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ON A RIEMANN BOUNDARY VALUE PROBLEM IN THE SPACE OF p-SUMMABLE FUNCTIONS WITH INFINITE INDEX

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Abstract. The paper considers the Riemann boundary value problem in the half-plane in the space $L^p(\rho)$, where weight function $\rho(x)$ has infinite number of zeros. A necessary and sufficient condition is obtained for the normal solvability and Noetherianness of the considered problem. If the problem is solvable, solutions are represented in an explicit form.

MSC2020 numbers: 30E25; 35Q15.

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1. Introduction

Let $\Pi^{\pm}$ be the upper and lower half-planes of the complex plane C, and let A be the class of functions Φ analytic in $\Pi^+ \cup \Pi^-$ satisfying the condition

$$|\Phi(z)| \le C|z|^{n_0}, \quad |Imz| \ge y_0 > 0,$$

where n_0 is a natural number, $y_0 > 0$ is arbitrary and C is a constant, possibly depending on y_0. By $L^p(\rho), 1 < p < \infty$ we define the following space

$$L^p(\rho) := \Big\{ f : \|f\|_{L^p(\rho)} := \int_{-\infty}^{+\infty} |f(x)|^p \rho(x) dx < \infty \Big\},$$

where

$$\rho(x) = \prod_{k=1}^{\infty} \Big| \frac{x - x_k}{x + i} \Big|^{\alpha_k}, \tag{1.1}$$

at that

$$\sum_{k=1}^{\infty} \alpha_k < \infty, \quad \text{and} \quad 0 < \alpha_k < 1, \, k = 1, 2, ...$$

We investigate the Riemann boundary value problem in the half-plane in the space $L^p(\rho), 1 < p < \infty$ in the following setting:

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Problem R_p. *Let $f \in L^p(\rho), 1 < p < \infty$. Determine an analytic in $\Pi^+ \cup \Pi^-$ function $\Phi \in A$ to satisfy the boundary condition:*

$$\lim_{y \to +0} \|\Phi^+(x+iy) - a(x)\Phi^-(x-iy) - f(x)\|_{L^p(\rho)} = 0, \; (1 < p < \infty), \tag{1.2}$$

where $\rho(x)$ is defined by (1.1), $a(x) \neq 0$ is an arbitrary function from the class $C^\delta(-\infty, +\infty), \delta > 0$ and $\Phi^\pm$ are the contractions of function Φ on $\Pi^\pm$ respectively.

The similar problem in $C(\rho)$ (the class of functions f continuous on the real axis with weight ρ) was investigated in the paper [19]. In that case it is shown that the homogeneous problem has one linearly independent solution. Note that a similar homogeneous problem in $L^1(\rho)$ has an infinite number of linearly independent solutions [20].

By T_p we denote

$$T_p = \{x_k : \; \alpha_k > \frac{1}{p}\}.$$

In this work, it is established that in the case $T_p = \emptyset$, the homogeneous problem R_p does not have a solution different from zero. When $T_p \neq \emptyset$ the homogeneous problem R_p has a finite number of linearly independent solutions.

2. Preliminary results

Let $\kappa = inda(t)$, $t \in (-\infty, +\infty)$,

$$\begin{aligned} S^+(z) &= exp\Big\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\ln a_1(t)dt}{t-z}\Big\}, \qquad z \in \Pi^+, \\ S^-(z) &= \Big(\frac{z+i}{z-i}\Big)^\kappa exp\Big\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\ln a_1(t)dt}{t-z}\Big\}, \qquad z \in \Pi^-, \end{aligned} \tag{2.1}$$

where

$$a_1(t) = \Big(\frac{t+i}{t-i}\Big)^\kappa a(t), \qquad inda_1(t) = 0.$$

In what follows, we assume that the sequence $\{x_k\}_1^\infty$ has a finite limit x_0.

Lemma 2.1. *Let the sequence $\{x_k\}_1^\infty$ satisfy the following conditions:*

$$\sum_{k=1}^\infty \alpha_k \ln|x_0 - x_k| > -\infty, \tag{2.2}$$

$$|x_k - x_j| > c|x_k - x_0|, \qquad j \neq k \tag{2.3}$$

for some fixed $c > 0$. Then

$$\inf \rho_m = \rho_0 > 0, \qquad m = 1, 2, ...,$$

where

$$\rho_m = \prod_{k \neq m}^\infty \Big|\frac{x_m - x_k}{x_m + i}\Big|^{\alpha_k}.$$

Proof. From condition (2.3) we have

$$\Big|\frac{x_j - x_k}{x_j + i}\Big|^{\alpha_k} > c^{\alpha_k}\Big|\frac{x_0 - x_k}{x_j + i}\Big|^{\alpha_k}$$

and

$$\prod_{k\neq j}^{\infty}\Big|\frac{x_j - x_k}{x_j + i}\Big|^{\alpha_k} > \prod_{k=1}^{\infty} c^{\alpha_k}\prod_{k\neq j}^{\infty}\Big|\frac{x_0 - x_k}{x_j + i}\Big|^{\alpha_k}.$$

According to the condition (2.2) there exists $\delta > 0$ such that $\inf \rho_m = \delta > 0$, $m = 1, 2, ...$ □

Let us denote

$$\delta_k(x) = \prod_{j\neq k}^{\infty}\Big|\frac{x - x_j}{x + i}\Big|^{\alpha_j} \tag{2.4}$$

and

$$\delta(x) = \delta_{k+1}(x) - \delta_k(x), \qquad x \in [x_k, x_{k+1}).$$

Here we state Lemmas 2.2 and 2.3, which were proved in [19].

Lemma 2.2. *There exist $x'_k \in [x_k, x_{k+1}),\ k = 1, 2, ...$ such that $\delta(x'_k) = 0$.*

Let $X_1 = (-\infty, x'_1)$ and $X_k = [x'_{k-1}, x'_k),\ k = 2, 3, ...$. It is clear that $X_k \cap X_{k+1} = \emptyset,\ k = 1, 2, 3, ...$.

Lemma 2.3. *Let the sequence of points $\{x_k\}_1^{\infty}$ satisfy either conditions* (2.2) *and* (2.3)*. Then there exists $\delta > 0$ such that for any $k = 1, 2, ...$:*

$$\inf_{x\in X_k} \delta_k(x) > \delta > 0.$$

Denote $\widetilde{\delta}(x) = \{\delta_k(x), x \in X_k\}$, $k = 1, 2, ...$ From Lemmas 2.2 and 2.3 it follows that function $\widetilde{\delta}(x)$ is continuous, and $\inf \widetilde{\delta}(x) > 0$, $x \in (-\infty, \infty)$.

Here we consider two cases:

1. We assume that $T_p = \emptyset$. Let $f(z) \in L^p(\rho)$. Define the function $\Phi(z)$ as follows

$$\Phi(z) = \frac{S(z)}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(t)dt}{S^+(t)(t - z)}, \qquad z \in \Pi^{\pm}. \tag{2.5}$$

Then $\Phi(z) \in H^p(\rho)$ (see [4], [5]).

2. Consider $T_p \neq \emptyset$. Define the function $\Phi_k(z)$ as follows

$$\Phi_k(z) = \frac{S(z)}{2\pi i(x_k - z)}\int_{X_k}\frac{f(t)(x_k - t)dt}{S^+(t)(t - z)},\ \ k = 1, 2, ... \qquad z \in \Pi^{\pm}. \tag{2.6}$$

Theorem 2.1. *The estimate*

$$\|\Phi_k^+(x + iy) - a(x)\Phi_k^-(x - iy)\|_{L^p(\rho)} \le C\|f\|_{L^p(\rho)},$$

where the constant C is independent of y and k, is true. The limit relation

$$\lim_{y\to+0}\|\Phi_k^+(x + iy) - a(x)\Phi_k^-(x - iy) - f(x)\|_{L^p(\rho)} = 0$$

also holds.

Proof: Consider

$$\Phi_k^+(x+iy)-a(x)\Phi_k^-(x-iy)=$$

$$=\frac{S(x+iy)}{2\pi i(x_k-x-iy)}\int_{X_k}\frac{f(t)}{S^+(t)}\frac{dt}{t-x-iy}-$$

$$-\frac{a(x)S(x-iy)}{2\pi i(x_k-x+iy)}\int_{X_k}\frac{f(t)}{S^+(t)}\frac{dt}{t-x+iy}=$$

$$=I_1(f,x,y)+I_2(f,x,y),$$

where

$$I_1(f,x,y)=\frac{y}{\pi}\frac{S(x+iy)}{x_k-x-iy}\int_{X_k}\frac{f(t)(x_k-t)dt}{S^+(t)((t-x)^2+y^2)},$$

$$I_2(f,x,y)=\frac{yT(x;y)}{2\pi i}\int_{X_k}\frac{f(t)(x_k-t)dt}{S^+(t)(t-x+iy)},$$

where

$$T(x;y)=\frac{S(x+iy)}{x_k-x-iy}-\frac{a(x)S(x-iy)}{x_k-i+iy}.$$

As

$$\int_{-\infty}^{+\infty}\frac{|x_k-x|^{\alpha_k}}{|x+i|^{\alpha_k}}\frac{y|dx|}{|x_k-x-iy|\,((t-x)^2+y^2)}\le const,$$

then

$$\|I_1(f,x,y)\|_{L^1(\rho)}=$$

$$=\widetilde{C_1}\int_{-\infty}^{+\infty}\frac{|x_k-x|^{\alpha_k}}{(1+|x|)^{\alpha}|x+i|^{\alpha_k}}\frac{y|dx|}{(x_k-x-iy)}\int_{X_k}\frac{|f(t)||x_k-t||dt|}{((t-x)^2+y^2)}$$

$$\le C_1\|f\|_{L^1(\rho)}\int_{-\infty}^{+\infty}\frac{|x_k-x|^{\alpha_k}}{(1+|x|)^{\alpha}|x+i|^{\alpha_k}}\frac{y|dx|}{(x_k-x-iy)}\le M_1'\|f\|_{L^1(\rho)},$$

where M_1' is a constant does not depend on y and k.
So

$$\|I_1(f,x,y)\|_{L^1(\rho)}\le M_1'\|f\|_{L^1(\rho)}.$$

Similarly we get

$$\|I_1(f,x,y)\|_{L^\infty(\rho)}\le M_1''\|f\|_{L^\infty(\rho)},$$

where $f\in L^\infty(\rho)$. By applying Riesz-Thorin interpolation theorem [3], we obtain

$$\|I_1(f,x,y)\|_{L^p(\rho)}\le M_1\|f\|_{L^p(\rho)},\quad 1<p<\infty. \tag{2.7}$$

As S^+ is bounded, then using the fact (Lemma 3 in [16]) that for sufficiently large R at $|x|>R$ the following estimate we have

$$|S^+(x+iy)-a(x)S^-(x-iy)|\le C_2|S^+(x+iy)|\frac{y}{|x+i|}, \tag{2.8}$$

where $C_2 > C_1 > 0$ some constant independent of y, we get

$$|T(x)| \leq \frac{Cy}{(x_k - x)^2 + y^2},$$

where $C > \max_k \{x_k\}$ is a constant.

Since

$$\int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{|x+i|^{\alpha_k}} \frac{y|dx|}{((x_k - x)^2 + y^2)} \leq const,$$

then

$$\|I_2(f,x,y)\|_{L^1(\rho)}$$
$$\leq \widetilde{C_2} \int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{|x+i|^{\alpha_k}} \frac{y|dx|}{((x_k - x)^2 + y^2)} \int_{X_k} \frac{|f(t)||x_k - t||dt|}{|(t - x + iy)|}$$
$$\leq C_2 \|f\|_{L^1(\rho)} \int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{|x+i|^{\alpha_k}} \frac{y|dx|}{((x_k - x)^2 + y^2)} \leq M_2' \|f\|_{L^1(\rho)},$$

where M_2' is a constant does not depend on y and k.

So

$$\|I_2(f,x,y)\|_{L^1(\rho)} \leq M_2' \|f\|_{L^1(\rho)}.$$

Similarly we get

$$\|I_2(f,x,y)\|_{L^\infty(\rho)} \leq M_2'' \|f\|_{L^\infty(\rho)},$$

where $f \in L^\infty(\rho)$. By applying Riesz-Thorin interpolation theorem, we get

$$\|I_2(f,x,y)\|_{L^p(\rho)} \leq M_2 \|f\|_{L^p(\rho)}, \;\; 1 < p < \infty. \tag{2.9}$$

Hence, from (2.7) and (2.9), we obtain

$$\|\Phi_k^+(x+iy) - a(x)\Phi_k^-(x-iy)\|_{L^p(\rho)} \leq M\|f\|_{L^p(\rho)}$$

where $1 < p < \infty$ and $M = \max\{M_1, M_2\}$ is a constant independent of y and k. The estimate of the theorem is proved.

Now let's prove the second statement of the theorem. Let $f_n(x) \in C^\alpha$ be a sequence of finite functions such that

$$\lim_{n\to\infty} \|f_n(x) - f(x)\|_{L^p(\rho)} = 0, 1 < p < \infty. \tag{2.10}$$

For any n we set

$$\widetilde{\Phi}_n(z) = \frac{S(z)}{2\pi i(x_k - z)} \int_{-\infty}^{+\infty} \frac{f_n(t)(x_k - t)dt}{S^+(t)(t-z)}, \qquad z \in \Pi^\pm.$$

and from Sokhotski-Plemelj formula (see [2]), we get

$$\lim_{y\to+0} \left\|\widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - f_n(x)\right\|_{L^1(\rho)} = 0. \tag{2.11}$$

Using the estimate of this theorem, we obtain

$$\lim_{y\to+0} \left\|\Phi_k^+(x+iy) - a(x)\Phi_k^-(x-iy) - f(x)\right\|_{L^p(\rho)}$$

$$\leq \left\|\widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - f_n(x)\right\|_{L^p(\rho)} + \|f_n(x) - f(x)\|_{L^p(\rho)}$$
$$+ \left\|\left(\widetilde{\Phi}_n^+(x+iy) - \Phi_k^+(x+iy)\right) - a(x)\left(\widetilde{\Phi}_n^-(x-iy) - \Phi_k^-(x-iy)\right)\right\|_{L^p(\rho)}$$
$$\leq \left\|\widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - f_n(x)\right\|_{L^p(\rho)} + 2\|f_n(x) - f(x)\|_{L^p(\rho)}.$$

Taking into account (2.10) and (2.11), we conclude

$$\lim_{y\to+0} \left\|\Phi_k^+(x+iy) - a(x)\Phi_k^-(x-iy) - f(x)\right\|_{L^p(\rho)} = 0.$$

Theorem is proved. □

3. The main result

3.1. **The problem R_p for $T_p = \emptyset$.**

Theorem 3.1. *Let $T_p = \emptyset$. Then the homogeneous problem R_p ($f \equiv 0$) does not have solution different from zero.*

Theorem 3.2. *Let $f \in L^p(\rho)$ and $T_p = \emptyset$. Also let the sequence of points $\{x_k\}_1^\infty$ satisfy the conditions (2.2), (2.3). Then the following assertions hold:*
a) If $\kappa \geq 0$, then the general solution of the inhomogeneous Problem R_p may be represented as

$$\Phi(z) = \frac{S(z)}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)dt}{S^+(t)(t-z)}, \quad z \in \Pi^+ \cup \Pi^-. \tag{3.1}$$

b) If $\kappa < 0$, then the inhomogeneous Problem R_p is solvable if and only if the function f satisfies the conditions

$$\int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)} \frac{dt}{(t+i)^j} = 0, \qquad j = 1, 2, ..., -\kappa - 1.$$

The general solution can be represented by (3.1).

Proof. The proof of the point a) follows from Lemma 2.3 and Theorem 3.1.

b) Let $\kappa < 0$. Denote

$$\Phi^+(x+iy) - a(x)\Phi^-(x-iy) = f_y(x). \tag{3.2}$$

Taking into account that $a(x) = \dfrac{S^+(x)}{S^-(x)}$, we get

$$\frac{\Phi^+(x+iy)}{S^+(x)} - \frac{\Phi^-(x-iy)}{S^-(x)} = \frac{f_y(x)}{S^+(x)}.$$

Denoting

$$\Phi_y^+(z) = \frac{\Phi^+(z+iy)}{S^+(z)},\ z \in \Pi^+, \qquad \Phi_y^-(z) = \frac{\Phi^-(z-iy)}{S^-(z)},\ z \in \Pi^-,$$

we get

$$\Phi_y^+(x) - \Phi_y^-(x) = \frac{f_y(x)}{S^+(x)}.$$

In the case $\kappa < 0$, the function $\Phi_y^-(z)$ has zero of order $|\kappa - 1|$ at the point of $z = -i$. Consequently, $f(x)$ satisfies the conditions

$$\int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)} \frac{dt}{(t+i)^j} = 0, \qquad j = 1, 2, ..., -\kappa - 1.$$

Theorem 3.2 is proved. □

3.2. **The problem R_p for $T_p \neq \emptyset$.**

Theorem 3.3. *Let $T_p \neq \emptyset$. Then the general solution of the homogeneous problem R_p $(f \equiv 0)$, can be represented as:*

$$\Phi_0(z) = S(z) \sum_{x_k \in T_p} \frac{A_k}{x_k - z},$$

where $\{A_k\} \in l^p$.

Proof. It is clear that the number of points $x_k \in T_p$ is finite and by n_p we denote those points. It is sufficient to establish that the function $r_k(z) = S(z)(x_k - z)^{-1}$ does not satisfy condition (1.2) if $x_k \notin T_p$. Indeed

$$|r_k(x+iy) - r_k(x-iy)| = |R_1(x,y) + R_2(x,y)|,$$

where

$$R_1(x,y) = \frac{(x_k - x)\left(S^+(x+iy) - a(x)S^-(x-iy)\right)}{(x_k-x)^2 + y^2},$$

$$R_2(x,y) = \frac{iy\left(S^+(x+iy) + a(x)S^-(x-iy)\right)}{(x_k-x)^2 + y^2}.$$

Using inequality (2.8), we get

$$\|R_1(x,y)\|_{L^p(\rho)}$$

$$\leq C_1 y^p \left(\int_{-\infty}^{+\infty} \frac{|x_k - x|^{p(1+\alpha_k)} dx}{|x+i|^p \, |1+|x||^{p\alpha_0} \left((x_k-x)^2 + y^2\right)^p} \right)^{\frac{1}{p}} < C.$$

On the other hand

$$\|R_2(x,y)\|_{L^p(\rho)} \geq C_0 \Big(2 \int_{|x-x_k|<\delta} \frac{y^p |x_k - x|^{p\alpha_k} dx}{\left((x_k-x)^2+y^2\right)^p} \Big)^{\frac{1}{p}} > C_1 > 0.$$

where C_1 does not depend on δ. So $\|r_k(x+iy) - r_k(x-iy)\|_{L^p(\rho)} \geq M > 0$. □

Theorem 3.4. *Let $f \in L^p(\rho)$ and $T_p \neq \emptyset$. Also let the sequence of points $\{x_k\}_1^\infty$ satisfy conditions (2.2), (2.3) and $\kappa \geq 0$. Then the general solution of the inhomogeneous Problem R_p may be represented as $\Phi(z) = \Phi_0(z) + \Phi_1(z)$ where Φ_0 is the general solution of the homogeneous problem and*

$$\Phi_1(z) = \sum_{k=1}^{\infty} \Phi_k(z), \tag{3.3}$$

where

$$\Phi_k(z) = \frac{S(z)}{2\pi i(x_k - z)} \int_{X_k} \frac{f(t)(x_k - t)dt}{(t-z)}, k = 1, 2, ...$$

Proof. Since $\kappa \geq 0$, then from Theorem 2.1 we have

$$\|\Phi_1^+(x+iy) - a(x)\Phi_1^-(x-iy)\|_{L^1(\rho)} \leq C\|f\|_{L^p(\rho)},$$

where the constant C is independent on y and k. Therefore, similar to the proof of the second part of Theorem 2.1, we obtain

$$\lim_{y\to+0} \|\Phi_1^+(x+iy) - a(x)\Phi_1^-(x-iy) - f(x)\|_{L^p(\rho)} = 0.$$

Taking into account Theorem 3.3 we get the proof of the theorem. □

Theorem 3.5. *Let $f \in L^p(\rho)$ and $T_p \neq \emptyset$. Also let the sequence of points $\{x_k\}_1^\infty$ satisfy conditions* (2.2), (2.3) *and $\kappa < 0$. Then the general solution of the in-homogeneous problem R_p may be represented as $\Phi(z) = \Phi_0(z) + \Phi_1(z)$, where $\Phi_1(z)$ is defined by* (3.3) *and*

$$\Phi_0(z) = S(z) \sum_{x_k \in T_p} \frac{A_k}{x_k - z},$$

here $\{A_k\}_{k=1}^\infty \in l^1$, $A_{-\kappa+1}, A_{-\kappa+2}, ...$ are arbitrary complex numbers, and the numbers $A_1, A_2, ..., A_\kappa$ are uniquely defined by the system of linear equations

$$(3.4) \qquad \begin{cases} \sum_{k=1}^\infty \frac{A_k}{(x_k+i)} = -\sum_{j=1}^\infty I_{1\,1} \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^2} = -\sum_{j=1}^\infty (I_{2\,1} + I_{1\,2}) \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^3} = -\sum_{j=1}^\infty (I_{3\,1} + 2I_{2\,2} + I_{1\,3}) \\ \dots \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^{-\kappa}} = -\sum_{j=1}^\infty \sum_{m=1}^{-\kappa} C_m^{-\kappa} I_{m\,-\kappa-m} \end{cases},$$

where C_m^n are the binomial coefficients and

$$I_{m\,n} = \frac{1}{2\pi i(x_k+i)^m} \int_{X_k} \frac{f(t)(x_k - t)}{S^+(t)(t+i)^n} dt, \quad m, n = 1, 2, ..., -\kappa\ .$$

Proof. In the case $\kappa < 0$, $S^-(z)$ has a pole of order $-\kappa(\kappa < 0)$ at the point $z = -i$. Hence in order $\Phi(z)$ to be solution of the in-homogeneous problem R_p, for $A_1, A_2, ..., A_{-\kappa}$ it must be hold (3.4). Note that the determinant of the linear system (3.4) is a Vandermonde determinant and is determined by the following formula:

$$det = \prod_{1\leq k<j\leq -\kappa} \left(\frac{1}{x_j + i} - \frac{1}{x_k + i} \right).$$

Since $\frac{1}{x_k+i}$, $k = 1, 2, ..., -\kappa$ are distinct, the determinant is non-zero. Hence the numbers $A_1, A_2, ..., A_{-\kappa}$ may be uniquely defined by the system of linear equations (3.4). □

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О ПРОИЗВЕДЕНИИ ПОДМНОЖЕСТВ В ПЕРЕОДИЧЕСКИХ ГРУППАХ

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Аннотация. Доказывается, что для любого конченого симметризованного подмножества S свободной бернсайдовой группы $B(m,n)$ имеет место неравенство $|S^t| \geq 4 \cdot 2.9^{t/(400s)^3}$, где s – наименьший нечетный делитель числа n, удовлетворяющий неравенству $s \geq 1003$.

MSC2020 number: 20F50; 20F05.

Ключевые слова: степень подмножеств; произведение множеств; бернсайдова группа; рост группы.

1. Введение

Пусть S конечное подмножество некоторой группы. Через S^t обозначим множество всевозможных произведений вида $a_1 \cdots a_t$, где $a_i \in S$. Мей-Чу Чанг (см. [1]) доказала, что для любого конечного подмножества свободной группы, не содержащегося ни в какой циклической подгруппе существуют константы $c, \delta > 0$ такие, что справедлива оценка $|S^3| > c|S|^{1+\delta}$. А. А. Разборов (см. [2]) улучшил эту оценку и показал, что существует константа $c > 0$ такая, что для любого конечного подмножества S свободной группы, не содержащегося ни в какой циклической подгруппе, справедливо неравенство $|S^3| > |S|^2/(log|S|)^c$. С.Р.Сафин [3] позже усилил этот результат, показав, что существуют такие константы $c_n > 0$, что для любого конечного подмножества S свободной группы, не содержащегося ни в какой циклической подгруппе, справедливо неравенство $|S^t| > c_t|S|^{[(t+1)/2]}$ при всех натуральных t. Другие результаты по аддитивной комбинаторике можно найти в [4].

[1]Исследование первого автора выполнено при финансовой поддержке Комитета по науке РА в рамках научного проекта № 21T-1A213. Исследование второго автора выполнено при финансовой поддержке КН РА и РФФИ (РФ) в рамках совместой научной программы 20RF-152 и 20-51-05010 соответственно.

Мы будем рассматривать степени конечных симметризованных подмножество S свободных бернсайдовых групп нечетного периода $n \geq 1003$. При этом подмножество S называется симметризованной, если из $a \in S$ следует $a^{-1} \in S$.

Нашей целью является следующая теорема.

Теорема 1.1. *Для любого конченого симметризованного подмножества S свободной бернсайдовой группы $B(m,n)$ имеет место неравенство $|S^t| \geq 4{\cdot}2.9^{t/(400s)^3}$, где s – наименьший нечетный делитель числа n, удовлетворяющий неравенству $s \geq 1003$.*

Напоминаем, что относительно свободная группа ранга m многообразия групп, удовлетворяющих тождеству $x^n = 1$, обозначается через $B(m,n)$ и называется свободной периодической или *свободной бернсайдовой группой* периода n и ранга m. Более просто

$$B(m,n) = \langle a_1, a_2, ..., a_m;\ x^n = 1\rangle.$$

Количество элементов множества S^t принято обозначать также через $\gamma_S(t)$. Число

$$\lambda(G,S) = \lim_{t\to\infty} \gamma_S(t)^{\frac{1}{t}} \leq \gamma_S(1) = |S|$$

называется степенью экспоненциального роста группы G относительно S.

Если $\inf\limits_S \lambda(G,S) > 1$, где инфимум берется по всем конечным порождающим множествам S, то скажем, что группа G имеет *равномерный экспоненциальный рост.* Из теоремы 1.1 вытекает

Следствие 1.1. *(см. [5], [6])Для любого $m \geqslant 2$ и нечетного $n \geqslant 1003$ любая конечно порожденная нециклическая подгруппа H свободной бернсайдовой группы $B(m,\, n)$ имеет равномерный экспоненциальный рост.*

Отметим, что вопрос имеют ли бесконечные свободные бернсайдовые группы $B(m,\, n)$ равномерный экспоненциальный рост был поставлен де ля Арпом в [7].

В дальнейшем изложении мы без специальных ссылок будем использовать обозначения и терминологию монографии [8]и статьи [9]. При ссылках на утверждения из [8] мы используем принятые в ней стандартные обозначения. Например, VI.2.4 [8] означает пункт 4 параграфа 2 главы VI монографии [8].

2. Доказательство теоремы 1.1

Пусть S - произвольное множество, порождающее нециклическую подгруппу $\langle S\rangle_{B(2,\,s)}$ группы $B(2,\, s)$, где $s \geq 1003$ фиксированное нечетное число. Согласно теореме С. И. Адяна VI.3.3[8], каждая абелевая подгруппа группы $B(m,s)$ - циклическая, следовательно подгруппа $\langle S\rangle_{B(2,\,s)}$ неабелева. Поэтому можно выбрать пару не перестановочных элементов X и Y из множества S.

Отправной точкой для нас служит следующее утверждение доказанное В. Атабекяном в работе [10].

Лемма 2.1. *(Теорема 2[10]). Пусть коммутатор $[A^d\,,\; Z^{-1}\,B^dZ]$ в группе $B(2,s,\,\alpha-1)$ равен элементарному периоду C ранга α, где A - элементарный период ранга γ, B - элементарный период ранга β, $Z\in\mathscr{M}_{\alpha-1}$ ($\gamma\leqslant\beta\leqslant\alpha-1$), $d=191$, $s\geqslant 1003$ - произвольное нечетное число и слова A^q и B^q входят в некоторые слова из множеств $\mathscr{M}_{\gamma-1}$ и $\mathscr{M}_{\beta-1}$ соответственно. Тогда слова*

$$u\rightleftharpoons C^{200}AC^{200}A^2\cdot\;\cdot\;\cdot\;A^{s-1}C^{200},\;\; v\rightleftharpoons C^{300}AC^{300}A^2\cdot\;\cdot\;\cdot\;A^{s-1}C^{300}$$

являются базисом свободной бернсайдовой подгруппы ранга 2 группы $B(2,s)$.

Чтобы пользоваться леммой 2.1 докажем, что произвольная нециклическая подгруппа $\Delta\rightleftharpoons\langle X,Y\rangle$ группы $B(2,s)$ содержит такую нециклическую подгруппу вида $U\,\langle A\,,\;C\rangle\,U^{-1}$, что C - элементарный период некоторого ранга α и $C\overset{B(m,s,\alpha-1)}{=}[A^d\,,\;Z^{-1}B^dZ]$, где A и B - минимизированные элементарные периоды некоторых рангов γ и β, $Z\in\mathscr{M}_{\alpha-1}$, $\gamma\leqslant\beta\leqslant\alpha-1$, $d=191$ и длины слов UAU^{-1} и UCU^{-1} относительно порождающих X и Y удовлетворяют неравенствам

$$|UAU^{-1}|_{\{X,Y\}}<(450s)^2 \text{ и } |UCU^{-1}|_{\{X,Y\}}<(450s)^2.$$

Заметим, что леммы 2.8, 3.2, 6.6 и 7.2 из работы [9] остаются справедливыми, если в их формулировках и доказательствах эквивалентность в ранге α заменить на эквивалентность в ранге α в смысле монографии [8], а равенство слов в группе Γ_α заменить на равенство в группе $B(2,s,\alpha)$.

В силу VI.2.4[8] и VI.1.2[8] для некоторых слов T, Z и минимизированных элементарных периодов F и E, имеющих ранги σ и ρ соответственно, имеют место равенства $X\overset{B(2,s)}{=}T\,F^{\,i}\,T^{-1}$ и $T^{-1}Y\,T\overset{B(2,s)}{=}Z^{-1}\,E^j\,Z$. Без ограничения общности, можем предположить, что $\sigma\leqslant\rho$, а в силу VI.2.4[8] и IV.1.13[8] можем считать, что $Z\in\mathscr{M}_\xi\cap\mathscr{A}_{\xi+1}$ для некоторого $\xi\geqslant\rho$. Пусть НОД$(i,s)=k$ и r такое целое число, что $|r|<s$ и $F^{i\,r}=F^k$. Выбрав число $s\rightleftharpoons[s/3k]$, получим $s/5<sk<s/3$. Таким образом $X^{rs}=T\,F^{\,i\,rs}\,T^{-1}=T\,F^{\,ks}\,T^{-1}$ и $186<ks<\frac{s+1}{2}-148$, поскольку $s\geqslant 1003$. Итак, для слова $X_1\rightleftharpoons X^{rs}$ имеем $X_1=T\,F^{\,ks}\,T^{-1}$ и $|X_1|_{\{X,Y\}}\leqslant|rs||X|_{\{X,Y\}}<s^2/3$. Аналогично можно найти такой $Y_1\in\langle Y\rangle_{B(2,s)}$

и такие числа t и l, что $186 < tl < \frac{s+1}{2} - 148$, $T^{-1} Y_1 T = Z^{-1} E^{tl} Z$ и $|Y_1|_{\{X,Y\}} < s^2/3$.

По теореме VI.3.1[8] $[X_1, Y_1] \neq 1$. В силу лемм 3.2[9], 7.2[9] и 2.8[9] коммутатор

$$[X_1, Y_1] = T\left[F^{ks}, Z^{-1} E^{tl} Z\right] T^{-1}$$

сопряжен в $B(m, s)$ некоторому минимизированному элементарному периоду D некоторого ранга $\delta \geqslant \rho + 1$. Пусть $T^{-1}[X_1, Y_1]\, T = Z_1^{-1} D\, Z_1$, где $Z_1 \in \mathscr{M}_\lambda \cap \mathscr{A}_{\lambda+1}$ для некоторого $\lambda \geqslant \delta$. Тогда вновь применив леммы 3.2[9], 7.2[9] и 2.8[9], получим, что коммутатор $\left[X_1, [X_1, Y_1]^{\,d}\right] = T\left[F^{ks}, Z_1^{-1} D^d Z_1\right] T^{-1}$ сопряжен в $B(2, s)$ некоторому минимизированному элементарному периоду B некоторого ранга $\mu \geqslant \delta + 1$. Допустим $T^{-1}\left[X_1, [X_1, Y_1]^{\,d}\right] T = Z_2^{-1} B Z_2$. Таким образом, в подгруппе $\Delta \rightleftharpoons \langle X, Y\rangle_{B(2,s)}$ содержатся элементы $[X_1, Y_1] = T Z_1^{-1} D\, Z_1 T^{-1}$ и $\left[X_1, [X_1, Y_1]^{\,d}\right] = T Z_2^{-1} B Z_2 T^{-1}$. Можно считать, что $Z_3^{-1} \rightleftharpoons Z_1 Z_2^{-1} \in \mathscr{M}_\nu \cap \mathscr{A}_{\nu+1}$, где $\nu \geqslant \mu$. По лемме 3.2[9] найдем приведенную форму C коммутатора $\left[D^d, Z_3^{-1} B^d Z_3\right]$. Согласно лемме 7.2[9] C - элементарный период некоторого ранга $\tau \geqslant \mu + 1$. В силу (3.6)[9] $C \overset{B(2,s,\mu)}{=} w\left[D^d, Z_3^{-1} B^d Z_3\right] w^{-1}$, где $w \in \Theta(D, D_1)$.

Рассмотрим элементарные периоды $A \rightleftharpoons w D w^{-1}$ и $C = w\left[D^d, Z_3^{-1} B^d Z_3\right] w^{-1}$. Из определений следует, что

$$A = w Z_1 T^{-1}[X_1, Y_1]\, T Z_1^{-1} w^{-1}$$

и

$$C = w Z_1 T^{-1}\left[[X_1, Y_1]^d, \left[X_1, [X_1, Y_1]^{\,d}\right]^d\right] T Z_1^{-1} w.$$

Значит, если $U \rightleftharpoons T Z_1^{-1} w^{-1}$, то $UAU^{-1} \in \Delta$, $UCU^{-1} \in \Delta$ и

$$|UAU^{-1}|_{\{X,Y\}} = |\,[X_1, Y_1]\,|_{\{X,Y\}} < \frac{s^2}{3}|\,[X, Y]\,|_{\{X,Y\}} = \frac{4}{3}s^2, \tag{2.1}$$

(2.2)

$$|UCU^{-1}|_{\{X,Y\}} = |\left[[X_1, Y_1]^d, \left[X_1, [X_1, Y_1]^{\,d}\right]^d\right]|_{\{X,Y\}} < \frac{s^2}{3}(8d + 2d(8d+2)).$$

Остается заметить, что $\frac{s^2}{3}(8d + 2d(8d+2)) < (450s)^2$.

Поскольку подгруппы $\langle u, v\rangle_{B(2,s)}$ и $\left\langle UuU^{-1}, UvU^{-1}\right\rangle_{B(2,s)}$ изоморфны, то в силу леммы 2.1 элементы UuU^{-1}, UvU^{-1} являются базисом свободной бернсайдовой подгруппы ранга 2 группы $B(2, s)$. Очевидно,

$$UuU^{-1} = (UCU^{-1})^{200}(UAU^{-1}) \cdot\ \cdot\ \cdot\ (UAU^{-1})^{s-1}(UCU^{-1})^{200},$$

$$UvU^{-1} = (UCU^{-1})^{300}(UAU^{-1}) \cdot\ \cdot\ \cdot\ (UAU^{-1})^{s-1}(UCU^{-1})^{300}.$$

По выбору имеем UuU^{-1}, $UvU^{-1} \in \langle X, Y\rangle_{B(m,s)}$. Используя неравенства (1) и (2), получаем

$$|UuU^{-1}|_{\{X,Y\}} < \frac{s(s-1)}{2}4s^2 + 200s(450s)^2;$$
$$|UvU^{-1}|_{\{X,Y\}} < \frac{s(s-1)}{2}4s^2 + 300s(450s)^2.$$

Заметим, что $|UuU^{-1}|_S \le |UuU^{-1}|_{\{X,Y\}}$, $|UvU^{-1}|_S \le |UvU^{-1}|_{\{X,Y\}}$, поскольку $\{X, Y\} \subseteq S$. В качестве L можно взять число $2s^2(s-1) + 300s(450s)^2 < (400s)^3$.

Перейдем к доказательству теоремы 1.1. В силу доказанных выше неравенств в группе $< X, Y >$ содержатся два элемента u, v длины $< L = (400s)^3$, порождающих свободную бернсайдову группу ранга 2. По известней теореме С. И. Адяна группы $B(2, s)$ имеют экспоненциалцный рост. Точнее, согласно теореме 2.15, гл. VI[8] в $\{u, v\}^k$ содержатся $\gamma(k) > 4 \cdot 2.9^{k-1}$ попарно различных элемента. Значит в множестве S^t, где $t \ge L$, содержатся $\gamma([t/L])$ попарно различных элемента, поскольку в шаре радиуса L содержатся два элемента порождающие подгруппу содержащую $\gamma([t/L])$ попорно различных элементов из S^t. Таким образом, $|S^t| \ge 4 \cdot 2.9^{[t/(400s)^3]}$. Чтобы зовершить доказательство остается заметить, что свободная бернсайдова группа $B(m, n)$ гомоморфно отображается на $B(m, s)$, если s делит n.

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О L^p-ГРИДИ УНИВЕРСАЛЬНЫХ ФУНКЦИЯХ ОТНОСИТЕЛЬНО ОБОБЩЕННОЙ СИСТЕМЫ УОЛША

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Аннотация. В настоящей работе построена функция $U \in L^1[0,1)$ которая обладает гриди универсальностью относительно обобщенной системе Уолша.

MSC2020 number: 42A65; 42A20.

Ключевые слова: обобщенная система Уолша; коэффициенты Фурье; свойство универсальности.

1. Введение

Пусть $a \geq 2$ фиксированное целое число и $\omega_a = e^{\frac{2\pi i}{a}}$.

Обобщенную систему Радемахера порядка a определяется следующим образом (см.[1]):

Положим для $n = 0$

$$\varphi_0(x) = \omega_a^k \quad x \in \left[\frac{k}{a}, \frac{k+1}{a}\right), \quad k = 0, 1, ..., a-1, \tag{1.1}$$

а для любого $n \geq 1$

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0(a^n x). \tag{1.2}$$

Тогда обобщенная система Уолша порядка a определяется так:

$$\psi_0(x) = 1,$$

и если $n = \alpha_1 a^{n_1} + ... + \alpha_s a^{n_s}$, где $n_1 > ... > n_s$, $0 \leq \alpha_j < a$, $j = 1, 2, ..., s$, тогда

$$\psi_n(x) = \varphi_{n_1}^{\alpha_1}(x) \cdot ... \cdot \varphi_{n_s}^{\alpha_s}(x). \tag{1.3}$$

Обозначим обобщенную систему Уолша порядка a через Ψ_a. Отметим, что Ψ_2 является классической системой Уолша, а система Ψ_a является частным случаем системы Виленкина.

[1]Исследование выполнено при финансовой поддержке Комитета по науке РА в рамках научного проекта N 21AG-1A066.

Основные свойства системы Ψ_a получены Г. Кристенсоном, Р. Пели, Ж. Файном, К. Ватари, Н. Виленкиным и другими математиками (см. [1]- [7]).

Отметим некоторые свойства системы Ψ_a, которые будем использовать в дальнейшем:

- Каждая n-ая функция Радемахера имеет период a^{-n} и

$$\varphi_n(x) = const \in \Omega_a = \{1, \omega_a, \omega_a^2, ..., \omega_a^{a-1}\}, \quad \Omega_a \subset \mathbb{C}, \tag{1.4}$$

  если $x \in \Delta_{n+1}^{(k)} = \left[\frac{k}{a^{n+1}}, \frac{k+1}{a^{n+1}}\right)$, $k = 0, ..., a^{n+1} - 1$, $n = 1, 2, ...$, где $\mathbb{C}$ множество комплексных чисел.
- $(\varphi_n(x))^k = (\varphi_n(x))^m$, $\forall n, k \in \mathbb{N}$, и $\quad m = k \ (mod\ a)$.
- $\psi_n(x)$ является конечным произведением функций Радемахера и принимает значения из Ω_a.
- Для всех $0 \le i\ ,\ j < a^s$ имеет место

$$\psi_i(x) \cdot \psi_j(a^s x) = \psi_{j \cdot a^s + i}(x). \tag{1.5}$$

- В частности, если $0 \le j \le a^k - 1$, то

$$\psi_{a^k + j}(x) = \varphi_k(x) \cdot \psi_j(x). \tag{1.6}$$

- Для всех $m = 1, 2, ...$ имеет место

$$\int_0^1 |\sum_{j=0}^{a^m - 1} \psi_j(x)| dx = 1. \tag{1.7}$$

- Ψ_a, $a \ge 2$ является полной ортонормированной системой в $L^2[0,1)$ (см. [2] стр. 5, 30) и базисом $L^p[0,1]$ при $p > 1$ (см. [7]).

Обозначим через $\chi_E(x)$ характеристическую функцию множества E, т.е.

$$\chi_E(x) = \begin{cases} 1\ , & \text{если } x \in E\ , \\ 0\ , & \text{если } x \notin E\ . \end{cases} \tag{1.8}$$

а через $L^p[0,1), p \ge 1$, класс всех функций $f(x)$ измеримых на $[0,1)$ и удовлетворяющие условию

$$\int_0^1 |f(x)|^p dx < \infty, \tag{1.9}$$

а через $\Psi_a = \{\psi_k(x)\}$ - обобщенную систему Уолша порядка a. Для заданной функции $f \in L^p[0,1)$, $p \ge 1$, обозначим $c_k(f)$ коэффициенты Фурье по системе Ψ_a, т.е.

$$c_k(f) = \int_0^1 f(x) \overline{\psi_k(x)} dx, \tag{1.10}$$

а N-ую частичную сумму ряда Фурье по системе Ψ_a определим следующим образом:

$$S_N(x,f)=\sum_{k=0}^{N} c_k(f)\psi_k(x). \tag{1.11}$$

Спектром функции $f(x)$ ($spec(f)$) назовем множество индексов, для которых коэффициенты $c_k(f)$ отличны от нуля, т.е.

$$spec(f)=\{k\in N, c_k(f)\neq 0\}. \tag{1.12}$$

Определение 1.1. *m - тый гриди аппроксимант элемента $f\in L^p[0,1), p\geq 1$, по системе Ψ_a называется следующая сумма:*

$$G_m(f,\phi)=\sum_{k\in\Lambda} c_k(f)\phi_k, \tag{1.13}$$

где $\Lambda\subset\{1,2,...\}$ произвольный набор индексов мощности m удовлетворяющий условию:

$$|c_n(f)|\geq|c_k(f)|, \text{ если } n\in\Lambda, \quad k\notin\Lambda.$$

Скажем, что гриди алгоритм функции $f\in L^p[0,1]$, $p\geq 0$, сходится относительно Ψ_a, если последовательность $G_m(x,f)$ сходится к $f(t)$ по L^p норме.

Существует много работ о сходимости гриди алгоритм по разным системам (см. [8]-[16]). В частности Т. В. Кернер построил функцию из L^2 (потом и непрерывную) гриди алгоритм которой по тригонометрической системе расходится почти всюду (см. [10]). Далее В. Н. Темляков построил пример функции $f\in L^p$, $p\in[1,2)$ (соотв. $p>2$), гриди алгоритм которой по тригонометрической системе расходится по мере (по норме L^p, $p>2$)(см. [11]). В работах [13] и [14] доказана, что существует функция $f\in L^p[0,1)$, $p\geq 1$, гриди алгоритма которой по системе Ψ_a, $a\geq 2$, расходится по норме $L^p[0,1]$.

В работах [15] - [20] рассматривались вопросы сходимости жадного алгоритма по классическим системам с точки зрения знаменитых теорем исправления Н. Н. Лузина [21] и Д. Е. Меньшова [22] .

В частности в работе [20] доказано следующее утверждение:

Теорема 1.1. *Пусть $p>2$, тогда для любых $\varepsilon>0$ и $f\in L^p[0,1)$ существует функция $g\in L^p[0,1)$, $|\{x\in[0,1)\ ;\ g\neq f\}|<\varepsilon$ такая, что гриди алгоритм функции g, по системе Ψ_a, $a\geq 2$, сходится в L^p.*

В настоящей работе мы усиливаем теорему 1.1, доказав следующую теорему:

Теорема 1.2. *Существует функция $U \in L^1[0,1)$ обладающая следующим свойством: для любых чисел $0 < \epsilon < 1$, $p > 1$, и для любой функции $f \in L^p[0,1)$, $p \geq 1$, можно найти функцию $\tilde{f} \in L^p[0,1)$, $mes\{x \in [0,1);\ \tilde{f} \neq f\} < \epsilon$, такую, что гриди алгоритм функции $\tilde{f}$, по системе Ψ_a, $a \geq 2$, сходится в L^p и $|c_k(\tilde{f})|\ = c_k(U)$, $\forall k \in spec(\tilde{f})$.*

Имеет место также следующее утверждение:

Теорема 1.3. *Существует функция $U \in L^1[0,1)$, обладающая следующим свойством: для любых $0 < \epsilon < 1$ и для любой функции $f \in \bigcap_{p>1} L^p$, можно найти функцию $\tilde{f} \in \bigcap_{p>1} L^p$, $mes\{x \in [0,1);\ \tilde{f} \neq f\} < \epsilon$, такую что гриди алгоритм функции $\tilde{f}$, по системе Ψ_a, $a \geq 2$, сходится по всем L^p нормам одновременно и $|c_k(\tilde{f})|\ = c_k(U)$, $\forall k \in spec(\tilde{f})$.*

Функция U обладающая свойствами теорем 1.1 и 1.2 назовем L^p-гриди универсальной относительно системы Ψ_a, $a \geq 2$. Отметим, что существуют много работ посвященные существованиям разным типам универсальных функциях по разным системам (см. [23] - [31]).

Ответ на следующий вопрос нам не известен:

Вопрос 1. Справедливы ли Теоремы 1.2 и 1.3 для тригонометрической системы?

2. Доказательство основных лемм

Мы воспользуемся Леммой 1, доказанной в [15].

Лемма 2.1. *Пусть даны интервал $\Delta = \Delta_m$ ранга a и числа $N_0 \in \mathbb{N}$, $\gamma \neq 0$, $\epsilon \in (0,1)$, $p \in [1,\infty)$. Тогда существуют измеримое множество $E \subset \Delta$ и полином Q по системе Ψ_a вида*

$$Q(x) = \sum_{k=N_0}^{N} a_k \psi_k(x)\ ,$$

удовлетворяющие условиям:

1) коэффициенты $\{a_k\}_{k=N_0}^{N}$ равны 0 или $\varkappa \cdot \gamma \cdot |\Delta|$, где $|\varkappa| = 1$, $\varkappa \in \mathbb{C}$,

2)$|E| > (1-\varepsilon)|\Delta|$,

3)$Q(x) = \begin{cases} \gamma : & \text{если } x \in E \\ 0 : & \text{если } x \notin \ \Delta \end{cases}$, *4)* $\max_{N_0\le M\le N}\|Q(x)\|_p \le \frac{a^2|\gamma|}{\epsilon^{1-\frac{1}{p}}}\ |\Delta|^{\frac{1}{p}}.$

Основным аппаратом для доказательства Теоремы 1.3 служит лемма 2.3, которую докажем с помощью следующей леммы:

Лемма 2.2. *Пусть даны интервал $\Delta=\Delta_m$ ранга a и числа $m_0\in\mathbb{N}$, $\gamma\ne 0$, $\delta\in(0,1)$, $\theta\in(0,\frac{|\gamma|}{\delta})$, $0<\theta<\frac{|\gamma|}{\delta}$. Тогда существуют функция $g(x)$, измеримое множество $E\subset\Delta$, полиномы $H(x)$ и $Q(x)$ по системе Ψ_a вида*

$$H(x)=\sum_{k=a^{m_0-1}}^{a^m} b_k\psi_k(x)\ ,$$

$$Q(x)=\sum_{k=a^{m_0}}^{a^m-1}\varepsilon_k b_k\psi_k(x)\ ,$$

удовлетворяющие следующим условиям:

1) $0<b_{k+1}<b_k<\theta$, $\forall k\in[a^{m_0},a^m)$,

2) $\varepsilon_k=0$ *или* $\varkappa_k$, *где* $|\varkappa_k|=1,\forall k\in[a^{m_0},a^m)$, $\varkappa\in\mathbb{C}$,

3) $\int_0^1|H(x)|dx<\theta$,

4) $|E|>(1-\delta)|\Delta|$,

5) $g(x)=\begin{cases}\gamma, & \text{если } x\in E\\ 0, & \text{если } x\notin\ \Delta\end{cases}$

6) $\left\|g(x)-Q(x)\right\|_p<\theta.$

Доказательство. Возьмем такое натуральное число $\nu_0>1$, чтобы

$$a^{-\nu_0}|\gamma|<\frac{\theta}{4}, \tag{2.1}$$

и представим данный интервал в виде объединения интервалов следующим образом

$$\Delta=\bigcup_{\nu=1}^{a^{\nu_0}}\Delta_\nu, \tag{2.2}$$

где $|\Delta_\nu|=a^{-\nu}|\Delta|$.

Последовательно применяя лемму 1, найдем множества $E_\nu\subset\Delta_\nu$ и полиномов

$$Q_\nu(x)=\sum_{j=a^{m_{\nu-1}}}^{a^{m_\nu}-1}a_j\psi_j(x), \tag{2.3}$$

где $a_j = 0$ или $\varkappa_\nu \gamma |\Delta_j|$, если $j \in [a^{m_{\nu-1}}, a^{m_\nu}), |\varkappa_\nu| = 1,\ \nu \in \{1, ..., a^{\nu_0}\}$, удовлетворяющие условиям:

$$|E_\nu| > (1-\epsilon)|\Delta_\nu|, \tag{2.4}$$

$$Q_\nu(x) = \begin{cases} \gamma\ : \quad \text{if } x \in\ \Delta_\nu \\ 0 \quad : \quad \text{if } x \notin\ \Delta_\nu \end{cases}, \tag{2.5}$$

$$\left\| Q_\nu(x) \right\|_p < \frac{a^2 |\ \gamma\ |}{\delta^{1-\frac{1}{p}}} |\Delta_\nu|^{\frac{1}{p}}. \tag{2.6}$$

Определим

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu, \tag{2.7}$$

$$b_k = \frac{|\ \gamma\ |}{a^{\nu_0}} |\Delta| + \frac{\theta}{2^{2k}},\ k \in [a^{m_0}, a^m), m = m_{\nu_0} - 1, \tag{2.8}$$

$$H(x) = \sum_{k=a^{m_0}}^{a^m - 1} b_k \psi_k(x)\ ,\ m = m_{\nu_0} - 1, \tag{2.9}$$

$$g(x) = \sum_{\nu=1}^{\nu_0} Q_\nu(x)\ = \sum_{\nu=1}^{\nu_0} \sum_{j=a^{m_{\nu-1}}}^{a^{m_\nu}-1} a_j \psi_j(x) = \sum_{j=a^{m_0}}^{a^m - 1} a_j \psi_j(x), \tag{2.10}$$

$$Q(x) = \sum_{\nu=1}^{\nu_0} \sum_{j=a^{m_{\nu-1}}}^{a^{m_\nu}-1} \varepsilon_j b_j \psi_j(x) = \sum_{j=a^{m_0}}^{a^m - 1} \varepsilon_j b_j \psi_j(x), \tag{2.11}$$

где

$$\varepsilon_j = \begin{cases} 0, \quad \text{если} \quad a_j = 0; \\ \varkappa_\nu, \quad \text{где} |\varkappa_\nu| = 1; \end{cases} \quad \forall j \in [a^{m_{\nu-1}}, a^{m_\nu}), \nu = 1, 2, ...a^{\nu_0}. \tag{2.12}$$

Из условий (14), (15), (17)-(21) следует

$$|E| > (1-\delta)|\Delta|, \tag{2.13}$$

$$0 < b_{k+1} < b_k < \theta, \forall k \in [a^{m_0}, a^m), \tag{2.14}$$

$$\varepsilon_k = 0, \quad \text{или} \quad \varkappa_\nu, \quad \text{где} \quad |\varkappa_\nu| = 1, k \in [a^{m_0}, a^m),$$

$$g(x) = \begin{cases} \gamma \quad : \quad \text{если } x \in E \\ 0 \quad : \quad \text{если } x \notin\ \Delta \end{cases},$$

$$\left\| g(x) - Q(x) \right\|_p < \theta.$$

Из (21), (22) имеем

$$H(x)=\frac{|\ \gamma\ |}{a^{\nu_0}}|\Delta|\left(\sum_{j=0}^{a^m-1}\psi_j(x)\ -\ \sum_{j=0}^{a^{m_0}-1}\psi_j(x)\right)+\theta\sum_{j=a^{m_0}}^{a^m-1}\frac{1}{2^{2j}}\psi_j(x)\ ,$$

Отсюда и из соотношений (7), (14) получим

$$\int_0^1|H(x)|dx\le\frac{2|\ \gamma\ |}{a^{\nu_0}}|\Delta|\ +\frac{\theta}{2}\le\theta\ .$$

Лемма 2.2 доказана.

Лемма 2.3. *Пусть даны числа* $m_0>1$, $\theta,\delta\in(0,1)$, *и полином* $f(x)$ *по системе* Ψ_a. *Тогда существуют функция* $g(x)$, *измеримое множество* $E\subset\Delta$ *и полиномы* $H(x)$, $Q(x)$ *по системе* Ψ_a *вида*

$$H(x)=\sum_{k=a^{m_0}}^{a^m-1}b_k\psi_k(x)\ ,$$

$$Q(x)=\sum_{k=a^{m_0}}^{a^m-1}\varepsilon_k b_k\psi_k(x),$$

удовлетворяющие условиям:

1) $$0<b_{k+1}<b_k<\theta,\quad\forall k\in[a^{m_0},a^m),$$

2) $$|\varepsilon_k|=0\quad\text{или}\quad 1,\quad\forall k\in[a^{m_0},a^m),$$

3) $$|E|>(1-\delta)|\Delta|,$$

4) $$\int_0^1|H(x)|dx<\theta,$$

5) $$g(x)=f(x),\quad\text{для всех}\quad x\in E,$$

6) $$\|g(x)\|_p<\frac{4\|f\|_p}{\delta^{1-\frac{1}{p}}},$$

7) $$\Big\|g(x)-Q(x)\Big\|_p<\theta.$$

Доказательство. *Пусть*

$$f(x) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \chi_{\Delta_\nu}(x), \quad \sum_{\nu=1}^{\nu_0} |\Delta_\nu| = 1 \ , \tag{2.15}$$

где Δ_ν *интервал ранга* a *.*

Последовательно применяя лемму 2.2, найдем множества $E_\nu \subset \Delta_\nu$, функции $g_\nu(x)$ и полиномы

$$H_\nu(x) = \sum_{k=a^{m_{\nu-1}}}^{a^{m_\nu}-1} b_k^{(\nu)} \psi_k(x) \ , 1 \le \nu \le \nu_0, \tag{2.16}$$

$$Q_\nu(x) = \sum_{k=a^{m_{\nu-1}}}^{a^{m_\nu}-1} \varepsilon_k b_k^{(\nu)} \psi_k(x), \tag{2.17}$$

где $\varepsilon_k^{(\nu)} = 0$, или $\varkappa_k \in \mathbb{C}, |\varkappa_k| = 1$, которые, для всех $1 \le \nu \le \nu_0$ и $k \in [a^{m_{\nu-1}}, a^{m_\nu})$, удовлетворяют следующим условиям:

$$0 < b_1^{(\nu+1)} < b_{k+1}^{(\nu)} < b_k^{(\nu)} < b_{k+1}^{(\nu)} < b_{a^{m_{\nu-1}-1}}^{(\nu)-1} < \theta, \quad \forall k \in [a^{m_{\nu-1}}, a^{m_\nu}), \tag{2.18}$$

$$|\varepsilon_k^{(\nu)}| = 0 \ \text{ или } \ 1, \forall k \in [a^{m_{\nu-1}}, a^{m_\nu}), \tag{2.19}$$

$$\int_0^1 |H_\nu(x)| dx < \frac{\theta}{2^\nu}, \tag{2.20}$$

$$|E_\nu| > (1-\delta)|\Delta_\nu|, \tag{2.21}$$

$$g_\nu(x) = \begin{cases} \gamma_\nu & : \quad \text{if } x \in E_\nu \\ 0 & : \quad \text{if } x \notin \ \Delta_\nu \end{cases}, \tag{2.22}$$

$$\Big\| g_\nu(x) - Q_\nu(x) \Big\|_p < \frac{\min\{\theta, ||f||_p\}}{2^\nu}, \tag{2.23}$$

$$\Big\| Q_\nu(x) \Big\|_p < \frac{3|\gamma_\nu|}{\delta^{1-\frac{1}{p}}}. \tag{2.24}$$

Определим

$$g(x) = \sum_{\nu=1}^{\nu_0} g_\nu(x) \ , \tag{2.25}$$

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu. \tag{2.26}$$

$$H(x) = \sum_{\nu=1}^{\nu_0} H_\nu(x) = \sum_{\nu=1}^{\nu_0} \sum_{k=a^{m_{\nu-1}}}^{a^{m_\nu}-1} b_k^{(\nu)} \psi_k(x) \ = \sum_{k=a^{m_0}}^{a^m-1} b_k \psi_k(x), \tag{2.27}$$

(2.28) $$Q(x)=\sum_{\nu=1}^{\nu_0}Q_\nu(x)=\sum_{\nu=1}^{\nu_0}\sum_{k=a^{m_{\nu-1}}}^{a^{m_\nu}-1}\varepsilon_k^{(\nu)}b_k^{(\nu)}\psi_k(x)=\sum_{k=a^{m_0}}^{a^m-1}\varepsilon_k b_k\psi_k(x),$$

где

(2.29) $$m=m_{\nu_0},\quad \varepsilon_k=\varepsilon_k^{(\nu)}\ \text{ и }\ b_k=b_k^{(\nu)}\text{при }\ k\in[a^{m_{\nu-1}},a^{m_\nu}),\quad 1\le\nu\le\nu_0.$$

Из условий (28), (31), (32), (34), (35), (38), (41) и (31) имеем

$$g(x)=f(x)\qquad\text{для } x\in E,$$

$$|E|>1-\delta,$$

$$0<b_{k+1}<b_k<\theta,|\varepsilon_k|=0\ \text{ или }\ 1,\forall k\in[a^{m_0},a^m).$$

Учитывая (35) и (36) для всех $\nu\in\{1,...,\nu_0\}$ получим

(2.30) $$\|Q_\nu(x)\|_p\ \le\frac{\|f\|_p}{2^\nu}\ ,\ x\notin\ \Delta_\nu.$$

Из условий (29), (33), (36) - (38) и (43) имеем

$$\int_0^1|H(x)|dx\le\sum_{\nu=1}^{\nu_0}\int_0^1|H_\nu(x)|dx<\sum_{\nu=1}^{\nu_0}\frac{\theta}{2^\nu}\le\theta.$$

$$\Big\|g(x)-Q(x)\Big\|_p<\sum_{\nu=1}^{\nu_0}\ \Big\|g_\nu(x)-Q_\nu(x)\Big\|_p\le\min\{\theta,\|f\|_p\}.$$

$$\|Q(x)\|_p<\frac{4\|f\|_\infty}{\delta^{1-\frac{1}{p}}}.$$

Учитывая (36) - (38) получим

$$\|g(x)\|_p<\frac{5\|f\|_p}{\delta^{1-\frac{1}{p}}}.$$

Лемма 2.3 доказана.

3. Доказательство теорем

Обозначим последовательность всех полиномов по системе Ψ_a с рациональными коэффициентами следующим образом:

(3.1) $$\{f_n(x)\}_{n=1}^\infty.$$

Пусть дана последовательность

(3.2) $$\{p_n\},\quad p_n\uparrow\infty.$$

Последовательно применяя лемму 3, для любого $n \geq 1$ можем определить последовательности функций $\{g_n^{(j)}(x)\}_{j=1}^n$, множеств $\{E_n^{(j)}\}_{j=1}^n$ и полиномов по системе Ψ_a вида

$$H_n^{(j)}(x) = \sum_{k=M_n^{(j-1)}}^{M_n^{(j)}-1} b_k^{(n,j)}\psi_k(x), \quad 1 \leq j \leq n, \quad b_k^{(n,j)} \searrow, \tag{3.3}$$

$$Q_n^{(j)}(x) = \sum_{k=M_n^{(j-1)}}^{M_n^{(j)}-1} \varepsilon_k^{(n,j)} b_k^{(n,j)}\psi_k(x), \quad 1 \leq j \leq n, \quad (|\varepsilon_k^{(n,j)}| = 1 \ \text{ или } \ 0), \tag{3.4}$$

где

$$M_n^{(j)} = a^{m_n^{(j)}}, 0 \leq M_1^{(0)} < M_1^{(1)} = M_2^{(0)} < M_2^{(1)} < M_2^{(2)} <$$

$$< M_{n-1}^{(n-1)} = M_n^{(0)} < M_n^{(1)} < .. < M_n^{(n)} = M_{n+1}^{(0)} < M_{n+1}^{(1)} \ldots, \tag{3.5}$$

удовлетворяющие условиям:

$$g_n^{(j)}(x) = f_n(x) \ \text{ при } \quad x \in E_n^{(j)}, \tag{3.6}$$

$$|E_n^{(j)}| = 1 - 2^{-j}, \tag{3.7}$$

$$||g_n^{(j)}(x) - Q_n^{(j)}(x)||_{p_n} < 2^{-4n}, \quad 1 \leq j \leq n, \tag{3.8}$$

$$||g_n^{(j)}(x)||_{p_n} < 5 \cdot 2^j ||f_n||_{p_n}, \quad 1 \leq j \leq n, \tag{3.9}$$

$$\left(\int_0^1 |H_n^{(j)}(x)| dx \right) < 4^{-(n+j)}, 1 \leq j \leq n. \tag{3.10}$$

Определим функцию $U(x)$ последовательность чисел b_k следующим образом:

$$U(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \left(H_n^{(j)}(x) \right) =$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \left(\sum_{k=M_n^{(j-1)}}^{M_n^{(j)}-1} b_k^{(n,j)}\psi_k(x) \right) = \sum_{k=0}^{\infty} b_k \psi_k(x), \tag{3.11}$$

где

$$b_k = b_k^{(n,j)} \ \text{ при } \ k = M_n^{(j-1)}, ..., M_n^{(j)} - 1, \quad j = 1, 2, ..., n, \quad n = 1, 2, ... \tag{3.12}$$

Нетрудно заметить, что

$$\int_0^1 |U(x)| dx \leq \sum_{n=1}^{\infty} \sum_{j=1}^{n} \left(\int_0^1 |H_n^{(j)}(x)| dx \right) < \sum_{n=1}^{\infty} \sum_{j=1}^{n} 4^{-(n+j)} < 1. \tag{3.13}$$

$$\left\|U-\sum_{k=0}^{M_n^{(n)}-1} b_k^{(n,j)}\psi_k(x)\right\|_1=\left\|U-\sum_{n=q}^{\infty}\sum_{j=0}^{n}H_n^{(j)}\right\|_1<2^{-q} \tag{3.14}$$

Отсюда и из (54) имеем

$$c_k(U)>0,\quad c_k(U)\downarrow 0,\quad k=0,1,2,..., \tag{3.15}$$

$$b_k=c_k(U)\ ,k=0,1,2,.... \tag{3.16}$$

Пусть $f(x)\in\bigcap_{p>1}L^p[0,1)$. Нетрудно заметить, что можно выбрать последовательность $\{f_{k_n}(x)\}_{n=1}^{\infty}$ из (44) так, что

$$\lim_{n\to\infty}\left\|\sum_{j=1}^{n}f_{k_j}(x)-f(x)\right\|_{p_n}=0\ , \tag{3.17}$$

$$||f_{k_n}(x)||_{p_n}\le 4^{-2n},\quad n>1, \tag{3.18}$$

где

$$k_1>j_0=[\log_{\frac{1}{2}}\delta]+1 \tag{3.19}$$

($[a]$- целая часть числа a).

Положим

$$Q_1(x)=Q_{k_1}^{(j_0+1)},\quad E_1=E_{k_1}^{(j_0+1)},\quad g_1=g_{k_1}^{(j_0+1)}.$$

Предположим, что числа $k_1=\nu_1<\ldots<\nu_{q-1}$, функции $f_{\nu_n}(x),\ g_n(x),\quad 1\le n\le q-1$, множества $E_n,\quad 1\le n\le q-1$ и полиномы

$$Q_n(x)=Q_{\nu_n}^{(n+j_0)}(x)=\sum_{k=M_{\nu_n}^{(n+j_0-1)}}^{M_{\nu_n}^{(n+j_0)}-1}\delta_k^{(\nu_n,n+j_0)}b_k\psi_k(x)$$

уже определены и для всех $1\le n\le q-1$ удовлетворяют условиям:

$$g_n(x)=f_{k_n}(x),\quad x\in E_n, \tag{3.20}$$

$$\left\|\sum_{k=1}^{n}[Q_k(x)-g_k(x)]\right\|_{p_n}<4^{-(n-1)}, \tag{3.21}$$

$$|E_n|>1-\delta 2^{-n}, \tag{3.22}$$

$$\|g_n(x)\|_{p_n}<5\delta^{-1}2^{-(n-8)}. \tag{3.23}$$

Нетрудно заметить, что можно выбрать функцию $f_{\nu_q}(x)$ $(\nu_q > \nu_{q-1})$ из последовательности (44) так, что

$$\left\| f_{\nu_q}(x) - \left(f_{k_q}(x) - \sum_{i=1}^{q-1} [Q_i(x) - g_i(x)] \right) \right\|_{p_q} < 4^{-2q}. \tag{3.24}$$

Из условий (61), (64) и (67) имеем

$$\|f_{\nu_q}\|_{p_q} \leq \left\| f_{\nu_q}(x) - \left(f_{k_q}(x) - \sum_{i=1}^{q-1} [Q_i(x) - g_i(x)] \right) \right\|_{p_q} +$$

$$+\|f_{k_q}\|_{p_q} + \left\| \sum_{i=1}^{q-1} [Q_i(x) - g_i(x)] \right\|_{p_q} < \lambda \cdot 4^{-(q-3)}. \tag{3.25}$$

Положим

$$g_q(x) = f_{k_q}(x) + [g_{\nu_q}^{(q+j_0)}(x) - f_{\nu_q}(x)], \tag{3.26}$$

$$Q_q(x) = Q_{\nu_q}^{(q+j_0)}(x) = \sum_{k=M_{\nu_q}^{(q+j_0-1)}}^{M_{\nu_q}^{(q+j_0)}-1} \delta_k^{(\nu_q, n+j_0)} b_k \psi_k(x), \tag{3.27}$$

$$E_q(x) = E_{\nu_q}^{(q+j_0)}. \tag{3.28}$$

Учитывая соотношения (49) и (63) получим

$$g_q(x) = f_{k_q}(x), \quad x \in E_q. \tag{3.29}$$

Из (51), (64) и (67) имеем

$$\left\| \sum_{j=1}^{q} [Q_j(x) - g_j(x)] \right\|_{p_q} = \left\| \sum_{j=1}^{q-1} [Q_j(x) - g_j(x)] + Q_q(x) - g_q(x) \right\|_{p_q} \leq$$

$$\leq \left\| f_{\nu_q}(x) - \left(f_{k_q}(x) - \sum_{i=1}^{q-1} [Q_i(x) - g_i(x)] \right) \right\|_{p_q} +$$

$$+||g_{\nu_q}^{(q+j_0)} - Q_{\nu_q}^{(q+j_0)}||_{p_q} < 4^{-(q-1)}. \tag{3.30}$$

Из соотношений (67) - (69), (73) следует, что

$$\|g_q(x)\|_{p_q} \leq \left\| f_{\nu_q}(x) - \left(f_{k_q}(x) - \sum_{i=1}^{q-1} [Q_i(x) - g_i(x)] \right) \right\|_{p_q} +$$

$$+ \left\| \sum_{j=1}^{q-1} [(Q_j(x)) - g_j(x)] \right\|_{p_q} + \|g_{\nu_q}^{(q+j_0)}\|_{p_q} <$$

$$< 4^{-2q} + 4^{-q+2} + B\cdot 2^{q+j_0}\|f_{\nu_q}(x)\|_{p_q} < \frac{B\lambda}{\varepsilon}\cdot 2^{-q+8}. \tag{3.31}$$

Таким образом, по индукции определяются последовательности функций $\{g_q(x)\}_{q=1}^{\infty}$, множеств $\{E_q\}_{q=1}^{\infty}$ и полиномов $\{Q_q(x)\}$, которые удовлетворяют (63) - (66) для всех $q \geq 1$.

Положим

$$E = \bigcap_{q=1}^{\infty} E_q. \tag{3.32}$$

Из (65) следует, что

$$|E| > 1 - \delta.$$

Пусть $p > 1$ любое число из (45), тогда существует q_0 такое, что $p < p_q$ для всех $q \geq q_0$. Тогда, учитывая (66), имеем

$$\left\|\sum_{q=q_0}^{\infty} g_q(x)\right\|_p \leq \sum_{q=1}^{\infty}\|g_q\|_{p_q} < \infty. \tag{3.33}$$

Определим функцию $\tilde{f}(x)$ и последовательность чисел $\{\varepsilon_k\}$ следующим образом:

$$\tilde{f}(x) = \sum_{q=1}^{\infty} g_q(x), \tag{3.34}$$

$$\varepsilon_k = \begin{cases}\varepsilon_k^{(\nu_q, q+j_0)}, k\in[M_{\nu_q}^{(q+j_0-1)}, M_{\nu_q}^{(q+j_0)}),\ q=1,2,... \\ 0,\quad k\notin\bigcup_{q=1}^{\infty}[M_{\nu_q}^{(q+j_0-1)}, M_{\nu_q}^{(q+j_0)})\end{cases}, \tag{3.35}$$

$$\tilde{f}(x) \in \bigcap_{p>1} L^p[0,1], \quad \tilde{f}(x) = f(x), \quad x\in E,$$

Из (63), (64), (68), (69) - (71), (75) - (77) для всех $q \geq q_0$ имеем

$$\left\|\sum_{k=0}^{M_{\nu_{q-1}}^{(q-1+j_0)}-1} \varepsilon_k b_k \psi_k(x) - \tilde{f}(x)\right\|_p \leq$$

$$\leq \left\|\sum_{k=0}^{M_{\nu_{q-1}}^{(q-1+j_0)}-1} \varepsilon_k b_k \psi_k(x) - \tilde{f}(x)\right\|_{p_q} = \left\|\sum_{n=1}^{q-1}\left(\sum_{k=M_{\nu_n}^{(n+j_0-1)}}^{M_{\nu_n}^{(n+j_0)}-1} \varepsilon_k^{(\nu_n, n+j_0)} b_k\psi_k(x)\right) - \tilde{f}(x)\right\|_{p_q} =$$

$$= \left\|\sum_{n=1}^{q-1} Q_n(x) - \tilde{f}(x)\right\|_{p_q} \leq \left\|\sum_{n=1}^{q-1}(Q_n(x) - g_n(x))\right\|_{p_q} + \sum_{n=q}^{\infty}\|g_n(x)\|_{p_q} \leq 5\delta^{-1}\cdot 2^{-(q-10)},$$

Учитывая (59), (74) и (77) получим

$$c_k(\tilde{f}) = \int_0^1 \tilde{f}(x)\overline{\psi_k(x)}dx = \varepsilon_k b_k = \varepsilon_k c_k(U), \quad k = 0,1,2,... .$$

Отсюда и из (59), (78) следует, что

$$|c_k(\tilde{f})| = c_k(U), \quad k \in spec(\tilde{f})$$

и следовательно, ненулевые коэффициенты Фурье исправленной функции $\tilde{f}$ по модулю убывают на $spec(\tilde{f})$. Учитывая определения гриди апроксиманта для любого $n \in \mathbb{N}$ можно найти подпоследовательность M_n такую, что $S_{M_n}(\tilde{f}) = G_n(\tilde{f})$ и поскольку система Ψ_a является базисом во всех L^p, то получаем, что $G_n(\tilde{f})$ сходится к $\tilde{f}$ по всем нормам L^p.

Если $f \in L^p$, при фиксированном $p > 1$, то подобными рассуждениями доказывается, что гриди алгоритм исправленной функции $\tilde{f}$ сходится к $\tilde{f}$ по норме L^p. Теоремы 1.2 и 1.3 доказаны.

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GEOMETRIC PROPERTIES OF NORMALIZED LE ROY-TYPE MITTAG-LEFFLER FUNCTIONS

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Abstract. The main focus of the present paper is to establish sufficient conditions for the parameters of the normalized form of the generalized Le Roy-type Mittag-Leffler function have certain geometric properties like close-to-convexity, univalency, convexity and starlikeness inside the unit disc. The results obtained are new and their usefulness is depicted by deducing several interesting corollaries. The results obtained improve some several results available in the literature for the Mittag-Leffler function.

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Keywords: Mittag-Leffler function; analytic function; univalent; starlike; convex; close-to-convex functions.

1. Introduction

To study the asymptotic behavior of the analytic continuation of certain power series, Édouard Le Roy considered the following example [18, Section 6]

$$\sum_{k=0}^{\infty} \frac{z^k}{(k!)^\gamma},\ \gamma > 0, \tag{1.1}$$

when $z \to \infty$ along the real axis. Recently, S. Gerhold [6] and, independently, R. Garra and F. Polito [5] introduced a generalization of (1.1) by

$$F^{(\gamma)}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[\Gamma(\alpha n + \beta)]^\gamma} \qquad (z \in \mathbb{C}, \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0), \tag{1.2}$$

which turns out to be an entire function of the complex variable z for all values of the parameters such that $\Re(\alpha) > 0, \beta \in \mathbb{R}$ and $\gamma > 0$.

Obvious specifications of parameters lead to a set of well-known special functions like the Mittag-Leffler $E_\alpha = F^{(1)}_{\alpha,1}$, two parameter Mittag-Leffler $E_{\alpha,\beta} = F^{(1)}_{\alpha,\beta}$, multi-parameter Mittag-Leffler function ($E^{\gamma}_{\alpha,\beta} = F^{(\gamma)}_{\alpha,\beta}, \gamma \in \mathbb{N}$) and their subsequent special cases.

Various geometric properties has been studied for different classes of special functions such as Mittag-Leffler function, Wright function, hypergeometric functions, Bessel functions, Fox-Wright function and some other related functions are an ongoing part of research in geometric function theory. We refer to some geometric properties of these functions [1, 17, 16, 27, 28, 13, 14, 3, 7, 8, 2, 11, 12] and references therein.

Let $\mathcal{H}$ denote the class of all analytic functions inside the unit disk $\mathcal{D} = \{z : |z| < 1\}$. Suppose that $\mathcal{A}$ is the class of all functions $f \in \mathcal{H}$ which are normalized by $f(0) = f'(0) - 1 = 0$ such that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, for all $z \in \mathcal{D}$.

A function $f \in \mathcal{A}$ is said to be a starlike function (with respect to the origin 0) in $\mathcal{D}$, if f is univalent in $\mathcal{D}$ and $f(\mathcal{D})$ is a starlike domain with respect to 0 in $\mathbb{C}$. This class of starlike functions is denoted by $\mathcal{S}^*$. The analytic characterization of $\mathcal{S}^*$ is given [3] below:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \;\; \forall z \in \mathcal{D} \quad \Longleftrightarrow \quad f \in \mathcal{S}^*.$$

If $f(z)$ is a univalent function in $\mathcal{D}$ and $f(\mathcal{D})$ is a convex domain in $\mathbb{C}$, then $f \in \mathcal{A}$ is said to be a convex function in $\mathcal{D}$. We denote this class of convex functions by $\mathcal{K}$. This class can be analytically characterized as follows:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \;\forall z \in \mathcal{D} \quad \Longleftrightarrow \quad f \in \mathcal{K}.$$

It is well-known that zf' is starlike if and only if $f \in \mathcal{A}$ is convex.

A function $f(z) \in \mathcal{A}$ is said to be close-to-convex in $\mathcal{D}$ if $\exists$ a starlike function $g(z)$ in $\mathcal{D}$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,$$

for all $z \in \mathcal{D}$. The class of all close-to-convex functions is denoted by $\mathcal{C}$.

A function $f \in \mathcal{A}$ is said to be uniformly convex (starlike) if for every circular arc γ contained in $\mathcal{D}$ with center $\zeta \in \mathcal{D}$ the image arc $f(\gamma)$ is convex (starlike w.r.t. the image $f(\zeta)$). The class of all uniformly convex (starlike) functions is denoted by UCV (UST) [20]. In [10, 9], A. W. Goodman introduced these classes. Later, F. Rönning [20] introduced a new class of starlike functions $\mathcal{S}_p$ defined by

$$\mathcal{S}_p(\mathcal{D}) := \{f : f(z) = zF'(z),\, F \in UCV\}.$$

The main focus of this paper is to study certain geometric properties including univalency, starlikeness, convexity and close-to-convexity in the open unit disk of the normalized Le Roy-type Mittag-Leffler function defined by

$$\begin{aligned} \mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) &= z[\Gamma(\beta)]^{\gamma}\, F^{(\gamma)}_{\alpha,\beta}(z) \\ &= \sum_{k=1}^{\infty}\left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^{\gamma} z^k =: z + \sum_{k=2}^{\infty} A_k(\alpha,\beta,\gamma) z^k. \end{aligned} \tag{1.3}$$

Geometric properties of normalized form of Mittag-Leffler function $\mathbb{F}^{(1)}_{\alpha,\beta}(z) := \mathbb{F}_{\alpha,\beta}(z)$ were discussed by Bansal and Prajapat in [1]. Recently, in [17, 16] geometric properties of normalized form of $\mathbb{F}_{\alpha,\beta}(z)$ were studied, which improve some results of [1]. The above results inspire us to study the geometric properties of Le Roy-type Mittag-Leffler function and improve the results available in the literature.

Each of the following definition will be used in our investigation.

Definition 1.1. (Mitrinović and Vasić [15]) A sequence of real numbers $\{a_n\},\ n = 0, 1, 2...$ satisfying the condition

$$2a_{n+1} \leq a_n + a_{n+2},\ n = 0, 1, 2... \tag{1.4}$$

is called convex sequence. Putting $\Delta a_n = a_n - a_{n+1}$ and $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$. Condition (1.4) may be written as $\Delta^2 a_n \geq 0, n = 0, 1, 2...$ It is well known that If $f(x)$ is convex function (of real variable) for $x \geq 0$, then the sequence $a_n = f(n), n = 0, 1, 2...$ is convex.

Definition 1.2. An infinite sequence $\{b_n\}_1^{\infty}$ of complex numbers will be called a subordinating factor sequence if whenever

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1.5}$$

is analytic, univalent and convex in $\mathcal{D}$, then

$$\left\{\sum_{n=1}^{\infty} a_n b_n z^n : z \in \mathcal{D}\right\} \subseteq f(\mathcal{D}),\ (a_1 = 1). \tag{1.6}$$

For more information on the various geometric properties involving subordination between analytic functions, we refer the reader to the earlier works [3, 23] and also to the references cited therei

2. Useful lemmas

In order to prove our results the following preliminary results will be helpful.

Lemma 2.1. *Let* $\min(\alpha,\gamma)\geq 1, \beta>0$ *such that* $\alpha+\beta\geq 2$. *Then the following inequality*

$$\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)\leq z+z\theta_{\alpha,\beta}^{(\gamma)}(e^z-1), \tag{2.1}$$

holds true for all $z>0$, *where*

$$\theta_{\alpha,\beta}^{(\gamma)}=\left[\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right]^{\gamma}. \tag{2.2}$$

Proof. First, we prove that the sequence

$$x_k:=\left\{\frac{\Gamma(k+1)}{[\Gamma(\alpha k+\beta)]^{\gamma}}\right\}_{k\geq 1}, \tag{2.3}$$

is decreasing. Let $\min(\alpha,\gamma)\geq 1$ and $\beta>0$, then we have

$$\begin{aligned}\frac{x_{k+1}}{x_k}&=\frac{(k+1)[\Gamma(\alpha k+\beta)]^{\gamma}}{[\Gamma(\alpha k+\alpha+\beta)]^{\gamma}}\\ &\leq\frac{(k+1)[\Gamma(\alpha k+\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta+1)]^{\gamma}}=\frac{k+1}{(\alpha k+\beta)^{\gamma}}\leq\frac{k+1}{\alpha k+\beta}.\end{aligned} \tag{2.4}$$

It is easy to proved that the function $\chi(\xi)$ defined by

$$\chi(\xi)=(\alpha-1)\xi+\beta-1,$$

is non-negative for all $\alpha\geq 1$ such that $\alpha+\beta\geq 2$.

This in turn implies that the sequence $(x_k)_{k\geq 1}$ monotonically decreases. Therefore, for $z>0$ we get

$$\begin{aligned}\frac{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}{z}&=1+\sum_{k=1}^{\infty}\frac{[\Gamma(\beta)]^{\gamma}\Gamma(k+1)}{[\Gamma(\alpha k+\beta)]^{\gamma}}\frac{z^k}{k!}\\ &\leq 1+\theta_{\alpha,\beta}^{(\gamma)}\sum_{k=1}^{\infty}\frac{z^k}{k!}=1+\theta_{\alpha,\beta}^{(\gamma)}(e^z-1).\end{aligned}$$

This proves (2.1). □

Lemma 2.2. (Ozaki [19]) *Let* $f(z)=z+\sum_{k=2}^{\infty}A_kz^k$. *If* $1\leq 2A_2\leq ...\leq nA_n\leq (n+1)A_{n+1}\leq ...\leq 2$, *or* $1\geq 2A_2\geq ...\geq nA_n\geq(n+1)A_{n+1}\geq ...\geq 0$, *then* f *is close-to-convex with respect to* $-\log(1-z)$.

Lemma 2.3. [11] *Let* $f\in\mathcal{A}$ *and* $|(f(z)/z)-1|<1$ *for each* $z\in\mathcal{D}$, *then* f *is univalent and starlike in* $\mathcal{D}_{1/2}=\{z:|z|<1/2\}$.

Lemma 2.4. [12] *Let* $f\in\mathcal{A}$ *and* $|f'(z)-1|<1$ *for each* $z\in\mathcal{D}$, *then* f *is convex in* $\mathcal{D}_{1/2}=\{z:|z|<1/2\}$.

Lemma 2.5. [24] *If $f \in \mathcal{A}$ and satisfy*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \quad z \in \mathcal{D},$$

where M is a solution of the equation $\cos M = M$, then $\Re(f'(z)) > 0$.

Lemma 2.6. [25] *Assume that $f \in \mathcal{A}$.*

(1) *If $\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}$, then $f \in UCV(\mathcal{D})$.*

(2) *If $\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{2}$, then $f \in \mathcal{S}_p(\mathcal{D})$.*

Lemma 2.7. (Féjer [4]). *If $A_n \geq 0$, $\{nA_n\}$ and $\{nA_n - (n+1)A_{n+1}\}$ both are nonincreasing, then the function $f(z) = z + \sum\limits_{n=2}^{\infty} A_n z^n$ is in $\mathcal{S}^*$.*

Lemma 2.8. (Féjer [4]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_1 = 1$, and that for $n \geq 2$ the sequence $\{a_n\}$ is a convex decreasing, i.e.*

$$a_1 - a_2 \geq \cdots \geq a_k - a_{k+1} \geq \cdots \geq 0.$$

Then

$$\Re\left(\sum_{n=1}^{\infty} a_n z^{n-1}\right) > 1/2, \quad z \in \mathcal{D}. \tag{2.5}$$

Lemma 2.9. (Wilf [29]). *The sequence $\{b_n\}_1^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0, \; z \in \mathcal{D}. \tag{2.6}$$

3. Main results

Theorem 3.1. *Let $\min(\alpha, \gamma) \geq 1$ and $\beta > 0$ such that $\alpha + \beta \geq 2$. Then the following assertions hold true:*

(a). *If $(e-1)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma$, then the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is starlike in $\mathcal{D}_{1/2}$.*

(b). *If $2(e-1)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma$ and $\beta \geq 2$, then the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is convex in $\mathcal{D}_{1/2}$.*

Proof. (a) In view of (2.1) and straightforward calculation would yield

$$\begin{aligned} \left|\frac{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}{z} - 1\right| &\leq \sum_{k=1}^{\infty} A_{k+1}|z|^k \\ &\leq \mathbb{F}_{\alpha,\beta}^{(\gamma)}(1) - 1 \\ &\leq \theta_{\alpha,\beta}^{(\gamma)}(e-1), \end{aligned} \tag{3.1}$$

for all $z \in \mathcal{D}$. Hence, under the given hypotheses we obtain

$$\left|\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z} - 1\right| < 1,\ z \in \mathcal{D},$$

and consequently the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}_{1/2}$ by the means of Lemma 2.3.
(b) A simple computation becomes

$$\begin{aligned}\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1 &= \sum_{k=1}^{\infty}(k+1)A_{k+1}z^k \\ &= \sum_{k=1}^{\infty}\frac{y_k z^k}{k!},\end{aligned} \tag{3.2}$$

where $(y_k)_k$ is defined by

$$y_k = \frac{[\Gamma(\beta)]^\gamma \Gamma(k+2)}{[\Gamma(\alpha k+\beta)]^\gamma},\ k \geq 1. \tag{3.3}$$

We define the function $\tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi)$ by

$$\tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi) = \frac{\Gamma(\xi+2)}{[\Gamma(\alpha\xi+\beta)]^\gamma},\ \xi > 0.$$

Therefore

$$(\tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi))' = \tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi)[\psi(\xi+2) - \alpha\gamma\psi(\alpha\xi+\beta)]. \tag{3.4}$$

Under the given conditions, we deduce that $\psi(\alpha\xi+\beta) \geq \psi(\xi+2)$ and consequently the function $\tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi)$ is decreasing on $[1,\infty)$. This implies that the sequence $(y_k)_{k\geq 1}$ monotonically decreases for all $\alpha \geq 1, \beta \geq 2$ and $\gamma \geq 1$. Therefore

$$\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1\right| < \sum_{k=1}^{\infty}\frac{y_1}{k!} = y_1(e-1). \tag{3.5}$$

This implies that

$$\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1\right| < 1,\ \ z \in \mathcal{D}.$$

Hence, the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$ by Lemma 2.4. This completes the proof of Theorem 3.1. □

On setting $\alpha = 1$ and $\gamma = 2$ in Theorem 3.1, we get the following results as follows:

Corollary 3.1. *The following assertions hold true:*
(a). If $\beta > \sqrt{e-1}$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is starlike in $\mathcal{D}_{1/2}$.
(b). If $\beta \geq 2$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$.

Remark 3.1. Theorem 3.1, indicates that the function $\mathbb{F}_{1,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$ if $\beta \geq 2$. It concludes that our result improve the result proved in [1, Theorem 2.4 (b)].

Theorem 3.2. *Suppose that* $\alpha,\beta,\gamma>0$ *such that* $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma$, *then the function* $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ *is close-to-convex with respect to starlike function* $-\log(1-z)$ *in* $\mathcal{D}$, *and consequently it is univalent in* $\mathcal{D}$.

Proof. To prove that $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$, it is sufficient to prove, in view of Lemma 2.2, that the sequence $\{kA_k\}_{k\geq 1}$ is decreasing. A simple computation gives

$$\begin{aligned} kA_k-(k+1)A_{k+1} &= k\left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^\gamma-(k+1)\left[\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\right]^\gamma \\ &= \frac{k[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha k+\beta)]^\gamma}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma-\frac{k+1}{k}\right]. \end{aligned}$$

By using the fact that the function $z\mapsto \frac{\Gamma(z+a)}{\Gamma(z)},\ a>0$ is increasing we deduce that the sequence

$$\left\{\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma\right\}_{k\geq 1},$$

is increasing provided that $\alpha>0$ and $\gamma>0$, on the other hand the sequence $\{\frac{k+1}{k}\}_{k\geq 1}$ is decreasing sequence. This implies that the sequence

$$\{v_k\}_{k\geq 1}:=\left\{\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma-\frac{k+1}{k}\right\}_{k\geq 1}$$

is increasing and consequently

$$v_k\geq v_1=\frac{[\Gamma(\alpha+\beta)]^\lambda-2[\Gamma(\beta)]^\lambda}{[\Gamma(\beta)]^\lambda},$$

which is non-negative under the given hypotheses. Hence

$$kA_k-(k+1)A_{k+1}\geq 0$$

for all $k\geq 1$. This completes the proof of the Theorem 3.2. □

Corollary 3.2. *For* $\alpha\geq 1,\gamma>0$ *and* $\beta\geq 2^{1/\gamma}$, *then the function* $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$, *close-to-convex with respect to starlike function* $-\log(1-z)$ *in* $\mathcal{D}$.

Theorem 3.3. *Let* $\alpha>0,\beta>0,\gamma>0$. *Assume that any one of the following conditions* $(H_1),(H_1^1)$ *or* (H_1^2) *hold true:*

$$(H_1):\begin{cases} (i). & \min(\alpha,\beta,\gamma)\geq 1, \alpha\gamma\geq 2, \\ (ii). & [\Gamma(\beta)]^\gamma(e-1)<[\Gamma(\alpha+\beta)]^\gamma \\ (iii). & \frac{e[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma}+\frac{4(e-2)[\Gamma(\beta)]^\gamma}{[\Gamma(2\alpha+\beta)]^\gamma}<1, \end{cases}$$

$$(H_1^1):\begin{cases}(i). & \min(\alpha,\gamma)\geq 1, \alpha+\beta\geq \max(4^{\frac{1}{\gamma}},2),\\ (ii). & \text{The function } L_{\alpha,\beta}^{\gamma}: z\mapsto (z+1)^2 - z(\alpha z+\beta)^{\gamma} \text{ is decreasing on } [1,\infty),\\ (iii). & \theta_{\alpha,\beta}^{(\gamma)} < \frac{1}{e},\end{cases}$$

$$(H_1^2):\ [\Gamma(\alpha+\beta)]^{\gamma}\geq 4[\Gamma(\beta)]^{\gamma},$$

Then the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is starlike in $\mathcal{D}$.

Proof. First we assume that the hypothesis (H_1) holds. By using the triangle inequality and using the fact that the sequence (x_n) is decreasing (see the proof of Lemma 2.1), then for all $z\in\mathcal{D}$ we get

$$\left|\frac{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right| \geq 1-\sum_{k=1}^{\infty}A_{k+1}|z|^k \geq 1-[\Gamma(\beta)]^{\gamma}\sum_{k=1}^{\infty}\frac{x_k}{k!}$$
$$(3.6)\qquad > 1-[\Gamma(\beta)]^{\gamma}\sum_{k=1}^{\infty}\frac{x_1}{k!} = 1-\theta_{\alpha,\beta}^{(\gamma)}(e-1) > 0,$$

where $\theta_{\alpha,\beta}^{(\gamma)}$ and $(x_k)_k$ are defined in (2.2) and (2.3) respectively. On the other hand, we have

$$(3.7)\qquad (\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))' - \frac{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}{z} = \sum_{k=1}^{\infty}\frac{B_k z^k}{k!},\ z\in\mathcal{D},$$

where $(B_k)_k$ is defined by

$$B_k = \frac{k\Gamma(k+1)[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}},\ k\geq 1.$$

The sequence $(B_k)_{k\geq 2}$ is monotonically decreases for all $\alpha\geq 1,\beta\geq 2$ and $\gamma\geq 1$ such that $\alpha\gamma\geq 2$. Indeed, for this we consider the function $g_{\alpha,\beta}^{\gamma}(z)$ defined by

$$f_{\alpha,\beta}^{(\gamma)}(\xi) = \frac{\xi\Gamma(\xi+1)}{[\Gamma(\alpha\xi+\beta)]^{\gamma}},\ \xi>0.$$

Then

$$(3.8)\qquad (f_{\alpha,\beta}^{(\gamma)}(\xi))' = f_{\alpha,\beta}^{(\gamma)}(\xi)\left[\frac{1}{\xi}+\psi(\xi+1)-\alpha\gamma\psi(\alpha\xi+\beta)\right].$$

Since the digamma function $\psi(\xi)$ is increasing on $(0,\infty)$, then for $\alpha\geq 1,\beta\geq 2$ and $\gamma\geq 1$ we have

$$\psi(\alpha\xi+\beta)\geq\psi(\xi+2),\ \xi\geq 1.$$

With the aid the functional relation

$$\psi(\xi+1)=\psi(\xi)+\frac{1}{\xi},\ \xi>0$$

combining with the above inequality and (3.8) we thus get

$$(f^{(\gamma)}_{\alpha,\beta}(\xi))' \leq f^{(\gamma)}_{\alpha,\beta}(\xi)\left[(1-\alpha\gamma)\psi(\xi)+\frac{2-\alpha\gamma}{\xi}-\frac{\alpha\gamma}{\xi+1}\right]\leq 0, \tag{3.9}$$

for all $\xi \geq 2$. Consequently, the sequence $(B_k)_{k\geq 2}$ is decreasing. It follows that

$$\begin{aligned}\left|(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))' - \frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z}\right| &\leq \sum_{k=1}^{\infty}\frac{B_k}{k!}\\ &\leq B_1 + B_2\sum_{k=2}^{\infty}\frac{1}{k!} = B_1 + B_2(e-2).\end{aligned} \tag{3.10}$$

Keeping (3.6) and (3.10) in mind, we get

$$\left|\frac{z(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'}{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)} - 1\right| \leq \frac{\theta^{(\gamma)}_{\alpha,\beta} + B_2(e-2)}{1-\theta^{(\gamma)}_{\alpha,\beta}(e-1)} < 1, \tag{3.11}$$

for all $z \in \mathcal{D}$, under the given hypothesis. This implies that

$$\Re\left(\frac{z(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'}{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}\right) > 0,$$

for all $z \in \mathcal{D}$ which implies that the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$ under the conditions (H_1). Now, we assume that (H_1^1) is valid. Since $\alpha \geq 1$ such that $\alpha+\beta \geq x^*$ we obtain

$$[\Gamma(\alpha k + \alpha + \beta)]^{\gamma} \geq [\Gamma(\alpha k + 1 + \beta)]^{\gamma}. \tag{3.12}$$

Thus, we get

$$\frac{B_{k+1}}{B_k} \leq \frac{(k+1)^2}{k(\alpha k+\beta)^{\gamma}}.$$

Moreover, since the function $z \mapsto L^{\gamma}_{\alpha,\beta}(z)$ is decreasing on $[1,\infty)$ such that $L^{\gamma}_{\alpha,\beta}(1) \leq 0$ we conclude that the sequence $(B_k)_{k\geq 1}$ is decreasing. Therefore, we have

$$\left|\frac{z(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'}{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)} - 1\right| \leq \frac{B_1}{1-\theta^{(\gamma)}_{\alpha,\beta}(e-1)} < 1, \tag{3.13}$$

for all $z \in \mathcal{D}$. Finally, we suppose that the hypothesis (H_1^2) is valid. In view of Lemma 2.7, we have to show that both $\{kA_k\}$ and $\{kA_k - (k+1)A_{k+1}\}$ are nonincreasing sequences for all $n \geq 1$. In Theorem 3.2 we have already proved that $\{kA_k\}$ is nonincreasing sequence for all $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^{\gamma} \geq 2[\Gamma(\beta)]^{\gamma}$. Now it remains to show that $\{kA_k - (k+1)A_{k+1}\}$ is nonincreasing or $\{kA_k\}$ is convex sequence (see Definition 1.1). That is $kA_k - 2(k+1)A_{k+1} + (k+2)A_{k+2} \geq 0$ (for all $k \geq 1$).

Neglecting the third term and taking difference of first two term, i.e.

$$kA_k - 2(k+1)A_{k+1} = \frac{k[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha k+\beta)]^\gamma}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma - \frac{2(k+1)}{k}\right].$$

As the function $z \mapsto \frac{\Gamma(z+a)}{\Gamma(z)}$, $a>0$ is increasing and hence the sequence

$$\left\{\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma\right\}_{k\geq 1} \tag{3.14}$$

is increasing provided that $\alpha > 0$ and $\gamma > 0$, on the other hand the sequence $\left\{\frac{2(k+1)}{k}\right\}_{k\geq 1}$ is decreasing sequence. This implies that the sequence

$$\{u_k\}_{k\geq 1} := \left\{\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma - \frac{2(k+1)}{k}\right\}_{k\geq 1} \tag{3.15}$$

is increasing and consequently

$$u_k \geq u_1 = \frac{[\Gamma(\alpha+\beta)]^\lambda - 4[\Gamma(\beta)]^\lambda}{[\Gamma(\beta)]^\lambda}$$

which is non-negative under the given hypotheses. Hence

$$kA_k - 2(k+1)A_{k+1} + (k+2)A_{k+2} \geq 0$$

for all $k \geq 1$. This completes the proof of Theorem 3.3. □

Corollary 3.3. *If $\beta > \sqrt{e} \approx 1.6487212707$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is starlike on $\mathcal{D}$.*

Proof. Upon setting $\alpha = 1$ and $\gamma = 2$ in the hypotheses (H_1^1) of Theorem 3.3. Then, the condition "$(H_1^1):(ii)$" and "$(H_1):(ii)$" hold true for all $\beta > 1$. In addition, the condition "$(H_1^)$: (iii)" holds true if and only if $\beta^2 > e$. □

Corollary 3.4. *If $\beta > 1.29$, then the function $\mathbb{F}_{2,\beta}(z)$ is starlike on $\mathcal{D}$.*

Proof. Specifying $\alpha = 2$ and $\gamma = 1$ in the conditions (H_1) of Theorem 3.3. Then the conditions "$(H_1):(i)$" and "$(H_1):(ii)$" are valid for all $\beta \geq 1$. Using mathematical software, we can verify that the condition "$(H_1):(iii)$" holds true for all $\beta > 1.29$. □

Remark 3.2. In [1, Example 2.1], the authors proved that the function $\mathbb{F}_{2,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq (-1+\sqrt{17})/2 \approx 1.5615...$ Further, according to [1, Theorem 2.2], $\mathbb{F}_{2,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq (3+\sqrt{17})/2 \approx 3.56155$. Moreover, [17, Theorem 6] indicates that $\mathbb{F}_{2,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq 3.214319744$. Hence, Corollary 3.4 provides results for $\mathbb{F}_{2,\beta}(z)$, better than the results available in [1, Theorem 2.1, Theorem 2.2] and [17, Theorem 6].

Theorem 3.4. *Let $\alpha, \beta > 0$ and γ be positive real numbers, and also let the following conditions (H_3) or (H_3^1) be satisfied:*

$$(H_2): \begin{cases} (i) & \alpha \geq 1, \beta \geq 3, \alpha\gamma \geq 2, \\ (ii) & 2(e-1)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma, \\ (iii) & \frac{2e[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma} + \frac{12(e-2)[\Gamma(\beta)]^\gamma}{[\Gamma(2\alpha+\beta)]^\gamma} < 1. \end{cases}$$

$$(H_2^1): [\Gamma(\alpha+\beta)]^\gamma \geq 8[\Gamma(\beta)]^\gamma,$$

then the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is convex function in $\mathcal{D}$.

Proof. It is well known that $f(z)$ is convex if and only if $zf'(z)$ is starlike. So in order to prove $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is convex it is sufficient to prove that the function

$$\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z) := z(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))'$$

is starlike. We have

$$(\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z))' - \mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)/z = \sum_{k=1}^{\infty} \frac{C_k z^k}{k!}, \tag{3.16}$$

where $(C_k)_{k\geq 1}$ is defined by

$$C_k = \frac{[\Gamma(\beta)]^\gamma k\Gamma(k+2)}{[\Gamma(\alpha k+\beta)]^\gamma},\ k \geq 1.$$

Next, we define the function $g_{\alpha,\beta}^{(\gamma)}$ by

$$g_{\alpha,\beta}^{(\gamma)}(\xi) = \frac{\xi\Gamma(\xi+2)}{[\Gamma(\alpha\xi+\beta)]^\gamma},\ \xi \geq 1.$$

Thus we get

$$(g_{\alpha,\beta}^{(\gamma)}(\xi))' = g_{\alpha,\beta}^{(\gamma)}(\xi)\left[\frac{1}{\xi} + \psi(\xi+2) - \alpha\gamma\psi(\alpha\xi+\beta)\right].$$

Again, by using the fact that the digamma function is increasing on $(0,\infty)$ we have

$$\psi(\alpha\xi+\beta) \geq \psi(\xi+3)$$

for all $\xi \geq 1, \alpha \geq 1$ and $\beta \geq 3$. Keeping in mind the above relations we obtain

$$\begin{aligned}(g_{\alpha,\beta}^{(\gamma)}(\xi))' &\leq g_{\alpha,\beta}^{(\gamma)}(\xi)\left[\frac{2-\alpha\gamma}{\xi} + \frac{1-\alpha\gamma}{\xi+1} - \frac{\alpha\gamma}{\xi+2} + (1-\alpha\gamma)\psi(\xi)\right] \\ &\leq 0,\end{aligned}$$

for all $\xi \geq 2$ and $\alpha\gamma \geq 2$. This implies that the sequences $(C_k)_{k\geq 2}$ is decreasing. Then, by (3.16) we get

$$\left|(\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z))' - \mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)/z\right| \leq C_1 + \sum_{k=2}^{\infty} \frac{C_2}{k!} = C_1 + C_2(e-2). \tag{3.17}$$

We see that the sequence $(y_k)_{k\geq 1}$ defined in (3.3) is also decreasing under the conditions (H_2). Therefore, by (3.2) we get

$$|\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)/z| = |(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'| \geq 1-\sum_{k=1}^{\infty}\frac{y_k z^k}{k!} \geq 1-y_1(e-1). \tag{3.18}$$

Having (3.18) and (3.17) in mind we obtain

$$\left|\frac{z(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z))'}{(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)}-1\right| \leq \frac{C_1+C_2(e-2)}{1-y_1(e-1)}. \tag{3.19}$$

The above inequality needs to be less than 1, this gives the conditions (H_3) : (iii). Thus we get

$$\Re\left(\frac{z(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z))'}{(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)}\right) > 0$$

for all $z \in \mathcal{D}$. This implies that the function $\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike on $\mathcal{D}$ and consequently the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is convex on $\mathcal{D}$ under the conditions (H_2). Now, assume that the condition (H_2^1) is valid.

$$\mathbb{G}^{\gamma}_{\alpha,\beta}(z) = z+\sum_{k=2}^{\infty}k\left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^{\gamma}z^k =: z+\sum_{k=2}^{\infty}\tilde{B}_k z^k. \tag{3.20}$$

In view of Lemma 2.7, we have to show that the sequence $\{k\tilde{B}_k\}$ is both decreasing and convex for all $k \geq 1$.

$$k\tilde{B}_k-(k+1)\tilde{B}_{k+1} = \frac{k^2[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^{\gamma}-\frac{(k+1)^2}{k^2}\right]. \tag{3.21}$$

Now using the same argument as in the proof of Theorem 3.3 under the conditions (H_1^1), we have $k\tilde{B}_k-(k+1)\tilde{B}_{k+1} \geq 0$ for all $k \geq 1$ under the condition $[\Gamma(\alpha+\beta)]^{\gamma} \geq 4[\Gamma(\beta)]^{\gamma}$, which is true under the hypothesis of Theorem 3.4. Now it remains to show that $\{k\tilde{B}_k\}$ is convex sequence. That is $k\tilde{B}_k-2(k+1)\tilde{B}_{k+1}+(k+2)\tilde{B}_{k+2} \geq 0$, for all $k \geq 1$. Neglecting the third term and taking difference of first two term i.e.

$$k\tilde{B}_k-2(k+1)\tilde{B}_{k+1} = \frac{k^2[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^{\gamma}-\frac{2(k+1)^2}{k^2}\right].$$

which is non-negative under the hypothesis that $[\Gamma(\alpha+\beta)]^{\gamma} \geq 8\Gamma(\beta)]^{\gamma}$. □

If we set $(\alpha = 1, \gamma = 2)$ and $(\alpha = 2, \gamma = 1)$ respectively in the second hypotheses of Theorem 3.4, we get the following results as follows:

Corollary 3.5. *The following assertions hold true:*
(a). If $\beta \geq 2\sqrt{2}$, *then the function* $\mathbb{F}^{(2)}_{1,\beta}(z)$ *is convex in* $\mathcal{D}$.
(b). If $\beta \geq \frac{-1+\sqrt{33}}{2} \approx 2.3722...$, *then* $\mathbb{F}_{2,\beta}(z)$ *is convex in* $\mathcal{D}$.

Remark 3.3. Recently, the authors [17, Theorem 7] proved that $\mathbb{F}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}$ if $\alpha \geq 1$ and $\beta \geq 3.56155281$. Therefore, the second assertions of Corollary 3.5 improve the results in [17] for $\alpha = 2$.

Theorem 3.5. *Let $\alpha \geq 1, \beta \geq 1, \gamma \geq 1$ such that $\alpha\gamma \geq 2$. Also, suppose that the following conditions*

$$[\Gamma(\beta)]^\gamma (e-1) < [\Gamma(\alpha+\beta)]^\gamma_; \text{ and } \frac{(1+M(e-1))[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma} + \frac{4(e-2)[\Gamma(\beta)]^\gamma}{[\Gamma(2\alpha+\beta)]^\gamma} < M,$$

are valid, where M is a solution of the equation $\cos(M) = M$. *Then*

$$\Re\left(\left[\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right]'\right) > 0.$$

Proof. The proof of this result go along the lines introduced in the proof of Theorem 3.3, when we used Lemma 2.5 such that the function

$$\frac{\theta^{(\gamma)}_{\alpha,\beta} + B_2(e-2)}{1-\theta^{(\gamma)}_{\alpha,\beta}(e-1)} < M,$$

where M is a solution of the equation $\cos(M) = M$, we omit the details. □

Theorem 3.6. *Let $\alpha \geq 1, \beta \geq 1, \gamma \geq 1$ such that $\alpha\gamma \geq 2$. Also, suppose that the following conditions*

$$[\Gamma(\beta)]^\gamma (e-1) < [\Gamma(\alpha+\beta)]^\gamma_; \text{ and } \frac{(e+1))[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma} + \frac{8(e-2)[\Gamma(\beta)]^\gamma}{[\Gamma(2\alpha+\beta)]^\gamma} < 1,$$

are valid. Then

$$\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) \in S_p(\mathcal{D}).$$

Proof. The proof of this result is very similar to the proof of Theorem 3.3 when we used the part (2) of Lemma 2.6, such that

$$\frac{\theta^{(\gamma)}_{\alpha,\beta} + B_2(e-2)}{1-\theta^{(\gamma)}_{\alpha,\beta}(e-1)} < 1/2,$$

thus, we omit the details in this case also. □

Theorem 3.7. *Let $\alpha \geq 1, \beta \geq 1$ and $\gamma \geq 1$ such that $\alpha + \beta \geq 3$. In addition, assume that the following conditions hold true:*

$$(6e-2)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma.$$

Then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is uniformly convex in $\mathcal{D}$.

Proof. Simple computation gives

$$(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))'' = \sum_{k=0}^{\infty} \frac{D_k z^k}{k!}, \tag{3.22}$$

where the sequence $(D_k)_{k\geq 0}$ is defined by

$$D_k = \frac{[\Gamma(\beta)]^\gamma \Gamma(k+3)}{[\Gamma(\alpha k + \beta + \alpha)]^\gamma},\ k \geq 0.$$

We define the function $h_{\alpha,\beta}^{(\gamma)}$ defined by

$$h_{\alpha,\beta}^{(\gamma)}(\xi) = \frac{\Gamma(\xi+3)}{[\Gamma(\alpha\xi + \beta + \alpha)]^\gamma},\ \xi > 0.$$

Therefore

$$(h_{\alpha,\beta}^{(\gamma)}(\xi))' = h_{\alpha,\beta}^{(\gamma)}(\xi) \left[\psi(\xi+3) - \alpha\gamma\psi(\alpha\xi + \beta + \alpha)\right], \xi \geq 0.$$

Again, by using the fact that the digamma function is increasing, we deduce that the function $h_{\alpha,\beta}^{(\gamma)}(\xi)$ is decreasing on $[0,\infty)$ for all $\alpha \geq 1, \beta \geq 1$ and $\gamma \geq 1$ such that $\alpha + \beta \geq 3$. This implies that the sequence $(D_k)_{k\geq 0}$ is decreasing. Then

$$\left|(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))''\right| \leq \frac{2e[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma},\ z \in \mathcal{D}. \tag{3.23}$$

We observe that the sequence $(y_k)_{k\geq 1} = (D_k/(k+2))_{k\geq 1}$ is also decreasing under the conditions of this Theorem. Then implies that the inequality (3.18) holds true. Now, bearing in mind the inequalities (3.18) and (3.23) we conclude

$$\left|\frac{z(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))''}{(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))'}\right| \leq \frac{2e\theta_{\alpha,\beta}^{(\gamma)}}{1-2(e-1)\theta_{\alpha,\beta}^{(\gamma)}},\ z \in \mathcal{D},$$

where $\theta_{\alpha,\beta}^{(\gamma)}$ is defined in (2.2). So, for the uniformly convex of the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ the above bound needs to be less than $\frac{1}{2}$, by the means of part (1) of Lemma 2.6. This gives the condition

$$(6e-2)\theta_{\alpha,\beta}^{(\gamma)} < 1,$$

or equivalently

$$(6e-2)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma.$$

With this, the proof of Theorem 3.7 is complete. □

Specifying $\alpha = 2$ and $\gamma = 1$ in Theorem 3.7, we conclude the following result as follows:

Corollary 3.6. *If* $\beta > \frac{-1+\sqrt{24e-7}}{2} \approx 3.3157163$, *then the function* $\mathbb{F}_{2,\beta}(z)$ *is uniformly convex in* $\mathcal{D}$.

Remark 3.4. In [16, Theorem 2.6], Noreen et al. proved that the function $\mathbb{F}_{2,\beta}(z)$ is uniformly convex in $\mathcal{D}$ if $\beta \geq 9.11125$. Hence, Corollary 3.6 improves Theorem 2.6 in [16].

Theorem 3.8. *For $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^{\gamma} \geq 2[\Gamma(\beta)]^{\gamma}$ we have*

$$\Re\left(\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z}\right) > \frac{1}{2}, \quad z \in \mathcal{D}. \tag{3.24}$$

Proof. In view of Lemma 2.8, it is sufficient to prove that the sequence $\{A_k\}_{k\geq 1}$, where A_k is defined by (1.3), is decreasing and convex.

$$A_k - A_{k+1} = \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\right)^{\gamma}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^{\gamma} - 1\right] \tag{3.25}$$

and

(3.26)

$$A_k - 2A_{k+1} + A_{n+2} = \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\right)^{\gamma}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^{\gamma} - 2 + \left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k+1)+\beta)}\right)^{\gamma}\right].$$

Now using the same argument as in Theorem 3.2, $A_k - A_{k+1} \geq 0$ for all $n \geq 1$ under the condition $[\Gamma(\alpha+\beta)]^{\gamma} \geq [\Gamma(\beta)]^{\gamma}$, which is true under the hypothesis of Theorem 3.8. Similarly $A_k - 2A_{k+1} \geq 0$ for all $k \geq 1$ (neglecting the third term) under the hypothesis that $[\Gamma(\alpha+\beta)]^{\gamma} \geq 2[\Gamma(\beta)]^{\gamma}$. □

Corollary 3.7. *For $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^{\gamma} \geq 2[\Gamma(\beta)]^{\gamma}$, the sequence*

$$\left\{\left(\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}\right)^{\gamma}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence for the class $\mathcal{K}$.

Proof. The result can be easily proved using Theorem 3.8 and Lemma 2.9, so we omit details here.

Theorem 3.9. *For $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^{\gamma} \geq 2[\Gamma(\beta)]^{\gamma}$*

$$\Re\left\{(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'\right\} > \frac{1}{2}, \quad z \in \mathcal{D}. \tag{3.27}$$

Proof. From (1.3), we get

$$(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))' = 1 + \sum_{k=2}^{\infty} \tilde{B}_k z^{k-1}, \tag{3.28}$$

where $(\tilde{B}_k)_k$ is defined in (3.20), and proceeding similarly as in Theorem 3.8, we achieve the desired result by the means of Lemma 2.8. □

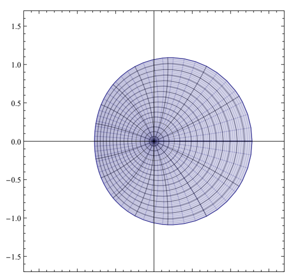
Corollary 3.8. *For $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma$, the sequence*

$$\left\{ \frac{(n+1)(\Gamma(\beta))^\gamma}{(\Gamma(\alpha n+\beta))^\gamma} \right\}_{n=1}^{\infty}$$

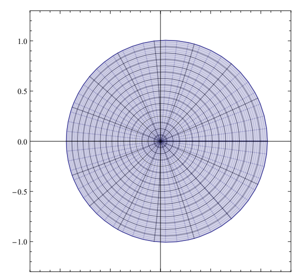
is a subordinating factor sequence for the class $\mathcal{K}$.

Proof. The claim follows by the means of Theorem 3.9 and Lemma 2.9, hence we omit details here.

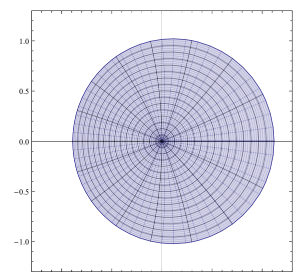
Remark 3.5. The following are graphs of the functions $\mathbb{F}^{(2)}_{1,2}(z), \mathbb{F}_{2,\frac{7}{2}}(z)$ and $\mathbb{F}_{2,\frac{5}{2}}(z)$ over $\mathcal{D}$. These figures depict the validity of our results.



Mapping of $\mathbb{F}^{(2)}_{1,2}(z)$ over $\mathcal{D}$



Mapping of $\mathbb{F}_{2,\frac{7}{2}}(z)$ over $\mathcal{D}$



Mapping of $\mathbb{F}_{2,\frac{5}{2}}(z)$ over $\mathcal{D}$

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MEROMORPHIC FUNCTIONS SHARING THREE VALUES WITH THEIR DERIVATIVES IN AN ANGULAR DOMAINS

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Abstract. In this paper, we investigate the uniqueness of transcendental meromorphic functions sharing three values with their derivatives in an arbitrary small angular domain including a Borel direction. The obtained results extend the corresponding results from Gundersenand and Mues-Steinmetz, Zheng and Li-Liu-Yi, Chen.

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Keywords: Meromorphic function; Shared value; Uniqueness theorems; Borel direction.

1. Introduction and main result

Let $f : C \to \hat{C} = C \bigcup \{\infty\}$ be a meromorphic function, where C is the complex plane. It is assumed that the reader is familiar withthe basic result and notations of the Nevanlinna's value distribution theory (see [6,14,15]), such as $T(r;f), N(r,f)$ and $m(r,f)$. Meanwhile, the lower order μ and the order λ of a meromorphic function f are in turn defined as follow

$$\mu := \mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

$$\lambda := \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Let f and g be nonconstant meromorphic functions in the domain $D \subseteq C$. If $f - c$ and $g - c$ have the same zeros with the same multiplicities in D, then $c \in C \bigcup \{\infty\}$ is called an CM shared value in a domain $D \subseteq C$ of two meromorphic functions f and g. If $f - c$ and $g - c$ only have the same zeros in D, then $c \in C \bigcup \{\infty\}$ is called an IM shared value in a domain $D \subseteq C$ of two meromorphic functions f and g. The zeros of $f - c$ imply the poles of f when $c = +\infty$.

In 1979, Gundersen [5] and Mues-Steinmetz [10] have considered the uniqueness of a meromorphic function f and its derivative f' and obtained the following result.

Theorem A: *Let f be a nonconstant meromorphic function in C, and let $a_j (j = 1, 2, 3)$ be three distinct finite complex numbers. If f and f' share $a_j (j = 1, 2, 3)$ IM. Then $f \equiv f'$.*

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Later on, Frank and Schwick [3] generalized the above results and proved the following result.

Theorem B: *Let f be a non-constant meromorphic function, and let k be a positive integer. If there exist three distinct finite complex numbers a, b and c such that f and $f^{(k)}$ share a, b, c IM, then $f \equiv f^{(k)}$.*

In 2004, Zheng [16] first considered the uniqueness question of meromorphic functions with shared values in an angular domain, and proved the following result (see [16, Theorem 3]):

Theorem C: *Let f be a transcendental meromorphic function of finite lower order and such that $\delta = \delta(a, f^{(p)}) > 0$ for some $a \in C \bigcup \{\infty\}$ and an integer $p \geq 0$. Let the pairs of real numbers $\{\alpha_j, \beta_j\} (j = 1, ..., q)$ be such that*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq ... \leq \alpha_q < \beta_q \leq \pi,$$

with $\omega = \max\{\frac{\pi}{\beta_j - \alpha_j} : 1 \leq j \leq q\}$, and

$$\sum_{j=1}^{q} (\alpha_{j+1} - \beta_j) < \frac{4}{\delta} \arcsin \sqrt{\delta(a, f^{(p)})/2},$$

where $\delta = \max\{\omega, \mu\}$. For a positive integer k, assume that f and $f^{(k)}$ share three distinct finite complex numbers $a_j (j = 1, 2, 3)$ IM in $X = \bigcup_{l=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f \equiv f^{(k)}$.

In 2015, Li, Liu, and Yi [9] observed that Theorem C is invalid for $q \geq 2$, and proved the following more general result, which extends Theorem C (see [9, p. 443]).

Theorem D: (see [9]). *Let f be a transcendental meromorphic function of finite lower order $\mu(f)$ in C and such that $\delta(a, f) > 0$ for some $a \in C$. Assume that $q \geq 2$ pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfy the conditions*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq ... \leq \alpha_q < \beta_q \leq \pi$$

with $\omega = \max\{\frac{\pi}{(\beta_j - \alpha_j)} : 1 \leq j \leq q\}$, and

$$\sum_{j=1}^{q} (\alpha_{j+1} - \beta_j) < \frac{4}{\delta} \arcsin \sqrt{\delta(a, f)/2},$$

where $\delta = \max\{\omega, \mu\}$. For a $k - th$ order linear differential polynomial $L[f]$ in f with constant coefficients given by

$$L[f] = b_k f^{(k)} + b_{k-1} f^{(k-1)} + \cdots + b_1 f', \tag{1.1}$$

where k is a positive integer, b_k, b_{k-1}, $\cdots$, b_1 are constants and $b_k \neq 0$, assume that f and $L[f]$ share $a_j (j = 1, 2, 3)$ IM in

$$X = \bigcup_{l=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\}.$$

where $a_j (j = 1, 2, 3)$ are three distinct finite complex numbers such that $a \neq a_j (j = 1, 2, 3)$. If $\lambda(f) \neq \omega$, then $f = L[f]$.

In 2019, J. F. Chen [2] proved the following result.

Theorem E: *Let f be a nonconstant meromorphic function of lower order $\mu(f) > 1/2$ in C, $a_j (j = 1, 2, 3)$ be three distinct finite complex numbers, and let $L[f]$ be given by Theorem D. Then there exists an angular domain $D = \{z : \alpha \leq argz \leq \beta\}$, where $0 \leq \beta - \alpha \leq 2\pi$, such that if f and $L[f]$ share $a_j (j = 1, 2, 3)$ CM in D, then $f = L[f]$.*

In theory of meromorphic functions, a function is uniquely determined by its value on a set with a accumulation point. It is natural to ask if we can prove similar results with the conditions

$$\bar{E}_D(f, a_j) = \bar{E}_D(f', a_j), \quad j = 1, 2, 3$$

for some typical set in C in steads of general angular domain in C, where $\bar{E}_D(a, f) = \{z : z \in D, f(z) = a\}$(as a set in C). In general, the answer of this question is negative. For $f(z) = e^{2z}$, it is clear that $f(z) \neq f'(z)$, but $|f(z)|$ is bounded by 1 on D being the left half plane. Thus

$$\bar{E}_D(f, n) = \bar{E}_D(f', n) = \emptyset \text{ for any } n > 1.$$

This example show us that if such angular domain D exists, it must be a region whose image under f should be dense in C.

Based on the theory on singular direction for a meromorphic function (see [14]) and the research results of shared values of a meromorphic function (see [8,12]), combining with the result of Theorem D and E we may conjecture that angular domain of the singular direction may be the right. The main result of this paper shows that it is true when D is a angular domain with the Borel direction as the center line for f with order $\lambda > 0$, which extend Theorems D and E.

In order to prove our main results, we introduce some notations about Ahlfors-Shimizu character of meromorphic function in C.

$$(1.2) \qquad T_0(r, f) = \int_0^r \frac{A(t)}{t} dt, \quad A(t) = \frac{1}{\pi} \int_0^{2\pi} \int_0^t (\frac{|f'(\rho e^{i\theta})|}{1 + |f(\rho e^{i\theta})|^2})^2 d\rho d\theta.$$

We recall the Nevanlinna theory on an angular domain.

Let f be a meromorphic function in $D = \{z : \alpha \leq argz \leq \beta\}$, where $0 \leq \beta - \alpha \leq 2\pi$. Nevanlinna [11] defined the following symbols (also see [4]).

$$A_{\alpha,\beta}(r, f) = \frac{\omega}{\pi} \int_1^r (\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}) \{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\} \frac{dt}{t},$$

$$B_{\alpha,\beta}(r, f) = \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(re^{i\theta})| \sin \omega(\theta - \alpha) d\theta$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_m|<r}(\frac{1}{|b_m|^\omega}-\frac{|b_m|^\omega}{r^{2\omega}})\sin\omega(\theta_m-\alpha),$$

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f)$$

where $\omega=\frac{\pi}{(\beta-\alpha)}$,and $b_m=|b_m|e^{i\theta_m}$ are the poles of f in D counting multiplicities

Throughout the paper, we denote by $R(r,*)$ a quantity satisfying

$$R(r,*)=O\{\log(rT(r,*))\},r\in E$$

where E denotes a set of positive real numbers with finite linear measure, which will not necessarily be the same in each occurrence. To state our result, we need the following theorem F and definitions .

Theorem F: (see [7]) *Let f be a meromorphic function of infinite order in C. Then there exists a function $\rho(r)$ such that:*

(i) $\rho(r)$ is continuous and non decreasing for $r\geq r_0$, and $\rho(r)\to\infty$ as $r\to+\infty$;

(ii) $U(r)=r^{\rho(r)}(r\geq r_0)$ satisfies the condition $\lim\limits_{r\to+\infty}\frac{\log U(R)}{\log U(r)}=1,R=r+\frac{r}{\log U(r)}$;

(iii) $\limsup\limits_{r\to\infty}\frac{\log T(r,f)}{\rho(r)\log r}=1$.

The fuction $\rho(r)$ is also called the precise order of f.

Definition 1.1. (see [13]). Let f be a meromorphic function of finite order $\lambda(f)>0$ in C. A direction $\arg z=\theta_0$ $(0\leq\theta_0<2\pi)$ is called a Borel direction of $f(z)$ of order $\lambda(f)$ if for arbitrary small positive ε the following relation holds:

$$\lim_{r\to\infty}\frac{\log n(r,\theta_0,\varepsilon,f=a)}{\log r}=\lambda(f)$$

for all $a\in\hat{C}=C\bigcup+\infty$ except at most two exceptional values, where $n(r,\theta_0,\varepsilon,f=a)$ denotes the number of the zeros of $f-a$ counting multiplicities in the sector $|argz-\theta_0|<\varepsilon$,$|z|\leq r$.

Definition 1.2. (see [7]). Let f be a meromorphic function of infinite order in C and let $\rho(r)$ be the precise order of f. A direction $\arg z=\theta_0$ $(0\leq\theta_0<2\pi)$ is called a Borel direction of $f(z)$ of with precise $\rho(r)$ if for arbitrary small positive ε the following relation holds:

$$\lim_{r\to\infty}\frac{\log n(r,\theta_0,\varepsilon,f=a)}{\rho(r)\log r}=1$$

for all $a\in\hat{C}$ except at most two exceptional values,where $n(r,\theta_0,\varepsilon,f=a)$ is as in definition 1.1.

In this paper we will prove the following theorem.

Theorem 1.1. *Let f be a meromorphic function of finite order $\lambda(f) > 0$ in C and ε be an arbitrary small positive number, and a direction $\arg z = \theta_0$ ($0 \le \theta_0 < 2\pi$) be a Borel direction of $f(z)$. Assume that f and f' share three distinct finite complex numbers $a_j (j = 1, 2, 3)$ IM in $A(\theta_0, \varepsilon)$, where $A(\theta_0, \varepsilon) = \{z : |\arg z - \theta_0| < \varepsilon\}$. Then $f \equiv f'$.*

Theorem 1.2. *Let f be a meromorphic function of infinite order in C and a direction $\arg z = \theta_0$ ($0 \le \theta_0 < 2\pi$) be a Borel direction of $f(z)$ with precise order $\rho(r)$. Then for arbitrary positive number ε, f and f' share two finite values IM at most in the angular region $\{z : |\arg z - \theta_0| < \varepsilon\}$.*

Theorem 1.3. *Let f be a meromorphic function of infinite order in C and $L[f]$ defined by (1.1), and $\arg z = \theta_0$ ($0 \le \theta_0 < 2\pi$) be a Borel direction of $f(z)$ with precise order $\rho(r)$. Then for arbitrary positive ε, f and $L[f]$ share two finite values CM at most in the angular region $\{z : |\arg z - \theta_0| < \varepsilon\}$.*

2. Preliminary

In this section, we will intrroduce and prove some lemmas that will be used in the proof of the main result.

Lemma 2.1. *([1,12]) Let $\mathcal{F}$ be a family of meromorphic functions such that for every function $f \in \mathcal{F}$ its zeros of multiplicity are at least k. If $\mathcal{F}$ is not a normal family at the origin 0, then for $0 \le \alpha \le k$, there exist*
(a) a real number r $(0 < r < 1)$;
(b) a sequence of complex numbers $z_n \to 0, |z_n| < r$;
(c) a sequence of functions $f_n \in \mathcal{F}$;
(d) a sequence of positive numbers $\rho_n \to 0$;
such that

$$g_n(z) = {\rho_n}^{-\alpha} f_n(z_n + \rho_n z)$$

converges locally uniformly with respect to spherical metric to a non-constant meromorphic function $g(z)$ on $\mathbf{C}$ and Moreover, g is of order at most two.

For convenience, we will use the following notation

$$LD(r, f : c_1, c_2) = c_1[m(r, \frac{f'}{f}) + \sum_{i=1}^{3} m(r, \frac{f'}{f - a_i})] + c_2[m(r, \frac{f''}{f'}) + \sum_{i=1}^{3} m(r, \frac{f''}{f' - ta_i})].$$

Lemma 2.2. *([12]) Let f be a meromorphic function in a domain $D = \{z : |z| < R\}$ and $a_j (j = 1, 2, 3)$ be three distinct finite complex numbers, and let t be a positive* *real number and* $a \in C$. *If*

$$\bar{E}_D(a_j, f) = \bar{E}_D(ta_j, f') \quad \text{for } j = 1, 2, 3;$$

and $a \neq a_j$ *and* $f(0) \neq a_j, \infty (j = 1, 2, 3,)$, $f'(0) \neq 0, at$ *and* $f''(0) \neq 0$, $f'(0) \neq tf(0)$, *then for* $0 < r < R$, *we have*

$$\begin{aligned} T(r, f) \leq LD(r, f : 2, 3) \quad &+ \quad \log \frac{\prod_{i=1}^{3} |f(0) - a_i|^2 |f'(0) - ta_i|^3}{|tf(0) - f'(0)|^5 |f'(0)|^2} \\ &+ \quad 3 \log \frac{1}{|f''(0)|} + (\log^+ t + m(r, \frac{f''}{f' - ta}) + 1) O(1). \end{aligned}$$

where $\bar{E}_D(a, f) = \{z : z \in D, f(z) = a\}$(*as a set in* $\mathbf{C}$). *and* $O(1)$ *is a complex number depending only on* a *and* $a_i (i = 1, 2, 3)$.

Lemma 2.3. *([14]). Let* $f(z)$ *be a meromorphic function with finite order* $\lambda > 0$ *and* $\arg z = \theta_0$ *is a Borel direction of* f. *Then there exist a series of circles*

$$\Gamma_j = \{z : |z - z_j| < \epsilon_j |z_j|\},$$

where $z_j = |z_j| e^{i\theta_0}$, *and* $\lim_{j \to \infty} |z_j| = +\infty$, $\lim_{j \to \infty} \epsilon_j = 0 (j = 1, 2, \cdots)$, *such that* f *take any complex number at least* $|z_j|^{\lambda - \delta_j}$ *times in every circle* Γ_j *with at most some exceptional values contained in two circles with spherical radius* 2^{-j}, *where* $\lim_{j \to \infty} |\delta_j| = 0$.

Lemma 2.4. *([14]). Let* $\mathcal{F}$ *be a family of meromorphic function on domain* D, *then* $\mathcal{F}$ *is normal on* D, *if and only if for every bounded closed domain* $K \subseteq D$, *there exists a positive number* M *such that every* $f \in \mathcal{F}$

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leq M.$$

Lemma 2.5. *([6],[17]). Let* m *be the normalized area measure on the Riemann sphere* S. *Then we have*

$$A(r, f) = \int_{\hat{C}} n(r, f = a) dm(a),$$

where $\hat{C} = C \bigcup \{\infty\}$.

Lemma 2.6. *([6], [17]) Let* $f(z)$ *be a meromorphic function in a domain* $D = \{z : |z| < R\}$. *If* $f(0) \neq \infty$, *then for* $0 < r < R$ *we have*

$$|T(t, f) - T_0(t, f) - \log^+ |f(0)|| \leq \frac{1}{2} \log 2.$$

where $\log^+ |f(0)|$ *will be replace by* $\log |c(0)|$ *when* $f(0) = \infty$, *and* $c(0)$ *is the coefficient of the Laurent series of* $f(z)$ *at 0, and* $T_0(t, f)$ *is defined as (1.2).*

Lemma 2.7. *([8]) Let $f(z)$ be a nonconstant meromorphic function in the complex plane, and a_1, a_2, a_3 are three distinct finite complex numbers. Assume that f and f' share the $a_i (i = 1, 2, 3)$ IM in $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ with $0 \le \alpha < \beta < 2\pi$. Then one of the following two cases holds: (i) $f \equiv f'$, or (ii) $S_{\alpha,\beta}(r, f) = Q(r, f)$, where $Q(r, f)$ is such a quantity that if $f(z)$ is of finite order, then $Q(r, f) = O(1)$ as $r \to \infty$. and if $f(z)$ is of infinite order, then $Q(r, f) = O(\log(rT(r, f))$ for $r \notin E$ and $r \to \infty$ and E denotes a set of positive real numbers with finite linear measure.*

Lemma 2.8. *([4,9]) Let f be a meromorphic function on $\overline{\Omega}(\alpha, \beta)$. If $S_{\alpha,\beta}(r, f) = O(1)$, then*

$$\log |f(re^{i\phi})| = r^{\omega} c \sin(\omega(\phi - \alpha)) + o(r^{\omega})$$

uniformly for $\alpha \le \phi \le \beta$ as $r \notin F$ and $r \to \infty$, where c is a positive constant, $\omega = \frac{\pi}{\beta-\alpha}$, and F is a set of finite logarithmic measure, and $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \le \arg z \le \beta\}$.

Lemma 2.9. *([13]) Let f be a meromorphic function of infinite order in C, and let $\rho(r)$ be a precise order of f. Then a direction $\arg z = \theta_0$ is a Borel direction of precise order $\rho(r)$ of f, if and only if for arbitrarily small $\varepsilon > 0$ we have*

$$\limsup_{r\to+\infty} \frac{\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, f)}{\rho(r) logr} = 1.$$

Lemma 2.10. *([2]) Let f be a meromorphic function of infinite order in C, $a_j (j = 1, 2, 3)$ be three distinct finite complex numbers and let $L[f]$ be given by(1.1). Suppose that f and $L[f]$ share $a_j (j = 1, 2, 3)$ CM in $D = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. If $f \not\equiv L[f]$, then $S_{\alpha,\beta}(r, f) = R(r, f)$.*

Lemma 2.11. *([14]) Let $f(z)$ be a meromorphic function in disc $D(0, R)$ centered at 0 with radius R. If $f(0) \ne 0, \infty$, then we have for $0 < r < \rho < R$*

$$m(r, \frac{f^{(k)}}{f}) < c_k\{1 + \log^+ \log^+ |\frac{1}{f(0)}| + \log^+ \frac{1}{r} + \log^+ \frac{1}{\rho - r} + \log^+ \rho + \log^+ T(\rho, f)\},$$

where k is a positive integer, c_k is a constant depending only on k.

Lemma 2.12. *([14]) Let $T(r)$ be a continuous, non-decreasing, non-negative function and $a(r)$ be a non-increasing, non-negative function on $[r_0, R] (0 < r_0 < R < \infty)$. If there exist constant b, c such that*

$$T(r) < a(r) + b \log^+ \frac{1}{\rho - r} + c \log^+ T(\rho),$$

for $r_0 < r < \rho < R$, then

$$T(r) < 2a(r) + B \log^+ \frac{2}{R - r} + C,$$

where B, C are two constants depending only on b, c.

Lemma 2.13. *Let $f(z)$ be a meromorphic function with finite order $\lambda > 0$ and $\arg z = \theta_0$ be a Borel direction of f, and $\Gamma_j = \{z : |z - z_j| < \epsilon_j |z_j|\}$ be a series of circles, where $z_j = |z_j|e^{i\theta_0}$, and $\lim_{j\to\infty} |z_j| = +\infty$, $\lim_{j\to\infty} \epsilon_j = 0 (j = 1, 2, \cdots)$. Suppose that f and f' share three distinct finite complex numbers $a_j (j = 1, 2, 3)$ IM in $A(\theta_0, \varepsilon)$, where $A(\theta_0, \varepsilon) = \{z : |\arg z - \theta_0| < \varepsilon\}$. If $f \not\equiv f'$, then for every sufficiently large $n (n \geq n_0)$,*

$$A(\varepsilon_n, z_n, f) \leq O(1)(1 + \log^+ |z_n|), \tag{2.1}$$

where $\varepsilon_n = |z_n|\epsilon_n$.

Proof. Set $f_n(z) = f(z_n + \varepsilon_n z)$. We distinguish two cases:

Case 1. Assume that $f_n(z)$ be normal at $|z| \leq 1$, by Lemma 2.4, implying that

$$\frac{|f_n'(z)|}{1 + |f_n(z)|^2} = \frac{\varepsilon_n |f'(z_n + \varepsilon_n z)|}{1 + |f(z_n + \varepsilon_n z)|^2} \leq M \; (n = 1, 2, ...)$$

in $|z| \leq 1$, where M is a positive numbers. Then we have

$$A(\varepsilon_n, z_n, f) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\varepsilon_n} (\frac{|f'(z_n + \rho e^{i\theta})|}{1 + |f(z_n + \rho e^{i\theta})|^2})^2 \rho d\rho d\theta \leq 2M^2.$$

So (2.1) holds.

Case 2. Assume that $f_n(z)$ be not normal at $|z| \leq 1$.

According to Lemma 2.1, there exist

1) a sequence of point $\{z_n'\} \subset \{|z| < 1\}$;

2) a subsequence of $\{f_n(z)\}_1^\infty$, without loss of generality, we still denote it by $\{f_n(z)\}$;

3) positive numbers ρ_n with $\rho_n \to 0 (n \to \infty)$; such that

$$h_n(z) = f_n(z_n' + \rho_n z) \to g(z) \tag{2.2}$$

in spherical metric uniformly on a compact subset of $\mathbf{C}$ as $n \to \infty$, where $g(z)$ is a non-constant meromorphic function. Thus for any positive integer k, we have

$$h_n^{(k)}(\xi) = {\rho_n}^k f_n^{(k)}(z_n' + \rho_n \xi) \to g^{(k)}(\xi).$$

We claim $g''(\xi) \not\equiv 0$. Otherwise, $g(z) = cz + d$, $(c, d \in \mathbf{C}$ and $c \neq 0)$. We can choose ξ_0, with $g(\xi_0) = a_1$. By Hurwitz's Theorem,there exists a sequence $\xi_n \to \xi_0$ such that

$$h_n(\xi_n) = f_n(z_n' + \rho_n \xi_n) = g(\xi_0) = a_1.$$

Notice that f and f' share a_1 IM in $\{z : |argz - \theta_0| < \varepsilon\}$, we have

$$c = g'(\xi_0) = \lim_{n\to\infty} h_n'(\xi_n) = \lim_{n\to\infty} \rho_n \varepsilon_n f'(z_n + \varepsilon_n(z_n' + \rho_n \xi_n))$$
$$= \lim_{n\to\infty} \rho_n \varepsilon_n f(z_n + \varepsilon_n(z_n' + \rho_n \xi_n)) = \lim_{n\to\infty} \rho_n \varepsilon_n a_1.$$

thus we have

$$\lim_{n\to\infty}\rho_n\varepsilon_n=\frac{c}{a_1}.$$

For finite complex number a_2, we can choose η_0 with $g(\eta_0)=a_2$. By Hurwitz's Theorem, there exists a sequence $\eta_n\to\eta_0$ such that

$$h_n(\eta_n)=f_n(z'_n+\rho_n\eta_n)=g(\eta_0)=a_2.$$

Likewise ,we get

$$\lim_{n\to\infty}\rho_n\varepsilon_n=\frac{c}{a_2},$$

this gives a contradiction.

For a sequence of positive numbers $\rho_n\varepsilon_n$, it is easy to know that there exist a subsequence, we still denoted by $\rho_n\varepsilon_n$, such that $\lim\limits_{n\to\infty}\rho_n\varepsilon_n=a_0$, where $a_0\in[0,+\infty)\bigcup\{+\infty\}$. Now we consider two cases: $a_0=0$ or $+\infty$ and $0<a_0<+\infty$.

Case 2.1 Assume that $\lim\limits_{n\to\infty}\rho_n\varepsilon_n=0$ or ∞.

We choose $\xi_0\in C$, such that

$$g(\xi_0)\neq 0,a_1,a_2,a_3,\infty, g'(\xi_0)\neq 0,\infty, g''(\xi_0)\neq 0,\infty.$$

Let $p_n(z)=f_n(z'_n+\rho_n\xi_0+z)$ for arbitrary small $\varepsilon>0$, in view of

$$\overline{E}_{A(\theta_0,\varepsilon)}(a_j,f)=\overline{E}_{A(\theta_0,\varepsilon)}(a_j,f'),\quad j=1,2,3,$$

and $\lim\limits_{n\to\infty}\epsilon_n=0$. and for sufficiently large n,

$$\Gamma_n=\{z|z-z_n|<\epsilon_n|z_n|,z_n=|z_n|e^{i\theta_0}\}\subseteq A(\theta_0,\varepsilon/2).$$

Therefor for every sufficiently large $n(n\geq n_0)$, we have

$$\bar{E}_D(a_i,p_n(z))=\bar{E}_D(\varepsilon_n a_i,p'_n(z))(i=1,2,3),$$

where $D=\{z:|z|<4\}$. Note that

$$p_n(0)=f_n(z'_n+\rho_n\xi_0)=h_n(\xi_0)\to g(\xi_0)\neq a_1,a_2,a_3,\infty,$$

$$p'_n(0)=f'_n(z'_n+\rho_n\xi_0)=\frac{h'_n(\xi_0)}{\rho_n},\quad h'_n(\xi_0)\to g'(\xi_0),$$

$$p''_n(0)=f''_n(z'_n+\rho_n\xi_0)=\frac{h''_n(\xi_0)}{\rho_n^2},\quad h''_n(\xi_0)\to g''(\xi_0),$$

$$\varepsilon_n p_n(0)-p'_n(0)=\frac{\varepsilon_n\rho_n h_n(\xi_0)-h'_n(\xi_0)}{\rho_n}.$$

Thus we have

$$\begin{aligned}(2.3)\qquad &\log\frac{\prod_{i=1}^3|p_n(0)-a_i|^2|p'_n(0)-\varepsilon_n a_i|^3}{|\varepsilon_n p_n(0)-p'_n(0)|^5|p'_n(0)|^2}+3\log\frac{1}{|p''_n(0)|}\\ &=\log\frac{\prod_{i=1}^3|p_n(0)-a_i|^2|p'_n(0)-\varepsilon_n a_i|^3}{|\varepsilon_n p_n(0)-p'_n(0)|^5|p'_n(0)|^2|p''_n(0)|^3}\\ &=4\log\rho_n+\log\frac{\prod_{i=1}^3|h_n(\xi_0)-a_i|^2|h'_n(\xi_0)-\rho_n\varepsilon_n a_i|^3}{|\rho_n\varepsilon_n h_n(\xi_0)-h'_n(\xi_0)|^5|h'_n(\xi_0)|^2|h''_n(\xi_0)|^3}.\end{aligned}$$

Since $\lim_{n\to\infty}\rho_n\varepsilon_n = 0$ or ∞. By simple calculation we can deduce for sufficiently large $n(n \geq n_0)$

$$\log\frac{\prod_{i=1}^{3}|h_n(\xi_0)-a_i|^2|h_n^{(k)}(\xi_0)-\rho_n^k\varepsilon_n a_i|^3}{|\rho_n^k\varepsilon_n h_n(\xi_0)-h_n^{(k)}(\xi_0)|^5|h_n^{(k)}(\xi_0)|^2|h_n^{(k+1)}(\xi_0)|^3} \leq O(1)\log^+|z_n|. \tag{2.4}$$

Applying Lemma 2.2 to $p_n(z)$ with properties (2.3), (2.4), we have

$$T(r,p_n) \leq LD(r,p_n;2,3) + O(1)(\log^+|z_n| + m(r,\frac{p_n''}{p_n'-\varepsilon_n a}) + 1)$$

for $0 < r \leq 3$ and sufficiently large n, where $a \neq a_j (j = 1, 2, 3)$ and $a \in C$.

By Lemma 2.11 and Lemma 2.12, we have

$$T(r,p_n) \leq O(1)(1+\log^+|z_n|).$$

In view of Lemma 2.6, we obtain

$$T_0(r,p_n) \leq O(1)(1+\log^+|z_n|).$$

Thus we get

$$T_0(3\varepsilon_n, z_n+\varepsilon_n(z_n'+\rho_n\xi_0), f) \leq O(1)(1+\log^+|z_n|).$$

It follows that

$$A(2\varepsilon_n, z_n+\varepsilon_n(z_n'+\rho_n\xi_0), f) \leq O(1)(1+\log^+|z_n|).$$

Note that $z_n'+\rho_n\xi_0 \to 0$,we get

$$\{z : |z-z_n| < \varepsilon_n\} \subseteq \{z : |z-z_n-\varepsilon_n(z_n'-\rho_n\xi_0)| < 2\varepsilon_n\}.$$

Therefor we have

$$A(\varepsilon_n, z_n, f) \leq O(1)(1+\log^+|z_n|).$$

Case 2.2. Assume that $\lim_{n\to\infty}\rho_n\varepsilon_n = a_0, a_0 \neq 0, \infty$. Now, we distinguish two subcases $a_0 g(z) \not\equiv g'(z)$ and $a_0 g(z) \equiv g'(z)$.

Case 2.2.1. $a_0 g(z) \not\equiv g'(z)$. We can choose $\xi_0 \in C$, such that

$$g(\xi_0) \neq 0, a_1, a_2, a_3, \infty, g'(\xi_0) \neq 0, \infty, g''(\xi_0) \neq 0, \infty, a_0 g(\xi_0) - g'(\xi_0) \neq 0, \infty.$$

Let

$$p_n(z) = f_n(z_n'+\rho_n\xi_0+z).$$

By the same arguments as in the case 2.1, we can get

$$A(\varepsilon_n, z_n, f) \leq O(1)(1+\log^+|z_n|).$$

Case 2.2.2. $a_0 g(z) \equiv g'(z)$ we can derive that $g(z) = e^{a_0 z+b_0}$, where $b_0 \in C$ .From (2.2), we obtain

$$h_n(z) = f_n(z_n'+\rho_n z) = f(z_n+\varepsilon_n(z_n'+\rho_n z)) = f(z_n+\varepsilon_n z_n'+\varepsilon_n\rho_n z) \to g(z). \tag{2.5}$$

On the other hand, Noting that f and f' share $a_i, i = 1, 2, 3$ in $A(\theta_0, \varepsilon)$, by Lemma 2.7, we have $S_{\theta-\varepsilon,\theta+\varepsilon}(r, f) = O(1)$. Therefore, applying Lamma 2.8 to f in $A(\theta_0, \varepsilon)$, we obtain

$$\log|f(re^{i\phi})| = r^{\omega} c \sin(\omega(\phi - \alpha)) + o(r^{\omega})$$

uniformly for $\theta_0 - \varepsilon = \alpha \le \phi \le \beta = \theta_0 + \varepsilon$ as $r \notin F$ and $r \to \infty$, where c is a positive constant,$\omega = \frac{\pi}{\beta-\alpha} = \frac{\pi}{2\varepsilon}$, and F is a set of finite logarithmic measure.

Noting that F is a set of finite logarithmic measure. Therefor, there exist a real number $R, 0 < R < \infty$ and a sequence of complex numbers $u_n, 0 < |u_n| < R$ for every sufficiently large n, such that

$$\log|f(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n)| = r_n^{\omega} c \sin(\omega(\phi - \alpha)) + o(r_n^{\omega}), \tag{2.6}$$

where $r_n = |z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n| \notin F$, $\phi_n = arg(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n)$, $\theta_0 - \varepsilon/2 \le \phi_n \le \theta_0 + \varepsilon/2$, and $\alpha = \theta_0 - \varepsilon$.

From (2.5), we get $\lim_{n\to\infty}(f(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n) - g(u_n)) = 0$. Noting that u_n is a bounded sequence, there exists convergent subsequence, we still denote it by u_n and set $u_n \to u_0 (n \to \infty)$. We have that $\lim_{n\to\infty} g(u_n) = \lim_{n\to\infty} e^{a_0 u_n + b_0} = e^{a_0 u_0 + b_0}$, it follows that

$$\lim_{n\to\infty} \frac{\log|f(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n)|}{r_n^{\omega}} = 0.$$

On the other hand, by the (2.6) we obtain that

$$\lim_{n\to\infty} \frac{\log|f(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n)|}{r_n^{\omega}} = \lim_{n\to\infty} c \sin\omega(\phi - \alpha) \ge csin\frac{\pi}{4} > 0$$

we obtain a contradiction and so Case 2.2 is false . This completes the proof of Lemma 2.13.

3. Proof of theorems

Proof of Theorem 1.1. Suppose that $f \not\equiv f'$, since $\arg z = \theta_0$ is a Borel direction of f, by the Lemma 2.3, there exist a series of circles

$$\Gamma_j = \{z : |z - z_j| < \epsilon_j |z_j|\},$$

where $z_j = |z_j|e^{i\theta_0}$, and $\lim_{j\to\infty} |z_j| = +\infty$, $\lim_{j\to\infty} \epsilon_j = 0 (j = 1, 2, \cdots)$, such that f take any complex number at least $|z_j|^{\lambda-\delta_j}$ times in every circle Γ_j with at most some exceptional values contained in two circles with spherical radius 2^{-j},where $\lim_{j\to\infty} |\delta_j| = 0$.We denote the two circles by Δ_{j1} and Δ_{j2}.

Therefore, by Lemma 2.5, we have

$$\begin{aligned} A(\epsilon_j|z_j|, z_j, f) &= \int_{\hat{C}} n(\epsilon_j|z_j|, z_j, f = a) dm(a) \\ &\ge \int_{\hat{C}-\Delta_{j1}-\Delta_{j2}} n(\epsilon_j|z_j|, z_j, f = a) dm(a) \ge \frac{1}{2}|z_j|^{\lambda-\delta_j}. \end{aligned} \tag{3.1}$$

On the other hand, from Lemma 2.13 the following inequality hold.

$$A(\varepsilon_n, z_n, f) \leq O(1)(1 + \log^+ |z_n|), \tag{3.2}$$

where $|z| \leq 1$ and $\varepsilon_n = |z_n|\epsilon_n$.

Combining with (3.1) and (3.2), we get

$$\frac{1}{2}|z_n|^{\lambda-\delta_n} \leq A(\varepsilon_n, z_n, f) \leq O(1)(1 + \log^+ |z_n|).$$

Noting that $\lambda > 0$ and $\lim_{n\to\infty} \delta_n = 0$. This contradicts with $\lim_{n\to\infty} |z_n| = +\infty$. The proof of the theorem 1.1 is complete.

Proof of Theorem 1.2. Suppose that f and f' share three distinct finite complex numbers $a_j (j = 1, 2, 3)$ IM in $A(\theta_0, \varepsilon)$, by Lemma 2.7, in view of f with infinite order and $f \not\equiv f'$, we have $S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, f) = R(r, f)$, implying that

$$S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, f) = O(\log U(r)), U(r) = r^{\rho(r)}.$$

On the other hand , $\arg z = \theta_0$ is a Borel direction of f with precise order $\rho(r)$. By Lemma 2.9, for arbitrarily small $\varepsilon > 0$, we have

$$\limsup_{r\to+\infty} \frac{\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, f)}{\rho(r)\log r} = 1.$$

Thus we arrive at a contradiction. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Suppose that f and $L[f]$ share three distinct finite complex numbers $a_j (j = 1, 2, 3)$ CM in $A(\theta_0, \varepsilon)$. using Lemma 2.10 and 2.9 in $A(\theta_0, \varepsilon)$, similar to Proof of Theorem 1.2, we can conclude a contradiction. This completes the proof of Theorem 1.3.

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ON THE IDEAL TRANSFORMS DEFINED BY AN IDEAL

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Abstract. Let R be a commutative Noetherian ring, I an ideal of R and M an R-module. The ambiguous structure of I-transform functor $D_I(-)$ makes the study of its properties attractive. In this paper we gather conditions under which, $D_I(R)$ and $D_I(M)$ appear in certain roles. It is shown, under these conditions that $D_I(R)$ is a Cohen-Macaulay ring, regular ring, $\cdots$ and $D_I(M)$ can be regarded as a Noetherian, flat, $\cdots$ R-module. We also present a primary decomposition of zero submodule of $D_I(M)$ and secondary representation of $D_I(M)$.

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1. Introduction

Throughout this paper, R will always denote a non-trivial commutative Noetherian ring with identity. For an R-module M, the local cohomology modules $H^i_I(M)$, $i = 0, 1, \ldots$ of an R-module M with respect to I were introduced by Grothendieck [6]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e., $\cup_{n=1}^{\infty}(0 :_M I^n)$. There is a natural isomorphism

$$H^i_I(M) \cong \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

Recall that for an R-module M, the *cohomological dimension* of M *with respect* to I is defined as

$$\operatorname{cd}(I, M) := \sup\left\{i \in \mathbb{Z} \,:\, H^i_I(M) \neq 0\right\}.$$

The cohomological dimension has been studied by several authors, see for example [6] and [7]. Also, for any proper ideal I of R, the *arithmetic rank* of I denoted by $\operatorname{ara}(I)$, is the least number of elements of R required to generate an ideal which has the same radical as I. For any ideal I of an arbitrary Noetherian ring R, the I-transform functor denoted by $D_I(-)$, is defined as:

$$D_I(-) = \varinjlim_{n\geq 1} \operatorname{Hom}_R(I^n, -).$$

If M is an R-module, then $D_I(M) = \varinjlim_{n\geq 1} \operatorname{Hom}_R(I^n, M)$ is the ideal transform of M with respect to I, or the I-transform of M for short. Recall from [3, Exercise 2.2.3(ii)] that $D_I(R)$ is a commutative ring with identity and also from [3, Exercise 2.2.10] that $\eta : R \to D_I(R)$ is a ring homomorphism. It is well known that the ring $D_I(R)$ has a finitely generated R-algebra structure, whenever the functor $D_I(-)$ is exact. We refer the reader to [3] for more details about ideal transform functor.

For every non-zero R-module M, we denote the set of all zero-divisors of M in R by $Z_R(M)$. Also, for any ideal I of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ by $V(I)$ and $\{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$ by $\sqrt{I}$. We recall that $\operatorname{grade}(I, R)$ is the common length of maximal regular R-sequences in ideal I. For any unexplained notation and terminology we refer the reader to [3] and [8].

This paper is devided into 3 sections. In the next section we gather some conditions to find affirmative answers to the questions: When is $D_I(M)$ a finitely generated R-module? When is it a flat R-module? See 2.1 and 2.2. Moreover in Theorem 2.3, we show that $D_I(R)$ is a projective R-module in case that I is a non-zero proper ideal of an arbitrary Noetherian domain with $\operatorname{Ann}_R(H^1_I(R)) \neq 0$ and $D_I(-)$ is an exact functor. Theorem 2.4 is a nice result that shows $D_I(R)$ is a Cohen-Macaulay ring under certain conditions. Next, in Theorem 2.5, it is seen that $D_I(R)$ is a regular ring whenever R is regular, $\operatorname{Ann}_R(I)$ is nilpotent and $\eta : R \to D_I(R)$ is a surjective ring homomorphism. In section 3 we present a minimal primary decomposition of zero submodule of $D_I(M)$ in case that M is a finitely generated R-module and $\operatorname{ara}(I) = 1$, see 3.1. An R-module M is said to be representable when it has a secondary representation, see [3, Definition 7.2.2]. In 3.2 we show that $D_I(M)$ is representable and $\operatorname{Att}_R(D_I(M)) \subseteq \operatorname{Att}_R(M)\backslash V(I)$ whenever M is a finitely generated representable R-module and $\operatorname{cd}(I, R) = 1$.

2. Some results

In this section we begin our investigations with the following Theorem.

Theorem 2.1. *Let R be a Noetherian ring and M be a non-zero finitely generated R-module. Let I be an ideal of R such that $0 \neq \operatorname{Ann}_R(H^1_I(M)) \nsubseteq Z_R(M)$. Then both $H^1_I(M)$ and $D_I(M)$ are Noetherian R-modules.*

Proof. By the assumption, there exists a non-zero element $x \in \operatorname{Ann}_R(H^1_I(M)) \setminus Z_R(M)$. So the exact sequence

$$0 \to M \xrightarrow{.x} M \to \frac{M}{xM} \to 0,$$

induces the long exact sequence

$$0 \to \Gamma_I(M) \stackrel{.x}{\to} \Gamma_I(M) \to \Gamma_I(\frac{M}{xM}) \stackrel{\beta}{\to} H^1_I(M) \stackrel{.x}{\to} H^1_I(M) \to \cdots.$$

Since $x.H^1_I(M) = 0$, it follows that $H^1_I(M)$ is a finitely generated R-module. Now, the exact sequence

$$0 \to \frac{M}{\Gamma_I(M)} \to D_I(M) \to H^1_I(M) \to 0,$$

leads that $D_I(M)$ is a Noetherian R-module. □

Corollary 2.1. *Let R be a Noetherian domain and I an ideal of R with $\mathrm{Ann}_R(H^1_I(R)) \neq 0$. Then $D_I(R)$ is a Noetherian R-module. In particular, it is a Noetherian integral extension of ring R.*

Proof. $D_I(R)$ is a Noetherian R-module by Theorem 2.1 and therefore it is a finitely generated R-module. Since R is a domain it follows that $\eta : R \longrightarrow D_I(R)$ is an injective ring homomorphism. Thus, by outlined Remark after [1, Corollary 5.3], $D_I(R)$ is an integral extension of R. Moreover it is a finitely generated R-algebra and so is a Noetherian ring. □

In the following we denote by $\mathrm{Att}_R(H^1_{\mathfrak{m}}(M))$ the set of all attached prime ideals of $H^1_{\mathfrak{m}}(M)$.

Corollary 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a non-zero finitely generated R-module with $0 \neq \mathrm{Ann}_R(H^1_{\mathfrak{m}}(M)) \nsubseteq Z_R(M)$. Then $\mathrm{Att}_R(H^1_{\mathfrak{m}}(M)) \subseteq \{\mathfrak{m}\}$.*

Proof. The assertion follows from Theorem 2.1, [3, Theorem 7.1.3] and [3, Corollary 7.2.12].

Lemma 2.1. *Let I be a non-nilpotent proper ideal of the Noetherian ring R and $D_I(-)$ an exact functor. Then $D_I(R)$ is a flat R-module.*

Proof. See [2, Theorem 3.11].

Theorem 2.2. *Let R be a Noetherian domain and I an ideal with $\mathrm{cd}(I, R) = 1$. If M is an R-module of finite projective dimension d and $\mathrm{Ass}_R M = \{0\}$, then $D_I(M)$ is a flat R-module.*

Proof. We proceed by induction on d. If $d = 0$, then M is projective and so, it is a direct summand of a free module. Thus the assertion follows from Lemma 2.1 and [3, Exercise 3.4.5]. Now assume that $d \geq 1$, and that

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \stackrel{\varepsilon}{\to} M \to 0,$$

is a projective resolution of M. Applying the exact functor $D_I(-)$ to the exact sequence

$$0 \to \ker \varepsilon \to P_0 \xrightarrow{\varepsilon} M \to 0,$$

we obtain the following exact sequence

$$0 \to D_I(\ker \varepsilon) \to D_I(P_0) \xrightarrow{\varepsilon} D_I(M) \to 0.$$

Since $\mathrm{pd}_R(\ker \varepsilon) = d-1$, by the inductive hypothesis, one can say $D_I(\ker \varepsilon)$ is a flat R-module. Moreover, $D_I(P_0)$ is flat because P_0 is a projective R-module. Thus for every ideal J of R we have $JD_I(P_0) \cong J \otimes_R D_I(P_0)$ and $JD_I(\ker \varepsilon) \cong J \otimes_R D_I(\ker \varepsilon)$. On the other hand, from hypothesis and [5, Proposition 2.10], we find that $Z_R(D_I(M)) = 0$. This guarantees the exactness of the bottom row, in the following commutative diagram.

$$\begin{array}{ccccccc}
J \otimes_R D_I(\ker \varepsilon) & \longrightarrow & J \otimes_R D_I(P_0) & \longrightarrow & J \otimes_R D_I(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
JD_I(\ker \varepsilon) & \longrightarrow & JD_I(P_0) & \longrightarrow & JD_I(M) & \longrightarrow & 0
\end{array}$$

Hence we have $JD_I(M) \cong J \otimes_R D_I(M)$. But $JD_I(M) \subseteq D_I(M)$. This means

$$J \otimes_R D_I(M) \to R \otimes_R D_I(M)$$

is an injective homomorphism and therefore $D_I(M)$ is a flat R-module. □

Lemma 2.2. *Let R be a Noetherian local ring and I a proper non-zero ideal of R. Then the I-transform functor $D_I(-)$ is exact if and only if* $\mathrm{cd}(I, R) \leq 1$.

Proof. It follows from [2, Lemma 3.2].

Theorem 2.3. *Let R be a Noetherian domain and I a non-zero proper ideal of R such that the I-transform functor $D_I(-)$ is exact. If $\mathrm{Ann}_R(H^1_I(R)) \neq 0$, then $D_I(R)$ is a projective R-module.*

Proof. In case that $(R, \mathfrak{m})$ is a Noetherian local ring, the assertion is clear by Corollary 2.1 and Lemma 2.1. Suppose that R is not local and assume the contrary that there exists an R-module M such that $\mathrm{Ext}^1_R(D_I(R), M) \neq 0$. Hence there exists a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $(\mathrm{Ext}^1_R(D_I(R), M))_{\mathfrak{p}} \neq 0$. By Corollary 2.1, $D_I(R)$ is a finitely generated R-module. Thus by [8, Exercise 7.7] and [3, Exercise 4.3.5, iii], $\mathrm{Ext}^1_{R_{\mathfrak{p}}}(D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}), M_{\mathfrak{p}}) \neq 0$ and so $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0$. In case that $I \nsubseteq \mathfrak{p}$, we have $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) = R_{\mathfrak{p}}$ because $IR_{\mathfrak{p}} = R_{\mathfrak{p}}$. This means that $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a projective $R_{\mathfrak{p}}$-module.

Now consider the case that $I \subseteq \mathfrak{p}$. Since $D_I(-)$ is an exact functor, it follows that $\mathrm{cd}(I, R) \leq 1$ by Lemma 2.2. Moreover, it is clear that $\mathrm{cd}(IR_{\mathfrak{p}}, R_{\mathfrak{p}}) \leq \mathrm{cd}(I, R)$.

Hence by using again Lemma 2.2, the $IR_{\mathfrak{p}}$-transform functor $D_{IR_{\mathfrak{p}}}(-)$ is exact. Since R is domain, it follows that the Noetherian ring $R_{\mathfrak{p}}$ is domain and so $IR_{\mathfrak{p}}$ is a non-nilpotent proper ideal of $R_{\mathfrak{p}}$. Hence by Lemma 2.1, the $R_{\mathfrak{p}}$-module $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is flat. Therefore $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a non-zero projective $R_{\mathfrak{p}}$-module because $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$ is a Noetherian local ring and $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a finitely generated $R_{\mathfrak{p}}$-module.
As it is seen, in both cases above $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a projective $R_{\mathfrak{p}}$-module which contradict the fact that $\operatorname{Ext}^1_{R_{\mathfrak{p}}}(D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}), M_{\mathfrak{p}}) \neq 0$. Thus for every R-module M we must have $\operatorname{Ext}^1_R(D_I(R), M) = 0$, i.e., $D_I(R)$ is a projective R-module. □

Theorem 2.4. *Let R be a Noetherian domain of dimension d and I be an ideal of R such that $I \subseteq J(R)$. Let $\operatorname{Ann}_R(H^1_I(R)) \neq 0$ and $H^i_I(R) = 0$ for each $1 < i < d$. Then $D_I(R)$ is a Noetherian Cohen-Macaulay ring.*

Proof. It follows from Corollary 2.1 that $D_I(R)$ is a Noetherian ring and it is integral over R. Consequently, we have $\dim D_I(R) = \dim R$. By [3, Corollary 2.2.10, iv] and [3, Theorem 4.2.1], one has $\Gamma_{ID_I(R)}(D_I(R)) = H^1_{ID_I(R)}(D_I(R)) = 0$. Moreover, one can find by [3, Corollary 2.2.10, v] and [3, Theorem 4.2.1] that $H^i_{ID_I(R)}(D_I(R)) = 0$ for every $1 < i < d$. Hence by view of [3, Theorem 6.2.7] we have $d \leq \operatorname{grade}(ID_I(R), D_I(R))$. On the other hand $\operatorname{grade}(ID_I(R), D_I(R)) \leq \dim D_I(R)$. These yield $\operatorname{grade}(ID_I(R), D_I(R)) = d$. Now let $\mathfrak{n} \in \operatorname{Max}(D_I(R))$. It follows from [1, Corollary 5.8] that $\mathfrak{m} := \mathfrak{n}^c$ is a maximal ideal of R. Since $I \subseteq \mathfrak{m}$, we have $ID_I(R) \subseteq \mathfrak{n}$ because of $ID_I(R) \subseteq \mathfrak{m}D_I(R) = \mathfrak{n}^{ce} \subseteq \mathfrak{n}$. Therefore

$$\operatorname{grade}(ID_I(R), D_I(R)) \leq \operatorname{grade}(\mathfrak{n}, D_I(R)) \leq \operatorname{grade}(\mathfrak{n}(D_I(R))_{\mathfrak{n}}, (D_I(R))_{\mathfrak{n}})$$
$$= \operatorname{depth}(D_I(R))_{\mathfrak{n}} \leq \dim(D_I(R))_{\mathfrak{n}} \leq \dim D_I(R) = d.$$

Thus for every $\mathfrak{n} \in \operatorname{Max}(D_I(R))$ we have $\operatorname{depth}(D_I(R))_{\mathfrak{n}} = \dim(D_I(R))_{\mathfrak{n}} = d$. □

Let I be an ideal of R such that $\operatorname{Ann}_R(I)$ is nilpotent. Then $IR_{\mathfrak{p}} \neq 0$ for every prime ideal $\mathfrak{p}$ of R, because $\operatorname{Ann}_R(I) \subseteq \mathfrak{p}$. In the following, we show that under certain assumptions, $D_I(R)$ is a regular ring. Recall that a Noetherian ring R is regular, if $R_{\mathfrak{p}}$ is a regular local ring for every prime ideal $\mathfrak{p}$ of R. For more details about regular local rings see [4, Section 2.2].

Theorem 2.5. *Let R be a Noetherian regular ring and I an ideal of R such that $\operatorname{Ann}_R(I)$ is nilpotent. Then $D_I(R)$ is a regular ring, provided $\eta : R \to D_I(R)$ is a surjective ring homohorphism.*

Proof. The assertion follows immediately in case that $\Gamma_I(R) = 0$ or I is a nilpotent ideal of R. Thus we may assume that $\Gamma_I(R) \neq 0$ and I is not nilpotent. Also, it should be mentioned that $D_I(R)$ is a Noetherian ring because R is a

Noetherian ring and $\eta : R \to D_I(R)$ is a surjective ring homohorphism. Now let $\mathfrak{q} \in \operatorname{Spec}(D_I(R))$ and $\mathfrak{p} := \mathfrak{q}^c$. Then the canonical map $\bar{\eta} : R_\mathfrak{p} \to (D_I(R))_\mathfrak{q}$ by $\bar{\eta}(\frac{r}{s}) = \frac{\eta(r)}{\eta(s)}$ for every $\frac{r}{s} \in R_\mathfrak{p}$, is a surjective ring homomorphism. Let $\frac{r}{s} \in \ker \bar{\eta}$. There exists $v \in R \setminus \mathfrak{p}$ such that $\eta(r)\eta(v) = 0$. Thus $rv \in \ker \eta = \Gamma_I(R)$. Hence $\frac{rv}{s} \in \Gamma_{IR_\mathfrak{p}}(R_\mathfrak{p})$. But $\operatorname{grade}(IR_\mathfrak{p}, R_\mathfrak{p}) \geq 1$ because $R_\mathfrak{p}$ is a domain and $IR_\mathfrak{p} \neq 0$ due to $\operatorname{Ann}_R(I) \subseteq \mathfrak{p}$. Therefore in view of [3, Exercise 1.3.9, (iii)], $\Gamma_{IR_\mathfrak{p}}(R_\mathfrak{p}) = 0$ and we get $\frac{r}{s} = 0$. This leads to $R_\mathfrak{p} \cong (D_I(R))_\mathfrak{q}$. □

3. Primary decomposition and secondary representation

Let R be a Noetherian ring and M a finitely generated R-module. It is well-known that every proper submodule of M has a primary decomposition. In the following Theorem, for an ideal I with $\operatorname{ara}(I) = 1$ we find a minimal primary decomposition of the zero submodule of $D_I(M)$.

Theorem 3.1. *Let R be a Noetherian ring and I an ideal of R with $\operatorname{ara}(I) = 1$. Let M be a finitely generated R-module and $0 = \bigcap_{i=1}^{t} N_i$ be a minimal primary decomposition of 0 in M. Then the zero submodule of $D_I(M)$ has a minimal primary decomposition in the form $0 = \bigcap_{i=1}^{s} D_I(N_i)$, $s \leq t$.*

Proof. Since $\operatorname{ara}(I) = 1$ there exists $x \in R$ such that $\sqrt{Rx} = \sqrt{I}$. By [3, Proposition 2.2.23] and [3, Theorem 2.2.19] we have

$$0 = D_I(0) \cong D_{Rx}(\bigcap_{i=1}^{t} N_i) \cong (\bigcap_{i=1}^{t} N_i)_x \cong \bigcap_{i=1}^{t} (N_i)_x \cong \bigcap_{i=1}^{t} D_{Rx}(N_i) = \bigcap_{i=1}^{t} D_I(N_i). \tag{3.1}$$

From the above intersection let us remove all $D_I(N_j), 1 \leq j \leq t$, such that

$$\bigcap_{\substack{i=1 \\ i \neq j}}^{t} D_I(N_i) \subseteq D_I(N_j). \tag{3.2}$$

Then we will get a minimal primary decomposition in the form $0 = \bigcap_{i=1}^{s} D_I(N_i), s \leq t$, provided that we show every $D_I(N_i), 1 \leq i \leq s$, is a primary submodule of $D_I(M)$.

In order to avoid inaccuracies, it is harmless to assume that the minimal primary decomposition $0 = \bigcap_{i=1}^{t} N_i$ is sorted in the following sense:

For every $1 \leq j \leq s$, $D_I(N_j)$ doesn't satisfy condition (3.2) and $D_I(N_j)$ satisfies condition (3.2) for every $j \geq s + 1$.

Since $\mathrm{cd}(I,R) \leq \mathrm{ara}(I) = 1$, it follows from [3, Lemma 6.3.1] that $D_I(-)$ is an exact functore and so
$$D_I\left(\frac{M}{N_i}\right) \cong \frac{D_I(M)}{D_I(N_i)}.$$
Hence in order to show that $D_I(N_i)$ is a primary submodule of $D_I(M)$ it is enough to show that the map $D_I(\frac{M}{N_i}) \xrightarrow{.r} D_I(\frac{M}{N_i})$ is either injective or nilpotent homomorphism for every $r \in R$. This is clear the fact because $D_I(-)$ is an R-linear functore. □

Before we discuss about representability of $D_I(M)$, we need the following preparative Lemma.

Lemma 3.1. *Let R be a Noetherian ring and I an ideal of R with $\mathrm{cd}(I,R) = 1$. Suppose that M_1, M_2 are submodules of a finitely generated R-module M such that $M = M_1 + M_2$. Then*
$$D_I(M) \cong D_I(M_1) + D_I(M_2).$$

Proof. First note that because $\mathrm{cd}(I,R) = 1$, by [3, Lemma 6.3.1] we find that $D_I(-)$ is an exact functor. Moreover, it is clear that I is not nilpotent. Therefore $D_I(R)$ is a flat R-module by Lemma 2.1. So by applying the functor $D_I(R) \otimes_R -$ to the exact sequence
$$0 \longrightarrow M_1 \cap M_2 \longrightarrow M_1 \oplus M_2 \longrightarrow M_1 + M_2 \longrightarrow 0,$$
we obtain the following exact sequence
$$0 \longrightarrow D_I(R)\otimes_R(M_1 \cap M_2) \longrightarrow D_I(R)\otimes_R(M_1 \oplus M_2) \longrightarrow D_I(R)\otimes_R(M_1 + M_2) \longrightarrow 0.$$
On the other hand, the following sequence is also exact
$$0 \longrightarrow (D_I(R) \otimes_R M_1) \cap (D_I(R) \otimes_R M_2) \longrightarrow (D_I(R) \otimes_R M_1) \oplus (D_I(R) \otimes_R M_2)$$
$$\longrightarrow (D_I(R) \otimes_R M_1) + (D_I(R) \otimes_R M_2) \longrightarrow 0.$$
Hence by view of [8, Theorem 7.4] and [1, Proposition 2.14], we have
$$D_I(R) \otimes_R (M_1 + M_2) \cong (D_I(R) \otimes_R M_1) + (D_I(R) \otimes_R M_2).$$
This fact together with [3, Exercise 6.1.9] concludes
$$D_I(M) = D_I(M_1 + M_2) \cong D_I(M_1) + D_I(M_2).$$
This completes the proof. □

Theorem 3.2. *Let R be a Noetherian ring and I an ideal of R with $\mathrm{cd}(I,R) = 1$. Suppose that the finitely generated R-module M is representable. Then, so is $D_I(M)$ and moreover $\mathrm{Att}_R(D_I(M)) \subseteq \mathrm{Att}_R(M) \setminus V(I)$.*

Proof. We may assume $D_I(M) \neq 0$. Let

$$M = M_1 + M_2 + \cdots + M_t \qquad \text{with } M_j \qquad \mathfrak{p}_j - \text{secondary} \, (1 \leq j \leq t),$$

be a minimal secondary representation of M. Then it follows from 3.1 that

$$D_I(M) \cong D_I(M_1) + D_I(M_2) + \cdots + D_I(M_t).$$

Note that, it may $D_I(M_j) = 0$ for some $1 \leq j \leq t$. Putting $T := \Sigma_{i=1}^{n} D_I(M_i), n \leq t$, where each $D_I(M_i)$, $1 \leq i \leq n$ is not zero, we have $D_I(M) \cong T$. So it is enough to show that T is representable; i.e., $\Sigma_{i=1}^{n} D_I(M_i)$ is a secondary representation of T. This is clearly the case because $D_I(-)$ is an R-linear functor. Therefore

$$\operatorname{Att}_R(D_I(M)) = \operatorname{Att}_R(T) \subseteq \{\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n\},$$

where $\mathfrak{q}_i = \sqrt{\operatorname{Ann}_R(D_I(M_i))} \in \operatorname{Spec} R$.

Now, let $1 \leq i \leq n$ and $r \in \mathfrak{q}_i$ be arbitrary. Then there exists $l \in \mathbb{N}$ such that $r^l D_I(M_i) = 0$. Since M_i is secondary, either $M_i = rM_i$ or $r \in \sqrt{\operatorname{Ann}_R(M_i)} := \mathfrak{p}$. It is not false to assume that the index of $\mathfrak{p}$ is i. In other words one can assume $\mathfrak{p} \in \operatorname{Att}_R(M)$ is exactly $\mathfrak{p}_i$ itself. (This is possible by rearranging the elements of set $\operatorname{Att}_R(M)$). If $r \notin \mathfrak{p}_i$, then we have $M_i = r^l M_i$. Consequently $D_I(M_i) = r^l D_I(M_i) = 0$ which contradicts the fact that $D_I(M_i) \neq 0$. Hence $\mathfrak{q}_i \subseteq \mathfrak{p}_i$. On the other hand, it is obvious that $\mathfrak{p}_i \subseteq \mathfrak{q}_i$. These yield that $\mathfrak{q}_i = \mathfrak{p}_i$ for all $1 \leq i \leq n$ and therefore $\operatorname{Att}_R(D_I(M)) \subseteq \operatorname{Att}_R(M)$. Finally we claim that $I \nsubseteq \mathfrak{q}_i$ for every $1 \leq i \leq n$. Otherwise, by [3, Corollary 2.2.10] we find $\Gamma_{\mathfrak{q}_i}(D_I(M_i)) \subseteq \Gamma_I(D_I(M_i)) = 0$. But $D_I(M_i)$ is an $\operatorname{Ann}_R(D_I(M_i))$-torsion module. Hence $D_I(M_i) = 0$ which is a contradiction. □

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SOME PROBLEMS OF CONVERGENCE OF GENERAL FOURIER SERIES

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Abstract. S. Banach [1] proved that good differential properties of function do not guarantee the a.e. convergence of the Fourier series of this function with respect to general orthonormal systems (ONS). On the other hand it is very well known that a sufficient condition for the a.e. convergence of an orthonormal series is given by the Menshov-Rademacher Theorem.

The paper deals with sequence of positive numbers (d_n) such that multiplying the Fourier coefficients $(C_n(f))$ of functions with bounded variation by these numbers one obtains a.e. convergent series of the form $\sum_{n=1}^{\infty} d_n C_n(f)\varphi_n(x)$. It is established that the resulting conditions are best possible.

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1. Some notations and theorems

Let (φ_n) be an orthonormal system (ONS) on $[0,1]$ and

$$C_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx, \quad n = 1,2 \tag{1.1}$$

be the Fourier coefficients of a function $f \in L_2(0,1)$.

We denote by $V(0,1)$ the class of all functions of bounded variation and write $V(f)$ for the total variation of a function f on $[0,1]$.

Let A be the class of all absolutely continuous functions f on $[0,1]$. This is a Banach space with the norm

$$\|f\|_A = \int_0^1 \left|f'(x)\right| dx + \|f(x)\|_C ,$$

where $C(0,1)$ is the class of all continuous functions f on $[0,1]$; $\|f(x)\|_C$ is the norm of f on $C(0,1)$.

Definition 1.1. A positive bounded sequence of numbers (d_n) is called a multiplier of convergence with respect to a function class E if

$$\sum_{k=1}^{\infty} d_k C_k(f)\varphi_k(x), \qquad \left(C_k(f) = \int_0^1 f(x)\varphi_k(x)\,dx\right)$$

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is convergence a.e. for all $f \in E$.

Theorem 1.1. *(see [4] ch.9.p332)(Menshov-Rademacher). If (φ_n) is an ONS on $[0,1]$ and a number sequence (c_n) satisfies the condition*

$$\sum_{n=1}^{\infty} c_n^2 \log_2^2 n < +\infty,$$

then the series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

converges a.e. on $[0,1]$.

Lemma 1.1 (see [5]). *If $f \in L_2(0,1)$ takes only finite values on $[0,1]$ and $g \in L_2(0,1)$ is an arbitrary function, then*

$$\int_0^1 f(x)\, g(x)\, dx = \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right) \int_0^{i/n} g(x)\, dx \tag{1.2}$$

$$+ \quad \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left(f(x) - f\left(\frac{i}{n}\right) \right) g(x)\, dx + f(1) \int_0^1 g(x)\, dx.$$

We have that $(\log n = \log_2 n)$

$$\sum_{k=1}^{\infty} d_k^2 C_k^2(f) \log^2 k = \sum_{k=1}^{\infty} d_k^2 C_k(f)\, C_k(f) \log^2 k \tag{1.3}$$

$$= \quad \int_0^1 f(x) \sum_{k=1}^{n} d_k^2 C_k(f) \log^2 k \varphi_k(x)\, dx = \int_0^1 f(x)\, P_n(d,a,x)\, dx.$$

Let

$$P_n(d,a,x) = \sum_{k=1}^{n} d_k^2 a_k \log^2 k \varphi_k(x).$$

Set

$$G_n(d,a) = \max_{1 \le i \le n} \left| \int_0^{i/n} P_n(d,a,x)\, dx \right| \tag{1.4}$$

and

$$T_n(d,a) = \left(\sum_{k=1}^{n} d_k^2 a_k^2 \log^2 k \right)^{1/2}, \tag{1.5}$$

where $(a_n) \in l_2$.

Lemma 1.2. *Let (d_n) be a positive, bounded sequence of numbers. Then for every $i, (i = 1, 2, \ldots, n)$*

$$\int_{(i-1)/n}^{i/n} |P_n(d,a,x)|\, dx = O(1)\, T_n(d,a).$$

Proof. If we use the Cauchy inequality and we mean that $D=\sup_k d_k$ we get

$$\begin{aligned}\int_{(i-1)/n}^{i/n}\left|P_n\left(d,a,x\right)\right|dx &\leq \frac{1}{\sqrt{n}}\left(\int_0^1 P_n^2\left(d,a,x\right)dx\right)^{1/2}\\ &= \frac{1}{\sqrt{n}}\left(\int_0^1\left(\sum_{k=1}^n d_k^2 a_k\log^2 k\varphi_k\left(x\right)\right)^2 dx\right)^{1/2}\\ &= \frac{1}{\sqrt{n}}\left(\sum_{k=1}^n d_k^4 a_k^2\log^4 k\right)^{1/2}\\ &\leq D\frac{\log n}{\sqrt{n}}\left(\sum_{k=1}^n d_k^2 a_k^2 log^2 k\right)^{1/2}=O\left(1\right)T_n\left(d,a\right).\end{aligned}$$

Lemma 1.2 is proved. □

2. Statement of the main problem

General ONS were studied by a lot of authors. We mention Gogoladze and Tsagareishvili [5]-[10], Kashin and Saakyan [4], Tsagareishvili and Tutberidze [17, 18]. Convergence and summability of Fourier series with respect to Walsh, Vilinkin, Haar and trigonometric systems were studied by Gogoladze and Tsagareishvili [18], Persson, Tephnadze and Tutberidze [11] (see also [2], [13]), Tephnadze [14]-[16], Tutberidze [19]-[22]. Similar problems for the two-dimensional case can be found in Goginava and Gogoladze [3], Persson, Tephnadze, and Wall [12].

From Banachs Teorem [1] it follows that if $f\in L_2(0,1)$, $(f\nsim 0)$ then there exists an ONS such that the Fourier series of this function f is not convergent on $[0,1]$ with respect to this system. Thus it is clear that the Fourier coefficients of functions of bounded variation in general do not satisfy condition of Theorem1.1.In the present paper we have studied the sequence (d_n) so that the Fourier coefficients of every function from $V(0,1)$ satisfy the condition

$$\sum_{n=1}^{\infty} d_n^2 C_n^2\left(f\right)\log^2 n<+\infty.$$

The similar results are obtained in [6] − [10].

3. The main results

Theorem 3.1. *Let (φ_n) be an ONS on $[0,1]$ and (d_n) is a given sequence of numbers. If for any $(a_n)\in l_2$*

$$G_n\left(d,a\right)=O\left(1\right)T_n\left(d,a\right), \tag{3.1}$$

then for every $f \in V(0,1)$

$$\sum_{n=1}^{\infty} d_n^2 C_n^2(f) \log^2 n < \infty.$$

Proof. By using Lemma1.1,when $g(x) = P_n(d,c,x)$ we have $(c = (C_k(f)))$

$$\begin{aligned}
&\int_0^1 f(x) P_n(d,c,x)\,dx \\
&= \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right) \int_0^{i/n} P_n(d,c,x)\,dx \\
&+ \sum_{i=1}^{n-1} \int_{(i-1)/n}^{i/n} \left(f(x) - f\left(\frac{i}{n}\right)\right) P_n(d,c,x)\,dx \\
&+ f(1) \int_0^1 P_n(d,c,x)\,dx.
\end{aligned} \tag{3.2}$$

Next (see 1.3)

$$\sum_{k=1}^{n} d_k^2 C_k^2(f) \log^2 k = \int_0^1 f(x) P_n(d,c,x)\,dx.$$

If $f \in V(0,1)$, then (see (3.2)), considering (3.1), we get

$$\begin{aligned}
&\left| \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right) \int_0^{i/n} P_n(d,c,x)\,dx \right| \\
&\le \sum_{i=1}^{n-1} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right| \max_{1\le i\le n} \left| \int_0^{i/n} P_n(d,c,x)\,dx \right| \\
&\le V(f)\, G_n(d,c) = O(T_n(c)).
\end{aligned} \tag{3.3}$$

Further, according to lemma1.2

$$\begin{aligned}
&\left| \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \left(f(x) - f\left(\frac{i}{n}\right)\right) P_n(d,c,x)\,dx \right| \\
&\le \sum_{i=1}^{n} \sup_{x\in\left[\frac{i-1}{n},\frac{i}{n}\right]} \left| f(x) - f\left(\frac{i}{n}\right)\right| \int_{(i-1)/n}^{i/n} |P_n(d,c,x)|\,dx \\
&= O(1) V(f)\, T_n(d,c) = O(1)\, T_n(d,c).
\end{aligned} \tag{3.4}$$

It is easy to see that (see (1.4))

$$\left| f(1) \int_0^1 P_n(d,c,x) \right| = O(1) G_n(d,c) = O(1) T_n(d,c).$$

Considering (3.3) and (3.4), from (3.2) and (3.3), we get

$$\sum_{k=1}^{n} d_k^2 C_k^2(f) \log^2 k = O(1)\, T_n(d,c) = O(1) \left(\sum_{k=1}^{n} d_k^2 C_k^2(f) \log^2 k \right)^{1/2}.$$

It follows that

$$\sum_{k=1}^{\infty} d_k^2 C_k^2 (f) \log^2 k < +\infty.$$

Theorem 3.1 is proved. □

Theorem 3.2. *Let* (φ_n) *be an ONS on* $[0,1]$ *and* (d_n) *is a given sequence of numbers. If for any* $(a_n) \in l_2$

$$G_n(d,a) = O(1)T_n(d,a),$$

then (d_n) *is a multiplier of convergence with respect to class* $V(0,1)$, *or the series*

$$\sum_{n=1}^{\infty} d_n C_n(f) \varphi_n(x)$$

converges a.e. on $[0,1]$ *for every* $f \in V(0,1)$.

Validity of Theorem 3.2 follows from Theorem 3.1 and 1.1.

Theorem 3.3. *Let* (φ_n) *be ONS on* $[0,1]$ *and* (d_n) *is a given bounded decreasing sequence of numbers. If for some* $(b_n) \in l_2$

$$\lim_{n\to\infty} \frac{G_n(b)}{T_n(b)} = +\infty.$$

Then, there exist function $f_0 \in A$, *such that*

$$\sum_{k=1}^{\infty} d_k^2 C_k^2 (f_0) \log^2 k = +\infty.$$

Proof. In first case we suppose

$$\lim_{n\to\infty} \frac{\left|\int_0^1 P_n(d,b,x)\,dx\right|}{T_n(b)} = +\infty.$$

If $f_0 = 1,\ x \in [0,1]$, then using the Cauchy inequality we get

$$\begin{aligned}
\left|\int_0^1 P_n(d,b,x)\,dx\right| &= \left|\sum_{k=1}^{n} d_k^2 b_k \log^2 k \int_0^1 \varphi_k(x)\right| = \left|\sum_{k=1}^{n} d_k^2 b_k \log^2 k C_k(f_0)\right| \\
&\le \left(\sum_{k=1}^{n} d_k^2 b_k^2 \log^2 k\right)^{1/2} \left(\sum_{k=1}^{n} d_k^2 C_k^2(f_0) \log^2 k\right)^{1/2} \\
&= T_n(b) \left(\sum_{k=1}^{n} d_k^2 C_k^2(f_0) \log^2 k\right)^{1/2}.
\end{aligned}$$

Consequently

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} d_k^2 C_k^2(f_0) \log^2 k\right)^{1/2} = \lim_{n\to\infty} \sup \frac{\left|\int_0^1 P_n(d,b,x)\,dx\right|}{T_n(b)} = +\infty.$$

As $f_0 \in A$ Theorem 2 holds.

Next we suppose that

$$\left|\int_0^1 P_n\left(d,b,x\right)dx\right| = O(1)T_n(b).$$

Let $1 \le i_n < n$ be an integer, such that

$$G_n\left(b\right) = \max_{1\le i\le n}\left|\int_0^{i/n} P_n\left(d,b,x\right)dx\right| = \left|\int_0^{i_n/n} P_n\left(d,c,x\right)dx\right|.$$

Suppose that for some sequence $b = (b_k) \in l_2$

$$\lim_{n\to\infty}\sup\frac{G_n\left(b\right)}{T_n\left(b\right)} = +\infty. \tag{3.5}$$

Consider the sequence of functions

$$f_n\left(x\right) = \begin{cases} 0, & \text{when} \quad x \in \left[0, \frac{i_n}{n}\right] \\ 1, & \text{when} \quad x \in \left[\frac{i_n+1}{n}, 1\right] \\ \text{continuous and linear,} & \text{when} \quad x \in \left[\frac{i_n}{n}, \frac{i_n+1}{n}\right]. \end{cases}$$

Let A be the class of absolutely continuous functions. Then

$$\left\|f_n\right\|_A = \int_0^1 \left|f_n^{'}\left(x\right)\right|dx + \left\|f_n\left(x\right)\right\|_C = 2.$$

Furthermore

$$\left|\sum_{i=1}^{n-1}\left(f_n\left(\frac{i}{n}\right) - f_n\left(\frac{i+1}{n}\right)\right)\int_0^{i/n} P_n\left(d,b,x\right)dx\right| \tag{3.6}$$
$$= \left|\int_0^{i_n/n} P_n\left(d,b,x\right)dx\right| = G_n\left(b\right).$$

Then if $x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$

$$\left|f_n\left(x\right) - f_n\left(\frac{i}{n}\right)\right|\begin{cases} \le 1, & \text{if} \quad i = i_n+1, \\ 0, & \text{if} \quad i \ne i_n+1, \end{cases}$$

we have (see lemma1.2)

$$\left|\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\left(f\left(x\right) - f\left(\frac{i}{n}\right)\right)P_n\left(d,b,x\right)dx\right| \tag{3.7}$$
$$\le \int_{i_n/n}^{(i_n+1)/n}\left|P_n\left(d,b,x\right)\right|dx = O\left(1\right)T_n\left(b\right).$$

Consequently from equality (3.2) when $f\left(x\right) = f_n\left(x\right)$ and $P_n\left(d,a,x\right) = P_n\left(d,b,x\right),$ considering (3.6) and (3.7), we get

$$\left|\int_0^1 f_n\left(x\right)P_n\left(d,b,x\right)\right|dx \ge G_n\left(b\right) - O\left(1\right)T_n\left(b\right).$$

From here and from (3.5) we have

$$\lim_{n\to\infty}\frac{\left|\int_0^1 f_n\left(x\right)P_n\left(d,b,x\right)dx\right|}{T_n\left(b\right)} = +\infty.$$

Since

$$U_n(f) = \frac{1}{T_n(b)} \int_0^1 f(x) P_n(d, b, x)\, dx$$

is a sequence of linear bounded functionals on A, then by the Banach-Steinhaus theorem, there exists a function $f_0 \in A$ such that

$$\lim_{n\to\infty} \frac{\left|\int_0^1 f_0(x) P_n(d, b, x)\, dx\right|}{T_n(b)} = +\infty.$$

Further using the Couchy inequality

$$\begin{aligned}
\left|\int_0^1 f_0(x) P_n(d, b, x)\, dx\right| &= \left|\sum_{k=1}^n d_k^2 b_k \log^2 k \int_0^1 f_0(x) \varphi_k(x)\, dx\right| \\
&= \left|\sum_{k=1}^n d_k^2 b_k \log^2 k C_k(f_0)\right| \\
&\le \left(\sum_{k=1}^n d_k^2 b_k^2 \log^2 k\right)^{1/2} \left(\sum_{k=1}^n d_k^2 C_k^2(f_0) \log^2 k\right)^{1/2} \\
&= T_n(b) \left(\sum_{k=1}^n d_k^2 C_k^2(f_0) \log^2 k\right)^{1/2}.
\end{aligned}$$

From here

$$\left(\sum_{k=1}^n d_k^2 C_k^2(f_0) \log^2 k\right)^{1/2} \ge \frac{\left|\int_0^1 f_0(x) P_n(d, b, x)\, dx\right|}{T_n(b)}$$

and therefore,

$$\sum_{k=1}^\infty d_k^2 C_k^2(f_0) \log^2 k = +\infty.$$

Theorem 3.3 is proved. □

Finally the following theorem holds:

Theorem 3.4. *Let* (φ_n) *be ONS on* $[0,1]$, $\int_0^1 \varphi_n(x)dx = 0$, $n = 1, 2, \ldots$, *such that uniformly for* $x \in [0,1]$

$$\int_0^x \varphi_n(y)dy = O\left(\frac{1}{n}\right) \tag{3.8}$$

and (d_n) *is an arbitrary non-decreasing sequence of numbers such that*

$$\lim_{n\to\infty} d_n = +\infty \text{ and } d_n = O\left(\frac{n^\gamma}{\log(n+1)}\right),\ 0 < \gamma < \frac{1}{2}.$$

Then for any $f \in V$ *the series*

$$\sum_{n=1}^\infty d_n C_n(f) \varphi_n(x)$$

is convergent a.e. on $[0,1]$.

Proof. According to the condition of Theorem 3.4 and using the Cauchy inequality we get (see (1.4))

$$
\begin{aligned}
(3.9)\left|\int_0^x P_n(d,c,y)dy\right| &= \left|\sum_{k=1}^n d_k^2 C_k(f)\log^2 k \int_0^x \varphi_k(y)dy\right| \\
&= O(1)\sum_{k=1}^n d_k^2\,|C_k(f)|\log^2 k\frac{1}{k} \\
&= O(1)\left(\sum_{k=1}^n d_k^2 C_k^2(f)\log^2 k\right)^{1/2}\left(\sum_{k=1}^n d_k^2\log^2 k\frac{1}{k^2}\right)^{1/2} \\
&= O(1)T_n(c)\left(\sum_{k=1}^n d_k^2\log^2 k\frac{1}{k^2}\right)^{1/2} \\
&= O(1)T_n(c)\left(\sum_{k=1}^n \frac{k^{2\gamma}}{k^2}\right)^{1/2} = O(1)T_n(c)
\end{aligned}
$$

Next as $f \in V$ by the Cauchy inequality (see (3.8))

$$
\begin{aligned}
&(3.10)\quad \left|\sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(f(x)-f\left(\frac{i}{n}\right)\right)P_n(d,c,x)\right| \\
&= O(1)\sum_{i=1}^n \sup_{x\in[0,1]}\left|f(x)-f\left(\frac{i}{n}\right)\right|\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}}|P_n(d,c,x)dx|\right)^{1/2} \\
&= O(1)\frac{1}{\sqrt{n}}\left(\sum_{k=1}^n d_k^4 C_k^2(f)\log^4 k\right)^{1/2} = O(1)\frac{d_n\log n}{\sqrt{n}}\left(\sum_{k=1}^n d_k^2 C_k^2(f)\log^2 k\right)^{1/2} \\
&= O(1)\frac{n^\gamma \log n}{\sqrt{n}\log(n+1)}T_n(d,c) = O(1)T_n(d,c).
\end{aligned}
$$

Using (1.2),(3.9) and (3.10) from (1.3) we receive

$$
\sum_{k=1}^n d_k^2 C_k^2(f)\log^2 k = O(1)T_n(d,c) = O(1)\left(\sum_{k=1}^n d_k^2 C_k^2(f)\log^2 k\right)^{1/2}
$$

From here we conclude

$$
\sum_{k=1}^\infty d_k^2 C_k^2(f)\log^2 k < +\infty.
$$

Finally according to the Menshov-Rademacher Theorem the series

$$
\sum_{k=1}^\infty d_k C_k(f)\varphi_k(x)
$$

converges a.e. on $[0,1]$. Theorem 3.4 is proved. □

It easy to see that Theorem 3.4 holds for trigonometric and Walsh systems (see [4], ch.4, p.117, p.150).

Theorem 3.5. *Let* (h_n) *be an increasing sequence of numbers such that* $\lim_{n\to\infty} h_n = +\infty$ *and* $h_n = O(1)\frac{\sqrt{n}}{\log(n+1)}$. *Then from any ONS* (φ_n) *one can insolate a subsequence* (φ_{n_k}) *such that for an arbitraty* $f \in V$

$$\sum_{k=1}^{\infty} h_k^2 C_{n_k}^2(f) \log^2 k < +\infty.$$

Proof. Without the loss of generality we can suppose that the ONS (φ_n) is a complet system. Then by the Parseval equality, for any $x \in [0,1]$

$$\sum_{n=1}^{\infty} \left(\int_0^x \varphi_n(u)du \right)^2 = x.$$

Consequently (Dini Theorem) for some sequence (n_k) of natural numbers

$$\sum_{n=n_k}^{\infty} \left(\int_0^x \varphi_n(u)du \right)^2 < \frac{1}{k^4}.$$

From here uniformly with respect to $x \in [0,1]$

$$\left| \int_0^x \varphi_{n_k}(u)du \right| < \frac{1}{k^2}. \tag{3.11}$$

We denote $((a_n) \in l_2)$

$$Q_m(h,a,x) = \sum_{k=1}^{m} h_k^2 a_k \log^2 k \varphi_{n_k}(x).$$

Using (3.11) and Cauchy inequality we get (see (3.9))

$$\begin{aligned} &\max_{1\le i\le m} \left| \int_0^{i/m} Q_m(h,a,x)dx \right| \\ &= \max_{1\le i\le m} \left| \sum_{k=1}^{m} h_k^2 a_k \log^2 k \int_0^{i/m} \varphi_{n_k}(x)dx \right| \\ &= O(1) \sum_{k=1}^{m} h_k^2 \, |a_k| \log^2 k \, \frac{1}{k^2} \\ &= O(1) \left(\sum_{k=1}^{m} h_k^2 a_k^2 \log^2 k \right)^{1/2} \left(\sum_{k=1}^{m} h_k^2 \log^2 k \, \frac{1}{k^4} \right)^{1/2} \\ &= O(1) \left(\sum_{k=1}^{m} h_k^2 a_k^2 \log^2 k \right)^{1/2} \left(\sum_{k=1}^{m} \frac{k \log^2 k}{\log^2(k+1) k^4} \right)^{1/2} \\ &= O(1) \left(\sum_{k=1}^{m} h_k^2 a_k^2 \log^2 k \right)^{1/2} \end{aligned} \tag{3.12}$$

Next, for any $i = 1, 2, \ldots, m$ (see (3.10))

$$(3.13)\ \int_{i-1/m}^{i/m} |Q_m(h,a,x)|\,dx \le \frac{1}{\sqrt{m}} \left(\int_0^1 Q_m^2(h,a,x)dx\right)^{1/2}$$
$$= O(1)\frac{1}{\sqrt{m}}\left(\sum_{k=1}^{m} h_k^4 a_k^2 \log^4 k\right)^{1/2} = O(1)\frac{h_m \log m}{\sqrt{m}}\left(\sum_{k=1}^{m} h_k^2 a_k^2 \log^2 k\right)^{1/2}$$
$$= O(1)\frac{\sqrt{m}\log m}{\log(m+1)\sqrt{m}}\left(\sum_{k=1}^{m} h_k^2 a_k^2 \log^2 k\right)^{1/2} = O(1)\left(\sum_{k=1}^{m} h_k^2 a_k^2 \log^2 k\right)^{1/2}.$$

Also (see (3.11) and (3.12))

$$(3.14)\qquad \left|\int_0^1 Q_m(h,a,x)dx\right| = O(1)\left(\sum_{k=1}^{m} h_k^2 a_k^2 \log^2 k\right)^{1/2}.$$

As it was shown in (1.3)

$$(3.15)\qquad \sum_{k=1}^{m} h_k^2 C_{n_k}^2(f)\log^2 k = \int_0^1 f(x)\sum_{k=1}^{m} h_k^2 C_{n_k}(f)\log^2 k\varphi_{n_k}(x)dx = \int_0^1 f(x)Q_m(h,c,x)dx.$$

Taking into account (3.2) and (3.15) where $Q_m(h,c,x) = P_n(d,c,x)$, $f \in V(0,1)$ and estimates (3.12), (3.13), (3.14) where $a = c$, $a_k = C_{n_k}(f)$, we obtain

$$\left|\sum_{k=1}^{m} h_k^2 C_{n_k}^2(f)\log^2 k\right| = \left|\int_0^1 f(x)Q_m(h,c,x)dx\right| = O(1)\left|V(f)+f(1)\right|\left(\sum_{k=1}^{m} h_k^2 C_{n_k}^2(f)\log^2 k\right)^{1/2}.$$

From here

$$\sum_{k=1}^{\infty} h_k^2 C_{n_k}^2(f)\log^2 k < +\infty.$$

Theorem 3.5 is completely proved. □

Theorem 3.6. *Let* (h_n) *be an increasing sequence of numbers such that*

$$\lim_{n\to\infty} h_n = +\infty \text{ and } h_n = O(1)\frac{\sqrt{n}}{\log(n+1)}.$$

Then from any ONS (φ_n) *one can insolate a subsequence* $(\varphi_{n_k}(x))$ *such that for an arbitrary* $f \in V$ *the series*

$$\sum_{k=1}^{\infty} h_k C_{n_k}(f)\varphi_{n_k}$$

is convergent a.e. on $[0,1]$.

The validity of Theorem 3.6 derives from Theorems 3.5 and 3.1.

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SOME ESTIMATES FOR RIESZ TRANSFORMS ASSOCIATED WITH SCHRÖDINGER OPERATORS

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Abstract. Let $\mathcal{L} = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^n$, where $n \geq 3$, and nonnegative potential V belongs to the reverse Hölder class RH_q with $n/2 \leq q < n$. Let $H^p_{\mathcal{L}}(\mathbb{R}^n)$ denote the Hardy space related to $\mathcal{L}$ and $BMO_{\mathcal{L}}(\mathbb{R}^n)$ denote the dual space of $H^1_{\mathcal{L}}(\mathbb{R}^n)$. In this paper, we show that $T_{\alpha,\beta} = V^{\alpha}\nabla\mathcal{L}^{-\beta}$ is bounded from $H^{p_1}_{\mathcal{L}}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$ for $\frac{n}{n+\delta'} < p_1 \leq 1$ and $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2(\beta-\alpha)}{n}$, where $\delta' = \min\{1, 2-n/q_0\}$, and q_0 is the reverse Hölder index of V. Moreover, we prove $T^*_{\alpha,\beta}$ is bounded on $BMO_{\mathcal{L}}(\mathbb{R}^n)$ when $\beta - \alpha = 1/2$.

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1. Introduction and results

The study of the theory of harmonic analysis related to Schrödinger operator is one of the most interesting topic, which has attracted a great deal of attention of many researchers; see [7][8],[12]-[16],[20]-[23] and references therein. The present paper investigate the boundedness of Riesz transform associated with Schrödinger operator on Hardy space and BMO space.

Let $\mathcal{L} = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^n$, where $n \geq 3$ and the nonnegative potential V belongs to reverse Hölder class RH_q for $q \geq n/2$. Recall that given $0 \leq V \in L^q_{loc}(\mathbb{R}^n)$ for $1 < q < \infty$, V is said to belong to the reverse Hölder class RH_q if there exists a constant $C = C(q, V) > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)dy$$

holds for every ball $B \subset \mathbb{R}^n$.

Clearly, if V belongs to $RH_q, q > 1$, then V is a Muckenhoup A_∞ weight; see [17]. From weight theory we know that $V(x)dx$ is a doubling measure and the class RH_q has self-improvement property; that is, if $V \in RH_q$ for some $q > 1$, then there exists $\epsilon > 0$ such that $V \in RH_{q+\epsilon}$. We define the reverse Hölder index of V as $q_0 = \sup\{q : V \in RH_q\}$. From now on, we always use δ' to denote $\min\{1, 2-n/q_0\}$.

As in [18], for a given potential $V \in RH_q$ with $q \geq n/2$, the auxiliary function is defined as

$$\rho(x) = \frac{1}{m(x,V)} = \sup\left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1 \right\}, \quad x \in \mathbb{R}^n.$$

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

The Hardy space related to $\mathcal{L}$ had been studied by Dziubański and Zienkiewicz in [5, 6]. Because $0 \leq V \in L^q_{loc}(\mathbb{R}^n)$, the Schrödinger operator $\mathcal{L}$ generates a (C_0) contraction semigroup $\{T^{\mathcal{L}}_s : s > 0\} = \{e^{-s\mathcal{L}} : s > 0\}$. The maximal function associated with $\{T^{\mathcal{L}}_s : s > 0\}$ is defined by $M^{\mathcal{L}} f(x) = \sup_{s>0} |T^{\mathcal{L}}_s f(x)|$. The Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is defined as follows.

Definition 1.1. We say that f is an element of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ if the maximal function $M^{\mathcal{L}} f$ belongs to $L^1(\mathbb{R}^n)$. The quasi-norm of f is defined by $\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} = \|M^{\mathcal{L}} f\|_{L^1(\mathbb{R}^n)}$.

The dual space of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is the BMO type space $BMO_{\mathcal{L}}(\mathbb{R}^n)$ (see [1]).

Definition 1.2. Let f be a locally function on $\mathbb{R}^n$ and $B = B(x,r)$. Set $f_B = \frac{1}{|B|}\int_B f(y)dy$ and $f(B,V) = f_B$ if $r < \rho(x)$; $f(B,V) = 0$ if $r \geq \rho(x)$. We say $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$ if

$$\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|} \int_B |f(y) - f(B,V)|dy < \infty.$$

It follows from [1] that $\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}$ is actually a norm which makes $BMO_{\mathcal{L}}(\mathbb{R}^n)$ a Banach space. Since $H^1(\mathbb{R}^n) \subset H^1_{\mathcal{L}}(\mathbb{R}^n)$, it conclude by duality that $BMO_{\mathcal{L}}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. Some papers have studied the boundedness of operators on $BMO_{\mathcal{L}}(\mathbb{R}^n)$ space; See[1, 3, 4].

To give the definition of Hardy space $H^p_{\mathcal{L}}(\mathbb{R}^n)$ for $\frac{n}{n+\delta'} < p < 1$, we introduce the Campanato type space.

Assume that $\frac{n}{n+\delta'} < p < 1$, and $1 \leq q' \leq \infty$. A locally integrable function f is said to be in the Campanato type space $\Lambda^L_{\frac{1}{p}-1,q'}$, if

$$\|f\|_{\Lambda^L_{\frac{1}{p}-1,q'}} = \sup_{B\subset\mathbb{R}^n} \left\{ |B|^{1-\frac{1}{p}} \left(\int_B |f(y) - f(B,V)|^{q'} \frac{dy}{|B|} \right)^{1/q'} \right\} < \infty.$$

For any $1 \leq q' \leq \infty$, the spaces $\Lambda^L_{\frac{1}{p}-1,q'}$ are mutually coincident with equivalent norms, it will be simply denoted by $\Lambda^L_{\frac{1}{p}-1}$. It can be proved that the maximal function $M^{\mathcal{L}} f$ is well defined for $f \in \left(\Lambda^L_{\frac{1}{p}-1}\right)^*$.

Definition 1.3. [2] For $\frac{n}{n+\delta'} < p < 1$, we say that $f \in \left(\Lambda^L_{\frac{1}{p}-1}\right)^*$ is an element of $H^p_{\mathcal{L}}(\mathbb{R}^n)$ if the maximal function $M^{\mathcal{L}} f$ belongs to $L^p(\mathbb{R}^n)$. The quasi-norm of f is defined by $\|f\|_{H^p_{\mathcal{L}}(\mathbb{R}^n)} = \|M^{\mathcal{L}} f\|_{L^p(\mathbb{R}^n)}$.

We consider the Riesz transform

$$T_{\alpha,\beta} = V^{\alpha}\nabla\mathcal{L}^{-\beta}, \quad 0 \le \alpha \le \frac{1}{2} \le \beta \le 1, \quad \beta - \alpha \ge \frac{1}{2}.$$

The boundedness of $T_{\alpha,\beta}$ has been studied under the condition $V \in RH_q$ for $n/2 \le q < n$. In [18], Shen showed that $T_{0,\frac{1}{2}}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < p_0$, $\frac{1}{p_0} = \frac{1}{s_0} - \frac{1}{n}$; he proved $T_{\frac{1}{2},1}$ is also bounded on $L^p(\mathbb{R}^n)$ for $1 < p < p_1$, $\frac{1}{p_1} = \frac{3}{2s_0} - \frac{1}{n}$. Li and Peng in [10] obtained that $T_{0,\frac{1}{2}}$ is bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$, Hu and Wang in [9] established the boundedness of commutator for $T_{\alpha,\alpha}$, Liu [11] obtained the boundedness of $T_{0,\beta}$ on $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

If $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta - \alpha \ge \frac{1}{2}, V \in RH_q$ for $n/2 \le q < n$, Sugano in [19] given the L^p-estimates for $T_{\alpha,\beta}$ and its adjoint operator $T^*_{\alpha,\beta}$.

Proposition 1.1. *Suppose that $V \in RH_q$ with $\frac{n}{2} \le q < n$. Let $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta - \alpha \ge \frac{1}{2}, \frac{1}{p_\alpha} = \frac{\alpha+1}{q} - \frac{1}{n}$.*

(i) *If $1 < p < \frac{1}{\frac{1}{p_\alpha} + \frac{2(\beta-\alpha)-1}{n}}$ and $\frac{1}{q} = \frac{1}{p} - \frac{2(\beta-\alpha)-1}{n}$, then*

$$\|T_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)};$$

(ii) *If $p'_\alpha < p < \frac{n}{2(\beta-\alpha)-1}$ and $\frac{1}{q} = \frac{1}{p} - \frac{2(\beta-\alpha)-1}{n}$, then*

$$\|T^*_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

In this paper, we investigate the boundedness of $T_{\alpha,\beta}$ and $T^*_{\alpha,\beta}$ on $H^p_{\mathcal{L}}$ space and $BMO_{\mathcal{L}}(\mathbb{R}^n)$ space respectively, and get the following results.

Theorem 1.1. *Suppose $V \in RH_q$ with $\frac{n}{2} \le q < n$. Let $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta-\alpha \ge \frac{1}{2}$. If $\frac{n}{n+\delta'} < p_1 \le 1$, and $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2(\beta-\alpha)-1}{n}$, then*

$$\|T_{\alpha,\beta}(f)\|_{L^{p_2}(\mathbb{R}^n)} \le C\|f\|_{H^{p_1}_{\mathcal{L}}(\mathbb{R}^n)}.$$

By Theorem 1.1 we get

Corollary 1.1. *Suppose $V \in RH_q$ with $\frac{n}{2} \le q < n$. Let $0 \le \alpha \le \frac{1}{2} < \beta \le 1, \beta-\alpha > \frac{1}{2}$. Then*

$$\|T^*_{\alpha,\beta}(f)\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \le C\|f\|_{L^{p_0}(\mathbb{R}^n)},$$

where $p_0 = \frac{n}{2(\beta-\alpha)-1}$.

When $\beta - \alpha = 1/2$, we have

Theorem 1.2. *Suppose $V \in RH_q$ with $\frac{n}{2} \le q < n$. Let $0 \le \alpha \le \frac{1}{2} < \beta \le 1$ and $\beta - \alpha = \frac{1}{2}$. Then*

$$\|T^*_{\alpha,\beta}(f)\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \le C\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Some preliminaries

Throughout this section we always assume $V \in RH_q$ with $\frac{n}{2} \leq q < n$.

2.1 Some results concerning the auxiliary function

Lemma 2.1. [5] *There exists constant $l_0 > 0$ such that*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \lesssim \left(1 + \frac{r}{\rho(x)}\right)^{l_0}.$$

Lemma 2.2. [18] *For $0 < r < R < \infty$, we have*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \lesssim \left(\frac{R}{r}\right)^{n/s-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

Lemma 2.3. [18] *There exist C and $k_0 \geq 1$ such that*

$$C^{-1}\left(1 + \frac{|x-y|}{\rho(y)}\right)^{\frac{1}{1+k_0}} \leq 1 + \frac{|x-y|}{\rho(x)} \leq C\left(1 + \frac{|x-y|}{\rho(y)}\right)^{1+k_0}$$

for all $x, y \in \mathbb{R}^n$.

A ball $B(x, \rho(x))$ is called critical. Assume that $Q = B(x_0, \rho(x_0))$, for $x \in Q$, the inequality above tell us that $\rho(x) \sim \rho(y)$, if $|x-y| < C\rho(x)$.

2.2 Atomic decomposition of Hardy space $H^p_{\mathcal{L}}(\mathbb{R}^n)$

Let $\frac{n}{n+\delta'} < p \leq 1 \leq q \leq \infty$ and $p \neq q$. A function $a \in L^2(\mathbb{R}^n)$ is called an $H^{p,q}_{\mathcal{L}}$-atom if $r < \rho(x_0)$ and the following conditions hold:

(i) supp $a \subset B(x_0, r)$,

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B(x_0, r)|^{1/q-1/p}$,

(iii) if $r < \rho(x_0)/4$, then $\int_{B(x_0,r)} a(x)dx = 0$.

By [5] and [6], the Hardy space $H^p_{\mathcal{L}}(\mathbb{R}^n)$ admits the following atomic decomposition:

Lemma 2.4. *Let $\frac{n}{n+\delta'} < p \leq 1 \leq q \leq \infty$. Then $f \in H^p_{\mathcal{L}}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^{p,q}_{\mathcal{L}}$- atoms , $\sum_j |\lambda_j|^p < \infty$, and the sum converges in the $H^p_{\mathcal{L}}(\mathbb{R}^n)$ quasi-norm. Moreover*

$$\|f\|_{H^p_{\mathcal{L}}(\mathbb{R}^n)} \sim \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all atomic decompositions of f into $H^{p,q}_{\mathcal{L}}$- atoms.

2.3 Estimates for the kernel functions

Suppose $\mathcal{W}_\beta = \nabla \mathcal{L}^{-\beta}$. Let $\mathcal{W}^*_\beta$ be the adjoint of $\mathcal{W}_\beta, K$ and K^* be the kernels of $\mathcal{W}_\beta$ and $\mathcal{W}^*_\beta$ respectively, then $K(x,z) = K^*(z,x)$ and we have the following estimates.

Lemma 2.5. [9] *Suppose* $1/2 < \beta \leq 1$.

(i) *For every N there exists a constant C such that*

$$|K^*(x,z)| \leq \frac{C_N}{\left(1+\frac{|x-z|}{\rho(x)}\right)^N} \frac{1}{|x-z|^{n-2\beta}} \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi - z|^{n-1}} d\xi + \frac{1}{|x-z|} \right).$$

Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

(ii) *For every N, and $\delta = 2 - n/q$, there exists a constant C_N such that*

$$|K^*(x,z) - K^*(y,z)| \leq \frac{C_N}{\left(1+\frac{|x-z|}{\rho(x)}\right)^N} \times \frac{|x-y|^\delta}{|x-z|^{n-2\beta+\delta}} \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi - z|^{n-1}} d\xi + \frac{1}{|x-z|} \right)$$

whenever $|x-y| < \frac{1}{16}|x-z|$. Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

2.4 Characterization of space $BMO_{\mathcal{L}}(\mathbb{R}^n)$

Lemma 2.6. [1] *Let $1 \leq p < \infty, B = B(x,r)$. If $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$, then*

$$\sup_B \left(\frac{1}{|B|} \int_B |f(y) - f(B,V)|^p dy \right)^{1/p} \leq C \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

A function $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$ if and only if there exists a suitable constant c_B depending on B and satisfying $c_B = 0$ whenever $r \geq \rho(x)$ such that

$$\sup_B \left(\frac{1}{|B|} \int_B |f(y) - c_B|^p dy \right)^{1/p} < \infty$$

and

$$\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \leq C \sup_B \left(\frac{1}{|B|} \int_B |f(y) - c_B|^p dy \right)^{1/p}.$$

3. Proof of main results

Proof of Theorem 1.1. By Lemma 2.4, we only need to prove

$$\|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \lesssim 1$$

holds for any $H_{\mathcal{L}}^{p_1,q_1}$-atom. Because $0 \leq \alpha \leq \frac{1}{2} < \beta \leq 1, \beta - \alpha \geq \frac{1}{2}$, we can choose q_1 and q_2 such that

$$1 < q_1 < \frac{1}{\frac{1}{p_\alpha} + \frac{2(\beta-\alpha)-1}{n}}$$

and

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{2(\beta-\alpha)-1}{n}.$$

Assume that supp$a \subset B(x_0,r), r < \rho(x_0)$. Then

$$\|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \leq \|\chi_{16B} T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} + \|\chi_{(16B)^c} T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} = I_1 + I_2.$$

By Hölder inequality, Proposition 1.1 and $\frac{1}{p_2}=\frac{1}{p_1}-\frac{2(\beta-\alpha)-1}{n}$, we have

$$\begin{aligned}
I_1 &= \|\chi_{16B}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)}\\
&\lesssim |16B|^{\frac{1}{p_2}-\frac{1}{q_2}}\left(\int_{\mathbb{R}^n}|T_{\alpha,\beta}a(x)|^{q_2}dx\right)^{1/q_2}\\
&\lesssim |16B|^{\frac{1}{p_2}-\frac{1}{q_2}}\left(\int_{B}|a(x)|^{q_1}dx\right)^{1/q_1}\\
&\lesssim\ |16B|^{\frac{1}{p_2}-\frac{1}{q_2}}|B|^{\frac{1}{q_1}-\frac{1}{p_1}}\lesssim 1.
\end{aligned}$$

We divided into two case for the estimate of $I_2:\ r\geq\rho(x_0)/4$ and $r<\rho(x_0)/4$.

Case I: $r\geq\rho(x_0)/4$. In this case, we have $r\sim\rho(x_0)$. It follows from Lemma 2.3 and Lemma 2.5 that

$$\begin{aligned}
\int_B \big|K(x,z)a(z)\big|dz \lesssim\ &\int_B \frac{|a(z)|dz}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N|x-z|^{n-2\beta+1}}\\
&+\int_B \frac{|a(z)|}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N|x-z|^{n-2\beta}}\int_{B(x,|x-z|/4)}\frac{V(\xi)}{|\xi-x|^{n-1}}d\xi dz.
\end{aligned}$$

For any $x\in C_k=\{x:2^kr<|x-z|\leq 2^{k+1}r\}, k\geq 4$, we have

$$\begin{aligned}
\int_B \big|K(x,z)a(z)\big|dz \lesssim\ &\frac{1}{2^{kN}(2^kr)^{n-2\beta+1}}\int_B|a(z)|dz\\
&+\frac{1}{2^{kN}(2^kr)^{n-2\beta}}\mathcal{I}_1(V\chi_{B(x_0,2^{k+1}r)})(x)\int_B|a(z)|dz,
\end{aligned}$$

where $\mathcal{I}_1(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)dy}{|x-y|^{n-1}}$. Then

$$I_2\leq\left(\sum_{k\geq 4}\int_{C_k}V(x)^{\alpha p_2}\left(\int_B\big|K(x,z)a(z)\big|dz\right)^{p_2}dx\right)^{1/p_2}\leq I_{21}+I_{22},$$

where

$$I_{21}=\left(\sum_{k\geq 4}\frac{(2^kr)^{(2\beta-n-1)p_2+n}}{2^{kNp_2}}\frac{1}{|2^kB|}\int_{2^kB}V(x)^{\alpha p_2}dx\right)^{1/p_2}\int_B|a(z)|dz,$$

and

$$I_{22}=\left(\sum_{k\geq 4}\frac{(2^kr)^{(2\beta-n)p_2+n}}{2^{kNp_2}}\frac{1}{|2^kB|}\int_{2^kB}V(x)^{\alpha p_2}(\mathcal{I}_1(V\chi_{2^{k+1}B})(x))^{p_2}dx\right)^{1/p_2}\int_B|a(z)|dz.$$

Notice

$$p_2\leq\frac{n}{n-2(\beta-\alpha)+1}<\frac{n}{2\alpha}<\frac{s}{\alpha}.$$

By Hölder inequality, $V \in RH_q$ and Lemma 2.1 we get

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{\alpha p_2}dx \lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x)^s dx\right)^{\alpha p_2/s}$$

$$\lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x)dx\right)^{\alpha p_2} \lesssim (2^kr)^{-2\alpha p_2}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{l_0\alpha p_2}. \tag{3.1}$$

Then

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{\alpha p_2}dx \lesssim (2^kr)^{-2\alpha p_2}2^{kl_0\alpha p_2}.$$

Because a is a $H_{\mathcal{L}}^{p_1,q_1}$- atom, so

$$\int_B |a(y)|dy \le |B|^{1-\frac{1}{p_1}}. \tag{3.2}$$

Note

$$2(\beta-\alpha)-1+n(\frac{1}{p_2}-\frac{1}{p_1}) = 0. \tag{3.3}$$

Then, by (3.1) and (3.2) we obtain

$$I_{21} \lesssim \left(\sum_{k\ge 4}\frac{1}{(2^k)^{\left(N-l_0\alpha-(2(\beta-\alpha)-1)+n\right)p_2-n}}\right)^{1/p_2}.$$

Taking N large enough such that $N > l_0\alpha + (2(\beta-\alpha)-1) - n + n/p_2$, we get $I_{21} \lesssim 1$.

It is easy to see $p_2 < p_\alpha$. Then

$$\frac{1}{p_2} > \frac{1}{p_\alpha} = \frac{\alpha}{q} + \frac{1}{t}, \quad \frac{1}{t} = \frac{1}{q} - \frac{1}{n}.$$

By Hölder inequality, the boundedness of fractional integral $\mathcal{I}_1 : L^q \to L^t$ with $\frac{1}{t} = \frac{1}{q} - \frac{1}{n}$ and $V \in RH_q$, we obtain

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{\alpha p_2}(\mathcal{I}_1(V\chi_{2^{k+1}B})(x))^{p_2}dx$$

$$\lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x)^q dx\right)^{p_2\alpha/q}\left(\frac{1}{|2^kB|}\int_{2^kB}(\mathcal{I}_1(V\chi_{2^{k+1}B})(x))^t dx\right)^{p_2/t}$$

$$\lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x) dx\right)^{p_2\alpha}\left(\frac{1}{|2^kB|}\int_{2^{k+1}B} V(x)^q dx\right)^{p_2/q}|2^kB|^{p_2(\frac{1}{q}-\frac{1}{t})}$$

$$\lesssim \left(\frac{1}{|2^kB|}\int_{2^kB} V(x) dx\right)^{p_2(\alpha+1)}(2^kr)^{p_2}$$

$$\lesssim (2^kr)^{-(2\alpha+1)p_2}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{p_2l_0(\alpha+1)}. \tag{3.4}$$

Then

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{\alpha p_2}(\mathcal{I}_1(V\chi_{2^{k+1}B})(x))^{p_2}dx \lesssim (2^kr)^{-(2\alpha+1)p_2}(2^k)^{p_2l_0(\alpha+1)}.$$

Noting (3.3) and taking N large enough such that $N > l_0(\alpha+1)+\big(2(\beta-\alpha)-1\big)-n+n/p_2$, we get

$$I_{22} \lesssim \left(\sum_{k\geq 4} \frac{1}{(2^k)^{p_2\left(N-l_0(\alpha+1)-\left(2(\beta-\alpha)-1\right)+n-n/p_2\right)}}\right)^{1/p_2} \lesssim 1.$$

Case II: $r < \rho(x_0)/4$. When $p=1$, by Lemma 2.3 and Lemma 2.5, for $\delta = 2-n/q$,

$$\begin{aligned}&|K(x,z)-K(x,x_0)|\\ &\lesssim \frac{1}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^{N'}} \frac{|z-x_0|^\delta}{|x-z|^{n+\delta-2\beta}} \left(\int_{B(x_0,|x-z|/4)} \frac{V(\xi)}{|\xi-x|^{n-1}}d\xi + \frac{1}{|x-z|}\right).\end{aligned}$$

Then, for $x \in C_k$, we have

$$\begin{aligned}|K(x,z)-K(x,x_0)| \leq& \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{r^\delta}{(2^k r)^{n+\delta-2\beta+1}}\\ &+ \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{r^\delta}{(2^k r)^{n+\delta-2\beta}} \int_{2^k B} \frac{V(\xi)}{|\xi-x|^{n-1}}d\xi.\end{aligned} \tag{3.5}$$

It follows from the vanishing condition of a, (3.5) and (3.2) that

$$I_2 = \left(\int_{(16B)^c} V(x)^{\alpha p_2}\left(\int_B |(K(x,z)-K(x,x_0))a(z)|dz\right)^{p_2} dx\right)^{1/p_2} \lesssim I'_{21}+I'_{22},$$

where

$$I'_{21} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta p_2}}{(2^k r)^{(n+\delta-2\beta+1)p_2}} \int_{C_k} V(x)^{\alpha p_2} dx\right)^{1/p_2},$$

and

$$I'_{22} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta p_2}}{(2^k r)^{(n+\delta-2\beta)p_2}} \int_{C_k} V(x)^{\alpha p_2}\left(\mathcal{I}_1(V\chi_{2^{k+1}B})(x)\right)^{p_2} dx\right)^{1/q}.$$

By (3.1), noting $\frac{1}{p_2}-1+\frac{2(\beta-\alpha)-1}{n}=0$ and taking $N \geq l_0\alpha$ we have

$$I'_{21} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2-l_0\alpha p_2}} \frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \lesssim \left(\sum_{k\geq 4}\frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \lesssim 1.$$

By (3.4) and taking $N \geq l_0(\alpha+1)$ we have

$$I'_{22} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2-l_0(\alpha+1)p_2}} \frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \lesssim \left(\sum_{k\geq 4}\frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \lesssim 1.$$

We consider the case $\frac{n}{n+\delta'} < p_1 < 1$. By $n/2 \leq q < n$ and the self-improvement property of the class RH_q, for some p_0 such that $\frac{n}{n+\delta'} < p_0 < p_1 < 1$, we have some $\delta_0 = 2 - n/q < 2 - n/q_0$ so that $p_0 = \frac{n}{n+\delta_0}$. Then

$$\frac{1}{p_2} = \frac{1}{p_1} - \frac{2(\beta-\alpha)-1}{n} < \frac{n+\delta_0-2(\beta-\alpha)+1}{n}.$$

So, we have $p_2 > \frac{n}{n+\delta_0-2(\beta-\alpha)+1}$.

By (3.5), for $x \in C_k$, we have

$$|K(x,z)-K(x,x_0)| \leq \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{r^{\delta_0}}{(2^k r)^{n+\delta_0-2\beta+1}} + \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{r^{\delta_0}}{(2^k r)^{n+\delta_0-2\beta}} \int_{2^k B} \frac{V(\xi)}{|\xi-x|^{n-1}} d\xi.$$

Thus,

$$I_2 = \left(\int_{(4B)^c} \left| T_{\alpha,\beta}(a)(x) \right|^{p_2} dx \right)^{1/p_2} \lesssim I_{21}^{''} + I_{22}^{''},$$

where

$$I_{21}^{''} = \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta_0 p_2}}{(2^k r)^{(n+\delta_0-2\beta+1)p_2}} \int_{2^k B} V(x)^{\alpha p_2} dx \right)^{1/p_2} \int_B |a(y)| dy,$$

and

$$I_{22}^{''} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta_0 p_2}}{(2^k r)^{(n+\delta_0-2\beta)p_2}} \int_{C_k} V(x)^{\alpha p_2} \left(\mathcal{I}_1(V\chi_{2^{k+1}B})(x) \right)^{p_2} dx \right)^{1/q} \times \int_B |a(y)| dy.$$

Note $p_2 > \frac{n}{n+\delta_0-2(\beta-\alpha)+1}$. Then, by (3.1),(3.2) and (3.3) and taking $N \geq l_0\alpha$, we get

$$I_{21}^{''} \lesssim \left(\sum_{k\geq 4} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{(N-l_0\alpha)p_2}} \frac{1}{\left(2^k\right)^{p_2(n+\delta_0-2(\beta-\alpha)+1)-n}} \right)^{1/p_2} \lesssim 1.$$

By (3.3), (3.4) and taking $N > l_0(\alpha+1)$, we get

$$I_2 \lesssim \left(\sum_{k\geq 3} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{(N-l_0(\alpha+1))p_2}} \frac{1}{\left(2^k\right)^{p_2(n+\delta_0-2(\beta-\alpha)+1)-n}} \right)^{1/p_2} \lesssim 1.$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. For $\beta - \alpha = 1/2$, it follows from Proposition 1.1 that

$$\|T^*_{\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for $p'_\alpha < p < \infty$.

Let $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$ and fix a ball $B = B(x_0, r)$. We consider two cases: $r \geq \rho(x_0)$ and $r < \rho(x_0)$. For the case $r \geq \rho(x_0)$, we write

$$f = f\chi_{B^*} + f\chi_{(B^*)^c} = f_1 + f_2,$$

where $B^* = 2B$. Owing to the fact that $T^*_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^n)$, we have

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|dx \lesssim \left(\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|^p dx\right)^{1/p} \lesssim \left(\frac{1}{|B^*|}\int_{B^*} |f(x)|^p dx\right)^{1/p}.$$

By Lemma 2.6 we get

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|dx \lesssim C\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Let $C_k = \{z : 2^k r < |x - z| \leq 2^{k+1} r\}, k \geq 1$. Then by Lemma 2.3 and Lemma 2.5,

$$\begin{aligned}
|T^*_{\alpha,\beta}(f_2)(x)| &\leq \int_{(B^*)^c} |K^*(x,z)|V(z)^\alpha |f(z)|dz \\
&\lesssim \int_{(B^*)^c} \frac{V(z)^\alpha}{\left(1 + \frac{|x-z|}{\rho(x_0)}\right)^N} \frac{|f(z)|}{|x-z|^{n-2\beta+1}} dz \\
&\quad + \int_{(B^*)^c} \frac{V(z)^\alpha}{\left(1 + \frac{|x-z|}{\rho(x_0)}\right)^N} \frac{|f(z)|}{|x-z|^{n-2\beta}} \int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi - z|^{n-1}} d\xi dz \\
&\lesssim \sum_{k=1}^\infty \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N} \frac{1}{(2^k r)^{n-2\beta+1}} \int_{2^k B} V(z)^\alpha |f(z)|dz \\
&\quad + \sum_{k=1}^\infty \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N} \frac{1}{(2^k r)^{n-2\beta}} \int_{2^k B} V(z)^\alpha |f(z)| \mathcal{I}_1(V\chi_{2^{k+1}B})(z)dz \\
&= J_1 + J_2.
\end{aligned}$$

Observe that $\frac{1}{p'_\alpha} + \frac{\alpha}{q} + \frac{1}{t} = 1, \frac{1}{t} = \frac{1}{q} - \frac{1}{n}$, by Hölder inequality and the boundedness of fractional integral $\mathcal{I}_1 : L^q \to L^t$ we get

$$\begin{aligned}
&\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^\alpha |f(z)| \mathcal{I}_1(V\chi_{2^{k+1}B})(z)dz \\
&\lesssim \left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^q dz\right)^{\alpha/q} \left(\frac{1}{(2^k r)^n}\int_{2^k B} |f(z)|^{p'_\alpha} dz\right)^{1/p'_\alpha} \\
&\quad \times \left(\frac{1}{(2^k r)^n}\int_{2^k B} |\mathcal{I}_1(V\chi_{2^{k+1}B})(z)|^t dz\right)^{1/t} \\
&\lesssim \left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z) dz\right)^{\alpha} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \\
&\quad \times \left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^q dz\right)^{1/q} (2^k r)^{\frac{n}{q} - \frac{n}{t}} \\
&\lesssim (2^k r)^{\frac{n}{q} - \frac{n}{t} - 2(\alpha+1)} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{l_0(\alpha+1)} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.
\end{aligned}$$

Due to $2\beta - 2(\alpha+1) + \frac{n}{s} - \frac{n}{t} = 0$, taking $N > l_0(\alpha+1)$ we get

$$J_2 \lesssim \sum_{k=1}^{\infty} \frac{1}{(2^k)^{N-l_0(\alpha+1)}} \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

By Hölder inequality,

$$\begin{aligned}
&\frac{1}{(2^k r)^n} \int_{2^k B} V(z)^\alpha |f(z)| dz \\
\lesssim\ & \left(\frac{1}{(2^k r)^n} \int_{2^k B} V(z)^q dz \right)^{\alpha/q} \left(\frac{1}{(2^k r)^n} \int_{2^k B} |f(z)|^{p'_\alpha} dz \right)^{1/p'_\alpha} \\
\lesssim\ & \left(\frac{1}{(2^k r)^n} \int_{2^k B} V(z) dz \right)^{\alpha} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \\
\lesssim\ & (2^k r)^{-2\alpha} \left(1 + \frac{2^k r}{\rho(x_0)} \right)^{\alpha l_0} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.
\end{aligned}$$

Then

$$\begin{aligned}
J_1 &= \sum_{k=1}^{\infty} \frac{(2^k r)^{2\beta-1}}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N} \frac{1}{(2^k r)^n} \int_{2^k B} V(z)^\alpha |f(z)| dz \\
&\lesssim \sum_{k=1}^{\infty} \frac{(2^k r)^{2\beta-1-2\alpha}}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N-\alpha l_0}} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{2^{k(N-l_0\alpha)}} \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, for any $x \in B(x_0, r)$, we get

$$|T^*_{\alpha,\beta}(f_2)(x)| \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Consequently,

$$\frac{1}{|B|} \int_B |T^*_{\alpha,\beta}(f_2)(x)| dx \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Let us consider the case $r < \rho(x_0)$. We set $B^\sharp = B(x_0, 2\rho(x_0))$ and write

$$f = f\chi_{B^\sharp} + f\chi_{(B^\sharp)^c} = f_1^\sharp + f_2^\sharp.$$

Similar to the estimates for $|T^*_{\alpha,\beta} f_2(x)|$, we have

$$|T^*_{\alpha,\beta}(f_2^\sharp)(x)| \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Then

$$\frac{1}{|B|} \int_B |T^*_{\alpha,\beta}(f_2^\sharp)(x) - (T^*_{\alpha,\beta}(f_2^\sharp))_B| dx \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

For any $x \in B(x_0, r)$, let $B_{x,k} = B(x, 2^{2-k}\rho(x_0))$. It is obvious that

$$f(B_{x,k}, V) = 0$$

for $k = 0, 1, 2$. Notice

$$|f(B_{x,3}, V) - f(B_{x,2}, V)| = |f(B_{x,3}, V)|$$
$$\lesssim \frac{1}{|B(x, \rho(x_0))|} \int_{B(x,\rho(x_0))} |f(z)|dz \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

So for $k = 3, 4, \cdots$, we have

$$|f(B_{x,k}, V) - f(B_{x,k-1}, V)| \lesssim \|f\|_{BMO(\mathbb{R}^n)} \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Then, for $k = 3, 4, \cdots$, we get

$$|f(B_{x,k}, V)| \lesssim k\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Hence, by Lemma 2.6, for any $p \geq 1$ and $k = 0, 1, 2, \cdots$, we get

$$\left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(z)|^p dz\right)^{1/p}$$
$$\lesssim \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(z) - f(B_{x,k}, V)|^p dz\right)^{1/p} + |f(B_{x,k}, V)|$$
$$\lesssim k\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

For any $x \in B$,

$$|T^*_{\alpha,\beta}(f_1^\sharp)(x)| \lesssim \int_{B^\sharp} |K^*(x, z)|V(z)^\alpha |f(z)|dz$$
$$\lesssim \sum_{k=0}^{\infty} \int_{B_{x,k} \backslash B_{x,k+1}} \frac{V(z)^\alpha}{\left(1 + \frac{|x-z|}{\rho(x_0)}\right)^N} \frac{|f(z)|}{|x - z|^{n-2\beta+1}} dz$$
$$+ \sum_{k=0}^{\infty} \int_{B_{x,k} \backslash B_{x,k+1}} \frac{V(z)^\alpha}{\left(1 + \frac{|x-z|}{\rho(x_0)}\right)^N} \frac{|f(z)|}{|x - z|^{n-2\beta}} \int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi - z|^{n-1}} d\xi dz$$
$$\lesssim \sum_{k=0}^{\infty} \frac{(2^{2-k}\rho(x_0))^{2\beta-1}}{|B_{x,k}|} \int_{B_{x,k}} V(z)^\alpha |f(z)|dz$$
$$+ \sum_{k=0}^{\infty} \frac{(2^{2-k}\rho(x_0))^{2\beta}}{|B_{x,k}|} \int_{B_{x,k}} V(z)^\alpha |f(z)|\mathcal{I}_1(V\chi_{B_{x,k-1}})(z)dz$$
$$= K_1 + K_2.$$

Notice $0 \leq \alpha \leq 1 < q$. By Hölder inequality and $\beta - \alpha = 1/2$, we get

$$K_1 \lesssim \sum_{k=0}^{\infty} (2^{2-k}\rho(x_0))^{2\alpha} \frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(y)^\alpha |f(y)|dy$$
$$\lesssim \sum_{k=0}^{\infty} (2^{2-k}\rho(x_0))^{2\alpha} \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(y)^q dy\right)^{\alpha/q} \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(y)|^{(\frac{s}{\alpha})'} dy\right)^{1/(\frac{s}{\alpha})'}$$
$$\lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \sum_{k=0}^{\infty} (k + 1) \left(\frac{1}{(2^{2-k}\rho(x_0))^{n-2}} \int_{B_{x,k}} V(y)dy\right)^\alpha.$$

It follows from Lemma 2.2 that

$$\frac{1}{(2^{2-k}\rho(x_0))^{n-2}}\int_{B_{x,k}} V(y)dy \lesssim 2^{-k\delta}$$

for $k = 3, \cdots$, where $\delta = 2 - n/q > 0$, and from Lemma 2.1 that

$$\frac{1}{(2^{2-k}\rho(x_0))^{n-2}}\int_{B_{x,k}} V(y)dy \lesssim 1$$

for $k = 0, 1, 2$. Then $K_1 \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}$.

Pay attention to $\frac{1}{p'_\alpha} + \frac{\alpha}{q} + \frac{1}{t} = 1, \frac{1}{t} = \frac{1}{q} - \frac{1}{n}$. By Hölder inequality and the boundedness of $\mathcal{I}_1$ we get

$$\begin{aligned}
&\frac{1}{|B_{x,k}|}\int_{B_{x,k}} V(z)^\alpha |f(z)| \mathcal{I}_1(V\chi_{B_{x,k-1}})(z)dz \\
&\lesssim \left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}} V(z)^q dz\right)^{\alpha/q} \left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}} |f(z)|^{p'_\alpha} dz\right)^{1/p'_\alpha} \\
&\quad\times \left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}} |\mathcal{I}_1(V\chi_{B_{x,k-1}})(z)|^t dz\right)^{1/t} \\
&\lesssim k\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}} V(z)dz\right)^{\alpha+1} |B_{x,k}|^{\frac{1}{s}-\frac{1}{t}} \\
&\lesssim k\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} (2^{2-k}\rho(x_0))^{-2(\alpha+1)} \left(\frac{1}{(2^{2-k}\rho(x_0))^{n-2}}\int_{B_{x,k}} V(z)dz\right)^{\alpha+1}
\end{aligned}$$

Then

$$K_2 \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)} \sum_{k=0}^{\infty} (k+1)2^{-k\delta 2(\alpha+1)} \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

Combining the estimates for K_1 and K_2, we have proved the inequality

$$|T^*_{\alpha,\beta}(f_1^\sharp)(x)| \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}$$

for any $x \in B(x_0, r), r < \rho(x_0)$. Thence

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1^\sharp)(x) - (T^*_{\alpha,\beta}(f_1^\sharp))_B| dx \lesssim \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.2.

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Contents