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GROUND STATES SOLUTIONS FOR A MODIFIED FRACTIONAL SCHRÖDINGER EQUATION WITH A GENERALIZED CHOQUARD NONLINEARITY

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Abstract. In this paper, using variational methods, we study the existence of ground states solutions to the modified fractional Schrödinger equations with a generalized Choquard nonlinearity.

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Keywords: Choquard equation; fractional Schrödinger equation; fractional laplacian; ground states solutions.

1. Introduction

In the present paper, we investigate the existence of ground states solutions for a modified fractional Schrödinger equation with a generalized Choquard nonlinearity
(1.1)

$$(-\triangle)^s u + \mu V(x)u + 2\left[(-\triangle)^s u^2\right] u = (I_\lambda * F(u))\, f(u) + \frac{|u|^{22_s^*(\beta)-2}u}{|x|^\beta},\ x \in \mathbb{R}^{\mathbb{N}},$$

where $\mathbb{N} \geq 3$, $s \in (0,1)$, $0 \leq \beta < 2s < \mathbb{N}$, μ is positive constant, $2_s^*(\beta) = \dfrac{2(n-\beta)}{n-2s}$ is the critical β-fractional Sobolev exponent, V(x) is a given potential, $f \in C(\mathbb{R},\mathbb{R})$ and $F \in C(\mathbb{R},\mathbb{R})$ with $F(u) = \int_0^u f(t)dt$, $I_\lambda(x) = |x|^{-\lambda}$ is the Rieze potential of order $\lambda \in (0,N)$ and $(-\triangle)^s$ denotes the fractional Laplacian of order s is defined as

$$(-\triangle)^s \varphi(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} dy, \qquad x \in \mathbb{R}^{\mathbb{N}},$$

with $\varphi \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$ and $B_\epsilon(x)$ denotes the ball of $\mathbb{R}^{\mathbb{N}}$ centered at $x \in \mathbb{R}^N$ and radius $\epsilon > 0$.

The study of existence and uniqueness of positive solutions for Choquard type equations attracted a lot of attention of researchers due to its vast applications in physical models [1]. Fractional Choquard equations and their applications is very interesting, we refer the readers to [2] –[11] and the references therein. The authors in [9], by using the Mountain Pass Theorem and the Ekeland variational principle obtained the existence of nonnegative solutions a Schrödinger-Choquard-

Kirchhoff-type fractional p-equation. Ma and Zhang [8] studied the fractional order Choquard equation and proved the existence and multiplicity of weak solutions. In [3], the authors investigated a class of Brézis-Nirenberg type problems of nonlinear Choquard equation involving the fractional Laplacian in bounded domain Ω. Wang and Yang [12] by using an abstract critical point theorem based on a pseudo-index related to the cohomological index studied the bifurcation results for the critical Choquard problems involving fractional p-Laplacian operator:

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + \left(\displaystyle\int_\Omega \frac{|u|^{p^*_{\mu,s}}}{|x-y|^\mu} dy \right) |u|^{p^*_{\mu,s}-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and λ is a real parameter. Also, in [13] – [15], the authors have studied the existence of multiple solutions for problem (1.2), when $p = 2$. For more works on the Brezis-Nirenberg type results on semilinear elliptic equations with fractional Laplacian, we refer to [16] – [17] and references therein.

On the other hand, Shao and Wang in [18] established the following Kirchhoff equations with Hardy-Littlewood-Sobolev critical nonlinearity:

$$\begin{cases} -\triangle u + V(x)u - u\triangle u^2 + \lambda \left(I_\alpha * |u|^p\right)|u|^{p-2}u = K(x)u^{-\gamma}, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $\alpha \in (0, N)$, $\lambda > 0$ and I_α is a Riesz potential. Under suitable assumption on K and V, the author obtained the existence of positive solutions for problem (1.3).

Zhang and Ji [19] studied the following problem

$$-\triangle u + V(x)u - u\triangle u^2 = \left(I_\alpha * G(u)\right) g(u), \qquad x \in \mathbb{R}^N, \tag{1.4}$$

where $\alpha \in (0, N)$, I_α is a Riesz potential and $V : \mathbb{R}^N \to \mathbb{R}$ is radial potential, and established the existence of ground state solutions for problem (1.4) by using the variational method. For more results on equations with Hardy-Littlewood-Sobolev critical nonlinearity and nonlocal fractional problems, we refer to [20] – [31] and references therein.

Recently, the authors in [32] studied the existence of ground state solutions for the following modified fractional Schrödinger equations

$$(-\triangle)^\alpha u + \mu u + \kappa \left[(-\triangle)^\alpha u^2\right] u = \sigma |u|^{p-1}u + |u|^{q-1}u, \quad x \in \mathbb{R}^N,$$

where $0 < \alpha < 1$, $\mu > 0$, $N \geq 2$, $\kappa > 0$, $2 < p + 1 < q < \frac{2N}{N-2\alpha}$.

Motivated by the above works, in this paper, we would like to study the existence of ground state solutions for problem (1.1).

Throughout the paper, we get the following conditions:
(V_1) $V(x) \geq 0$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\Omega := int\left(V^{-1}(0)\right)$ is non-empty with smooth boundary;
(V_2) There exists $M > 0$ such that meas$(x \in \mathbb{R}^{\mathbb{N}} | V(x) \leq M) < \infty$, where meas (.) denotes the Lebesgue measure;
(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
(f_2) $\lim_{t\to\infty} \frac{f(t)}{t^{q-1}} = 0$ for some $\frac{2N-\lambda}{N} \leq q \leq \frac{2N-\lambda}{N-2s}$;
(f_3) There exists $\alpha \in (4, 22_s^*(\beta))$ that $0 < \alpha F(t) < tf(t)$, for all $t \in \mathbb{R}$.
Also, we introduce the following fractional Choquard equation:

$$
\begin{cases} (-\triangle)^s u + 2\left[(-\triangle)^s u^2\right] u = (I_\lambda * F(u)) f(u) + \frac{|u|^{22^*_\alpha(\beta)-2}u}{|x|^\beta}, & x \in \Omega, \\ u = 0, \qquad x \in \mathbb{R}^{\mathbb{N}} \setminus \Omega, \end{cases} \tag{1.5}
$$

where Ω is defined in (V_1). The main results are as follows:

Theorem 1.1. *Let $0 < \mu < \min\{N, 4s\}$. Assume that $(f_1) - (f_3)$ and $(V_1) - (V_2)$ hold. Then there exists $\mu^* > 0$ such that* (1.1) *has a least a ground state solution for any $\mu > \mu^*$.*

Theorem 1.2. *Under the assumptions of Theorem 1.1, assume that u_{μ_n} be a ground state of problem (1.1) with $\mu_n \to \infty$. Then, up to a subsequence, $u_{\mu_n} \to u$ in $H^s(R^N)$ as $n \to \infty$. Moreover, u is a ground state solution of problem* (1.5).

The paper is organized as follows. In Section 2, we recall some basic definitions of fractional Sobolev space and Hardy-Littlewood-Sobolev Inequality, and we give some useful auxiliary lemmas. In Section 3, we give the proof of the main results.

2. Preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof to the main results. The fractional Sobolev space $H^s(\mathbb{R}^N)$ $(0 < s < 1)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{\psi \in L^2(\mathbb{R}^N) : \| (-\triangle)^{\frac{s}{2}} \psi\|^2 < \infty\right\},$$

with the norm

$$\|\psi\|_{H^s(\mathbb{R}^N)} = \left(\|\psi\|_2^2 + \| (-\triangle)^{\frac{s}{2}} \psi\|^2\right)^{\frac{1}{2}},$$

where

$$\| (-\triangle)^{\frac{s}{2}} \psi\| = \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}}.$$

The space $D^{s,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$[\psi]_{s,2} = \| (-\triangle)^{\frac{s}{2}} \psi\|.$$

Let S be the best Sobolev constant

$$S := \inf_{\psi\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\psi\|^2}{\left(\int_{\mathbb{R}^N} |\psi|^{2_s^*(\alpha)}dx\right)^{\frac{2}{2_s^*(\alpha)}}}. \tag{2.1}$$

Also, define the space

$$E = \left\{\psi \in H^s(\mathbb{R}^{\mathbb{N}}) | \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x)\psi^2 dx < +\infty\right\},$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \mu V(x)u^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Let us recall the following results.

Lemma 2.1. *(see [33, Lemma 1]) $(E, ||.||)$ is a uniformly convex Banach space.*

Lemma 2.2 ([34])**.** *Suppose that V satisfies (V_2) and $\mu^* > 0$ be a fixed constant. Then the embedding $E \hookrightarrow L^\nu(\mathbb{R}^{\mathbb{N}})$ is continuous for all $\mu > \mu^*$ and $\nu \in [2, 2_s^*(\beta))$. Moreover, for any $R > 0$ and $\nu \in [1, 2_s^*(\beta)]$ the embedding $E \hookrightarrow L^\nu(B_R(0))$ is compact.*

Proof. The proof is similar to that of Lemma 1 in [34], so we omit it here.

Now, we state the following fractional Hardy-Sobolev inequality

Lemma 2.3. *([35, Lemma 2]) Assume that $\alpha \in [0, 2s]$ with $2s < N$. Then there exists a positive constant C such that*

$$\left(\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx\right)^{\frac{1}{2_s^*(\alpha)}} \leq C\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}} \text{ for every } u \in H^s(\mathbb{R}^N).$$

Lemma 2.4. *(Hardy-Littlewood-Sobolev Inequality, [36, Theorem 4.3]) Suppose that $r, t \in (1, \infty)$, $\lambda \in (0, N)$ with*

$$\frac{1}{t} + \frac{1}{r} + \frac{\lambda}{N} = 2.$$

So there exists a sharp constant $C(N, \lambda, r, t) > 0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|\zeta(x)|.|\eta(y)|}{|x-y|^\lambda}dxdy \leq C(N, \lambda, r, t)\|\zeta\|_r\|\eta\|_t,$$

for all $\zeta \in L^r(\mathbb{R}^{\mathbb{N}})$ and $\eta \in L^t(\mathbb{R}^{\mathbb{N}})$.

If $F \in L^t(\mathbb{R}^{\mathbb{N}})$ for some $t > 1$ with $\frac{2}{t} + \frac{\lambda}{N} = 2$, then by Lemma 2.4,

$$\iint_{\mathbb{R}^{2N}} \frac{|F(u(x))|.|F(u(y))|}{|x-y|^\lambda}dxdy$$

is well defined.

We mean by a weak solution of (1.1), any $u \in E$ such that

$$\int_{\mathbb{R}^{\mathbb{N}}} (-\triangle)^{\frac{s}{2}} u.(-\triangle)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x) u\varphi dx + 2\int_{\mathbb{R}^{\mathbb{N}}} (-\triangle)^{\frac{s}{2}} u^2.(-\triangle)^{\frac{s}{2}} u\varphi dx$$
$$= \int_{\mathbb{R}^{\mathbb{N}}} (I_\lambda * F(u)) f(u)\varphi dx + \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22^*_s(\beta)-2}u.\varphi}{|x|^\beta} dx,$$

for any $\varphi \in E$. The energy function corresponding to (1.1) is

$$I_\mu(u) = \frac{1}{2}[u]^2_{s,2} + \frac{\mu}{2}\int_{\mathbb{R}^{\mathbb{N}}} V(x)|u|^2 dx + \frac{1}{2}[u^2]^2_{s,2} -$$
$$\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\lambda} dxdy - \frac{1}{22^*_s(\beta)}\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22^*_s(\beta)}}{|x|^\beta},$$

and energy function corresponding to (1.5) is

$$\begin{aligned} I_0(u) =& \frac{1}{2}[u]^2_{s,2} + \frac{1}{2}[u^2]^2_{s,2} \\ &- \frac{1}{2}\iint_{\Omega\times\Omega} \frac{F(u(x))F(u(y))}{|x-y|^\lambda} dxdy - \frac{1}{22^*_s(\beta)}\int_{\Omega} \frac{|u|^{22^*_s(\beta)}}{|x|^\beta}. \end{aligned}$$

Set $X := \left\{\zeta \in E : \zeta^2 \in E\right\}$ with $\|\zeta\|_X = \|\zeta\|_E$ and

$$X_0 := \left\{\zeta \in H^s(\mathbb{R}^{\mathbb{N}}) : \zeta^2 \in H^s(\mathbb{R}^{\mathbb{N}}), \quad u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}.$$

Now, we show that $X \neq \emptyset$. For simplicity, we assume N$=1$. Let

$$u(x) := \begin{cases} \sqrt{|\sin(x)|} & x \in [1, 2\pi], \\ \\ 0 & x \in \mathbb{R} \setminus [1, 2\pi]. \end{cases}$$

and

$$V(x) := \begin{cases} \frac{|x|-1}{|x|^3} & x \in \mathbb{R} \setminus (-1, 1), \\ \\ 0 & x \in (-1, 1). \end{cases}$$

$$\begin{aligned} \iint_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}} dxdy &= \iint_{[1,2\pi]\times[1,2\pi]} \frac{|\sqrt{|\sin(x)|} - \sqrt{|\sin(y)|}|^2}{|x-y|^{1+2s}} dxdy \\ &\le \iint_{[1,2\pi]\times[1,2\pi]} \frac{|\sqrt{|\sin(x)-\sin(y)|}|^2}{|x-y|^{1+2s}} dxdy \\ &\le C_1 \iint_{[1,2\pi]\times[1,2\pi]} \frac{1}{|x-y|^{1+2s}} dxdy < \infty, \end{aligned}$$

where $C_1 \geq 0$ and

$$\int_{\mathbb{R}} \mu V(x)|u(x)|^2 dx \le \int_{\mathbb{R}} \mu V(x) dx < \infty,$$

then $u(x) \in E$. In addition, we have

$$\begin{aligned} \iint_{\mathbb{R}^2} \frac{|u^2(x)-u^2(y)|^2}{|x-y|^{1+2s}} dxdy &= \iint_{[1,2\pi]\times[1,2\pi]} \frac{||\sin(x)| - |\sin(y)||^2}{|x-y|^{1+2s}} dxdy \\ &\le C_2 \iint_{[1,2\pi]\times[1,2\pi]} \frac{1}{|x-y|^{1+2s}} dxdy < \infty, \end{aligned}$$

where $C_2 \geq 0$ and

$$\int_{\mathbb{R}} \mu V(x)|u^2(x)|^2 dx \leq \int_{\mathbb{R}} \mu V(x) dx < \infty,$$

then $u^2(x) \in E$ and $u(x) \in X$. Then $X \neq \emptyset$.

Also, $I_\mu(u)$ is well defined on X and $I_0(u)$ is well defined on X_0. Under the assumation (V_1) asnd (V_2), I_μ, I_0 are well defined and $I_\mu, I_0 \in C^1(X, \mathbb{R}^N)$.

Let

$$\mathbb{J}(u) = \iint_{\mathbb{R}^{2N}} \frac{|u^2(x) - u^2(y)|^2}{|x-y|^{N+2s}} dxdy.$$

We have

(2.2)

$$\prec \mathbb{J}'(u), v \succ = \frac{d}{dt}\mathbb{J}(u+tv)\mid_{t=0} = \frac{d}{dt} \iint_{\mathbb{R}^{2N}} \frac{|(u(x)+tv(x))^2 - (u(y)+tv(y))^2|^2}{|x-y|^{N+2s}} dxdy$$

$$= 2 \iint_{\mathbb{R}^{2N}} \frac{\Big((u(x)+tv(x))^2 - (u(y)+tv(y))^2\Big)}{|x-y|^{N+2s}} \times \Big(2(u(x)+tv(x))v(x) - 2(u(y)+tv(y))v(y)\Big) dxdy \mid_{t=0} \tag{2.3}$$

$$= 4 \iint_{\mathbb{R}^{2N}} \frac{\left(u^2(x) - u^2(y)\right)\left(u(x)v(x) - u(y)v(y)\right)}{|x-y|^{N+2s}} dxdy.$$

So by (2.2), we can easily check that

$$\begin{aligned}\left\langle I_\mu'(u), \varrho \right\rangle = &\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varrho(x)-\varrho(y))}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x) u(x)\varrho(x) dx \\ &+ 2\iint_{\mathbb{R}^{2N}} \frac{(u^2(x)-u^2(y))(u(x)\varrho(x)-u(y)\varrho(y))}{|x-y|^{N+2s}} dxdy \\ &- \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))\varrho(x)}{|x-y|^\lambda} dxdy - \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22_s^*(\beta)-2}u(x)\varrho(x)}{|x|^\beta} dx,\end{aligned}$$

for all $u, \varrho \in X$ and

$$\begin{aligned}\left\langle I_0'(u), \varrho \right\rangle = &\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varrho(x)-\varrho(y))}{|x-y|^{N+2s}} dxdy \\ &+ 2\iint_{\mathbb{R}^{2N}} \frac{(u^2(x)-u^2(y))(u(x)\varrho(x)-u(y)\varrho(y))}{|x-y|^{N+2s}} dxdy \\ &- \iint_{\Omega\times\Omega} \frac{F(u(y))f(u(x))\varrho(x)}{|x-y|^\lambda} dxdy - \int_{\Omega} \frac{|u|^{22_s^*(\beta)-2}u(x)\varrho(x)}{|x|^\beta} dx,\end{aligned}$$

for all $u, \varrho \in X_0$.

Lemma 2.5. *Assume that (f_1) and (f_2), we have*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^\lambda} f(u(x))u(x) dxdy \right| \leq C([u]_{s,2}^4 + [u]_{s,2}^{2q}), \tag{2.4}$$

and

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(y))F(u(x))}{|x-y|^\lambda} dxdy \right| \leq C([u]_{s,2}^4 + [u]_{s,2}^{2q}). \tag{2.5}$$

Proof. The proof is similar to that of Lemma 2.5 in [37], so we omit it here.

Lemma 2.6. *Assume that* $\{u_n\} \subset E$ *such that* $u_n \rightharpoonup u$ *in* E. *From* $(f_1), (f_2)$ *and* $0 < \lambda < \min\{N, 4S\}$, *we have*

$$\int_{\mathbb{R}^N} (I_\lambda * F(u_n))F(u_n)dx \to \int_{\mathbb{R}^N} (I_\lambda * F(u))F(u)dx,$$
$$\int_{\mathbb{R}^N} (I_\lambda * F(u_n))f(u_n)\varphi dx \to \int_{\mathbb{R}^N} (I_\lambda * F(u))f(u)\varphi dx.$$

as $n \to \infty$.

Proof. The proof is similar to that of the proof of Lemma 2.6 in [37], so we omit it here. Set

$$m_\mu := \inf_{u\in\Sigma} I_\mu(u), \qquad m_0 := \inf_{u\in\Sigma_0} I_0(u),$$

where

$$\Sigma := \left\{ u \in X \setminus \{0\} \mid < I_\mu'(u), u >= 0 \right\}, \qquad \Sigma_0 := \left\{ u \in X_0 \setminus \{0\} \mid < I_0'(u), u >= 0 \right\}.$$

We know that to prove our main results, we should check that m_μ is achieved by a critical point of I_μ for $\mu > \mu^*$.

Lemma 2.7. $\Sigma_0 \neq \varnothing$.

Proof. Let $u_0 \in X \setminus \{0\}$ with $u_0 \geq 0$ and $\kappa(t) = \zeta\left(\frac{tu_0}{[u_0]_{s,2}}\right)$, where

$$\zeta(u) = \iint_{\Omega\times\Omega} \frac{F(u(y))F(u(x))}{|x-y|^\lambda} dxdy.$$

From (f_3), we have

$$\frac{\alpha}{t} \leq \frac{\kappa'(t)}{\kappa(t)}, \quad \forall\, t > 0.$$

Consequently, by integrating from the above inequality over $[1, t[u_0]_{s,2}]$ with $t > \frac{1}{[u_0]_{s,2}}$, one can get

$$\zeta(tu_0) \geq \zeta\left(\frac{u_0}{[u_0]_{s,2}}\right) t^\alpha [u_0]_{s,2}^\alpha.$$

So, we get

$$I_0(t_0u_0) \leq \frac{t_0^2}{2}[u_0]_{s,2}^2 + \frac{t_0^4}{2}[u_0^2]_{s,2}^2 - \frac{\lambda}{2}\zeta(\frac{u_0}{[u_0]_{s,2}})t^\alpha [u_0]_{s,2}^\alpha,$$

since $\alpha > 4$, if $t_0 \to +\infty$, we have $I_0(t_0u_0) \to -\infty$.

On the other hand,

$$
\begin{aligned}
I_0(t_0u_0) &= \frac{t_0^2}{2}[u_0]_{s,2}^2 + \frac{t_0^4}{2}[u_0^2]_{s,2}^2 - \frac{1}{2}\iint_{\Omega\times\Omega}\frac{F(t_0u_0(x))F(t_0u_0(y))}{|x-y|^\lambda}dxdy \\
&- \frac{t_0^{22_s^*(\beta)}}{22_s^*(\beta)}\int_\Omega \frac{|u|^{22_s^*(\beta)}}{|x|^\beta} \geq \frac{t_0^2}{2}[u_0]_{s,2}^2 + \frac{t_0^4}{2}[u_0^2]_{s,2}^2 \\
&- C_1\left(t_0^4[u_0]_{s,2}^4 + t_0^{2q}[u_0]_{s,2}^{2q}\right) - C_2 t_0^{22_s^*(\beta)}[u_0^2]_{s,2}^{2_s^*(\beta)},
\end{aligned}
$$

which implies that for small $t_0 > 0$, $I_0(t_0u_0) > 0$. Then, there exists $t > 0$ such that $\frac{d}{dt}|_{t_0=t}I_0(tu_0) = 0$, means, $tu_0 \in \Sigma_0$, then we have the conclusion. □

Lemma 2.8. *There exists $K > 0$ such that $m_\mu \geq K$.*

Proof. We divide the proof into the following three steps.

Step 1: $\Sigma_0 \subset \Sigma$ and $m_0 \geq m_\mu$.

For any $u \in \Sigma_0$, by the definition of Ω, one has

$$
\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx = 0.
$$

Consequently,

$$
< I_\mu^{'}(u), u > = < I_0^{'}(u), u > + \int_{\mathbb{R}^N} \mu V(x)|u|^2 dx,
$$

hence, $u \in \Sigma$ and $\Sigma_0 \subset \Sigma$, $\Sigma \neq \varnothing$. Similarly, we can prove that $I_\mu(u) = I_0(u)$, and then we get

$$
m_\mu = \inf_{u\in\Sigma} I_\mu(u) \leq \inf_{u\in\Sigma_0} I_\mu(u) = \inf_{u\in\Sigma_0} I_0(u) = m_0.
$$

Step 2: m_μ is bounded from below.

From (f_3), for any $u \in \Sigma$, we get

$$
\begin{aligned}
I_\mu(u) &= I_\mu(u) - \frac{1}{\alpha} < I_\mu^{'}(u), u > \\
&= \left(\frac{1}{2} - \frac{1}{\alpha}\right)[u]_{s,2}^2 + \left(\frac{1}{2} - \frac{1}{\alpha}\right)\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx \\
&+ \left(\frac{1}{2} - \frac{2}{\alpha}\right)[u^2]_{s,2}^2 - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(y))F(u(x))}{|x-y|^\lambda}dxdy \\
&+ \frac{1}{\alpha}\iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))u(x)}{|x-y|^\lambda}dxdy - \left(\frac{1}{22_s^*(\beta)} - \frac{1}{\alpha}\right)\int_{\mathbb{R}^N} \frac{|u|^{22_s^*(\beta)}}{|x|^\beta}dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)[u]_{s,2}^2 + \left(\frac{1}{2} - \frac{1}{\alpha}\right)\int_{\mathbb{R}^N} \mu V(x)|u|^2 dx \\
&+ \left(\frac{1}{2} - \frac{2}{\alpha}\right)[u^2]_{s,2}^2 - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(y))F(u(x))}{|x-y|^\lambda}dxdy
\end{aligned}
$$

$$
\begin{aligned}
(2.6)\qquad & +\frac{1}{2\alpha}\iint_{\mathbb{R}^{2N}}\frac{F(u(y))f(u(x))u(x)}{|x-y|^{\lambda}}dxdy-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u|^{22_s^*(\beta)}}{|x|^{\beta}}dx\\
&=\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u]_{s,2}^2+\left(\frac{1}{2}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\mu V(x)|u|^2dx+\left(\frac{1}{2}-\frac{2}{\alpha}\right)[u^2]_{s,2}^2\\
&-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u|^{22_s^*(\beta)}}{|x|^{\beta}}dx,
\end{aligned}
$$

since $\alpha\in(4,22_s^*(\beta))$, then $(\frac{1}{2}-\frac{1}{\alpha})>0,(\frac{1}{\alpha}-\frac{1}{22_s^*(\beta)})>0$, consequently, $I_\mu(u)\geq 0$. This result implies that $m_\mu\geq 0$.

Step 3: m_μ have positive uniform bounded from below.

Let $\{u_n\}$ be a minimizing sequence of m, then $I_\mu(u_n)\to m$ and $I_\mu^{'}(u_n)\to 0$. According to the proof of the (2.6), we have

$$
\begin{aligned}
m_0+o_n(1)&\geq m_\mu+o_n(1)\\
&\geq\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u_n]_{s,2}^2+\left(\frac{1}{2}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\mu V(x)|u_n|^2dx+\left(\frac{1}{2}-\frac{2}{\alpha}\right)[u_n^2]_{s,2}^2\\
&-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u_n|^{22_s^*(\beta)}}{|x|^{\beta}}dx\geq\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u_n]_{s,2}^2\\
&+\left(\frac{1}{2}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\mu V(x)|u_n|^2dx.
\end{aligned}\tag{2.7}
$$

Thus

$$
m_0+o_n(1)\geq m_\mu+o_n(1)\geq C_1||u_n||^2,\tag{2.8}
$$

where $C_1=(\frac{1}{2}-\frac{1}{\alpha})$. From fractional Hardy-Sobolev inequality and lemma 2.5, there exist two constants $C_2,C_3>0$ such that

$$
\begin{aligned}
\|u_n\|^2&\leq\|u_n\|^2+[u_n^2]_{s,2}^2\\
&=\iint_{\mathbb{R}^{2N}}\frac{F(u_n(y))f(u_n(x))\varphi}{|x-y|^{\lambda}}dxdy+\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u_n|^{22_s^*(\beta)-2}u_n\varphi}{|x|^{\beta}}dx\\
&\leq C_2([u_n]_{s,2}^4+[u_n]_{s,2}^{2q})+C_3[u_n]_{s,2}^{22_s^*(\beta)}\\
&\leq C_2(\|u_n\|^4+\|u_n\|^{2q})+C_3\|u_n\|^{22_s^*(\beta)}.
\end{aligned}
$$

So, we may choose a constant $C_4>0$ such that

$$
\|u_n\|^2\geq C_4.\tag{2.9}
$$

From (2.8) and (2.9), there exist $K:=C_1\times C_4>0$, such that

$$
m_\mu\geq\|u_n\|^2\geq K.
$$

Therefore, we have the conclusion. $\square$

3. Proof of the main theorems

In this section, we prove our main results.

Proof of Theorem 1.1. Fix $\mu > \mu^*$ and take a sequence $\{u_n\} \subset \Sigma$, that is $I_\mu(u_n) \to m_\mu$. Then, by (2.8), $\{u_n\}$ is bounded in X. Hence, $u_n \rightharpoonup u$, $u_n^2 \rightharpoonup u^2$ in E up to subsequence, and thus by Lemma 2.2,

$$(3.1)\quad \begin{cases} u_n \to u, \ \ u_n^2 \to u^2 \ \text{ in } L^s_{loc}(\mathbb{R}^{\mathbb{N}}) \ \ (1 \le s < 2_s^*(\beta)), \\ u_n \to u, \ \ \text{a.e. in } \mathbb{R}^N, \\ \frac{|u_n|}{|x|^\beta} \to \frac{|u|}{|x|^\beta} \quad in \quad L^r(\mathbb{R}^{\mathbb{N}}, \frac{dx}{|x|^\beta}) \quad for\ 2 \le r < 2_s^*(\beta)\ and\ 0 \le \beta < 2s. \end{cases}$$

Let $\psi \in H^s(\mathbb{R}^{\mathbb{N}})$ and we define a linear functional on X as follows

$$B_\psi(\varphi) = \iint_{\mathbb{R}^{2N}} \frac{(\psi^2(x) - \psi^2(y))(\psi(x)\varphi(x) - \psi(y)\varphi(y))}{|x-y|^{N+2s}} dxdy, \ \ \forall\, \varphi \in X.$$

Hence, one has

$$(3.2)\qquad \lim_{n\to\infty} B_u(u_n - u) = 0.$$

Let $\xi \in X$ be fixed and Φ_υ be the linear functional on X defined by

$$\Phi_\xi(\upsilon) = \iint_{\mathbb{R}^{\mathbb{N}}} \frac{(\xi(x) - \xi(y))(\upsilon(x) - \upsilon(y))}{|x-y|^{N+2s}} dxdy, \ \ \forall\, \upsilon \in X.$$

Since $I'_\mu(u_n) \to 0$, one can get

$$\lim_{n\to\infty} < I'_\mu(u_n) - I'_\mu(u), u_n - u >= 0.$$

Consequently,

$$\begin{aligned} o(1) =< I'_\mu(u_n) - I'_\mu(u), u_n - u > &= \Phi_{u_n}(u_n - u) - \Phi_u(u_n - u) + 2B_{u_n}(u_n - u) \\ &+ \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x)|u_n(x) - u(x)|^2 dx - \iint_{\mathbb{R}^{2N}} \frac{F(u_n(y))f(u_n(x))(u_n(x) - u(x))}{|x-y|^\lambda} dxdy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))(u_n(x) - u(x))}{|x-y|^\lambda} dxdy \\ &- \int_{\mathbb{R}^{\mathbb{N}}} [\frac{|u_n|^{22_s^*(\beta)-2}u_n - |u|^{22_s^*(\beta)-2}u}{|x|^\beta}](u_n - u)dx. \end{aligned}$$

From Lemma 2.6, we have

$$(3.3)\quad \iint_{\mathbb{R}^{2N}} \frac{(F(u_n(y))f(u_n(x)) - F(u(y))f(u(x)))(u_n(x) - u(x))}{|x-y|^\lambda} dxdy \to 0, \ \text{as}\, n \to \infty.$$

Also, in view of (3.1), we get

$$(3.4)\qquad \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x)|u_n(x) - u(x)|^2 dx \to 0, \quad \text{as} \quad n \to \infty.$$

By similare method of proof Lemma 3.4. in [37], we have

$$(3.5)\qquad \frac{|u_n|^{22_s^*(\beta)}}{|x|^\beta} \to \frac{|u|^{22_s^*(\beta)}}{|x|^\beta}.$$

Moreover, from (3.5) and Brezis-Lieb Lemma [38], we get

(3.6)
$$\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n - u|^{22^*_s(\alpha)}}{|x|^\beta} dx = \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n|^{22^*_s(\alpha)}}{|x|^\beta} dx - \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22^*_s(\alpha)}}{|x|^\beta} dx + o(1) \to 0, \text{ as } n \to \infty.$$

So, by (3.6) and the Hölder inequality, we have

$$\int_{\mathbb{R}^{\mathbb{N}}} \left[\frac{|u_n|^{22^*_s(\beta)-2} u_n}{|x|^\beta} - \frac{|u|^{22^*_s(\beta)-2} u}{|x|^\beta} \right] (u_n - u) dx \to 0 \quad as\ n \to \infty. \tag{3.7}$$

Hence, in view of the Hölder inequality, one can get

$$\Phi_{u_n}(u_n - u) - \Phi_u(u_n - u) \geq \left([u_n]_{s,2} - [u]_{s,2}\right)^2 \geq 0 \tag{3.8}$$

From $(3.3) - (3.8)$ and $B_{u_n}(u_n - u) \geq 0$, we have $\|u_n\| \to \|u\|$. Since X uniformly convex Banach space, then the weak convergence and norm convergence imply strong convergence. In view of $I_\mu \in C(X, R)$, $I_\mu(u) = m_\mu$ and $I^{'}(u) = 0$. Hence, we have the conclusion. □

Proof of Theorem 1.2. Take u_{μ_n} be a ground state of I_{μ_n} as $\mu_n \to \infty$, that is, $I_{\mu_n}(u_{\mu_n}) = m_{\mu_n}$ and $I^{'}_{\mu_n}(u_{\mu_n}) = 0$. For notion simplicity, we denote u_{μ_n} by u_n. We may suppose that $\mu_n > \mu^*$ for all n without loss of generality. In view of (2.7), we get

$$m_0 \geq m_{\mu_n} \geq (\frac{1}{2} - \frac{1}{\alpha})[u]^2_{s,2} + (\frac{1}{2} - \frac{1}{\alpha}) \int_{\mathbb{R}^{\mathbb{N}}} \mu V(x)|u|^2 dx.$$

In view of Lemma 2.2, we can get

$$\begin{cases} u_n \rightharpoonup u, u_n^2 \rightharpoonup u^2, \text{ in } H^s(\mathbb{R}^{\mathbb{N}}), \\ u_n \to u, \ u_n^2 \to u^2 \text{ in } L^s_{loc}(\mathbb{R}^{\mathbb{N}}) \ (1 \leq s < 2^*_s(\beta)), \\ u_n \to u, \text{ a.e. in } \mathbb{R}^N, \\ \frac{|u_n|}{|x|^\beta} \to \frac{|u|}{|x|^\beta} \quad in \quad L^r(\mathbb{R}^{\mathbb{N}}, \frac{dx}{|x|^\beta}) \quad for\ 2 \leq r < 2^*_s(\beta)\ and\ 0 \leq \beta < 2s. \end{cases} \tag{3.9}$$

We divide the proof into the following three steps:

Step 1: $u(x) = 0$ a.e in $\mathbb{R}^{\mathbb{N}} \setminus \Omega$.

By (2.7), we get

$$\int_{\mathbb{R}^{\mathbb{N}}} V(x)|u_n|^2 dx \leq \frac{Cm_0}{\mu_n} \to 0, \quad \text{as } n \to \infty.$$

Also, the Fatou's Lemma implies that

$$\int_{\mathbb{R}^{\mathbb{N}} \setminus \Omega} V(x)|u|^2 dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^{\mathbb{N}}} V(x)|u_n|^2 dx = 0.$$

Hence, we have $u(x) = 0$ a.e in $\mathbb{R}^{\mathbb{N}} \setminus \Omega$.

Step 2: u is a critical point of I_0. Since $I^{'}_{\mu_n}(u_n) = 0$, we have

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\zeta(x)-\zeta(y))}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^{\mathbb{N}}} \mu_n V(x) u_n \zeta(x) dx$$
$$+2\iint_{\mathbb{R}^{2N}} \frac{(u_n^2(x)-u_n^2(y))(u_n(x)\zeta(x)-u_n(y)\zeta(y))}{|x-y|^{N+2s}}dxdy$$
$$-\iint_{\mathbb{R}^{2N}} \frac{F(u_n(y))f(u_n(x))\zeta(x)}{|x-y|^{\lambda}}dxdy - \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n|^{22_s^*(\beta)-2}u_n\zeta(x)}{|x|^{\beta}}dx = 0,$$

for all $\zeta \in H^s(\mathbb{R}^{\mathbb{N}})$. Now, in view of (3.9) and $V(x)=0$ in Ω,

(3.10)
$$\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\zeta(x)-\zeta(y))}{|x-y|^{N+2s}}dxdy \to \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\zeta(x)-\zeta(y))}{|x-y|^{N+2s}}dxdy,$$

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n^2(x)-u_n^2(y))(u_n(x)\zeta(x)-u_n(y)\zeta(y))}{|x-y|^{N+2s}}dxdy \to$$
$$\iint_{\mathbb{R}^{2N}} \frac{(u^2(x)-u^2(y))(u(x)\zeta(x)-u(y)\zeta(y))}{|x-y|^{N+2s}}dxdy, \tag{3.11}$$

as $n\to\infty$, and

$$\lim_{n\to\infty} \int_{\mathbb{R}^{\mathbb{N}}} \mu_n V(x) u_n \zeta(x) dx = 0, \tag{3.12}$$

for all $\varphi \in H^s(\mathbb{R}^{\mathbb{N}})$. From Lemma 2.6, we have

(3.13)
$$\iint_{\mathbb{R}^{2N}} \frac{F(u_n(y))f(u_n(x))\zeta(x)}{|x-y|^{\lambda}}dxdy \to \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))\zeta(x)}{|x-y|^{\lambda}}dxdy,\ \forall \zeta \in H^s(\mathbb{R}^{\mathbb{N}}),$$

similarly to (3.7), we get

$$\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n|^{22_s^*(\beta)-2}u_n\zeta(x)}{|x|^{\beta}}dx \to \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u|^{22_s^*(\beta)-2}u\zeta(x)}{|x|^{\beta}}dx,\ \forall \zeta \in H^s(\mathbb{R}^{\mathbb{N}}). \tag{3.14}$$

Then, $(3.10)-(3.14)$ and step 1 imply that

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\zeta(x)-\zeta(y))}{|x-y|^{N+2s}}dxdy$$
$$+2\iint_{\mathbb{R}^{2N}} \frac{(u^2(x)-u^2(y))(u(x)\zeta(x)-u(y)\zeta(y))}{|x-y|^{N+2s}}dxdy$$
$$-\iint_{\Omega\times\Omega} \frac{F(u(y))f(u(x))\zeta(x)}{|x-y|^{\lambda}}dxdy - \int_{\Omega} \frac{|u|^{22_s^*(\beta)-2}u\zeta}{|x|^{\beta}}dx = 0,\ \forall \zeta \in H^s(\mathbb{R}^{\mathbb{N}}),$$

which implies that u is a critical point of I_0.

Step 3: $u_n \to u$ in $L^s(\mathbb{R}^{\mathbb{N}})$ for $2 \le s < 2_s^*(\beta)$.

From (3.9), by decay of the lebesgue integral, there exist $R>0$, such that

$$\int_{\mathbb{R}^{\mathbb{N}}\setminus B_R(0)} |u(x)|^2 dx < \epsilon. \tag{3.15}$$

Let $\omega_1 := \left\{x \in \mathbb{R}^{\mathbb{N}} : |x| > R' \quad \text{and} \quad V(x) \le M\right\}$,

$$\omega_2 := \left\{x \in \mathbb{R}^{\mathbb{N}} : |x| > R' \quad \text{and} \quad V(x) > M\right\}.$$

From (V_2), we have

$$\lim_{R^{'}\to\infty} \text{meas}(\omega_1(R^{'})) = 0. \tag{3.16}$$

By the Hölder inequality and the Sobolev embedding theoream, we can get

$$\begin{aligned}\int_{\omega_1(R^{'})} |u_n(x)|^2 dx &\leq \left(\text{meas}(\omega_1(R^{'})\right)^{\frac{2s-\beta}{N-\beta}} \left(\int_{\omega_1(R^{'})} |u_n(x)|^{2_s^*(\beta)} dx\right)^{\frac{2}{2_s^*(\beta)}} \\ &\leq C\left(\text{meas}(\omega_1(R^{'})\right)^{\frac{2s-\beta}{N-\beta}}. \end{aligned}\tag{3.17}$$

On the other hand

$$\int_{\omega_2(R^{'})} |u_n(x)|^2 dx \leq \frac{1}{\mu M}\int_{\omega_2(R^{'})} \mu M|u_n(x)|^2 dx \leq \frac{C}{\mu M}. \tag{3.18}$$

From $(3.15)-(3.18)$, for any $\varepsilon>0$, we may choose $\mu_0>0$ and $R^{'}>0$ such that

$$\int_{\mathbb{R}^{\mathbb{N}}\setminus B_{R^{'}}(0)} |u_n(x)|^2 dx < \epsilon \quad \text{for} \quad \mu\geq\mu_0. \tag{3.19}$$

Take $R_0=\max\{R,R'\}$,

$$\begin{aligned}\int_{\mathbb{R}^{\mathbb{N}}} |u_n-u|^2 dx &= \int_{B^c_{R_0}(0)} |u_n-u|^2 dx + \int_{B_{R_0}(0)} |u_n-u|^2 dx \\ &\leq 2\int_{B^c_{R_0}(0)} |u_n|^2 dx + 2\int_{B^c_{R_0}(0)} |u|^2 dx + \int_{B_{R_0}(0)} |u_n-u|^2 dx \\ &\leq 4\varepsilon + \int_{B_{R_0}(0)} |u_n-u|^2 dx.\end{aligned}$$

Also, by Lemma 2.2, we get $u_n\to u$ in $L^2(\mathbb{R}^{\mathbb{N}})$ as $n\to\infty$. Since $u_n\rightharpoonup u$ in E and $u_n\to u$ in $L^2(\mathbb{R}^{\mathbb{N}})$, one can get $u_n\to u$ in $L^s(\mathbb{R}^{\mathbb{N}})$ for $2\leq s<2_s^*(\beta)$.

Step 4: m_0 is achieved by u. Moreover, $u_n\to u$ in $H^s(\mathbb{R}^{\mathbb{N}})$.

By the lower semi-continuity, we have

$$\liminf_{n\to\infty}[u_n]^2_{s,2}\geq[u]^2_{s,2}, \qquad \liminf_{n\to\infty}[u_n^2]^2_{s,2}\geq[u^2]^2_{s,2}. \tag{3.20}$$

In the other hand, by similar method in (2.6), we can obtain

$$\begin{aligned}m_0 &\geq \lim_{n\to\infty} m_{\mu_n} = \lim_{n\to\infty}\left(I_{\mu_n}(u_n) - \frac{1}{\alpha} < I^{'}_{\mu_n}(u_n), u_n >\right) \\ &= \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u_n]^2_{s,2} + \left(\frac{1}{2}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}} \mu_n V(x)|u_n|^2 dx\right. \\ &+ \left(\frac{1}{2}-\frac{2}{\alpha}\right)[u_n^2]^2_{s,2} - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u_n(x))F(u_n(y))}{|x-y|^\lambda} dxdy \\ &\left. + \frac{1}{\alpha}\iint_{\mathbb{R}^{2N}} \frac{F(u_n(y))f(u_n(x))u_n(x)}{|x-y|^\lambda} dxdy - \left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}} \frac{|u_n|^{22_s^*(\beta)}}{|x|^\beta} dx\right\} \\ &\geq \left(\frac{1}{2}-\frac{1}{\alpha}\right)[u]^2_{s,2} + \left(\frac{1}{2}-\frac{2}{\alpha}\right)[u^2]^2_{s,2} - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\lambda} dxdy\end{aligned}$$

$$+\frac{1}{\alpha}\iint_{\mathbb{R}^{2N}}\frac{F(u(y))f(u(x))u(x)}{|x-y|^\lambda}dxdy-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\mathbb{R}^{\mathbb{N}}}\frac{|u|^{22_s^*(\beta)}}{|x|^\beta}dx$$
$$=\left(\frac{1}{2}-\frac{1}{\alpha}\right)[u]_{s,2}^2+\left(\frac{1}{2}-\frac{2}{\alpha}\right)[u^2]_{s,2}^2-\frac{1}{2}\iint_{\Omega\times\Omega}\frac{F(u(x))F(u(y))}{|x-y|^\lambda}dxdy$$
$$+\frac{1}{\alpha}\iint_{\Omega\times\Omega}\frac{F(u(y))f(u(x))u(x)}{|x-y|^\lambda}dxdy-\left(\frac{1}{22_s^*(\beta)}-\frac{1}{\alpha}\right)\int_{\Omega}\frac{|u|^{22_s^*(\beta)}}{|x|^\beta}dx=I_0(u)\geq m_0.$$

which implies that $I_0(u)=m_0$, $\quad \lim_{n\to\infty} m_{\mu_n}=m_0$, and

$$\liminf_{n\to\infty}[u_n]_{s,2}^2=[u]_{s,2}^2,\qquad \liminf_{n\to\infty}[u_n^2]_{s,2}^2=[u^2]_{s,2}^2. \tag{3.21}$$

By step 3 and (3.21), we have $\|u_n\|_{H^s(\mathbb{R}^N)}\to\|u\|_{H^s(\mathbb{R}^N)}$. This together with the fact that $u_n\rightharpoonup u$ in $H^s(\mathbb{R}^N)$, we get $u_n\to u$ in $H^s(\mathbb{R}^N)$. $\square$

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ON THE SOLITARY SOLUTIONS FOR THE NONLINEAR KLEIN-GORDON EQUATION COUPLED WITH BORN-INFELD THEORY

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Abstract. The aim of this paper is to prove the existence of the nonlinear Klein-Gordon equations coupled with Born-Infeld theory by using variational methods.

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1. Introduction

In recent years, the Born-Infeld nonlinear electromagnetism has become more and more attractive and regained its importance due to its relevance in the theory of superstring and membranes. Mathematically, some people considered the system coupled Klein-Gordon equation with Born-Infeld theory through using variational methods. Furthermore, by variational methods, the existence of solitary wave solution has been studied in different systems, see References [1, 2, 5, 15, 17].

The Born-Infeld (BI) electromagnetic theory [12] was originally proposed as a nonlinear correction of the Maxwell theory in order to overcome the problem of infiniteness in the classical electrodynamics of point particles. The Born–Infeld geometric theory of electromagnetism is a nonlinear generalization of the classical Maxwell theory. The underlying idea was to simply modify the classical theory not to have physical quantities of infinities, that is the principle of finiteness. It was to replace the original Lagrangian density for the Maxwell electrodynamics with a square root form with a parameter b, by which the finiteness of electric fields is ensured.

This paper can be deduced by the search for solutions of the following nonlinear Klein–Gordon equation:

$$\psi_{tt} - \Delta\psi + m^2\psi - |\psi|^{q-2}\psi = 0 \tag{1.1}$$

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with Born-Infeld theory [12]

$$\Sigma_{BI} = \frac{b^2}{4\pi}\left(1 - \sqrt{1 - \frac{1}{b^2}\left(|E|^2 - |B|^2\right)}\right), \tag{1.2}$$

where $\psi = \psi(x,t) \in C, x \in \mathbb{R}^3, t \in \mathbb{R}$, m is a real constant, $b \gg 1$ is the so-called Born–Infeld parameter. It is well known that the classical theory has two difficulties arising from the divergence of energy (see the first section of [11]). Born and Infeld suggested a way to overcome such difficulties, thus introduced the Lagrangian density. Moreover equation (1.2) can be used to develop the theory of electrically charged fields [10]. In addition, E is the the electric field and B is the magnetic induction field. The electromagnetic field is described by the gauge potential (ϕ, A) :

$$\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, \qquad A : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3,$$

from (ϕ, A), we get the electric field

$$E = -\nabla\phi - A_t$$

and the magnetic induction field $B = \nabla \times A$.

Suppose that ψ is a charged field and let e denote the eletric charge. The interaction of ψ with the electro-magnetic field is described by the minimal coupling rule, that is, the formal substitution

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + ie\phi, \nabla \to \nabla - ieA \tag{1.3}$$

into the Lagrangian density relative equation (1.1) given by

$$\Sigma_0 = \frac{1}{2}\left[|\frac{\partial\psi}{\partial t}|^2 - |\nabla\psi|^2 - m^2|\psi|^2\right] + \frac{1}{q}|\psi|^q, \tag{1.4}$$

where e denotes the electric charge.
Then equation (1.3) becomes

$$\Sigma_0 = \frac{1}{2}\left[|\frac{\partial\psi}{\partial t} + ie\phi\psi|^2 - |\nabla\psi - ieA\psi|^2 - m^2|\psi|^2\right] + \frac{1}{q}|\psi|^q. \tag{1.5}$$

The total action of the system is $\Xi = \iint \left(\Sigma_{BI} + \Sigma_0\right) \mathrm{d}x\mathrm{d}t$.

In [11], the authors considered the second order expansion of equation (1.2) for

$$\beta = \frac{1}{2b^2} \to 0^+,$$

then they got

$$\Sigma_{BI}^{'} = \frac{1}{4\pi}\left[\frac{1}{2}\left(|E|^2 - |B|^2\right) + \frac{\beta}{4}\left(|E|^2 - |B|^2\right)^2\right],$$

the total action given by $\Xi = \iint \left(\Sigma_{BI}^{'} + \Sigma_0\right) \mathrm{d}x\mathrm{d}t$. Under the electrostatic solitary wave ansatz

$$\psi(x,t) = u(x)e^{i\omega t}, \phi = \phi(x), A = 0,$$

and $e=1$, where u and ϕ are real valued functions defined on $\mathbb{R}^3$ and ω is a positive frequency parameter, so now

$$\Sigma_{BI}^{'} = \frac{1}{4\pi}\left[\frac{1}{2}\left(|E|^2-|B|^2\right)+\frac{\beta}{4}\left(|E|^2-|B|^2\right)^2\right] = \frac{1}{8\pi}\left|\nabla\phi\right|^2+\frac{\beta}{16\pi}\left|\nabla\phi\right|^4,$$

therefore the Euler–Lagrange equations associated with the total action Ξ take the the following form

$$\begin{cases} -\Delta u+\left[m^2-(\omega+\phi)^2\right]u=|u|^{p-2}u, & x\in\mathbb{R}^3, \\ \Delta\phi+\beta\Delta_4\phi=4\pi\left(\omega+\phi\right)u^2, & x\in\mathbb{R}^3, \end{cases} \tag{1.6}$$

this type of equations has been found via modern variational methods under various hypotheses on the nonlinear term, see [7, 8, 9, 13, 15]. In [9] the authors found the existence of infinitely many radially symmetric solutions for this problem when $4<p<6$ and $|m|>\omega$, in [13] the range $p\in(2,4]$ was also covered provided $\sqrt{\left(\frac{p}{2}-1\right)}|m|>\omega$.

Then Chen and Li [7] got the existence of multiple solutions for problem

$$\begin{cases} -\Delta u+\left[m^2-(\omega+\phi)^2\right]u=|u|^{p-2}u+h(x), & x\in\mathbb{R}^3, \\ \Delta\phi+\beta\Delta_4\phi=4\pi\left(\omega+\phi\right)u^2, & x\in\mathbb{R}^3, \end{cases} \tag{1.7}$$

when $4<p<6$ and $|m|>\omega$ or $2<p\leqslant 4$ and $\sqrt{\left(\frac{p}{2}-1\right)}|m|>\omega$.

Later Teng and Zhang [15] got that problem

$$\begin{cases} -\Delta u+\left[m^2-(\omega+\phi)^2\right]u=|u|^{p-2}u+|u|^{2^*-2}u, & x\in\mathbb{R}^3, \\ \Delta\phi+\beta\Delta_4\phi=4\pi\left(\omega+\phi\right)u^2, & x\in\mathbb{R}^3, \end{cases} \tag{1.8}$$

has at least a nontrivial solution when $4<p<6$ and $m>\omega$ under the electrostatic solitary wave ansatz by using variational methods.

On the other hand, by shrinking the area in problem (1.6), Teng [14] showed some existence and multiple results for the following nonlinear Klein-Gordon equation coupled with Born-Infeld theory in a bounded domain with smooth boundary

$$\begin{cases} -\Delta u+\left[m^2-(\omega+\phi)^2\right]u=f(x,u), & \text{in } \Omega, \\ \Delta\phi+\beta\Delta_4\phi=4\pi\left(\omega+\phi\right)u^2, & \text{in } \Omega, \\ u=\phi=0, & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $m^2>\frac{\mu}{\mu-2}\omega^2-\lambda_1$ and f satisfies the following conditions:

(f_1) $f\in C\left(\overline{\Omega}\times\mathbb{R}\right)$ and $f(x,0)=0$,

(f_2) There are constants $a_1,a_2>0$ such that $|f(x,t)|\leqslant a_1+a_2|t|^s$, where $1<s<\frac{n+2}{n-2}(n\geqslant 3)$,

(f_3) $\lim_{t\to 0} \frac{f(x,t)}{t} = 0,$

(f_4) There exists $\mu > 2$ and $\mathbb{R} \geqslant 0$ such that $tf(x,t) \geqslant \mu F(x,t) > 0$ for $|t| \geqslant \mathbb{R}$ and $x \in \Omega$,

or $m^2 > \frac{\mu}{\mu-2}\omega^2 - \lambda_1$ and f satisfies the conditions above and an extra condition:

(f_5) $f(x,-u) = -f(x,u)$ for all $u \in \mathbb{R}$ and $x \in \Omega$.

In addition, the authors in [1] proved the existence of nontrivial ground state solution for the following nonlinear Klein–Gordon equation coupled with Born–Infeld theory in $\mathbb{R}^2$ involving unbounded or decaying radial potentials

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] V(|x|)u = K(|x|)f(u), & \text{in } \mathbb{R}^2, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi\,(\omega+\phi)\, V(|x|)u^2, & \text{in } \mathbb{R}^2, \end{cases} \tag{1.10}$$

where $V, K : \mathbb{R}^2 \to \mathbb{R}$ are radial potentials which may be unbounded, singular at the origin or vanishing at infinity and the nonlinear term $f(s)$ is allowed to enjoy a critical exponential growth.

Recently, Che and Chen in [6] proved the existence of infinitely many negative-energy solutions for the following system via the genus properties in critical point theory

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\,\phi u = f\,(x,u)\,, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi\,(\omega+\phi)\,u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.11}$$

where the functions $V(x)$ and $f(x,u)$ satisfy the following hypotheses.

(V_1) $V \in C\left(\mathbb{R}^3\right)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geqslant a > 0$, where $a > 0$ is a constant. Moreover, for any $M > 0, meas\{x \in \mathbb{R}^3 : V(x) \leqslant M\} < \infty$, where $meas$ denotes the Lebesgue measure in $\mathbb{R}^3$.

(1) $f \in C\left(\mathbb{R}^3 \times \mathbb{R}\right)$ and there exists $1 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 2, m \in \mathbb{N}, m \geqslant 1, c_i(x) \in L^{\frac{2}{2-\alpha_i}}\left(\mathbb{R}^3, \mathbb{R}^+\right)$ such that

$$|f(x,u)| \leqslant \sum_{i=1}^{m} \alpha_i c_i(x)|u|^{\alpha_i - 1}, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(2) There exists a bounded open set $J \subset \mathbb{R}^3$ and three constants $a_1, a_2 > 0$ and $a_3 \in (1,2)$ such that

$$F(x,u) \geqslant a_2|u|^{a_3}, \qquad \forall (x,u) \in J \times [-a_1, a_1]\,,$$

where $F(x,u) = \int_0^u f(x,s)ds$.

(3) $f(x,u) = -f(x,-u)$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

Immediately after the previous equation, Wen, Tang and Chen in [16] proved the existence of infinitely many solutions and least energy solutions for the nonhomogeneous Klein-Gordon equation coupled with Born-Infeld theory.

For general potential $a(x)$ and the nonlinearity $f(x,u)=\lambda K(x)|u|^{q-2}u+g(x)|u|^{p-2}u$, Chen and Song in [8] studied this system

$$\begin{cases} -\Delta u+a(x)u-(2\omega+\phi)\,\phi u=\lambda K(x)|u|^{q-2}u+g(x)|u|^{p-2}u, & x\in\mathbb{R}^3,\\ \Delta\phi+\beta\Delta_4\phi=4\pi\,(\omega+\phi)\,u^2, & x\in\mathbb{R}^3,\end{cases} \tag{1.12}$$

and proved the existence of multiple solutions for Klein–Gordon equation with concave and convex nonlinearities coupled with Born–Infeld theory when a,k,g are measurable functions satisfying the following conditions:

(a_1) $a(x)\in C(\mathbb{R}^3)$ satisfying $a_0:=\inf\limits_{x\in\mathbb{R}^3}a(x)>0$.

(k) $k(x)\in L^{\frac{12q}{(6-q)(1+q)}}\left(\mathbb{R}^3\right),k(x)\geqslant 0$ for $a.e.$ $x\in\mathbb{R}^3$ and $k(x)\neq 0$.

(g) $g(x)\in L^{\infty}\left(\mathbb{R}^3\right),g(x)\geqslant 0$ for $a.e.$ $x\in\mathbb{R}^3$ and $g(x)\neq 0$.

The main idea of this paper is to establish the existence of solitary wave solutions of the following Klein-Gordon equation coupled with Born-Infeld theory:

$$\begin{cases} -\Delta u+\eta(x)u-(2\omega+\phi)\,\phi u=\mu K(x)|u|^{q-2}u+|u|^{2^*-2}u,\\ \Delta\phi+\beta\Delta_4\phi=4\pi\,(\omega+\phi)\,u^2,\end{cases} \tag{1.13}$$

where ω and μ are positive constants, $\beta\gg 1$ is a constant, $\eta(x)\in C(\mathbb{R}^3),K(x)\in L^{\infty}(\mathbb{R}^3),4\leqslant q<2^*=\frac{2N}{N-2}$. Since we define in three-dimensional space in this paper, after that $2^*=6$.

In this case, the functional F corresponding to problem (1.13) defined by

$$\begin{aligned} F(u,\phi)=\int\bigg[&\frac{1}{2}|\nabla u|^2+\frac{1}{2}\eta(x)u^2-\frac{1}{2}(2\omega+\phi)\phi u^2\\ &-\frac{1}{8\pi}\left|\nabla\phi\right|^2-\frac{\beta}{16\pi}\left|\nabla\phi\right|^4-\frac{\mu}{q}K(x)|u|^q-\frac{1}{6}|u|^6\bigg], \end{aligned} \tag{1.14}$$

which by a standard argument is C^1 on $H(\mathbb{R}^3)\times D(\mathbb{R}^3)$, the definitions of $H(\mathbb{R}^3)$ and $D(\mathbb{R}^3)$ will be given later. Here and hereafter, $\int\cdot$ denotes $\int_{\mathbb{R}^3}\cdot\,\mathrm{d}x$.

Remark 1.1. The functional F is strongly indefinite, i.e. unbounded from below and from above on infinite subspaces. In order to avoid this indefiniteness, we can borrow the reduction methods.

2. Main results

Firstly, assume that the system (1.13) satisfies the following conditions:

(i) $\eta(x)\geqslant 0$ is a radial function, that is, $\eta(x)=\eta(r),r=|x|$,

(ii) $K:\mathbb{R}^3\to\mathbb{R}$ is a radial function, moreover $0\leqslant K(x)<\Lambda$ and $K(x)\not\equiv 0$ for $a.e.$ $x\in\mathbb{R}^3$, where $\Lambda>0$ is a constant.

Next some notations are given. For all $1\leqslant s\leqslant+\infty$, $L^s(\mathbb{R}^3)$ denotes a Lebesgue space with the norm given by $|\cdot|_{L^s}$.

Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_{D^{1,2}}^2 = \int |\nabla u|^2.$$

The space $H^1(\mathbb{R}^3)$ is endowed with the norm

$$\|u\|_{H^1}^2 = \int \left(|\nabla u|^2 + u^2\right).$$

$D(\mathbb{R}^3)$ denotes the the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D = \left(\int |\nabla u|^2\right)^{\frac{1}{2}} + \left(\int |\nabla u|^4\right)^{\frac{1}{4}}.$$

Define

$$H = \{u \in H^1(\mathbb{R}^3) : \int \left[|\nabla u|^2 + \eta(x)u^2\right] < \infty\}$$

is a Hilbert space, whose inner product and norm are given, respectively

$$(u, v) = \int \left(\nabla u \cdot \nabla v + \eta(x)uv\right), \quad \|u\|^2 = (u, u).$$

Obviously, by the Poincaré inequality, the embedding $H(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3)$ is continuous and $D(\mathbb{R}^3)$ is continuously embedded in $D^{1,2}(\mathbb{R}^3)$. Moreover, from Sobolev's imbedding theorem (see [11]), $D(\mathbb{R}^3)$ is continuously embedded in$L^\infty(\mathbb{R}^3)$.

In this paper, we show the following results about the system (1.13):

Theorem 2.1. *Suppose (i)-(ii) hold, if $4 < q < 6$, then for each $\mu > 0$ the problem* (1.13) *admits a radially symmetric solution.*

Theorem 2.2. *Suppose (i)-(ii) hold, if $q = 4$, then for sufficiently large $\mu > 0$, the problem* (1.13) *still possesses a radially symmetric solution.*

Moreover, we have the following lemma about the second equation of problem (1.13).

Lemma 2.1. (*a*) *For any $u \in H(\mathbb{R}^3)$, there exists a unique $\Phi(u) = \phi \in D(\mathbb{R}^3)$ which satisfies $\Delta\Phi(u) + \beta\Delta_4\Phi(u) = 4\pi\left(\omega + \Phi(u)\right)u^2$,*
(*b*) *If u is radially symmetric, then $\Phi(u)$ is also radially symmetric,*
(*c*) *For any $u \in H(\mathbb{R}^3)$, it results in $\Phi(u) \leqslant 0$. Moreover, $\Phi(u)(x) \geqslant -\omega$, provided $u(x) \neq 0$.*

The results were proved by Lemma 3 in [9], Lemma 5 in [9], Lemma 2.3 in [13], respectively. Similar to the Proposition 1.1 in Reference [5], we have the following lemma.

Lemma 2.2. *The map ϕ is C^1 and $G_\phi = \{(u, \phi) \in H(\mathbb{R}^3) \times D(\mathbb{R}^3) \big| F_\phi'(u, \phi) = 0\}$.*

Proof. Since

$$F(u,\Phi(u)) = \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 - \frac{1}{2}\left(2\omega + \Phi(u)\right)\Phi(u)u^2 - \frac{1}{8\pi}\left|\nabla\Phi(u)\right|^2\right] + \int \left[-\frac{\beta}{16\pi}\left|\nabla\Phi(u)\right|^4 - \frac{\mu}{q}K(x)|u|^q - \frac{1}{6}|u|^6\right],$$

then

$$F'_\phi\left(u,\Phi(u)\right) = \int \left[-\frac{1}{4\pi}\left|\nabla\Phi(u)\right|^2 - \frac{\beta}{4\pi}\left|\nabla\Phi(u)\right|^4 - \omega\Phi(u)u^2 - \Phi^2(u)u^2\right]. \tag{2.1}$$

On the other hand, from the second equation in problem (1.13), one gets

$$-\int \left(\left|\nabla\Phi(u)\right|^2 + \beta\left|\nabla\Phi(u)\right|^4\right) = \int 4\pi\left(\omega + \Phi(u)\right)\Phi(u)u^2,$$

i.e.,

$$\int \left[\frac{1}{4\pi}\left|\nabla\Phi(u)\right|^2 + \frac{\beta}{4\pi}\left|\nabla\Phi(u)\right|^4\right] = \int \left[-\omega\Phi(u)u^2 - \Phi^2(u)u^2\right]. \tag{2.2}$$

Therefore, according to equation (2.1), $F'_\phi\left(u,\phi\right) = 0$. So now we define $I(u) = F(u,\phi)$ in $H(\mathbb{R}^3)$. □

By Lemma 2.2, we have

$$I'(u) = F'_u\left(u,\Phi(u)\right) + F'_\phi\left(u,\Phi(u)\right)\Phi'(u) = F'_u\left(u,\Phi(u)\right),$$

and if $u, v \in H(\mathbb{R}^3)$, one gets

$$I'(u)v = \int \left[\nabla u\cdot\nabla v + \eta(x)uv - (2\omega+\phi)\phi uv - \mu K(x)|u|^{q-2}uv - |u|^4uv\right]. \tag{2.3}$$

Lemma 2.3. *The following statements are equivalent:*
(a) $(u,\phi) \in H(\mathbb{R}^3)\times D(\mathbb{R}^3)$ *is a solution of system* (1.13)*,*
(b) u *is a critical point for* I *and* $\phi = \Phi(u)$.

Proof. $(b) \Longrightarrow (a)$ Obviously.
$(a) \Longrightarrow (b)$ Suppose $F'_u\left(u,\phi\right)$ and $F'_\phi\left(u,\phi\right)$ denote the partial derivatives of F at $(u,\phi) \in H(\mathbb{R}^3)\times D(\mathbb{R}^3)$. Then for every $v \in H(\mathbb{R}^3)$ and $\psi \in D(\mathbb{R}^3)$, one gets

$$F'_u\left(u,\phi\right)[v] = \int \left[\nabla u\cdot\nabla v + \eta(x)uv - (2\omega+\phi)\phi uv - \mu K(x)|u|^{q-2}uv - |u|^4uv\right], \tag{2.4}$$

$$F'_\phi\left(u,\phi\right)[\psi] = \int \left[-\frac{1}{2\pi}\nabla\phi\nabla\psi - \frac{\beta}{\pi}\left|\nabla\phi\right|^2\phi\psi - \omega\psi u^2 - 2\phi\psi u^2\right]. \tag{2.5}$$

By the standard computations, we can prove that $F'_u\left(u,\phi\right)$ and $F'_\phi\left(u,\phi\right)$ are continuous. From equations (2.4) and (2.5), it is easy to obtain that its critical points are solutions of problem (1.13), by (a) of Lemma 2.1, one has $\phi = \Phi(u)$. □

Due to the presence of the critical growth, the Sobolev embedding $H(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)(2 \leqslant p \leqslant 6)$ is not compact and then it is usually difficult to prove that a Palais–Smale sequence is strongly convergent when we seek solutions of problem

(1.13) by variational methods. A standard tool to overcome the problem is to restrict ourselves to radial functions, namely we look at the functional I on the subspace $H_r(\mathbb{R}^3) = \{u \in H(\mathbb{R}^3) \big| u(x) = u(|x|)\}$, compactly embedded in $L_r^p(\mathbb{R}^3)$ for $2 < p < 6$. Moreover, from [2], for all $u \in H(\mathbb{R}^3)$, for any $g \in O(3)$, we have

$$I(T_g u) = I(u).$$

By standard arguments, one sees that if a critical point $u \in H_r(\mathbb{R}^3)$ for the functional $I\big|_{H_r(\mathbb{R}^3)}$ is also a critical point of I.

3. The Proof of Theorem 2.1

Firstly, we prove the functional I possesses the Mountain-Pass geometry. From the second equation of system (1.13), one obtains equation (2.2), combining equation (1.14) with (2.2), one gets

$$\begin{aligned} I(u) = F(u,\phi) &= \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 - \frac{1}{2}(2\omega+\phi)\phi u^2\right] \\ &\quad + \int \left[-\frac{1}{8\pi}|\nabla\phi|^2 - \frac{\beta}{16\pi}|\nabla\phi|^4 - \frac{\mu}{q}K(x)|u|^q - \frac{1}{6}|u|^6\right] \\ &= \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 + \frac{1}{8\pi}|\nabla\phi|^2 + \frac{3\beta}{16\pi}|\nabla\phi|^4 + \frac{1}{2}\phi^2u^2\right] \\ &\quad - \int \left[\frac{\mu}{q}K(x)|u|^q + \frac{1}{6}|u|^6\right]. \end{aligned} \tag{3.1}$$

By the Sobolev inequality, one has $I(u) \geqslant C_1\|u\|^2 - C_2\|u\|^q - C_3\|u\|^6$, then there exists $\alpha, \rho > 0$ such that $\inf\limits_{\|u\|=\rho} I(u) > \alpha$. In addition, from equation (1.14), there exists a function $u \in H_r(\mathbb{R}^3) \setminus \{0\}$, it is easy to obtain

$$\begin{aligned} \lim_{t\to+\infty} I(tu) &= \int \left[\frac{t^2}{2}|\nabla u|^2 + \frac{t^2}{2}\eta(x)u^2 - \frac{t^2}{2}(2\omega+\Phi(tu))\Phi(tu)u^2 - \frac{1}{8\pi}|\nabla\Phi(tu)|^2\right] \\ &\quad + \int \left[-\frac{\beta}{16\pi}|\nabla\Phi(tu)|^4 - \frac{\mu t^q}{q}K(x)|u|^q - \frac{t^6}{6}|u|^6\right] \\ &\leqslant \frac{t^2}{2}\int \left[|\nabla u|^2 + \eta(x)u^2 - 2\omega\Phi(tu)u^2 - \frac{2\mu t^{q-2}}{q}K(x)|u|^q - \frac{t^4}{3}|u|^6\right] \\ &\leqslant -\infty, \end{aligned}$$

which implies that $I(u) \to -\infty$, as $\|u\| \to \infty$. In particular, there exists $u_1 \in H_r(\mathbb{R}^3)$, $\|u_1\| > \rho$ such that $I(u_1) < 0$. Define

$$c = \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} I(\gamma(t)), \tag{3.2}$$

where $\Gamma = \{\gamma \in C\left([0,1], H_r(\mathbb{R}^3)\right) \big| \gamma(0) = 0, \gamma(1) = u_1\}$ is the MP level. Obviously, $c \geqslant \alpha > 0$. There exists a $(PS)_c$ sequence $\{u_n\} \subset E$ such that

$$\begin{aligned} &I(u_n) \to c, \\ &I'(u_n) \to 0, \quad n \to \infty. \end{aligned} \tag{3.3}$$

Lemma 3.1. *The* $(PS)_c$ *sequence* $\{u_n\} \subset E$ *given in equation* (3.3) *is bounded.*

Proof. There is a positive constant M such that

$$
\begin{aligned}
& M + o(1)\|u_n\| \geqslant I(u_n) - \frac{1}{q}(I'(u_n), u_n) \\
(3.4) \quad & = \left(\frac{1}{2} - \frac{1}{q}\right) \int \left[|\nabla u_n|^2 + \eta(x) u_n^2\right] + \frac{1}{8\pi} \int |\nabla \Phi(u_n)|^2 + \frac{3\beta}{16\pi} \int |\nabla \Phi(u_n)|^4 \\
& \quad + \left(\frac{1}{2} + \frac{1}{q}\right) \int \Phi^2(u_n) u_n^2 + \left(\frac{1}{q} - \frac{1}{6}\right) \int |u_n|^6 + \frac{2}{q} \int \omega \Phi(u_n) u_n^2.
\end{aligned}
$$

Substituting equation (2.2) into equation (3.4), we get

$$
\begin{aligned}
M + o(1)\|u_n\| & \geqslant I(u_n) - \frac{1}{q}(I'(u_n), u_n) \\
& = \left(\frac{1}{2} - \frac{1}{q}\right) \int \left[|\nabla u_n|^2 + \eta(x) u_n^2\right] + \left(\frac{1}{8\pi} - \frac{1}{2q\pi}\right) \int |\nabla \Phi(u_n)|^2 \\
& \quad + \left(\frac{1}{2} - \frac{1}{q}\right) \int \Phi^2(u_n) u_n^2 \\
& \quad + \left(\frac{3\beta}{16\pi} - \frac{\beta}{2q\pi}\right) \int |\nabla \Phi(u_n)|^4 + \left(\frac{1}{q} - \frac{1}{6}\right) \int |u_n|^6 \geqslant C_4 \|u_n\|^2.
\end{aligned}
$$

Since $4 < q < 6$, as a consequence of the above inequality, $\{u_n\}$ is bounded in $H_r(\mathbb{R}^3)$. □

Furthermore, by equation (2.2), one gets

$$
(3.5) \qquad \int \left(|\nabla \Phi(u_n)|^2 + \beta |\nabla \Phi(u_n)|^4\right) = -4\pi \int \omega \Phi(u_n) u_n^2 - 4\pi \int \Phi^2(u_n) u_n^2.
$$

Then by Hölder inequality and Sobolev inequality, one obtains

$$
\int \left(|\nabla \Phi(u_n)|^2 + \beta |\nabla \Phi(u_n)|^4\right) \leqslant C_5 \|\Phi(u_n)\|_{D_r} \|u_n\|_{H_r}^2.
$$

So $\{\Phi(u_n)\}$ is bounded (even uniformly). Up to subsequence, we may assume that there exists $u \in H_r(\mathbb{R}^3)$ and $\varphi \in D_r(\mathbb{R}^3)$ such that

$$
(3.6) \qquad u_n \rightharpoonup u \quad \text{in } H_r(\mathbb{R}^3),
$$

$$
(3.7) \qquad u_n \to u \quad \text{in } L_r^s(\mathbb{R}^3) \text{ for } 2 < s < 6,
$$

$$
(3.8) \qquad \Phi(u_n) \rightharpoonup \varphi \quad \text{in } D_r(\mathbb{R}^3).
$$

Lemma 3.2. $\varphi = \Phi(u)$ *and* $\Phi(u_n) \to \Phi(u)$ *in* $D_r(\mathbb{R}^3)$.

Proof. First we prove the uniqueness. For every fixed $u \in H_r(\mathbb{R}^3)$, we consider the following minimizing problem $\inf_{\phi \in D_r} E_u(\phi)$, where $E_u : D_r \to \mathbb{R}$ defined as energy functional of the second equation in system (1.13).

$$
E_u(\phi) = \int \left[\frac{1}{8\pi} |\nabla \phi|^2 + \frac{\beta}{16\pi} |\nabla \phi|^4 + \omega \phi u^2 + \frac{1}{2} \phi^2 u^2\right].
$$

In fact, by the proof of Lemma 2.1 in [17], one can know

$$\Phi(u_n) \to \varphi, \text{ locally uniformly in } \mathbb{R}^3,$$

so we obtain

$$\int \Phi(u_n)u_n^2 \to \int \varphi u^2, \qquad \int \Phi^2(u_n)u_n^2 \to \int \varphi^2 u^2.$$

From the weak lower semicontinuity of the norm in D_r and the convergence above, one has

$$E_u(\varphi) \leqslant \liminf_{n\to\infty} E_{u_n}(\Phi(u_n)) \leqslant \liminf_{n\to\infty} E_{u_n}(\Phi(u)) = E_u(\Phi(u)),$$

then by (a) of Lemma 2.1, $\varphi = \Phi(u)$.

Next we prove that $\{\Phi(u_n)\}$ converges strongly in D_r. Since $\Phi(u_n)$ and $\Phi(u)$ satisfy the second equation in problem (1.13).

$$\begin{cases} \int \left[\nabla\Phi(u_n)\cdot\nabla v + \beta\left|\nabla\Phi(u_n)\right|^3\cdot\nabla v\right] = \int \left[-4\pi\omega u_n^2 v - 4\pi\Phi(u_n)u_n^2 v\right], \\ \int \left[\nabla\Phi(u)\cdot\nabla v + \beta\left|\nabla\Phi(u)\right|^3\cdot\nabla v\right] = \int \left[-4\pi\omega u^2 v - 4\pi\Phi(u)u^2 v\right], \end{cases}$$

then we take the difference for Φ to have

(3.9)

$$\begin{aligned} &\int \left[\nabla\left(\Phi(u_n)-\Phi(u)\right)\cdot\nabla v + \beta\left(\left|\nabla\Phi(u_n)\right|^2\nabla\Phi(u_n) - \left|\nabla\Phi(u)\right|^2\nabla\Phi(u)\right)\cdot\nabla v\right] \\ &= -4\pi\int \left[\omega\left(u_n^2-u^2\right)v + \left(\Phi(u_n)u_n^2 - \Phi(u)u^2\right)v\right], \qquad v \in D_r(\mathbb{R}^3). \end{aligned}$$

Testing with $v = (\Phi(u_n)-\Phi(u))$ the following holds:

$$\begin{aligned} &C_6\left(\left|\nabla\Phi(u_n)-\nabla\Phi(u)\right|_{L_r^2}^2 + \left|\nabla\Phi(u_n)-\nabla\Phi(u)\right|_{L_r^4}^4\right) \\ &\leqslant -4\pi\int \left[w(u_n^2-u^2)v + \left(\Phi(u_n)u_n^2 - \Phi(u)u^2\right)v\right] \\ &= -4\pi\int \left[w(u_n^2-u^2)v + u_n^2\left(\Phi(u_n)-\Phi(u)\right)v + (u_n^2-u^2)\Phi(u)v\right], \end{aligned}$$

the above equation holds since we have inequality

$$\left[\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y)\right] \geqslant C_p|x-y|^p, \qquad x,y\in\mathbb{R}^N, p\geqslant 2.$$

By Hölder inequality and Sobolev inequality, one has

$$\begin{aligned} &\left|\nabla\Phi(u_n)-\nabla\Phi(u)\right|_{L_r^2}^2 + \left|\nabla\Phi(u_n)-\nabla\Phi(u)\right|_{L_r^4}^4 \\ &\leqslant |4\pi\omega|\int\left[\left|u_n^2-u^2\right|\left|\Phi(u_n)-\Phi(u)\right|\right] + 4\pi\int\left[\left|u_n^2-u^2\right|\left|\Phi(u)\right|\left|\Phi(u_n)-\Phi(u)\right|\right] \\ &\leqslant |4\pi\omega|\left|\Phi(u_n)-\Phi(u)\right|_{L_r^6}\left|u_n^2-u^2\right|_{L_r^{\frac{6}{5}}} \\ &\quad + 4\pi\left|\Phi(u)\right|_{L_r^6}\left|\Phi(u_n)-\Phi(u)\right|_{L_r^6}\left|u_n^2-u^2\right|_{L_r^{\frac{3}{2}}} \leqslant C_7\left|u_n-u\right|_{L_r^{\frac{12}{5}}} + C_8\left|u_n-u\right|_{L_r^3}. \end{aligned}$$

Thus $\Phi(u_n)\to\Phi(u)$ strongly in $D_r(\mathbb{R}^3)$. $\square$

Lemma 3.3. *The weak limit* $(u, \Phi(u))$ *solves problem* (1.13).

Proof.

$$(I'(u_n), v) = \int \left[\nabla u_n \cdot \nabla v + \eta(x) u_n v - (2\omega + \Phi(u_n))\, \Phi(u_n) u_n v\right] \tag{3.10}$$
$$- \int \left[\mu K(x)|u_n|^{q-2} u_n v + |u_n|^4 u_n v\right].$$

All convergences in the sequel must be understood passing to a subsequence if necessary. Since $\{u_n\}$ is bounded in $L^6_r(\mathbb{R}^3)$, it follows

$$|u_n|^4 u_n \rightharpoonup |u|^4 u, \qquad \textit{in } (L^6_r(\mathbb{R}^3))^*.$$

Moreover by Lemma 3.2, one gets

$$\int u_n \Phi^2(u_n) v + 2\omega \int \Phi(u_n) u_n v \to \int u \Phi^2(u) v + 2\omega \int \Phi(u) u v, \qquad v \in H_r(\mathbb{R}^3).$$

In fact one obtains

(3.11)
$$\int |\Phi(u)u - \Phi(u_n)u_n|\,|v| \leqslant |\Phi(u) - \Phi(u_n)|_{L^6_r} |u|_{L^3_r} |v|_{L^2_r} + |\Phi(u_n)|_{L^6_r} |v|_{L^2_r} |u_n - u|_{L^3_r}$$

and

$$\int |u_n \Phi^2(u_n) - u\Phi^2(u)||v| \leqslant |u_n - u|_{L^3_r} |\Phi(u_n)|^2_{L^6_r} |v|_{L^3_r} + \tag{3.12}$$
$$|\Phi(u_n) - \Phi(u)|_{L^6_r} |\Phi(u_n) + \Phi(u)|_{L^6_r} |u|_{L^6_r} |v|_{L^2_r}.$$

The compactness of the embedding $H_r(\mathbb{R}^3) \hookrightarrow L^q_r(\mathbb{R}^3)$ the lemma follows. $\square$

Due to the lack of compactness, which prevents us to prove that u_n converges strongly in $H_r(\mathbb{R}^3)$, we do not know yet whether $u \neq 0$. In order to overcome this difficulty, we need let c denote the MP level.

Lemma 3.4. *Since functions are defined in dimension* $N = 3$, *then we can get* $c < \frac{1}{3} S^{\frac{3}{2}}$, *where* S *corresponds to the best constant for the Sobolev embedding* $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, *precisely,*

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int |\nabla u|^2}{\left(\int |u|^6\right)^{\frac{1}{3}}}.$$

Proof. Now given ε, we consider the Talenti function [3] $u_\varepsilon \in D^{1,2}(\mathbb{R}^3)$ defined by

$$u_\varepsilon = C \frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}},$$

where $C > 0$ is a normalized constant. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leqslant \varphi \leqslant 1$, and there exists $R > 0$ such that $\varphi\big|_{B_R} \equiv 1$, $supp\varphi \subset B_{2R}$. Set $W_\varepsilon = \varphi u_\varepsilon$ and define $V_\varepsilon := \frac{W_\varepsilon}{|W_\varepsilon|_{L^6_r}}$. From the estimates obtained in [4] we get, as $\varepsilon \to 0$,

$$X_\varepsilon := |\nabla V_\varepsilon|^2_{L^2_r} \leqslant S + O\left(\varepsilon^{\frac{1}{2}}\right), \tag{3.13}$$

$$(3.14) \qquad |V_\varepsilon|^2_{L^2_r} = O\left(\varepsilon^{\frac{1}{2}}\right).$$

Since as $t \to +\infty, I(tV_\varepsilon) \to -\infty$, we may assume that $\sup_{t\geqslant 0} I(tV_\varepsilon) = I(t_\varepsilon V_\varepsilon)$ and without loss of generality that $t_\varepsilon \geqslant C_0 > 0$, for all $\varepsilon > 0$ (otherwise we could find a sequence $\varepsilon_n \to 0$ such that $t_{\varepsilon_n} V_{\varepsilon_n} \to 0$ contradicting that $c > 0$). Next for any $\varepsilon > 0$ small enough, the following estimate holds

$$(3.15) \qquad t_\varepsilon \leqslant \left(X_\varepsilon + \int \left(\eta(x) + 2\omega^2\right) V_\varepsilon^2\right)^{\frac{1}{4}} = t_0.$$

Let $f(t) = I(tV_\varepsilon)$ and compute

$$\begin{aligned}
f'(t) &= (I'(tV_\varepsilon), V_\varepsilon) \\
&= \int \left[t|\nabla V_\varepsilon|^2 + \eta(x)tV_\varepsilon^2 - (2\omega + \Phi(tV_\varepsilon))\, \Phi(tV_\varepsilon)tV_\varepsilon^2 - \mu t^{q-1}K(x)|V_\varepsilon|^q - t^5|V_\varepsilon|^6\right] \\
&\leqslant \int \left[t|\nabla V_\varepsilon|^2 + \eta(x)tV_\varepsilon^2 - 2\omega\Phi(tV_\varepsilon)tV_\varepsilon^2 - t^5|V_\varepsilon|^6\right] \\
&\leqslant \int \left[t|\nabla V_\varepsilon|^2 + \eta(x)tV_\varepsilon^2 + 2\omega^2 tV_\varepsilon^2 - t^5|V_\varepsilon|^6\right] \\
&= t\int \left[|\nabla V_\varepsilon|^2 + \eta(x)V_\varepsilon^2 + 2\omega^2 V_\varepsilon^2\right] - t^5 = tt_0^4 - t^5 \leqslant 0, \qquad t \geqslant t_0.
\end{aligned}$$

Thus equation (3.15) holds true. From the second equation in system (1.13), one gets

$$(3.16) \qquad \int \left(\frac{1}{16\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4\right) = -\frac{1}{4}\int (\omega + \phi)\,\phi u^2.$$

Now substituting this equation into the functional $I(u)$, one has

(3.17)

$$I(u) = \int \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\eta(x)u^2 - \frac{3}{4}\omega\phi u^2 - \frac{1}{4}\phi^2u^2 - \frac{1}{16\pi}|\nabla\phi|^2 - \frac{\mu}{q}K(x)|u|^q - \frac{1}{6}|u|^6\right].$$

In view of equation (3.16), we have

$$(3.18) \qquad -\frac{1}{4}\int \phi^2u^2 \leqslant \int \omega^2u^2$$

and the function $j(t) = \frac{1}{2}t^2t_0^4 - \frac{1}{6}t^6$ is increasing on $[0, t_0)$, then by equations (3.13), (3.17), (3.18) and (c) of Lemma 2.1, one obtains

$$\begin{aligned}
I(t_\varepsilon V_\varepsilon) &= \int \left[\frac{t_\varepsilon^2}{2}\left(|\nabla V_\varepsilon|^2 + \eta(x)V_\varepsilon^2\right) - \frac{t_\varepsilon^2}{4}\Phi^2(t_\varepsilon V_\varepsilon)V_\varepsilon^2 - \frac{1}{16\pi}|\nabla\Phi(t_\varepsilon V_\varepsilon)|^2\right] \\
&\quad + \int \left[-\frac{3t_\varepsilon^2}{4}\omega\Phi(t_\varepsilon V_\varepsilon)V_\varepsilon^2 - \frac{\mu t_\varepsilon^q}{q}K(x)|V_\varepsilon|^q - \frac{t_\varepsilon^6}{6}|V_\varepsilon|^6\right] \\
&\leqslant \int \left[\frac{t_\varepsilon^2}{2}\left(|\nabla V_\varepsilon|^2 + \left(\eta(x) + 2\omega^2\right)V_\varepsilon^2\right)\right] + \int \left[-\frac{3t_\varepsilon^2}{4}\omega\Phi(t_\varepsilon V_\varepsilon)V_\varepsilon^2 - \frac{\mu t_\varepsilon^q}{q}K(x)|V_\varepsilon|^q - \frac{t_\varepsilon^6}{6}|V_\varepsilon|^6\right] \\
&\leqslant \frac{1}{3}\left(S + O\left(\varepsilon^{\frac{1}{2}}\right) + \int \left(\eta(x) + 2\omega^2\right)V_\varepsilon^2\right)^{\frac{3}{2}} + \frac{3t_\varepsilon^2}{4}\omega^2\int V_\varepsilon^2 - \frac{\mu t_\varepsilon^q}{q}\int K(x)|V_\varepsilon|^q,
\end{aligned}$$

then using the inequality $(a+b)^{\delta} = a^{\delta} + \delta (a+b)^{\delta-1} b$, for all $\delta \geqslant 1, a, b \geqslant 0$, we get

$$I(t_\varepsilon V_\varepsilon) \leqslant \frac{1}{3} S^{\frac{3}{2}} + O\left(\varepsilon^{\frac{1}{2}}\right) + C_1(\varepsilon) \int V_\varepsilon^2 - \mu C_2(\varepsilon) \int |V_\varepsilon|^q,$$

with constants $C_i(\varepsilon) \geqslant C_0 > 0\,(i=1,2)$. On the other hand, we may get the conclusion that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\frac{1}{2}}} \int \left(V_\varepsilon^2 - \mu |V_\varepsilon|^q\right) = -\infty \textit{ for } \varepsilon \textit{ small enough.} \tag{3.19}$$

In fact, by the definition of W_ε, since for $\varepsilon \to 0$, as in [3],

$$\int_{B_{2R}} |W_\varepsilon|^6 \mathrm{d}x = \int_{B_{2R}} |\varphi u_\varepsilon|^6 \mathrm{d}x = C \int \frac{1}{(1+|x|^2)^3} + O\left(\varepsilon^{\frac{3}{2}}\right). \tag{3.20}$$

It suffices to evaluate (3.19) with W_ε in place of V_ε, one has for $p \geqslant 1$,

(3.21)

$$|u_\varepsilon|_{L_r^p}^p = \int_{B_R} \frac{\varepsilon^{\frac{p}{4}}}{(\varepsilon + |x|^2)^{\frac{p}{2}}} \mathrm{d}x = C \int_0^R \frac{\varepsilon^{-\frac{p}{4}} s^2}{\left(1 + \left(\frac{s}{\sqrt{\varepsilon}}\right)^2\right)^{\frac{p}{2}}} \mathrm{d}s = C \varepsilon^{\frac{6-p}{4}} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^{\frac{p}{2}}} \mathrm{d}r,$$

while

$$\int_{B_{2R} \setminus B_R} |W_\varepsilon|^p \mathrm{d}x = O\left(\varepsilon^{\frac{p}{4}}\right), \qquad \varepsilon \to 0, \tag{3.22}$$

and therefore, one has for $4 < q < 6$, as $\varepsilon \to 0$,

$$\int_{B_{2R}} W_\varepsilon^2 \mathrm{d}x - \mu \int_{B_{2R}} W_\varepsilon^q \mathrm{d}x \leqslant C_9 \varepsilon^{\frac{1}{2}} - C_{10} \mu \varepsilon^{\frac{6-q}{4}}, \tag{3.23}$$

where $C_i > 0\,(i = 9, 10)$ are independent from ε. According to equations (3.20) and (3.23), we conclude the proof of equation (3.19).

Now we only need prove $u \neq 0$. Assume that the lemma holds true, by contradiction, $u = 0$, (and hence $\Phi(u) = 0$). Since, as $n \to \infty, (I'(u_n), u_n) \to 0, u_n \to 0$ in $L_r^s(\mathbb{R}^3)$. Obviously, $\int \left[u_n^2 \Phi^2(u_n) + 2\omega \Phi(u_n) u_n^2\right] \to 0$. Next we may assume

$$\int \left[|\nabla u_n|^2 + \eta(x) u_n^2\right] \to l, \qquad l \geqslant 0.$$

$$\int |u_n|^6 \to l, \qquad n \to \infty.$$

So $I(u_n) \to \frac{1}{3} l, n \to \infty$. In view of $c > 0$, then $l > 0$, by the definition of S,

$$S \leqslant \frac{\int \left[|\nabla u_n|^2 + \eta(x) u_n^2\right]}{\left(\int |u_n|^6\right)^{\frac{1}{3}}} \to l^{\frac{2}{3}},$$

so one has

$$c = \left(\frac{1}{2} - \frac{1}{6}\right) l \geqslant \frac{1}{3} S^{\frac{3}{2}}, \tag{3.24}$$

which makes a controdiction with the lemma. $\square$

4. The Proof of Theorem 2.2

We can observe that as in [3], if $q = 4$, in the equation (3.23) one can stress the parameter choosing $\mu = \varepsilon^{-\delta}$, $\delta > 0$, then to get equation (3.19), the rest proof of Theorem 2.2 is similar to proof of Theorem 2.1.

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COVARIOGRAM AND ORIENTATION-DEPENDENT CHORD LENGHT DISTRIBUTION FUNCTION FOR OBLIQUE PRISM

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Abstract. In the present paper we obtain explicit form of covariogram and oriented-dependent chord length distribution function for oblique prism when we know the covariogram of base. Additional we calculate the covariogram and oriented-dependent chord length distribution function in the case if the base is any trapezoid.

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Keywords: Stochastic geometry; chord length distribution function; covariogram; bounded convex body.

1. Introduction

Blaschke formulated the question whether the chord length distribution function characterizes a set [15]. The answer to this question is negative. Mallows and Clark presented non-congruent convex polygons with the same chord length distribution function[11]. There are many articles ([6],[7],[16]) where for solving this problem it is considered that a subclass of the class of convex bodies for which the chord length distribution function is not equal for non-congruent members.

A convex body in R^n is a compact convex set K with non empty interior. Denote by L_n n-dimensional Lebesgue measure on R^n. If $x \in R^n$, D+x denote the translate of D by x, i.e.,

$$D + x = \{y + x, y \in D\}$$

If $D \subset R^n$ is a convex body, then its covariogram $C_D(x)$ is the function defined for $x \in R^n$ by

$$C_D(x) = L_n(D \cap (D + x)).$$

G. Matheron posed in [12] the following question.

Covariogram Problem. Does the covariogram determine a convex body D in R^n, among all convex bodies, up to translation and reflection?

Reflection in this paper always means reflection at a point. Matheron problem is true if n=2 and it is false for $n \geq 4$, but for $n = 3$ it is still open. In [12] Matheron showed that for every $t > 0$ and $\phi \in S^{n-1}$ (S^{n-1} is (n-1)-dimensional unit sphere centered at the origin)

$$\frac{\partial C_D(t\phi)}{\partial t} = -L_{n-1}(\{y \in \phi^{\perp} : L_1(D \cap (l_\phi + y)) \geq t\}) \tag{1.1}$$

where $l_\phi + y$ denotes the line parallel to ϕ through the point y, while $\phi^{\perp}$ denotes the hyperplane in R^n with normal direction $\phi \in S^{n-1}$.

Let G be the space of all lines in the Euclidean plane R^2, $g \in G$ and (p, ϕ) is the polar coordinates of the foot of the perpendicular to g from the origin, $p \geq 0$, $\phi \in S^1$. For a closed bounded convex domain $D \subset R^2$ we denote by $S_D(\phi)$ the support function in direction $\phi \in S^1$ defined by

$$S_D(\phi) = max\{p \geq 0 : g(p, \phi) \cap D \neq \emptyset\}$$

For a bounded convex domain $D \subset R^2$ we denote by $b_D(\phi)$ the breadth function in direction $\phi \in S^1$, that is, the distance between two support lines to the boundary of D that are perpendicular to ϕ. We have

$$b_D(\phi) = S_D(\phi) + S_D(\phi + \pi)$$

Note that $b_D(\phi)$ is a periodic function with period π [15].

For a bounded convex domain D the chord length distribution function in direction ϕ, denoted by $F_D(x, \phi)$, is defined to be the probability of having chord $\chi(g) = g \cap D$ with length at most x in the bundle of lines parallel to ϕ. A random line which is parallel to ϕ and intersects D has an intersection point (denoted by y) with the line $\phi^{\perp}$. The intersection point y is uniformly distributed on the segment $[0, b_D(\phi)]$. Thus, we have

$$F_D(x, \phi) = \frac{L_1(y \in \Pi_D(\phi) : \chi(l_\phi + y) \leq x)}{b_D(\phi)} \tag{1.2}$$

It is not difficult to verify that for n = 2 formula (1.1) is equivalent to

$$\frac{\partial C_D(t\phi)}{\partial t} = -b_D(\phi)(1 - F_D(t, \phi)) \tag{1.3}$$

Denote by Γ the space of lines γ in R^3. Let $\Pi_D(\omega)$ denote the projection of a bounded convex body $D \subset R^3$ in direction $\omega \in S^2$ and let $s_D(\omega)$ be its area. Every line which is parallel to ω and intersects D has an intersection with $\Pi_D(\omega)$. Denote that point by y and that line by $l_\omega + y$. The intersection point y is uniformly distributed on $\Pi_D(\omega)$. The chord length distribution function of D in direction $\omega \in S^2$ is defined by

$$F_D(x, \omega) = \frac{L_2\{y : \chi(l_\omega + y) \leq x\}}{s_D(\omega)} \tag{1.4}$$

It is easy to verify that for n = 3 formula (1.1) is equivalent to

$$(1.5) \qquad \frac{\partial C_D(t\phi)}{\partial t} = -s_D(\phi)(1 - F_D(t, \phi))$$

This article aims to calculate covariogram and orientation-dependent chord length distribution functions (see [1],[2],[5],[8],[9],[13],[14]).

In this paper, we obtain the following results

1)The calculation of the covariogram and Orientation-dependent chord length distribution function for any trapezoid. This is a generalization of the result of [14].

2)Relationships between the covariogram and the orientation-dependent chord length distribution function of an oblique prism and those of its base.

3) Explicit forms of the covariogram and the orientation-dependent chord length distribution function of an oblique prism with cyclic, elliptical, trapezoid and triangular bases. The second and third results are a generalization of [9].

2. Computation of chord length distribution function of an oblique prism

Consider the oblique prism U with base B (not necessarily convex), the length of prism generator is equal to d and angle between prism generator and base is equal to β. It is obvious that the domain $U \cap (U + x)$ is also a prism. If we denote by t the length of x and by $\omega = (\phi, \theta)$, (ϕ, θ) is the cylindrical parametrization of ω; $\phi \in S_1, \theta \in [-\pi/2, \pi/2]$ the direction of x, then the base of the prism $U \cap (U + x)$ will be the domain $B \cap (B + y)$, where y is a planar vector of length $\frac{t\sin(\beta-\theta)}{\sin\beta}$ and direction ϕ, and the height of the new prism will be $d\sin\beta - t\sin\theta$(due to the symmetry we consider only the case $\theta \in [0, \pi/2]$). We can say that

$$C_U(x) = C_U(x\omega) = L_3(U \cap (U + t\omega)) = L_2(B \cap (B + \left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi)(d\sin\beta - t\sin\theta)$$

Implying that

$$(2.1) \qquad C_U(t\omega) = C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi)(d\sin\beta - t\sin\theta)$$

Differentiating both sides of equation (2.1) with respect to t, we get

(2.2)

$$\frac{\partial C_U(t\omega)}{\partial t} = -\sin\theta C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi) + (d\sin\beta - t\sin\theta)\frac{\partial C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi)}{\partial t}$$

Using equation (1.3)

$$(2.3) \qquad -\frac{\partial C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi)}{\partial t} = b_B(\phi)\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)(1 - F_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right), \phi))$$

If we integrate both parts of equation (1.3) from 0 to $\frac{t\sin(\beta-\theta)}{\sin\beta}$, we get

(2.4)

$$C_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\phi) = ||B|| - b_B(\phi)\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right)\int_0^t (1 - F_B(\left(\frac{u\sin(\beta-\theta)}{\sin\beta}\right), \phi))du$$

where $||B||$ is the area of B. Using equations (1.5),(2.3),(2.4) we can transform equation (2.2) the following way

$$s_U(\omega)(1 - F_U(t,\omega)) = \sin\theta(||B|| - b_B(\phi)\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)\int_0^t (1 - F_B(\left(\frac{u\sin(\beta-\theta)}{\sin\beta}\right), \phi))du) +$$

$$+(d\sin\beta - t\sin\theta)b_B(\phi)\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)(1 - F_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right), \phi))$$

We can say that

$$s_U(\omega) = ||B||\sin\theta + b_B(\phi)d\sin\beta\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)$$

Using above mentioned we can formulate the following theorem

Theorem 2.1. *For oblique prism U with base B (not necessarily convex), with prism generator d and angle between prism generator and base β the orineted-dependent chord length distribution given by the following formula*

(2.5)

$$F_U(t,\omega) = \begin{cases} 0, & \textit{if } t \le 0 \\ \frac{b_B(\phi)\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)}{||B||\sin\theta + b_B(\phi)d\sin\beta\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)} \times & \\ \times\Big(t\sin\theta + \sin\theta\int_0^t (1 - F_B(\left(\frac{u\sin(\beta-\theta)}{\sin\beta}\right), \phi))du + & \\ +(d\sin\beta - t\sin\theta)F_B(\left(\frac{t\sin(\beta-\theta)}{\sin\beta}\right), \phi)\Big), & \textit{if } 0 \le t \le t_{max}(\omega) \\ 1, & \textit{if } t \ge t_{max}(\omega) \end{cases}$$

Where $t_{max}(\omega)$ is

(2.6)

$$t_{max}(\omega) = \begin{cases} \frac{\sin\beta x_{max}(\phi)}{|\sin(\beta-\theta)|}, & \text{if } \theta \in [-\arctan\frac{d\sin\beta}{x_{max}(\phi) - d\sin\beta}, \arccos\frac{d\sin\beta}{d\sin\beta + x_{max}(\phi)}] \\ \frac{d\sin\beta}{|\sin\theta|}, & \text{otherwise} \end{cases}$$

when $d\cos\beta < x_{max}(\phi)$ and

(2.7)

$$t_{max}(\omega) = \begin{cases} \frac{\sin\beta x_{max}(\phi)}{|\sin(\beta-\theta)|}, & \text{if } \theta \in [0, \arctan\frac{d\sin\beta}{x_{max}(\phi) + d\sin\beta}] \\ \frac{d\sin\beta}{|\sin\theta|}, & \text{if } \theta \in [\arctan\frac{d\sin\beta}{x_{max}(\phi) + d\sin\beta}, \arctan\frac{d\sin\beta}{d\sin\beta - x_{max}(\phi)}] \\ \frac{\sin\beta x_{max}(\phi)}{|\sin(\theta-\beta)|}, & \text{if } \theta \in [\arctan\frac{d\sin\beta}{d\sin\beta - x_{max}(\phi)}], \pi/2] \cup [-\pi/2, 0] \end{cases}$$

when $d\cos\beta > x_{max}(\phi)$.

3. Chord length distribution in a trapezoid

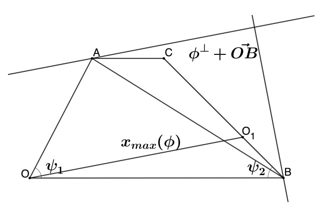
Let $T \subset R^2$ be a trapezoid with bases a and b and the angle between longer base and legs are ψ_1, ψ_2. Without loss of generality we can assume that $0 < \psi_1 \leq \pi/2$, $\psi_1 \leq \psi_2 < \pi$ and $b \leq a$. We can translate and rotate trapezoid so that the longer base be on X-axis.

It is obvious that, the height of trapezoid is equal to $h = (a-b)\frac{sin\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$, the side OA is equal to $l_{OA} = (a-b)\frac{sin\psi_2}{sin(\psi_1+\psi_2)}$ and the side CB is equal to $l_{CB} = (a-b)\frac{sin\psi_1}{sin(\psi_1+\psi_2)}$. From here we can say that the vertices of trapezoid are O(0,0), A($(a-b)\frac{cos\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$,$(a-b)\frac{sin\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$),B(a,0),C(b+$(a-b)\frac{cos\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$,$(a-b)\frac{sin\psi_1 sin\psi_2}{sin(\psi_1+\psi_2)}$). If we take the square or rectangle we should know height and side instead of the above mentioned quantities.

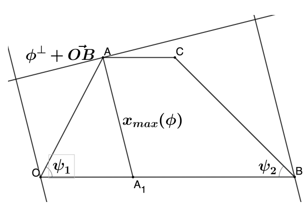
For calculating the orientation-dependent chord length distribution function of a trapezoid, we firstly need explicit form of breadth function of the trapezoid.

Lemma 3.1. *Let $T \subset R^2$ be trapezoid with bases a and b and the angle between longer base and legs are ψ_1 ψ_2. We can assume that the longer leg is equal to a and $\psi_1 \leq \psi_2$. Then the breadth function has the following form*

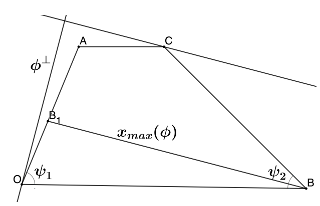
$$b_T(\phi) = \begin{cases} l_{CB} sin(\phi+\psi_2) + b sin\phi, & \text{if } 0 \leq \phi \leq \psi_1 \\ a sin\phi, & \text{if } \psi_1 \leq \phi \leq \pi - \psi_2 \\ b sin(\phi) + l_{OA} sin(\phi - \psi_1), & \text{if } \pi - \psi_2 \leq \phi \leq \pi \end{cases} \tag{3.1}$$



(a) $0 \leq \phi \leq \psi_1$



(b) $\psi_1 \leq \phi \leq \pi - \psi_2$



(c) $\pi - \psi_2 \leq \phi \leq \pi$

Рис. 1

Proof. To prove this lemma firstly we should understand which two vertices have the last intersection with lines in direction ϕ. This means that we should find the $l_\phi + y$ for every Vertex and take the two vertices for which y has the minimum and the maximum value.

(Case i) for $0 \leq \phi < \psi_1$ two vertices are A and B. That means the $b_T(\phi)$ is equal to the projection of AB diagonal onto $\phi^\perp$.

$$b_T(\phi) = L_1(\Pi_{AB}(\phi)) = l_{CB} sin(\phi + \psi_2) + b sin\phi$$

(Case ii) for $\psi_1 \le \phi < \pi - \psi_2$ two vertices are O and B. That means the $b_T(\phi)$ is equal to the projection of OB base onto $\phi^{\perp}$.

$$b_T(\phi) = L_1(\Pi_{OB}(\phi)) = asin\phi$$

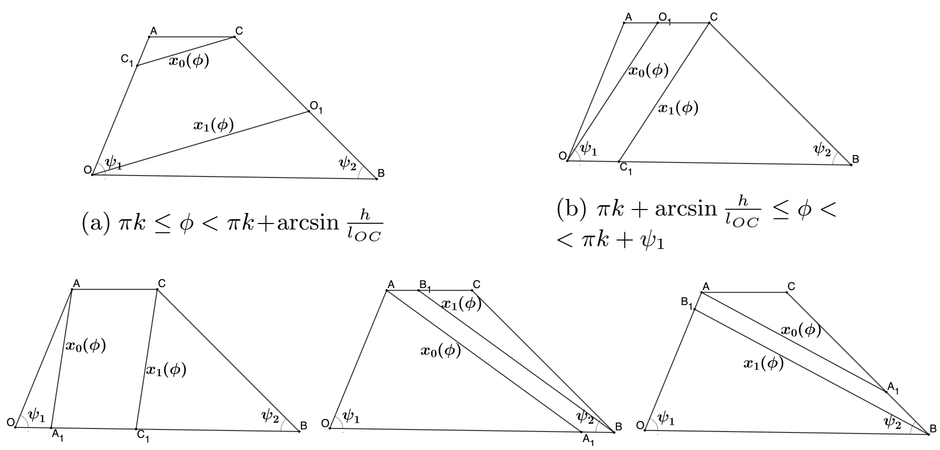
(Case iii) for $\pi - \psi_2 \le \phi < \pi$ two vertices are C and O. That means the $b_T(\phi)$ is equal to the projection of OC diagonal onto $\phi^{\perp}$.

$$b_T(\phi) = L_1(\Pi_{CO}(\phi)) = bsin(\phi) + l_{OA}sin(\phi - \psi_1). \qquad \square$$

We denote the lines $x_0(\phi)$ and $x_1(\phi)$ which has ϕ angle with X-axis, pass through a vertex of trapezoid and make a chord of positive Lebesgue measure,

$$x_0(\phi) = min\chi(l_\phi + y) \text{ and } x_1(\phi) = max\chi(l_\phi + y)$$

Figure 2 shows all cases of above mentioned quantities.



(a) $\pi k \le \phi < \pi k + \arcsin \frac{h}{l_{OC}}$

(b) $\pi k + \arcsin \frac{h}{l_{OC}} \le \phi <$ $< \pi k + \psi_1$

(c) $\psi_1 + \pi k \le \phi < \pi(k+1) -$ $-\psi_2$

(d) $\pi(k+1) - \psi_2 \le \phi < \pi(k+1) -$ $- \arcsin \frac{h}{l_{AB}}$

(e) $\pi(k+1) - \arcsin \frac{h}{l_{AB}} \le \phi <$ $< \pi(k+1)$

Рис. 2

Lemma 3.2. *$x_1(\phi) = x_{max}(\phi)$ for any angle ϕ. If we choose some $k \in Z$ we should have the following cases for $x_0(\phi)$ and $x_1(\phi)$*

(i) If $\pi k \le \phi < \psi_1 + \pi k$

$$x_0(\phi) = \begin{cases} \frac{b \sin \psi_1}{|\sin(\psi_1 - \phi)|}, & \text{if } \pi k \le \phi < \pi k + \arcsin \frac{h}{l_{OC}} \\ \frac{h}{|\sin \phi|}, & if \pi k + \arcsin \frac{h}{l_{OC}} \le \phi < \pi k + \psi_1 \end{cases}$$

$$x_{max}(\phi) = \begin{cases} \frac{a \sin \psi_2}{|\sin(\psi_2 + \phi)|}, & \text{if } \pi k \le \phi < \pi k + \arcsin \frac{h}{l_{OC}} \\ \frac{h}{|\sin \phi|}, & if \pi k + \arcsin \frac{h}{l_{OC}} \le \phi < \pi k + \psi_1 \end{cases}$$

(ii) If $\psi_1 + \pi k \le \phi < \pi(k+1) - \psi_2$

$$x_0(\phi) = x_1(\phi) = \frac{h}{|\sin\phi|}$$

(iii) If $\pi(k+1) - \psi_2 \le \phi < \pi(k+1)$ *, and* $l_{OA}\sin\psi_1 < a$

$$x_0(\phi) = \begin{cases} \frac{h}{|\sin\phi|}, & if\ \pi(k+1) - \psi_2 \le \phi < \pi(k+1) - \arcsin\frac{h}{l_{AB}} \\ \frac{b\sin\psi_2}{|\sin(\phi+\psi_2)|}, & if \pi(k+1) - \arcsin\frac{h}{l_{AB}} \le \phi < \pi(k+1) \end{cases}$$

$$x_{max}(\phi) = \begin{cases} \frac{h}{|\sin(\phi)|}, & if\ \pi(k+1) - \psi_2 \le \phi < \pi(k+1) - \arcsin\frac{h}{l_{AB}} \\ \frac{a\sin\psi_1}{|\sin(\phi-\psi_1)|}, & if \pi(k+1) - \arcsin\frac{h}{l_{AB}} \le \phi < \pi(k+1) \end{cases}$$

(iv) If $\pi(k+1) - \psi_2 \le \phi < \pi(k+1)$ *, and* $l_{OA}\cos\psi_1 > a$

$$x_0(\phi) = \begin{cases} \frac{h}{|\sin\phi|}, & if\ \pi(k+1) - \psi_2 \le \phi < \arcsin\frac{h}{l_{AB}} \\ \frac{b\sin\psi_2}{|\sin(\phi+\psi_2)|}, & if\ \arcsin\frac{h}{l_{AB}} \le \phi < \pi(k+1) \end{cases}$$

$$x_{max}(\phi) = \begin{cases} \frac{h}{|\sin(\phi)|}, & if\ \pi(k+1) - \psi_2 \le \phi < \arcsin\frac{h}{l_{AB}} \\ \frac{a\sin\psi_1}{|\sin(\phi-\psi_1)|}, & if\ \arcsin\frac{h}{l_{AB}} \le \phi < \pi(k+1) \end{cases}$$

Proof. A chord of maximal length in a convex polygon with direction ϕ, also known as ϕ-diameter of the polygon, is not necessarily unique but for any given ϕ exists a ϕ-diameter such that at least one endpoint of the chord coincides with a vertex of the given polygon.

Case (i) sub-case 1 ($\pi k \le \phi < \pi k + \arcsin\frac{h}{l_{OC}}$) From Figure 2a it can be seen that $x_0(\phi) = CC_1$ and $x_1(\phi) = x_{max}(\phi) = OO_1$. By Sine Rule

$$x_0(\phi) = \frac{b\sin(180 - \psi_1)}{\sin(\psi_1 - \phi + \pi k)} = \frac{b\sin\psi_1}{|\sin(\psi_1 - \phi)|}$$

$$x_1(\phi) = x_{max}(\phi) = \frac{a\sin\psi_2}{\sin(180 - \psi_2 - \phi + \pi k)} = \frac{a\sin\psi_2}{|\sin(\psi_2 + \phi)|}$$

Case (i) sub-case 2 ($\pi k + \arcsin\frac{h}{l_{OC}} \le \phi < \pi k + \psi_1$) From Figure 2b it shows that $x_0(\phi) = x_1(\phi) = x_{max}(\phi) = CC_1$. By Sine Rule

$$x_0(\phi) = x_1(\phi) = x_{max}(\phi) = \frac{h}{|\sin\phi|}$$

Case (ii) ($\psi_1 + \pi k \le \phi < \pi(k+1) - \psi_2$) From Figure 2c it can be seen that $x_0(\phi) = x_1(\phi) = x_{max}(\phi) = CC_1$. By Sine Rule

$$x_0(\phi) = x_1(\phi) = x_{max}(\phi) = \frac{h}{|\sin\phi|}$$

Case (iii) sub-case 1 ($\pi(k+1) - \psi_2 \le \phi < \pi(k+1) - \arcsin\frac{h}{l_{AB}}$) From Figure 2d it shows that $x_0(\phi) = BB_1$ and $x_1(\phi) = x_{max}(\phi) = AA_1$. By Sine Rule

$$x_0(\phi) = \frac{l_{CB}\sin(180 - \psi_2)}{\sin(180 - \phi + \pi k)} = \frac{h}{|\sin\phi|}$$

$$x_1(\phi) = x_{max}(\phi) = \frac{h}{\sin(180 - \phi + \pi k)} = \frac{h}{|\sin\phi|}$$

Case (iii) sub-case 2 ($\pi(k+1) - \arcsin\frac{h}{l_{AB}} \leq \phi < \pi(k+1)$) From Figure 2e it can be seen that $x_0(\phi) = AA_1$ and $x_1(\phi) = x_{max}(\phi) = BB_1$. By Sine Rule

$$x_0(\phi) = \frac{b\sin(180-\psi_2)}{\sin(\phi+\psi_2-180+\pi k)} = \frac{b\sin\psi_2}{|\sin(\phi+\psi_2)|}$$

$$x_1(\phi) = x_{max}(\phi) = \frac{a\sin\psi_1}{|\sin(\phi-\psi_1)|}$$

The proof of case (iv) has the same steps as case(iii). □

Theorem 3.1. *$F_T(x,\phi) = 0$ if $x < 0$ and $F_T(x,\phi) = 1$ if $x > x_{max}(\phi)$.Now we discuss the non-trivial cases when $0 < x < x_{max}$. Because this is π periodic function we can assume that k is equal to 0.*

(i) For $0 \leq \phi < \psi_1$

$$F_T(x,\phi) = \begin{cases} \frac{x\sin\phi(\sin(\psi_1-\phi)\sin\psi_2+\sin\psi_1\sin(\phi+\psi_2))}{b_T(\phi)\sin\psi_1\sin\psi_2}, & \text{if } 0 \leq x < x_0(\phi) \\ \frac{1}{b_T(\phi)}(b\sin\phi + \frac{(x-x_0(\phi))\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{\sin(\psi_1+\psi_2)} + \\ +\frac{x\sin(\phi+\psi_2)\sin\phi}{\sin\psi_2}), & \text{if} x_0(\phi) \leq x < x_{max}(\phi) \end{cases}$$

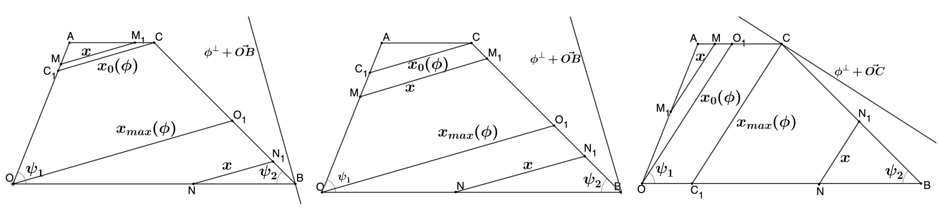
(ii) For $\psi_1 \leq \phi < \pi - \psi_2$

$$F_T(x,\phi) = \frac{x\sin\phi}{b_T(\phi)}\left(\frac{\sin(\psi_2+\phi)\sin\psi_1 + \sin(\phi-\psi_1)\sin\psi_2}{\sin\psi_1\sin\psi_2}\right)$$

(iii) For $\pi - \psi_2 \leq \phi < \pi$

$$F_T(x,\phi) = \begin{cases} \frac{-x\sin\phi(\sin\psi_1\sin(\phi+\psi_2)-\sin(\phi-\psi_1)\sin\psi_2)}{b_T(\phi)\sin\psi_1\sin\psi_2}, & \text{if } 0 \leq x \leq x_0(\phi) \\ \frac{1}{b_T(\phi)}(b\sin\phi - \frac{(x-x_0(\phi))sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)} + \\ +\frac{x\sin\phi\sin(\phi-\psi_1)}{\sin\psi_1}), & if x_0(\phi) \leq x < x_{max}(\phi) \end{cases}$$

Proof.



(a) $0 \leq \phi < \arcsin\frac{h}{l_{OC}}$, $0 \leq x < x_0(\phi)$

(b) $0 \leq \phi < \arcsin\frac{h}{l_{OC}}$, $x_0(\phi) \leq x < x_{max}(\phi)$

(c) $\arcsin\frac{h}{l_{OC}} \leq \phi < \psi_1$, $0 \leq x < x_{max}(\phi)$

Рис. 3

Case (i) sub-case 1 let $0 \leq \phi < \arcsin\frac{h}{l_{OC}}$ and $0 \leq x < x_0(\phi)$. In Figure 3a $|MM_1| = |NN_1| = x < x_0(\phi) = |CC_1| < |OO_1| = x_{max}(\phi)$. For this we can say that $F_T(x,\phi) = \frac{1}{b_T(\phi)}(b_{\Delta AMM_1}(\phi) + b_{\Delta BNN_1}(\phi))$. Here $b_{\Delta AMM_1}(\phi)$ and $b_{\Delta BNN_1}(\phi)$

are equal to the height of triangle AMM_1 (with base MM_1) and BNN_1 (with base NN_1)

$$b_{\Delta AMM_1}(\phi) = \frac{x\sin(\psi_1 - \phi)\sin\phi}{sin\psi_1}$$

$$b_{\Delta BNN_1}(\phi) = \frac{x\sin(\psi_2 + \phi)\sin\phi}{sin\psi_2}$$

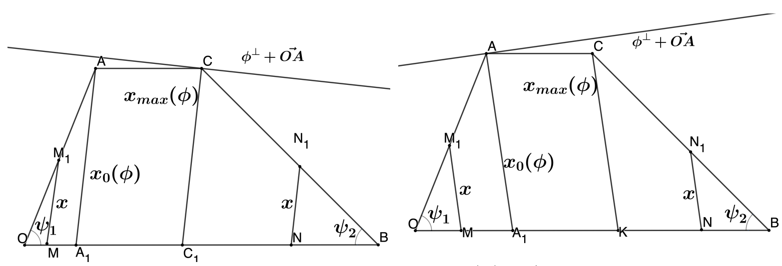
Case (i) sub-case 2 let $0 \le \phi < \arcsin\frac{h}{l_{OC}}$ and $x_0(\phi) \le x < x_{max}(\phi)$. In Figure 3b $x_0(\phi) = CC_1 < x = MM_1 = NN_1 < x_{max}(\phi)$. In this case we have $F(x,\phi) = \frac{1}{b_T(\phi)}(b_{ACMM_1}(\phi) + b_{\Delta BNN_1}(\phi)) = \frac{1}{b_T(\phi)}(b_{ACC_1}(\phi) + b_{\Delta BNN_1}(\phi)) + b_{MCC_1M_1}) = \frac{1}{b_T(\phi)}(b\sin\phi + \frac{x\sin(\psi_2+\phi)\sin\phi}{sin\psi_2} + b_{MCC_1M_1})$. We should calculate the height of trapezoid MCC_1M_1

$$b_{MCC_1M_1} = \frac{\sin(\psi_1 - \phi)\sin(\psi_2 + \phi)(x - x_0(\phi))}{\sin(\psi_1 + \psi_2)}$$

Case (i) sub-case 3 let $\arcsin\frac{h}{l_{OC}} \le \phi < \psi_1$ and $0 \le x < x_{max}(\phi)$. In Figure 3c $x = |NN_1| = |MM_1| < |CC_1| = |OO_1| = x_0(\phi) = x_{max}(\phi)$. Computations of this case are identical as in the previous case (1) sub-case 1. Completing the above we can say that for any $\phi \in [0, \psi_1]$ it brings to

$$F_T(x,\phi) = \begin{cases} \frac{1}{b_T(\phi)}(\frac{x\sin(\psi_1-\phi)\sin\phi}{sin\psi_1} + \frac{x\sin(\psi_2+\phi)\sin\phi}{sin\psi_2}), & \text{if } 0 \le x < x_0(\phi) \\ \frac{1}{b_T(\phi)}(b\sin\phi + \frac{\sin(\psi_1-\phi)\sin(\psi_2+\phi)(x-x_0(\phi)}{\sin(\psi_1+\psi_2)} + & \\ + \frac{x\sin(\phi+\psi_2)\sin\phi}{\sin\psi_2}), & if x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

Case (ii) sub-case 1 $\psi_1 \le \phi < \pi/2$ and $0 \le x < x_{max}(\phi)$. Here $F_t(x,\psi) =$



(a) $\psi_1 \le \phi < \pi/2$, $0 \le x < x_{max}(\phi)$

(b) $\pi/2 \le \phi < \pi - \psi_2$, $0 \le x < x_{max}(\phi)$

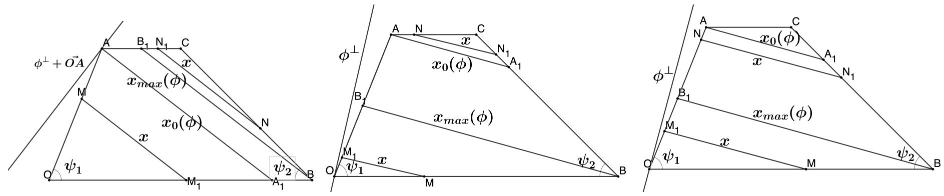
Рис. 4

$\frac{1}{b_T(\phi)}(\frac{x\sin(\phi-\psi_1)\sin\phi}{sin\psi_1} + \frac{x\sin(\psi_2+\phi)\sin\phi}{sin\psi_2})$.

Case (ii) Sub-case 2 $\pi/2 \le \phi < \pi - \psi_2$ and $0 \le x < x_{max}(\phi)$. In Figure 4b $x = |NN_1| = |MM_1| < |CC_1| = |AA_1| = x_0(\phi) = x_{max}(\phi)$ we have

$$F_T(x,\phi) = \frac{1}{b_T(\phi)}(b_{\Delta OMM_1} + B_{\Delta BNN_1}) =$$

$$= \frac{1}{b_T(\phi)}(\frac{x\sin(\phi - \psi_1)\sin\phi}{sin\psi_1} + \frac{x\sin(\psi_2 + \phi)\sin\phi}{sin\psi_2})$$



(a) $\pi - \psi_2 \leq \phi < \pi - \psi_2 - \arcsin \frac{h}{l_{AB}}$, $0 \leq x \leq x_{max}(\phi)$

(b) $\pi - \psi_2 - \arcsin \frac{h}{l_{AB}} \leq \phi < \pi$, $0 \leq x < x_0(\phi)$

(c) $\pi - \psi_2 - \arcsin \frac{h}{l_{AB}} \leq \phi < \pi$, $x_0(\phi) \leq x < x_{max}(\phi)$

Рис. 5

Case (iii) sub-case 1 $\pi - \psi_2 \leq \phi < \pi - \psi_2 - \arcsin \frac{h}{l_{AB}}$ In Figure 5a $x = |NN_1| = |MM_1| < |AA_1| = |BB_1| = x_0(\phi) = x_{max}(\phi)$

$$F_t(x, \phi) = \frac{1}{b_T(\phi)}(b_{\Delta OMM_1} + b_{\Delta CNN_1}) =$$

$$= \frac{1}{b_T(\phi)}\left(\frac{-x\sin(\psi_2 + \phi)\sin\phi}{\sin\psi_2} + \frac{x\sin(\phi - \psi_1)\sin\phi}{\sin\psi_1}\right)$$

Case (iii) sub-case 2 $\pi - \psi_2 - \arcsin \frac{h}{l_{AB}} \leq \phi < \pi$ and $0 \leq x < x_0(\phi)$. In Figure 5b $x = |MM_1| = |NN_1| = x < |AA_1| = x_0(\phi) < |BB_1| = x_{max}(\phi)$

$$F_T(x, \phi) = \frac{1}{b_T(\phi)}(b_{\Delta CNN_1}(\phi) + b_{\Delta OMM_1}(\phi)) =$$

$$= \frac{1}{b_T(\phi)}\left(\frac{-x\sin(\psi_2 + \phi)\sin\phi}{\sin\psi_2} + \frac{x\sin(\phi - \psi_1)\sin\phi}{\sin\psi_1}\right)$$

Case (iii) sub-case 3 $\pi - \psi_2 - \arcsin \frac{h}{l_{AB}} \leq \phi < \pi$ and $x_0(\phi) \leq x < x_{max}(\phi)$

$$F_T(x, \phi) = \frac{1}{b_T(\phi)}(b_{ACNN_1} + b_{\Delta OMM_1}) = \frac{1}{b_T(\phi)}(b_{\Delta CAA_1} + b_{AN_1A_1N} + b_{\Delta OMM_1})$$

$$= \frac{1}{b_T(\phi)}\left(b\sin\phi - \frac{(x - x_0(\phi))sin(\psi_2 + \phi)\sin(\phi - \psi_1)}{\sin(\psi_1 + \psi_2)} + \frac{x\sin\phi\sin(\phi - \psi_1)}{\sin\psi_1}\right). \qquad \square$$

Object	The angles ψ_1, ψ_2	The basis a, b and height h	Article
Square	$\psi_1 = \psi_2 = \pi/2$	a=b=h	[13]
Rectangle	$\psi_1 = \psi_2 = \pi/2$	a=b $\neq h$	[14]
Parallelogram	$\psi_1 = \pi - \psi_2$	a=b	[4]
Right trapezoid	$\psi_2 = \pi/2$	a>b	[14]

We can use theorem 3.1 and obtain the known results of orientation-dependent chord length distribution function (for square and rectangle instead of two bases we should know height and one base). In the table above we show how to do that.

4. Computation of covariogram and chord length distribution function of oblique prism

4.1. **The case of a cyclic oblique prism.** Let L_r be an oblique prism with radius (of the base) r, side d and sides lean over at the base is β. The covariogram of a disc with radius r is

$$C_r(t,\phi) = \begin{cases} 2r^2 \arccos \frac{t}{2r} - \frac{t}{2}\sqrt{4r^2 - t^2}, & \text{if } 0 \leq t \leq 2r \\ 0, & \text{otherwise} \end{cases}$$

Using equation (2.1) for the covariogram L_r we obtain

$$C_{L_r}(t,\omega) = \begin{cases} (d\sin\beta - t\sin\theta)2r^2 \arccos \frac{t\sin(\theta-\beta)}{2r\sin\beta} - \\ - \frac{t\sin(\theta-\beta)}{2\sin\beta}\sqrt{4r^2 - \frac{t\sin(\theta-\beta)}{\sin\beta}}, & \text{if } 0 \leq t \leq \chi_{max}(\omega) \\ 0, & \text{otherwise} \end{cases}$$

where $\chi_{max}(\omega)$ we calculate using (2.6) or (2.7)

For the orientation-dependent chord length distribution function we have

$$F_r(t,\phi) = \begin{cases} 0, & \text{if } t < 0 \\ 1 - \sqrt{1 - \frac{t^2}{4r^2}}, & \text{if } 0 \leq t < 2r \\ 1, & \text{if } t \geq 2r \end{cases}$$

Using equation (2.5) and knowing that $\chi_{max}(\phi) = 2r$, we obtain

$$F_{L_r}(t,\phi) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{2\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)}{\pi r\sin\theta + 2d\sin(\beta-\theta)}\Bigg(d\sin\beta - (d\sin\beta - \\ - \frac{3t\sin\theta)}{2})\left(\sqrt{1 - \left(\frac{t\sin(\beta-\theta)}{2r\sin\beta}\right)^2}\right) + \\ + \frac{r\sin\theta\sin\beta}{\sin(\theta-\beta)}(\arcsin(\frac{t\sin(\beta-\theta)}{2r\sin\beta})) & \text{if } 0 \leq t < \chi_{max}(\omega) \\ 1, & \text{if } t \geq \chi_{max}(\omega) \end{cases}$$

4.2. **The case of an elliptic oblique prism.** Consider a prism L_e with prism generator d, the angle with prism generator and base is β and base as an ellipse with semi-major axes a and b. The covariogram of an ellipse with semi-major axes a and b has the form [10]:

$$C_r(t,\phi) = \begin{cases} 2ab\left(\frac{\pi}{2} - \frac{t}{\chi_{max}(\phi)}\sqrt{1 - \frac{t^2}{\chi_{max}(\phi)}} - \arcsin\frac{t}{\chi_{max}(\phi)}\right), & \text{if } 0 \leq t < \chi_0(\phi) \\ 0, & \text{otherwise} \end{cases}$$

where

$$\chi_{max}(\phi) = \frac{2ab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}$$

is the maximum chord in direction ϕ.

From (2.1) we get

$$C_{L_r}(t\omega) = 2ab\biggl(\frac{\pi}{2} - \frac{t\sin(\beta-\theta)}{\chi_{max}(\phi)\sin\beta}\sqrt{1-\frac{t^2\sin^2(\beta-\theta)}{\chi_{max}(\phi)}\sin^2\beta} -$$

-arcsin $\frac{t\sin(\beta-\theta)}{\chi_{max}(\phi)\sin\beta}\biggr)\biggl(d\sin\beta - t\sin\theta\biggr)$

where $\chi_{max}(\omega)$ we can calculate using equation (2.6) or (2.7)

For the orientation-dependent chord length distribution function we have [10].

$$F_e(t,\phi) = \begin{cases} 0, & \text{if } t < 0 \\ 1 - \sqrt{1 - \frac{t^2}{\chi_{max}(\phi)}}, & \text{if } 0 \le t < \chi_{max}(\phi) \\ 1, & \text{if } t \ge \chi_{max}(\phi) \end{cases}$$

Using equation (2.5) we get

$$F_{L_e}(t,\phi) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{b_e(\phi)\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)}{\pi ab\sin\theta + b_e(\phi)d\sin(\beta-\theta)}\biggl(d\sin\beta - (d\sin\beta - & \\ -\frac{3t\sin\theta)}{2})\left(\sqrt{1-\left(\frac{t\sin(\beta-\theta)}{\chi_{max}(\phi)\sin\beta}\right)^2}\right) + & \\ +\frac{\chi_{max}(\phi)\sin\theta\sin\beta}{2\sin(\theta-\beta)}(\arcsin(\frac{t\sin(\beta-\theta)}{\chi_{max}(\phi)\sin\beta})) & \text{if } 0 \le t < \chi_{max}(\omega) \\ 1, & \text{if } t \ge \chi_{max}(\omega) \end{cases}$$

and $b_e(\phi)$ is equal to

$$b_e(\phi) = \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}$$

4.3. **The case of a triangle oblique prism.** Let L_Δ denote an oblique prism with triangular base Δ. We consider the side of Δ that lies on the X axes. Let a be the length of that side, and ψ_1 and ψ_2 be the corresponding adjacent angles. In [3] it is shown that the covariogram of Δ is given by

$$C_\Delta(t,\phi) = \begin{cases} S_\Delta\left(1 - \frac{t}{\chi_{max}(\phi)}\right)^2, & \text{if } 0 \le t < \chi_{max}(\phi) \\ 0, & \text{otherwise} \end{cases}$$

where S_Δ is the area of the triangle Δ, while $\chi_{max}(\phi)$ is defined by the following formula

$$\chi_{max}(\phi) = \begin{cases} a\sin\psi_2, & \text{if } 0 \le \phi < \psi_1 \\ a\sin\psi_1\sin\psi_2, & \text{if } \psi_1 \le \phi < \pi - \psi_2 \\ a\sin\psi_1, & \text{if } \pi - \psi_2 \le \phi < \pi \end{cases}$$

Taking into account (2.1), we obtain

$$C_{L_\Delta}(t,\phi) = \begin{cases} S_\Delta\left(1 - \frac{t\sin(\beta-\theta)}{\sin\beta\,\chi_{max}(\phi)}\right)^2\biggl(s\sin\beta - t\sin\theta\biggr), & \text{if } 0 \le t < \chi_{max}(\omega) \\ 0, & \text{otherwise} \end{cases}$$

where $\chi_{max}(\phi)$ is defined by (2.6) or (2.7). Again from [3] we have

$$F_\Delta(t,\phi) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{t}{\chi_{max}(\phi)}, & \text{if } 0 \le t < \chi_{max}(\phi) \\ 1, & \text{if } t \ge \chi_{max}(\phi) \end{cases}$$

Using equation (2.5) we get

$$(4.1) \quad F_U(t,\omega) = \begin{cases} 0, & \text{if } t \le 0 \\ \frac{b_B(\phi)t\left(\frac{\sin(\beta-\theta)}{\sin\beta}\right)}{S_\Delta \sin\theta + b_B(\phi) d \sin\beta\left(\frac{2\sin(\beta-\theta)}{\sin\beta}\right)} \times & \\ \times\left(2\sin\theta - \frac{3tsin\theta\sin(\beta-\theta)}{2\sin\beta\chi_{max}(\phi)} + \frac{d\sin(\beta-\theta)}{\chi_{max}(\phi)}\right), & \text{if } 0 \le t \le t_{max}(\omega) \\ 1, & \text{if } t \ge t_{max}(\omega) \end{cases}$$

If for the three sub-sections above we take $\beta = \pi/2$ then we have same results as in [9].

4.4. **The case of a trapezoidal oblique prism.** Denote by D_T the oblique prism with tapezoidal base Using Matheron's formula we can say that

$$\frac{\partial C_T(t,\phi)}{\partial t} = -b_T(\phi)(1 - F_T(t,\phi))$$

If we integrate both parts the last equations yields

$$(4.2) \qquad C_T(t,\phi) = C_T(0,\phi) - b_T(\phi)\int_0^t (1 - F_T(u,\phi))du$$

Using equation (2.1) and Theorem 3.1 we come to explicit formula for $C_T(\phi)$. It is enough to compute for $\phi \in [0,\pi]$ because $C(\cdot,\phi)$ is π-periodic function.

$$C_T(t,\phi) = \frac{h(a+b)}{2} - tb_T(\phi) + b_T(\phi)\int_0^t F_T(u,\phi)du = \frac{h(a+b)}{2} - tb_T(\phi) +$$

$$\begin{cases} \frac{t^2\sin\phi(\sin(\psi_1-\phi)\sin\psi_2 + \sin\psi_1\sin(\phi+\psi_2))}{2\sin\psi_1\sin\psi_2}, & \text{if } 0 \le \phi \le \psi_1,\ 0 \le t < x_0(\phi) \\ tb\sin\phi + \frac{t^2\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{2\sin(\psi_1+\psi_2)} - & \\ -\frac{tx_0(\phi)\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{\sin(\psi_1+\psi_2)} + \frac{t^2\sin(\phi+\psi_2)\sin\phi}{2\sin\psi_2}, & \text{if } 0 \le \phi \le \psi_1,\ x_0(\phi) \le t < x_{max}(\phi) \\ t^2\sin\phi\left(\frac{\sin(\psi_2+\phi)\sin\psi_1 + \sin(\phi-\psi_1)\sin\psi_2}{2\sin\psi_1\sin\psi_2}\right), & \text{if } \psi_1 \le \phi \le \pi - \psi_2,\ 0 \le t \le t_{max}(\phi) \\ \frac{-t^2\sin\phi(\sin\psi_1\sin(\phi+\psi_2) - \sin(\phi-\psi_1)\sin\psi_2)}{2\sin\psi_1\sin\psi_2}, & \text{if } \pi - \psi_2 \le \phi \le \pi,\ 0 \le t < x_0(\phi) \\ tb\sin\phi - \frac{t^2 sin(\psi_2+\phi)\sin(\phi-\psi_1)}{2\sin(\psi_1+\psi_2)} + & \\ +\frac{tx_0(\phi)sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)} + \frac{t^2\sin\phi\sin(\phi-\psi_1)}{2\sin\psi_1}, & \text{if } \pi - \psi_2 \le \phi \le \pi,\ x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

Using equation (2.5) we can find explicit form of orientation-dependent chord length distribution function of oblique prism with trapezoid base.

Denote by

$$m_1(\phi) = \frac{\sin\phi(\sin(\psi_1-\phi)\sin\psi_2 + \sin\psi_1\sin(\phi+\psi_2))}{b_T(\phi)\sin\psi_1\sin\psi_2},$$

$$c_1(\phi) = \frac{1}{b_T(\phi)}(b\sin\phi - \frac{x_0(\phi)(\sin(\psi_1-\phi)\sin(\psi_2+\phi))}{\sin(\psi_1+\psi_2)})$$
$$m_2(\phi) = \frac{1}{b_T(\phi)}(\frac{(\sin(\psi_1-\phi)\sin(\psi_2+\phi)}{\sin(\psi_1+\psi_2)} + \frac{\sin(\phi+\psi_2)\sin\phi}{\sin\psi_2})$$
$$m_3(\phi) = \frac{\sin\phi}{b_T(\phi)}\left(\frac{\sin(\psi_2+\phi)\sin\psi_1 + \sin(\phi-\psi_1)\sin\psi_2}{\sin\psi_1\sin\psi_2}\right)$$
$$m_4(\phi) = \frac{-\sin\phi(\sin\psi_1\sin(\phi+\psi_2) - \sin(\phi-\psi_1)\sin\psi_2)}{b_T(\phi)\sin\psi_1\sin\psi_2}$$
$$c_2(\phi) = \frac{1}{b_T(\phi)}(b\sin\phi + \frac{x_0(\phi)sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)})$$
$$m_5(\phi) = \frac{1}{b_T(\phi)}(-\frac{sin(\psi_2+\phi)\sin(\phi-\psi_1)}{\sin(\psi_1+\psi_2)} + \frac{\sin\phi\sin(\phi-\psi_1)}{\sin\psi_1})$$

Using the notations above we can rewrite Theorem 3.1

Theorem 3.1(rewrite) $F_T(x,\phi) = 0$ if $x < 0$ and $F_T(x,\phi) = 1$ if $x > x_{max}(\phi)$. Now we discuss the non-trivial cases when $0 < x < x_{max}(\phi)$. Because this is π periodic function we can assume that k is equal to 0.

(i) For $0 \le \phi < \psi_1$

$$F_T(x,\phi) = \begin{cases} xm_1(\phi), & \text{if } 0 \le x < x_0(\phi) \\ xm_2(\phi) + c_1(\phi), & \text{if } x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

(ii) For $\psi_1 \le \phi < \pi - \psi_2$

$$F_T(x,\phi) = xm_3(\phi)$$

(iii)For $\pi - \psi_2 \le \phi < \pi$

$$F_T(x,\phi) = \begin{cases} xm_4(\phi), & \text{if } 0 \le x \le x_0(\phi) \\ xm_5(\phi) + c_2(\phi), & \text{if } x_0(\phi) \le x < x_{max}(\phi) \end{cases}$$

Lemma 4.1. *For oblique prism with trapezoid base we have chord length distribution function as (for shortness denote by* $c = \frac{\sin(\beta-\theta)}{\sin\beta}$*)*

(i) If $\pi k \le \phi \le \psi_1 + \pi k$ and $x_0(\phi) \ge \frac{x_{max}(\omega)|\sin(\beta-\theta)|}{\sin\beta}$

$$F_{D_t}(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$
$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)tcm_1(\phi) - \frac{t^2c\sin\theta m_1(\phi)}{2}\right)$$

(ii) If $\pi(k+1) - \psi_2 \le \phi \le \pi(k+1)$ and $x_0(\phi) \le \frac{x_{max}(\omega)|\sin(\beta-\theta)|}{\sin\beta}$

For this case we have 2 sub-cases for calculating $F_T(uc,\phi)$

$$F_T(uc,\phi) = \begin{cases} um_1(\phi)c, & \text{if } u < \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} \\ um_2(\phi)c + c_1(\phi), & \text{if } \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} \le u \le \chi_{max}(\omega) \end{cases}$$

Therefore we get

$$F_{D_t}(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\Bigg(2t\sin\theta + (d\sin\beta - t\sin\theta)(tcm_2(\phi) - c_1(\phi)) - \sin\theta\int_0^{\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}} um_1(\phi)cdu +$$

$$-\sin\theta\int_{\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}}^{t} ucm_2(\phi) - c_1(\phi)du = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\Bigg(2t\sin\theta + (d\sin\beta - t\sin\theta)(tcm_2(\phi) - c_1(\phi)) - \frac{\sin\theta}{2}\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}\right)^2 m_1(\phi)c -$$

$$-\frac{\sin\theta}{2}(\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}\right)^2 - t^2)cm_2(\phi) + c_1(\phi)(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} - t)$$

Case (iii) If $\psi_1 + \pi k \le \phi \le \pi(k+1) - \psi_2$ and $0 \le t \le t_{max}(\omega)$

$$F_U(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)tcm_3(\phi) - \frac{t^2c\sin\theta m_3(\phi)}{2}\right)$$

Case (iv) If $\pi(k+1) - \psi_2 \le \phi \le \pi(k+1)$ and $t \le \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}$

$$F_{D_t}(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\left(2t\sin\theta + (d\sin\beta - t\sin\theta)tcm_4(\phi) - \frac{t^2c\sin\theta m_4(\phi)}{2}\right)$$

(v) If $\pi k \le \phi \le \psi_1 + \pi k$ and $\frac{x_0(\phi)\sin\beta}{||\sin(\beta-\theta)||} \le t \le t_{max}(\omega)$

For this case we have 2 sub-cases for calculating $F_T(uc,\phi)$

$$F_T(uc,\phi) = \begin{cases} um_4(\phi)c, & \text{if } u < \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} \\ um_5(\phi)c + c_2(\phi), & \text{if } \frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} \le u \le \chi_{max}(\omega) \end{cases}$$

Therefore we get

$$F_{D_t}(t,\omega) = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\Bigg(2t\sin\theta + (d\sin\beta - t\sin\theta)(tcm_5(\phi) - c_2(\phi)) - \sin\theta\int_0^{\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}} um_4(\phi)cdu +$$

$$-\sin\theta\int_{\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}}^{t} ucm_5(\phi) - c_2(\phi)du = \frac{b_B(\phi)c}{||B||\sin\theta + b_B(\phi)d\sin\beta c}$$

$$\Bigg(2t\sin\theta + (d\sin\beta - t\sin\theta)(tcm_5(\phi) - c_2(\phi)) - \frac{\sin\theta}{2}\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}\right)^2 m_4(\phi)c -$$

$$-\frac{\sin\theta}{2}(\left(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|}\right)^2 - t^2)cm_5(\phi) + c_2(\phi)(\frac{x_0(\phi)\sin\beta}{|\sin(\beta-\theta)|} - t)$$

where $\chi_{max}(\phi)$ is defined by (2.6) or (2.7).

If we take $\beta = \pi/2$ then we have same results as in [14].

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ZERO-FREE REGIONS FOR LACUNARY TYPE POLYNOMIALS

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Abstract. This paper aims to set an account of zero-free regions for lacunary type polynomials whose coefficients or their real and imaginary parts are subjected to certain restrictions. We also find bounds concerning the number of zeros in a specific annular region.

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Keywords: Eneström-Kakeya theorem, polynomial, zeros.

1. Introduction

Deriving zero bounds for real and complex zeros of polynomials is a classical problem that has been proven essential in various disciplines such as engineering, mathematics, and mathematical chemistry. As indicated, there is a large body of literature dealing with the problem of providing disks in the complex plane representing so called inclusion radii (bounds) where all zeros of an univariate complex polynomial are situated. A review on the location of zeros of polynomials, where the polynomials can be factored over disks in complex plane can be found in ([13],[8],[17],[16]). In accordance with, the following first result which describes the inclusion radii where all zeros of an univariate complex polynomial are scattered is due to Cauchy [3]. All the zeros of a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n, \quad a_n \neq 0$$

lie in the disk

$$|z| < 1 + M,$$

where $M = \max\limits_{0 \le j \le n-1} \left| \frac{a_j}{a_n} \right|$.
Cauchy type polynomials have been studied extensively in the past more than one-century. The research associated with this has sprawled into several directions and generates a plethora of publications for example see ([10], [12], [18], [13]). The research on mathematical objects associated with polynomials and relative position of their zeros has been active over a period; there are many research papers published in a variety of journals each year and different approaches have been taken for different purposes. The present article is concerned with zero free regions and

particularly the number of zeros of a polynomial in a given disk. The following result establishes the improvement of above Cauchy bound under the assumption that the coefficients satisfy monotonicity condition.

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0.$$

Then $P(z)$ has all its zeros in $|z| \leq 1$. This elegant result is known as Eneström-Kakeya Theorem, (for reference see section 8.3 of [18]). In the literature, there exist various extensions and generalizations of Eneström-Kakeya Theorem ([2],[5], [6], [8], [10], [12], [13], [15], [16], [18]). Following analogous result established by Joyal et al.[10], the foremost and the most cited one after Eneström-Kakeya Theorem which acts as a generalization of it.

Let

$$a_n \geq a_{n-1} \geq a_{n-2} ... \geq a_1 \geq a_0.$$

Then the polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ of degree n has all its zeros in

$$|z| \leq \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}.$$

Several years later Aziz and Zargar [2] relaxed the hypothesis in several ways and among other things proved the following result.

Let

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

be a polynomial of degree n with real coefficients such that, for some $k \geq 1$ and for some $0 < \rho \leq 1$ we have

$$k a_n \geq a_{n-1} \geq ... \geq \rho a_0 \geq 0,$$

then $P(z)$ has all its zeros in

$$|z + k - 1| \leq k + \frac{2a_0(1-\rho)}{a_n}.$$

These results proved to be, each in its own way, enabling the growth of sophisticated techniques and critical practices are foundational in the development of the geometry of the zeros of univariate complex polynomial.

Up till now, we have precisely reviewed the regions containing all the zeros of a polynomial $P(z)$ under restricted coefficients. Since the motivation of this article is about the zero free regions and the number of zeros for special family of polynomials and in view of that it is significant to deal with some preliminary results related to zero free regions. The following result is due to Zargar [20].

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some real number $k \geq 1$

$$0 < a_n \leq a_{n-1} \leq ... \leq a_1 \leq k a_0,$$

then $P(z)$ does not vanish in the disk $|z| < \frac{1}{2k-1}$.

Generally speaking, the methods and techniques to develop the zero free and zero containing regions are different and are satisfactory for the readers. The theory on zero free regions for the univariate complex polynomials has been well established ([20], [9], [1], [4], [11]), while somewhat is known for analytic functions. This article describes zero free regions for lacunary type polynomials and this approach is new in comparison with previously published material in the study of zero free regions.

Next we move to the number of zeros of a polynomial in a given disk, the following result concerning the number of zeros of a polynomial in a closed disk can be found in Titchmarsh's classic "The Theory of Functions (see [19],page 171, 2nd edition).

Theorem 1.1. *Let $F(z)$ be analytic in $|z| \leq R$. Let $|F(z)| \leq M$ in $|z| \leq R$ and suppose $F(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disk $|z| \leq R\delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|}.$$

Regarding the number of zeros of a polynomial in $|z| \leq \frac{1}{2}$ and under the same Eneström -Kakeya type restrictions on the coefficients. Mohammad [15] used a special case of Theorem 1.1 in order to establish the following result.

Theorem 1.2. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0,$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

This result has been refined and generalized in different ways (see [5], [7], [8], [16]). Recently Mir et al. [14] imposed certain conditions on the moduli of coefficients and among other things of the Lacunary type polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ and proved the following results.

Theorem 1.3. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \neq 0$, *where for some* $t > 0$ *and some* $\mu \le k \le n$,

$$t^{\mu}|a_{\mu}| \le ... \le t^{k-1}|a_{k-1}| \le t^k|a_k| \ge t^{k+1}a_{k+1} \ge ... \ge t^{n-1}|a_{n-1}| \ge t^n|a_n|$$

and $|arg\ a_j - \beta| \le \alpha \le \frac{\pi}{2}$ *for* $\mu \le j \le n$, *for some real* α *and* β. *Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in* $|z| \le \delta t$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{\mathcal{M}}{|a_0|},$$

where $\mathcal{M} = 2|a_0|t + (|a_{\mu}|t^{\mu+1} + |a_n|t^{n+1})(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha$.

Theorem 1.4. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \neq 0$, *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $t > 0$ *and some* k *with* $\mu \le k \le n$,

$$t^{\mu}\alpha_{\mu} \le ... \le t^{k-1}\alpha_{k-1} \le t^k\alpha_k \ge t^{k+1}\alpha_{k+1} \ge ... \ge t^{n-1}\alpha_{n-1} \ge t^n\alpha_n.$$

Then for $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in* $|z| \le \delta t$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{\mathcal{M}}{|a_0|},$$

where $\mathcal{M} = 2(|\alpha_0|+|\beta_0|)t+(|\alpha_{\mu}|-\alpha_{\mu})t^{\mu+1}+2\alpha_k t^{k+1}+(|\alpha_n|-\alpha_n)t^{n+1}+2\sum_{j=\mu}^{n}|\beta_j|t^{j+1}$.

Theorem 1.5. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \neq 0$, *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $t > 0$ *,for some* k *with* $\mu \le k \le n$, *we have*

$$t^{\mu}\alpha_{\mu} \le ... \le t^{k-1}\alpha_{k-1} \le t^k\alpha_k \ge t^{k+1}\alpha_{k+1} \ge ... \ge t^{n-1}\alpha_{n-1} \ge t^n\alpha_n,$$

and for some $\mu \le l \le n$, *we have*

$$t^{\mu}\beta_{\mu} \le ... \le t^{l-1}\alpha_{l-1} \le t^l\beta_l \ge t^{l+1}\beta_{l+1} \ge ... \ge t^{n-1}\beta_{n-1} \ge t^n\beta_n.$$

Then for $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in* $|z| \le \delta t$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{\mathcal{M}}{|a_0|},$$

where

$$\mathcal{M} = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_{\mu}| - \alpha_{\mu} + |\beta_{\mu}| - \beta_{\mu})t^{\mu+1} +$$
$$+2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_{\mu}| - \beta_{\mu})t^{n+1}.$$

2. Main results

The purpose of this paper is to obtain zero free regions for the lacunary type polynomials whose coefficients satisfy certain monotonicity conditions. We shall also establish the annular region so that number of zeros of $P(z)$ in this region does not exceed any given real number. Also the parameters can be adapted appropriately to the intensity required. In fact we prove the following results.

Theorem 2.1. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu < n,\ a_0 \neq 0,$ *where for some* $t > 0$ *and some* $\mu \leq k \leq n$,

$$t^{\mu}|a_{\mu}| \leq ... \leq t^{k-1}|a_{k-1}| \leq t^{k}|a_k| \geq t^{k+1}|a_{k+1}| \geq ... \geq t^{n-1}|a_{n-1}| \geq t^{n}|a_n|$$

and $|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $\mu \leq j \leq n$, *for some real* α *and* β. *Then no zero of* $P(z)$ *lies in*

$$|z| < \frac{t^2|a_0|}{|a_0|t + (|a_{\mu}|t^{\mu+1} + |a_n|t^{n+1})(1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha}.$$

Theorem 2.1 in conjunction with Theorem 1.3, immediately leads to the following result.

Corollary 2.1. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu < n,\ a_0 \neq 0,$ *where for some* $t > 0$ *and some* $\mu \leq k \leq n$,

$$t^{\mu}|a_{\mu}| \leq ... \leq t^{k-1}|a_{k-1}| \leq t^{k}|a_k| \geq t^{k+1}a_{k+1} \geq ... \geq t^{n-1}|a_{n-1}| \geq t^{n}|a_n|$$

and $|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $\mu \leq j \leq n$, *for some real* α *and* β. *Then for* $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in* $\frac{t^2|a_0|}{\mathcal{M}_1} \leq |z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{\mathcal{M}}{|a_0|},$$

where

$$\mathcal{M} = 2|a_0|t + (|a_{\mu}|t^{\mu+1} + |a_n|t^{n+1})(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha$$

$$\mathcal{M}_1 = |a_0|t + (|a_{\mu}|t^{\mu+1} + |a_n|t^{n+1})(1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n}|a_j|t^{j+1}\sin\alpha.$$

Notice that when $t = 1$ in Theorem 2.1, it produces the following result.

Corollary 2.2. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu < n,\ a_0 \neq 0,$ *where for some* $\mu \leq k \leq n$.

Theorem 2.2. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $t > 0$ *and some* k *with* $\mu \le k \le n$,

$$t^{\mu}\alpha_{\mu} \le ... \le t^{k-1}\alpha_{k-1} \le t^{k}\alpha_{k} \ge t^{k+1}\alpha_{k+1} \ge ... \ge t^{n-1}\alpha_{n-1} \ge t^{n}\alpha_{n}.$$

Then no zero of $P(z)$ *lies in*

$$|z| < \frac{t^2(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n}|\beta_j| t^{j+1}}.$$

On combining Theorem 2.2 and Theorem 1.4, we get the following result.

Corollary 2.3. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $t > 0$ *and some* k *with* $\mu \le k \le n$, *we have*

$$t^{\mu}\alpha_{\mu} \le ... \le t^{k-1}\alpha_{k-1} \le t^{k}\alpha_{k} \ge t^{k+1}\alpha_{k+1} \ge ... \ge t^{n-1}\alpha_{n-1} \ge t^{n}\alpha_{n}.$$

Then for $0 < \delta < 1$, *the number of zeros of* $P(z)$ *in* $\frac{t^2(|\alpha_0|+|\beta_0|)}{\mathcal{M}_2} \le |z| \le \delta t$ *does not exceed*

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{\mathcal{M}}{|a_0|},$$

where

$$\mathcal{M} = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n}|\beta_j| t^{j+1}$$

and

$$\mathcal{M}_2 = (|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n}|\beta_j| t^{j+1}.$$

Taking $t = 1$ in Theorem 2.2, we get the following result.

Corollary 2.4. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* $\mu \le k \le n$, *we have*

$$\alpha_{\mu} \le ... \le \alpha_{k-1} \le \alpha_{k} \ge \alpha_{k+1} \ge ... \ge \alpha_{n-1} \ge \alpha_{n}.$$

Then $P(z)$ *does not vanish in*

$$|z| < \frac{(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2\sum_{j=\mu}^{n}|\beta_j|}.$$

Finally, we put the monotonicity conditions on the real and imaginary parts of the coefficients of $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ in order to obtain zero free region and an annular region onwards. More precisely, we prove the following results.

Theorem 2.3. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu < n,\ a_0 \neq 0,$ where $Re\ a_j = \alpha_j$ and $Im\ a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some $t > 0$,for some k with $\mu \leq k \leq n$,*

$$t^{\mu}\alpha_{\mu} \leq ... \leq t^{k-1}\alpha_{k-1} \leq t^{k}\alpha_{k} \geq t^{k+1}\alpha_{k+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^{n}\alpha_{n}$$

and for some $\mu \leq l \leq n$,

$$t^{\mu}\beta_{\mu} \leq ... \leq t^{l-1}\alpha_{l-1} \leq t^{l}\beta_{l} \geq t^{l+1}\beta_{l+1} \geq ... \geq t^{n-1}\beta_{n-1} \geq t^{n}\beta_{n}.$$

Then $P(z)$ does not vanish in

$$|z| < \frac{t^2(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + k\, t^{n+1}}.$$

where $k = |\alpha_n| - \alpha_n + |\beta_n| - \beta_n$.

Theorem 2.3 in conjunction with Theorem 1.5 yields the following result.

Corollary 2.5. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu < n,\ a_0 \neq 0,$ where $Re\ a_j = \alpha_j$ and $Im\ a_j = \beta_j$ for $\mu \leq j \leq n$. Suppose that for some $t > 0$,for some k with $\mu \leq k \leq n$,*

$$t^{\mu}\alpha_{\mu} \leq ... \leq t^{k-1}\alpha_{k-1} \leq t^{k}\alpha_{k} \geq t^{k+1}\alpha_{k+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^{n}\alpha_{n}$$

and for some $\mu \leq l \leq n$,

$$t^{\mu}\beta_{\mu} \leq ... \leq t^{l-1}\alpha_{l-1} \leq t^{l}\beta_{l} \geq t^{l+1}\beta_{l+1} \geq ... \geq t^{n-1}\beta_{n-1} \geq t^{n}\beta_{n}.$$

Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $\frac{t^2(|\alpha_0|+|\beta_0|)}{\mathcal{M}_3} \leq |z| \leq \delta t$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{\mathcal{M}}{|a_0|},$$

where

$$\begin{aligned}\mathcal{M} = {} & 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} \\ & + 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_\mu| - \beta_\mu)t^{n+1},\end{aligned}$$

$$\begin{aligned}\mathcal{M}_3 = {} & (|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} \\ & + 2(\alpha_k t^{k+1} + \beta_l t^{l+1}) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1}.\end{aligned}$$

Here it is interesting to note that Theorem 2.3 gives us several corollaries under the monotonicity conditions on real and imaginary parts. Taking $t = 1$ in Theorem 2.3, we get the following result.

Corollary 2.6. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$. *Suppose that for some* k *with* $\mu \le k \le n$, *we have*

$$\alpha_\mu \le ... \le \alpha_{k-1} \le \alpha_k \ge \alpha_{k+1} \ge ... \ge \alpha_{n-1} \ge \alpha_n$$

and for some $\mu \le l \le n$,

$$\beta_\mu \le ... \le \alpha_{l-1} \le \beta_l \ge \beta_{l+1} \ge ... \ge \beta_{n-1} \ge \beta_n.$$

Then no zero of $P(z)$ *lies in*

$$|z| < \frac{(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + 2(\alpha_k + \beta_l) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)}.$$

Fix $t = 1$ and $k = l = n$ in Theorem 2.3, we immediately obtain the following result.

Corollary 2.7. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$ *such that*

$$\alpha_\mu \le ... \le \alpha_{n-1} \le \alpha_n$$

and

$$\beta_\mu \le ... \le \beta_{n-1} \le \beta_n.$$

Then no zero of $P(z)$ *lies in*

$$|z| < \frac{(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + (|\alpha_n| + \alpha_n + |\beta_\mu| + \beta_\mu)}.$$

Set $t = 1$ and $k = l = \mu$ in Theorem 2.3, we get the following result.

Corollary 2.8. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu < n,\ a_0 \ne 0,$ *where* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$ *for* $\mu \le j \le n$ *such that*

$$\alpha_\mu \ge ... \ge \alpha_{n-1} \ge \alpha_n$$

and

$$\beta_\mu \ge ... \ge \beta_{n-1} \ge \beta_n.$$

Then no zero of $P(z)$ *lies in*

$$|z| < \frac{(|\alpha_0| + |\beta_0|)}{(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu + |\beta_\mu| + \beta_\mu) + (|\alpha_n| - \alpha_n + |\beta_\mu| - \beta_\mu)}.$$

3. Proofs of theorems

For the proofs of our main results, we need the following auxiliary result.

Lemma 3.1. *Let $P(z)$ be a polynomial of degree n. If for some real α and β, $|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $0 \leq j \leq n$ and for any $t > 0$ such that, $|a_j| \geq |a_{j-1}|, 0 \leq j \leq n$, then $\left|ta_j - a_{j-1}\right| \leq \big(t|a_j| - |a_{j-1}|\big)cos\alpha + \big(t|a_j| + |a_{j-1}|\big)sin\alpha$.*

The above lemma is due to Govil and Rahman [8].

Proof of Theorem 2.1 Consider the polynomial

$$F(z) = (t-z)P(z) = (t-z)\bigg(a_0 + \sum_{j=\mu}^{n} a_j z^j\bigg).$$

This implies,

$$F(z) = a_0 t + \sum_{j=\mu}^{n} ta_j z^j - a_0 z - \sum_{j=\mu}^{n} a_j z^{j+1} = a_0(t-z) + \sum_{j=\mu}^{n} ta_j z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j$$

i.e., $F(z) = a_0(t-z) + ta_\mu z^\mu + \sum\limits_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1} = a_0 t + R(z)$, where $R(z) = -a_0 z + ta_\mu z^\mu + \sum\limits_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1}$. On $|z| = t$, we have

$$\begin{aligned}|R(z)| =& \bigg| - a_0 z + ta_\mu z^\mu + \sum_{j=\mu+1}^{n} (ta_j - a_{j-1})z^j - a_n z^{n+1}\bigg| \\ \leq& |a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{n} |ta_j - a_{j-1}|t^j + |a_n|t^{n+1}.\end{aligned}$$

Equivalently,

$$|R(z)| \leq t|a_0| + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} |ta_j - a_{j-1}|t^j + \sum_{j=k+1}^{n} |ta_j - a_{j-1}|t^j + |a_n|t^{n+1}.$$

Using lemma 3.1, we get

$$\begin{aligned}|R(z)| \leq t|a_0| + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} \{(|a_j|t - |a_{j-1}|)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\}t^j \\ + \sum_{j=k+1}^{n} \{(|a_{j-1}| - |a_j|t)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\}t^j + |a_n|t^{n+1} \\ = t|a_0| + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} |a_j|t^{j+1}\cos\alpha - \sum_{j=\mu+1}^{k} |a_{j-1}|t^j\cos\alpha \\ + \sum_{j=\mu+1}^{k} |a_j|t^{j+1}\sin\alpha + \sum_{j=\mu+1}^{k} |a_{j-1}|t^j\sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}|t^j\cos\alpha\end{aligned}$$

$$- \sum_{j=k+1}^{n} |a_j| t^{j+1} \cos\alpha + \sum_{j=k+1}^{n} |a_{j-1}| t^{j} \sin\alpha + \sum_{j=k+1}^{n} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1}.$$

This gives,

$$|R(z)| \le |a_0|t + |a_\mu|t^{\mu+1} - |a_\mu|t^{\mu+1}\cos\alpha + |a_k|t^{k+1}\cos\alpha + |a_\mu|t^{\mu+1}\sin\alpha$$
$$+ |a_k|t^{k+1}\sin\alpha + 2\sum_{j=\mu+1}^{k-1} |a_j|t^{j+1} + |a_k|t^{k+1}\cos\alpha - |a_n|t^{n+1}\cos\alpha + |a_k|t^{k+1}\sin\alpha$$
$$+ |a_n|t^{n+1}\sin\alpha + 2\sum_{j=k+1}^{n-1} |a_j|t^{j+1}\sin\alpha + |a_n|t^{n+1} = |a_0|t + (|a_\mu|t^{\mu+1} + |a_n|t^{n+1})$$
$$\times (1 - \sin\alpha - \cos\alpha) + 2|a_k|t^{k+1}\cos\alpha + 2\sum_{j=\mu}^{n} |a_j|t^{j+1}\sin\alpha = \mathcal{M}_1.$$

Applying Schwarz lemma to $R(z)$, we get $|R(z)| \le \dfrac{\mathcal{M}_1|z|}{t}, \quad |z| \le t$. Hence

$$|F(z)| = |a_0 t + R(z)| \ge |a_0|t - |R(z)| \ge |a_0|t - \frac{\mathcal{M}_1|z|}{t} > 0 \qquad for\ |z| \le t,$$

if $|a_0|t - \dfrac{\mathcal{M}_1|z|}{t} > 0$. That is, if $|z| < \dfrac{t^2|a_0|}{\mathcal{M}_1}$. This shows that $F(z)$ and hence $P(z)$ has no zero in $|z| < \frac{t^2|a_0|}{\mathcal{M}_1}$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 We consider

$$F(z) = (t-z)P(z) = a_0(t-z) + t a_\mu z^\mu + \sum_{j=\mu+1}^{n} (t a_j - a_{j-1}) z^j - a_n z^{n+1}.$$

Equivalently,

$$F(z) = (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu) t z^\mu + \sum_{j=\mu+1}^{n} (\alpha_j t - \alpha_{j-1}) z^j$$
$$+ i \sum_{j=\mu+1}^{n} (\beta_j t - \beta_{j-1}) z^j - (\alpha_n + i\beta_n) z^{n+1} = (\alpha_0 + i\beta_0)t + R(z).$$

For $|z| = t$, we have

$$|R(z)| \le (|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{n} |\alpha_j t - \alpha_{j-1}| t^j$$
$$+ \sum_{j=\mu+1}^{n} |\beta_j t - \beta_{j-1}| t^j + (|\alpha_n| + |\beta_n|)t^{n+1}$$
$$= (|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_j t - \alpha_{j-1}) t^j$$
$$+ \sum_{j=k+1}^{n} (\alpha_{j-1} - \alpha_j t) t^j + \sum_{j=\mu+1}^{n} \big(|\beta_j| t + |\beta_{j-1}|\big) t^j + (|\alpha_n| + |\beta_n|)t^{n+1} =$$

$$= (|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|-\alpha_\mu)t^{\mu+1}+2\alpha_k t^{k+1}+(|\alpha_n|-\alpha_n)t^{n+1}+2\sum_{j=\mu}^{n}|\beta_j|t^{j+1} = \mathcal{M}_2.$$

Applying Schwarz lemma to the polynomial $R(z)$, we get

$$|R(z)| \leq \frac{\mathcal{M}_2|z|}{t}, \qquad for\ |z| \leq t.$$

Hence $|F(z)| = |a_0 t + R(z)| \geq |a_0|t - |R(z)| \geq |a_0|t - \dfrac{\mathcal{M}_2|z|}{t} > 0$, $|z| \leq t$, if $|a_0|t - \dfrac{\mathcal{M}_2|z|}{t} > 0$, that is, if $|z| < \dfrac{t^2|a_0|}{\mathcal{M}_2}$. This shows that $F(z)$ and hence $P(z)$ has no zero in $|z| < \frac{t^2|a_0|}{\mathcal{M}_2}$ and the proof of Theorem 2.2 is complete. □

Proof of Theorem 2.3 As in the proof of Theorem 2.2,

$$\begin{aligned}F(z) =&(\alpha_0+i\beta_0)(t-z)+(\alpha_\mu+i\beta_\mu)tz^\mu+\sum_{j=\mu+1}^{n}(\alpha_j t-\alpha_{j-1})z^j\\&+i\sum_{j=\mu+1}^{n}(\beta_j t-\beta_{j-1})z^j-(\alpha_n+i\beta_n)z^{n+1}=(\alpha_0+i\beta_0)t+R(z).\end{aligned}$$

where

$$\begin{aligned}R(z)=&-(\alpha_0+i\beta_0)z+(\alpha_\mu+i\beta_\mu)tz^\mu+\sum_{j=\mu+1}^{n}(\alpha_j t-\alpha_{j-1})z^j\\&+i\sum_{j=\mu+1}^{n}(\beta_j t-\beta_{j-1})z^j-(\alpha_n+i\beta_n)z^{n+1}.\end{aligned}$$

For $|z| = t$, we have

$$\begin{aligned}|R(z)| \leq& (|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}+\sum_{j=\mu+1}^{n}|\alpha_j t-\alpha_{j-1}|t^j\\&+\sum_{j=\mu+1}^{n}|\beta_j t-\beta_{j-1}|t^j+(|\alpha_n|+|\beta_n|)t^{n+1}=(|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}\\&+\sum_{j=\mu+1}^{k}(\alpha_j t-\alpha_{j-1})t^j+\sum_{j=k+1}^{n}(\alpha_{j-1}-\alpha_j t)t^j+\sum_{j=\mu+1}^{k}(\beta_j t-\beta_{j-1})t^j\\&+\sum_{j=k+1}^{n}(\beta_{j-1}-\beta_j t)t^j+(|\alpha_n|+|\beta_n|)t^{n+1}=(|\alpha_0|+|\beta_0|)t\\&+(|\alpha_\mu|-\alpha_\mu+|\beta_\mu|-\beta_\mu)t^{\mu+1}+2(\alpha_k t^{k+1}+\beta_l t^{l+1})\\&+(|\alpha_n|-\alpha_n+|\beta_n|-\beta_n)t^{n+1}=\mathcal{M}_3.\end{aligned}$$

Applying Schwarz lemma to the polynomial $R(z)$, we get

$$|R(z)| \leq \frac{\mathcal{M}_2|z|}{t}, \qquad for\ |z| \leq t.$$

Hence

$$\begin{aligned}|F(z)| =&|(\alpha_0 + i\beta_0)t + R(z)| \geq (|\alpha_0| + |\beta_0|)t - |R(z)|\\ &\geq (|\alpha_0| + |\beta_0|)t - \frac{\mathcal{M}_3|z|}{t} > 0, \qquad |z| \leq t,\end{aligned}$$

if

$$(|\alpha_0| + |\beta_0|)t - \frac{\mathcal{M}_3|z|}{t} > 0.$$

That is, if

$$|z| < \frac{(|\alpha_0| + |\beta_0|)t^2}{\mathcal{M}_3}.$$

This shows that $F(z)$ and hence $P(z)$ has no zero in $|z| < \frac{(|\alpha_0|+|\beta_0|)t^2}{\mathcal{M}_3}$. This completes the proof of Theorem 2.3. □

4. Examples

Since the present article is concerned with newly developed approach to obtain the zero free regions and the number of zeros for the lacunary type polynomials in a given disk. From this point of view, the comparison of the bounds obtained with the previous bounds appropriately have no scope within this type of study. Instead of comparing the bounds, we point out few examples which may be helpful to be examined.

Example 4.1. Let $P(z) = 2z^5 + 2.5z^4 + 4z^3 + 3z^2 + 2z + 1$. Clearly, here $\mu = 1$ and $n = 5$. We take $k = 3$, $\alpha = \pi/2$ and $t = 1$. In view of Theorem 2.1 and due to this type of intensity of parameters the radius of given disk comes out to be 0.0357. Since the appropriate zeros of $P(z)$ are $-0.358+0.9154i$, $-0.358-0.9154i$, $0.0756+0.8657i$, $0.0756-0.8657i$, -0.6853. Then one can see that $P(z)$ does not vanish in $|z| < 0.0357$.

Since corollary 2.1.1 is the union of Theorem 2.1 and Theorem 1.3. Under the same example it is clear that all the zeros of $P(z) = 2z^5+2.5z^4+4z^3+3z^2+2z+1$ lie in $|z| \geq 0.0357$. If we set $\delta = 0.7$, the upper bound of the annular region in corollary 2.1.1 comes out to be 0.7 as $t = 1$. In this case, we found that the number of zeros of underlying polynomial $P(z)$ in $0.0357 \leq |z| \leq 0.7$ does not exceed $\frac{1}{\log\frac{1}{0.7}} \log(29) \approx 9.4524$. Hence we conclude that $P(z)$ has at most one zero in $0.0357 \leq |z| \leq 0.7$ and of course, $P(z)$ has exactly one zero in $0.0357 \leq |z| \leq 0.7$. All above discussion demonstrates one thing, which is beauty to say, that the bound in Theorem 2.1 becomes the lower bound of the annular region in corollary 2.1.1.

Example 4.2. Let $P(z) = 2z^5+3z^4+4z^3+2z^2+1.5z+1$. Clearly, here $\mu = 1$ and $n = 5$. Setting $k = 3$ and $t = 1$. In view of Theorem 2.2 the radius comes out to be

0.1111. Numerically the appropriate zeros of $P(z)$ are $-0.6193+1.0343i,\ -0.6193-1.0343i,\ 0.2089+0.6804i,\ 0.2089-0.6804i,\ 0.6792$. It is clear from these zeros that $P(z)$ does not vanish in $|z|<0.1111$.

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A NEW PROOF OF THE GASCA - MAEZTU CONJECTURE FOR $n = 5$

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Abstract. An n-correct node set $\mathcal{X}$ is called GC_n set if the fundamental polynomial of each node is a product of n linear factors. In 1982 Gasca and Maeztu conjectured that for every GC_n set there is a line passing through $n+1$ of its nodes. So far, this conjecture has been confirmed only for $n \leq 5$. The case $n = 4$, was first proved by J. R. Busch [3]. Several other proofs have been published since then. For the case $n = 5$ there is only one proof by H. Hakopian, K. Jetter and G. Zimmermann (Numer Math 127:685–713, 2014). Here we give a second proof, which largely follows the first one but is much shorter and simpler.

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1. Introduction

Denote by Π_n the space of bivariate polynomials of total degree at most n :

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij} x^i y^j \right\}, \quad N := \dim \Pi_n = \binom{n+2}{2}.$$

Consider a set of distinct nodes $\mathcal{X}_s = \{(x_1, y_1), (x_2, y_2), \dots, (x_s, y_s)\}$.

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(x_i, y_i) = c_i, \qquad i = 1, 2, \dots s, \tag{1.1}$$

is called *interpolation problem.*

Definition 1.1. A set of nodes $\mathcal{X}_s$ is called *n-poised* if for any data $\{c_1, \dots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions (1.1).

A necessary condition of n-poisedness is: $\#\mathcal{X}_s = s = N$. If this latter equality takes place then the following holds:

Proposition 1.1. *A set of nodes $\mathcal{X}_N$ is n-poised if and only if*

$$p \in \Pi_n, \ p(x_i, y_i) = 0 \quad i = 1, \dots, N \implies p = 0.$$

A polynomial $p \in \Pi_n$ is called an *n-fundamental polynomial* for a node $A = (x_k, y_k) \in \mathcal{X}_s$ if

$$p(x_i, y_i) = \delta_{ik}, \ i = 1, \ldots, s,$$

where δ is the Kronecker symbol. We denote the n-fundamental polynomial of $A \in \mathcal{X}_s$ by $p_A^\star = p_{A,\mathcal{X}}^\star$.

Definition 1.2. A set of nodes $\mathcal{X}_s$ is called *n-independent* if each node has n-fundamental polynomial. Otherwise, $\mathcal{X}_s$ is called *n-dependent.* A set of nodes $\mathcal{X}_s$ is called *essentially n-dependent* if none of its nodes has n-fundamental polynomial.

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-indepen-dence is $\#\mathcal{X}_s = s \le N$.

One can readily verify that a node set $\mathcal{X}_s$ is n-independent if and only if the interpolation problem (1.1) is solvable, meaning that for any data $\{c_1, \ldots, c_s\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

A *plane algebraic curve* is the zero set of some bivariate polynomial of degree ≥ 1. To simplify notation, we shall use the same letter, say p, to denote the polynomial p and the curve given by the equation $p(x, y) = 0$. In particular, by ℓ, we denote a linear polynomial $\ell \in \Pi_1$ and the line defined by the equation $\ell(x, y) = 0$.

Definition 1.3. Let $\mathcal{X}$ be an n-poised set. We say, that a node $A \in \mathcal{X}$ *uses a line* ℓ, if ℓ is a factor of the fundamental polynomial $p_A^\star$, i.e.,

$$p_A^\star = \ell q, \tag{1.2}$$

where $q \in \Pi_{n-1}$.

Since the fundamental polynomial of a node in an n-poised set is unique we get

Lemma 1.1 ([9], Lemma 2.5)**.** *Suppose $\mathcal{X}$ is a poised set and a node $A \in \mathcal{X}$ uses a line ℓ. Then ℓ passes through at least two nodes from $\mathcal{X}$, at which q from* (1.2) *does not vanish.*

Definition 1.4. Let $\mathcal{X}$ be a set of nodes. We say, that a line ℓ is a *k-node line* if it passes through exactly k nodes of $\mathcal{X}$: $\ \ell \cap \mathcal{X} = k$.

The following proposition is well-known (see e.g. [8] Proposition 1.3):

Proposition 1.2. *Suppose that a polynomial $p \in \Pi_n$ vanishes at $n + 1$ points of a line ℓ. Then we have that $p = \ell r, \ $ where $r \in \Pi_{n-1}$.*

From here we readily get that at most $n+1$ nodes of an n-poised set $\mathcal{X}_N$ can be collinear. In view of this an $(n+1)$-node line ℓ is called a *maximal line* [2].

Next, let us bring the Cayley-Bacharach theorem (see e.g. [6], Th. CB4; [8], Prop. 4.1).

Theorem 1.1. *Assume that two algebraic curves of degree m and n, respectively, intersect at mn distinct points. Then the set $\mathcal{X}$ of these intersection points is essentially $(m+n-3)$-dependent.*

We are going to consider a special type of n-poised sets defined by Chung and Yao:

Definition 1.5 ([5])**.** An n-poised set $\mathcal{X}$ is called *GC_n set*, if the n-fundamental polynomial of each node $A \in \mathcal{X}$ is a product of n linear factors.

Now we are in a position to present the Gasca-Maeztu conjecture.

Conjecture 1.1 ([7])**.** *For any GC_n set $\mathcal{X}$ there is a maximal line, i.e., a line passing through its $n+1$ nodes.*

Since now the Gasca-Maeztu conjecture was proved to be true only for $n \leq 5$. The case $n=2$ is trivial, and the case $n=3$ is easy to verify. The case $n=4$ first was proved by J. R. Busch [3]. Several other proofs have been published since then (see e.g. [4], [9], [1]). For the case $n=5$ there is only one proof by H. Hakopian, K. Jetter and G. Zimmermann [10]. Here we give a second proof, which largely follows the first one but is much shorter and simpler.

1.1. **The m-distribution sequence of a node.** In this section we bring a number of concepts, properties and results from [10].

Suppose that $\mathcal{X}$ is a GC_n set. Consider a node $A \in \mathcal{X}$ together with the set of n used lines denoted by $\mathcal{L}_A$. The $N-1$ nodes of $\mathcal{X} \setminus \{A\}$ belong to the lines of $\mathcal{L}_A$.

Let us order the lines of $\mathcal{L}_A$ in the following way:

The line ℓ_1 is a line in $\mathcal{L}_A$ that passes through maximal number of nodes of $\mathcal{X}$, denoted by $k_1 : \mathcal{X} \cap \ell_1 = k_1$.

The line ℓ_2 is a line in $\mathcal{L}_A \setminus \{\ell_1\}$ that passes through maximal number of nodes of $\mathcal{X} \setminus \ell_1$, denoted by $k_2 : (\mathcal{X} \setminus \ell_1) \cap \ell_2 = k_2$.

In the general case the line $\ell_s,\ s = 1, \ldots, n$, is a line in $\mathcal{L}_A \setminus \{\ell_1, \ldots, \ell_{s-1}\}$ that passes through maximal number of nodes of the set $\mathcal{X} \setminus \cup_{i=1}^{s-1} \ell_i$, denoted by k_s : $(\mathcal{X} \setminus \cup_{i=1}^{s-1} \ell_i) \cap \ell_s = k_s$.

A correspondingly ordered line sequence

$$\mathcal{S} = (\ell_1, \dots, \ell_n)$$

is called a *maximal line sequence* or briefly an *m-line sequence* if the respective sequence $(k_1, \dots, k_n)$ is the maximal in the lexicographic order [10]. Then the latter sequence is called a *maximal distribution sequence* or briefly an *m-d sequence.*

Evidently, for the m-d sequence we have that

$$k_1 \geq k_2 \geq \dots \geq k_n \text{ and } k_1 + \dots + k_n = N - 1. \tag{1.3}$$

Though the m-distribution sequence for a node A is unique, it may correspond to several m-line sequences.

Note that, an intersection point of several lines of $\mathcal{L}_A$ is counted for the line containing it which appears in $\mathcal{S}$ first. Each node in $\mathcal{X}$ is called a *primary* node for the line it is counted for, and a *secondary* node for the other lines containing it.

According to Lemma 1.1, every used line contains at least two primary nodes, i.e.,

$$k_i \geq 2 \quad \text{for } i = 1, \dots, n\,. \tag{1.4}$$

Let $\mathcal{S} = (\ell_1, \dots, \ell_n)$ be an m-line sequence with the associated m-d sequence $(k_1, \dots, k_n)$.

Lemma 1.2 ([10], Lemma 2.5)**.** *Assume that $k_i = k_{i+1} =: k$ for some i. If the intersection point of lines ℓ_i and ℓ_{i+1} belongs to $\mathcal{X}$, then it is a secondary node for both ℓ_i and ℓ_{i+1}. Moreover, interchanging ℓ_i and ℓ_{i+1} in $\mathcal{S}$ still yields an m-line sequence.*

We say that a polynomial has $(s_i, \dots, s_j)$ *primary zeroes* in the lines $(\ell_i, \dots, \ell_j)$ if the zeroes are primary nodes in the respective lines. From Proposition 1.2 we get

Corollary 1.1. *If a polynomial $p \in \Pi_{m-1}$ has $(m, m-1, \dots, m-k)$ primary zeroes in the lines $(\ell_{m-k}, \ell_{m-k+1} \dots, \ell_m)$ then we have that $p = \ell_m \ell_{m-1} \cdots \ell_{m-k} r$, where $r \in \Pi_{m-k}$.*

In some cases a particular line $\widetilde{\ell}$ used by a node is fixed and then the properties of the other factors of the fundamental polynomial are studied.

In this case in the corresponding m-line sequence, called $\widetilde{\ell}$*-m-line sequence*, one takes as the first line ℓ_1 the line $\widetilde{\ell}$, no matter through how many nodes it passes. Then the second and subsequent lines are chosen, as in the case of the m-line sequence.

Thus the line ℓ_2 is a line in $\mathcal{L}_A \setminus \{\widetilde{\ell}_1\}$ that passes through maximal number of nodes of $\mathcal{X} \setminus \widetilde{\ell}_1$, and so on.

Correspondingly the $\widetilde{\ell}$*-m-distribution sequence* is defined.

2. The Gasca-Maeztu conjecture for $n = 5$

Let us formulate the Gasca-Maeztu conjecture for $n = 5$ as:

Theorem 2.1. *For any GC_5 set $\mathcal{X}$ of 21 nodes there is a maximal line, i.e., a 6-node line.*

To prove the theorem assume by way of contradiction the following.

Assumption 2.1. The set $\mathcal{X}$ is a GC_5 set with no maximal line.

In view of (1.3) and (1.4) the only possible m-d sequences for any node $A \in \mathcal{X}$ are

$$(5,5,5,3,2); \quad (5,5,4,4,2); \quad (5,5,4,3,3); \quad (5,4,4,4,3); \quad (4,4,4,4,4). \tag{2.1}$$

The results from [10] below show how many times a line can be used, depending the number of nodes it passes through. In each statement it is assumed that $\mathcal{X}$ is a GC_5 set with no maximal line.

Proposition 2.1 ([10], Prop. 2.11)**.** *Suppose that $\widetilde{\ell}$ is a 2-node line. Then $\widetilde{\ell}$ can be used by at most one node of $\mathcal{X}$.*

Proposition 2.2 ([10], Prop. 2.12)**.** *Suppose that $\widetilde{\ell}$ is a 3-node line and is used by two nodes A, $B \in \mathcal{X}$. Then there exists a third node C using $\widetilde{\ell}$. Furthermore, A, B, and C share three other lines, each passing through five primary nodes. For each of the three nodes, the m-d sequence is $(5,5,5,3,2)$, and the other two nodes are the primary nodes in the respective fifth line. In particular, $\widetilde{\ell}$ is used exactly three times.*

Proposition 2.3 ([10], Prop. 2.13)**.** *Suppose that a line $\widetilde{\ell}$ is used by three nodes A, B, $C \in \mathcal{X}$. Then $\widetilde{\ell}$ passes through at least three nodes of $\mathcal{X}$.*

If $\widetilde{\ell}$ is a 4-node line, then A, B, and C share $\widetilde{\ell}$ and three other lines, ℓ_2 and ℓ_3 passing through five and ℓ_4 through four primary nodes. For each of the three nodes, the $\widetilde{\ell}$-m-distribution sequence with respect to $\widetilde{\ell}$ is $(4,5,5,4,2)$. $\widetilde{\ell}$ can only be used by A, B, and C, i.e., it is used exactly three times.

Corollary 2.1 ([10], Cor. 2.14)**.** *Suppose that a line $\widetilde{\ell}$ is used by four nodes in $\mathcal{X}$. Then $\widetilde{\ell}$ is a 5-node line.*

Proposition 2.4 ([10], Prop. 2.15)**.** *Suppose that a line $\widetilde{\ell}$ is used by five nodes in $\mathcal{X}$. Then $\widetilde{\ell}$ is a 5-node line, and it is actually used by exactly six nodes in $\mathcal{X}$. These six nodes form a GC_2 set and share two more lines with five primary nodes each, i.e., each of these six nodes has the m-d sequence* $(5, 5, 5, 3, 2)$.

At the end we bring a (part of a) table from [10] which follows from Propositions 2.1, 2.2, 2.3, 2.4 and Corollary 2.1. It shows under which conditions a k-node line $\widetilde{\ell},\ 2 \leq k \leq 5$, can be used at most how often, provided that the considered GC_5 set has no maximal line.

(2.2)

total # of nodes in $\widetilde{\ell}$	maximal # of nodes using $\widetilde{\ell}$: in general	maximal # of nodes using $\widetilde{\ell}$: no node uses $(5, 5, 5, 3, 2)$ m-d sequence
5	6	4
4	3	3
3	3	1
2	1	1

2.1. **The case** $(5\,,5\,,5\,,3\,,2)$**.** In this and the following sections, we will prove the following

Proposition 2.5. *Assume that $\mathcal{X}$ is a GC_5 set with no maximal line. Then for no node in $\mathcal{X}$ the m-d sequence is* $(5, 5, 5, 3, 2)$.

Assume by way of contradiction the following.

Assumption 2.2. $\mathcal{X}$ contains a node for which an m-line sequence $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$ implies the m-d sequence $(5, 5, 5, 3, 2)$.

Set $\mathcal{X} = \mathcal{A} \cup \mathcal{B}$ (see Fig. 2.1), with

$$\mathcal{A} = \mathcal{X} \cap \{\ell_1 \cup \ell_2 \cup \ell_3\}, \quad \#\mathcal{A} = 15, \quad \text{and} \quad \mathcal{B} = \mathcal{X} \setminus \mathcal{A}, \quad \#\mathcal{B} = 6.$$

Denote $\mathcal{L}_3 := \{\ell_1, \ell_2, \ell_3\}$. Note that no intersection point of the three lines of $\mathcal{L}_3$ belongs to $\mathcal{X}$.

Below we bring a simple proof for

Lemma 2.1 ([10], Lemma 3.2)**.**

(i) *The set $\mathcal{B}$ is a GC_2 set, and each node $B \in \mathcal{B}$ uses the three lines of $\mathcal{L}_3$ and the two lines it uses within $\mathcal{B}$, i.e.,*

$$p^{\star}_{B,\mathcal{X}} = \ell_1\, \ell_2\, \ell_3\, p^{\star}_{B,\mathcal{B}}\,. \tag{2.3}$$

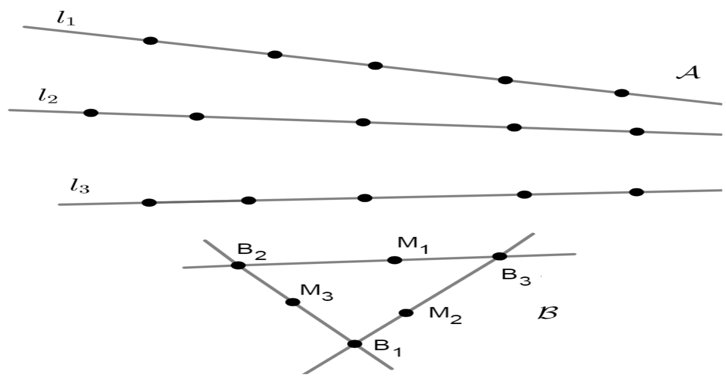

Рис. 2.1. The case $(5,5,5,3,2)$ with $\mathcal{X} = \mathcal{A} \cup \mathcal{B}$.

(ii) *No node in* $\mathcal{A}$ *uses any of the lines of* $\mathcal{L}_3$.

Proof. (i) Suppose by way of contradiction that the set $\mathcal{B}$ is not 2-poised, i.e., it is a subset of a conic $\mathcal{C}$. Then $\mathcal{X}$ is a subset of the zero set of the polynomial $\ell_1\,\ell_2\,\ell_3\,\mathcal{C}$, which contradicts Proposition 1.1. Then we readily obtain the formula (2.3).

(ii) Without loss of generality assume that $A \in \ell_1$ uses the line ℓ_2. Then $p_A^\star = \ell_2\, q$, where $q \in \Pi_4$. It is easily seen that q has (5,4) primary zeros in the lines (ℓ_3, ℓ_1). Therefore, in view of Corollary 1.1, we obtain that $p_A^\star = \ell_2\,\ell_3\,\ell_1\, r$, which is a contradiction. □

Evidently, any node in a GC_2 set uses a maximal line, i.e., 3-node line. Hence we conclude readily that any GC_2 set, including also $\mathcal{B}$, possesses at least three maximal lines (see Figure 2.1).

A node $A \in \mathcal{X}$ is called a 2_m-*node* if it is the intersection point of two maximal lines. Note that the nodes $B_i,\ i = 1,2,3$, in Fig. 2.1, are 2_m-nodes for $\mathcal{B}$.

Definition 2.1. We say, that a line ℓ is a $k_{\mathcal{A}}$-*node* line if it passes through exactly k nodes of $\mathcal{A}$.

Lemma 2.2. *(i) Assume that a line* $\widetilde{\ell} \notin \mathcal{L}_3$ *does not intersect a line* $\ell \in \mathcal{L}_3$ *at a node in* $\mathcal{X}$. *Then the line* $\widetilde{\ell}$ *can be used at most by one node from* $\mathcal{A}$. *Moreover, this latter node belongs to* $\ell \cap \mathcal{A}$.

(ii) If a line ℓ *is* $0_{\mathcal{A}}$ *or* $1_{\mathcal{A}}$*-node line then no node from* $\mathcal{A}$ *uses the line* ℓ.

(iii) If a line ℓ *is* $2_{\mathcal{A}}$*-node line then* ℓ *can be used by at most one node from* $\mathcal{A}$.

(iv) Suppose ℓ *is a maximal line in* $\mathcal{B}$. *Then* ℓ *can be used by at most one node from* $\mathcal{A}$.

Proof. (i) Without loss of generality assume that $\ell = \ell_1$ and $A \in \ell_2$ uses $\widetilde{\ell}$:

$$p_A^\star = \widetilde{\ell}\, q,\ \ q \in \Pi_4.$$

It is easily seen that q has $(5,4,3)$ primary zeros in the lines (ℓ_1, ℓ_3, ℓ_2). Therefore, in view of Corollary 1.1, we conclude that $p_A^\star = \widetilde{\ell}\, \ell_1\, \ell_2\, \ell_3\, r,\ \ r \in \Pi_1$, which is a contradiction.

Now assume conversely that $A, B \in \ell_1 \cap \mathcal{X}$ use the line $\widetilde{\ell}$. Choose a point $C \in \ell_2 \setminus (\widetilde{\ell} \cup \mathcal{X})$. Then choose numbers α and β, with $|\alpha| + |\beta| \neq 0$, such that $p(C) = 0$, where $p := \alpha p_A^\star + \beta p_B^\star$. It is easily seen that $p = \widetilde{\ell}\, q,\ q \in \Pi_4$ and the polynomial q has $(5,4,3)$ primary zeros in the lines (ℓ_2, ℓ_3, ℓ_1). Therefore $p = \widetilde{\ell}\ell_1\, \ell_2\, \ell_3\, q$, where $q \in \Pi_1$. Thus $p(A) = p(B) = 0$, implying that $\alpha = \beta = 0$, which is a contradiction.

The items (ii) and (iii) readily follow from (i). The item (iv) readily follows from (iii). □

Denote by ℓ_{AB} the line passing through the points A and B.

Proposition 2.6. *Let $\ell_{B_1M_1}$ be 5-node line, which is used by all the six nodes of a subset $\mathcal{A}_6 \subset \mathcal{A}$. Suppose also that ℓ is a 4-node line passing through B_1. If the line ℓ is used by three nodes from $\mathcal{A}$ then all these three nodes belong to $\mathcal{A}_6$.*

Proof. The six nodes of $\mathcal{A}_6$ use the 5-node line $\ell_{B_1M_1}$. Therefore, in view of Proposition 2.4, these six nodes share also two more lines passing through five primary nodes. It is easily seen that these latter two lines are the lines $\ell_{B_2M_2}$ and $\ell_{B_3M_3}$. Assume by way of contradiction that the nodes $D_1, D_2, D_3 \in \mathcal{A}$ are using the line ℓ and $D_1 \notin \mathcal{A}_6$. According to Proposition 2.3 these three nodes share also two lines passing through five primary nodes.

In view of Lemma 2.2, (iv), these latter two lines cannot be maximal lines in $\mathcal{B}$. Therefore they belong to the set $\{\ell_{B_2M_2}, \ell_{B_3M_3}, \ell_{M_1M_2}, \ell_{M_2M_3}, \ell_{M_1M_3}\}$. One of them should be $\ell_{B_2M_2}$ or $\ell_{B_3M_3}$, since any two lines from $\{\ell_{M_1M_2}, \ell_{M_2M_3}, \ell_{M_1M_3}\}$ share a node. Therefore one of them will be used by seven nodes, namely by D_1 and the nodes of $\mathcal{A}_6$. This contradicts Proposition 2.4. □

2.2. **The proof of Proposition 2.5.** Consider all the lines passing through $B := B_1$ and at least one more node of $\mathcal{X}$. Denote the set of these lines by $\mathcal{L}(B)$. Let $m_k(B),\ k = 1,2,3$, be the number of $k_{\mathcal{A}}$-node lines from $\mathcal{L}(B)$.

We have that

$$1m_1(B) + 2m_2(B) + 3m_3(B) = \#\mathcal{A} = 15. \tag{2.4}$$

Lemma 2.3. *Suppose that a line ℓ, passing through B and different from the line ℓ_{BM_1}, is a $3_{\mathcal{A}}$-node line. Then ℓ can be used by at most three nodes from $\mathcal{A}$.*

Proof. Note that ℓ is not a maximal line for $\mathcal{B}$, since otherwise ℓ will be a maximal line for $\mathcal{X}$. Therefore ℓ is a 4-node line and Proposition 2.3 completes the proof. □

Lemma 2.4. *We have that $m_3(B) \leq 4$.*

Proof. The equality (2.4) implies that $m_3(B) \leq 5$. Assume by way of contradiction that five lines pass through B and three nodes in $\mathcal{A}$. Therefore these five lines intersect the three lines ℓ_1, ℓ_2, ℓ_3, at the 15 nodes of $\mathcal{A}$. Then, by Theorem 1.1, these 15 nodes are $5 + 3 - 3 = 5$-dependent, which is a contradiction. □

Proof of Proposition 2.5. In view of Proposition 2.4 we divide the proof into three cases.

Case 1. Suppose that ℓ_{BM_1} is 5-node line used by six nodes from $\mathcal{A}$.

Denote the set of these six nodes by $\mathcal{A}_6 \subset \mathcal{A}$. We have that any node from $\mathcal{A}$ uses at least one line from $\mathcal{L}(B)$. Proposition 2.6 implies that all $3_{\mathcal{A}}$-node lines from $\mathcal{L}(B)$, except ℓ_{BM_1}, can be used by at most two nodes from $\mathcal{A} \setminus \mathcal{A}_6$.

From Lemma 2.2, we have that

$$15 - 6 \leq 0m_1(B) + 1m_2(B) + 2(m_3(B) - 1). \tag{2.5}$$

In view of (2.4) we get

$$m_1(B) + 2m_2(B) + 3m_3(B) - 6 \leq 1m_2(B) + 2m_3(B) - 2. \tag{2.6}$$

Therefore we conclude that $m_1(B)+m_2(B)+m_3(B) \leq 4$, or, in other words, $3m_1(B)+3m_2(B) + 3m_3(B) \leq 12$, which contradicts (2.4).

Case 2. Suppose that ℓ_{BM_1} is 5-node line used by at most four nodes of $\mathcal{A}$.

In this case we have that

$$15 \leq 1m_2(B) + 3(m_3(B) - 1) + 4.$$

In view of (2.4) we get

$$m_1(B) + 2m_2(B) + 3m_3(B) \leq 1m_2(B) + 3m_3(B) + 1. \tag{2.7}$$

Hence $2m_1(B) + 2m_2(B) \leq 2$. Now, by using (2.4) again, we conclude that

$$3m_3(B_1) \geq 13, \tag{2.8}$$

which contradicts Lemma 2.4.

Case 3. Suppose that ℓ_{BM_1} is not 5-node line.

Then, in view of the table (2.2), it can be used by at most three nodes of $\mathcal{A}$. From Lemmas 2.2 and 2.3, (ii),(iii), we have that

$$15 \leq 1m_2(B) + 3m_3(B). \tag{2.9}$$

In view of (2.4) we get

$$m_1(B) + 2m_2(B) + 3m_3(B) \leq m_2(B) + 3m_3(B). \tag{2.10}$$

Hence $m_1(B) = m_2(B) = 0$ and $m_3(B) \geq 5$, which contradicts Lemma 2.4. $\square$

2.3. **The cases** $(5\,,5\,,4\,,4\,,2)$ **,** $(5\,,5\,,4\,,3\,,3)$, **and** $(5\,,4\,,4\,,4\,,3)$**.** Let us fix a node $A \in \mathcal{X}$ and consider the set of lines $\mathcal{L}(A)$. Let $n_k(A)$ be the number of $(k+1)$-node lines from $\mathcal{L}_A$. In view of Assumption 2.1 we have that

$$1n_1(A) + 2n_2(A) + 3n_3(A) + 4n_4(A) = \#\big(\mathcal{X} \setminus \{A\}\big) = 20. \tag{2.11}$$

Next we bring a result from [10]. We present also the proof for the convenience.

Lemma 2.5 ([10], Lemma 3.13)**.** *Assume that $\mathcal{X}$ is a GC_5 set with no maximal line. By Proposition 2.5, for no node of $\mathcal{X}$ the m-d sequence is* $(5, 5, 5, 3, 2)$*. Then the following hold.*

(i) *There is no* 3*-node line and m-node line is used exactly $m-1$ times, where* $m = 2, 4, 5$.

(ii) *No two lines used by the same node intersect at a node in $\mathcal{X}$.*

Proof. (i) Consider all the lines in $\mathcal{L}(A)$. From the third column of the table in (2.2), it follows that for the total number $M(A)$ of uses of these lines, we have that

$$M(A) \leq 1\,n_1(A) + 1\,n_2(A) + 3\,n_3(A) + 4\,n_4(A)\,. \tag{2.12}$$

Since each node in $\mathcal{X}\setminus\{A\}$ uses at least one line through A, we must have $M(A) \geq 20$. In view of the equality (2.11) we conclude that $M(A) = 20$ and $n_2(A) = 0$.

Moreover, we deduce that any line containing m nodes including A has to be used exactly $m-1$ times, where $m = 2, 4, 5$. Since the node A is arbitrary, this is true for all lines containing at least two nodes of $\mathcal{X}$.

(ii) Assume conversely that two lines ℓ_1, ℓ_2, used by a node $A \in \mathcal{X}$ intersect at a node $B \in \mathcal{X}$. Then each of the nodes in $\mathcal{X} \setminus \{A, B\}$ uses at least one line through B, while the node A uses at least two lines. Thus we have $M(A) \geq 21$, which is a contradiction. $\square$

Corollary 2.2. *For no node in $\mathcal{X}$ the m-d sequence is* $(5, 5, 4, 3, 3)$ *or* $(5, 4, 4, 4, 3)$.

Proof. Suppose, that for a node $A \in \mathcal{X}$, the m-d sequence is $(5,5,4,3,3)$ or $(5,4,4,4,3)$. In view of Lemma 2.5, (ii), there are no secondary nodes in the used lines. Thus the presence of 3 the m-d sequence implies presence of a 3-node line in an m-line sequence, which contradicts Lemma 2.5, (i). □

Proposition 2.7. *For no node in $\mathcal{X}$ the m-d sequence is* $(5,5,4,4,2)$.

Proof. Assume that for a node $A \in \mathcal{X}$ some m-line sequence $(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)$ implies the m-d sequence $(5,5,4,4,2)$. In view of Lemma 2.5, (ii), the lines $\ell_1,...,\ell_5$, contain exactly $5,5,4,4,2$, nodes, respectively. Denote by B and C the two nodes in the line ℓ_5. Then we have

$$p_B^\star = \ell_1\,\ell_2\,\ell_3\,\ell_4\,\ell_{AC} \quad \text{and} \quad p_C^\star = \ell_1\,\ell_2\,\ell_3\,\ell_4\,\ell_{AB}.$$

In view of Lemma 2.5 the line ℓ_1 is used by exactly four nodes of $\mathcal{X}$. Therefore, there exists a node $D \in \mathcal{X} \setminus \{A,B,C\}$, which is using the line ℓ_1.

In view of (2.1), Proposition 2.5, and Corollary 2.2, for the node $D \in \mathcal{X}$ some m-line sequence $(\ell_1,\ell_2',\ell_3',\ell_4',\ell_5')$ yields the m-d sequence $(5,5,4,4,2)$.

Now, as above, we have that the two nodes in the line ℓ_5' use the line ℓ_1. In view of Proposition 2.1, the line ℓ_5', used by the node D, cannot coincide with the lines ℓ_{AB},ℓ_{AC} or ℓ_{BC}. Therefore ℓ_5' contains a node different from A,B,C,D. Hence, the line ℓ_1 is used at least five times, which is a contradiction. □

2.4. **Proof of theorem 2.1.** What is left to complete the proof of Theorem 2.1 is the following

Proposition 2.8. *For no node in $\mathcal{X}$ the m-d sequence is* $(4,4,4,4,4)$.

Proof. Let us fix a node $A \in \mathcal{X}$. In view of (2.1), Propositions 2.5, 2.7 and Corollary 2.2, for the node A, m-d sequence is $(4,4,4,4,4)$. Thus, in view of Lemma 2.5, (ii), all used lines are 4-node lines. Therefore, in view of Lemma 2.5, (i), we conclude that $n_1(A) = n_2(A) = n_4(A) = 0$. Now, the equality (2.11) implies that $3n_3(A) = 20$, which is not possible. □

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BERGMAN-TYPE AND Q_k-TYPE SPACES OF p-HARMONIC FUNCTIONS

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Abstract. In this paper, we extend a Hardy-Littlewood type theorem to the exponentially p-harmonic Bergman space on the real unit ball $\mathbb{B}$ in $\mathbb{R}^n$. As an application, we characterize exponentially p-harmonic Bergman spaces in terms of Lipschitz type conditions. Furthermore, some derivative-free characterizations for n-harmonic Q_k spaces are established.

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1. Introduction and main results

For $n \geq 2$, let $\mathbb{R}^n$ denote the usual real vector space of dimension n. For two column vectors $x, y \in \mathbb{R}^n$, we use $\langle x, y\rangle$ to denote the inner product of x and y. The ball in $\mathbb{R}^n$ with center a and radius r is denoted by $\mathbb{B}(a, r)$. In particular, we write $\mathbb{B} = \mathbb{B}(0, 1)$ and $\mathbb{B}_r = \mathbb{B}(0, r)$. Let dv be the normalized volume measure on $\mathbb{B}$ and $d\sigma$ the normalized surface measure on the unit sphere $\mathbb{S} = \partial\mathbb{B}$.

The purpose of this paper is to investigate p-harmonic functions whose definition is as follows.

Definition 1.1. Let $p > 1$ and Ω be a domain in $\mathbb{R}^n$. A continuous function $u \in W^{1,p}_{loc}(\Omega)$ is *p-harmonic* if

$$\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = 0$$

in the weak sense, i.e.,

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\eta\rangle dv(x) = 0$$

for each $\eta \in C_0^\infty(\Omega)$.

p-harmonic functions are natural extensions of harmonic functions from a variational point of view. It has been extensively studied because of its various interesting

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features and applications. By a well-known regularity result due to Tolksdorf, p-harmonic functions are $C^1(\Omega)$. Moreover $u \in W^{2,2}_{loc}(\Omega)$ if $p \geq 2$ and $u \in W^{2,p}_{loc}(\Omega)$ if $1 < p < 2$ (cf. [12, 20]).

Let $p > 1$, we denote by $h_p(\mathbb{B})$ the set of all p-harmonic functions on the real unit ball $\mathbb{B}$ in $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$ and $\beta > 0$, the so-called exponential weighted function $\omega_{\alpha,\beta}$, introduced by Aleman and Siskakis [2], is defined as

$$\omega_{\alpha,\beta}(x) = (1 - |x|)^{\alpha} \exp\Big(\frac{-1}{(1 - |x|)^{\beta}}\Big), \quad x \in \mathbb{B},$$

and the associated weighted volume measure is denoted by

$$dv_{\alpha,\beta}(x) = \omega_{\alpha,\beta}(x) dv(x).$$

For $1 < s < \infty$, $\alpha \in \mathbb{R}$ and $\beta > 0$, the *exponentially weighted p-harmonic Bergman space* $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$ is defined as

$$\mathcal{A}^s_{\alpha,\beta}(\mathbb{B}) = \Big\{ u \in h_p(\mathbb{B}) : \|u\|^s_{\mathcal{A}^s_{\alpha,\beta}} = \int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x) < \infty \Big\}.$$

In particular, if $\beta = 0$, then $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$ becomes the weighted p-harmonic Bergman space, which is denoted by $\mathcal{A}^s_{\alpha}(\mathbb{B})$.

For $0 < s < \infty$, $\alpha > -1$, let f be a holomorphic function on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. The famous Hardy-Littlewood theorem for holomorphic Bergman spaces asserts that

$$(1.1) \int_{\mathbb{D}} |f(z)|^s (1 - |z|^2)^{\alpha} dA(z) \approx |f(0)|^s + \int_{\mathbb{D}} |f'(z)|^s (1 - |z|^2)^{s+\alpha} dA(z),$$

where dA is the area measure on $\mathbb{C}$ normalized so that $A(\mathbb{D}) = 1$ (cf. [10]).

It is well-known that integral estimate (1.1) plays an important role in the theory of holomorphic functions. For the generalizations and applications of (1.1) to the spaces of holomorphic functions, harmonic functions, and solutions to certain PDEs, see [3, 4, 5, 9, 15, 11, 14, 21, 25] and the references therein. In [18], Siskakis extended (1.1) to the setting of exponentially weighted Bergman space of holomorphic functions for $1 \leq s < \infty$. For the further generalizations of (1.1) to holomorphic Bergman spaces with some general differential weights, see [15, 19]. By applying these results, Cho and Park characterized exponentially weighted Bergman space in terms of Lipschitz type conditions([5, Theorem A], [6, Theorem 3.1]).

In [11], Kinnunen et al. pointed out that (1.1) is also true for p-harmonic functions. More precisely, they obtained the following integral estimate.

Theorem A. *Let $\alpha > -1$, $1 < s < \infty$, then*

$$\int_{\mathbb{B}} |u(x)|^s (1-|x|)^\alpha dv(x) \approx |u(0)|^s + \int_{\mathbb{B}} |\nabla u(x)|^s (1-|x|)^{s+\alpha} dv(x) \tag{1.2}$$

for all $u \in h_p(\mathbb{B})$.

With developing of theory on the standard (weighted) Bergman space, more general spaces such as weighted Bergman spaces with exponential type weights have been extensively studied (see [2, 4, 5, 6, 8, 16]). As the first aim of this paper, we consider an analogue of (1.2) in the setting of exponentially weighted p-harmonic Bergman space $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$. The following is our result in this line.

Theorem 1.1. *Let $1 < s < \infty$, $\alpha \in \mathbb{R}$ and $\beta \geq s-1$, then*

$$\int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x) \approx |u(0)|^s + \int_{\mathbb{B}} |\nabla u(x)|^s (1-|x|)^s dv_{\alpha,\beta}(x) \tag{1.3}$$

for all $u \in h_p(\mathbb{B})$.

To state our next results, let us recall the following notion.

The weighted hyperbolic distance d_λ, due to Dall'Ara [7], is induced by the metric $\lambda(x)^{-2} dx \otimes dx$, i.e,

$$d_\lambda(x,y) = \inf_\gamma \int_0^1 \frac{|\gamma'(t)|}{\lambda(\gamma(t))} dt, \quad x, y \in \mathbb{B},$$

where $\lambda(x) = (1-|x|^2)^2$ and $\gamma : [0,1] \to \mathbb{B}$ is a parametrization of a piecewise C^1 curve with $\gamma(0) = x$ and $\gamma(1) = y$. By [7], it was shown that $d_\lambda(x,y) \approx \frac{|x-y|}{[x,y]^2}$ when x, y are close sufficiently in $\mathbb{B}$, see Section 4 in [7] for details.

As an application of Theorem 1.1, we obtain a Lipschitz type characterization for exponentially weighted p-harmonic Bergman space $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$.

Theorem 1.2. *Let $1 < s < \infty$, $\alpha \in \mathbb{R}$, $\beta \geq 2s-1$ and $u \in h_p(\mathbb{B})$. Then the following statements are equivalent:*

(a) $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$;

(b) There exists a positive continuous function $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ such that

$$|u(x) - u(y)| \leq \frac{|x-y|}{[x,y]^2} \big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$;

(c) There exists a positive continuous function $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ such that

$$|u(x) - u(y)| \leq d_\lambda(x,y) \big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$;

(d) *There exists a positive continuous function* $h \in L^s(\mathbb{B}, dv_{\alpha+2s,\beta})$ *such that*

$$|u(x) - u(y)| \le |x - y|\big(h(x) + h(y)\big)$$

for all $x, y \in \mathbb{B}$.

Remark 1.1. Theorem 1.2 is a generalization of [5, Theorem A] to the setting of p-harmonic functions.

In recent years a special class of Möbius invariant function spaces in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, the so-called holomorphic $\mathbb{Q}_k$ space, has attracted much attention. See [23, 24] for a summary of recent research about $\mathbb{Q}_k$ spaces in the unit disk $\mathbb{D}$. Recall that for $0 < k < \infty$, a holomorphic function f is said to belong to the $\mathbb{Q}_k$ space if

$$\|f\|_{\mathbb{Q}_k} = \sup_{a\in\mathbb{D}} \int_{\mathbb{B}} |f'(z)|^2(1 - |\varphi_a(z)|^2)^k dA(z) < \infty.$$

It is well-known that $\mathbb{Q}_k = \mathcal{B}$, the holomorphic Bloch space if $k > 1$ and $\mathbb{Q}_k =$ BMOA if $k = 1$.

In our final results, we focus on the borderline case $p = n$. It is known that n-harmonic functions are Möbius invariant, and thus we are able to generalize some properties of holomorphic $\mathbb{Q}_k$ spaces to the n-harmonic setting.

Definition 1.2. For $0 < k < \infty$, the Q_k space consists of all $u \in h_n(\mathbb{B})$ such that

$$\|u\|_{Q_k} = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\nabla u(x)|^n(1 - |\varphi_a(x)|^2)^k dv(x) < \infty,$$

where φ_a is the Möbius transformation on the real unit ball $\mathbb{B}$ that interchanges the points 0 and a (see the definition in Section 2).

In [13], Latvala characterized n-harmonic Q_k and $BMO(\mathbb{B})$ spaces by means of certain Möbius invariant weighted Dirichlet integrals. Motivated by the results in [13, 22], we show a derivative-free characterization of Q_k as follows.

Theorem 1.3. *Let* $0 < k < n$ *and* $u \in h_p(\mathbb{B})$. *Then* $u \in Q_k$ *if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|u(x) - u(y)|^n}{[x, y]^{2n}}(1 - |\varphi_a(x)|^2)^k dv(x)dv(y) < \infty.$$

For $0 < r < 1$ and $u \in h_n(\mathbb{B})$, we define the oscillation of u at x in the pesudo-hyperbolic metric as $o_r(u)(x)$ which is given by

$$o_r(u)(x) = \sup_{y\in E(x,r)} |u(x) - u(y)|.$$

Similarly, define another oscillation of u at x as

$$\widehat{o}_r(u)(x) = \sup_{y\in E(x,r)} |\widehat{u}_r(x) - u(y)|,$$

where

$$\widehat{u}_r(x) = \frac{1}{|E(x,r)|}\int_{E(x,r)} u(y)dv(y).$$

Theorem 1.4. *Let $0 < r < 1$ and $u \in h_n(\mathbb{B})$. Then the following statements are equivalent:*

(a) $\quad u \in Q_k$;

(b) $\quad \sup\limits_{a\in\mathbb{B}} \int_{\mathbb{B}} |o_r(u)(x)|^n (1-|\varphi_a(x)|^2)^k d\tau(x) < \infty,$

(c) $\quad \sup\limits_{a\in\mathbb{B}} \int_{\mathbb{B}} |\widehat{o}_r(u)(x)|^n (1-|\varphi_a(x)|^2)^k d\tau(x) < \infty,$

where $d\tau(x) = (1-|x|^2)^{-n}dv(x)$ is the invariant measure on $\mathbb{B}$.

The rest of this paper is organized as follows. In Section 2, some necessary terminology and notation will be introduced. In Section 3, we shall prove Theorem 1.1. The proof of Theorem 1.2 will be presented in Section 4 by applying Theorem 1.1. The final Section 5 is devoted to the proofs of Theorems 1.3 and 1.4. Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. For nonnegative quantities X and Y, $X \lesssim Y$ means that X is dominated by Y times some inessential positive constant. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$.

2. Preliminaries

In this section, we introduce notation and collect some preliminary results that involve Möbius transformations and p-harmonic functions.

Let $a \in \mathbb{R}^n$, we write a in polar coordinate by $a = |a|a'$. For $a, b \in \mathbb{R}^n$, let

$$[a,b] = \Big||a|b - a'\Big|.$$

The symmetric lemma shows

$$[a,b] = [b,a].$$

For any $a \in \mathbb{B}$, denote by φ_a the Möbius transformation in $\mathbb{B}$. It's an involution of $\mathbb{B}$ such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$, which is of the form

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{[x,a]^2}, x \in \mathbb{B}.$$

An elementary computation gives

$$|\varphi_a(x)| = \frac{|x-a|}{[x,a]}.$$

In terms of φ_a, the *pseudo-hyperbolic metric* ρ is given by

$$\rho(a,b) = |\varphi_a(b)|, \quad a,b \in \mathbb{B}.$$

The *pseudo-hyperbolic ball* with center a and radius r is denoted by

$$E(a,r) = \{x \in \mathbb{B} : \rho(a,x) < r\}.$$

However, $E(a,r)$ is also a Euclidean ball with center c_a and radius r_a given by

$$c_a = \frac{(1-r^2)a}{1-|a|^2r^2} \quad \text{and} \quad r_a = \frac{r(1-|a|^2)}{1-|a|^2r^2}, \tag{2.1}$$

respectively (cf. [1, 17]).

Following [5], we define a positive value function ϱ in $\mathbb{B}$ as

$$\varrho(a,b) = \frac{|a-b|}{[a,b]^2}, \quad a,b \in \mathbb{B}.$$

The ball $B_r(a)$ associated with ϱ is given by

$$B_r(a) = \{x \in \mathbb{B} : \varrho(a,x) < r\}.$$

Obviously, one see that $\varrho(a,b) < r$ implies $\rho(a,b) < 2r$ for a small positive r.

Lemma 2.1. *Let r be a small positive number and $x \in B_r(a)$ (resp. $E(a,r)$). Then*

$$1-|x|^2 \approx 1-|a|^2 \approx [a,x], \qquad d_\lambda(a,x) \approx \varrho(a,x)$$

and

$$|B_r(a)| \approx (1-|a|^2)^{2n}, \qquad (resp.\ |E(a,r)| \approx (1-|a|^2)^n)$$

where $|B_r(a)|$ and $|E(a,r)|$ denote the Euclidean volume of $B_r(a)$ and $E(a,r)$, respectively.

Proof. It is obvious from [17, Lemma 2.1].

By Lemma 2.1, the following comparable results can be easily derived.

Lemma 2.2. *For a small $r > 0$, there exist two positive constants r_1, r_2 such that*

$$\mathbb{B}(a, r_1(1-|a|^2)^2) \subseteq B_r(a) \subseteq \mathbb{B}(a, r_2(1-|a|^2)^2), \quad a \in \mathbb{B}.$$

Let $u \in h_p(\mathbb{B})$, for convenience, we denote

$$\fint_{\mathbb{B}(x,r)} u(y)dv(y) = \frac{1}{|\mathbb{B}(x,r)|}\int_{\mathbb{B}(x,r)} u(y)dv(y).$$

We end this section with some useful inequalities concerning p-harmonic functions which are crucial for our investigations (cf. [11]).

Lemma 2.3. *Assume that $u \in h_p(\mathbb{B})$. Then we have the following inequalities.*
(1) *For each $\delta > 1$, there is a positive constant C such that*

$$\int_{\mathbb{B}(x,r)} |\nabla u(y)|^p dv(y) \le \frac{C}{r^p} \int_{\mathbb{B}(x,\delta r)} |u(y)|^p dv(y),$$

whenever $\mathbb{B}(x, \delta r) \subset \mathbb{B}$.
(2) *For each $\delta > 1$ and $0 < s \le t$, there is a positive constant C such that*

$$|u(x)| \le C\Big(⨍_{\mathbb{B}(x,r)} |u(y)|^t dv(y)\Big)^{\frac{1}{t}} \le C\Big(⨍_{\mathbb{B}(x,\delta r)} |u(y)|^s dv(y)\Big)^{\frac{1}{s}},$$

whenever $\mathbb{B}(x, \delta r) \subset \mathbb{B}$.
(3) *For each $\delta > 1$ and $0 < s \le t$, there is a positive constant C such that*

$$|\nabla u(x)| \le C\Big(⨍_{\mathbb{B}(x,r)} |\nabla u(y)|^t dv(y)\Big)^{\frac{1}{t}} \le C\Big(⨍_{\mathbb{B}(x,\delta r)} |\nabla u(y)|^s dv(y)\Big)^{\frac{1}{s}},$$

whenever $\mathbb{B}(x, \delta r) \subset \mathbb{B}$.
(4) *For each $t > 0$ and $\delta > 1$, there is a positive constant C such that*

$$osc_{x\in\mathbb{B}(y,r)} u(x) \le C\Big(⨍_{\mathbb{B}(y,\delta r)} |\nabla u(y)|^t dv(y)\Big)^{\frac{1}{t}},$$

whenever $\mathbb{B}(y, \delta r) \subset \mathbb{B}$.

3. Proof of Theorem 1.1

Proposition 3.1. *Let $1 < s < \infty$, $\alpha \in \mathbb{R}$ and $\beta > 0$, then*

$$|u(0)|^s + \int_{\mathbb{B}} (1-|x|)^s |\nabla u(x)|^s dv_{\alpha,\beta}(x) \lesssim \int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x) \tag{3.1}$$

for all $u \in h_p(\mathbb{B})$.

Proof. By Lemma 2.3, we have

$$|u(0)| \le C\Big(\int_{\mathbb{B}_{\frac{1}{2}}} |u(x)|^s dv_{\alpha,\beta}(x)\Big)^{\frac{1}{s}} \lesssim \Big(\int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x)\Big)^{\frac{1}{s}}.$$

Hence it is sufficient to prove without the term $|u(0)|^s$. It follows from Lemma 2.3 again that for each fixed $x \in \mathbb{B}$,

$$\begin{aligned} |\nabla u(x)| &\le C\Big(⨍_{\mathbb{B}(x,\frac{(1-|x|)}{4})} |\nabla u(y)|^p dv(y)\Big)^{\frac{1}{p}} \\ &\lesssim \Big((1-|x|)^{-p} ⨍_{\mathbb{B}(x,\frac{(1-|x|)}{3})} |u(y)|^p v(y)\Big)^{\frac{1}{p}} \\ &\lesssim (1-|x|)^{-1} \Big(⨍_{\mathbb{B}(x,\frac{(1-|x|)}{2})} |u(y)|^s v(y)\Big)^{\frac{1}{s}}. \end{aligned}$$

Combing this with Lemma 2.1 and Fubini's theorem, we conclude that

$$\begin{aligned}\int_{\mathbb{B}} |\nabla u(x)|^s (1-|x|)^s dv_{\alpha,\beta}(x) &\lesssim \int_{\mathbb{B}} \fint_{\mathbb{B}(x,\frac{(1-|x|)}{2})} |u(y)|^s dv(y) dv_{\alpha,\beta}(x)\\ &\lesssim \int_{\mathbb{B}} \fint_{\mathbb{B}(x,\frac{(1-|x|)}{2})} |u(y)|^s dv_{\alpha,\beta}(y) dv(x)\\ &\lesssim \int_{\mathbb{B}} |u(y)|^s \fint_{\mathbb{B}(y,\frac{(1-|y|)}{2})} dv(x) dv_{\alpha,\beta}(y)\\ &\lesssim \int_{\mathbb{B}} |u(y)|^s dv_{\alpha,\beta}(y).\end{aligned}$$

This proves the result.

Proposition 3.2. *Let $1 < s < \infty$, $\alpha \in \mathbb{R}$ and $\beta \geq s-1$, then*

$$\int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x) \lesssim |u(0)|^s + \int_{\mathbb{B}} |\nabla u(x)|^s (1-|x|)^s dv_{\alpha,\beta}(x) \tag{3.2}$$

for all $u \in h_p(\mathbb{B})$.

Proof. Assume that $u(0) = 0$. We divide the integral on the left-hand side of (3.2) into two parts:

$$\int_{\mathbb{B}} |u(x)|^s dv_{\alpha,\beta}(x) = \int_{\mathbb{B}_{\frac{1}{3}}} + \int_{\mathbb{B}\setminus\mathbb{B}_{\frac{1}{3}}}.$$

It is easy to see that the integral over $\mathbb{B}_{\frac{1}{3}}$ is dominated by

$$\begin{aligned}\int_{\mathbb{B}_{\frac{1}{3}}} |u(x)|^s dv_{\alpha,\beta}(x) &\lesssim \left(osc_{x\in\mathbb{B}_{\frac{1}{3}}} u(x)\right)^s\\ &\lesssim \int_{\mathbb{B}_{\frac{1}{2}}} |\nabla u(x)|^s (1-|x|)^s dv_{\alpha,\beta}(x)\\ &\lesssim \int_{\mathbb{B}} |\nabla u(x)|^s (1-|x|)^s dv_{\alpha,\beta}(x).\end{aligned}$$

We now estimate the integral over $\mathbb{B}\setminus\mathbb{B}_{\frac{1}{3}}$. Since u is $C^1(\mathbb{B})$, for $\zeta \in \mathbb{S}$, we have

$$|u(r\zeta) - u(\frac{1}{3}\zeta)| \lesssim C\int_{\frac{1}{3}}^{r} |\nabla u(t\zeta)| dt.$$

Thus

$$\begin{aligned}\int_{\mathbb{B}\setminus\mathbb{B}_{\frac{1}{3}}} |u(x)|^s dv_{\alpha,\beta}(x) &= \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1} nr^{n-1} |u(r\zeta)|^s \omega_{\alpha,\beta}(r) dr d\sigma(\zeta)\\ &\lesssim \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1} r^{n-1}\Big(|u(r\zeta) - u(\frac{1}{3}\zeta)|^s + |u(\frac{1}{3}\zeta)|^s\Big)\omega_{\alpha,\beta}(r) dr d\sigma(\zeta).\end{aligned}$$

Note that the integral

$$\int_{\mathbb{S}}\int_{\frac{1}{3}}^{1} r^{n-1} |u(\frac{1}{3}\zeta)|^s \omega_{\alpha,\beta}(r) dr d\sigma(\zeta) \lesssim \int_{\mathbb{B}} |\nabla u(x)|^s (1-|x|)^s dv_{\alpha,\beta}(x)$$

by the same reasoning as the above integral estimate over $\mathbb{B}_{\frac{1}{3}}$. It follows from Lemma 2.3 and Hölder's inequality that

$$\begin{aligned} I &= \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1} r^{n-1}|u(r\zeta)-u(\frac{1}{3}\zeta)|^{s}\omega_{\alpha,\beta}(r)drd\sigma(\zeta) \\ &= \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1} r^{n-1}\Big(\int_{\frac{1}{3}}^{r}|\nabla u(t\zeta)|dt\Big)^{s}\omega_{\alpha,\beta}(r)drd\sigma(\zeta) \\ &\lesssim \int_{\mathbb{S}}\int_{\frac{1}{3}}^{1}\Big(\int_{0}^{r} t^{(n-1)/s}|\nabla u(t\zeta)|dt\Big)^{s}\omega_{\alpha,\beta}(r)drd\sigma(\zeta) \\ &\lesssim \int_{\mathbb{S}}\int_{0}^{1}\int_{0}^{r} t^{n-1}|\nabla u(t\zeta)|^{s}dt\omega_{\alpha,\beta}(r)drd\sigma(\zeta) \\ &\lesssim \int_{\mathbb{S}}\int_{0}^{1} t^{n-1}|\nabla u(t\zeta)|^{s}dt\int_{t}^{r}\omega_{\alpha,\beta}(r)drd\sigma(\zeta). \end{aligned}$$

Observe that

$$\int_{s}^{1}\omega_{\alpha,\beta}(r)dr \lesssim (1-s)^{\beta+1}\omega_{\alpha,\beta}(s), \quad 0<s<1$$

from [18, Example 3.2], we obtain

$$\begin{aligned} I &\lesssim \int_{\mathbb{S}}\int_{0}^{1} t^{n-1}|\nabla u(t\zeta)|^{s}\omega_{\alpha,\beta}(t)(1-|t|)^{s}dt(r)d\sigma(\zeta) \\ &\lesssim \int_{\mathbb{B}}|\nabla u(x)|^{s}(1-|x|)^{s}dv_{\alpha,\beta}(x) \end{aligned}$$

from the assumption $\beta \geq s-1$.

To remove the restriction $u(0)=0$, let $u(x)=u(0)+u_1(x)$ with $\nabla u=\nabla u_1$ and $u_1(0)=0$. Therefore,

$$\begin{aligned} \int_{\mathbb{B}}|u(x)|^{s}dv_{\alpha,\beta}(x) &= \int_{\mathbb{B}}|u(0)+u_1(x)|^{s}dv_{\alpha,\beta}(x) \\ &\lesssim |u(0)|^{s}+\int_{\mathbb{B}}|u_1(x)|^{s}dv_{\alpha,\beta}(x) \\ &\lesssim |u(0)|^{s}+\int_{\mathbb{B}}(1-|x|)^{s}|\nabla u(x)|^{s}dv_{\alpha,\beta}(x) \end{aligned}$$

as desired. □

Proof of Theorem 1.1. Gathering Propositions 3.1 and 3.2, the assertion (1.3) follows. By a slight modification on the proof of Proposition 3.2, we can also obtain the following corollary which can view as an extension of [5, Proposition 2.10] into p-harmonic setting.

Corollary 3.1. *Let $1<s<\infty$, $\alpha\in\mathbb{R}$ and $\beta\geq 2s-1$, then*

$$\int_{\mathbb{B}}|u(x)|^{s}dv_{\alpha,\beta}(x)\approx|u(0)|^{s}+\int_{\mathbb{B}}|\nabla u(x)|^{s}(1-|x|)^{2s}dv_{\alpha,\beta}(x) \tag{3.3}$$

for all $u\in h_p(\mathbb{B})$.

4. Lipschitz type characterizations for $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$

In this section, we discuss Lipschitz type characterizations of the space $\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$ by applying Corollary 3.1.

Proof of Theorem 1.2. We first prove $(b) \Rightarrow (a)$. Assume that (b) holds. Then for each fixed x and all y sufficiently close to x

$$\Big|\frac{u(x)-u(y)}{x-y}\Big| \leq \frac{1}{[x,y]^2}\big(g(x)+g(y)\big), \quad x \neq y.$$

By letting y approach x in the direction of each real coordinate axis, we see that

$$(1-|x|)^2|\nabla u(x)| \leq Cg(x)$$

for all $x \in \mathbb{B}$. It follows from the assumption $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ that

$$\int_{\mathbb{B}} (1-|x|)^{2s}|\nabla u(x)|^s dv_{\alpha,\beta}(x) < \infty.$$

Thus $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$ by Corollary 3.1.

For the converse, we assume $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$. Fix a small $r > 0$ and consider any two points $x, y \in \mathbb{B}$ with $\varrho(x,y) < r$. By Lemma 2.1, it is given that

$$\begin{aligned}
|u(x)-u(y)| &= \Big|\int_0^1 \frac{du}{dt}(ty+(1-t)x)dt\Big| \\
&\leq C|x-y|\int_0^1 |\nabla u(ty+(1-t)x)|dt \\
&\leq C\varrho(x,y)\sup\{(1-|\zeta|)^2|\nabla u(\zeta)| : \zeta \in B_r(x)\} \\
&\leq \varrho(x,y)h(x),
\end{aligned}$$

where

$$h(x) = C(r)\sup\{(1-|\zeta|)^2|\nabla u(\zeta)| : \zeta \in B_r(x)\}.$$

If $\varrho(x,y) \geq r$, the triangle inequality implies

$$\begin{aligned}
|u(x)-u(y)| &\leq |u(x)|+|u(y)| \\
&\leq \varrho(x,y)\Big(\frac{|u(x)|}{r}+\frac{|u(y)|}{r}\Big).
\end{aligned}$$

Letting $g(x) = h(x) + \frac{|u(x)|}{r}$, then

$$|u(x)-u(y)| \leq \varrho(x,y)\big(g(x)+g(y)\big)$$

for all $x, y \in \mathbb{B}$. Note that $g(x) = h(x) + \frac{|u(x)|}{r}$ is the desired function provided that $h \in L^s(\mathbb{B}, dv_{\alpha,\beta})$.

Since r is a small positive number, by Lemma 2.2, we see that $B_r(\zeta) \subset \mathbb{B}(x, \frac{(1-|x|)^2}{4})$ for every $\zeta \in B_r(x)$. It follows from Lemma 2.3 that

$$\begin{aligned}\sup_{\zeta \in B_r(x)} |\nabla u(\zeta)| &\leq C\Big(⨍_{\mathbb{B}(x, \frac{(1-|x|)^2}{4})} |\nabla u(y)|^p dv(y)\Big)^{\frac{1}{p}} \\ &\lesssim \Big((1-|x|)^{-2p} ⨍_{\mathbb{B}(x, \frac{(1-|x|)^2}{3})} |u(y)|^p v(y)\Big)^{\frac{1}{p}} \\ &\lesssim (1-|x|)^{-2}\Big(⨍_{\mathbb{B}(x, \frac{(1-|x|)^2}{2})} |u(y)|^s v(y)\Big)^{\frac{1}{s}}.\end{aligned}$$

Hence by Fubini's theorem and Lemma 2.1,

$$\begin{aligned}\|h\|^s_{\mathcal{A}^s_{\alpha,\beta}} &\lesssim \int_{\mathbb{B}} (1-|x|)^{-2n} \omega_{\alpha,\beta}(x) \int_{\mathbb{B}(x, \frac{(1-|x|)^2}{2})} |u(y)|^s dv(y) dv(x) \\ &\lesssim \int_{\mathbb{B}} |u(y)|^s \omega_{\alpha,\beta}(y) \int_{\mathbb{B}(y, \frac{(1-|y|)^2}{2})} (1-|x|)^{-2n} dv(y) dv(x) \lesssim \|u\|^s_{\mathcal{A}^s_{\alpha,\beta}},\end{aligned}$$

which implies $h \in L^s(\mathbb{B}, dv_{\alpha,\beta})$. This proves $(a) \Leftrightarrow (b)$.

$(a) \Leftrightarrow (c)$. It follows from Lemmas 2.1, 2.2 and a discussion similar to the above, the assertion follows.

$(a) \Leftrightarrow (d)$. Assume that (d) holds. Then it can be deduced that

$$(1-|x|)^2 |\nabla u(x)| \leq C(1-|x|)^2 h(x)$$

for all $x \in \mathbb{B}$. The assumption $h \in L^s(\mathbb{B}, dv_{\alpha+2s,\beta})$ implies $(1-|x|)|^2 \nabla u(x)| \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ and thus, according to Corollary 3.1, means that $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$.

Conversely, suppose that $u \in \mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$. Then (b) implies that there exists a positive continuous function $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$ such that

$$|u(x) - u(y)| \leq C \frac{|x-y|}{[x,y]^2} \big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$. Since for $x, y \in \mathbb{B}$,

$$[x,y] \geq 1 - |x|, \quad [x,y] \geq 1 - |y|,$$

we see that

$$\begin{aligned}|u(x) - u(y)| &\leq C|x-y| \Big(\frac{g(x)}{(1-|x|)^2} + \frac{g(y)}{(1-|y|)^2}\Big) \\ &\leq |x-y| \big(h(x) + h(y)\big), \quad x, y \in \mathbb{B},\end{aligned}$$

where

$$h(x) = \frac{Cg(x)}{(1-|x|)^2}.$$

Hence $h \in L^s(\mathbb{B}, dv_{\alpha+2s,\beta})$ from the assumption $g \in L^s(\mathbb{B}, dv_{\alpha,\beta})$. □

In the following, we consider a symmetric lifting operator L which is defined as

$$Lu(x,y)=\frac{u(x)-u(y)}{x-y},\quad x\neq y$$

where $u\in h_p(\mathbb{B})$.

As an application of Theorem 1.2, we can obtain the boundedness of operator L as follows.

Theorem 4.1. *Let $1<s<\infty$, $\alpha\in\mathbb{R},\beta\geq 2s-1$. Then $L:\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})\to L^s(\mathbb{B}\times\mathbb{B},dv_{\alpha+s,\beta}\times dv_{\alpha+s,\beta})\cap h_p(\mathbb{B}\times\mathbb{B})$ is bounded.*

Proof. Let $u\in\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})$. Then there exists a positive continuous function $g\in L^s(\mathbb{B},dv_{\alpha,\beta})$ such that

$$|Lu(x,y)|^s=\Big|\frac{u(x)-u(y)}{x-y}\Big|^s\lesssim\frac{|g(x)|^s+|g(y)|^s}{[x,y]^{2s}},\quad x\neq y,$$

by Theorem 1.2. Applying Fubini's Theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{B}}\int_{\mathbb{B}}|Lu(x,y)|^s dv_{\alpha+s,\beta}(x)dv_{\alpha+s,\beta}(y)\\
\leq\;& 2C\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|g(x)|^s}{[x,y]^{2s}}dv_{\alpha+s,\beta}(x)dv_{\alpha+s,\beta}(y)\\
\lesssim\;& \int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|g(x)|^s}{(1-|x|)^s(1-|y|)^s}dv_{\alpha+s,\beta}(x)dv_{\alpha+s,\beta}(y)\\
\lesssim\;& \int_{\mathbb{B}}|g(x)|^s dv_{\alpha,\beta}(x)<\infty.
\end{aligned}$$

Consequently, $L:\mathcal{A}^s_{\alpha,\beta}(\mathbb{B})\to L^s(\mathbb{B}\times\mathbb{B},dv_{\alpha+s,\beta}\times dv_{\alpha+s,\beta})\cap h_p(\mathbb{B}\times\mathbb{B})$ is bounded. □

5. Characterizations of Q_k spaces

In this section, we discuss some derivative-free characterizations for Q_k spaces of n-harmonic functions on the real unit ball $\mathbb{B}$ in $\mathbb{R}^n$.

Lemma 5.1. *Let $0<k<\infty$ and $u\in h_n(\mathbb{B})$. Then there exists a constant $C>0$ such that*

$$\int_{\mathbb{B}}|\nabla u(x)|^n(1-|x|^2)^k dv(x)\leq C\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|u(x)-u(y)|^n}{[x,y]^{2n}}(1-|x|^2)^k dv(x)dv(y).$$

Proof. Write

$$K=\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|u(x)-u(y)|^n}{[x,y]^{2n}}(1-|x|^2)^k dv(x)dv(y).$$

Making the change of variables $y \mapsto \varphi_x(y)$ leads to

$$\begin{aligned} K &= \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|u(x) - u\circ\varphi_x(y)|^n}{[x,\varphi_x(y)]^{2n}}(1-|x|^2)^k J\varphi_x(y)dv(x)dv(y) \\ &= \int_{\mathbb{B}}\int_{\mathbb{B}} |u\circ\varphi_x(0) - u\circ\varphi_x(y)|^n(1-|x|^2)^{k-n}dv(x)dv(y) \\ &= \int_{\mathbb{B}}(1-|x|^2)^{k-n}dv(x)\int_{\mathbb{B}}|u\circ\varphi_x(0) - u\circ\varphi_x(y)|^n dv(y). \end{aligned}$$

Note that $u\circ\varphi_x \in h_n(\mathbb{B})$, it follows from (1.2) that

$$\int_{\mathbb{B}}|u\circ\varphi_x(0) - u\circ\varphi_x(y)|^n dv(y) \approx \int_{\mathbb{B}}|\nabla(u\circ\varphi_x)(y)|^n(1-|y|^2)^n dv(y).$$

It deduces from [13, Lemma 4.4] that

$$\begin{aligned} K &\approx \int_{\mathbb{B}}(1-|x|^2)^{k-n}dv(x)\int_{\mathbb{B}}|\nabla(u\circ\varphi_x)(y)|^n(1-|y|^2)^n dv(y) \\ &\approx \int_{\mathbb{B}}(1-|x|^2)^{k-n}dv(x)\int_{\mathbb{B}}|\nabla u(y)|^n(1-|\varphi_x(y)|^2)^n dv(y) \\ &\geq C\int_{\mathbb{B}}(1-|x|^2)^{k-n}dv(x)\int_{E(x,\frac{1}{2})}|\nabla u(y)|^n(1-|\varphi_x(y)|^2)^n dv(y) \\ &\geq C\int_{\mathbb{B}}(1-|x|^2)^{k}dv(x)\fint_{E(x,\frac{1}{2})}|\nabla u(y)|^n dv(y) \\ &\geq C\int_{\mathbb{B}}|\nabla u(x)|^n(1-|x|^2)^k dv(x). \end{aligned}$$

□

Lemma 5.2. *Let $0 < k < n$ and $u \in h_n(\mathbb{B})$. Then there exists a constant $C > 0$ such that*

$$K = \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|u(x) - u(y)|^n}{[x,y]^{2n}}(1-|x|^2)^k dv(x)dv(y) \leq C\int_{\mathbb{B}}|\nabla u(x)|^n(1-|x|^2)^k dv(x).$$

Proof. From the proof of Lemma 5.1, we see that

$$K \approx \int_{\mathbb{B}}|\nabla u(y)|^n dv(y)\int_{\mathbb{B}}(1-|\varphi_x(y)|^2)^n(1-|x|^2)^{k-n}dv(x)$$

It follows from the assumption $0 < k < n$ and [17, Lemma 2.4] that

$$\begin{aligned} \int_{\mathbb{B}}(1-|\varphi_x(y)|^2)^n(1-|x|^2)^{k-n}dv(x) &= \int_{\mathbb{B}}\frac{(1-|x|^2)^k(1-|y|^2)^n}{[x,y]^{2n}}dv(x) \\ &\lesssim (1-|y|^2)^k, \end{aligned}$$

as desired. □

Proof of Theorem 1.3. By [13, Lemmas 2.3 and 4.4], we know that $u \in Q_k$ if and only if

$$\sup_{a\in\mathbb{B}}\int_{\mathbb{B}}|\nabla(u\circ\varphi_a)(x)|^n(1-|x|^2)^k dv(x) < \infty.$$

This together with Lemmas 5.1 and 5.2, the assertion follows.

Proof of Theorem 1.4. The proof will follow by the routes $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$. Let $u \in Q_k$. By Lemma 2.3, for $0 < r < 1$ and a fixed $x \in \mathbb{B}$,

$$|o_r(u)(x)|^n \quad \lesssim \quad \frac{1}{|E(x,r')|} \int_{E(x,r')} |u(x) - u(y)|^n dv(y),$$

where $r < r' < 1$. From Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \frac{1}{|E(x,r')|} & \int_{E(x,r')} |u(x) - u(y)|^n dv(y) \\ & \lesssim \int_{E(x,r')} |u(x) - u(y)|^n \frac{(1-|x|^2)^n}{[x,y]^{2n}} dv(y) \\ & = \int_{\mathbb{B}(0,r')} |u \circ \varphi_x(0) - u \circ \varphi_x(y)|^n dv(y) \\ & \lesssim \int_{\mathbb{B}(0,r')} |\nabla(u \circ \varphi_x)(y)|^n (1-|y|^2)^n dv(y). \end{aligned}$$

By making the change of variables and [13, Lemma 4.3],

$$|o_r(u)(x)|^n \quad \lesssim \quad \int_{E(x,r')} |\nabla u(y)|^n dv(y),$$

from which we see that

$$\begin{aligned} & \int_{\mathbb{B}} |o_r(u)|^n (1-|\varphi_a(x)|^2)^k d\tau(x) \\ \lesssim & \int_{\mathbb{B}} (1-|\varphi_a(x)|^2)^k d\tau(x) \int_{E(x,r')} |\nabla u(y)|^n dv(y) \\ \lesssim & \int_{\mathbb{B}} |\nabla u(x)|^n (1-|\varphi_a(x)|^2)^k dv(x), \end{aligned}$$

for each $a \in \mathbb{B}$. Hence (a) implies (b).

$(b) \Rightarrow (c)$. By Lemma 2.3, for $0 < r < 1$,

$$\begin{aligned} \sup_{y \in E(x,r)} |\widehat{u}_r(x) - u(y)| & \lesssim \sup_{y \in E(x,r)} \frac{1}{|E(x,r)|} \int_{E(x,r)} |u(y) - u(z)| dv(z) \\ & \lesssim \sup_{y \in E(x,r)} \sup_{z \in E(x,r)} |u(y) - u(z)| \\ & \lesssim \sup_{y \in E(x,r)} |u(x) - u(y)|. \end{aligned}$$

Thus

$$\widehat{o}_r(u)(x) \lesssim o_r(u)(x),$$

from which $(b) \Rightarrow (c)$ follows.

$(c) \Rightarrow (a)$. For $0 < r < 1$ and $x \in \mathbb{B}$, we have

$$\begin{aligned}(1-|x|^2)^n|\nabla u(x)|^n &\lesssim \frac{1}{|E(x,r)|}\int_{E(x,r)}|u(y)-\widehat{u}_r(x)|^n dv(y)\\ &\lesssim \Big(\sup_{y\in E(x,r)}|\widehat{u}_r(x)-u(y)|\Big)^n\end{aligned}$$

by Lemma 2.3. Consequently,

$$\sup_{a\in\mathbb{B}}\int_{\mathbb{B}}|\nabla u(x)|^n(1-|\varphi_a(x)|^2)^k dv(x) \lesssim \sup_{a\in\mathbb{B}}\int_{\mathbb{B}}|\widehat{o}_r(u)(x)|^n(1-|\varphi_a(x)|^2)^k d\tau(x).$$

The proof of this theorem is complete.

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