First suppose $a(z)\not\equiv 0$ with $N\left(r,\frac{1}{f-a(z)}\right)=T(r,f)+S(r,f)$. By *Lemma 3.5* and then applying the First Fundamental Theorem we have,

$$\begin{aligned}T(r,\gamma(z)) &= m(r,\gamma(z))+S(r,f)\\ &\leq m\left(r,\frac{g-L(a(z))}{f-a(z)}\right)+m\left(r,\frac{L(a(z))-a(z)}{f-a(z)}\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{f-a(z)}\right)+S(r,f)=T(r,f)-N\left(r,\frac{1}{f-a(z)}\right)+S(r,f),\end{aligned}$$

which implies

$$T(r,\gamma(z))=S(r,f). \tag{4.2}$$

If $a(z)\equiv 0$, then in view of *Lemma 3.5* from (4.1) we automatically get (4.2).

Let z_0 be a zero of $f-b(z)$ such that it is not a zero of $b(z)-a(z)$. Since g and f share $b(z)$ IM, so z_0 is also a zero of $g-b(z)$. Therefore from (4.1) we have,

$$\gamma(z_0)=\frac{b(z_0)-a(z_0)}{b(z_0)-a(z_0)}=1,$$

which yields all zeros of $f-b(z)$ are zeros of $\gamma(z)-1$ as long as they are not zeros of $b(z)-a(z)$. Suppose $g\not\equiv f$. So $\gamma(z)\not\equiv 1$. Therefore we can write

$$\begin{aligned}\overline{N}\left(r,\frac{1}{g-b(z)}\right) &= \overline{N}\left(r,\frac{1}{f-b(z)}\right)\\ &\leq N\left(r,\frac{1}{\gamma(z)-1}\right)+N\left(r,\frac{1}{b(z)-a(z)}\right)\\ &\leq T(r,\gamma(z))+S(r,f)=S(r,f).\end{aligned} \tag{4.3}$$

Set $d_{\gamma}(z)=a(z)-\frac{a(z)-b(z)}{\gamma(z)}$. Then it is obvious that $d_{\gamma}(z)\not\equiv a(z)$ as well as $\not\equiv b(z)$. Rewriting (4.1) we have,

$$g-b(z)=\gamma(z)[f-d_{\gamma}(z)].$$

Therefore,

$$\overline{N}\left(r,\frac{1}{f-d_{\gamma}(z)}\right)=\overline{N}\left(r,\frac{1}{g-b(z)}\right)+S(r,f)=S(r,f). \tag{4.4}$$

Let us consider the same auxiliary function H_{f}^{g} as defined in *Lemma 3.8*. Since $E(0,f-a(z))\subseteq E(0,g-a(z))$ and f, g share $b(z)$ IM, so by *Lemma 3.8*

$$\begin{aligned} T(r,H_{f}^{g}) &\leq \overline{N}\left(r,\frac{1}{f-b(z)}\right)+\overline{N}_{\otimes}(r,0,f-a(z);g-a(z))+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{f-b(z)}\right)+N\left(r,\frac{1}{\gamma(z)}\right)+S(r,f)\end{aligned}$$

Using (4.2) and (4.3) we obtain that

$$T(r,H_{f}^{g})=S(r,f). \tag{4.5}$$

Now we consider two cases:

Case 1: $H_{f}^{g}\equiv 0$. Since $g\not\equiv f$, so proceeding in a similar way as in case (iii) of *Lemma 3.8* we have $\overline{N}\left(r,\frac{1}{f-d(z)}\right)=S(r,f)$. Also, one can easily check that $d(z)\not\equiv d_{\gamma}(z)$. So, by *Lemma 3.4*, from (4.3) and (4.4) we obtain

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r,\frac{1}{f-b(z)}\right)+\overline{N}\left(r,\frac{1}{f-d_{\gamma}(z)}\right)+\overline{N}\left(r,\frac{1}{f-d(z)}\right)+S(r,f)\\ &= S(r,f),\end{aligned}$$

which is a contradiction.

Case 2: $H_{f}^{g}\not\equiv 0$. Here $d_{\frac{1}{\gamma}}(z)=a(z)-(a(z)-b(z))\gamma(z)$. Then it is obvious that $d_{\frac{1}{\gamma}}(z)\not\equiv a(z)$ as well as $\not\equiv b(z)$. Rewriting (4.1) we have,

$$f-b(z)=\frac{1}{\gamma(z)}[g-d_{\frac{1}{\gamma}}(z)].$$

Therefore,

$$\overline{N}\left(r,\frac{1}{g-d_{\frac{1}{\gamma}}(z)}\right)=\overline{N}\left(r,\frac{1}{f-b(z)}\right)+S(r,f)=S(r,f). \tag{4.6}$$

Since $E(\infty,g)\subseteq E(\infty,f)$ with $N_{=1}(r,g)=S(r,f)$, so in view of (4.5) we get

$$\overline{N}(r,g)\leq N\left(r,\frac{1}{H_{f}^{g}}\right)+S(r,f)=S(r,f).$$

Now, by *Lemma 3.4*, (4.3) and (4.6) we obtain

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g-b(z)}\right)+\overline{N}\left(r,\frac{1}{g-d_{\frac{1}{\gamma}}(z)}\right)+S(r,g)\\ &= S(r,f),\end{aligned}$$

which is a contradiction. Hence $g \equiv f$ holds. □

Proof of Theorem 2.2. Suppose $g_1 = (L_c^0 f(z))^{(n)}$ and $g_1 \not\equiv f$. Since f and g_1 share $a(z)$ CM, so there exist a meromorphic function $h(z)$ such that

$$\frac{g_1 - a(z)}{f - a(z)} = h(z). \tag{4.7}$$

Here $h \not\equiv 0, 1$. Clearly $N(r, f) = S(r, f)$ with *Lemma 3.3*, implies $N(r, g_1) = S(r, f)$, which yields $N(r, h) = S(r, f)$. As f and g_1 share $a(z)$ CM and 0 IM, so by *Lemma 3.4* and then by applying *Lemma 3.5* we have

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - a(z)}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{g_1 - f}\right) + S(r, f) \leq T\left(r, \frac{1}{g_1 - f}\right) + S(r, f) \\ &\leq T(r, g_1 - f) + S(r, f) \leq m(r, g_1 - f) + S(r, f) \\ &\leq m(r, f) + m\left(r, \frac{g_1 - f}{f}\right) + S(r, f) \\ &\leq m(r, f) + m\left(r, \frac{g_1}{f}\right) + S(r, f) \leq T(r, f) + S(r, f). \end{aligned}$$

Therefore

$$\begin{aligned} T(r, f) &= \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - a(z)}\right) + S(r, f) \\ &= T(r, g_1 - f) + S(r, f) = N\left(r, \frac{1}{g_1 - f}\right) + S(r, f) \end{aligned} \tag{4.8}$$

and so

$$m\left(r, \frac{1}{g_1 - f}\right) = S(r, f).$$

From (4.7), in view of *Lemma 3.5* using (4.8) we can obtain

$$\begin{aligned} T(r, h) &= m(r, h) + N(r, h) = m(r, h) + S(r, f) \\ &\leq m\left(r, \frac{g_1 - (L_c^0 a(z))^{(n)}}{f - a(z)}\right) + m\left(r, \frac{1}{f - a(z)}\right) + S(r, f) \\ &\leq m\left(r, \frac{1}{f - a(z)}\right) + S(r, f) = m\left(r, \frac{h - 1}{g_1 - f}\right) + S(r, f) \\ &\leq m(r, h) + m\left(r, \frac{1}{g_1 - f}\right) + S(r, f) \leq T(r, h) + S(r, f). \end{aligned}$$

So,

$$T(r, h) = m\left(r, \frac{1}{f - a(z)}\right) + S(r, f). \tag{4.9}$$

Since f and g_1 share 0 IM, from (4.7) we can say that all zeros of f are 1-point of $h(z)$ or zeros of $a(z)$. Hence, in view of (4.9) we can write

$$\overline{N}\left(r,\frac{1}{f}\right) \leq N\left(r,\frac{1}{h-1}\right)+S(r,f) \leq T(r,h)+S(r,f) = m\left(r,\frac{1}{f-a(z)}\right)+S(r,f). \tag{4.10}$$

By the First Main Theorem and then using (4.8), (4.10) we get

$$\begin{aligned} m\left(r,\frac{1}{f-a(z)}\right)+N\left(r,\frac{1}{f-a(z)}\right) &= T(r,f)+S(r,f)\\ &= \overline{N}\left(r,\frac{1}{f-a(z)}\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{f-a(z)}\right)+m\left(r,\frac{1}{f-a(z)}\right)+S(r,f), \end{aligned}$$

that yields

$$N\left(r,\frac{1}{f-a(z)}\right)=\overline{N}\left(r,\frac{1}{f-a(z)}\right)+S(r,f)$$

and consequently in view of (4.10) we have

$$m\left(r,\frac{1}{f-a(z)}\right)=\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)=T(r,h)+S(r,f). \tag{4.11}$$

Now we consider two auxiliary functions as follows:

$$\alpha(z)=\frac{P(f)(g_1-f)}{f(f-a(z))}, \qquad \beta(z)=\frac{P(g_1)(g_1-f)}{g_1(g_1-a(z))},$$

where $P(f)$ is as defined in *Lemma 3.7* together with $b(z)\equiv 0$. Here $\alpha(z)$ as well as $\beta(z)\not\equiv 0$. As f and g_1 share $a(z)$ CM, 0 IM and $N(r,f)=N(r,g_1)=S(r,f)$, so $N(r,\alpha(z))=S(r,f)$ and $N(r,\beta(z))=S(r,f)$. In a similar way as in page 11 of [1] we can easily have $m(r,\alpha(z))=S(r,f)$. Thus,

$$T(r,\alpha(z))=S(r,f). \tag{4.12}$$

Following the same logic of construction of the auxiliary function H_f^g in *Lemma 3.8*, here we define $H_f^{g_1}$ with $b(z)\equiv 0$. Since f and g_1 share $a(z)$ CM, 0 IM, from *Lemma 3.8* we have

$$T(r,H_f^{g_1})\leq \overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \tag{4.13}$$

As $a(z)$ is periodic small function with period c, so $(L_c^0 a(z))^{(n)}=0$ which in view of *Lemma 3.5* gives $m\left(r,\frac{g_1}{f-a(z)}\right)=m\left(r,\frac{g_1-(L_c^0a(z))^{(n)}}{f-a(z)}\right)=S(r,f)$. Rewriting (4.7) and then using (4.9) we can obtain

$$\begin{aligned}(4.14)\qquad m\left(r,\frac{g_1-f}{g_1-a(z)}\right) &= m\left(r,\frac{h-1}{h}\right)\\ &\leq T(r,h)+O(1)=m\left(r,\frac{1}{f-a(z)}\right)+S(r,f)\\ &\leq m\left(r,\frac{g_1}{f-a(z)}\right)+m\left(r,\frac{1}{g_1}\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{g_1}\right)+S(r,f).\end{aligned}$$

Now we distinguish in two cases on the consideration of $H_f^{g_1}$.

Case 1. Suppose $H_f^{g_1} \equiv 0$. Then by *Lemma 3.8* in view of (4.8) we get a contradiction.

Case 2. Next suppose $H_f^{g_1} \not\equiv 0$. Since $N(r,f)=S(r,f)$ and $N(r,g_1)=S(r,f)$, so from (4.8), we can write

$$\begin{aligned}T(r,f)=m(r,f)+S(r,f) &= m(r,g_1-f)+S(r,f)=m\left(r,\frac{H_f^{g_1}(g_1-f)}{H_f^{g_1}}\right)+S(r,f)\\ &= m\left(r,\frac{\alpha(z)-\beta(z)}{H_f^{g_1}}\right)+S(r,f)\\ &\leq m\left(r,\alpha(z)-\beta(z)\right)+m\left(r,\frac{1}{H_f^{g_1}}\right)+S(r,f).\end{aligned}$$

Now, using (4.12), (4.13), (4.14) and *Lemmas 3.7 and 3.5* we can have

$$\begin{aligned}T(r,f) &\leq m(r,\beta)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq m\left(r,\frac{P(g_1)}{g_1}\right)+m\left(r,\frac{g_1-f}{g_1-a(z)}\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{g_1}\right)+\overline{N}\left(r,\frac{1}{g_1}\right)+S(r,f)\\ &\leq T(r,g_1)+S(r,f)\leq m(r,g_1)+S(r,f)\\ &\leq m(r,f)+m\left(r,\frac{g_1}{f}\right)+S(r,f)=m(r,f)+S(r,f)\leq T(r,f)+S(r,f).\end{aligned}$$

Noting that $N(r,\beta(z))=S(r,f)$, also from the above we see that i.e.,

$$(4.15)\qquad T(r,f) = T(r,\beta)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)$$

and

$$(4.16)\qquad T(r,g_1)=T(r,f)+S(r,f).$$

Now our claim is $T(r,\beta(z))=S(r,f)$. Putting $b(z)\equiv 0$, using (4.16), in a similar manner as used in Page 12-14 of [1] we can easily establish our claim. Therefore, (4.15) yields

$$T(r,f)=\overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \tag{4.17}$$

Using (4.17), from (4.8), we get $\overline{N}\left(r,\frac{1}{f-a(z)}\right)=S(r,f)$. As f and g_1 share $a(z)$ CM, so $\overline{N}\left(r,\frac{1}{g_1-a(z)}\right)=S(r,f)$. Again according to the sharing hypothesis of f and g_1 using (4.16) from (4.8) we have

$$\begin{aligned} T(r,g_1) &= \overline{N}\left(r,\frac{1}{g_1}\right)+\overline{N}\left(r,\frac{1}{g_1-a(z)}\right)+S(r,f) \\ &= \overline{N}\left(r,\frac{1}{g_1}\right)+S(r,f)\le N\left(r,\frac{1}{g_1}\right)+S(r,f)\le T(r,g_1)+S(r,f), \end{aligned}$$

that implies

$$T(r,g_1)=N\left(r,\frac{1}{g_1}\right)+S(r,f) \text{ and so } m\left(r,\frac{1}{g_1}\right)=S(r,f).$$

Now from (4.17), using (4.11) and (4.14) we have

$$T(r,f)=\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)=m\left(r,\frac{1}{f-a(z)}\right)+S(r,f)\le m\left(r,\frac{1}{g_1}\right)+S(r,f).$$

Hence from the above two lines we get $T(r,f)=S(r,f)$, a contradiction. Hence $g_1\equiv f$. □

Proof of Theorem 2.3. Set $g_2=\tilde{L}(f(z))$. Since $E(0,f_c-a(z))\subseteq E(0,g_2-a(z))$, $E(\infty,g_2)\subseteq E(\infty,f_c)$, so

$$\frac{g_2-a(z)}{f_c-a(z)}=\gamma_1(z), \tag{4.18}$$

where $\gamma_1(z)$ is a meromorphic function such that $N(r,\gamma_1(z))=S(r,f)$.
First suppose $a(z)\not\equiv 0$ with $N\left(r,\frac{1}{f_c-a(z)}\right)=T(r,f)+S(r,f)$. Now, by *Lemma 3.5* and then applying the First Fundamental Theorem we have,

$$\begin{aligned} T(r,\gamma_1(z)) &= m(r,\gamma_1(z))+S(r,f) \\ &\le m\left(r,\frac{g_2-\tilde{L}(a(z-c))}{f_c-a(z)}\right)+m\left(r,\frac{\tilde{L}(a(z-c))-a(z)}{f_c-a(z)}\right)+S(r,f) \\ &\le m\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f)=T(r,f)-N\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f), \end{aligned}$$

that implies

$$T(r,\gamma_1(z))=S(r,f). \tag{4.19}$$

If $a(z)\equiv 0$, then in view of *Lemma 3.5* from (4.18) we automatically get (4.19). By similar argument as used in *Theorem 2.1*, we can have

$$\overline{N}\left(r,\frac{1}{f_c-b(z)}\right)=\overline{N}\left(r,\frac{1}{g_2-b(z)}\right)=S(r,f). \tag{4.20}$$

Clearly $d_{\gamma_1}(z) \not\equiv a(z)$ as well as $\not\equiv b(z)$. Rewriting (4.18) we have,

$$g_2 - b(z) = \gamma_1(z)[f_c - d_{\gamma_1}(z)].$$

Therefore,

$$\overline{N}\left(r, \frac{1}{f_c - d_{\gamma_1}(z)}\right) = \overline{N}\left(r, \frac{1}{g_2 - b(z)}\right) + S(r, f) = S(r, f). \tag{4.21}$$

Here as usual we can define $H_{f_c}^{g_2}$ like *Lemma 3.8*. Since f_c, g_2 share $b(z)$ IM and $E(0, f_c - a(z)) \subseteq E(0, g_2 - a(z))$, by the similar argument as used in *Theorem 2.1* we can get

$$T(r, H_{f_c}^{g_2}) = S(r, f). \tag{4.22}$$

Now we consider two cases:

Case 1: $H_{f_c}^{g_2} \equiv 0$. Then proceeding in a similar manner as in *Case 1* of *Theorem 2.1* we can reach up to a contradiction.

Case 2: $H_{f_c}^{g_2} \not\equiv 0$. Since $E(\infty, g_2) \subseteq E(\infty, f_c)$ and g_2 has no simple poles, so in view of (4.22) we get

$$\overline{N}(r, g_2) \leq N\left(r, \frac{1}{H_{f_c}^{g_2}}\right) = S(r, f).$$

After that, following *Case 2* of *Theorem 2.1* we can again reach up to a contradiction.

□

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CORRIGENDUM TO 'ON WEIGHTS WHICH ADMIT REPRODUCING KERNEL OF SZEGÖ TYPE'

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In *Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)* volume 55 (5), pages 320 – 327, 2020, the paper 'On weights which admit reproducing kernel of Szegö type' was published. The author found a mistake which he wants to fix.

Theorem 5.2. is miscited. Instead of

$$\int_{\partial\Omega_2} f \mathrm{d}S = \int_{\partial\Omega_1} (f \circ \Phi) \det |J_{\mathbb{C}}\Phi|^{\frac{2N}{N+1}} \mathrm{d}S$$

we should have

$$\int_{\partial\Omega_2} f \mathrm{d}\sigma_{F_2} = \int_{\partial\Omega_1} (f \circ \Phi) \det |J_{\mathbb{C}}\Phi|^{\frac{2N}{N+1}} \mathrm{d}\sigma_{F_1},$$

where we integrate using Fefferman measure instead of Lebesgue measure.

Theorem 5.2. Let Ω_1, Ω_2 be domains of one of types 1-3 introduced above and $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Then for any integrable function $f : \partial\Omega_2 \to \mathbb{C}$ we have

$$\int_{\partial\Omega_2} f \mathrm{d}\sigma_{F_2} = \int_{\partial\Omega_1} (f \circ \Phi) |\det J_{\mathbb{C}}\Phi|^{\frac{2N}{N+1}} \mathrm{d}\sigma_{F_1},$$

where $J_{\mathbb{C}}\Phi$ is the complex Jacobian matrix of Φ.

Theorem 5.3. remains true, since integrating in Lebesgue measure and integrating in Fefferman measure define the same topologies, i.e. for any domain Ω_j which satisfies assumptions of the theorem, there exist positive constants d_j, D_j, such that for any positive almost everywhere f we have

$$d_j \int_{\partial\Omega_j} f \mathrm{d}\sigma_F \leq \int_{\partial\Omega_j} f \mathrm{d}S \leq D_j \int_{\partial\Omega_j} f \mathrm{d}\sigma_F. \tag{0.1}$$

The proof however needs some changes.

Theorem 5.3.: Let Ω_1, Ω_2 be of type 1, 2 or 3. Let $\Phi : \Omega_1 \to \Omega_2$ be a biholomorphism. Then

(i) for any g measurable and non-negative almost everywhere we have:

$$\int_{\partial\Omega_2} g\mu \mathrm{d}S < \infty \Leftrightarrow \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi) \mathrm{d}S < \infty$$

In particular, $g \in L^2H(\partial\Omega_2, \mu)$ if and only if $g \circ \Phi \in L^2H(\partial\Omega_1, \mu \circ \Phi)$.

(ii) μ is S-admissible on $\partial\Omega_2$ if and only if $\mu \circ \Phi$ is S-admissible on $\partial\Omega_1$.

Proof: (i) By the fact that $u := |\det J_{\mathbb{C}}\Phi|^{\frac{2N}{N+1}}$ is smooth function on compact set $\overline{\Omega_1}$, we have

$$C_1 \int_{\partial\Omega_1} (g\circ\Phi)(\mu\circ\Phi)\mathrm{d}S \leq \int_{\partial\Omega_1} (g\circ\Phi)(\mu\circ\Phi)|\det J_{\mathbb{C}}\Phi|^{\frac{2N}{N+1}}\mathrm{d}S \leq C_2 \int_{\partial\Omega_1} (g\circ\Phi)(\mu\circ\Phi)\mathrm{d}S,$$

where $C_1 := \min_{w\in\overline{\Omega}} u(w) > 0$ and $C_2 := \max_{w\in\overline{\Omega}} u(w)$. By theorem 5.2. and inequality (0.1) we have

$$\int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi)\mathrm{d}S \leq D_1 \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi)\mathrm{d}\sigma_{F_1} \leq \frac{D_1}{C_1} \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi)u\mathrm{d}\sigma_{F_1}$$
$$= \frac{D_1}{C_1} \int_{\partial\Omega_2} g\mu\mathrm{d}\sigma_{F_2} \leq \frac{D_1}{C_1 d_2} \int_{\partial\Omega_2} g\mu\mathrm{d}S.$$

Similarly

$$\int_{\partial\Omega_2} g\mu\mathrm{d}S \leq D_2 \int_{\partial\Omega_2} g\mu\mathrm{d}\sigma_{F_2} = D_2 \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi)u\mathrm{d}\sigma_{F_1}$$
$$\leq D_2C_2 \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi)\mathrm{d}\sigma_{F_2} \leq \frac{D_2C_2}{d_1} \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi)\mathrm{d}S.$$

So we showed that there exist positive constants c, C, such that

$$c \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi)\mathrm{d}S \leq \int_{\partial\Omega_2} g\mu\mathrm{d}S \leq C \int_{\partial\Omega_1} (g \circ \Phi)(\mu \circ \Phi)\mathrm{d}S \tag{0.2}$$

If the integral on the right-hand side is finite, then the integral in the middle is also finite. If the integral in the middle is finite, then the integral on the left-hand side is also finite.

To complete the proof of (i) we just need to recall that composition of two holomorphic functions is also a holomorphic function.

(ii) Since Φ is biholomorphism, we need only to show implication in one direction.

If μ is S-admissible on $\partial\Omega_2$, then for any compact set $X \subset \Omega_2$, $w \in X$ and any $f \in \tilde{B}(\partial\Omega_2, \mu)$ we have

$$|f(w)| \leq C_X \sqrt{\int_{\partial\Omega_2} |f|^2\mu\mathrm{d}S}. \tag{0.3}$$

By using (0.2) for inequality (0.3) we gain

$$|(f \circ \Phi)(\tilde{w})| \leq C_X\sqrt{C}\sqrt{\int_{\partial\Omega_1} |f \circ \Phi|^2(\mu \circ \Phi)\mathrm{d}S},$$

for $\Omega_1 \supset Y := \Phi^{-1}(X)$, $\tilde{w} := \Phi^{-1}(w) \in Y$, so (CB) is satisfied for $C_Y := C_X\sqrt{C_2}$. ■

Also in an example of non-admissible weight all instances of $\overline{\Omega} \setminus A_n$ should be replaced with $\overline{\Omega \setminus A_n}$.

Other parts of the text should remain unchanged.

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