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THE WALSH-FOURIER TRANSFORM ON THE REAL LINE

G. GÁT AND U. GOGINAVA

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University of Debrecen, Debrecen, Hungary[1]
United Arab Emirates University, Al Ain, Abu Dhabi, UAE
E-mails: *gat.gyorgy@science.unideb.hu; zazagoginava@gmail.com; ugoginava@uaeu.ac.ae*

Abstract. The element of the Walsh system, that is the Walsh functions map from the unit interval to the set $\{-1, 1\}$. They can be extended to the set of nonnegative reals, but not to the whole real line. The aim of this article is to give an Walsh-like orthonormal and complete function system which can be extended on the real line.

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Keywords: Walsh-Fourier transform; inverse transform.

1. The triadic field

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$. Let $\mathbf{F}$ denote the set of double infinite sequences

$$x = (x_n : n \in \mathbb{Z})$$

where $x_n = 0, 1$ or -1 and $x_n \to 0$ as $n \to -\infty$. Thus, to each $x \in \mathbf{F}$ with $x \neq 0$ there corresponds an integer $s(x) \in \mathbb{Z}$ such that

$$x_{s(x)} \neq 0 \text{ but } x_n = 0 \text{ for } n < s(x).$$

Let $x = (x_n : n \in \mathbb{Z})$ and $y = (y_n : n \in \mathbb{Z})$ be elements of $\mathbf{F}$. Define the sum of x and y by

$$x + y = ((x_n + y_n) \bmod 3 : n \in \mathbb{Z}).$$

Notice that $(\mathbf{F}, +)$ is an Abelian group. The algebra $\mathbf{F}$ is normed. Indeed, for $x = (x_n : n \in \mathbb{Z}) \in \mathbf{F}$ define

$$|x| := \sum_{n \in \mathbb{Z}} \frac{|x_n|}{3^{n+1}}.$$

It is easy to see that $|x| \geqslant 0$,

$$|x + y| \leqslant |x| + |y|.$$

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A character on $\mathbf{F}$ is a continuous complex-valued map which satisfies $(x, y \in \mathbf{F})$

$$\varphi(x+y) = \varphi(x)\varphi(y) \quad \text{and} \quad |\varphi(x)| = 1.$$

It is evident that $\varphi(0) = 1$. Let e_j denote the element $x = (x_n : n \in \mathbb{Z})$ which satisfies $x_j = 1$ for some $j \in \mathbb{Z}$ and $x_n = 0$ for $n \neq j \ (n \in \mathbb{Z})$. Since φ is continuous on $\mathbf{F}$ and $e_j \to 0$ in $\mathbf{F}$ as $j \to \infty$, we have

$$\varphi(e_j) \to \varphi(0) = 1$$

as $j \to \infty$. On the other hand,$e_j + e_j + e_j = 0$. Hence

$$1 = \varphi(0) = \varphi(e_j + e_j + e_j) = \varphi^3(e_j),$$
$$\varphi(e_j) = \sqrt[3]{1} = e^{\frac{2\pi ik}{3}}, k = -1, 0, 1.$$

Consequently, there is a sequence $y = \{y_j : j \in \mathbb{Z}, y_j = -1, 0, 1\}$ such that for every $-j-1 > s(y)$ and

$$\varphi_y(e_j) = e^{\frac{2}{3}\pi i y_{-j-1}}.$$

It is easy to show that

$$\varphi(x_j e_j) = (\varphi(e_j))^{x_j}.$$

Then from the continuity of φ we can write

$$\varphi_y(x) = \prod_{j\in\mathbb{Z}} (\varphi(e_j))^{x_j} = e^{\frac{2}{3}\pi i \sum\limits_{j\in\mathbb{Z}} y_{-j-1}x_j \cdot} \qquad (x, y \in \mathbf{F}).$$

The functions $\varphi_y \ (y \in \mathbf{F})$ exhaut the character of the additive group $(\mathbf{F}, +)$.

2. The Walsh-Fourier transform in L_1

For a given $f \in L_1(\mathbf{F})$ the Walsh-Fourier transform of f is the function on $\mathbf{F}$ will be defined by

$$\widehat{f}(y) := \frac{1}{9}\int_{\mathbf{F}} f(x)\overline{\varphi}_y(x)\, d\mu(x) \qquad (y \in \mathbf{F}).$$

It is quite well-known that a clasical Walsh function maps from the unit interval to the set $\{-1, 1\}$ and also that it can be extended to the set of nonnegative real numbers (see e.g. [6], [4]). But cannot to the whole set of reals. Besides, the same situation hold for the Vilenkin functions (see e.g. [1]). Therefore, in order to involve the real line we must do something different. Basically the "problem" with the Walsh functions is that we can "stay" or "step right". We should be "able to step left" also. Next, we introduce Walsh-like functions on the real line as follows. It is easy to prove that every real number $y \in \mathbb{R}$ can be expressed by the following sum.

$$y = \sum_{k=-\infty}^{+\infty} \frac{y_k}{3^k},$$

where $y_k \in \{-1, 0, 1\}$ for all $k \in \mathbb{N}$. The digits $-1, 0, 1$ mean that we're going to the left, or getting nowhere, or we're going to the right by $1/3^k$. There is no convergence problems, since y is a finite real, and consequently, $y_k = 0$ for k's small enough ($\lim_{k \to -\infty} y_k = 0$). We can identify y be the sequence $(y_k, k \in \mathbb{Z})$. Unfortunately, this identification is not always a bijection, since for all $j \in \mathbb{N}$ the numbers

$$\begin{aligned} \frac{6j+1}{2 \cdot 3^{n+1}} &= \frac{j}{3^n} + \frac{1}{2 \cdot 3^{n+1}} = \frac{j}{3^n} + \frac{1}{3^{n+1}} + \sum_{k=n+2}^{+\infty} \frac{-1}{3^k} \\ &= \frac{j}{3^n} + \frac{0}{3^{n+1}} + \sum_{k=n+2}^{+\infty} \frac{1}{3^k} \end{aligned}$$

have two corresponding $-1, 0, 1$ sequences. The set of these numbers is called the set of triadic rationals. In this situation we choose the one terminates in -1's. Anyhow, the set of these reals is countable ($j, n \in \mathbb{N}$), and consequently of minor importance.

Define the addition $\oplus : \{-1, 0, 1\}^2 \to \{-1, 0, 1\}$ as the mod 3 addition. (E.g. $1 \oplus 1 = -1, (-1) \oplus (-1) = 1$.) Then define the addition $\oplus$ on $\mathbb{R}$ as $x \oplus y := (x_k \oplus y_k, k \in \mathbb{Z})$. The inverse operation is denoted by $\ominus$

Introduce the set of triadic intervals on $\mathbb{R}$. Let $n \in \mathbb{Z}$, and $t \in \mathbb{R}$. Then the set

$$I_n(t) := \{y \in \mathbb{R} : y_i = t_i \text{ for } i \leqslant n\}$$

is called a (triadic) interval. We also use the notation $I_{-\infty}(t) = \mathbb{R}$. It is easy to have

$$I_n(t) = \left[t_{(n)} - \frac{1}{2 \cdot 3^n}, t_{(n)} + \frac{1}{2 \cdot 3^n}\right) := \left[\frac{2k-1}{2 \cdot 3^n}, \frac{2k+1}{2 \cdot 3^n}\right),$$

where

$$t_{(n)} = \sum_{l=-\infty}^{n} t_l / 3^l$$

and

$$k = \sum_{l=-\infty}^{n} t_l 3^{n-l}. \tag{2.1}$$

The Lebesgue measure of an interval: $\operatorname{mes}(I_n(t)) = 3^{-n}$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$.

Introduce the generalized Walsh function in the following way. Let $x, y \in \mathbb{R}$, and

$$\omega(x, y) := r^{\sum_{k=-\infty}^{\infty} x_k y_{-k-1}}, \quad r = \exp\left(2\pi\imath\frac{1}{3}\right), \imath = \sqrt{-1}.$$

Some properties of the Walsh function:

$$|\omega| = 1, \quad \omega(x, y) = \omega(y, x), \quad \omega(x \oplus z, y) = \omega(x, y)\omega(z, y)$$

for $x, y, z \in \mathbb{R}$, and $x \oplus z$ is triadic irrational.

The Walsh-Fourier transform of an $f \in L_1(\mathbb{R})$ is defined by

$$\widehat{f}(y) := \frac{1}{9} \int_{\mathbb{R}} f(x) \overline{\omega}(x,y)\, dx \quad (y \in \mathbb{R}).$$

3. Inversion of the Walsh-Fourier transform

For each $f \in L_1(\mathbb{R})$ and $t > 0$ define the Walsh-Dirichlet integral by

$$S_t(x; f) := \int_{-t}^{t} \widehat{f}(y) \omega(x,y)\, dy.$$

By Fubin's theorem it is evident that

$$S_t(x; f) = \int_{-t}^{t} \frac{1}{9} \left(\int_{\mathbb{R}} f(u) \overline{\omega}(u,y)\, du \right) \omega(x,y)\, dy$$

$$= \frac{1}{9} \int_{\mathbb{R}} \left(\int_{-t}^{t} \overline{\omega}(u,y) \omega(x,y)\, dy \right) f(u)\, du = \frac{1}{9} \int_{\mathbb{R}} \left(\int_{-t}^{t} \omega(u \ominus x, y)\, dy \right) f(u)\, du$$

$$= \frac{1}{9} \int_{\mathbb{R}} f(u) D_t(u \ominus x)\, du,$$

where

$$D_t(x) = \int_{-t}^{t} \omega(x,y)\, dy$$

for $t \in \mathbb{R}_+$ and $f \in L_1(\mathbb{R})$.

Theorem 3.1. *Let $N \in \mathbb{Z}$. Then*

$$D_{\frac{3^N}{2}}(x) = 3^N \mathbb{I}_{I_{N-2}(0)}(x),$$

where $\mathbb{I}_E(x)$ is the characteristic function of the set E.

Proof. Let $x \in I_{N-2}(0)$ and $y \in \left[-\frac{3^N}{2}, \frac{3^N}{2}\right)$. Then $x_k = 0, k \leqslant N-2$ and $y_k = 0$, $k \leqslant -N$. Hence,

$$\omega(x,y) = e^{\frac{2}{3}\pi i \sum_{k=-\infty}^{\infty} x_k y_{-k-1}} = 1. \tag{3.1}$$

Now, we suppose that $x \notin I_{N-2}(0)$. Then there exists $l \in \mathbb{Z}$ such that $l \leqslant N-2$ and $x_l \in \{-1, 1\}$. We can write

$$
\begin{aligned}
& D_{\frac{3^N}{2}}(x) \\
= & \int_{-\frac{3^N}{2}}^{\frac{3^N}{2}} \omega(x,y)\,dy \\
= & \sum_{\substack{y_m \in \{-1,0,1\},\\ m \in \{-N+1,\dots,-l-2\}}} \int_{I_{-l-1}(0,\dots,0,y_{-N+1},\dots,y_{-l-2},-1)} e^{-\frac{2}{3}\pi i x_l} e^{\frac{2}{3}\pi i \sum_{k=-\infty,k\neq l}^{N-2} x_k y_{-k-1}} dy \\
& + \sum_{\substack{y_m \in \{-1,0,1\},\\ m \in \{-N+1,\dots,-l-2\}}} \int_{I_{-l-1}(0,\dots,0,y_{-N+1},\dots,y_{-l-2},0)} e^{\frac{2}{3}\pi i x_l \cdot 0} e^{\frac{2}{3}\pi i \sum_{k=-\infty,k\neq l}^{N-2} x_k y_{-k-1}} dy \\
& + \sum_{\substack{y_m \in \{-1,0,1\},\\ m \in \{-N+1,\dots,-l-2\}}} \int_{I_{-l-1}(0,\dots,0,y_{-N+1},\dots,y_{-l-2},1)} e^{\frac{2}{3}\pi i x_l} e^{\frac{2}{3}\pi i \sum_{k=-\infty,k\neq l}^{N-2} x_k y_{-k-1}} dy \\
= & \sum_{\substack{y_m \in \{-1,0,1\},\\ m \in \{-N+1,\dots,-l-2\}}} \int_{I_{-l-1}(0,\dots,0,y_{-N+1},\dots,y_{-l-2},-1)} \left(e^{-\frac{2}{3}\pi i x_l} + e^{\frac{2}{3}\pi i x_l \cdot 0} + e^{\frac{2}{3}\pi i x_l}\right) \\
& \times e^{\frac{2}{3}\pi i \sum_{k=-\infty,k\neq l}^{N-2} x_k y_{-k-1}} dy
\end{aligned}
$$

Since

$$
e^{-\frac{2}{3}\pi i x_l} + e^{\frac{2}{3}\pi i x_l 0} + e^{\frac{2}{3}\pi i x_l} = 0, x_l \in \{-1, 1\}
$$

we obtain that

$$
D_{\frac{3^N}{2}}(x) = 0, x \neq I_{N-2}(0). \tag{3.2}
$$

Combining (3.1) and (3.2) we complete the proof of Theorem 3.1. □

Now, we prove some inversion result for the Walsh-Fourier transform.

Theorem 3.2. *Let $f \in L_1(\mathbb{R})$ be W-continuous on $\mathbb{R}$. If $\widehat{f} \in L_1(\mathbb{R})$ then*

$$
f(y) = \int_{\mathbb{R}} \widehat{f}(x)\,\omega(x,y)\,dx.
$$

Proof. We can write

$$
\begin{aligned}
& \int_{\mathbb{R}} \widehat{f}(x)\,\omega(x,y)\,dx \\
= & \int_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \widehat{f}(x)\,\omega(x,y)\,dx + \int_{\mathbb{R}\setminus\left[-\frac{3^n}{2}, \frac{3^n}{2}\right]} \widehat{f}(x)\,\omega(x,y)\,dx := I + II.
\end{aligned}
\tag{3.3}
$$

Since $\widehat{f}$ is integrable for II we get

$$|II| \leqslant \int\limits_{R\setminus\left[-\frac{3^n}{2},\frac{3^n}{2}\right]} \left|\widehat{f}(x)\right| dx \to 0 \quad (n\to\infty). \tag{3.4}$$

We can write

$$\int\limits_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \widehat{f}(x)\,\omega(x,y)\,dx - f(y) = \frac{1}{9}\int\limits_{\mathbb{R}} f(y\oplus u)\,D_{\frac{3^n}{2}}(u)\,du - f(y)$$

$$= 3^{n-2}\int\limits_{I_{n-2}(0)} \left[f(y\oplus u)-f(y)\right]du.$$

Hence,

$$\left|\int\limits_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \widehat{f}(x)\,\omega(x,y)\,dx - f(y)\right| \leqslant 3^{n-2}\int\limits_{I_{n-2}(0)} \left|f(y\oplus u)-f(y)\right|du.$$

Lrt $\varepsilon>0$, fix $y\in\mathbb{R}$, and choose an integer $n>0$ such that

$$|f(y\ominus u)-f(y)|<\varepsilon$$

for all $y\in\mathbb{R}$ which satisfy $u\in I_{n-2}(0)$. Then we obtain

$$\left|\int\limits_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \widehat{f}(x)\,\omega(x,y)\,dx - f(y)\right| < \varepsilon. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5) we complete the proof of Theorem 3.2. □

4. Generalized Walsh function

Theorem 4.1. *Let $n\in\mathbb{Z}$. Then the system*

$$\left\{3^{n/2}\omega\left(x,3^{n+2}j\right), j\in\mathbb{Z}\right\}$$

is orthonormal and complete in $L_2(I_n(0))$.

Proof. Proof of the orthonormality. For the sake of brevity we prove Theorem 4.1 for $L_2(I_{-n}(0))$ instead of $L_2(I_n(0))$. That is, we discuss the system $\{3^{-n/2}\omega\left(x,3^{-n+2}j\right), j\in\mathbb{Z}\}$. Recall that $I_{-n}(0)=[-3^n/2,3^n/2)$. Since to see the normality it is trivial, then we can suppose that $j\neq k$, $j,k\in\mathbb{Z}$. We are to prove

$$\int_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \omega(x,3^{-n+2}j)\bar{\omega}(x,3^{-n+2}k)dx=0. \tag{4.1}$$

We can write

$$\int_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \omega(x,3^{-n+2}j)\bar{\omega}(x,3^{-n+2}k)dx = \int_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \omega(x,3^{-n+2}\left(j\ominus k\right))dx$$
$$= D_{\frac{3^n}{2}}\left(3^{-n+2}\left(j\ominus k\right)\right).$$

It is easy to see that$3^{-n+2}\left(j\ominus k\right)\notin I_{n-2}\left(0\right) \quad (j\neq k)$. Then, from Theorem 3.1 we prove (4.1). The proof of the orthonormality is complete. Completeness is discused later. □

Define the Dirichlet kernel functions with respect to the system $(3^{n/2}\omega(x,3^{n+1}j)$, $j\in\mathbb{Z})$ as:

$$\mathbf{D}_N(x) := \sum_{\{j\in\mathbb{Z}:|j|<N\}} 3^n\omega(x,3^{n+2}j) \quad N\in\mathbb{N}, n\in\mathbb{Z}, x\in\mathbb{R}.$$

We prove a formula for the Dirichlet kernels $\mathbf{D}_{\frac{3^N+1}{2}}$.

Theorem 4.2. *Let $x\in I_n\left(0\right)$ and $N\in\mathbb{N}$. Then*

$$\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right) = \begin{cases} 3^{n+N}, & \text{if } x\in I_{n+N}\left(0\right) \\ 0, & \text{if } x\notin I_{n+N}\left(0\right) \end{cases}.$$

Proof. Integrate the function $|\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2$ on the interval $I_n\left(0\right)$. By the help of Theorem 4.1, that is, by orthonormality we have:

$$\int_{I_n(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx = \int_{I_n(0)} \left|\sum_{\left\{j\in\mathbb{Z}:|j|<\frac{3^N+1}{2}\right\}} 3^n\omega(x,3^{n+2}j)\right|^2 dx$$
$$= \sum_{\left\{j,k\in\mathbb{Z}:|k|,|j|<\frac{3^N+1}{2}\right\}} 3^{2n}\int_{I_n(0)} \omega(x,3^{n+2}j)\bar{\omega}(x,3^{n+2}k)dx$$
$$= \sum_{\left\{j\in\mathbb{Z}:|j|<\frac{3^N+1}{2}\right\}} 3^{2n}\int_{I_n(0)} 1dx = 3^{n+N}.$$

It is easy to see that $\omega(x,3^{n+2}j)=1, x\in I_{n+N}\left(0\right), |j|<\frac{3^N+1}{2}$. On the other hand, this also gives:

$$3^{n+N} = \int_{I_n(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx = \int_{I_{n+N}(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx$$
$$+\int_{I_n(0)\setminus I_{n+N}(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx = 3^{n+N} + \int_{I_n(0)\setminus I_{n+N}(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx.$$

This means:

$$\int_{I_n(0)\setminus I_{n+N}(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx = 0.$$

Consequently, the function $\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)$ equals with zero for almost every x on the set $I_n\left(0\right)\setminus I_{n+N}\left(0\right)$. Since the Walsh-like function $\omega(x,3^{n+2}j)$ is continuous, then so does the Dirichlet kernel $\mathbf{D}_{\frac{3^N+1}{2}}(x)$. That is, this function is the constant zero function on the set $I_n(0) \setminus I_{n+N}(0)$. $\square$

Define the Fourier coefficients of the integrable function $f : I_n(0) \to \mathbb{C}$ as

$$\hat{f}(j) := \int_{I_n(0)} f(x) 3^{n/2} \bar{\omega}(x, 3^{n+2} j) dx,$$

where $j \in \mathbb{Z}$. It is easy to have for the partial sums of the Fourier series

$$\begin{aligned} S_N f(y) \quad : \quad &= \sum_{\{j \in \mathbb{Z} : |j| < N\}} \hat{f}(j) 2^{n/2} \omega(y, 3^{n+2} j) \\ &= \int_{I_n(0)} f(x) \mathbf{D}_N(y \ominus x) dx. \end{aligned}$$

Consequently,

$$S_{\frac{3^N+1}{2}} f(y) = 3^{n+N} \int_{I_{n+N}(y)} f(x) dx.$$

By this equality in the standard way one can prove that the system $(\omega(x, 3^{n+2} j), j \in \mathbb{Z})$ is complete in the Banach space of the integrable functions on the interval $I_n(0)$. It is also of interest, that the Dirichlet kernel are integer valued functions. The reason of this fact is that if $\omega(x, 3^{n+2} j)$ occurs as an addend in the kernel function, then so does its conjugate $\omega(x, -3^{n+2} j)$. The sum of these two things is 2 $(1 + 1 = 2)$ or -1 $(r + \bar{r} = -1)$.

In the sequel we discuss the uniform convergence of these partial sums of the Fourier series of continuous functions. Denote by the (triadic) modulus of continuity of the function $f : I_n(0) \to \mathbb{C}$ by

$$w(I_n(0), N, f) := \sup_{h \in I_{n+N}, x \in I_n(0)} |f(x) - f(x \oplus h)| \quad (N \in \mathbb{N}).$$

If it does not cause any misunderstood, then we write $w(N, f)$ simply. This is a monotone decreasing nonnegative sequence. It is not difficult to prove, that a function f on $I_n(0)$ is continuous if and only if it modulus of continuity converges to zero. We have

$$\begin{aligned} \left| S_{\frac{3^N+1}{2}} f(y) - f(y) \right| &\leqslant 3^{n+N} \int_{I_{n+N}(y)} |f(x) - f(y)| \, dx \\ \leqslant \quad & \left| 3^{n+N} \int_{I_{n+N}(0)} f(y \oplus h) - f(y) dh \right| \leqslant w(n+N, f). \end{aligned}$$

Remark 4.1. It seems also to be very interesting to discuss some other materials with respect to this system, and harmonic analysis. Dirichlet kernels $\mathbf{D}_N$, the norm (and pointwise) convergence of the partial sums S_N, the Fejér kernels and means. We suppose that there are many similarities with the ordinary Walsh system, since the function $\omega(x, 2^{n+1} j) + \omega(x, -2^{n+1} j)$ can take the values $+2$ and -1.

5. The Walsh-Fourier transform in $L_p\,(1<p\leqslant 2)$

It this section we obtain that in cases when $f\in L_p(\mathbb{R})\,(1<p\leqslant 2)$ the Walsh-Fourier transform is as a limit of truncated Walsh-Fourier transforms.

Theorem 5.1. *Let $f\in L_p(\mathbb{R})\,(1<p\leqslant 2)$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\widehat{f}(x,a):=\int_{-a}^{a} f(y)\,\omega(x,y)\,dy$$

converges in L_p norm. Moreover,

$$\int_{\mathbb{R}}\left|\int_{\mathbb{R}} f(y)\,\omega(x,y)\,dy\right|^q dx\leqslant 9\left(\int_{\mathbb{R}}|f(x)|^p\,dx\right)^{\frac{1}{p-1}}.$$

Proof. Set

$$\Delta_k^{(n)}:=\left[\frac{2k-1}{2\cdot 3^n},\frac{2k+1}{2\cdot 3^n}\right),\quad \alpha_k:=\int_{\Delta_k^{(n-2)}} f(u)\,du,$$

$$\Phi_m(x):=\sum_{|k|<m}\alpha_k\omega\left(x,3^{-n+2}k\right),$$

$$\Phi(x):=3^{-n/2}\Phi_m(x)=\sum_{|k|<m}\alpha_k 3^{-n/2}\omega\left(x,3^{-n+2}k\right),$$

where

$$m:=\left[a\cdot 3^{n-2}\right],a>0.$$

Applying Riesz's inequality [5] we can write

$$\left(\int_{-\frac{3^n}{2}}^{\frac{3^n}{2}}|\Phi(x)|^q\,dx\right)^{1/q}\leqslant M^{\frac{2}{p}-1}\left(\sum_{|k|<m}|\alpha_k|^p\right)^{1/p},$$

where $M:=3^{-n/2}$. Hence,

$$\int_{-\frac{3^n}{2}}^{\frac{3^n}{2}}|\Phi_m(x)|^q\,dx\quad\leqslant\quad 9\left(\sum_{|k|<m}\int_{\Delta_k^{(n-2)}}|f(u)|^p\,du\right)^{\frac{1}{p-1}}\leqslant 9\left(\int_{-a}^{a}|f(u)|^p\,du\right)^{\frac{1}{p-1}}.$$

For any fixed $A<\frac{3^n}{2}$ we obtain

$$\int_{-A}^{A}|\Phi_m(x)|^q\,dx\leqslant 9\left(\int_{-a}^{a}|f(u)|^p\,du\right)^{\frac{1}{p-1}}. \tag{5.1}$$

It is easy to see that

$$\omega(x,y)=\omega\left(x,3^{-n+2}k\right)$$

if $x \in \left[-\frac{3^n}{2}, \frac{3^n}{2}\right) = I_{-n}(0)$ and $y \in \Delta_k^{(n-2)} := I_{n-2}(t)$. Indeed,

$$3^{-n+2}k = \sum_{l=-\infty}^{n-2} \frac{t_l}{3^l}, x = \sum_{k=-n+1}^{\infty} \frac{x_k}{3^k}.$$

Then

$$\begin{aligned} \omega\left(x, 3^{-n+2}k\right) &= e^{\frac{2}{3}\pi i(x_{-n+1}t_{n-2}+x_{-n+2}t_{n-3}+\cdots)} \\ &= e^{\frac{2}{3}\pi i(x_{-n+1}y_{n-2}+x_{-n+2}y_{n-3}+\cdots)} = \omega\left(x, y\right). \end{aligned}$$

Hence

$$\Phi_m(x) = \sum_{|k|<m} \left(\int_{\Delta_k^{(n-2)}} f(u)\, du \right) \omega\left(x, 3^{-n+2}k\right)$$

$$= \sum_{|k|<m} \int_{\Delta_k^{(n-2)}} f(u)\, \omega(x,u)\, du = \int_{(-2m-3)/(2\cdot 3^{n-2})}^{(2m-1)/\left(2\cdot 3^{n-2}\right)} f(u)\, \omega(x,u)\, du,$$

$$\left| \Phi_m(x) - \int_{-a}^{a} f(u)\, \omega(x,u)\, du \right|$$

$$\leqslant \int_{-a}^{(-2m-3)/\left(2\cdot 3^{n-2}\right)} |f(u)|\, du + \int_{(2m-1)/(2\cdot 3^{n-2})}^{a} |f(u)|\, du \to 0$$

as $n \to \infty$. Then from (5.1) we obtain

$$\int_{-A}^{A} \left| \int_{-a}^{a} f(u)\, \omega(x,u)\, du \right|^q dx \leqslant 9 \left(\int_{-a}^{a} |f(u)|^p\, du \right)^{\frac{1}{p-1}}.$$

Consequently, when $A \to \infty$ we have

$$\int_{-\infty}^{\infty} \left| \int_{-a}^{a} f(u)\, \omega(x,u)\, du \right|^q dx \leqslant 9 \left(\int_{-a}^{a} |f(u)|^p\, du \right)^{\frac{1}{p-1}}. \tag{5.2}$$

Set

$$f(x,a) := f(x)\, \mathbb{I}_{(-\infty,-a]\cup[a,\infty)}(x).$$

For $b > a$ we have

$$\int_{\mathbb{R}} \left| \int_{-b}^{b} f(y,a)\, \omega(x,y)\, dy \right|^q dx \leqslant 9 \left(\int_{-b}^{b} |f(y,a)|^p\, dy \right)^{\frac{1}{p-1}}$$

$$= 9 \left(\int_{[-b,b]\setminus[-a,a]} |f(y)|^p\, dy \right)^{\frac{1}{p-1}} \to 0$$

as $a, b \to \infty$. On the other hand,

$$\begin{aligned}
&\int\limits_{\mathbb{R}} \left| \int\limits_{-b}^{b} f(y,a)\,\omega(x,y)\,dy \right|^q dx \\
= &\int\limits_{\mathbb{R}} \left| \int\limits_{-b}^{b} f(y)\,\omega(x,y)\,dy - \int\limits_{-a}^{a} f(y)\,\omega(x,y)\,dy \right|^q dx \\
= &\int\limits_{\mathbb{R}} \left| \widetilde{f}(x,b) - \widetilde{f}(x,a) \right|^q dx \to 0
\end{aligned}$$

as $a, b \to \infty$. Hence, there exists a function $\widetilde{f} \in L_p(\mathbb{R})$ such that

$$\lim_{a\to\infty} \left\| \widetilde{f}(\cdot,a) - \widetilde{f}(\cdot) \right\|_p = 0. \tag{5.3}$$

Since (see (5.2))

$$\int\limits_{-\infty}^{\infty} \left| \widetilde{f}(x,b) \right|^q dx \leqslant 9 \left(\int\limits_{-b}^{b} |f(u)|^p\,du \right)^{\frac{1}{p-1}}$$

from (5.3) we conclude that

$$\int\limits_{-\infty}^{\infty} \left| \int\limits_{-\infty}^{\infty} f(y)\,\omega(x,y) \right|^q dx \leqslant 9 \left(\int\limits_{-\infty}^{\infty} |f(u)|^p\,du \right)^{\frac{1}{p-1}}.$$

Theorem 5.1 is proved. □

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О РЯДАХ ФУРЬЕ, ПОЧТИ УНИВЕРСАЛЬНЫХ В КЛАССЕ ИЗМЕРИМЫХ ФУНКЦИЙ

М. Г. Григорян

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Ереванский государственный университет[1]
E-mail: *gmarting@ysu.am*

Аннотация. В работе построен такой универсальный тригонометрический ряд, что после умножения членов этого ряда на некоторую последовательность знаков $\{\delta_k = \pm 1\}_{k=0}^{\infty}$ его можно превратить в ряд Фурье некоторой интегрируемой функции.

MSC2020 number: 42C10; 43A15.

Ключевые слова: универсальный тригонометрический ряд; ряд Фурье; сходимость.

Существование функций и рядов, универсальных в том или ином смысле в различных классах функций, изучалось многими математиками, и публикации по этой тематике регулярно появляются в математической литературе. Понятие универсального ряда (как по классическим, так и по общим ортонормальным системам) восходит к работам Меньшова и Талаляна. Наиболее общие результаты были получены ими и их учениками.

Первой работой, где построены универсальные в обычном смысле тригонометрические ряды в классе всех измеримых функций в смысле сходимости почти всюду является работа [1] Меньшова.

Ряды по любой ортонормированной полной системе, универсальные в классе всех измеримых функций в смысле сходимости почти всюду, были построены в работе [2] Талаляном.

В [3] Гроссе - Эрдман доказал существование универсального ряда Тейлора в классе всех непрерывных на $[-1, 1]$ функций $f(x)$ с $f(0) = 0$.

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Замечание 1. Нетрудно видеть, что из известной теоремы Колмогорова [4] (ряд Фурье каждой интегрируемой функции по тригонометрической системе сходится в $L^p[-\pi,\pi]$, $p \in (0,1)$) следует, что не существует функции $U \in L^1[-\pi,\pi]$ ряд Фурье которой по тригонометрической системе (а также по системе Уолша) был бы универсальным в классе всех измеримых функций.

Значит в классе измеримых функций не существует универсального ряда Фурье (по тригонометрической системе), но тем не менее можно построить универсальный тригонометрический ряд $\sum_{k=1}^{\infty} \alpha_k \cos kx + \beta_k \sin kx$ такой, что после выбора подходящих знаков $\{\delta_k = \pm 1\}_{k=0}^{\infty}$ для его коэффициентов можно достичь того, что $\sum_{k=1}^{\infty} \delta_k(\alpha_k \cos kx + \beta_k \sin kx)$ уже будет рядом Фурье некоторой интегрируемой функции (см. Теорему 1 и определение 3).

Возникает следующий вопрос ответ на который нам не известен.

Вопрос 1. Существует ли ограниченная ортонормированная система $\{\varphi_n(x)\}_{k=0}^{\infty}$ такая, что в классе измеримых функций можно было бы построить универсальный ряд Фурье по системе $\{\varphi_n(x)\}_{k=0}^{\infty}$?

Отметим, что в работах [6] – [13] были получены некоторые результаты, связанные с существованием и описанием структуры функций, ряды Фурье которых (по системе Уолша либо по тригонометрической системе) универсальны в том или ином смысле в различных функциональных классах.

Пусть $|E|$-мера Лебега измеримого множества $E \subseteq [-\pi,\pi]$ и $\mathbf{N}-$ совокупность натуральных чисел.

Пусть $C(E)-$класс непрерывных на $E \subseteq [0,1]$ функций и $M[-\pi,\pi]$ совокупность (не обязательно конечных) измеримых функций. Под сходимостью в $M[-\pi,\pi]$ мы будем подразумевать сходимость почти всюду, а под сходимостью в $C(E)$ -равномерную сходимость.

Пусть $c_k(U) := \int_a^b U(x)\varphi_k(x)dx$ – коэффициенты Фурье по заданной на $[a,b]$ ортонормированной системе $\{\varphi_k(x)\}_{k=0}^{\infty}$ функции $U \in L^1[a,b]$ и пусть $b_k(U) := \frac{1}{\pi}\int_{-\pi}^{\pi} U(x)\sin kxdx$, $a_k(U) := \frac{1}{\pi}\int_{-\pi}^{\pi} U(x)\cos kxdx$, $k = 0,1,2...$ - последовательность коэффициентов Фурье по тригонометрической системе.

Спектр ряда $\sum := \sum_{k=0}^{\infty} c_k\varphi_k(x)$ будем обозначать через

$$\Lambda = \Lambda(\sum) := spec\left(\sum\right) = \{k \in \mathbf{N} \cup \{0\}; \quad c_k \neq 0\},$$

а через $\#(\omega)-$ число точек конечного множества $\omega \subset \mathbf{N}$.

Пусть метрическое пространство $S-$ какое-нибудь из пространств $M[a,b]$, $L^p[a,b]$, $p \geq 0, C(E)$ и пусть $f_k \in S$.

Определение 1. Пусть $\Omega \subset \Lambda \subseteq \mathbf{N} \cup \{0\}$

$$\rho(\Omega)_\Lambda := \overline{\lim_{m\to\infty}} \frac{\#(\Omega \cap [0,n))}{\#(\Lambda \cap [0,n))} \tag{1}$$

$\rho(\Omega)_\Lambda-$ называется плотностью подмножества Ω относительно множества Λ.

Определение 2. Ряд

$$\sum_{k=0}^{\infty} f_k(x) \tag{2}$$

называется универсальным в S, если для каждой функции $f \in S$ существует последовательность возрастающих натуральных чисел n_k такая, что последовательность частичных сумм ряда (2) с номерами n_k сходится к $f(x)$ в S.

Определение 3. Универсальный в S ряд

$$\sum := \sum_{k=0}^{\infty} c_k \varphi_k(x)$$

называется

а) условно универсальным рядом Фурье, если после умножения коэффициентов этого ряда на последовательность знаков $\{\delta_k = \pm 1\}_{k=0}^{\infty}$ его можно превратить в ряд Фурье некоторой интегрируемой функции,

б) почти универсальным рядом Фурье по ортонормированной системе $\{\varphi_k(x)\}_{k=0}^{\infty}$ в S, если существует последовательность знаков $\{\delta_k = \pm 1\}_{k=0}^{\infty}$ с $\rho(\Omega)_\Lambda = 1$ (здесь $\Omega = \{k \in \Lambda := spec(\sum);\ \ \delta_k = 1\}$) такая, что ряд $\sum_{k=0}^{\infty} \delta_k c_k \varphi_k(x)$ был бы рядом Фурье некоторой интегрируемой функции.

Теорема 1. *Существуют тригонометрический ряд*

$$\frac{\alpha_0}{2} + \sum_{k=0}^{\infty} (\alpha_k \cos kx + \beta_k \sin kx),$$

который является условно универсальным рядом Фурье в $M[-\pi,\pi]$.

Замечание 1. Теорема 1 окончательна в следующем смысле: нетрудно видеть, что из известной теоремы Колмогорова [4] (ряд Фурье каждой интегрируемой функции по тригонометрической системе сходится по мере на $[-\pi,\pi]$) следует, что не существует функции $U \in L^1[-\pi,\pi\ ]$ ряд Фурье которой по тригонометрической системе был бы универсальным в классе $M[-\pi,\pi\ ]$ (даже в случае, сходимости по мере).

Теорема 1 следует из следующей теоремы.

Теорема 2. *Существуют тригонометрический ряд*

$$\frac{\alpha_0}{2} + \sum_{k=0}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx) \tag{3}$$

и совокупность замкнутых множеств $\{ F_n\}_{n=1}^{\infty}$ *с* $F_1 \subset ... \subset F_n \subset F_{n+1} \subset ... \subset [-\pi, \pi]$ *и* $\lim_{n\to\infty} |F_n| = 2\pi$, *такие, что*

1. ряд (3) *универсален во всех* $C(F_k)$, $k \geq 1$, *более того, для каждой функции* $f \in C[-\pi, \pi]$ *найдется подпоследовательность натуральных чисел* $\{N_j\} \nearrow \infty$ *такая, что равномерно на каждом* F_k, $k \geq 1$,

2. ряд (3) *является* ***почти универсальным рядом Фурье*** *в* $M[-\pi, \pi]$.

Замечание 2. Нетрудно видеть, что Теорема 3 окончательна в следующем смысле:

1. в $C[-\pi, \pi\ ]$ не существует **условно универсального ряда Фурье** по тригонометрической системе

2. не существует функции $U \in L^1[-\pi, \pi\ ]$, ряд Фурье которой по тригонометрической системе был бы универсальным во всех $C(F_n)$, $n = 1, 2, ..$

Замечание 3. Теоремы 1 и 2 верны и для системы Уолша $\{W_k(x)\}_{k=0}^{\infty}$: нетрудно убедится, что повторяя рассуждения, приведенные при доказательстве теоремы 2 и вместо ниже сформулированной леммы 1 применяя лемму 3 работы [6] для системы Уолша можно построить ряд по системе Уолша вида

$$\sum_{k=0}^{\infty} d_k W_k(x), \quad 0 < |d_{k+1}| < |d_k|, \quad k = 0, 1, 2, ...,$$

который является **почти универсальным рядом Фурье** по системе Уолша в $M[0, 1]$ (во всех $C(F_n)$, $F_n \subset F_{n+1} \subset [0, 1]$, $n = 1, 2, .$ и $\lim_{n\to\infty} |F_n| = 1$).

Иинтересно было бы выяснить ответ на следующий вопрос.

Вопрос 2. Существует ли **почти универсальный ряд Фурье** по тригонометрической системе с по модулю монотонно убывающими коэффициентами?

Замечание 4. Метод доказательства Теоремы 2 (см. также доказательство теоремы 4 работы [12]) позволяет получить новый подход для построения универсальных рядов: любую измеримую, почти всюду конечную функцию путем изменения ее значений на некотором множестве сколь угодно малой меры можно превратить в такую функцию, что после выбора соответствующих знаков для членов ряда Фурье (как по тригонометрической системе так и по системе Уолша) измененной функции можно достичь того, что полученный ряд уже будет универсальным рядом в $M[0, 1]$.

При доказательстве теоремы 2 воспользуемся следующей леммой, доказанной в работе [9].

Лемма 1. *Пусть даны числа* $N_0 \in N, \varepsilon,\ \theta \in (0,1)$ *и ступенчатая функция* $f(x)$. *Тогда можно найти функцию* $g(x)$, *измеримое замкнутое множество* $E \subset [-\pi, \pi\,]$ *с мерой* $|E| > 2\pi - \theta$ *и полиномы по тригонометрической системе*

$$H(x) = \sum_{k=N_0}^{N} (a_k \cos kx + b_k \sin kx), \quad Q(x) = \sum_{k=N_0}^{N} \delta_k (a_k \cos kx + b_k \sin kx), \ \delta_k = \pm 1,$$

удовлетворяющие следующим условиям:

$$\int_0^{2\pi} |g(x)| dx < 4 \int_{-\pi}^{\pi} |f(x)| dx < \varepsilon, \qquad g(x) = f(x), \quad x \in E \qquad |E| > 2\pi - \theta$$

$$\int_0^{2\pi} |H(x)| dx < \varepsilon, \qquad \int_{-\pi}^{\pi} |f(x) - g(x)| dx < \varepsilon.$$

Из этой леммы вытекает

Лемма 2. *Пусть даны числа* $N_0 \in N, \varepsilon,\ \theta \in (0,1)$ *и тригонометрический полином* $f(x)$. *Тогда можно найти измеримое замкнутое множество* $E \subset [-\pi, \pi\,]$ *с мерой* $|E| > 2\pi - \theta$ *и полиномы по тригонометрической системе*

$$H(x) = \sum_{k=N_0}^{N} a_k \cos kx + b_k \sin kx, \quad Q(x) = \sum_{k=N_0}^{N} \delta_k (a_k \cos kx + b_k \sin kx), \ \delta_k = \pm 1,$$

удовлетворяющие следующим условиям:

$$\int_{-\pi}^{\pi} |H(x)| dx < \varepsilon, \qquad \int_E |f(x) - Q(x)| dx < \varepsilon, \qquad |E| > 2\pi - \theta$$

Доказательство Теоремы 2. Пусть

$$F = \{f_n(x)\}_{n=1}^{\infty} \tag{4}$$

есть последовательность всех полиномов по тригонометрической системе с рациональными коэффициентами. Нетрудно видеть, что последовательно применив лемму 2, по индукции можем найти последовательности замкнутых множеств $\{E_n^{(j)}\}_{j=1}^{\lambda_n}$, и полиномов $\{P_n^{(j)}(x)\}_{j=1}^{\lambda_n}$; $\{Q_n^{(j)}(x)\}_{j=1}^{\lambda_n}$, $n \geq 1$ вида

$$P_n^{(j)}(x) = \sum_{k=l_n^{(j-1)}}^{l_n^{(j)}-1} \left(a_k^{(n,j)} \cos kx + b_k^{(n,j)} \sin kx \right), \qquad l_1^{(0)} = 1, \tag{5}$$

$$Q_n^{(j)}(x) = \sum_{k=l_n^{(j-1)}}^{l_n^{(j)}-1} \delta_k^{(n,j)} (a_k^{(n,j)} \cos kx + b_k^{(n,j)} \sin kx), \quad \delta_k^{(n,j)} = \pm 1, \quad n = 1, 2, . \tag{6}$$

$$1 < l_1^{(1)} = l_2^{(0)} < l_2^{(1)} < l_2^{(2)} < l_{n-1}^{(\lambda_{n-1})} = l_n^{(0)} < l_n^{(1)} < .. < l_n^{(\lambda_n)} = l_{n+1}^{(0)} < l_{n+1}^{(1)} \ldots, \tag{7}$$

которые для всех $1 \leq j \leq \lambda_n,\ \ j \in [1, \lambda_n]$ удовлетворяют условиям:

(8) $$\delta_k^{(n,j)} = \pm 1,\ k \in [l_n^{(j-1)}, l_n^{(j)}),\ 1 \leq j \leq \lambda_n\ , n = 1, 2, .,$$

(9) $$\lambda_n = 2^n l_n^{(1)},\ \ |E_n| > 2\pi - 4^{-2n}, n = 1, 2, .,$$

(10) $$\int_0^1 |P_n^{(j)}(x)|dx < 2^{-3(n+j)},\ 1 \leq j \leq \lambda_n,\ \ n = 1, 2, .,$$

(11) $$\int_{E_n} \left| f_n(x) - \sum_{k=1}^{n} \left(\sum_{j=2}^{\lambda_k} P_k^{(j)}(x) + Q_k^{(1)}(x) \right) \right| dx < \frac{1}{2^{3(n+2)}}, n = 1, 2, ..$$

Положим

(12) $$G_n^{(j)} = \left\{ x \in [0, 1]\, ,\ \ \left| P_n^{(j)}(x) \right| < 2^{-2(n+j)} \right\},$$

(13) $$G_n = \left\{ x \in E_n;\ \ \left| f_n(x) - \sum_{j=1}^{n} \left(\sum_{j=2}^{\lambda_k} P_k^{(j)}(x) + Q_k^{(1)}(x) \right) \right| \leq 2^{-n} \right\}.$$

Из (11) and (13) следует

$$2^{-n}|E_n \backslash G_n| \leq \int_{E_n \backslash G_n} \left| f_n(x) - \sum_{k=1}^{n} \left(\sum_{j=2}^{\lambda_k} P_k^{(j)}(x) + Q_k^{(1)}(x) \right) \right| dx \leq$$

$$\leq \int_{E_n} \left| f_n(x) - \sum_{k=1}^{n} \left(\sum_{j=2}^{\lambda_k} P_k^{(j)}(x) + Q_k^{(1)}(x) \right) \right| dx \leq 2^{-3n-6}.$$

Отсюда и из (9) вытекает

(14) $$|G_n| \geq |E_n| - 2^{-2n-1} \geq 2\pi - 2^{-2n-2}\ .$$

Аналогично в силу (10) и (12) будем иметь

(15) $$\left| G_n^{(j)} \right| \geq 2\pi - 2^{-(n+j)},\ \forall\ j \in [1, \lambda_n]\ .$$

Ясно, что (см (10))

(16) $$\sum_{n=1}^{\infty} \sum_{j=1}^{\lambda_n} \left(\int_{-\pi}^{\pi} \left| P_n^{(j)}(x) \right| dx \right) < \sum_{n=1}^{\infty} 2^{-n}.$$

Определим последовательность замкнутых множеств $F_1,\ \ F_2, ..\ .F_k...$, функцию $U(x)$ и последовательности чисел $\{a_k\}\ \ \ \{b_k\}$ следующим образом

(17) $$F_k = \bigcap_{n=k}^{\infty} G_n \left(\bigcap_{j=1}^{\lambda_n} G_n^{(j)} \right),\ \ k = 1, 2, 3, ...$$

$$(18)\qquad U(x)=1+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\lambda_n}P_n^{(j)}(x)\right)=1+\sum_{k=1}^{\infty}a_k\cos kx+b_k\sin kx,$$

где

$$(19)\qquad a_k=a_k^{(n,j)},\quad b_k=b_k^{(n,j)},\quad k\in[l_n^{(j-1)},l_n^{(j)}),\ 1\le j\le\lambda_n\ ,n=1,2,.$$

Отсюда и из (5), (14) – (19) получим

$$(20)\qquad U(x)\in L^1[-\pi,\pi],\quad F_1\subset F_2\subset ..\ \subset F_k\subset ....\subset[-\pi,\pi\ ],\quad \lim_{k\to\infty}|F_k|=2\pi$$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\left|1+\sum_{k=1}^{l_n^{(n)}}(a_k\cos kx+b_k\sin kx)-U(x)\right|dx=$$

$$=\lim_{n\to\infty}\int_{-\pi}^{\pi}\left|\sum_{m=n+1}^{\infty}\left(\sum_{j=1}^{\lambda_m}P_n^{(j)}(x)\right)\right|dx=0$$

и, следовательно,

$$(21)\qquad a_0(U)=2,\ a_k=a_k(U),\quad b_k=b_k(U),\quad k\ge 1$$

Положим

$$(22)\qquad \delta_k=\begin{cases}1,\ k\in\bigcup\limits_{n=1}^{\infty}[l_n^{(1)},l_n^{(\lambda_n)})\cup\{0\}\\ \delta_k^{(n,j)},\ k\in[l_n^{(0)},l_n^{(1)}),n=1,2,3\end{cases}.$$

$$(23)\qquad \alpha_k=\delta_k a_k(U),\ k=0,1,2...,\quad \beta_k=\delta_k b_k(U),\ k=1,2...,\ ,$$

Докажем, что ряд

$$(24)\qquad \sum:=\frac{\alpha_0}{2}+\sum_{k=1}^{\infty}(\alpha_k\cos kx+\beta_k\sin kx)$$

универсален во всех пространствах $C(F_k)$, $k\ge 1$, следовательно, и в $M[-\pi,\pi]$. Более того докажем, что для каждой функции $f\in C[-\pi,\pi]$, $f(x+2\pi)=f(x)$ существует возрастающая подпоследовательность $N_q\nearrow$ такая, что ряд для каждого $m\in N$

$$(25)\qquad \lim_{q\to\infty}\left\|\frac{\alpha_0}{2}+\sum_{k=1}^{N_q}(\alpha_k\cos kx+\beta_k\sin kx)\ -f(x)\right\|_{C(F_m)}=0.$$

Пусть $f(x)$ произвольная функция из $C[-\pi,\pi]$. Нетрудно видеть, что можно выбрать подпоследовательность $\{f_{n_q}(x)\}_{q=1}^{\infty}$ из последовательности (4) такую, что

$$(26)\qquad \lim_{q\to\infty}\|\ f_{n_q}(x)-(f(x)-1)\ \|_{C[-\pi,\pi]}=0.$$

Обозначая через $N_q = l_{n_q}^{(1)} - 1$ в силу (5), (6), (21) - (24) и (26) для каждого фиксированного $m \in N$ и для всех $q \geq q_0$ $(m_{q_0} \geq m)$ имеем

$$\left\| \frac{\alpha_0}{2} + \sum_{k=1}^{N_q} (\alpha_k \cos kx + \beta_k \sin kx) - f(x) \right\|_{C(F_m)} \leq \| f_{n_q}(x) - (f(x) - 1) \|_{C[-\pi,\pi]} +$$

$$+ \left\| f_{n_q}(x) - \sum_{k=1}^{n_q} \left(\sum_{j=2}^{\lambda_k} P_k^{(j)}(x) + Q_k^{(1)}(x) \right) \right\|_{C(F_m)} .$$

Отсюда и из (13), (17) и (26) будем иметь (25).

Теперь докажем, что универсаленый ряд (24) является почти универсальным рядом Фурье, как в $M[-\pi, \pi]$ так и во всех пространствах $C(F_k)$, $k \geq 1$.

Ясно, что (см.(18), (20) и (22)- (24))

$$(27) \qquad \delta_k \alpha_k = a_k(U),\ k = 0, 1, 2...,\quad \delta_k \beta_k = b_k(U),\ k = 1, 2... .$$

Положим (см. (24))

$$(28) \qquad \Lambda := \Lambda(\sum) = spec(\sum), \qquad \Omega := \{k \in \Lambda,\ \delta_k = 1\}.$$

Принимая во внимание неравенство (см. (9) и (28))

$$\frac{\neq \left(\Omega \bigcap [0, l_n^{(\lambda_n)})\right)}{\neq \left(\Lambda \bigcap [0, l_n^{(\lambda_n)})\right)} \geq \frac{\neq \left(\Lambda \bigcap (0, l_n^{(\lambda_n)})\right) - l_n^{(1)}}{\neq \left(\Lambda \bigcap (0, l_n^{(\lambda_n)})\right)} \geq 1 - \frac{l_n^{(1)}}{\lambda_n} \geq 1 - \frac{1}{2^n}.$$

Откуда вытекает

$$\rho(\Omega)_\Lambda = \overline{\lim_{n\to\infty}} \frac{\neq (\Omega \bigcap [0, n))}{\neq (\Lambda \bigcap [0, n))} = 1.$$

Теорема 2 доказана.

Abstract. In this work, an universal trigonometric series is constructed such that, after multiplying the terms of this series by some sequence of signs, it can be turned into the Fourier series of some integrable function.

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SHARP WEIGHTED ESTIMATES FOR STRONG-SPARSE OPERATORS

G. MNATSAKANYAN

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Institute of Mathematics of Armenian National Academy of Sciences
E-mail: *mnatsakanyan_g@yahoo.com*

Abstract. We prove the sharp weighted-L^2 bounds for the strong-sparse operators introduced in [3]. The main contribution of the paper is the construction of a weight that is a lacunary mixture of dual power weights. This weight helps to prove the sharpness of the trivial upper bound of the operator norm.

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1. Introduction

The theory of weighted inequalities started with the seminal work of Muckenhoupt [10], where he proved that the Hardy-Littlewood maximal operator is bounded on $L^p(w)$, $1 < p < \infty$, for positive measurable $w : \mathbb{R} \to \mathbb{R}$ if and only if

$$[w]_{A_p} := \sup_I \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\right)^{p-1} < \infty, \tag{1.1}$$

where the supremum is taken over all intervals and $|I|$ denotes the Lebesgue measure of the interval. If (1.1) holds, then w is said to be in the Muckenhoupt class A_p and the quantity $[w]_{A_p}$ is called its A_p characteristic. Later, Buckley [11] obtained the sharp dependence of the norm of the maximal operator on the A_p characteristic. Namely, he proved that

$$\|M\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\frac{1}{p}}, \tag{1.2}$$

$$\|M\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}, \tag{1.3}$$

and these are sharp in the sense of the theorems below.

The problem of the sharp dependence of the $L^2(w) \to L^2(w)$ norm of the Caldéron-Zygmund operator on the A_2 characteristic of w is known as the A_2-conjecture. It was first proved by Hytönen [7, 6]. A simpler proof was given by Lerner [8, 9] proving that the Caldéron-Zygmund operators can be dominated by the simple sparse operators. Later, it was proved that a number of operators in harmonic analysis admit pointwise or norm domination by the sparse operators

[12, 13, 2, 1, 8, 9]. On the other hand, L^p and weighted-L^p bounds for the sparse operators are fairly easy to obtain[1].

Let us have a family $\mathcal{S}$ of intervals in $\mathbb{R}$ and $0<\gamma<1$. $\mathcal{S}$ is called γ-sparse, or just sparse, if there exists pairwise disjoint subsets $E_A \subset A$, $A \in \mathcal{S}$, such that $|E_A| \geq \gamma |A|$. Let us set for an interval B

$$\langle f\rangle_B := \frac{1}{|B|}\int_B |f|, \quad M_B f := \sup_{A \text{ intervals: } A \supset B} \langle f\rangle_A.$$

For a sparse family $\mathcal{S}$, we define the sparse and the strong-sparse operators as

$$\mathcal{A}_{\mathcal{S}} f(x) := \sum_{A\in\mathcal{S}} \langle f\rangle_A \cdot \mathbf{1}_A(x), \tag{1.4}$$

$$\mathcal{A}^*_{\mathcal{S}} f(x) := \sum_{A\in\mathcal{S}} (M_A f) \cdot \mathbf{1}_A(x), \tag{1.5}$$

respectively. The sharp weighted bound for the sparse operator[1] is as follows

$$\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max(1,\frac{1}{p-1})}. \tag{1.6}$$

The strong-sparse operators were introduced by Karagulyan and the author in [3], where L^p and weak-L^1 estimates are proved in the setting of an abstract measure space with ball-basis. In this paper, we obtain the sharp dependence of the weighted-L^2 norm of the strong-sparse operator on the A_2 characteristic of the weight.

Theorem 1.1. *For an A_2 weight w we have the bound*

$$\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w)\to L^{2,\infty}(w)} \lesssim [w]_{A_2}^{\frac{3}{2}}. \tag{1.7}$$

The inequality is sharp in the following sense: there exist a sparse family $\mathcal{S}$ and a sequence of weights w_α such that

$$[w_\alpha]_{A_2} \to \infty, \text{ as } \alpha \to 0, \tag{1.8}$$

and for any function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(x)/x^{\frac{3}{2}} \to 0$ as $x \to \infty$, we have

$$\frac{\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w_\alpha)\to L^{2,\infty}(w_\alpha)}}{\phi([w_\alpha]_{A_2})} \to \infty, \text{ as } \alpha \to 0. \tag{1.9}$$

Theorem 1.2. *For an A_2 weight w we have the bound*

$$\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}^2. \tag{1.10}$$

The inequality is sharp in the following sense: there exist a sparse family $\mathcal{S}$ and a sequence of weights w_α such that

$$[w_\alpha]_{A_2} \to \infty, \text{ as } \alpha \to 0, \tag{1.11}$$

and for any function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(x)/x^2 \to 0$ as $x \to \infty$, we have

$$\frac{\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w_\alpha)\to L^2(w_\alpha)}}{\phi([w_\alpha]_{A_2})} \to \infty, \text{ as } \alpha \to 0. \tag{1.12}$$

On the other hand, we have the following simple partial improvement for the strong bound. For this theorem we assume that all the intervals in the statement, proof and in the definition of the strong-sparse operator are dyadic.

Theorem 1.3. *Let the sparse family $\mathcal{S}$ be such that for any two $A, B \in \mathcal{S}$ either $A \subset B$ or $B \subset A$. Then, we have*

$$\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w)\to L^2(w)} \lesssim [w]^{\frac{3}{2}}_{A_2}. \tag{1.13}$$

Looking at the definition of the strong-sparse operators, we see that $M_B f \leq Mf(x)$ for any $x \in B$. Thus, $M_B f \leq \langle Mf \rangle_B$ and we obtain

$$\mathcal{A}^*_{\mathcal{S}} f(x) \leq \mathcal{A}_{\mathcal{S}}(Mf). \tag{1.14}$$

Then, one can try to black-box the sharp weighted bounds (1.2), (1.3) and (1.6) for Theorem 1.1 and Theorem 1.2. As it will be shown in Section 2, the weighted weak-L^2 bound for the sparse operator is the same as for the strong one. Thus, Theorem 1.1 will not follow from such a black-box. Instead, we will decompose the operator according to the magnitude of the $M_B f$ for the sparse intervals B, then, we will use the weighted weak bound of the maximal function (1.2). We will do this in Section 2.

As for Theorem 1.2, we see that by black-boxing the above mentioned inequalities we trivially get the upper bound, i.e.

$$\begin{aligned}\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w)\to L^2(w)} &\leq \|\mathcal{A}_{\mathcal{S}} \circ M\|_{L^2(w)\to L^2(w)} \\ &\lesssim \|\mathcal{A}_{\mathcal{S}}\|_{L^2(w)\to L^2(w)} \|M\|_{L^2(w)\to L^2(w)} \lesssim [w]^2_{A_2}.\end{aligned}$$

Thus, the interesting thing about Theorem 1.2 is to obtain the sharpness of this estimate. For that we will construct a weight which is a lacunary mixture of the dual power weights $x^{\alpha-1}$ and $x^{1-\alpha}$. We will do this in Section 3.

In Section 4, we will prove Theorem 1.3.

We say $a \lesssim b$ if there is an absolute constant c, maybe depending on the sparse parameter γ, such that $a \leq c \cdot b$. Furthermore, we say $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

2. The upper bound of Theorem 1.1

2.1. **A well-known property of A_∞ weights.** Following [5, 4], we say that w is an A_∞ weights if

$$[w]_{A_\infty} := \sup_I \frac{1}{w(I)} \int_I M(w\mathbf{1}_I)(x)dx < \infty. \tag{2.1}$$

It is well-known that any A_p weight is also an A_∞ weights and that a reverse Hölder inequality holds for in the latter class. The following theorem with sharp constants is due to Hytönen, Pérez and Rela[4].

Theorem 2.1. *If w is an A_∞ weight and $\epsilon = \frac{1}{4[w]_{A_\infty}}$, then $\langle w^{1+\epsilon}\rangle_I \le 2\left(\langle w\rangle_I\right)^{1+\epsilon}$, for any interval I.*

This implies the following lemma.

Lemma 2.1. *For any cube Q and measurable subset $E \subset Q$, we have*

$$w(E) \le 2w(Q)\left(\frac{|E|}{|Q|}\right)^{c/[w]_{A_\infty}},$$

where c is an absolute constant.

Proof. Let ϵ be as before.

$$\begin{aligned}
\int_E w &\le \left(\int_E w^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \cdot |E|^{\epsilon/(1+\epsilon)} \quad \text{(Hölder)}\\
&\le \langle w^{1+\epsilon}\rangle_Q^{\frac{1}{1+\epsilon}} \cdot |E|^{\epsilon/(1+\epsilon)} \cdot |Q|^{1/(1+\epsilon)}\\
&\le 2\langle w\rangle_Q |E|^{\epsilon/(1+\epsilon)} \cdot |Q|^{1/(1+\epsilon)} \quad \text{(Reverse Hölder)}\\
&= 2w(Q)\left(\frac{|E|}{|Q|}\right)^{c/[w]_{A_\infty}}.
\end{aligned}$$

2.2. **The proof of the weak bound.** The idea is to group $M_B f$'s, $B \in \mathcal{S}$, according to their magnitude and estimate each group applying Lemma 2.1 and the weighted weak bound for the maximal operator (1.2). Denote $\alpha := \frac{1}{[w]_\infty}$, and for $\lambda > 0$ let

$$\begin{aligned}
A_0 &:= \{B \in \mathcal{S} \ : \ M_B f > \alpha\lambda\},\\
A_j &:= \{B \in \mathcal{S} \ : \ 2^{-j+1}\alpha\lambda \ge M_B f > 2^{-j}\alpha\lambda\},
\end{aligned}$$

for $j = 1, 2, \ldots$. Thus, A_j's partition $\mathcal{S}$. We write

$$\begin{aligned}
w\{\mathcal{A}_{\mathcal{S}}^* f > \lambda\} &\le \sum_{j=0}^{\infty} w\Big\{\sum_{B\in A_j} (M_B f)\chi_B > \lambda 2^{-j/2} C\Big\}\\
&\le w\left(\bigcup_{B\in A_0} B\right) + \sum_{j=1}^{\infty} w\Big\{\sum_{B\in A_j} \chi_B > \frac{1}{\alpha} 2^{j/2} C\Big\}\\
&\le w\big\{Mf > \lambda\alpha\big\} + \sum_{j=1}^{\infty} 2w\left(\bigcup_{B\in A_j} B\right) \frac{|\{\sum_{B\in A_j}\chi_B > \frac{1}{\alpha}2^{j/2}C\}|^{c/[w]_\infty}}{|\bigcup_{B\in A_j} B|^{c/[w]_\infty}}\\
&\le w\big\{Mf > \lambda\alpha\big\} + 2\sum_{j=1}^{\infty} w\left(\bigcup_{B\in A_j} B\right) 2^{-2^{j/2}C\alpha\cdot\frac{c}{\alpha}}\\
&\le w\big\{Mf > \lambda\alpha\big\} + 2\sum_{j=1}^{\infty} w\big\{Mf > 2^{-j}\lambda\alpha\big\} 2^{-cC2^{j/2}}\\
&\lesssim \frac{[w]_{A_\infty}^2}{\lambda^2}\|M\|_{L^2\to L^{2,\infty}}^2,
\end{aligned}$$

where the first line is due to the triangle inequality, the third inequality follows from Lemma 2.1 and the fourth one from the fact, that A_j is a sparse collection. It reimains to apply the bound (1.2) to get the upper bound of Theorem 1.1.

2.3. **The lower bound of Theorem 1.1.** Let $w = |x|^{\alpha-1}$ and $\sigma = |x|^{1-\alpha}$ be the dual power weights, $0 < \alpha < 1$. We know, for example from [11], that

$$[w]_{A_2} = [\sigma]_{A_2} \sim \frac{1}{\alpha}. \tag{2.2}$$

Let $\mathcal{S} := \{[0, 2^{-k}) : \text{ for } k \in \mathbb{N}\}$ be a sparse family. Then, we claim

$$\|\mathcal{A}^*_{\mathcal{S}}(\sigma \mathbf{1}_{[0,1)})\|_{L^{2,\infty}(w)} \sim [w]^{3/2}_{A_2} \|\sigma \mathbf{1}_{[0,1)}\|_{L^2(w)}. \tag{2.3}$$

The square of the right-hand side of (2.3) equals $\frac{1}{(2-\alpha)\alpha^3}$. On the other hand,

$$\begin{aligned} &\|\mathcal{A}^*_{\mathcal{S}}(\sigma \mathbf{1}_{[0,1)})\|^2_{L^{2,\infty}(w)} \geq \frac{1}{\alpha^2} w\{\mathcal{A}^*_{\mathcal{S}}(\sigma \mathbf{1}_{[0,1)}) > \frac{1}{\alpha}\} \\ &= \frac{1}{\alpha^2} w\{\sum_{k=1}^{\infty} \mathbf{1}_{[0,2^{-k})} \gtrsim \frac{1}{\alpha}\} = \frac{1}{\alpha^2} w([0, 2^{-\frac{c}{\alpha}})) \sim \frac{2^{-\frac{c}{\alpha}\alpha}}{\alpha^3}. \end{aligned}$$

So the proof of Theorem 1.1 is complete.

3. The lower bound of Theorem 1.2

3.1. **Construction of the weight.** Let $0 < \alpha < 1$ be small enough integer power of 2, i.e. $\alpha = 2^{-a}$ for large enough integer a. Let us define the weight $\sigma : \mathbb{R} \to [0, \infty)$ to be even and

$$\sigma(x) := \begin{cases} \frac{2^{2k(1-\alpha)}}{\alpha}(x - 2^{-(k+1)})^{1-\alpha}, \quad x \in [2^{-(k+1)}, (1+\alpha)2^{-(k+1)}) \text{ for } k \in \mathbb{N} \\ x^{\alpha-1}, \quad x \in [(1+\alpha)2^{-(k+1)}), (1-\alpha)2^{-k}) \text{ for } k \in \mathbb{N} \\ \frac{2^{2k(1-\alpha)}}{\alpha}(2^{-k} - x)^{1-\alpha}, \quad x \in [(1-\alpha)2^{-k}, 2^{-k}) \text{ for } k \in \mathbb{N} \\ x^{\alpha-1}, \quad x \in [\frac{1}{2}, \infty). \end{cases} \tag{3.1}$$

The dual weight to σ is $w(x) := \sigma(x)^{-1}$. We will prove that

$$\sup_I \frac{1}{|I|^2} \Big(\int_I w\Big) \cdot \Big(\int_I \sigma\Big) \sim \frac{1}{\alpha}, \tag{3.2}$$

that is, $\sigma \in A_2$ with $[\sigma]_{A_2} \sim \frac{1}{\alpha}$.

First, we show that (3.2) holds for dyadic intervals. Let us partition all dyadic intervals into three groups.

a. $I=[0,2^{-k})$ for some $k\in\mathbb{N}_0$. Then, we compute

$$\int_{2^{-(k+1)}}^{2^{-k}} w(x)dx = \int_{(1+\alpha)2^{-(k+1)}}^{(1-\alpha)2^{-k}} x^{1-\alpha}dx + \alpha 2^{2k(\alpha-1)} \int_{(1-\alpha)2^{-k}}^{2^{-k}} (2^{-k}-x)^{\alpha-1}dx$$
$$+\alpha 2^{2k(\alpha-1)} \int_{2^{-(k+1)}}^{(1+\alpha)2^{-(k+1)}} (x-2^{-(k+1)})^{\alpha-1}dx$$
$$=\frac{(1-\alpha)^{2-\alpha}2^{-(2-\alpha)k}-(1+\alpha)^{2-\alpha}2^{-(2-\alpha)(k+1)}}{2-\alpha}$$
$$+\alpha 2^{2k(\alpha-1)}\cdot\frac{\alpha^{\alpha}(2^{-k\alpha}+2^{-(k+1)\alpha})}{\alpha}=c(\alpha)2^{-k(2-\alpha)}. \tag{3.3}$$

In the above computations and below $c(\alpha)$ is a constant depending on α absolutely bounded and away from 0. It will be different at each occurence. Next, we have

$$\int_0^{2^{-k}} w(x)dx=\sum_{j=k}^{\infty}\int_{2^{-(j+1)}}^{2^{-j}} w(x)dx=\sum_{j=k}^{\infty}c(\alpha)2^{-j(2-\alpha)}=c(\alpha)\cdot 2^{-k(2-\alpha)}. \tag{3.4}$$

For σ we have

$$\int_{2^{-(k+1)}}^{2^{-k}} \sigma(x)dx = \int_{(1+\alpha)2^{-(k+1)}}^{(1-\alpha)2^{-k}} \sigma(x)dx + \int_{(1-\alpha)2^{-k}}^{2^{-k}} \sigma(x)dx + \int_{2^{-(k+1)}}^{(1+\alpha)2^{-(k+1)}} \sigma(x)dx$$
$$=\frac{(1-\alpha)^{\alpha}2^{-k\alpha}-(1+\alpha)^{\alpha}2^{-(k+1)\alpha}}{\alpha}+\frac{2^{2k(1-\alpha)}}{\alpha}\cdot$$
$$\cdot\left(\int_{2^{-k}(1-\alpha)}^{2^{-k}}(2^{-k}-x)^{1-\alpha}dx+\int_{2^{-(k+1)}}^{(1+\alpha)2^{-(k+1)}}(x-2^{-(k+1)})^{1-\alpha}dx\right)$$
$$=c(\alpha)2^{-k\alpha}+\frac{2^{2k(1-\alpha)}}{\alpha}\cdot\frac{\alpha^{2-\alpha}(2^{-k(2-\alpha)}+2^{-(k+1)(2-\alpha)})}{2-\alpha}$$
$$=c(\alpha)\frac{2^{-k\alpha}}{\alpha}+\alpha 2^{-k\alpha}=c(\alpha)\frac{2^{-k\alpha}}{\alpha}. \tag{3.5}$$

Then, we have

$$\int_0^{2^{-k}}\sigma(x)dx=\sum_{j=k}^{\infty}\int_{2^{-(j+1)}}^{2^{-j}}\sigma(x)dx=\sum_{j=k}^{\infty}c(\alpha)\frac{2^{-j\alpha}}{\alpha}=c(\alpha)\frac{2^{-k\alpha}}{\alpha}. \tag{3.6}$$

Combining the two computations above, we have for (3.2)

$$2^{2k}\Big(\int_0^{2^{-k}} w\Big)\cdot\Big(\int_0^{2^{-k}}\sigma\Big)=c(\alpha)2^{2k}2^{-k(2-\alpha)}\frac{2^{-k\alpha}}{\alpha}\sim\frac{1}{\alpha}. \tag{3.7}$$

b. One of the following holds: for some $k \in \mathbb{N}_0$, $I \subset [2^{-(k+1)}, (1+\alpha)2^{-(k+1)})$, $I \subset [(1+\alpha)2^{-(k+1)}, (1-\alpha)2^{-k})$ or $I \subset [(1-\alpha)2^{-k}, 2^{-k})$. On these intervals, the weights w and σ are just rescaled versions of the power weights. Thus, we immediately have

$$\frac{1}{|I|^2}\Big(\int_I w\Big)\cdot\Big(\int_I \sigma\Big) \lesssim \frac{1}{\alpha}, \tag{3.8}$$

by the A_2 characteristic of the power weights (2.2).

c. $I \subset [2^{-(k+1)}, 2^{-k})$ and either $[(1-\alpha)2^{-k}, 2^{-k}) \subsetneq I$ or $[2^{-(k+1)}, (1+\alpha)2^{-(k+1)}) \subsetneq I$ for some $k \in \mathbb{N}_0$. This is the intermediate case between the above two. The computation for the choice of the last two conditions is identical, so we consider only one of them. Let $|I| = 2^{-m}$ so that $I = [2^{-k} - 2^{-m}, 2^{-k})$ and $k+2 \le m < k+a$, where we recall $\alpha = 2^{-a}$. We start calculating

$$\begin{aligned}
\int_{2^{-k}-2^{-m}}^{2^{-k}} w(x)dx &= \int_{2^{-k}-2^{-m}}^{(1-\alpha)2^{-k}} x^{1-\alpha}dx + \alpha 2^{2k(\alpha-1)} \int_{(1-\alpha)2^{-k}}^{2^{-k}} (2^{-k}-x)^{\alpha-1}dx \\
&= \frac{(1-2^{-a})^{2-\alpha}2^{-(2-\alpha)k} - 2^{-(2-\alpha)k}(1-2^{k-m})^{2-\alpha}}{2-\alpha} \\
&\quad + \alpha\cdot 2^{2k(\alpha-1)} \int_{2^{-k}(1-\alpha)}^{2^{-k}} (2^{-k}-x)^{\alpha-1}dx \\
&= c(\alpha,m)2^{-k(2-\alpha)}\Big((1+\frac{2^{k-m}-2^{-a}}{1-2^{k-m}})^{2-\alpha} - 1\Big) + \alpha^\alpha 2^{-k(2-\alpha)} \\
&= c(\alpha,m)2^{-k(2-\alpha)}2^{k-m} + \alpha^\alpha 2^{-k(2-\alpha)} = c(\alpha,m)2^{-k(2-\alpha)}.
\end{aligned}$$

As before $c(\alpha,m)$ is a positive constant bounded from above and away from 0. For σ we write

$$\begin{aligned}
\int_{2^{-k}-2^{-m}}^{2^{-k}} \sigma(x)dx &= \int_{2^{-k}-2^{-m}}^{(1-\alpha)2^{-k}} \sigma(x)dx + \int_{(1-\alpha)2^{-k}}^{2^{-k}} \sigma(x)dx \\
&= \frac{(1-\alpha)^\alpha 2^{-k\alpha} - 2^{-k\alpha}(1-2^{k-m})^\alpha}{\alpha} \\
&\quad + \frac{2^{2k(1-\alpha)}}{\alpha} \int_{2^{-k}(1-\alpha)}^{2^{-k}} (2^{-k}-x)^{1-\alpha}dx \\
&= c(\alpha,m)2^{-k\alpha}\frac{(1+\frac{2^{k-m}-2^{-a}}{1-2^{k-m}})^\alpha - 1}{\alpha} + \alpha 2^{-k\alpha} \\
&= c(\alpha,m)2^{-k\alpha}\cdot 2^{k-m} + 2^{-a}\cdot 2^{-k\alpha} = c(\alpha,m)2^{-k\alpha+k-m}.
\end{aligned}$$

Here, in the penultimate equality we used the the Taylor expansion

$$(1+x)^{\beta} - 1 \sim \beta x, \text{for } 0 < x < 1. \tag{3.9}$$

Thus, for (3.2) we have

$$\frac{1}{|I|^2}\Big(\int_I w\Big)\cdot\Big(\int_I \sigma\Big) = 2^{2m}\cdot c(\alpha,m)2^{-k\alpha+k-m}\cdot 2^{-k(2-\alpha)} \sim 2^{m-k} \lesssim 2^{a} = \frac{1}{\alpha}.$$

We conclude, that the dyadic A_2 characteristic of w is $\frac{c}{\alpha}$. It is important here, that the supremum is attained at a large number of dyadic intervals and not only on one chain.

We turn to the case of a general interval I. First of all, the arguments for the case **b** are also true for all intervals I due to the A_2 characteristic of power weights. On the other hand, if I can be covered by a dyadic interval of a comparable size, then again (3.8) holds. Otherwise, let k be such that $I \subset [0, 2^{-(k-1)})$, $I \not\subset [0, 2^{-(k+1)})$ and $|I| \lesssim 2^{-k}$. We distinguish two cases.

(i) One of the following holds: $(1+\alpha)2^{-(k+1)} \in I$, $(1-\alpha)2^{-k} \in I$, $(1+\alpha)2^{-k} \in I$, $(1-\alpha)2^{-(k-1)} \in I$. All four cases are similar, so we only consider the second one. For σ we have

$$\int_I \sigma(x)dx \sim 2^{-k(\alpha-1)}|I|. \tag{3.10}$$

As for w we write

$$\int_I w(x)dx \sim \big((1-\alpha)2^{-k} - l(I)\big)2^{-k(1-\alpha)} + \int_{(1-\alpha)2^{-k}}^{r(I)} w(x)dx, \tag{3.11}$$

where $l(I)$ and $r(I)$ are the left and right endpoints of I.

(i.1) If $r(I) < (1-\alpha)2^{-k} + \alpha 2^{-(k+1)}$, then we have

$$\int_I w(x)dx \sim |I|2^{-k(1-\alpha)}, \tag{3.12}$$

and so

$$\frac{1}{|I|^2}\Big(\int_I w\Big)\cdot\Big(\int_I \sigma\Big) \lesssim 1. \tag{3.13}$$

(i.2) If $(1-\alpha)2^{-k} + \alpha 2^{-(k+1)} < r(I)$, then using the computation in (3.3), we have

$$\int_{(1-\alpha)2^{-k}}^{r(I)} w(x)dx \lesssim 2^{-k(2-\alpha)}. \tag{3.14}$$

Hence, we obtain

$$\frac{1}{|I|^2}\Big(\int_I w\Big)\cdot\Big(\int_I \sigma\Big) \lesssim \frac{1}{|I|^2}2^{-k(2-\alpha)}\cdot |I|2^{-k(1-\alpha)} \lesssim \frac{2^{-k}}{|I|} \lesssim \frac{1}{\alpha},$$

where the last step is due to $l(r) < (1-\alpha)2^{-k} < (1-\alpha)2^{-k} + \alpha 2^{-(k+1)} < r(I)$.

(ii) Let us have $(1-\alpha)2^{-k} \notin I$, $(1+\alpha)2^{-k} \notin I$ and $2^{-k} \in I$. Without loss of generality we can assume $r(I) - 2^{-k} \leq 2^{-k} - l(I)$. Then, we have $|I| \sim (2^{-k} - l(I))$. Furthermore,

$$\int_I \sigma(x)dx \sim \int_{l(I)}^{2^{-k}} \sigma(x)dx, \quad \text{and} \quad \int_I w(x)dx \sim \int_{l(I)}^{2^{-k}} w(x)dx.$$

Thus, as w and σ are just power weights on $[(1-\alpha)2^{-k}, 2^k)$, and the estimate (3.2) holds.

3.2. **Construction of the sparse family.** Let us take the following sparse family:

$$\mathcal{S} := \{[2^{-k} - 2^{-j}, 2^{-k}) \ : \ \text{for all } k, j \in \mathbb{N} \text{ and } j \geq a + k\}. \tag{3.15}$$

We also denote $B_{k,j} := [2^{-k} - 2^{-j}, 2^{-k})$. Using (3.6), we have

$$M_{B_{k,j}}(\sigma) \sim 2^k \int_0^{2^{-k}} \sigma(x)dx \sim \frac{2^{k(1-\alpha)}}{\alpha}, \tag{3.16}$$

and the corresponding strong-sparse operator is

$$\mathcal{A}^*_{\mathcal{S}} f(x) := \sum_{k=1}^{\infty} \frac{2^{k(1-\alpha)}}{\alpha} \sum_{j=a+k}^{\infty} \mathbf{1}_{B_{j,k}}(x). \tag{3.17}$$

3.3. **The lower bound.** We claim that

$$\int_0^1 \mathcal{A}^*_{\mathcal{S}}(\sigma)(x)^2 w(x)dx \sim \frac{1}{\alpha^4} \int_0^1 \sigma(x)dx. \tag{3.18}$$

By (3.17), we can write

$$\int_0^1 S^*(\sigma)(x)^2 w(x)dx \sim \sum_{k=1}^{\infty} \frac{2^{2k(1-\alpha)}}{\alpha^2} \int_0^1 \Big(\sum_{j=k+a}^{\infty} \mathbf{1}_{B_{k,j}}(x) \Big)^2 w(x)dx. \tag{3.19}$$

We make a change of variables in the integral and see that it realizes the sharp constant for the regular sparse operator. Putting $y = \frac{x-(1-\alpha)2^{-k}}{2^{-k}\alpha}$, we can write

$$\begin{aligned} \int_0^1 \Big(\sum_{j=k+a}^{\infty} \mathbf{1}_{B_{k,j}}(x) \Big)^2 w(x)dx &= \alpha 2^{\alpha k - 2k} \int_0^1 \Big(\sum_{j=1}^{\infty} \mathbf{1}_{[0,2^{-j})}(y) \Big)^2 y^{\alpha-1} dy \\ &\sim \alpha 2^{\alpha k - 2k} \cdot \frac{1}{\alpha^3} = \frac{2^{\alpha k - 2k}}{\alpha^2}, \end{aligned}$$

where the penultimate estimate is a direct computation. Plugging this into (3.19), we obtain

$$\int_0^1 \mathcal{A}^*_{\mathcal{S}}(\sigma)(x)^2 w(x)dx \sim \sum_{k=1}^{\infty} \frac{2^{2k(1-\alpha)}}{\alpha^2} \cdot \frac{2^{\alpha k - 2k}}{\alpha^2} \sim \frac{1}{\alpha^5} \sim \frac{1}{\alpha^4} \int_0^1 \sigma.$$

This finishes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

We can assume that the intervals in the sparse family are in some bounded interval, and the general case will follow by a limiting argument. Let us enumerate the intervals of the sparse family $\mathcal{S}$.

$$B_1 \supset B_2 \supset \cdots \supset B_k \supset \cdots .$$

Let $g \in L^2(w)$. We inductively choose $\pi(B_i) \supset B_i$ such that it is the largest interval with $M_{B_i}(g) \leq 2\langle g\rangle_{\pi(B_i)}$ and $\pi(B_i) \subset \pi(B_{i-1})$. We can enumerate $\{\pi(B_i)\}$ by $A_1 \supsetneq A_2 \supsetneq \ldots$. Note, that there can be many B_i with $\pi(B_i) = A_j$. Moreover, recalling that A_i are dyadic we see that $\{A_i\}_i$ is again a sparse family.

Consider the following function

$$\tilde{g}(x) = \begin{cases} \frac{1}{|A_i \setminus A_{i+1}|}\int_{A_i\setminus A_{i+1}} g, & x \in A_i \setminus A_{i+1} \text{ for some } i \in \mathbb{N}, \\ g(x), \text{ otherwise.} \end{cases}$$

First of all, it is clear that for all i

$$\int_{A_i} g = \int_{A_i} \tilde{g}. \tag{4.1}$$

Let $B \in \mathcal{S}$ be such that $A_i = \pi(B)$. Then, $A_{i+1} \subsetneq B$ due to the choice of $\pi(B)$. Then, by (4.1) and by the definition of $\tilde{g}$, we have

$$\begin{aligned}\langle g\rangle_{A_i} &= \frac{1}{|A_i|}\Big(\int_{A_{i+1}} g + \int_{A_i\setminus A_{i+1}} g\Big) \lesssim \frac{1}{|A_i|}\int_{A_{i+1}} \tilde{g} + \frac{1}{|B\setminus A_{i+1}|}\int_{B\setminus A_{i+1}} \tilde{g} \\ &\lesssim \frac{1}{|B|}\int_B \tilde{g} = \langle \tilde{g}\rangle_B.\end{aligned}$$

We conclude, that for all x

$$\mathcal{A}^*_{\mathcal{S}} g(x) \lesssim \mathcal{A}_{\mathcal{S}}\tilde{g}(x). \tag{4.2}$$

We turn to the norm of $\tilde{g}$.

$$\begin{aligned}\int_{\mathbb{R}} \tilde{g}^2 w &= \sum_i \int_{A_i\setminus A_{i+1}} \tilde{g}^2 w + \int_{\mathbb{R}\setminus\cup(A_i\setminus A_{i+1})} g^2 w \\ &\leq \sum_i \left(\frac{1}{|A_i \setminus A_{i+1}|}\int_{A_i\setminus A_{i+1}} g\right)^2 w(A_i \setminus A_{i+1}) + \int_{\mathbb{R}\setminus\cup(A_i\setminus A_{i+1})} g^2 w \\ &\leq \sum_i \frac{w(A_i \setminus A_{i+1})\cdot\sigma(A_i \setminus A_{i+1})}{|A_i \setminus A_{i+1}|^2}\int_{A_{i+1}\setminus A_i} g^2 w + \int_{\mathbb{R}\setminus\cup(A_i\setminus A_{i+1})} g^2 w \\ &\leq \sum_i \frac{w(A_i)\cdot\sigma(A_i)}{|A_i|^2}\int_{A_{i+1}\setminus A_i} g^2 w + \int_{\mathbb{R}\setminus\cup(A_i\setminus A_{i+1})} g^2 w \lesssim [w]_{A_2}\int_{\mathbb{R}} g^2 w.\end{aligned}$$

Combining the last estimate, (4.2) and the sparse bound (1.6) we conclude

$$\|\mathcal{A}^*_{\mathcal{S}} g\|_{L^2(w)} \lesssim \|\mathcal{A}_{\mathcal{S}}\tilde{g}\|_{L^2(w)} \lesssim [w]_{A_2}\|\tilde{g}\|_{L^2(w)} \lesssim [w]_{A_2}^{\frac{3}{2}}\|g\|_{L^2(w)}.$$

And the proof of Theorem 1.3 is complete.

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UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING THEIR DERIVATIVES AND SHIFTS WITH PARTIALLY SHARED VALUES

W.-J. CHEN, Z.-G. HUANG
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Suzhou University of Science and Technology, Suzhou, P. R. China[1]
E-mail: *2659238103@qq.com, alexehuang@sina.com*

Abstract. The uniqueness problems of the j-th derivative of a meromorphic function $f(z)$ and the k-th derivative of its shift $f(z+c)$ are investigated in this paper, where j,k are integers with $0 \leqslant j < k$. We show that when $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share one IM value and two partially shared values CM, the uniqueness result remains valid under some additional hypotheses. With one CM value and two partially shared values CM, a uniqueness theorem about the j-th derivative of $f(z)$ and the k-th derivative of its shift $f(z+c)$ is also proved.

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Keywords: meromorphic function; difference operator; uniqueness theorem; partially shared value CM.

1. Introduction and main results

Nevanlinna value distribution theory of meromorphic functions has been extensively applied to the uniqueness theory of meromorphic functions, see [23]. Given a meromorphic function f, recall that a meromorphic function α is said to be a small functions of f, if $T(r,\alpha(z)) = S(r,f)$ where $S(r,f)$ is used to denote any quantity that satisfies $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside of a set of r of finite logarithmic measure. Let $\hat{S}(f) = S(f)\bigcup\{\infty\}$. For each $a \in \hat{S}(f)$, we say that two meromorphic functions $f(z)$ and $g(z)$ share a IM(ignoring multiplicities) if $f(z)-a$ and $g(z)-a$ have the same zeros, and we say that $f(z)$ and $g(z)$ share a CM(counting multiplicities) provided that $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities.

Rubel and Yang[20] considered the uniqueness of a nonconstant entire function when it shares two values with its first derivative. Mues, Steinmetz [17] and Gundersen [12] improved the result to the case of meromorphic functions and obtained the following result.

Theorem A.[20] Let f be a nonconstant meromorphic function, and let a and b be two distinct finite values. If f and f' share a and b CM, then $f \equiv f'$.

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Gundersen[12] showed, by a counter-example, that two shared values CM in Theorem A cannot be reduced to 1CM+1IM. However, 2CM is able to be replaced by 3IM, see[10, 17]. Moreover, Frank and Weissenborn[8] proved the conclusion is still valid by replacing f' by a higher order derivative $f^{(k)}$.

Theorem B.[8] Let f be a nonconstant entire function and $k \geqslant 2$ be a positive integer. If f shares two distinct finite values a and b CM with $f^{(k)}$, then $f \equiv f^{(k)}$.

Later on, there are many related results about the uniqueness of meromorphic functions with their first derivative f' or their k-th derivative $f^{(k)}$ [1, 2, 8, 21]. In recent decade, Halburd and Korhonen[13]and, independently, Chiang and Feng[6] developed a parallel difference version of classical Nevanlinna theory for meromorphic functions. Then, many scholars tried to investigate the uniqueness of a meromorphic function $f(z)$ taking into account with its shift $f(z + c)$ or difference operator $\Delta_c f(z) = f(z + c) - f(z)$ where c is a complex constant, see[14, 15, 18, 22]. For instance, Heittokangas et.al[14] considered the problem of a meromorphic function f of finite order with its shift $f(z + c)$ sharing two values CM and one value IM.

Theorem C.[14] Let $f(z)$ be a meromorphic function of finite order, and let $a_1, a_2, a_3 \in \widehat{S}(f)$ be three distinct periodic functions with period c, where $c \in \mathbb{C}\setminus\{0\}$ is a constant. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) \equiv f(z+c)$.

Regarding Theorem A and Theorem C, one may ask a question: What can be said when the shift or difference operator of a meromorphic function $f(z)$ shares some values with its derivative? For a transcendental entire function $f(z)$, Qi et.al[18] proved the uniqueness result still remains true if $f'(z)$ and $f(z+c)$ share two values CM.

Theorem D.[18] Let $f(z)$ be a transcendental entire function of finite order and a be a nonzero complex constant. If $f'(z)$ and $f(z + c)$ share 0, a CM, then $f'(z) \equiv f(z + c)$.

In 2018, Chen[4] considered the question above using the notation of partially shared values by some ingenious methods.

Definition 1.1. Denote by $E(a, f)$ the set of all zeros of $f - a$, where each zero with multiplicity m times is counted m times. Similarly, we denote by $\overline{E}(a, f)$ the set of zeros of $f - a$, where each zero is counted only once. If $\overline{E}(a, f) \subseteq \overline{E}(a, g)$, then we say that $f(z)$ partially shares a with $g(z)$. If $E(a, f) \subset E(a, g)$, then we can say that f and g partially share a CM.

Theorem E.[4] Let $f(z)$ be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \neq 0 \in \mathbb{C}$. If $\Delta_c f$ and $f(z)$ share value 1 CM and satisfy $E(0, f(z)) \subset E(0, \Delta_c f)$ and $E(\infty, \Delta_c f) \subset E(\infty, f(z))$, then $f(z) \equiv \Delta_c(f)$ for all $z \in \mathbb{C}$.

In [5], Chen et.al extended the result to the case of n-th order differences $\Delta_c^n f(z)$. More recently, for $f'(z)$ and $f(z+c)$, Qi et.al[19] proved the following result.

Theorem F.[19] Let $f(z)$ be a nonconstant meromorphic function of finite order, and $a \in \mathbb{C} \setminus \{0\}$. If $f'(z)$ and $f(z+c)$ share a CM, and satisfy $E(0, f(z+c)) \subset E(0, f'(z))$, $E(\infty, f'(z)) \subset E(\infty, f(z+c))$, then $f'(z) \equiv f(z+c)$. Further, $f(z)$ is a transcendental entire function.

In this paper we consider the uniqueness of $f^{(j)}(z)$ and the k-th derivative of shift $f(z+c)$ under the conditions of one shared value IM and two partially shared values $0, \infty$ CM. Actually, we obtain the following Theorem 1.1 by a different method from those mentioned above.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let c be a nonzero finite complex number and j, k be integers with $0 \leqslant j < k$. Suppose that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a finite value $a \neq 0$ IM and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$. If $N(r, \frac{1}{f^{(j)}(z)}) + \overline{N}(r, \frac{1}{f(z)}) = S(r, f)$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

If we remove the hypothesis "$N(r, \frac{1}{f^{(j)}(z)}) + \overline{N}(r, \frac{1}{f(z)}) = S(r, f)$"and replace IM by CM, then the conclusion still holds.

Theorem 1.2. *Let $f(z)$ be a nonconstant meromorphic function of finite order, a be a nonzero finite complex number and j, k be integers with $0 \leqslant j < k$. If $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a CM, and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

2. Some lemmas

To prove our result, we recall some notations and results. Let k be a positive integer, we use $N_{k)}(r, \frac{1}{f-a})$ to denote the counting function of a points of f with multiplicity $\leqslant k$ and use $N_{(k+1}(r, \frac{1}{f-a})$ to denote the counting function of a points of f with multiplicity $> k$, where each a point is counted on the basis of its multiplicity. Similarly, we define $\overline{N}_{k)}(r, \frac{1}{f-a})$ and $\overline{N}_{(k+1}(r, \frac{1}{f-a})$ where in counting the a points of f we ignore the multiplicities.

Lemma 2.1. [6] *Let $f(z)$ be a meromorphic function of finite order and $c \in \mathbb{C}$. Then we have*

$$m(r, \frac{f(z+c)}{f(z)}) + m(r, \frac{f(z)}{f(z+c)}) = S(r, f),$$

where $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set E with finite linear measure.

Lemma 2.2. [23] *Let $f(z)$ be a nonconstant meromorphic function in the complex plane and k be a positive integer. Set*

$$\Psi(z)=\sum_{k=0}^{n}a_k(z)f^{(k)}(z),$$

where $a_k(z)(k=0,1,...,n)$ are all small functions of $f(z)$. Then

$$\begin{aligned} T(r,\Psi) &\leqslant T(r,f)+k\overline{N}(r,f)+S(r,f)\\ &\leqslant (k+1)T(r,f)+S(r,f),\\ N(r,\frac{1}{\Psi}) &\leqslant N(r,\frac{1}{f})+k\overline{N}(r,f)+S(r,f). \end{aligned}$$

Lemma 2.3. [6] *Let $f(z)$ be a nonconstant meromorphic function of finite order and $c\in\mathbb{C}$. Then*

$$T(r,f(z+c))=T(r,f)+S(r,f),$$

$$N(r,f(z+c))=N(r,f)+S(r,f),\qquad N(r,\frac{1}{f(z+c)})=N(r,\frac{1}{f(z)})+S(r,f),$$

and

$$\overline{N}(r,f(z+c))=\overline{N}(r,f)+S(r,f),\qquad \overline{N}(r,\frac{1}{f(z+c)})=\overline{N}(r,\frac{1}{f(z)})+S(r,f).$$

Lemma 2.4. *Suppose that $f(z)$ is a nonconstant meromorphic function of finite order in $|z|<R$ and $a_t(t=1,2,...,q)$ are $q(\geqslant 2)$ distinct finite complex numbers. Let j,k be integers with $0\leqslant j<k$. Then for $0<r<R$, we have*

$$m(r,f^{(j)}(z))+\sum_{t=1}^{q}m(r,\frac{1}{f^{(j)}(z)-a_t})\leqslant 2T(r,f^{(j)}(z))-N_{pair}(r,f)+S(r,f),$$

where

$$N_{pair}(r,f)=2N(r,f^{(j)}(z))-N(r,f^{(k)}(z+c))+N(r,\frac{1}{f^{(k)}(z+c)})+S(r,f).$$

Proof. Set $F(z)=\sum\limits_{t=1}^{q}\frac{1}{f^{(j)}(z)-a_t}$, then

$$G(z)=F(z)f^{(k)}(z+c)=\sum_{t=1}^{q}\frac{f^{(k)}(z+c)}{f^{(j)}(z)-a_t}.$$

It follows from the lemma of logarithmic derivatives that

$$m(r,G(z))=m(r,\sum_{t=1}^{q}\frac{f^{(k)}(z+c)}{f^{(j)}(z)-a_t})\leqslant\sum_{t=1}^{q}m(r,\frac{f^{(k)}(z+c)}{f^{(j)}(z)-a_t})+S(r,f)=S(r,f).$$

Thus

$$m(r,F(z))=m(r,G(z)\frac{1}{f^{(k)}(z+c)})\leqslant m(r,G(z))+$$

$$+m(r,\frac{1}{f^{(k)}(z+c)})=m(r,\frac{1}{f^{(k)}(z+c)})+S(r,f). \tag{2.1}$$

Next, by Nevanlinna's first fundamental theorem, we get from (2.1) that

$$\begin{aligned}T(r,f^{(k)}(z+c)) &= T(r,\frac{1}{f^{(k)}(z+c)})+O(1)\\ &= m(r,\frac{1}{f^{(k)}(z+c)})+N(r,\frac{1}{f^{(k)}(z+c)})+S(r,f)\\ &\geqslant m(r,F(z))+N(r,\frac{1}{f^{(k)}(z+c)})+S(r,f)\\ &= m(r,\sum_{t=1}^{q}\frac{1}{f^{(j)}(z)-a_t})+N(r,\frac{1}{f^{(k)}(z+c)})+S(r,f). \qquad (2.2)\end{aligned}$$

Then by (2.2), we have

$$\begin{aligned}\sum_{t=1}^{q}m(r,\frac{1}{f^{(j)}(z)-a_t}) &= m(r,\sum_{t=1}^{q}\frac{1}{f^{(j)}(z)-a_t})+O(1)\\ &\leqslant T(r,f^{(k)}(z+c))-N(r,\frac{1}{f^{(k)}(z+c)})+S(r,f). \qquad (2.3)\end{aligned}$$

Hence, it is easy to deduce from (2.3) that

$$\begin{aligned}(2.4)\qquad & m(r,f^{(j)}(z))+\sum_{t=1}^{q}m(r,\tfrac{1}{f^{(j)}(z)-a_t})\\ &\leqslant m(r,f^{(j)}(z))+T(r,f^{(k)}(z+c))-N(r,\tfrac{1}{f^{(k)}(z+c)})+S(r,f)\\ &= T(r,f^{(j)}(z))-N(r,f^{(j)}(z))+m(r,f^{(k)}(z+c))+N(r,f^{(k)}(z+c))-N(r,\tfrac{1}{f^{(k)}(z+c)})\\ &+S(r,f)\leqslant T(r,f^{(j)}(z))-N(r,f^{(j)}(z))+m(r,f^{(j)}(z))+m(r,\tfrac{f^{(k)}(z+c)}{f^{(j)}(z)})\\ &+N(r,f^{(k)}(z+c))-N(r,\tfrac{1}{f^{(k)}(z+c)})+S(r,f)\\ &= 2T(r,f^{(j)}(z))-2N(r,f^{(j)}(z))+N(r,f^{(k)}(z+c))-N(r,\tfrac{1}{f^{(k)}(z+c)})+S(r,f)\\ &= 2T(r,f^{(j)}(z))-[2N(r,f^{(j)}(z))-N(r,f^{(k)}(z+c))+N(r,\tfrac{1}{f^{(k)}(z+c)})]+S(r,f).\end{aligned}$$

We use $N_p(r,\frac{1}{f^{(k)}-a})$ to denote the counting function of the zeros of $f-a$ where a $p-$ folds zero is counted m times if $m\leqslant p$ and p times if $m>p$.

Lemma 2.5. [24, Lemma 2.4] *Let f be a non-constant transcendental meromorphic function. If $f^{(k)}\not\equiv 0$, we have $N_p(r,\frac{1}{f^{(k)}})\leqslant T(r,f^{(k)})-T(r,f)+N_{p+k}(r,\frac{1}{f})+S(r,f)$.*

Lemma 2.6. *Let f be a non-constant transcendental meromorphic function and $j\geqslant 0$ is an integer. If $\overline{N}(r,\frac{1}{f(z)})=S(r,f)$, then $S(r,f^{(j)})=S(r,f)$.*

Proof. By Lemma 2.5,

$$\begin{aligned}T(r,f) &\leqslant T(r,f^{(j)})-N_p(r,\frac{1}{f^{(j)}})+N_{1+j}(r,\frac{1}{f})+S(r,f)\\ &\leqslant T(r,f^{(j)})+(1+j)\overline{N}(r,\frac{1}{f})+S(r,f)\leqslant T(r,f^{(j)})+S(r,f).\end{aligned}$$

Also by Lemma 2.2, $T(r,f^{(j)})\leqslant(j+1)T(r,f)+S(r,f)$. This completes the proof.

3. Proof of Theorem 1.1

Without loss of generality, we assume that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share $a=1$ IM. For a general case, we can consider substituting $\frac{1}{a}f(z)$ for $f(z)$. Suppose on the contrary that $f^{(j)}(z)\not\equiv f^{(k)}(z+c)$.

Set $h(z)=f^{(j)}(z)$ and $g(z)=f^{(k)}(z+c)$. By the assumption that $E(0,h(z))\subset E(0,g(z))$ and $E(\infty,g(z))\subset E(\infty,h(z))$, we have

$$\frac{f^{(k)}(z+c)}{f^{(j)}(z)}=\frac{g(z)}{h(z)}=G(z), \tag{3.1}$$

where $G(z)$ is an entire function.

From (3.1), the lemma of logarithmic derivative and Lemma 2.1 it follows that

$$m(r,G(z))=m(r,\frac{f^{(k)}(z+c)}{f^{(j)}(z)})\leqslant m(r,\frac{f^{(k)}(z+c)}{f^{(k)}(z)})+m(r,\frac{f^{(k)}(z)}{f^{(j)}(z)})=S(r,f). \tag{3.2}$$

Since $G(z)$ is an entire function, we know that

$$N(r,G(z))=0. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$T(r,G(z))=m(r,G(z))+N(r,G(z))=S(r,f). \tag{3.4}$$

Set

$$F=\frac{1}{h}(\frac{g'}{g-1}-\frac{h'}{h-1})=\frac{g}{h}(\frac{g'}{g-1}-\frac{g'}{g})-(\frac{h'}{h-1}-\frac{h'}{h}). \tag{3.5}$$

From the lemma of logarithmic derivative again, (3.2) and (3.5) it follows that

$$m(r,F)=m(r,\frac{g}{h}(\frac{g'}{g-1}-\frac{g'}{g})-(\frac{h'}{h-1}-\frac{h'}{h}))=S(r,f). \tag{3.6}$$

By (3.5), we see that the possible poles of $F(z)$ can occur at the zeros of $h(z)$, the 1 points of $h(z)$ and $g(z)$, and the poles of $h(z)$ and $g(z)$. If z_0 is a 1 point of $h(z)$, then by a short calculation with Laurent series and (8) we know that z_0 is a simple pole of $F(z)$. And hence, the 1 points of $g(z)$ are also the simple pole of $F(z)$. If z_0 is a pole of $h(z)$ with multiplicity $p\geqslant 1$, by $E(\infty,g(z))\subset E(\infty,h(z))$, we have $F(z)=O((z-z_0)^{p-1})$. Similarly, the poles of $g(z)$ are not the poles of $F(z)$. Therefore, the poles of $F(z)$ can occur at the 1 point of $h(z)$, the 1 point of $g(z)$ and the zeros of $h(z)$. From (3.1), (3.4), the hypothesis $N(r,\frac{1}{f^{(j)}})=S(r,f)$ and h shares 1 IM with g, we can find that

$$\begin{aligned} N(r,F) &\leqslant \overline{N}(r,\frac{1}{h-1})+\overline{N}(r,\frac{1}{g-1})+N(r,\frac{1}{h}) \\ &\leqslant N(r,\frac{1}{G-1})+N(r,\frac{1}{G-1})+N(r,\frac{1}{f^{(j)}}) \\ &\leqslant 2T(r,G)+S(r,f)=S(r,f). \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we conclude that

$$T(r,F)=m(r,F)+N(r,F)=S(r,f). \tag{3.8}$$

If $F\equiv 0$, then by (3.5) we find that $g-1=A(h-1)$, with $A\neq 0$ being constant. We assert that $A=1$. Otherwise, if $A\neq 1$, then $m(r,\frac{1}{h})=\frac{1}{1-A}m(r,\frac{g}{h}-A)=S(r,f)$. Due to $N(r,\frac{1}{h})=N(r,\frac{1}{f^{(j)}})=S(r,f)$, it is easy to deduce that $T(r,\frac{1}{h})=m(r,\frac{1}{h})+N(r,\frac{1}{h})=S(r,f)$, and then by the first fundamental theorem, we have $T(r,h)=T(r,\frac{1}{h})+O(1)=S(r,f)$. Noting that $h=f^{(j)}$, by Lemma 2.6 we have $S(r,f)=S(r,h)$, and hence $T(r,h)=S(r,h)$, which is a contradiction. Then $F\not\equiv 0$. And so we can know from (3.5) and (3.8) that

$$m(r,h)\leqslant m(r,\frac{1}{F})+m(r,\frac{g'}{g-1}-\frac{h'}{h-1})\leqslant T(r,F)+S(r,f)=S(r,f). \tag{3.9}$$

Set

$$H(z)=\frac{g'(h-1)}{h'(g-1)}=(\frac{g'}{g-1}-\frac{g'}{g})\frac{g}{h'}(h-1). \tag{3.10}$$

It follows from the lemma of logarithmic derivative, (12) and (13) that

$$m(r,H)=m(r,\frac{g'(h-1)}{h'(g-1)})\leqslant m(r,\frac{g'}{g-1}-\frac{g'}{g})+m(r,\frac{g}{h'})+m(r,h-1)=S(r,f). \tag{3.11}$$

We now estimate the poles of $H(z)$. Obviously, the poles of $H(z)$ can only occur at the 1 point of g, the poles of h and g', and the zeros of h'. Since $h(z)$ and $g(z)$ share 1 IM, then by Laurent series we know that $H(z)$ is analytic at the 1 point of g. If h has a pole z_∞ with multiplicity $p\geqslant 2$, then by a short calculation with Laurent series and (3.10) we see that the poles of h are not poles of $H(z)$. Similarly, the poles of g' are not poles of $H(z)$. Let z_0 be a zero of h' with multiplicity q, if z_0 is also a zero of h (respectively $h-1$) with multiplicity $q+1$, then from these and (3.10) it is easy to see that z_0 is a pole of $H(z)$ with multiplicity at most q. Thus

$$N(r,H)\leqslant N_0(r,\frac{1}{h'})+S(r,f), \tag{3.12}$$

where $N_0(r,\frac{1}{h'})$ denotes the zeros of h' which are not zeros of $h-1$. From (3.11) and (3.12), we deduce that

$$T(r,H)=m(r,H)+N(r,H)\leqslant N_0(r,\frac{1}{h'})+S(r,f), \tag{3.13}$$

Next, we consider the simple poles of $h(z)$. Let z_0 be a simple pole of h. Since $E(\infty,h(z))\supset E(\infty,g(z))$, we need to discuss two cases:

Case 1. z_0 is not a simple pole of g. We set

$$h(z)=\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots \tag{3.14}$$

and

$$g(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots, \tag{3.15}$$

where $a_j (j=-1,0,1,\cdots)$ and $b_j (j=0,1,\cdots)$ are the coefficients of the Laurent series of $h(z)$ and $g(z)$ respectively. Differentiating (3.14) and (3.15), we obtain

$$h'(z) = -\frac{a_{-1}}{(z-z_0)^2} + a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots$$

and

$$g'(z) = b_1 + 2b_2(z-z_0) + 3b_3(z-z_0)^2 + \cdots.$$

By (3.10) it follows that

$$H(z) = \frac{g'(h-1)}{h'(g-1)} = \frac{[b_1 + 2b_2(z-z_0) + \cdots][\frac{a_{-1}}{z-z_0} + a_0 - 1 + a_1(z-z_0) + \cdots]}{[-\frac{a_{-1}}{(z-z_0)^2} + a_1 + 2a_2(z-z_0) + \cdots][b_0 - 1 + b_1(z-z_0) + \cdots]}.$$

Thus $H(z_0) = 0$. If $H(z) \equiv 0$, then we have $g'(h-1) \equiv 0$. By integration, we can get $f(z)$ is a nonconstant polynomial, this contradicts with the fact that $f(z)$ is a transcendental function. Thus $H \not\equiv 0$, and so

$$N_{1)}(r,h) \leqslant N(r, \frac{1}{H}). \tag{3.16}$$

Case 2. z_0 is a simple pole of g. Similarly as in Case 1, let

$$h(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

and

$$g(z) = \frac{d_{-1}}{z-z_0} + d_0 + d_1(z-z_0) + d_2(z-z_0)^2 + \cdots.$$

Then

$$h'(z) = -\frac{c_{-1}}{(z-z_0)^2} + c_1 + 2c_2(z-z_0) + 3c_3(z-z_0)^2 + \cdots$$

and

$$g'(z) = -\frac{d_{-1}}{(z-z_0)^2} + d_1 + 2d_2(z-z_0) + 3d_3(z-z_0)^2 + \cdots.$$

By (3.5), it follows that

$$F(z) = \frac{1}{h}(\frac{g'}{g-1} - \frac{h'}{h-1}) = \frac{1}{\frac{c_{-1}}{z-z_0} + c_0 + \cdots}(\frac{-\frac{d_{-1}}{(z-z_0)^2} + d_1 + \cdots}{\frac{d_{-1}}{z-z_0} + d_0 - 1 + \cdots} - \frac{-\frac{c_{-1}}{(z-z_0)^2} + c_1 + \cdots}{\frac{c_{-1}}{z-z_0} + c_0 - 1 + \cdots})$$

Thus $F(z_0) = 0$. If $F(z) \equiv 0$, then we have $g - 1 = t(h-1)$ with $t \neq 0$ constant. Similarly, we can assert that $t = 1$, then $g \equiv h$, this contradicts with the assumption $g \not\equiv h$. Thus $F \not\equiv 0$, and so

$$N_{1)}(r,h) \leqslant N(r, \frac{1}{F}). \tag{3.17}$$

Combining (3.8), (3.13), (3.16) and (3.17), we have

$$\begin{aligned} N_{1)}(r,h) &\leqslant N(r,\frac{1}{F}) + N(r,\frac{1}{H}) \leqslant T(r,F) + T(r,H) \\ &\leqslant S(r,f) + N_0(r,\frac{1}{h'}) + S(r,f) = N_0(r,\frac{1}{h'}) + S(r,f). \end{aligned} \tag{3.18}$$

Since h and g share 1 IM, it follows from (3.1) and (3.4) that

$$\overline{N}(r,\frac{1}{h-1})\leqslant N(r,\frac{1}{G-1})\leqslant T(r,G)=S(r,f). \tag{3.19}$$

Combining (3.18), (3.19), the second fundamental theorem, $N(r,\frac{1}{h})=N(r,\frac{1}{f^{(j)}})=S(r,f)$ and $S(r,h)=S(r,f)$, we have

$$\begin{aligned} T(r,h) &\leqslant N(r,\frac{1}{h})+\overline{N}(r,h)+\overline{N}(r,\frac{1}{h-1})-N_0(r,\frac{1}{h'})+S(r,h)\\ &\leqslant S(r,f)+N_0(r,\frac{1}{h'})-N_0(r,\frac{1}{h'})+S(r,h)\\ &= S(r,f)+S(r,h)=S(r,h), \end{aligned} \tag{3.20}$$

which is impossible. Therefore, $f^{(j)}(z)\equiv f^{(k)}(z+c)$.

4. Proof of Theorem 1.2

Firstly, we prove that $T(r,f^{(j)}(z))$ and $T(r,f^{(k)}(z+c))$ can be restricted by each other. It follows from Lemma 2.2 that

$$\begin{aligned} T(r,f^{(j)}(z)) &\leqslant T(r,f(z))+j\overline{N}(r,f(z))+S(r,f(z))\\ &\leqslant (j+1)T(r,f(z))+S(r,f(z)) \end{aligned} \tag{4.1}$$

On the other hand, by Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned} T(r,f^{(k)}(z+c)) &= T(r,f^{(k)}(z))+S(r,f)\\ &\leqslant T(r,f(z))+k\overline{N}(r,f(z))+S(r,f(z))\\ &\leqslant (k+1)T(r,f(z))+S(r,f(z)). \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$S(r,f^{(j)}(z))=S(r,f^{(k)}(z+c))=S(r,f).$$

Set

$$H(z)=\frac{f^{(k)}(z+c)}{f^{(j)}(z)}, \tag{4.3}$$

From the assumption $E(0,f^{(j)}(z))\subset E(0,f^{(k)}(z+c))$ and $E(\infty,f^{(k)}(z+c))\subset E(\infty,f^{(j)}(z))$, we can deduce that $H(z)$ is an entire function. That is to say,

$$N(r,H(z))=0. \tag{4.4}$$

Case 1 If $H(z)\equiv 1$, then $f^{(j)}(z)\equiv f^{(k)}(z+c)$.

Case 2 We suppose on the contrary that the result of Theorem 1.2 is not valid, i.e., $H(z) \not\equiv 1$. By Lemma 2.1 and the lemma of logarithmic derivative, we know that

$$\begin{aligned} m(r, H(z)) &= m(r, \frac{f^{(k)}(z+c)}{f^{(j)}(z)}) \leqslant m(r, \frac{f^{(k)}(z+c)}{f^{(k)}(z)}) + \\ &+ m(r, \frac{f^{(k)}(z)}{f^{(j)}(z)}) = S(r, f). \end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we can obtain that

$$T(r, H(z)) = m(r, H(z)) + N(r, H(z)) = S(r, f). \tag{4.6}$$

Without loss of generality, we assume that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share $a = 1$ CM. For a general situation, we can consider replacing $f(z)$ by $\frac{1}{a}f(z)$. As a result of the hypothesis that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share 1 CM, we find that

$$\begin{aligned} \overline{N}(r, \frac{1}{f^{(j)}(z)-1}) &\leqslant N(r, \frac{1}{\frac{f^{(k)}(z+c)}{f^{(j)}(z)} - 1}) = \\ = N(r, \frac{1}{H-1}) &\leqslant T(r, H) + S(r, f) = S(r, f). \end{aligned} \tag{4.7}$$

Secondly, we shall estimate the counting functions of the zeros of $f^{(j)}(z) - 1$ whose multiplicities are not less than 2.

Differentiating (4.3), we have

$$H'(z) = (\frac{f^{(k)}(z+c)}{f^{(j)}(z)})' = \frac{f^{(k+1)}(z+c)f^{(j)}(z) - f^{(k)}(z+c)f^{(j+1)}(z)}{[f^{(j)}(z)]^2}. \tag{4.8}$$

It follows from (4.3) and (4.8) that

$$\begin{aligned} \frac{H'(z)}{H(z)} &= \frac{f^{(k+1)}(z+c)f^{(j)}(z) - f^{(k)}(z+c)f^{(j+1)}(z)}{[f^{(j)}(z)]^2} \cdot \frac{f^{(j)}(z)}{f^{(k)}(z+c)} \\ &= \frac{f^{(k+1)}(z+c)f^{(j)}(z) - f^{(k)}(z+c)f^{(j+1)}(z)}{f^{(j)}(z)f^{(k)}(z+c)} \\ &= \frac{f^{(k+1)}(z+c)}{f^{(k)}(z+c)} - \frac{f^{(j+1)}(z)}{f^{(j)}(z)}. \end{aligned} \tag{4.9}$$

Let z_0 be a 1 point of $f^{(j)}(z)$ with multiplicity $m \geqslant 2$. Since $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share 1 CM, we obtain that z_0 is also a 1 point of $f^{(k)}(z+c)$ with multiplicity $m \geqslant 2$. Then by (4.9) and calculation with Laurent series, we see that z_0 is also a zero of $\frac{H'(z)}{H(z)}$ with multiplicity at least $m-1$. Thus by Lemma 2.2 we can get

$$\begin{aligned} N_{(2}(r, \frac{1}{f^{(j)}(z)-1}) &\leqslant 2N(r, \frac{1}{\frac{H'}{H}}) \leqslant 2N(r, H) + 2N(r, \frac{1}{H'}) \\ &\leqslant 2N(r, H) + 2[N(r, \frac{1}{H}) + \overline{N}(r, H) + S(r, f)] \\ &\leqslant 6T(r, H) + S(r, f) = S(r, f). \end{aligned} \tag{4.10}$$

Together (4.7) with (4.10), we have

$$N(r,\frac{1}{f^{(j)}(z)-1})=\overline{N}(r,\frac{1}{f^{(j)}(z)-1})+N_{(2}(r,\frac{1}{f^{(j)}(z)-1})\leqslant S(r,f). \tag{4.11}$$

By the assumption that $E(0,f^{(j)}(z))\subset E(0,f^{(k)}(z+c))$ and $E(\infty,f^{(k)}(z+c))\subset E(\infty,f^{(j)}(z))$ again, we deduce that

$$N(r,\frac{1}{f^{(j)}(z)})-N(r,\frac{1}{f^{(k)}(z+c)})\leqslant 0,\quad N(r,f^{(k)}(z+c))-N(r,f^{(j)}(z))\leqslant 0. \tag{4.12}$$

From Lemma 2.4, we get

$$\begin{aligned}&m(r,f^{(j)}(z))+m(r,\frac{1}{f^{(j)}(z)})+m(r,\frac{1}{f^{(j)}(z)-1})\\ &\leqslant 2T(r,f^{(j)}(z))-N_{pair}(r,f)+S(r,f).\end{aligned} \tag{4.13}$$

Adding $N(r,f^{(j)}(z))+N(r,\frac{1}{f^{(j)}(z)})+N(r,\frac{1}{f^{(j)}(z)-1})$ on both sides of (4.13) at the same time and by (4.12), we obtain

$$\begin{aligned}T(r,f^{(j)}(z)) &\leqslant N(r,f^{(j)}(z))+N(r,\frac{1}{f^{(j)}(z)})+N(r,\frac{1}{f^{(j)}(z)-1})-N_{pair}(r,f)+S(r,f)\\ &= N(r,\frac{1}{f^{(j)}(z)-1})+[N(r,\frac{1}{f^{(j)}(z)})-N(r,\frac{1}{f^{(k)}(z+c)})]\\ &+ [N(r,f^{(k)}(z+c))-N(r,f^{(j)}(z))]+S(r,f)\\ &\leqslant N(r,\frac{1}{f^{(j)}(z)-1})+S(r,f)\leqslant S(r,f),\end{aligned}$$

which yields a contradiction.

Therefore, $H(z)\equiv 1$. Then we have $f^{(j)}(z)\equiv f^{(k)}(z+c)$.

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INFINITELY MANY SOLUTIONS FOR KIRCHHOFF TYPE EQUATIONS INVOLVING DEGENERATE OPERATOR

J. CHEN, L. LI, SH. CHEN

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Chongqing Technology and Business University, Chongqing, China[1]
E-mails: *995446428@qq.com; lilin420@gmail.com; 11183356@qq.com*

Abstract. In this paper, we study the existence of infinitely many nontrivial solutions for a class of nonlinear Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)\Delta_\lambda u+V(x)u=f(x,u),\ \text{ in } \mathbb{R}^N$$

where constants $a>0,\ b>0$, Δ_λ is a strongly degenerate elliptic operator, and f is a function with a more general superlinear conditions or sublinear conditions.

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Keywords: Kirchhoff type equation; symmetric mountain pass theorem; strongly degenerate elliptic operator.

1. Introduction

This paper is concerned with a class of nonlinear Kirchhoff type equations

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)\Delta_\lambda u+V(x)u=f(x,u),\quad \text{in } \mathbb{R}^N \tag{1.1}$$

where constants $a,b>0$, $N\geqslant 1$, $V\in C(\mathbb{R}^N,\mathbb{R})$, $\nabla_\lambda=(\lambda_1\partial_{x_1}u,...,\lambda_N\partial_{x_N}u)$ and Δ_λ is a strongly degenerate elliptic operator of the following form

$$\Delta_\lambda:=\sum_{i=1}^N\partial_{x_i}(\lambda_i^2\partial_{x_i}),\quad \lambda=(\lambda_1,\cdots,\lambda_N):\mathbb{R}^N\to\mathbb{R}^N.$$

Kogoj and Lanconelli in [7] firstly introduced the strongly degenerate elliptic operator Δ_λ. After that, a growing attention has been devoted to Δ_λ-Laplacians. Kogoj and Lanconelli in [7] assume that the operator is homogeneous of degree two with respect to a group dilation in $\mathbb{R}^N$. Kogoj and Sonner [8] showed that global well-posedness and long-time behavior of solutions of semilinear degenerate parabolic involving the Δ_λ-Laplacians, and this result was extended in [9], where hyperbolic problems were considered. Ahn and My [2] proved that Liouville-type theorems for elliptic inequalities involving the Δ_λ-Laplacians. Finally, Kogoj and Sonner remark that the Δ_λ-Laplacians belong to the more general class of $X-$

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elliptic operators. The Δ_λ operator contains many degenerate elliptic operators such as the Grushin-type operator

$$G_\alpha = \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha > 0,$$

where (x, y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $N_1 + N_2 = N$, and the operator of the form

$$P_{\alpha,\beta,\gamma} = \Delta_x + |x|^{2\alpha}\Delta_y + |x|^{2\beta}|y|^{2\gamma}\Delta_z, \ \ (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}, \ N_1 + N_2 + N_3 = N,$$

where α, β and γ are real positive constants. We can refer the readers to [1] for some important properties of this operator.

In the last decades, Δ_λ elliptic equations

$$\begin{cases} -\Delta_\lambda u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, has been studied by many authors. See [3, 1, 7, 13, 14, 19] and the references therein. The nonlinear term f satisfies the Ambrosetti-Rabinowitz(AR) condition is studied in [7]. The (AR) condition guarantees the boundedness of the Palais-Smale(PS) sequence of the energy functional, which is essential for the application of the critical point theorem. When f does not satisfy the (AR) condition is studied in [3, 1, 14]. At present, some authors began to consider problem (1.2) on unbounded domain $\mathbb{R}^N$. The main difficulty in $\mathbb{R}^N$ is lack of compactness of Sobolev embedding. For this reason, some authors work on the subspace of Sobolev space to overcome this difficulty. Luyen and Tri [15] considered that $V(x)$ is a coercive potential, which ensures that the weighted Sobolev space embedding is compactness. They proved that Δ_λ equation possess infinity many solutions with the nonlinear term has (AR) condition.

Recently, a class of Kirchhoff-type elliptic equation

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

has received extensive attention and research by many authors. Cheng and Wu [4] proved the existence result of positive solutions to Kirchhoff-type problems with the variational method. Mao and Zhang [16] shows that in the absence of (PS) condition, the minimax methods and invariant sets of descent flow are used to study multiple solutions of Kirchhoff type problems. The problem (1.3) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla_x u|^2 dx\right)\Delta_x u = g(x, u) \tag{1.4}$$

which was proposed by Kirchhoff in 1883 as a generalization of the well-known d'Alembert's wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = g(x,u)$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Problem (1.4) models several physical systems, where u describes a process which depends on the average of itself. A parabolic version of equation (1.4) be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed dependent on the "energy" of the entire system with u being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria) which gives rise to equations of the type $u_t - a(\int_\Omega u dx)\Delta u = h$.

In this paper, we want to use the idea of [21] to study the existence of infinitely many nontrivial solutions for the Kirchhoff type problem with Δ_λ type operator. Now, we give the following assumptions on potential $V(x)$:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

(V_2) There exists a constant $R > 0$ such that

$$\int_{|x|\geqslant R} V^{-1} dx < \infty.$$

For the nonlinearity f, we give the following assumptions:

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $C_1, C_2 > 0$ and $p \in (2, 2^*_\lambda)$ such that

$$|f(x,u)| \leqslant C_1|u| + C_2|u|^{p-1}, \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

where $2^*_\lambda = \frac{2Q}{Q-2}$ and Q denotes the homogeneous dimension of $\mathbb{R}^N$ with respect to a group of dilations(see Section 2 for more details).

(f_2) $f(x,u) = -f(x,-u)$, $\forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$.

(f_3) $\lim_{|u|\to\infty} \frac{|F(x,u)|}{|u|^4} = \infty$, uniformly in $x \in \mathbb{R}^N$, $Q < 4$, and there exists $r_0 \geqslant 0$, such that $F(x,u) \geqslant 0$, $\forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$, $|u| \geqslant r_0$, where $F(x,u) := \int_0^u f(x,t)dt$.

(f_4) There exist $\beta \geqslant 0$ such that $F(x,u) \leqslant \frac{1}{4} f(x,u)u + \beta u^2$, $\forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$.

(f_5) $F(x,u) \geqslant 0, \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$ and $G(x,h) \leqslant G(x,l)$ whenever $(h,l) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $h \leqslant l$, where $G(x,u) := \frac{1}{4} f(x,u)u - F(x,u)$.

In the following theorem, we give the multiplicity result of the solution of problem (1.1) when f satisfies the superlinear condition.

Theorem 1.1. *Assume that the potential $V(x)$ satisfies $(V_1),(V_2)$ and nonlinearity $f(x,u)$ satisfies $(f_1)-(f_4)$. Then the problem(1.1) has possesses infinitely many nontrivial solutions $\{u_k\}$ such that*

$$\lim_{k\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\left(a|\nabla_\lambda u_k|^2+V(x)u_k^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_k|^2dx\right)^2-\int_{\mathbb{R}^N}F(x,u_k)dx=+\infty.$$

Theorem 1.2. *Assume that the potential $V(x)$ satisfies $(V_1),(V_2)$ and nonlinearity $f(x,u)$ satisfies $(f_1)-(f_3)$ and (f_5). Then the problem(1.1) has possesses infinitely many nontrivial solutions $\{u_k\}$ such that*

$$\lim_{k\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\left(a|\nabla_\lambda u_k|^2+V(x)u_k^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_k|^2dx\right)^2-\int_{\mathbb{R}^N}F(x,u_k)dx=+\infty.$$

Next, in addition to discussing the above results, we also consider the multiplicity result that can still obtain a solution of problem (1.1) when f satisfies the sublinear.

(f_6) $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, there exist constant $1<q_1<q_2<2$, such that

$$|f(x,t)|\leqslant q_1|t|^{q_1-1}+q_2|t|^{q_2-1}.$$

(f_7) There exist a bounded open set $\tilde{B}\subset\mathbb{R}^N$ and constants $\delta,\xi>0,\ q_3\in(1,2)$ such that

$$F(x,u)\geqslant\xi|u|^{q_3},\ \forall(x,u)\in\tilde{B}\times[-\delta,\delta].$$

Now, we give the second result:

Theorem 1.3. *Assume that the potential $V(x)$ satisfies $(V_1),(V_2)$ and nonlinearity $f(x,u)$ satisfies $(f_2),(f_6),(f_7)$. Then the problem (1.1) has possesses infinitely many nontrivial solutions $\{u_k\}$.*

Remark 1.1. Compared with problem (1.1), we extend the equation to operator Δ_λ, because operator Δ_λ is more complicated with the addition of function λ. As can be seen from [7], when the function λ is smooth, then Δ_λ is the general operator class studied by Hömander in [5], and is hypoelliptic. The typical example is the Grushin-type operator, which means that Δ_λ is a generalization of Grushin-type operator. Later, Δ_λ belongs to the more general $X-$*elliptic operators* introduced in [10], and has some of the same important homogeneity as the classical Laplacian. Therefore, it is meaningful for us to extend the problem (1.3) to a more general Kirchhoff-type equation, and it is applicable to more environments.

Now, we give an example that satisfies all the assumptions of Theorem 1.1, as follows $f(x,u)=u|\sin x|+|u|^3u|\cos x|$, obviously, $F(x,u)=\frac{1}{2}u^2|\sin x|+\frac{1}{5}|u|^5|\cos x|$.

Of course, there is also f that satisfies the sublinear condition, such as $f(x,u) = \frac{4}{3}|u|^{-\frac{2}{3}}u\sin^2 x + \frac{5}{4}|u|^{-\frac{3}{4}}\cos^2 x$, and $F(x,u) = |u|^{\frac{4}{3}}\sin^2 x + |u|^{\frac{5}{4}}\cos^2 x$. Through simple calculations, it can be verified that the assumptions of each theorem are satisfied.

The main structure of this article is as follows. In the second section, we give some preliminary knowledge and main theorems. In the third section, we use the symmetric mountain pass theorem to prove Theorems 1.1 and 1.2. In the fourth section, we apply the theorem in [18], to get the multiplicity result of the solution.

2. Preliminaries

We recall the functional setting in [7, 3]. We consider the operator of the form

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}(\lambda_i^2 \partial_{x_i}),$$

where $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, N$. Here the function $\lambda_i : \mathbb{R}^N \to \mathbb{R}$ are continuous, strictly positive and of C^1 outside the coordinate hyperplane, i.e. $\lambda_i > 0, i = 1, \ldots, N$ in $\mathbb{R}^N \setminus \prod$, where $\prod = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : \prod_{i=1}^N x_i = 0\}$. As in [7], we assume that λ_i satisfy the following properties:

(1) $\lambda_1(x) \equiv 1,\ \lambda_i(x) = \lambda_i(x_1,\ \ldots, x_{i-1}),\ i = 2, \ldots N$;

(2) For every $x \in \mathbb{R}^N$, $\lambda_i(x) = \lambda_i(x^*),\ i = 1, \ldots, N$, where $x^* = (|x_1|, \ldots, |x_N|)$ if $x = (x_1, \ldots, x_N)$;

(3) There exists a constant $\rho \geqslant 0$ such that

$$0 \leqslant x_k \partial_{x_k} \lambda_i(x) \leqslant \rho \lambda_i(x), \quad \forall k \in \{1, \ldots, i-1\}, i = 2, \ldots, N,$$

and for every $x \in \mathbb{R}^N_+ := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_i \geqslant 0,\ \forall i = 1, \ldots, N\}$;

(4) Exists a group of dilations $\{\delta_t\}_{t>0}$

$$\delta_t : \mathbb{R}^N \to \mathbb{R}, \quad \delta_t(x) = \delta_t(x_1, \ldots, x_N) = (t^{\epsilon_1} x_1, \ldots, t^{\epsilon_N} x_N),$$

where $1 \leqslant \epsilon_1 \leqslant \epsilon_2 \leqslant \ldots \leqslant \epsilon_N$, such that λ_i is $\delta_t -$ homogeneous of degree $\epsilon_i - 1$, i.e.

$$\lambda_i(\delta_t(x)) = t^{\epsilon_i - 1}\lambda_i(x), \quad \forall x \in \mathbb{R}^N,\ t > 0,\ i = 1, \ldots, N.$$

This implies that the operation Δ_λ is $\delta_t -$ homogeneous of degree two, i.e.

$$\Delta_\lambda(u(\delta_t(x))) = t^2 (\Delta_\lambda u)(\delta_t(x)), \quad \forall u \in C^\infty(\mathbb{R}^N).$$

We denote by Q the homogeneous dimension of $\mathbb{R}^N$ with respect to group of dilations $\{\delta_t\}_{t>0}$, i.e.

$$Q := \epsilon_1 + \cdots + \epsilon_N.$$

The homogeneous Q plays a crucial role, both in the geometry and the functional associated to the operator Δ_λ.

Now, we denote by $W_\lambda^{1,2}(\mathbb{R}^N)$ the closure of $C_0^1(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{W_\lambda^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\nabla_\lambda u|^2 + u^2)dx\right)^{\frac{1}{2}},$$

is Hilbert space with the inner product

$$(u,v) = \int_{\mathbb{R}^N} (\nabla_\lambda u \nabla_\lambda v + uv)dx.$$

Under the hypotheses (V_1), we define space

$$E = \left\{u \in W_\lambda^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2dx < +\infty\right\},$$

with the inner product

$$(u,v) = \int_{\mathbb{R}^N} (\nabla_\lambda u \nabla_\lambda v + V(x)uv)\, dx,$$

and the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla_\lambda u|^2 + V(x)u^2)dx.$$

Here, we denote $\|\cdot\|_p$ as the norm of Lebesgue space $L^p(\mathbb{R}^N)$.

Proposition 2.1. *Under the assumptions (V_1) and (V_2), the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for every $p \in [1, 2_\lambda^*)$.*

Proof. In [15], we know that under the assumption of (V_1), the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $p \in [2, 2_\lambda^*]$, and $E \hookrightarrow L^p_{loc}(\mathbb{R}^N)$ is compact for $p \in [1, 2_\lambda^*)$. Then there are constant C_p such that

$$\|u\|_p \leqslant C_p\|u\|,\ \forall u \in E. \tag{2.1}$$

When we want to embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in [1, 2_\lambda^*)$ under the assumption of (V_1) and (V_2), it suffices to prove the result for $p = 1$. Assume $u_n \rightharpoonup u$ in E. For any $R > 0$, write

$$\int_{\mathbb{R}^N} |u_n - u|dx = \int_{|x|\leqslant R} |u_n - u|dx + \int_{|x|>R} |u_n - u|dx. \tag{2.2}$$

By the Hölder inequality to obtain that

$$\int_{|x|>R} |u_n - u|dx \leqslant \left(\int_{|x|>R} V|u_n - u|^2dx\right)^{\frac{1}{2}} \left(\int_{|x|>R} V^{-1}dx\right)^{\frac{1}{2}} = o_R(1), \tag{2.3}$$

where $o_R(1)$ is a quantity that converges to 0 as $R \to \infty$ uniformly for n. Then $u_n \to u$ strongly in $L^1(\mathbb{R}^N)$ since $u_n \to u$ in $L^1_{loc}(\mathbb{R}^N)$. □

Remark 2.1. Several important problems arising in many research fields such as physics and differential geometry lead to consider semilinear variational elliptic equations defined on unbounded domains of the Euclidean space and a great deal of work has been devoted to their study. From the mathematical point of view, probably the main interest relies on the fact that often the tools of nonlinear functional analysis, based on compactness arguments, can not be used, at least in a straight forward way, and some new techniques have to be developed. The seminal paper [11] by Lions has inspired a (nowadays usual) way to overcome the lack of compactness by exploiting symmetry. This approach is fruitful in the study of variational elliptic problems in presence of a suitable continuous action of a topological group on the Sobolev space where the solutions are being sought.

Here, we use another skill following the idea of Rabinowitz [17] to get the Sobolev embedding is compact by the potential V. Luyen and Tri [15] use the idea of Rabinowitz to get the Sobolev compact embedding, but they only obtained the embedding map from E into $L^p(\mathbb{R}^N)$ is compact for $2 \leqslant p < 2^*_\lambda$. We want to study the sublinear case, so we give a wider interval for the Sobolev embedding. Moreover, Assumption (V_2) makes V look like a well-shaped potential.

Now, we define the following energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}(a|\nabla_\lambda u|^2 + V(x)u^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)^2 - \int_{\mathbb{R}^N}F(x,u)dx,\ u \in E.$$

Obviously, given constant $a > 0$, $\int_{\mathbb{R}^N}(|\nabla_\lambda u|^2 + V(x)u^2)dx$ is equivalent to $\int_{\mathbb{R}^N}(a|\nabla_\lambda u|^2 + V(x)u^2)dx$. Hence, the norm of u in E denoted by

$$\|u\| = \left(\int_{\mathbb{R}^N}(a|\nabla_\lambda u|^2 + V(x)u^2)dx\right)^{\frac{1}{2}}$$

that is,

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)^2 - \int_{\mathbb{R}^N}F(x,u)dx. \tag{2.4}$$

Definition 2.1. A sequence $\{u_n\} \subset E$ is said to be a $(C)_c-$*sequence* if $J(u_n) \to c$ and $J'(u_n)(1 + \|u_n\|) \to 0$. J is said to satisfy the $(C)_c -$ *condition* if any $(C)_c -$ *sequence* has a convergent subsequence.

Definition 2.2. A sequence $\{u_n\} \subset E$ is said to be a $(PS)-$*sequence* if $J(u_n) \leqslant c$ and $J'(u_n) \to 0,\ n \to \infty$. J is said to satisfy $(PS)-$*condition* if any $(PS)-$*sequence* has a convergent subsequence.

Definition 2.3. Let X be a Banach space, $J \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. Set

$$\Sigma = \{A \subset X - \{0\} : A \text{ is closed in } X \text{ and symmetric with respect to } 0\},$$
$$K_c = \{u \in X : J(u) = c, J'(u) = 0\},\ J^c = \{u \in X : J(u) \leqslant c\},$$

for $A \in \Sigma$, we say genus of A is n denoted by $\gamma(A) = n$ if there is an odd map $\phi \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property.

Theorem 2.1. ([18]) *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $J \in C^1(X, \mathbb{R})$ satisfies $(C)_c$ − condition for all $c > 0$, and*

- (J_1) *$J(0) = 0, J(-u) = J(u)$ for all $u \in X$;*
- (J_2) *there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho \cap Z} > \alpha$;*
- (J_3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $J(u) \leqslant 0$ on $\tilde{X} \setminus B_R$.*

Then J possesses an unbounded sequence of critical values.

Theorem 2.2. ([18]) *Let X be a Banach space, J be an even C^1 functional on X and satisfy the (PS) − condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \geqslant n\},\ c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} J(u).$$

- (i) *If $\Sigma_n \neq 0$ and $c_n \in \mathbb{R}$, then c_n is critical value of J;*
- (ii) *If there exists $k \in \mathbb{N}$ such that $c_n = c_{n+1} = \cdots = c_{n+k} = c \in \mathbb{R}$, and $c \neq J(0)$, then $\gamma(K_c) \geqslant k + 1$.*

3. The superlinear case

Lemma 3.1. *Assume (V_1), (V_2) and (f_1) are satisfied. Then $J(u)$ is well-defined and of class $C^1(E, \mathbb{R})$ and*

(3.1)
$$\langle J'(u), v\rangle = (u, v) + b\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)\int_{\mathbb{R}^N} \nabla_\lambda u \nabla_\lambda v dx - \int_{\mathbb{R}^N} f(x, u) v dx,\ u, v \in E.$$

And, the critical points of $J(u)$ in E are also solutions of problem (1.1).

Proof. We can get from (f_1), one has

$$|F(x, u)| \leqslant \frac{C_1}{2}|u|^2 + \frac{C_2}{p}|u|^p, \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.2}$$

for $2 \leqslant p < 2^*_\lambda$, where $F(x, u) = \int_0^u f(x, t)dt$. It can be known from Proposition 2.1 and the above formula, $J(u)$ defined by (2.4) is well-defined on E.

Let $H(u)=\int_{\mathbb{R}^N}F(x,u)dx$. For all $u,v\in E$ and $0<|t|<1$, by the Mean Value Theorem and (f_1), there exist $\theta\in(0,1)$ such that

$$\begin{aligned}&\frac{|F(x,u(x)+tv(x))-F(x,u(x))|}{|t|}=|f(x,u(x)+\theta tv(x))v(x)|\\ \leqslant&C_1|u(x)||v(x)|+C_1|v(x)|^2+C_2|u(x)+\theta tv(x)|^{p-1}|v(x)|\\ \leqslant&C_1|u(x)||v(x)|+C_1|v(x)|^2+2^{p-1}C_2(|u(x)|^{p-1}|v(x)|+|v(x)|^p).\end{aligned}$$

The Hölder inequality implies that

$$\begin{aligned}\int_{\mathbb{R}^N}|u(x)||v(x)|dx\leqslant&\|u(x)\|_p\|v(x)\|_{\frac{p}{p-1}},\\ \int_{\mathbb{R}^N}|v(x)|^2dx\leqslant&\|v(x)\|_p\|v(x)\|_{\frac{p}{p-1}},\\ \int_{\mathbb{R}^N}|u(x)|^{p-1}|v(x)|dx\leqslant&\|u(x)\|_p^{p-1}\|v(x)\|_p,\\ \int_{\mathbb{R}^N}|v(x)|^pdx\leqslant&\|v(x)\|_{\frac{p^2}{2}}^{p/2}\|v(x)\|_{\frac{p^2}{2(p-1)}}^{p/2(p-1)}.\end{aligned}$$

Hence,

$$\nu(x):=C_1|u(x)||v(x)|+C_1|v(x)|^2+2^{p-1}C_2(|u(x)|^{p-1}|v(x)|+|v(x)|^p)\in L^1(\mathbb{R}^N).$$

which implies $H(u)\in C^1(E,\mathbb{R})$. By Lebesgue's Dominated Convergence Theorem and Mean Value Theorem, we obtain

$$\begin{aligned}\langle H'(u),v\rangle=&\lim_{t\to0^+}\frac{H(u+tv)-H(u)}{t}=\lim_{t\to0^+}\int_{\mathbb{R}^N}\frac{F(x,u+tv)-F(x,u)}{t}dx.\\ =&\lim_{t\to0^+}\int_{\mathbb{R}^N}f(x+t\theta v)vdx=\int_{\mathbb{R}^N}f(x,u)vdx.\end{aligned}$$

Next, we prove the continuity of H'. Let $u_n\to u$ in E, then $u_n\to u$ in $L^p(\mathbb{R}^N)$ by Proposition 2.1 for $p\in[1,2_\lambda^*)$. Note that

$$\begin{aligned}&\|H'(u_n)-H'(u)\|=\sup_{\|v\|\leqslant1}|\langle H'(u_n)-H'(u),v\rangle|\\ =&\sup_{\|v\|\leqslant1}\left|\int_{\mathbb{R}^N}[f(x,u_n)-f(x,u)]vdx\right|\leqslant\sup_{\|v\|\leqslant1}\int_{\mathbb{R}^N}\big|f(x,u_n)-f(x,u)\big||v|dx.\end{aligned}$$

By the Hölder inequality

$$\begin{aligned}&\sup_{\|v\|\leqslant1}\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)||v|dx\\ \leqslant&\sup_{\|v\|\leqslant1}\left(\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}v^pdx\right)^{\frac{1}{p}}\to0,\end{aligned}$$

as $n\to\infty$. Hence, H' is continuous. This shows that (3.1) holds. Moreover, by a standard argument, it is easy to show that the critical points of J in E are solutions of problem (1.1). □

Lemma 3.2. *Assume that* $(V_1),(V_2),(f_1),(f_3),(f_4)$ *are satisfied. Then any* $(C)_c-$ *sequence* $\{u_n\}$ *of* J *is bounded in* E.

Proof. We will use the contradiction method to prove the boundness of $\{u_n\}$, assume that $\|u_n\| \to \infty,$ as $n \to \infty$. Let $\{u_n\} \subset E$ be $(C)_c - $*sequence* such that

$$J(u_n) \to c,\ (1+\|u_n\|)J'(u_n) \to 0, \tag{3.3}$$

then we have

$$c+1 \geqslant J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle. \tag{3.4}$$

Setting $v_n := \frac{u_n}{\|u_n\|}$, then $\|v_n\|$=1. And assume that

$$\begin{aligned} &v_n \rightharpoonup v \text{ in } E,\\ &v_n \to v \text{ in } L^p(\mathbb{R}^N),\ \text{for } 1 \leqslant p < 2^*_\lambda,\\ &v_n(x) \to v(x) \text{ a.e. } x \in \mathbb{R}^N. \end{aligned}$$

If $v=0$, then $v_n \to 0$ in $L^p(\mathbb{R}^N),\ \forall p \in [1, 2^*_\lambda)$, and $v_n(x) \to 0$ a.e. in $\mathbb{R}^N$. By (f_4) and (3.4), we have

$$\begin{aligned} \frac{c+1}{\|u_n\|^2} &\geqslant \frac{1}{\|u_n\|^2}\left(J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle\right)\\ &= \frac{1}{\|u_n\|^2}\left(\frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^N} \frac{1}{4} f(x,u_n)u_n - F(x,u_n)dx\right)\\ &\geqslant \frac{1}{4} - \beta\int_{\mathbb{R}^N} \frac{u_n^2}{\|u_n\|^2}dx = \frac{1}{4} - \beta\int_{\mathbb{R}^N} v_n^2 dx, \end{aligned} \tag{3.5}$$

as $n \to \infty$, which implies $\frac{1}{4} \leqslant 0$. Thus, it is a contradiction.

If $v \neq 0$. For $0 \leqslant \delta_0 < \delta_1$, let $A_n(\delta_0, \delta_1) = \{x \in \mathbb{R}^N : \delta_0 \leqslant |u_n| < \delta_1\}$. Setting $B := \{x \in \mathbb{R}^N : v(x) \neq 0\}$. Thus, $meas(B) > 0$. For almost every $x \in B$, we have $\lim_{n\to\infty} |v_n(x)| = \infty$. Hence, $B \subset A_n(r_0, \infty)$ for large $n \in \mathbb{N}$, where r_0 is given in (f_3). By (f_3), we have

$$\lim_{n\to\infty} \frac{|F(x,u_n)|}{\|u_n\|^4} = \lim_{n\to\infty} \frac{|F(x,u_n)|}{|u_n|^4}|v_n|^4 = \infty.$$

From Fatou's Lemma, (3.2) and (3.3) we can get

$$\begin{aligned}
0 &= \lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|^4} = \lim_{n\to\infty}\frac{J(u_n)}{\|u_n\|^4}\\
&= \lim_{n\to\infty}\frac{1}{\|u_n\|^4}\left(\frac{1}{2}\|u_n\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)^2-\int_{\mathbb{R}^N}F(x,u_n)dx\right)\\
&=\frac{b}{4}\lim_{n\to\infty}\frac{(\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx)^2}{\|u_n\|^4}+\lim_{n\to\infty}\frac{1}{\|u_n\|^4}\left(\frac{1}{2}\|u_n\|^2\right.\\
&\quad\left.-\int_{A_n(0,r_0)}F(x,u_n)dx-\int_{A_n(r_0,+\infty)}F(x,u_n)dx\right)\\
&\leqslant\frac{b}{4}+\lim_{n\to\infty}\left(\frac{1}{2\|u_n\|^2}-\int_{A_n(0,r_0)}\frac{F(x,u_n)}{|u_n|^2}\frac{|v_n|^2}{|u_n|^2}|v_n|^2dx-\int_{A_n(r_0,+\infty)}\frac{F(x,u_n)}{\|u_n\|^4}dx\right)\\
&\leqslant\frac{b}{4}+\limsup_{n\to\infty}\left[\frac{1}{2\|u_n\|^2}+\left(\frac{C_1}{2}+\frac{C_2}{p}r_0^{p-2}\right)\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N}|v_n|^2dx-\int_{A_n}\frac{F(x,u_n)}{|u_n|^4}|v_n|^4dx\right]\\
&\leqslant\frac{b}{4}+C_3-\liminf_{n\to\infty}\int_{A_n}\frac{F(x,u_n)}{|u_n|^4}|v_n|^4dx=-\infty,
\end{aligned} \tag{3.6}$$

which is a contradiction. Thus, $\{u_n\}$ is bounded in E. □

Lemma 3.3. *Assume that $(V_1),(V_2),(f_1)-(f_3)$ and (f_5) are satisfied. Then any $(C)_c-$ sequence $\{u_n\}$ of J is bounded in E.*

Proof. The proof method is similar to Lemma 3.2, also assuming that $\|u_n\|\to\infty$, as $n\to\infty$. We may assume that $v_n\rightharpoonup v$ in E, by Proposition 2.1, $v_n\to v$ in $L^p(\mathbb{R}^N)$ for $1\leqslant p<2^*_\lambda$, and $v_n(x)\to v(x)$ a.e. $x\in\mathbb{R}^N$.

If $v=0$, we define

$$J(t_nu_n)=\max_{t\in[0,1]}J(tu_n).$$

For any $K>0$, set $\overline{v}_n=\sqrt{4K}\frac{u_n}{\|u_n\|}=\sqrt{4K}v_n$, then $\|\overline{v}_n\|^2=4K$. By (3.2) and Proposition 2.1, we have

$$\left|\int_{\mathbb{R}^N}F(x,\overline{v}_n)dx\right|\leqslant\frac{C_1}{2}\int_{\mathbb{R}^N}|\overline{v}_n|^2dx+\frac{C_2}{p}\int_{\mathbb{R}^N}|\overline{v}_n|^pdx\to 0,\ n\to\infty.$$

Therefore, for a sufficiently large n such that

$$J(t_nu_n)\geqslant J(\overline{v}_n)=\frac{1}{2}\|\overline{v}_n\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla_\lambda\overline{v}_n|^2dx\right)^2-\int_{\mathbb{R}^N}F(x,\overline{v}_n)dx\geqslant K. \tag{3.7}$$

Hence, by$(f_3),(f_5)$, we obtain

$$\begin{aligned} J(u_n)-\frac{1}{4}\left\langle J'(u_n),u_n\right\rangle =&\frac{1}{4}\|u_n\|^2+\int_{\mathbb{R}^N}\left(\frac{1}{4}f(x,u_n)u_n-F(x,u_n)\right)dx\\ \geqslant&\frac{1}{4}\|t_nu_n\|^2+\int_{\mathbb{R}^N}\left(\frac{1}{4}f(x,t_nu_n)t_nu_n-F(x,t_nu_n)\right)dx\\ =&J(t_nu_n)-\frac{1}{4}\left\langle J'(t_nu_n),t_nu_n\right\rangle. \end{aligned}$$

According to (3.7), which implies $\lim_{n\to\infty}J(t_nu_n)=\infty$, and due to the choice of t_n we know $\langle J'(t_nu_n),t_nu_n\rangle=0$. That is, $J(u_n)-\frac{1}{4}\langle J'(u_n),u_n\rangle\geqslant\infty$, which contradicts with (3.4).

If $v\neq 0$, contradictions can be obtained by similar argument as (3.6). The proof is complete. □

Lemma 3.4. ([20]) *Assume that $p_1,p_2>1,r,q\geqslant 1$ and $\Omega\subseteq\mathbb{R}$. Let $g(x,t)$ be a Carathéodory function on $\mathbb{R}^N\times\mathbb{R}$ and satisfy*

$$|g(x,t)|\leqslant a_1|t|^{(p_1-1/r)}+a_2|t|^{(p_2-1/r)},\ \forall(x,t)\in\mathbb{R}^N\times\mathbb{R}, \tag{3.8}$$

where $a_1,a_2\geqslant 0$. If $u_n\to u$ in $L^{p_1}(\mathbb{R}^N)\cap L^{p_2}(\mathbb{R}^N)$, $u_n(x)\to u(x)$ a.e. $x\in\mathbb{R}^N$, then for any $v\in L^{p_1q}(\mathbb{R}^N)\cap L^{p_2q}(\mathbb{R}^N)$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|g(x,u_n)-g(x,u)|^r|v|^qdx=0. \tag{3.9}$$

Lemma 3.5. ([20]) *Assume that $p_1,p_2>1,r\geqslant 1$ and $\Omega\subseteq\mathbb{R}$. Let $g(x,t)$ be a Carathéodory function on $\mathbb{R}^N\times\mathbb{R}$ and satisfy (3.8). If $u_n\to u$ in $L^{p_1}(\mathbb{R}^N)\cap L^{p_2}(\mathbb{R}^N)$, $u_n(x)\to u(x)$ a.e. $x\in\mathbb{R}^N$, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|g(x,u_n)-g(x,u)|^r|u_n-u|dx=0. \tag{3.10}$$

Lemma 3.6. *Assume that $(V_1),(V_2),(f_1),(f_3)$ and (f_4) or (f_5) are satisfied. Then any $(C)_c-$ sequence $\{u_n\}$ has a convergent subsequence in E.*

Proof. By the previous lemma, we know that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can suppose that $u_n\rightharpoonup u$ in E. By Proposition 2.1, $u_n\to u$ in $L^p(\mathbb{R}^N)$ for $1\leqslant p<2^*_\lambda$, and together with by Lemma 3.5, one has

$$\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)||u_n-u|dx\to 0,n\to\infty. \tag{3.11}$$

Observe that,

$$
\begin{aligned}
\langle J'(u_n)-J'(u),u_n-u\rangle =&\|u_n-u\|^2+b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u_n\nabla_\lambda(u_n-u)dx\\
&-b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx\\
&-\int_{\mathbb{R}^N}\big[f(x,u_n)-f(x,u)\big](u_n-u)dx\\
=&\|u_n-u\|^2+b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda|u_n-u|^2dx\\
&-b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx\\
&-\int_{\mathbb{R}^N}\big[f(x,u_n)-f(x,u)\big](u_n-u)dx\\
\geqslant&\|u_n-u\|^2-\int_{\mathbb{R}^N}\big[f(x,u_n)-f(x,u)\big](u_n-u)dx\\
&-b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx.
\end{aligned}
\tag{3.12}
$$

It is clear that,

$$
\begin{aligned}
\|u_n-u\|^2\leqslant&\langle J'(u_n)-J'(u),u_n-u\rangle+\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx\\
&+b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx.
\end{aligned}
\tag{3.13}
$$

By the definition of weak convergence, we have

$$\langle J'(u_n)-J'(u),u_n-u\rangle\to 0,\ n\to\infty. \tag{3.14}$$

Set $\overline{E}=\{u\in L^2(\mathbb{R}^N):\nabla_\lambda u\in L^2(\mathbb{R}^N)\}$ with the norm $\|u\|_{\overline{E}}=\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)^{\frac{1}{2}}$. Then the embedding $E\hookrightarrow\overline{E}$ is continuous. Hence, $u_n\rightharpoonup u$ in $\overline{E}$. According to the boundedness of $\{u_n\}$ in E, one has

$$b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx\to 0,\ \text{as } n\to\infty. \tag{3.15}$$

From the (3.11)-(3.15) we can get $u_n\to u$ in $E,$ as $n\to\infty.$ □

Let $\{e_j\}$ is an orthonormal basis of E and define $X_j=\mathbb{R}e_j$,

$$Y_k=\oplus_{j=1}^{k}X_j,\ Z_k=\overline{\oplus_{j=k+1}^{\infty}X_j},\ k\in\mathbb{Z}.$$

Lemma 3.7. *Assume that* (V_1) *and* (V_2) *are satisfied. Then*

$$\beta_k:=\sup_{u\in Z_k,\|u\|=1}\|u\|_p\to 0,\ k\to\infty,\ p\in[1,2_\lambda^*).$$

Proof. It is clear that $0<\beta_{k+1}\leqslant\beta_k$, so that $\beta_k\to\beta\geqslant 0,\ k\to\infty$. For every $k\in\mathbb{N}$, there exists $u_k\in Z_k$ such that $\|u_k\|_2>\frac{\beta_k}{2}$ and $\|u_k\|=1$. We denote

$v = \sum_{j=1}^{\infty} c_j e_j$, for any $v \in E$ by the Cauchy-Schwarz inequality, one has

$$\begin{aligned} |(u_k, v)| &= \left| \left(u_k, \sum_{j=1}^{\infty} c_j e_j \right) \right| = \left| \left(u_k, \sum_{j=k}^{\infty} c_j e_j \right) \right| \leqslant \|u_k\| \left\| \sum_{j=k}^{\infty} c_j e_j \right\| \\ &\leqslant \left(\sum_{j=k}^{\infty} c_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=k}^{\infty} e_j^2 \right)^{\frac{1}{2}} = \left(\sum_{j=k}^{\infty} c_j^2 \right)^{\frac{1}{2}} \to 0, \text{ as } k \to \infty, \end{aligned}$$

which implies that $u_k \rightharpoonup 0$ in E. By Proposition 2.1, we have $u_k \to 0$ in $L^p(\mathbb{R}^N)$. Hence, letting $k \to \infty$, we get $\beta = 0$. □

Lemma 3.8. *Assume that* (V_1), (V_2) *and* (f_1) *are satisfied, there exist constants* $\rho,\ \alpha \geqslant 0$ *such that* $J_\lambda|_{\partial B_\rho \cap Z_m} \geqslant \alpha$.

Proof. By Lemma 3.7, we can choose an integer $m \geqslant 1$ such that

$$\|u\|_2^2 \leqslant \frac{1}{2C_1}\|u\|^2,\ \|u\|_p^p \leqslant \frac{p}{4C_2}\|u\|^p,\ \forall u \in Z_m. \tag{3.16}$$

According to (2.4) (3.2) and (3.16), for $u \in Z_m$, we have

$$\begin{aligned} J(u) =& \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx \right)^2 - \int_{\mathbb{R}^N} F(x,u)dx \\ \geqslant& \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \geqslant \frac{1}{2}\|u\|^2 - \frac{C_1}{2}\|u\|_2^2 - \frac{C_2}{p}\|u\|_p^p \\ \geqslant& \frac{1}{4}(\|u\|^2 - \|u\|^p) = \frac{2^{p-2}-1}{2^{p+2}} := \alpha \geqslant 0, \end{aligned}$$

choosing $\rho = \|u\| = \frac{1}{2}$. □

Lemma 3.9. *Assume that* (V_1), (V_2), (f_1) *and* (f_3) *are satisfied. Then for any finite dimensional subspace* $\overline{E} \subset E$, *there is* $R = R(\overline{E}) > 0$ *such that*

$$J(u) \leqslant 0,\ \forall u \in \overline{E} \setminus B_R.$$

Proof. For any $\overline{E} \subset E$, there is a positive integral number m such that $\overline{E} \subset E_m$. Since all norms are equivalent in finite dimensional space, there is a constant $\eta > 0$ such that

$$\|u\|_4 \geqslant \eta\|u\|,\ \forall u \in E_m. \tag{3.17}$$

By (f_1) and (f_3), one has

$$F(x,u) \geqslant \delta|u|^4 - C_\delta|u|^2,\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.18}$$

for any $\delta > \frac{b}{4C_4^4}$ and constant $C_\delta > 0$. Hence, by (3.17) and (3.18), we have

$$J(u) \leqslant \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \delta\|u\|_4^4 + C_\delta\|u\|_2^2 \leqslant \left(\frac{1}{2} + C_\delta C_2^2 \right)\|u\|^2 - \left(\delta C_4^4 - \frac{b}{4} \right)\|u\|^4, \forall u \in E_m.$$

Hence, there is a large $R = R(\overline{E}) > 0$ such that $J(u) \leqslant 0$ for all $u \in \overline{E} \setminus B_R$. □

Proof of Theorem 1.1. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. Obviously, $J(0) = 0$ and (f_2) implies J is even. By Lemmas 3.2, 3.6, 3.8 and 3.9, all conditions of Theorem 2.1 are satisfied. Thus, problem (1.1) possesses infinitely many nontrivial sequence solutions $\{u_k\}$ such that $J(u_k) \to \infty$ as $k \to \infty$. □

Proof of Theorem 1.2. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. Obviously, $J(0) = 0$ and (f_2) implies J is even. By Lemmas 3.3, 3.6, 3.8 and 3.9, all conditions of Theorem 2.1 are satisfied. Thus, problem (1.1) possesses infinitely many nontrivial sequence solutions $\{u_k\}$ such that $J(u_k) \to \infty$ as $k \to \infty$. □

4. The sublinear case

Lemma 4.1. *Assume that* (V_1),(V_2),(f_2), (f_6),(f_7) *are satisfied. Then the* J *satisfies the* (PS)*-condition.*

Proof. Obviously, from (V_1), (f_6), we know the functional $J \in C^1$ and also have the derivative functional (3.1). According to the (f_6), one has

$$|F(x,u)| \leqslant |u|^{q_1} + |u|^{q_2},\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{4.1}$$

From the above formula, for $1 < q_1 < q_2 < 2$, we can get

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geqslant \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \geqslant \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left(|u|^{q_1} + |u|^{q_2}\right)dx \\ &\geqslant \frac{1}{2}\|u\|^2 - C_1'\left(\|u\|^{q_1} + \|u\|^{q_2}\right) \to \infty. \end{aligned} \tag{4.2}$$

as $\|u\| \to \infty$. Hence J is bounded from below. Next we show that J satisfies (PS)-condition. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset E$ is (PS)-sequence. Therefore, according to (2.1), there exist a constant $\eta > 0$, such that

$$\|u\|_2 \leqslant C_2\|u\| < \eta. \tag{4.3}$$

By Proposition 2.1 let a subsequence still denoted by $\{u_n\}$, such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } E, \\ &u_n \to u \text{ in } L^p(\mathbb{R}^N),\ \text{for } 1 \leqslant p < 2^*_\lambda. \end{aligned}$$

It follows from (f_6) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \big|f(x,u_n)-f(x,u)\big|\big|u_n-u\big|dx \\
&\leqslant \int_{\mathbb{R}^N} \Big|q_1(|u_n|^{q_1-1}-|u|^{q_1-1})+q_2(|u_n|^{q_2-1}-|u|^{q_2-1})\Big|\Big|u_n-u\Big|dx \\
&\leqslant q_1\left(\int_{\mathbb{R}^N}|u_n-u|^{q_1}dx\right)^{\frac{1}{q_1}}\left[\left(\int_{\mathbb{R}^N}|u_n|^{q_1}dx\right)^{\frac{q_1-1}{q_1}}-\left(\int_{\mathbb{R}^N}|u|^{q_1}dx\right)^{\frac{q_1-1}{q_1}}\right] \\
&\quad+q_2\left(\int_{\mathbb{R}^N}|u_n-u|^{q_2}dx\right)^{\frac{1}{q_2}}\left[\left(\int_{\mathbb{R}^N}|u_n|^{q_2}dx\right)^{\frac{q_2-1}{q_2}}-\left(\int_{\mathbb{R}^N}|u|^{q_2}dx\right)^{\frac{q_2-1}{q_2}}\right]\to 0,
\end{aligned}
\tag{4.4}
$$

as $n\to\infty$. According to (3.12), we know

$$
\begin{aligned}
\|u_n-u\|^2 \leqslant&\langle J'(u_n)-J'(u),u_n-u\rangle+\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx \\
&+b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx.
\end{aligned}
\tag{4.5}
$$

It follow from (3.14), (3.15), (4.4) and (4.5), we have $\|u_n-u\|\to 0$, as $n\to\infty$. $\square$

Proof of Theorem 1.3. We take n disjoint open sets $\tilde{B}_i$ for any $n\in\mathbb{N}$, such that $\bigcup_{i=1}^n \tilde{B}_i\subset\tilde{B}$. Let $u_i\in\big(W_0^{1,2}(\tilde{B}_i)\bigcap E\big)\setminus\{0\}$ and $\|u_i\|_E=1$, $i=1,2,...,n$, and

$$\Lambda_1=span\{u_1,u_2,\cdots,u_n\},\ \Lambda_2=\{u\in\Lambda_1:\|u\|_E=1\}.$$

For any $u\in\Lambda_1$, there exist $\tau_i\in\mathbb{R},\ i=1,2,...,n$ such that

$$u(x)=\sum_{i=1}^n\tau_iu_i(x),\ x\in\mathbb{R}^N. \tag{4.6}$$

Hence,

$$
\begin{aligned}
\|u\|_{q_3}&=\left(\int_{\mathbb{R}^N}|u|^{q_3}dx\right)^{\frac{1}{q_3}}=\left(\int_{\mathbb{R}^N}\left|\sum_{i=1}^n\tau_iu_i(x)\right|^{q_3}dx\right)^{\frac{1}{q_3}} \\
&=\left(\sum_{i=1}^n|\tau_i|^{q_3}\int_{\tilde{B}_i}|u_i(x)|^{q_3}dx\right)^{\frac{1}{q_3}},
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
\|u\|^2&=\int_{\mathbb{R}^N}(a|\nabla_\lambda u|^2+V(x)u^2)dx=\sum_{i=1}^n\tau_i^2\int_{\tilde{B}_i}(a|\nabla_\lambda u_i|^2+V(x)u_i^2)dx \\
&=\sum_{i=1}^n\tau_i^2\|u_i\|^2=\sum_{i=1}^n\tau_i^2,
\end{aligned}
\tag{4.8}
$$

which together with (4.7) implies there exists a constant $\kappa>0$ such that

$$\kappa\|u\|\leqslant\|u\|_{q_3},\ u\in\Lambda_1. \tag{4.9}$$

It follows from (4.6) − (4.9) and (f_7), we have

$$\begin{aligned}
J(tu) &= t^2\|u\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,tu)dx \\
&\leqslant t^2\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \sum_{i=1}^{n}\int_{\tilde{B}_i} F(x,t\tau_i u_i)dx \\
&\leqslant t^2\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \xi t^{q_3}\sum_{i=1}^{n}|\tau_i|^{q_3}\int_{\tilde{B}_i} |u_i|^{q_3}dx \\
&= t^2\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \xi t^{q_3}\|u\|_{q_3}^{q_3} \\
&\leqslant t^2\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \xi (t\kappa)^{q_3}\|u\|^{q_3} \\
&= t^2 + \frac{bt^4}{4} - \xi(t\kappa)^{q_3} := -\sigma,\ u \in \Lambda_2.
\end{aligned}$$

Hence, there exist $0 < t < 1$ and $\sigma > 0$ such that $J(tu) < -\sigma$, $u \in \Lambda_2$. Let

$$\Lambda_2' = \{tu : u \in \Lambda_2\},\ \tilde{B} = \left\{(\tau_1, \tau_2, \cdots, \tau_n) \in \mathbb{R}^n : \sum_{i=1}^{n}\tau_i^2 < t^2\right\}.$$

Therefore $J(u) < -\sigma,\ u \in \Lambda_2'$. And by (f_2), we know J is even and $J(0) = 0$, can deduce $\Lambda_2' \subset J^{-\sigma} \in \Sigma$. Also, in view of $(4.6), (4.8)$, there exist an odd mapping $\varphi \in C(\Lambda_2', \partial\tilde{B})$. By properties of the genus, we obtain that

$$\gamma(J^{-\sigma}) \geqslant \gamma(\Lambda_2') = n. \tag{4.10}$$

Hence, we get for any $n \in \mathbb{N}$, there exists $\sigma > 0$ such that $\gamma(J^{-\sigma}) \geqslant n$. Now let

$$c_n = \inf_{A\in\Sigma_n}\sup_{u\in A} J(u).$$

In view of J is bounded below on E and (4.10), one has

$$-\infty < c_n < -\sigma < 0. \tag{4.11}$$

In other words, for any $n \in \mathbb{N},\ c_n$ is negative real number. Thus, we can apply the Theorem 2.2 to get that problem (1.1) has infinitely many solutions.

□

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INTEGRAL INEQUALITIES FOR THE GROWTH AND HIGHER DERIVATIVE OF POLYNOMIALS

N. A. RATHER, A. BHAT, M. SHAFI

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Department of Mathematics, University of Kashmir, Srinagar, India
E-mails: *dr.narather@gmail.com; aijazb824@gmail.com; wanishafi1933@gmail.com*

Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| \leqslant 1$, it was proved by S. Gulzar [Anal Math 42, 339-352 (2016). https://doi.org/10.1007/s10476-016-0403-7] that

$$\left\| z^s P^{(s)}(z) + \beta \frac{n(n-1)...(n-s+1)}{2^s} P(z) \right\|_p \leqslant n(n-1)...(n-s+1) \left\| \left(1+\frac{\beta}{2^s}\right) z + \frac{\beta}{2^s} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}$$

for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $1 \leqslant s \leqslant n$ and $0 \leqslant p < \infty$. In this paper we extend the above result to the growth of polynomials and also generalize the above and other related results in this direction.

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1. Introduction

Let $\mathcal{P}_n$ denote the space of all polynomials of degree at most n over the field of complex numbers. The subject of inequalities for polynomials and related classes of functions plays an important and crucial role in obtaining inverse theorems in Approximation Theory. The extremal problems of analytic functions and the results were some approaches to obtaining the classical inequalities are developed on using various methods of the geometric function theory are known for various norms and for many classes of functions such as polynomials with various constraints and in various regions of the complex plane. A classical result due to Bernstein [4] is that, for two polynomials $P(z)$ and $T(z)$ with degree of $P(z)$ not exceeding that of $T(z)$ and $T(z) \neq 0$ for $|z| > 1$, the inequality $|P(z)| \leqslant |T(z)|$ on the unit circle $|z| = 1$ implies the inequality of their derivatives $|P'(z)| \leqslant |T'(z)|$ on $|z| = 1$. In particular, for $T(z) = z^n \max_{|z|=1} |P(z)|$ gives a famous Bernstein inequality namely, if $P(z)$ is a polynomial of degree n then

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1.1}$$

On the other hand, concerning the growth of polynomials we have for $P \in \mathcal{P}_n$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then $|Q(z)| = |P(z)|$ for $|z| = 1$. This implies $|Q(z)| \leqslant \max_{|z|=1} |P(z)|$ for $|z| = 1$. This further implies, by using maximum modulus theorem, that $|Q(z)| \leqslant \max_{|z|=1}|P(z)|$ for $|z| \leqslant 1$ or equivalently $|z^n\overline{P(\frac{1}{\bar{z}})}| \leqslant \max_{|z|=1}|P(z)|$. If we take $z = e^{i\theta}/R$ where $\theta \in [0, 2\pi)$ and $R \geqslant 1$, we get $|(e^{in\theta}/R^n)\overline{P(Re^{i\theta})}| \leqslant \max_{|z|=1}|P(z)|$. Hence, the growth estimate for $|P(z)|$ over a large cricle $|z| = R$ in comparsion with its maximum modulus over the unit circle $|z| = 1$ is given by

$$\max_{|z|=R}|P(z)| \leqslant R^n \max_{|z|=1}|P(z)|, \qquad R > 1. \tag{1.2}$$

These inequalities (1.1) and (1.2) are related with each other and have been the starting point of a considerable literature in polynomial approximations and these inequalities were generalized and extended in several directions, in different norms and for different classes of functions.

Define the standard Hardy space norm for $P \in \mathcal{P}_n$ by

$$\|P\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right)^{1/p}, \quad 0 < p < \infty;$$

and the Mahler measure by

$$\|P\|_0 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|P(e^{i\theta})|d\theta\right).$$

It is well known that $\lim_{p\to 0+}\|P\|_p = \|P\|_0$. We also note that the supremum norm of the space H^∞ satisfies $\|P\|_\infty := \lim_{p\to\infty}\|P\|_p = \max_{|z|=1}|P(z)|$.

If $P \in \mathcal{P}_n$, then

$$\|P'(z)\|_p \leqslant n\|P(z)\|_p, \qquad p \geqslant 1, \tag{1.3}$$

and for $R \geqslant 1$

$$\|P(Rz)\|_p \leqslant R^n\|P(z)\|_p, \qquad p > 0. \tag{1.4}$$

The inequality (1.3) is due to Zygmund [16], whereas the inequality (1.4) is a simple consequences of a result due to Hardy [8]. Arestov [2] verified that (1.3) remains true for $0 \leqslant p < 1$ as well. Also inequalities (1.3) and (1.4) are further generalized by Aziz and Rather [3] as

$$\left\|zP'(z) + \beta\frac{n}{2}P(z)\right\|_p \leqslant n\left|1 + \frac{\beta}{2}\right|\|P(z)\|_p, \qquad p > 0, \tag{1.5}$$

and

$$\left\|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right\|_p \leqslant \left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|\|P(z)\|_p, \qquad p > 0, \tag{1.6}$$

respectively for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $R \geqslant 1$. For $p = \infty$, inequalities (1.5) and (1.6) are due to Jain [10].

The inequalities (1.3) and (1.4) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact if $P(z) \neq 0$ for $|z| < 1$, the inequality (1.3) can be replaced by

$$\|P'(z)\|_p \leqslant n\frac{\|P(z)\|_p}{\|1+z\|_p}, \qquad 0 \leqslant p \leqslant \infty, \tag{1.7}$$

whereas the inequality (1.4) can be replaced by

$$\|P(Rz)\|_p \leqslant n\frac{\|1+R^n z\|_p}{\|1+z\|_p}\|P(z)\|_p, \qquad 0 \leqslant p \leqslant \infty. \tag{1.8}$$

For $p \geqslant 1$, inequality (1.7) is due to de Brujin [6] and inequality (1.8) is due to Boas and Rahman. Rahman and Schmeisser [14] extended both for $0 \leqslant p < 1$. For $p = \infty$, inequality (1.7) was conjectured by Erdös and later verified by Lax [12] and inequality (1.8) by Ankeny and Rivlin [1]. Inequalities (1.7) and (1.8) are further generalized by Aziz and Rather [[3] corollary 5, 6] as

$$\left\|zP'(z) + \beta\frac{n}{2}P(z)\right\|_p \leqslant n\left\|\left(1+\frac{\beta}{2}\right)z + \frac{\beta}{2}\right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}, \qquad p > 0, \tag{1.9}$$

and

$$\begin{aligned}&\left\|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right\|_p \\ &\leqslant \left\|\left(R^n + \beta\left(\frac{R+1}{2}\right)^n\right)z + 1 + \beta\left(\frac{R+1}{2}\right)^n\right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}, \qquad p > 0,\end{aligned} \tag{1.10}$$

respectively for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $R \geqslant 1$. For $p = \infty$, inequalities (1.9) and (1.10) are due to Jain [10] which were further generalized by Hans and Lal [9] for s^{th} derivative of polynomials. Recently S. Gulzar [7] obtained an L_p version of Hans and Lal [9] results and proved following theorems:

Theorem A. *If $P \in \mathcal{P}_n$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $1 \leqslant s \leqslant n$, and $0 \leqslant p < \infty$*

$$\left\|z^s P^{(s)}(z) + \beta\frac{n_s}{2^s}P(z)\right\|_p \leqslant n_s\left|1+\frac{\beta}{2^s}\right|\|P(z)\|_p, \tag{1.11}$$

where $n_s = n(n-1)(n-2)\dots(n-s+1)$.

Theorem B. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| \leqslant 1$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $1 \leqslant s \leqslant n$, and $0 \leqslant p < \infty$*

$$\left\|z^s P^{(s)}(z) + \beta\frac{n_s}{2^s}P(z)\right\|_p \leqslant n_s\left\|\left(1+\frac{\beta}{2^s}\right)z + \frac{\beta}{2^s}\right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}, \tag{1.12}$$

where $n_s = n(n-1)(n-2)\dots(n-s+1)$.

2. Main results

In this paper, we first present the following interesting result which is compact generalization of inequalities (1.3) – (1.6) and (1.11).

Theorem 2.1. *If $P \in \mathcal{P}_n$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\left\| z^s P^{(s)}(Rz) + \beta (R+1)^{n-s} \frac{s!C(n,s)}{2^n} P(z) \right\|_p$$

$$\leqslant s!C(n,s) \left| R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right| \|P(z)\|_p . \tag{2.1}$$

The result is best possible and equality in (2.1) *holds for* $P(z) = cz^n$, $c \neq 0$.

For taking $R = 1$ in (2.1) we obtain (1.11). The following result is obtained by letting $p \to \infty$ in (2.1).

Corollary 2.1. *If $P \in \mathcal{P}_n$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\left\| z^s P^{(s)}(Rz) + \beta (R+1)^{n-s} \frac{s!C(n,s)}{2^n} P(z) \right\|_\infty$$

$$\leqslant s!C(n,s) \left| R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right| \|P(z)\|_\infty . \tag{2.2}$$

The result is best possible and equality holds for $P(z) = cz^n$, $c \neq 0$.

Taking $\beta = 0$ in (2.1), we get the following compact generalization of inequalities of (1.3) and (1.4).

Corollary 2.2. *If $P \in \mathcal{P}_n$, then for $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\left\| z^s P^{(s)}(Rz) \right\|_p \leqslant s!C(n,s) R^{n-s} \|P(z)\|_p . \tag{2.3}$$

For taking both $s = 1$ and $R = 1$ in (2.3), we get inequality (1.3) and for taking $s = 0$, inequality (2.3) reduces to (1.4).

Remark 1. Inequality (1.5) can be obtained by putting $s = 1$ and $R = 1$ in (2.1) and for $s = 0$, inequality (2.1) reduces to (1.6).

Next, we present the following compact generalization of the inequalities (1.7), (1.8), (1.9), (1.10) and (1.12).

Theorem 2.2. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| \leqslant 1$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

(2.4)

$$\left\| z^s P^{(s)}(Rz) + (R+1)^{n-s} \beta \frac{s!C(n,s)}{2^n} P(z) \right\|_p$$

$$\leqslant s!C(n,s) \left\| \left(R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right) z + \frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\beta}{2^n} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p} .$$

The result is best possible and equality in (2.4) *holds for* $P(z) = az^n + b$, $|a| = |b| = 1$.

Remark 2. By letting $p \to \infty$ in (2.4), we obtain a result due to Jain [[11], Theorem 3]. Inequality (1.12) can be obtained by putting $R = 1$ in (2.4).
The following is compact generalization of inequalities (1.7) and (1.8) is obtained by putting $\beta = 0$ in (2.4).

Corollary 2.3. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| \leqslant 1$, then for $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\left\| z^s P^{(s)}(Rz) \right\|_p \leqslant s! C(n,s) \left\| R^{n-s} z + \frac{d^s(1)}{dz^s} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}. \tag{2.5}$$

For $s = 1$ and $R = 1$, inequality (2.5) reduces to (1.7) and inequality (1.8) is obtained by putting $s = 0$ in (2.5). Also for $s = 1$ and $R = 1$ in (2.4), we obtain (1.9) and inequality (1.10) can be obtained by putting $s = 0$ in (2.4).

Finally, we establish the following result for self-inversive polynomials.

Theorem 2.3. *If $P \in \mathcal{P}_n$ and $P(z)$ is a self-inversive polynomial, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

(2.6)

$$\left\| z^s P^{(s)}(Rz) + (R+1)^{n-s} \beta \frac{s! C(n,s)}{2^n} P(z) \right\|_p$$
$$\leqslant s! C(n,s) \left\| \left(R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right) z + \frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\beta}{2^n} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}.$$

If we let $p \to \infty$ in (2.6) , we obtain the following result:

Corollary 2.4. *If $P \in \mathcal{P}_n$ and $P(z)$ is a self-inversive polynomial, then for $R \geqslant 1$, and $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$*

(2.7)

$$\left\| z^s P^{(s)}(Rz) + (R+1)^{n-s} \beta \frac{s! C(n,s)}{2^n} P(z) \right\|_\infty$$
$$\leqslant \frac{s! C(n,s)}{2} \left\{ \left| R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right| + \left| \frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\beta}{2^n} \right| \right\} \|P(z)\|_\infty .$$

3. Lemmas

For the proof of these theorems, we need the following lemmas. The first lemma is the following well known-result ([[13] Theorem 14.1.2 and its proof , corollary 12.1.3] and [[6] Theorem 1 and its proof]).

Lemma 3.1. *Let $F \in \mathcal{P}_n$ and let P be a polynomial of degree at most n, such that $|P(z)| \leqslant |F(z)|$ for $|z| = 1$. If $F(z) \neq 0$ for $|z| < 1$ (resp. $|z| > 1$) and for every $z \in \mathbb{C}$ and every $\alpha, P(z) \neq e^{i\alpha} F(z)$, then*
(i) $|P(z)| \leqslant |F(z)|$ for $|z| < 1$ (resp. $|z| > 1$),

(ii) $F(z)+\beta P(z)\neq 0$ *for* $|z|<1$ *(resp.* $|z|>1$*) and* $\beta\in\mathbb{C}$ *with* $|\beta|\leqslant 1$ *and*
(iii)$P(z)+\lambda F(z)\neq 0$ *for* $|z|<1$ *(resp.* $|z|>1$*) and* $\lambda\in\mathbb{C}$ *with* $|\lambda|\geqslant 1$.

Lemma 3.2. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *have all its zeros in* $|z|\leqslant 1$*, then for every* $R>1$*, and* $|z|=1$*,*

$$|P(Rz)|\geqslant\left(\frac{R+1}{2}\right)^n|P(z)|.$$

Proof. Since all the zeros of $P(z)$ lie in $|z|\leqslant 1$, we write

$$P(z)=c\prod_{j=1}^{n}\left(z-r_je^{i\theta_j}\right),$$

where $r_j\leqslant 1$. Now for $0\leqslant\theta<2\pi$, $R>1$, we have

$$\begin{aligned}\left|\frac{Re^{i\theta}-r_je^{i\theta_j}}{e^{i\theta}-r_je^{i\theta_j}}\right|&=\left\{\frac{R^2+r_j^2-2Rr_j\cos(\theta-\theta_j)}{1+r_j^2-2r_j\cos(\theta-\theta_j)}\right\}^{1/2}\\&\geqslant\left\{\frac{R+r_j}{1+r_j}\right\}\geqslant\left\{\frac{R+1}{2}\right\},\quad\text{for }j=1,2,\cdots,n.\end{aligned}$$

Hence

$$\left|\frac{P(Re^{i\theta})}{P(e^{i\theta})}\right|=\prod_{j=1}^{n}\left|\frac{Re^{i\theta}-r_je^{i\theta_j}}{e^{i\theta}-r_je^{i\theta_j}}\right|\geqslant\prod_{j=1}^{n}\left(\frac{R+1}{2}\right)=\left(\frac{R+1}{2}\right)^n$$

for $0\leqslant\theta<2\pi$. This implies for $|z|=1$ and $R>1$,

$$|P(Rz)|\geqslant\left(\frac{R+1}{2}\right)^n|P(z)|,$$

which completes the proof of Lemma 3.2. □

By applying lemma 3.2 to the polynomial $P^s(z)$, $(1\leqslant s\leqslant n)$, we obtain

Lemma 3.3. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *have all its zeros in* $|z|\leqslant 1$*, then for* $1\leqslant s\leqslant n$

$$\left|P^{(s)}(Rz)\right|\geqslant\left(\frac{R+1}{2}\right)^{n-s}\left|P^{(s)}(z)\right|,\qquad R\geqslant 1\quad\textit{and}\quad|z|=1.$$

Lemma 3.4. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *have all its zeros in* $|z|\leqslant 1$*, then for* $0\leqslant s\leqslant n$*,*

$$|z^sP^{(s)}(z)|\geqslant\frac{s!C(n,s)}{2^s}|P(z)|,\qquad R\geqslant 1\quad\textit{and}\quad|z|=1.$$

The above lemma is simply consequences of repeated application of Turán theorem [15].

Lemma 3.4 along with lemma (3.3) leads to following lemma:

Lemma 3.5. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *have all its zeros in* $|z|\leqslant 1$*, then for* $0\leqslant s\leqslant n$*,*

$$|z^sP^{(s)}(Rz)|\geqslant(R+1)^{n-s}\frac{s!C(n,s)}{2^n}|P(z)|,\qquad R\geqslant 1\quad\textit{and}\quad|z|=1,$$

and for every $\beta\in\mathbb{C}$ *with* $|\beta|\leqslant 1$*, the zeros of polynomial*

$$z^sP^{(s)}(Rz)+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}P(z)$$

lies in $|z| \leqslant 1$.

The second part of above lemma is the consequences of lemma 3.1.
The next lemma is due to Jain [11].

Lemma 3.6. *Let* $F(z)$ *be a polynomial of degree* n *having all its zeros in* $|z| \leqslant 1$ *and* $P(z)$ *be a polynomial of degree not exceeding that of* $F(z)$ *such that*

$$|P(z)| \leqslant |F(z)|, \qquad |z| = 1,$$

then for $R \geqslant 1$, $0 \leqslant s \leqslant n$, *and* $|\beta| \leqslant 1$

$$\left|z^s P^{(s)}(Rz) + \beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}P(z)\right| \leqslant$$
$$\left|z^s F^{(s)}(Rz) + \beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}F(z)\right| \quad for \quad |z| \geqslant 1.$$

The next lemma follows immediately from lemma 3.6 by taking $F(z) = Q(z)$ where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Lemma 3.7. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, *and* $R \geqslant 1$

$$\left|z^s P^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(z)\right| \tag{3.1}$$
$$\leqslant \left|z^s Q^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}Q(z)\right| \quad for |z| \geqslant 1,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Lemma 3.8. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$ *and* $Q(z) = z^n\overline{P(1/\overline{z})}$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, *and* α *real*

$$\left(z^s P^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(z)\right)e^{i\alpha} + z^n\overline{M(1/\overline{z})} \neq 0 \qquad for \quad |z| < 1,$$

where $M(z) = z^s Q^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}Q(z)$.

Proof. Since $P(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish in $|z| < 1$, therefore by lemma 3.7 for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $|z| = 1$, we have

$$\left|z^s P^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(z)\right| \leqslant \left|z^s Q^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}Q(z)\right|$$
$$= |M(z)| = |z^n\overline{M(1/\overline{z})}|.$$

Since $P(0) \neq 0$ implies deg$Q(z) = n$. Moreover $Q(z) \neq 0$ for $|z| > 1$ and then lemma 3.5 implies that $M(z) \neq 0$ for $|z| > 1$. Therefore $z^n\overline{M(1/\overline{z})} \neq 0$ for $|z| < 1$. Then by lemma 3.1 for $|z| < 1$

$$\left(z^s P^{(s)}(Rz) + \beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}P(z)\right)e^{i\alpha} + z^n\overline{M(1/\overline{z})} \neq 0. \qquad \square$$

Next we describe a result of Arestov [2].
For $\delta = (\delta_0, \delta_1, \cdots, \delta_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$, we define

$$\Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j.$$

The operator Λ_δ is said to be admissible if it preserves one of the following properties:
(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leqslant 1\}$,
(ii) $P(z)$ has all its zeros in$\{z \in \mathbb{C} : |z| \geqslant 1\}$.
The result of Arestov [2] may now be stated as follows.

Lemma 3.9. [2, Theorem 4] *Let $\phi(x) = \psi(logx)$ where ψ is a convex non decreasing function on $\mathbb{R}$. Then for all $P \in \mathcal{P}_n$ and each admissible operator Λ_δ,*

$$\int_0^{2\pi} \phi(|\Lambda_\delta P(e^{i\theta})|) d\theta \leqslant \int_0^{2\pi} \phi(A(\delta, n)|P(e^{i\theta})|) d\theta,$$

where $A(\delta, n) = max(|\delta_0|, |\delta_n|)$.

In particular, Lemma 3.9 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$. Therefore, we have

$$\left\{\int_0^{2\pi} (|\Lambda_\delta P(e^{i\theta})|^p) d\theta\right\}^{1/p} \leqslant A(\delta, n) \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{3.2}$$

From lemma 3.9, we deduce the following result:

Lemma 3.10. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, α real, and $p > 0$,*

$$\int_0^{2\pi} \left| \left(e^{is\theta} P^{(s)}(Re^{i\theta}) + \beta (R+1)^{n-s} \frac{s!C(n,s)}{2^n} P(e^{i\theta}) \right) e^{i\alpha} + e^{in\theta} \overline{M(e^{i\theta})} \right|^p d\theta$$

(3.3)

$$\leqslant (s!C(n,s))^p \left| \left(R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right) e^{i\alpha} + \frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\overline{\beta}}{2^n} \right|^p \int\limits_0^{2\pi} |P(e^{i\theta})|^p d\theta,$$

where $M(z) = z^s Q^{(s)}(Rz) + \beta (R+1)^{n-s} \frac{s!C(n,s)}{2^n} Q(z)$.

Proof. Since $P(z) = \sum\limits_{j=o}^{n} a_j z^j$ does not vanish in $|z| < 1$. Therefore by lemma 3.8, the polynomial

$$\Lambda_\delta P(z) = \left(z^s P^{(s)}(Rz) + \beta (R+1)^{n-s} \frac{s!C(n,s)}{2^n} P(z) \right) e^{i\alpha} + z^n \overline{M(1/\overline{z})}$$

$$= s!C(n,s) \left\{ \left(R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right) e^{i\alpha} + \frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\overline{\beta}}{2^n} \right\} a_n z^n +$$

$$... + s!C(n,s) \left\{ \left(\frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\beta}{2^n} \right) e^{i\alpha} + R^{n-s} + (R+1)^{n-s} \frac{\overline{\beta}}{2^n} \right\} a_0$$

does not vanish in $|z| < 1$ for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and α real. Therefore Λ_δ is an admissible operator. Applying (3.2) we get desired result for $p > 0$. This completes the proof of lemma 3.10. □

4. Proofs of the theorems

Proof of Theorem 2.1. By hypothesis $P \in \mathcal{P}_n$, we can write

$$P(z) = P_1(z)P_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n} (z - z_j), \ \ k \geqslant 1,$$

where the zeros $z_1, z_2, \ldots, z_k$ of $P_1(z)$ lie in $|z| \leqslant 1$ and the zeros $z_{k+1}, z_{k+2}, \ldots, z_n$ of $P_2(z)$ lie in $|z| > 1$. Since all the zeros of $P_2(z)$ lie in $|z| > 1$, the polynomial $Q_2(z) = z^{n-k}\overline{P_2(1/\overline{z})}$ has all its zeroes in $|z| < 1$ and $|Q_2(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$T(z) = P_1(z)Q_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n} (1 - z\overline{z_j}),$$

then all the zeros of $T(z)$ lie in $|z| \leqslant 1$, and for $|z| = 1$,

$$|T(z)| = |P_1(z)|\,|Q_2(z)| = |P_1(z)|\,|P_2(z)| = |P(z)|\,.$$

Now on applying lemma 3.6 we get for $R \geqslant 1$, $0 \leqslant s \leqslant n$, and $|\beta| \leqslant 1$

$$\begin{aligned} &\left| z^s P^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(z)\right| \\ &\leqslant \left| z^s T^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(z)\right| \quad \text{for} \quad |z| \geqslant 1, \end{aligned}$$

which in particular gives for $p > 0$,

$$\begin{aligned} &\int_0^{2\pi} \left| e^{is\theta} P^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(e^{i\theta})\right|^p d\theta \\ &\leqslant \int_0^{2\pi} \left| e^{is\theta} T^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(e^{i\theta})\right|^p d\theta. \end{aligned} \tag{4.1}$$

Since all the zeros of $T(z)$ lies in $|z| \leqslant 1$, by lemma 3.5 the polynomial

$$z^s T^{(s)}(Rz) + \beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}T(z),$$

has also all its zeros in $|z| \leqslant 1$ for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$. Therefore if $T(z) = c_n z^n + c_{n-1} z^{n-1} + ... + c_1 z + c_0$, then the operator Λ_δ defined by

$$\begin{aligned}\Lambda_\delta T(z) &= z^s T^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(z)\\ &= s!C(n,s)\left(R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right)c_n z^n + ...\\ &+ s!C(n,s)\left(\frac{d^s(1)}{dz^s} + (R+1)^{n-s}\frac{\beta}{2^n}\right)c_0,\end{aligned}$$

is admissible. Hence by (3.2) of lemma 3.9 for each $p > 0$, we have

(4.2)

$$\int_0^{2\pi}\left|e^{is\theta}T^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(e^{i\theta})\right|^p d\theta \leqslant (c(\delta))^p \int_0^{2\pi}|T(e^{i\theta})|^p d\theta,$$

where $c(\delta) = max\left(s!C(n,s)\left|R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|, \quad s!C(n,s)\left|\frac{d^s(1)}{dz^s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|\right)$. For every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $R \geqslant 1$, it can be easily verified that $c(\delta) = s!C(n,s)\left|R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|$. Thus from (4.2), we have

$$\begin{aligned}(4.3)\qquad &\int_0^{2\pi}\left|e^{is\theta}T^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(e^{i\theta})\right|^p d\theta\\ &\leqslant (s!C(n,s))^p\left|R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|^p\int_0^{2\pi}|T(e^{i\theta})|^p d\theta.\end{aligned}$$

Combining inequalities (4.1) and (4.3) and noting that $|T(e^{i\theta})| = |P(e^{i\theta})|$, we obtain

$$\begin{aligned}&\int_0^{2\pi}\left|e^{is\theta}P^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(e^{i\theta})\right|^p d\theta\\ &\leqslant (s!C(n,s))^p\left|R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta.\end{aligned}$$

This proves theorem (2.1) for $p > 0$. To obtain this result for $p = 0$, we simply make $p \to 0+$. □

Proof of Theorem 2.2. By hypothesis $P(z)$ does not vanish in $z < 1$, therefore by lemma 3.6 for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $0 \leqslant \theta \leqslant 2\pi$

$$\begin{aligned}(4.4)\qquad &\left|e^{is\theta}P^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(e^{i\theta})\right|\\ &\leqslant\left|e^{is\theta}Q^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}Q(e^{i\theta})\right|,\end{aligned}$$

where $Q(z)=z^n\overline{P(1/\overline{z})}$. Also by lemma 3.10,

(4.5)

$$\int_0^{2\pi}\left|e^{i\alpha}F(\theta)+e^{in\theta}\overline{M(e^{i\theta})}\right|^p d\theta$$

$$\leqslant (s!C(n,s))^p\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta,$$

where

$$F(\theta)=e^{is\theta}P^{(s)}(Re^{i\theta})+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}P(e^{i\theta})$$

and

$$M(e^{i\theta})=e^{is\theta}Q^{(s)}(Re^{i\theta})+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}Q(e^{i\theta}).$$

Integrating both sides of (4.5) with respect to α from 0 to 2π, we get for each $p>0$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|e^{i\alpha}F(\theta)+e^{in\theta}\overline{M(e^{i\theta})}\right|^p d\theta d\alpha$$

(4.6)

$$\leqslant (s!C(n,s))^p\int_0^{2\pi}\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p d\alpha\int_0^{2\pi}|P(e^{i\theta})|^p d\theta.$$

Now for every real α, $t\geqslant 1$ and $p>0$, we have

$$\int_0^{2\pi}|t+e^{i\alpha}|^p d\alpha\geqslant\int_0^{2\pi}|1+e^{i\alpha}|^p d\alpha.$$

If $F(\theta)\neq 0$, we take $t=|M(e^{i\theta})/F(\theta)|$, then by (4.4), $t\geqslant 1$ and we get

$$\int_0^{2\pi}\left|e^{i\alpha}F(\theta)+e^{in\theta}\overline{M(e^{i\theta})}\right|^p d\alpha=|F(\theta)|^p\int_0^{2\pi}\left|e^{i\alpha}+\frac{e^{in\theta}\overline{M(e^{i\theta})}}{F(\theta)}\right|^p d\alpha$$

$$=|F(\theta)|^p\int_0^{2\pi}\left|e^{i\alpha}+\left|\frac{M(e^{i\theta})}{F(\theta)}\right|\right|^p d\alpha\geqslant|F(\theta)|^p\int_0^{2\pi}|1+e^{i\alpha}|^p d\alpha.$$

For $F(\theta)=0$, this inequality is trivially true. Using this in (4.6) , we conclude that for every $\beta\in\mathbb{C}$ with $|\beta|\leqslant 1$

(4.7)

$$\int_0^{2\pi}|F(\theta)|^p d\theta\int_0^{2\pi}|1+e^{i\alpha}|^p d\alpha$$

$$\leqslant (s!C(n,s))^p\int_0^{2\pi}\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p d\alpha\int_0^{2\pi}|P(e^{i\theta})|^p d\theta.$$

Since

$$\int_0^{2\pi}\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p d\alpha$$

$$=\int_0^{2\pi}\left|\left|R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right|e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\left|\frac{\overline{\beta}}{2^n}\right|\right|^p d\alpha$$

$$=\int_0^{2\pi}\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\beta}{2^n}\right|^p d\alpha, \tag{4.8}$$

the desired result follows immediately by combining (4.7) and (4.8). This proves Theorem 2.2 for $p > 0$. To establish this result for $p = 0$, we simply make $p \to 0$. □

Proof of Theorem 2.3. Since $P(z)$ is a self-inversive polynomial, we have $P(z) = uQ(z)$ for all $z \in \mathbb{C}$ where $|u| = 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$. Therefore for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$,

$$\left|e^{is\theta}P^{(s)}(Re^{i\theta})+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}P(e^{i\theta})\right| =$$
$$\left|e^{is\theta}Q^{(s)}(Re^{i\theta})+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}Q(e^{i\theta})\right|,$$

for all $z \in \mathbb{C}$. Using (4.4) and proceeding similarly as in the proof of Theorem 2.2 we get the desired result. This completes the proof of Theorem 2.3. □

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