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ON THE SUMMABILITY OF FOURIER SERIES BY THE GENERALIZED CESÁRO METHOD

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Abstract. The analogous of Lebesgue-Gergen convergence test for generalized Cesáro means of Fourier trigonometric series is given.

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1. Notations and formulation of the Main Theorem

Let f be a 2π-periodic locally integrable function and

$$S_n(x,f)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$$

be the partial sums of the Fourier series of f with respect to the trigonometric system.

Let (α_n) be a sequence of real numbers, where $\alpha_n>-1,\ \ n=1,2,\ldots$. Suppose

$$\sigma_n^{\alpha_n}(x,f)=:\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha_n-1}S_\nu(x,f)/A_n^{\alpha_n},$$

where

$$A_k^{\alpha_n}=(\alpha_n+1)\,(\alpha_n+2)\ldots(\alpha_n+k)\,/k!.$$

These means (generalized Cesáro (C,α_n) means) were introduced by Kaplan [7]. The author compared the methods of summability (C,α_n) and (C,α) for number series, and obtained necessary and sufficient conditions, in terms of the α_n, for the inclusion $(C,\alpha_n)\subset(C,\alpha)$, and sufficient conditions for $(C,\alpha)\subset(C,\alpha_n)$. Later Akhobadze ([1]-[5]) and Tetunashvili [10]-[15] investigated problems of (C,α_n) summability of trigonometric Fourier series.

If (α_n) is a constant sequence ($\alpha_n=\alpha,\ n=1,2,\ldots$) then $\sigma_n^{\alpha_n}(x,f)$ coincides with the usual Cesáro $\sigma_n^{\alpha}(x,f)-$means [18, Ch. III].

One of the most general test of convergence of Fourier series at a point was given by Lebesgue [8].

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Theorem 1.1 (Lebesgue). *Let f be 2π-periodic locally integrable function ($f \in L([0, 2\pi])$) and at a point x the following conditions are fulfilled:*

$$h^{-1} \int_0^h |\varphi(x,t)|\, dt = o(1) \tag{1.1}$$

and

$$\int_h^\pi t^{-1} |\varphi(x,t) - \varphi(x,t+h)|\, dt = o(1), \;\; h \to +0, \tag{1.2}$$

where

$$\varphi(x,t) = f(x+t) + f(x-t) - 2f(x). \tag{1.3}$$

Then the trigonometric Fourier series convergence at the point x.

In 1930 Gergen [6] improved the last Lebesgue statement. In particular, he proved

Theorem 1.2 (Gergen). *Let*

$$\Phi(x,t) = \int_0^t \varphi(x,u) du.$$

If $f \in L([0, 2\pi])$ and at a point x relations (1.2) *and*

$$h^{-1}\Phi(x,h) = o(1), \; h \to +0, \tag{1.4}$$

are valid, then the Fourier series of f convergence at the point x.

In 1981 Sahney and Waterman [9] proved

Theorem 1.3 (Sahney, Waterman). *Let $-1 < \alpha < 0$. Suppose that assumption* (1.1) *holds true and*

$$\int_\eta^\pi t^{-1-\alpha} |\varphi(x,t) - \varphi(x,t+\eta)|\, dt = o(n^\alpha), \;\; \eta = \pi/(n + (\alpha+1)/2) \to +0.$$

Moreover, let

$$\Phi(x,\pi) - \Phi(x, \pi - h) = o(h^{-\alpha}), \;\; h \to +0. \tag{1.5}$$

Then the trigonometric Fourier series is (C, α)-summable at x.

Long ago (in 1964) Zhizhiashvili ([16]; see, also, [17, Theorem 2.2.1]), proved more strong result then the last theorem. In particular, he showed that condition (1.5) is not necessary.

Theorem 1.4 (Zhizhiashvili). *Suppose* $-1 < \alpha < 1$. *Then under assumptions* (1.4) *and*

$$h^{\alpha}\int_{h}^{\pi} t^{-1-\alpha}\left|\varphi(x,t)-\varphi(x,t+h)\right|dt = o(1),\ h \to +0, \tag{1.6}$$

the Fourier series of f *is* (C,α)*-summable at point* x.

The object of this paper is to generalize the above result for (C,α_n)-summability method.

Theorem 1.5. *Let* $-1 < \alpha_n < 1,\ n = 1,2,...,$ *and*

$$\overline{\Phi}(x,t) = \sup_{0\le u\le t}\left|\Phi(x,u)\right|.$$

Suppose that

$$\frac{1}{(1+\alpha_n)n}\int_{\frac{\pi}{n}}^{\pi}\frac{\overline{\Phi}(x,t)}{t^3}dt = o(1), \tag{1.7}$$

$$\frac{1}{(1+\alpha_n)n^{\alpha_n}}\sup_{0<h\le\frac{\pi}{n}}\int_{\frac{\pi}{n}}^{\pi} t^{-1-\alpha_n}\left|\varphi(x,t)-\varphi(x,t+h)\right|dt = o(1),\ n\to\infty, \tag{1.8}$$

hold true. Then the trigonometric Fourier series is (C,α_n) *-summable at* x. *Summability is uniform over any closed interval inside interval of continuity where* (1.7) *and* (1.8) *are satisfied uniformly.*

Using the last statement it is easy to prove

Corollary 1.1. *Let* $\alpha_0 \in [0,1)$ *and for all* n *natural number* $\alpha_n \in (\alpha_0,1)$. *Then for almost all* x *the trigonometric Fourier series is* $(C,-\alpha_n)$*-summable at point* x.

Corollary 1.2. *Theorem* 1.4 *in the case* $-1 < \alpha \le 0$ *is a consequence of Theorem* 1.5.

2. Auxiliary statements

Let $K_n^{\alpha_n}(t)$ be the kernel of the (C,α_n)-summability method.

Lemma 2.1. [3, Lemma 2] *For every natural* n *and* $\alpha_n \in (-1,1)$

$$\left|K_n^{\alpha_n}(x)\right| \le \frac{n}{1+\alpha_n}+\frac{1}{2}. \tag{2.1}$$

Lemma 2.2. *If* k,n *and* i *are natural numbers then*

$$C_1(i)(i+\alpha_n)(i+1+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < C_2(i)(i+\alpha_n)(i+1+\alpha_n)k^{\alpha_n},$$

$\alpha_n \in (-i-1, -i)$.[2]

This lemma actually was proved in [3, Lemma 2].

Lemma 2.3. *For every natural n and $\alpha_n \in (-1,1)$*

$$\left|(K_n^{\alpha_n}(x))'\right| \le \frac{4n^2}{1+\alpha_n}.$$

Proof. The proof of this lemma is a simple consequence of Jackson's well-known inequality (see, e.g., [18, Ch. III, Lemma (13.16)]) and Lemma 2.1. □

Using representation (1.12) (see [19, Ch. XI]) for sequence (α_n) we get

$$K_n^{\alpha_n}(t) = \varphi_n^{\alpha_n}(t) + r_n^{\alpha_n}(t),$$

where

$$\varphi_n^{\alpha_n}(t) = \frac{\sin[(n+1/2+\alpha_n/2)t - \alpha_n\pi/2]}{A_n^{\alpha_n}(2\sin(t/2))^{1+\alpha_n}}$$

and

$$r_n^{\alpha_n}(t) = -Im\left\{\frac{e^{-i\frac{t}{2}}}{2A_n^{\alpha_n}\sin\frac{t}{2}}\sum_{j=1}^{3}\frac{A_n^{\alpha_n-j}}{(1-e^{-it})^j} + \frac{e^{i(n+1/2)t}}{2A_n^{\alpha_n}\sin\frac{t}{2}}\frac{\sum\limits_{\nu=n+1}^{\infty}A_\nu^{\alpha_n-4}e^{-i\nu t}}{(1-e^{-it})^3}\right\} =$$

$$(2.2)\qquad -\frac{1}{A_n^{\alpha_n}}Im\left\{\sum_{j=1}^{3}i^{-j}\left(2\sin\frac{t}{2}\right)^{-j-1}e^{i(j-1)t/2}A_n^{\alpha_n-j} + i\sum_{\nu=n+1}^{\infty}A_\nu^{\alpha_n-4}\left(2\sin\frac{t}{2}\right)^{-4}e^{i(n+2-\nu)t}\right\}.$$

Lemma 2.4. *For every natural n and $\alpha_n \in (-1,1)$*

$$\left|[r_n^{\alpha_n}(t)]'\right| \le \frac{C}{(1+\alpha_n)\,nt^3}.$$

Proof. Using representation (2.2) we get

$$(2.3)\quad [r_n^{\alpha_n}(t)]' = \frac{1}{A_n^{\alpha_n}}Im\left\{\sum_{j=1}^{3}i^{-j}\left(2\sin\frac{t}{2}\right)^{-j-2}(j+1)\cos\frac{t}{2}e^{i(j-1)t/2}A_n^{\alpha_n-j} - \frac{1}{2}\sum_{j=1}^{3}i^{-j+1}\left(2\sin\frac{t}{2}\right)^{-j-1}(j-1)e^{i(j-1)t/2}A_n^{\alpha_n-j} - 4i\sum_{j=n+1}^{\infty}A_\nu^{\alpha_n-4}\left(2\sin\frac{t}{2}\right)^{-5}\cos\frac{t}{2}e^{i(n+2-\nu)t} - \sum_{\nu=n+1}^{\infty}A_\nu^{\alpha_n-4}\left(2\sin\frac{t}{2}\right)^{-4}(n+2-\nu)e^{i(n+2-\nu)t}\right\} =: \sum_{k=1}^{4}N_k.$$

[2] In what follows by $C_1(i), C_2(i), C, ...$ we denote positive constants, respectively, absolute or dependent on parameters and indices which are, in general different in different formulas.

By Lemma 2.2 it is easy verify that for $t \in [\pi/n, \pi]$

$$N_1 = O\left(\frac{1}{1+\alpha_n}\sum_{j=1}^{3}(nt)^{-j}t^{-2}\right) = O\left(\frac{1}{(1+\alpha_n)\,nt^3}\right), \tag{2.4}$$

(2.5)

$$N_2 = O\left\{\sum_{j=1}^{3}\frac{1}{t^{1+j}}\frac{A_n^{\alpha_n-j}}{A_n^{\alpha_n}}\right\} = O\left\{\frac{1}{1+\alpha_n}\sum_{j=1}^{3}\frac{1}{(nt)^j\,t}\right\} = O\left(\frac{1}{(1+\alpha_n)\,nt^3}\right),$$

(2.6)

$$N_3 = O\left\{\frac{1}{A_n^{\alpha_n}}\sum_{\nu=n+1}^{\infty}\frac{\nu^{\alpha_n-4}}{t^5}\right\} = O\left\{\frac{1}{A_n^{\alpha_n}t^5}(n+1)^{\alpha_n-3}\right\} = O\left(\frac{1}{(1+\alpha_n)\,nt^3}\right).$$

Furthermore, it is easy to see that

$$N_4 = O\left\{\frac{1}{A_n^{\alpha_n}}\sum_{\nu=n+1}^{\infty}\frac{\nu^{\alpha_n-4}|n+2-\nu|}{t^4}\right\} = \tag{2.7}$$

$$O\left\{\frac{1}{(1+\alpha_n)(nt)^4} + \frac{n^{\alpha_n-4}}{A_n^{\alpha_n}t^4}\sum_{\nu=n+3}^{2n}(\nu-n-2) + \frac{1}{A_n^{\alpha_n}t^4}\sum_{\nu=2n+1}^{\infty}\nu^{\alpha_n-3}\right\} =$$

$$O\left\{\frac{1}{(1+\alpha_n)(nt)^4} + \frac{n^{\alpha_n-2}}{A_n^{\alpha_n}t^4} + \frac{n^{\alpha_n-2}}{A_n^{\alpha_n}t^4}\right\} =$$

$$O\left(\frac{1}{(1+\alpha_n)\,n^2t^4}\right) = O\left(\frac{1}{(1+\alpha_n)\,nt^3}\right).$$

Therefore, according to (2.3) - (2.7) the lemma follows. □

3. Proofs of the results

Proof of Theorem 1.5. Let $-1 < \alpha_n < 1, n = 1, 2, 3, \dots$. We have (see [18, Ch. III, (5.4)])

(3.1)

$$\sigma_n^{-\alpha_n}(x,f) - f(x) = \frac{1}{\pi}\int_0^{\pi/n}\varphi(x,t)K_n^{-\alpha_n}(t)dt + \frac{1}{\pi}\int_{\pi/n}^{\pi}\varphi(x,t)K_n^{-\alpha_n}(t)dt =: I_1 + I_2,$$

where $\varphi(x,t)$ is defined by (1.3). Using Lemmas 2.1 and 2.3, by the formula for integration by parts, we get

$$|I_1| = \left|\frac{1}{\pi}\left[\Phi(x,t)K_n^{-\alpha_n}(t)\right]\Big|_0^{\pi/n} - \frac{1}{\pi}\int_0^{\pi/n}\Phi(x,t)\left[K_n^{\alpha_n}(x)\right]'dt\right| \leq \tag{3.2}$$

$$\frac{1}{\pi}\left|\Phi\left(x,\frac{\pi}{n}\right)\right|\left|K_n^{-\alpha_n}\left(\frac{\pi}{n}\right)\right| + \frac{1}{\pi}\int_0^{\pi/n}\sup_{0\leq t\leq\pi/n}|\Phi(x,t)|\left|\left[K_n^{\alpha_n}(x)\right]'\right|dt \leq$$

$$\frac{1}{\pi}\overline{\Phi}\left(x,\frac{\pi}{n}\right)\left(\frac{n}{1-\alpha_n} + \frac{1}{2} + \frac{\pi}{n}\cdot\frac{4n^2}{1-\alpha_n}\right) < \frac{5n}{1-\alpha_n}\overline{\Phi}\left(x,\frac{\pi}{n}\right).$$

On the other hand by the well-known representation [18, Ch. III, (5.14)] we have

$$I_2 = \frac{1}{\pi A_n^{-\alpha_n}} \int_{\pi/n}^{\pi} \varphi(x,t) \frac{\sin\left[(n+1/2-\alpha_n/2)\,t+\alpha_n\pi/2\right]}{\left(2\sin\frac{t}{2}\right)^{1-\alpha_n}} dt + \tag{3.3}$$

$$\frac{1}{\pi} \int_{\pi/n}^{\pi} \varphi(x,t) r_n^{-\alpha_n}(t) dt =: I_2^{(1)} + I_2^{(2)},$$

where

$$r_n^{-\alpha_n}(t) = -\frac{\alpha_n}{n} \cdot \frac{\theta_n(t)}{\left(2\sin\frac{t}{2}\right)^2}, \;\; |\theta_n(t)| \le 1. \tag{3.4}$$

Besides,

$$\left|I_2^{(2)}\right| = \left| \frac{1}{\pi}\left[\Phi(x,t) r_n^{-\alpha_n}(t)\right]\Big|_{\pi/n}^{\pi} - \frac{1}{\pi}\int_{\pi/n}^{\pi} \Phi(x,t)\left[r_n^{-\alpha_n}(x)\right]' dt \right| \le$$

$$\frac{1}{\pi n}\left|\Phi\left(x,\pi\right)\right| + \frac{1}{\pi}\left|\Phi\left(x,\frac{\pi}{n}\right)\right|\left|r_n^{-\alpha_n}\left(\frac{\pi}{n}\right)\right| + \frac{C}{\pi\left(1-\alpha_n\right)n}\int_{\pi/n}^{\pi}\frac{|\Phi(x,t)|}{t^3}dt.$$

Therefore, by (3.4) and (1.7) we can conclude that for the estimation I_2 it suffices (see (3.3)) to consider

$$M =: \frac{n^{\alpha_n}}{1-\alpha_n}\int_{\pi/n}^{\pi} \varphi(x,t) g_n(t) cosntdt,$$

where

$$g_n(t) = \cos\frac{1-\alpha_n}{2}t\left(\sin\frac{t}{2}\right)^{\alpha_n-1}. \tag{3.5}$$

It may be easily verified that

$$2M = \frac{n^{\alpha_n}}{1-\alpha_n}\left\{ \int_{\pi/n}^{\pi-\pi/n} \left[\varphi(x,t)-\varphi(x,t+\pi/n)\right] g_n(t)\cos ntdt + \right. \tag{3.6}$$

$$\int_{\pi/n}^{\pi-\pi/n} \varphi(x,t+\pi/n)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt -$$

$$\left. \int_0^{\pi/n} \varphi(x,t+\pi/n) g_n(t+\pi/n)\cos ntdt + \int_{\pi-\pi/n}^{\pi} \varphi(x,t) g_n(t)\cos ntdt \right\} =: \sum_{i=1}^{4} M_i.$$

By the condition (1.8) of Theorem 1.5 we obtain

$$M_1 = o(1), \;\; n\to\infty. \tag{3.7}$$

Now from (3.6), taking into account the estimation

$$|g_n(t+\pi/n)-g_n(t)| \le \frac{C(1-\alpha_n)}{nt^{2-\alpha_n}} \tag{3.8}$$

and condition (1.8), we get

$$\left| M_2 - \frac{n^{\alpha_n}}{1-\alpha_n}\int\limits_{\pi/n}^{\pi-\pi/n}\varphi(x,t)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt\right| \le \tag{3.9}$$

$$\frac{n^{\alpha_n}}{1-\alpha_n}\int\limits_{\pi/n}^{\pi-\pi/n}|\varphi(x,t+\pi/n)-\varphi(x,t)|\,|g_n(t)-g_n(t+\pi/n)|\,dt \le$$

$$Cn^{\alpha_n-1}\int\limits_{\pi/n}^{\pi-\pi/n}|\varphi(x,t+\pi/n)-\varphi(x,t)|\,\frac{dt}{t^{2-\alpha_n}} \le$$

$$Cn^{\alpha_n}\int\limits_{\pi/n}^{\pi-\pi/n}|\varphi(x,t+\pi/n)-\varphi(x,t)|\,\frac{dt}{t^{1-\alpha_n}} = o(1),\ n\to\infty.$$

Therefore, instead of M_2 it suffices to estimate

$$M_2^\star = \frac{n^{\alpha_n}}{1-\alpha_n}\int\limits_{\pi/n}^{\pi-\pi/n}\varphi(x,t)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt.$$

Analogously to the representation of (3.6) we have

$$2M_2^\star = \frac{n^{\alpha_n}}{1-\alpha_n}\left\{\int\limits_{\pi/n}^{2\pi/n}\varphi(x,t)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt+\right. \tag{3.10}$$

$$\int\limits_{2\pi/n}^{\pi-2\pi/n}\left[\varphi(x,t)-\varphi(x,t+\pi/n)\right]\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt+$$

$$\int\limits_{2\pi/n}^{\pi-2\pi/n}\varphi(x,t+\pi/n)\left[g_n(t)-2g_n(t+\pi/n)+g_n(t+2\pi/n)\right]\cos ntdt-$$

$$\int\limits_{0}^{2\pi/n}\varphi(x,t+\pi/n)\left[g_n(t+\pi/n)-g_n(t+2\pi/n)\right]\cos ntdt+$$

$$\left.\int\limits_{\pi-2\pi/n}^{\pi-\pi/n}\varphi(x,t)\left[g_n(t)-g_n(t+\pi/n)\right]\cos ntdt\right\} =: \sum_{i=5}^{9} M_i.$$

It is easy verify that for $t\in[\pi/n,\pi]$

$$|g_n''(t)| \le \frac{C(1-\alpha_n)}{t^{3-\alpha_n}}$$

and

$$\left|g_n^{(3)}(t)\right| \le \frac{C(1-\alpha_n)}{t^{4-\alpha_n}}.$$

Hence, using Lagrange theorem repeatedly, we obtain

$$|g_n'(t) - g_n'(t+\pi/n)| \le \frac{C(1-\alpha_n)}{nt^{3-\alpha_n}}, \tag{3.11}$$

$$|g_n(t) - 2g_n(t+\pi/n) + g_n(t+2\pi/n)| \le \frac{C(1-\alpha_n)}{n^2t^{3-\alpha_n}}, \tag{3.12}$$

$$|g_n'(t) - 2g_n'(t+\pi/n) + g_n'(t+2\pi/n)| \le \frac{C(1-\alpha_n)}{n^2t^{4-\alpha_n}}. \tag{3.13}$$

Now applying formula for integration by parts and take into account piece wise monotonicity of $\cos n\tau$ on $[0, \frac{2\pi}{n}]$ it follows by (1.7), (3.8), (3.11)

$$|M_5| \le \frac{n^{\alpha_n}}{1-\alpha_n}\left\{\left|g_n\left(\frac{2\pi}{n}\right) - g_n\left(\frac{3\pi}{n}\right)\right|\left|\int\limits_0^{2\pi/n}\varphi(x,t)\cos ntdt\right| + \right. \tag{3.14}$$

$$\left|g_n\left(\frac{\pi}{n}\right) - g_n\left(\frac{2\pi}{n}\right)\right|\left|\int\limits_0^{\pi/n}\varphi(x,t)\cos ntdt\right| +$$

$$\left|\int\limits_{\pi/n}^{2\pi/n}\int\limits_0^t\varphi(x,\tau)\cos n\tau d\tau\cdot[g_n'(t) - g_n'(t+\pi/n)]\,dt\right| \le$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\cdot\frac{(1-\alpha_n)\,n^{2-\alpha_n}}{n}\overline{\Phi}\left(x,\frac{2\pi}{n}\right) +$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\cdot\frac{1-\alpha_n}{n}\int\limits_{\pi/n}^{2\pi/n}\frac{\overline{\Phi}\,(x,t)}{t^{3-\alpha_n}}dt = o(1),\ n\to\infty.$$

On the other hand, by (3.8) and (1.8) we can conclude

$$|M_6| \le \frac{Cn^{\alpha_n}}{1-\alpha_n}\cdot\frac{1-\alpha_n}{n}\int\limits_{2\pi/n}^{\pi-2\pi/n}\frac{|\varphi(x,t) - \varphi\,(x,t+2\pi/n)|}{t^{2-\alpha_n}}dt \le \tag{3.15}$$

$$Cn^{\alpha_n}\int\limits_{2\pi/n}^{\pi-2\pi/n}\frac{|\varphi(x,t) - \varphi\,(x,t+2\pi/n)|}{t^{1-\alpha_n}}dt = o(1),\ n\to\infty.$$

Let

$$b_n(t) =: g_n(t) - 2g_n\left(t+\frac{\pi}{n}\right) + g_n\left(t+\frac{2\pi}{n}\right).$$

Then

$$|M_7| = \frac{n^{\alpha_n}}{1-\alpha_n}\left[b_n\left(\pi-\frac{2\pi}{n}\right)\cos(\pi n - 2\pi)\int\limits_0^{\pi-2\pi/n}\varphi\left(x,\tau+\frac{\pi}{n}\right)d\tau - \right.$$

$$b_n\left(\frac{2\pi}{n}\right)\cos(2\pi)\int\limits_0^{2\pi/n}\varphi\left(x,\tau+\frac{\pi}{n}\right)d\tau-$$

$$\left.\int\limits_{2\pi/n}^{\pi-2\pi/n}\int\limits_0^{t}\varphi\left(x,\tau+\frac{\pi}{n}\right)d\tau\left(b_n'(t)\cos nt-nb_n(t)\sin nt\right)dt\right].$$

Now taking into account (3.12), (3.13) and (1.7), we get

(3.16)

$$|M_7|\le\frac{Cn^{\alpha_n}}{1-\alpha_n}\left[\frac{(1-\alpha_n)}{n^2\left(\pi-\frac{2\pi}{n}\right)^{3-\alpha_n}}\left|\int\limits_{\pi/n}^{\pi-\pi/n}\varphi(x,\tau)\,d\tau\right|+\frac{1-\alpha_n}{n^2n^{\alpha_n-3}}\left|\int\limits_{\pi/n}^{3\pi/n}\varphi(x,\tau)\,d\tau\right|+\right.$$

$$\left.\int\limits_{2\pi/n}^{\pi-2\pi/n}\left|\int\limits_{\pi/n}^{t+\pi/n}\varphi(x,\tau)\,d\tau\right|\cdot\left(\frac{1-\alpha_n}{n^2t^{4-\alpha_n}}+\frac{n(1-\alpha_n)}{n^2t^{3-\alpha_n}}\right)dt\right]=o(1),\ n\to\infty.$$

In the same manner we can see that

$$M_8=o(1),\ n\to\infty. \tag{3.17}$$

Besides, by (3.8) simply obtain

$$M_9=o(1),\ n\to\infty. \tag{3.18}$$

Now applying (3.10), (3.14) - (3.18) we get that

$$M_2=o(1),\ n\to\infty. \tag{3.19}$$

On the other hand, for M_3 (see (3.6)) as well easily we have

$$|M_3|\le\frac{Cn^{\alpha_n}}{1-\alpha_n}\int\limits_0^{\pi/n}\overline{\Phi}\left(x,\frac{2\pi}{n}\right)n\cdot n^{1-\alpha_n}dt=o(1),\ n\to\infty. \tag{3.20}$$

Furthermore, since $g_n(t)$ is a decreasing function, by the second mean-value theorem it follows $\left(\xi_n\in\left(\pi-\frac{\pi}{n},\pi\right)\right)$ (see (3.6))

$$|M_4|=\left|\frac{n^{\alpha_n}}{1-\alpha_n}\int\limits_{\pi-\pi/n}^{\pi}\varphi(x,t)\,g_n(t)dt\right|=\left|\frac{n^{\alpha_n}}{1-\alpha_n}g_n\left(\pi-\frac{\pi}{n}\right)\int\limits_{\pi-\pi/n}^{\xi_n}\varphi(x,t)\,dt\right|\le$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\left|\int\limits_{2\pi/n}^{\xi_n}\varphi(x,t)\,dt-\int\limits_{2\pi/n}^{\pi-\pi/n}\varphi(x,t)\,dt\right|=$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\left|\int\limits_{\pi/n+(\pi-\xi_n)}^{\pi-\pi/n}\varphi\left(x,t+\xi_n+\frac{\pi}{n}-\pi\right)dt-\int\limits_{2\pi/n}^{\pi-\pi/n}\varphi(x,t)\,dt\right|\le$$

$$\frac{Cn^{\alpha_n}}{1-\alpha_n}\left\{\int\limits_{2\pi/n}^{\pi-\pi/n}\left|\varphi\left(x,t+\xi_n+\frac{\pi}{n}-\pi\right)-\varphi\left(x,t\right)\right|dt+\left|\int\limits_{2\pi/n}^{\pi/n+\pi-\xi_n}\varphi\left(x,t\right)dt\right|\right\}.$$

Hence according to the conditions of Theorem 1.5 we obtain

$$M_4=o(1),\ \ n\to\infty. \tag{3.21}$$

Finally, on the base of (3.1), (3.2), (3.6), (3.7), (3.9), (3.10), (3.14) - (3.21) the proof of the first part of Theorem 1 is complete. It is easy see that in corresponding restrictions uniform (C,α_n)-summability of trigonometric Fourier series can be proved similarly. □

Proof of Corollary 1.2. By definition of $\overline{\Phi}(x,t)$ there exists a $t_0\in[0,t]$ such that

$$\overline{\Phi}(x,t)=\left|\int\limits_0^{t_0}\varphi(x,u)du\right|.$$

Thus

$$\frac{1}{t}\overline{\Phi}(x,t)\le\frac{1}{t_0}\left|\int\limits_0^{t_0}\varphi(x,u)du\right|.$$

Hence (1.4) implies

$$\overline{\Phi}(x,t)=o(t),\ \ t\to+0,$$

and for constant sequence α_n ($\alpha_n=\alpha\in(-1,0]$) this in turn implies (1.7).

Let $\alpha\in(-1,0]$ and $h\in(0,\pi/n]$. There exists $h_0\in(0,\pi/n]$ such that

$$\sup_{0<h\le\pi/n}\int\limits_{\pi/n}^{\pi}t^{-1-\alpha}|\varphi(x,t)-\varphi(x,t+h)|dt=$$

$$\int\limits_{\pi/n}^{\pi}t^{-1-\alpha}|\varphi(x,t)-\varphi(x,t+h_0)|dt.$$

We have

$$\left(\frac{\pi}{n}\right)^{\alpha}\sup_{0<h\le\frac{\pi}{n}}\int\limits_{\frac{\pi}{n}}^{\pi}t^{-1-\alpha}\left|\varphi(x,t)-\varphi(x,t+h)\right|dt\le$$

$$h_0^{\alpha}\int\limits_{h_0}^{\pi}t^{-1-\alpha}\left|\varphi(x,t)-\varphi(x,t+h_0)\right|dt.$$

Hence (1.6) implies (1.8). □

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EXPONENTIAL POLYNOMIALS AS SOLUTIONS OF NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. Exponential polynomials, an important subclass of finite order entire functions, as solutions of differential or difference or differential-difference equations are considered in [5, 10, 19, 20]. The critical domains of zeros and the quotients of exponential polynomials are considered in [6]. In this paper, we proceed to consider the exponential polynomials as solutions of some general complex differential-difference equations and extend existence results.

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1. Introduction

Assume that the reader is familiar with the standard notation and fundamental results of Nevanlinna theory [4, 8, 22]. A meromorphic function $f(z)$ means meromorphic in the complex plane. If a meromorphic function $f(z)$ has at least one pole, then $f(z)$ is called a properly meromorphic function. Recall the definitions of the order and the hyper-order for a meromorphic function $f(z)$ as follows

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Exponential polynomials, an important subclass of finite order entire functions with the form

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)} \tag{1.1}$$

where $P_j(z)$ and $Q_j(z)$ $(j = 1, 2, \cdots, k)$ are polynomials in z. It is easy to find that $\sigma(f) = \max\{\deg Q_j\}$ in (1.1). Exponential polynomials are the generalizations of exponential sums which implies that $\max\{\deg Q_j\} = 1$ in (1.1). Recently, the exponential polynomial solutions of complex differential or difference or differential-difference equations are considered in [5, 10, 19, 20]. The critical domains of zeros of exponential polynomials and the quotients of exponential polynomials are also

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considered in [6]. More details on value distribution of exponential sums and exponential polynomials could be seen in [1], [11]–[15].

Let

$$q = \max\{\deg(Q_j) : Q_j(z) \not\equiv 0\},$$

and let $\omega_1, \cdots, \omega_m$ be pairwise different leading coefficients of the polynomials $Q_j(z)$ with the maximum degree q. Thus, (1.1) can be written as

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}, \tag{1.2}$$

where $H_j(z)$ are either exponential polynomials of degree $< q$ or ordinary polynomials in z. To express the characteristic function of (1.2), we recall the definition of convex hull below.

We fix the notations $W = \{\overline{\omega_1}, \cdots, \overline{\omega_m}\}$, $W_0 = \{0, \overline{\omega_1}, \cdots, \overline{\omega_m}\}$. The convex hull of a set $W \subset \mathbb{C}$, denoted by $co(W)$, is the intersection of all convex sets containing W. If W contains only finitely many elements, then $co(W)$ is obtained as an intersection of finitely closed half-planes, and hence $co(W)$ is either a compact polygon (with a non-empty interior) or a line segment. We denote the perimeter of $co(W)$ by $C(co(W))$. If $co(W)$ is a line segment, then $C(co(W))$ equals to twice the length of this line segment. The following result for exponential polynomials is given by Steinmetz [14].

Theorem A. Let f be given by (1.2). Then

$$T(r, f) = C(co(W_0))\frac{r^q}{2\pi} + o(r^q). \tag{1.3}$$

Yang and Laine [21] investigated the existence of finite order entire solutions $f(z)$ of non-linear differential-difference equations of the form

$$f(z)^n + L(z, f) = h(z),$$

where $L(z, f)$ is a linear differential-difference polynomial, $n \geqslant 2$ is an integer. In particular, Yang and Laine [21] showed that the equation

$$f(z)^2 + q(z)f(z+1) = P(z), \tag{1.4}$$

has no transcendental entire solutions of finite order, where $P(z)$, $q(z)$ are polynomials. Thus, there does not exist exponential polynomial solutions on (1.4). However, if we replace $q(z)$ with $q(z)e^{Q(z)}$ in (1.4), there exist transcendental entire solutions of finite order. Wen, Heittokangas and Laine [19] studied and classified the finite order entire solutions f of non-linear difference equation

$$f(z)^n + q(z)e^{Q(z)}f(z+c) = P(z). \tag{1.5}$$

For the statement on the properties of transcendental entire solutions below, we give the following notations. Denote

$\Gamma_1 = \{e^{\alpha(z)} + d : d \in \mathbb{C}$ and $\alpha(z)$ is a non-constant polynomial$\}$,

$\Gamma_0 = \{e^{\alpha(z)}$: $\alpha(z)$ is a non-constant polynomial$\}$,

$\Gamma_1' = \{p(z)e^{\alpha(z)} + h(z)$: $p(z) \not\equiv 0$, $h(z)$ are polynomials and $\alpha(z)$ is a non-constant polynomial$\}$,

$\Gamma_0' = \{p(z)e^{\alpha(z)}$: $p(z)$ is a non-zero polynomial and $\alpha(z)$ is a non-constant polynomial$\}$.

Theorem B.[19] *Let $n \geqslant 2$ be an integer, let $c \in \mathbb{C}\backslash\{0\}$ and $q(z)$, $Q(z)$, $P(z)$ be polynomials such that $Q(z)$ is not a constant and $q(z) \not\equiv 0$. Then the finite order transcendental entire solution f of (1.5) satisfies the follows:*

(a) Every solution f satisfies $\sigma(f) = deg(Q)$ and is of mean type.

(b) Every solution f satisfies $\lambda(f) = \sigma(f)$ if and only if $P(z) \not\equiv 0$.

(c) A solution f belongs to Γ_0 if and only if $P(z) \equiv 0$. In particular, this is the case if $n \geqslant 3$.

(d) If a solution f belongs to Γ_0 and if g is any other finite order entire solution to (1.5), then $f = \eta g$, where $\eta^{n-1} = 1$.

(e) If f is an exponential polynomial solution of the form (1.1), then $f \in \Gamma_1$. Moreover, if $f \in \Gamma_1\backslash\Gamma_0$, then $\sigma(f) = 1$.

Results in the spirit of Theorem B have been obtained by Li and Yang [9] for more generalized complex difference equation of the form

$$f(z)^n + a_{n-1}f(z)^{n-1} + \cdots + a_1 f(z) + q(z)e^{Q(z)}f(z+c) = P(z), \tag{1.6}$$

where $q(z)$, $P(z)$, $Q(z)$ are polynomials, $n \geqslant 2$ is an integer and $Q(z)$ is not a constant, $q(z) \not\equiv 0$, $c \in \mathbb{C}\backslash\{0\}$ and $a_1, \cdots, a_{n-1} \in \mathbb{C}$.

Note that (1.5) and (1.6) are complex non-linear difference equations. Motivated by (1.5), Liu [10] has classified the finite order entire solutions f of non-linear differential-difference equations of the form

$$f(z)^n + q(z)e^{Q(z)}f^{(k)}(z+c) = P(z), \tag{1.7}$$

where $q(z)$, $P(z)$, $Q(z)$ are polynomials. The results in [19] regarding (1.5) concern the classes Γ_0 and Γ_1. Meanwhile, the results in [10] regarding (1.7) concern the classes Γ_0' and Γ_1'.

We have two motivations as follows and will present some results and discussions in the last two sections.

Motivation 1: Can the results regarding the solutions of (1.5), (1.6) and (1.7) be extended to differential-difference equations, where $f(z+c)$ or $f^{(k)}(z+c)$ is replaced with a differential-difference polynomial?

Motivation 2: How to classify the properly meromorphic solutions of these differential-difference equations?

For the discussions of Motivation 1, it should be very difficult for arbitrary differential-difference polynomials. In this paper, we consider a complex k-homogeneous differential-difference polynomial

$$L(z,f)=\sum_{i=1}^{m}\varphi_i(z)[f^{(\nu_{i1})}(z+c_{i1})]^{k_{i1}}\cdots[f^{(\nu_{in})}(z+c_{in})]^{k_{in}}, \tag{1.8}$$

where $k_{i1}+\cdots+k_{in}=k$, $i=1,\cdots,m$ and $\varphi_i(z)(i=1,\cdots,m)$ are polynomials. We also say $L(z,f)$ has the same shifts, if $c_{i1}=c_{i2}=\cdots=c_{in}$, $i=1,\cdots,m$. For example, $f(z+c)$, $f'(z+c)-f(z+c)$, $f^{(t)}(z+c)$ are 1-homogeneous differential-difference polynomials with the same shifts, $f(z+c)f'(z+c)+f''(z+c)f'''(z+c)$ is a 2-homogeneous differential-difference polynomial with the same shifts.

2. Lemmas

Given a meromorphic function $f(z)$, recall that $\alpha(z)\not\equiv 0,\infty$ is a small function with respect to $f(z)$, if $T(r,\alpha)=S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f)=o(T(r,f))$, and $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure. The following lemma can be seen as the differential-difference analogue of the logarithmic derivative lemma which is a combination [7, Lemma 2.2] with the lemma on the logarithmic derivative.

Lemma 2.1. *Let f be a transcendental meromorphic function with finite order $\sigma(f)$, let c,h be two complex numbers, $\varepsilon>0$. Then*

$$m\left(r,\frac{f^{(k)}(z+h)}{f(z+c)}\right)=O(r^{\sigma(f)-1+\varepsilon})+O(\log r)=S(r,f). \tag{2.1}$$

Furthermore, if $L(z,f)$ is a k-homogeneous differential-difference polynomial, then

$$m\left(r,\frac{L(z,f)}{f(z)^k}\right)=O(r^{\sigma(f)-1+\varepsilon})+O(\log r)=S(r,f). \tag{2.2}$$

Lemma 2.2. [2, 3] *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f)<\infty$, and let c be a fixed non-zero constant. Then, for each $\varepsilon>0$, we have*

$$T(r,f(z+c))=T(r,f(z))+O(r^{\sigma(f)-1+\varepsilon})+O(\log r).$$

$$N(r,f(z+c))=N(r,f(z))+O(r^{\sigma(f)-1+\varepsilon})+O(\log r).$$

Recall the following two results on the zeros of the 1-homogeneous differential-difference polynomials $f^{(k)}(z)$ and $f(z+c)$.

Lemma 2.3. [22, Theorem 1.24] *Let $f(z)$ be a transcendental meromorphic function and k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leqslant T(r, f^{(k)}) - T(r, f) + N(r, \frac{1}{f}) + S(r, f)$$

and

$$N\left(r, \frac{1}{f^{(k)}}\right) \leqslant N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.4. [9, Lemma 2.3] *Let $f(z)$ be a transcendental meromorphic function with $\sigma_2(f) < 1$, and $c \in \mathbb{C}\backslash\{0\}$. Then*

$$N(r, 1/f(z+c)) = N(r, 1/f) + S(r, f).$$

Related to the zeros of k-homogeneous differential-difference polynomials, we obtain the result below.

Lemma 2.5. *Let $f(z)$ be a transcendental meromorphic function with finite order and $L(z, f)$ be a k-homogeneous differential-difference polynomial. Then*

$$N\left(r, \frac{1}{L(z, f)}\right) \leqslant T(r, L(z, f) - T(r, f) + kN(r, \frac{1}{f}) + S(r, f),$$

and

$$N\left(r, \frac{1}{L(z, f)}\right) \leqslant 2kN(r, \frac{1}{f}) + AN(r, f) + S(r, f),$$

where A is a constant.

Proof. From Lemma 2.1, then

$$m\left(r, \frac{1}{f^k}\right) \leqslant m\left(r, \frac{L(z, f)}{f^k}\right) + m\left(r, \frac{1}{L(z, f)}\right) \leqslant m\left(r, \frac{1}{L(z, f)}\right) + S(r, f). \tag{2.3}$$

Using the first main theorem of Nevanlinna theory, we have

$$T(r, f^k) - N\left(r, \frac{1}{f^k}\right) \leqslant T(r, L(z, f)) - N\left(r, \frac{1}{L(z, f)}\right) + S(r, f).$$

Using Lemma 2.2, we obtain

$$
\begin{aligned}
N\left(r, \frac{1}{L(z,f)}\right) &\leqslant T(r, L(z,f)) - T(r, f^k) + N\left(r, \frac{1}{f^k}\right) + S(r,f)\\
&= T\left(r, \frac{L(z,f)}{f^k} f^k\right) - T(r, f^k) + N\left(r, \frac{1}{f^k}\right) + S(r,f)\\
&\leqslant T\left(r, \frac{L(z,f)}{f^k}\right) + kN\left(r, \frac{1}{f}\right) + S(r,f)\\
&\leqslant N\left(r, \frac{L(z,f)}{f^k}\right) + kN\left(r, \frac{1}{f}\right) + S(r,f)\\
&\leqslant N(r, L(z,f)) + 2kN\left(r, \frac{1}{f}\right) + S(r,f)\\
&\leqslant AN(r,f) + 2kN\left(r, \frac{1}{f}\right) + S(r,f).
\end{aligned}
$$

Remark. (1) If f is a transcendental entire function in Lemma 2.5, then

$$N\left(r, \frac{1}{L(z,f)}\right) \leqslant kN(r, \frac{1}{f}) + S(r,f),$$

using the similar reason as the above.

(2) The constant A depends on the expression of $L(z,f)$, which can be obtained by the second equality of Lemma 2.2 and the trivial inequality $N(r, f^{(k)}) \leqslant (k+1)N(r,f) + S(r,f)$. For example, $A = 3$ for $L(z,f) = f'(z+c) - f(z+c)$.

Lemma 2.6. [4] *Let f be a meromorphic function. Suppose that*

$$\Psi(z) := a_n f(z)^n + \cdots + a_0(z)$$

has small meromorphic coefficients $a_j(z), a_n \neq 0$ in the sense of $T(r, a_j) = S(r,f)$. Moreover, assume that

$$\overline{N}(r, \frac{1}{\Psi}) + \overline{N}(r,f) = S(r,f).$$

Then

$$\Psi = a_n \left(f + \frac{a_{n-1}}{na_n}\right)^n.$$

Lemma 2.7. [19] *Let $q \in \mathbb{N}$, $a_0(z), \cdots, a_n(z)$ be either exponential polynomials of degree $< q$ or ordinary polynomials in z, and let $b_1, \cdots, b_n \in \mathbb{C}\backslash\{0\}$ be distinct constants. Then*

$$\sum_{j=1}^{n} a_j(z) e^{b_j z^q} = a_0(z)$$

holds only when $a_0(z) \equiv a_1(z) \equiv \cdots \equiv a_n(z) \equiv 0$.

3. The exponential polynomials solutions on a general equation

Remark that the equations (1.5), (1.6) and (1.7), the last term on the left hand side has only one term $f(z+c)$ or $f^{(k)}(z+c)$. It is natural to ask what will happen $f(z+c)$ or $f^{(k)}(z+c)$ is replaced with differential-difference polynomials. We mainly consider the non-linear differential-difference equations

$$f(z)^n + a_{n-1}f(z)^{n-1} + \cdots + a_s f(z)^s + q(z)e^{Q(z)}[L(z,f)]^t = P(z), \tag{3.1}$$

where $L(z,f)$ is a k-homogeneous differential-difference polynomial. We will assume that $q(z)$, $P(z)$, $Q(z)$ are polynomials, $k \geqslant 1$ is an integer and t is a positive integer, $n > s \geqslant tk \geqslant 1$ and $Q(z)$ is not a constant, $q(z) \not\equiv 0$ and $a_s, \cdots, a_{n-1} \in \mathbb{C}$. It is easy to see that both equations (1.5) and (1.6) have no polynomial solutions, since $Q(z)$ is not a constant. However, there exist polynomial solutions with degree less than k in (1.7) and (3.1), resp. For example, $f(z) = z$ is a solution of

$$f(z)^n - q(z)e^{Q(z)}f''(z+1) = z^n.$$

Recent results on complex differential-difference equations also can be found in [16, 17, 18]. In this paper, we mainly consider the transcendental solutions in (3.1) and obtain the following result.

Theorem 3.1. *The finite order transcendental entire solution f of* (3.1) *should satisfy the following conclusions:*

(*a*) *Every solution f satisfies $\sigma(f) = \deg(Q)$ and is of mean type.*

(*b*) *If $\lambda(f) < \sigma(f)$, then $a_{n-1} = \cdots = a_s = 0$ and $P(z) \equiv 0$.*

(*c*) *If $P(z) \equiv 0$, then $z^{n-s} + a_{n-1}z^{n-s-1} + \cdots + a_s = (z + \frac{a_{n-1}}{n})^{n-s}$. Furthermore, if there exists $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$, then all the $a_j (j = s, \ldots, n-1)$ must be zeros as well and $\lambda(f) < \sigma(f)$; otherwise $\lambda(f) = \sigma(f)$.*

Furthermore, the following conclusions are true for a k-homogeneous differential-difference polynomial $L(z,f)$ with the same shifts.

(*d*) *$f \in \Gamma_0'$ if and only if $P(z) \equiv 0$ and there exists $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$.*

(*e*) *If the solution f belongs to Γ_0, then $\sigma(f) = 1$. What's more, if $g \in \Gamma_0$, then $f = \eta g$, where $\eta^{n-kt} = 1$.*

(*f*) *If there exists $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$. Then*

$$f \in \Gamma_0' \quad \text{and} \quad P(z) \equiv 0 = a_s = \cdots = a_{n-1}$$

provided that one of the following holds:

1) *$s \geqslant k + 2$;*

2) *$s = k + 1$ and $z^{n-s} + \cdots + a_s = 0$ has at least one zero with multiplicity 2;*

3) *$s < k+1$ and either $z^{n-s}+\cdots+a_s = 0$ has at least two zeros with multiplicity 2 or at least one zero with multiplicity 3.*

Remark 3.1. (1) Transcendental entire solutions with finite order of (3.1) exist. For example, the function $f(z) = e^z + \alpha$ solves

$$f(z)^2 - 2\alpha f(z) - 2e^z f'(z - \log 2) = -\alpha^2.$$

(2) From Theorem B (e), we can not get $\sigma(f) = 1$ when f belongs to Γ_0. For example, the function$f(z) = e^{z^2}$ solves

$$f(z)^2 - e^{z^2-2z-1} f(z+1) = 0.$$

However, from Theorem 3.1 (e), if f belongs to Γ_0, the solutions of the equation

$$f(z)^n + a_{n-1}f(z)^{n-1} + \cdots + a_s f(z)^s + q(z)e^{Q(z)}[L(z,f)]^t = 0,$$

must satisfy $\sigma(f) = 1$. Such solutions exist, for example, the function $f(z) = e^z$ solves $f(z)^2 - e^{z-1}f'(z+1) = 0$.

Proof of Theorem 3.1 (*a*). Assume that $f(z)$ is a finite order transcendental entire solution of (3.1). From Valiron-Mohon'ko theorem and Lemma 2.1, we obtain

$$\begin{aligned} nT(r,f) &= T(r, f^n + \cdots + a_s f^s) + S(r,f) = T(r, P(z) - q(z)e^{Q(z)}[L(z,f)]^t) + S(r,f) \\ &= m(r, P(z) - q(z)e^{Q(z)}[L(z,f)]^t) + S(r,f) \\ &\leqslant m(r, P(z)) + m(r, q(z)) + m(r, e^{Q(z)}) + m(r, [L(z,f)]^t) + S(r,f) \\ &\leqslant m(r, e^{Q(z)}) + m\left(r, \left(\frac{L(z,f)}{f(z)^k}\right)^t f(z)^{tk}\right) + S(r,f) \\ &\leqslant m(r, e^{Q(z)}) + tkT(r, f(z)) + S(r,f). \end{aligned}$$

Since $n > tk$, then

$$(n - tk)T(r,f) \leqslant m(r, e^{Q(z)}) + S(r,f),$$

which implies that $\sigma(f) \leqslant \deg(Q(z))$. If $\sigma(f) < \deg(Q(z))$, then $\sigma(L(z,f)) < \deg(Q(z))$, which is impossible for (3.1). Hence, $\sigma(f) = \deg(Q(z))$. From the definition of type, we get

$$\tau(f) = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\deg(Q(z))}} \in (0, +\infty),$$

which implies that $f(z)$ is of mean type.

Proof of Theorem 3.1 (*b*). If $\lambda(f) < \sigma(f)$, that is the value 0 is a Borel exceptional value of $f(z)$, then f is of regular growth (or normal growth) [22, Theorem 2.11]. Thus, $N(r, \frac{1}{f}) = S(r,f)$ follows by [22, Theorem 1.18]. Let

(3.2) $G(z) := f(z)^n + a_{n-1}f(z)^{n-1} + \cdots + a_s f(z)^s - P(z) = -q(z)e^{Q(z)}[L(z,f)]^t.$

From the Remark after Lemma 2.5 and f is an entire function, we have

$$N\left(r,\frac{1}{G(z)}\right)=N\left(r,\frac{1}{q(z)[L(z,f)]^t}\right)$$
$$\leqslant N\left(r,\frac{1}{q(z)}\right)+tN\left(r,\frac{1}{L(z,f)}\right)\leqslant tkN(r,\frac{1}{f})+S(r,f)=S(r,f). \tag{3.3}$$

Thus $\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,f)=S(r,f)$. Lemma 2.6 implies that

$$G(z)=\left(f+\frac{a_{n-1}}{n}\right)^n. \tag{3.4}$$

If $a_{n-1}\neq 0$, using the second main theorem of Nevanlinna theory, we have

$$\begin{aligned} T(r,f) &\leqslant \overline{N}\left(r,\frac{1}{f+\frac{a_{n-1}}{n}}\right)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)\\ &\leqslant \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)=S(r,f), \end{aligned}$$

a contradiction. Thus $a_{n-1}=0$. Therefore, from (3.4) and (3.2), we have

$$a_{n-1}=\cdots=a_s=0\equiv P(z).$$

Proof of Theorem 3.1 (c). Since $P(z)\equiv 0$ and $s\geqslant tk$, then (3.1) can be written as

$$\begin{aligned} H(z):&=f(z)^{n-kt}+a_{n-1}f(z)^{n-kt-1}+\cdots+a_sf(z)^{s-kt}\\ &=-q(z)e^{Q(z)}\left[\frac{L(z,f)}{f(z)^k}\right]^t. \end{aligned} \tag{3.5}$$

From (3.5), we have

$$tN\left(r,\frac{L(z,f)}{f(z)^k}\right)\leqslant N\left(r,\frac{1}{q(z)}\right)=S(r,f). \tag{3.6}$$

Combining (3.6) with Lemma 2.1, we obtain

$$T\left(r,\frac{L(z,f)}{f(z)^k}\right)=S(r,f). \tag{3.7}$$

Using the first main theorem of Nevanlinna theory, we have

$$T\left(r,\frac{1}{\frac{L(z,f)}{f(z)^k}}\right)=T\left(r,\frac{L(z,f)}{f(z)^k}\right)+O(1)=S(r,f). \tag{3.8}$$

From (3.5) and (3.8), we obtain

$$\overline{N}\left(r,\frac{1}{H(z)}\right)+\overline{N}(r,f)\leqslant N\left(r,\frac{1}{\frac{L(z,f)}{f(z)^k}}\right)+N\left(r,\frac{1}{q(z)}\right)=S(r,f).$$

Lemma 2.6 implies that

$$H(z)=\left(f(z)+\frac{a_{n-1}}{n-kt}\right)^{n-kt}=-q(z)e^{Q(z)}\left(\frac{L(z,f)}{f(z)^k}\right)^t. \tag{3.9}$$

Case 1. There exists $i_0 \in \{s, \cdots, n-1\}$ such that $a_{i_0} = 0$. From (3.9), we have $a_{n-1} = \cdots = a_s = 0$. Thus, (3.1) can be reduced to the follows form

$$f(z)^{n-kt} = -q(z)e^{Q(z)} \left(\frac{L(z,f)}{f(z)^k}\right)^t. \tag{3.10}$$

Since $n > kt$, then $N(r, \frac{1}{f}) = S(r,f)$. So $\lambda(f) < \sigma(f)$.

Case 2. There does not exist $i_0 \in \{s, \cdots, n-1\}$ such that $a_{i_0} = 0$. From (3.8) and (3.9), we have

$$\overline{N}\left(r, \frac{1}{f(z) + \frac{a_{n-1}}{n-kt}}\right) \leqslant \overline{N}\left(r, \frac{1}{\frac{L(z,f)}{f(z)^k}}\right) + \overline{N}\left(r, \frac{1}{q(z)}\right) = S(r,f).$$

Using the second main theorem, we have

$$\begin{aligned} T(r,f) &\leqslant \overline{N}\left(r, \frac{1}{f(z) + \frac{a_{n-1}}{n-kt}}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}(r, f(z)) + S(r,f) \\ &= \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f). \end{aligned}$$

Therefore, $\lambda(f) = \sigma(f)$.

Proof of Theorem 3.1 (d). If $f \in \Gamma_0'$, then $\lambda(f) < \sigma(f)$ follows. From Theorem 3.1 (b), we have $a_{n-1} = \cdots = a_s = 0 \equiv P(z)$.

On the other hand, we will prove that if $P(z) \equiv 0$ and there exists $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$, then $f \in \Gamma_0'$. The condition $P(z) \equiv 0$ implies that (3.1) reduces to

$$f(z)^n + q(z)e^{Q(z)}[L(z,f)]^t = 0. \tag{3.11}$$

Since $P(z) \equiv 0$ and there exists an $i_0 \in \{s, \ldots, n-1\}$ such that $a_{i_0} = 0$, from Theorem 3.1 (c), then $\lambda(f) < \sigma(f)$. The Hadamard factorization theorem implies that

$$f(z) = H(z)e^{\alpha(z)}, \tag{3.12}$$

where $\alpha(z)$ is a non-constant polynomial with $\deg(\alpha(z)) = \deg(Q(z))$ and $H(z)$ is an entire function satisfying

$$\lambda(H(z)) = \sigma(H(z)) = \lambda(f) < \sigma(f).$$

In the following, we will prove $H(z)$ is a non-zero polynomial. Otherwise, if $H(z)$ is a transcendental entire function, from (3.12), then

$$f^{(v_{ij})}(z+c) = H_{ij}(z+c)e^{\alpha(z+c)} \quad (j \in 1, 2, \cdots, n),$$

where $H_{ij}(z+c)$ is 1-homogeneous differential-difference polynomial in $H(z+c)$. Remark that $L(z,f)$ is a k-homogeneous differential-difference polynomial with the same shifts in (1.8), then

$$\begin{aligned} L(z,f) &= \sum_{i=1}^{m} H_{i1}(z+c)\cdots H_{in}(z+c)e^{k\alpha(z+c)} \\ &= e^{k\alpha(z+c)}\sum_{i=1}^{m} H_{i1}(z+c)\cdots H_{in}(z+c) =: e^{k\alpha(z+c)}H_k(z) \end{aligned}$$

where $\sigma(H_k(z)) < \sigma(f)$ and $H_k(z)$ is a k-homogeneous differential-difference polynomial.

Substituting (3.12) and (3.13) into (3.11), we have

$$H(z)^n e^{n\alpha(z)} + q(z)e^{Q(z)+kt\alpha(z+c)}H_k(z)^t = 0, \tag{3.13}$$

so

$$H(z)^{n-kt} + q(z)e^{Q(z)+kt\alpha(z+c)-n\alpha(z)}\left(\frac{H_k(z)}{H(z)^k}\right)^t = 0. \tag{3.14}$$

By Lemma 2.1, we have

$$m\left(r, \frac{H_k(z)}{H(z)^k}\right) = O(r^{\sigma(H)-1+\varepsilon}). \tag{3.15}$$

From (3.14), the poles of $\frac{H_k(z)}{H(z)^k}$ are the zeros of $q(z)$, then

$$N\left(r, \frac{H_k(z)}{H(z)^k}\right) = O(\log r). \tag{3.16}$$

Thus,

$$T\left(r, \frac{H_k(z)}{H(z)^k}\right) = O(r^{\sigma(H)-1+\varepsilon}) + O(\log r). \tag{3.17}$$

Hence, we have $N\left(\frac{1}{\frac{H_k(z)}{H(z)^k}}\right) = O(r^{\sigma(H)-1+\varepsilon}) + O(\log r)$. Since $n > kt$, so the zeros of $H(z)$ are the zeros of $\frac{H_k(z)}{H(z)^k}$ or $q(z)$, thus $\lambda(H) < \sigma(H)$, which is a contradiction with $\lambda(H) = \sigma(H)$. So $H(z)$ is a polynomial. Hence, $f \in \Gamma_0'$.

Proof of Theorem 3.1 (*e*). If $f, g \in \Gamma_0$, then $a_{n-1} = \cdots = a_s = 0 \equiv P(z)$ follows by Theorem 3.1 (*b*) and $H(z) \equiv 1$ in (3.12). From (3.13), we have $q(z)$, $H_k(z)$ are also constants. If $H_k(z)$ is a constant, then $\alpha(z)$ must be a linear polynomial and $\varphi_i(z)$ are also constants φ_i. We may assume that $q(z) = q \in \mathbb{C}$ and $f(z) = e^{b_1 z + d_1}$ and $g(z) = e^{b_2 z + d_2}$, where $b_i (\neq 0), d_i$ $(i = 1, 2)$ are constants. Substituting $f(z)$ and $g(z)$ into (3.11), we can get

$$e^{nb_1 z + nd_1} + q(z)e^{Q(z)}L_1(b_1)e^{ktb_1 z + kt(d_1 + c)} = 0, \tag{3.18}$$

and

$$e^{nb_2 z + nd_2} + q(z)e^{Q(z)}L_1(b_2)e^{ktb_2 z + kt(d_2 + c)} = 0, \tag{3.19}$$

where $L_1(z) = \sum_{i=1}^{m} \varphi_i z^{k_{i1}v_{i1}} \cdots z^{k_{in}v_{in}}$. Combining (3.18) and (3.19), we have

$$L_1(b_1)e^{(kt-n)b_1 z + (kt-n)d_1} \equiv L_1(b_2)e^{(kt-n)b_2 z + (kt-n)d_2}.$$

Thus, we have $b_1 = b_2$ and $e^{(n-kt)(d_1-d_2)} = 1$, which implies that $f = \eta g$, where $\eta^{n-kt} = 1$.

Proof of Theorem 3.1 (f). If $P(z) \not\equiv 0$, from Lemma 2.1, Lemma 2.2 and the second main theorem of Nevanlinna theory, then

(3.20)

$$\begin{aligned}
&nT(r,f) = T\left(r, f^n + \cdots + a_s f^s\right) + S(r,f)\\
&\leqslant \overline{N}\left(r, \frac{1}{f^n + \cdots + a_s f^s - P(z)}\right) + \overline{N}\left(r, \frac{1}{f^n + \cdots + a_s f^s}\right) + \overline{N}\left(r, f^n + \cdots + a_s f^s\right) + S(r,f)\\
&\leqslant \overline{N}\left(r, \frac{1}{f^n + \cdots + a_s f^s}\right) + \overline{N}\left(r, \frac{1}{q(z)[L(z,f)]^t}\right) + S(r,f)\\
&\leqslant kT(r,f) + \overline{N}\left(r, \frac{1}{f^n + \cdots + a_s f^s}\right) + S(r,f).
\end{aligned}$$

We will get a contradiction from (3.20) in every case of 1), 2), 3) to show $P(z) \equiv 0$ below. Thus, the conclusion of (f) follows by Theorem 3.1 (c) and (d).

Case 1). If $s \geqslant k+2$, from (3.20), we have

$$\begin{aligned}
(n-k)T(r,f) &\leqslant \overline{N}\left(r, \frac{1}{f(z)^s(f(z)^{n-s} + \cdots + a_s)}\right) + S(r,f)\\
&\leqslant (n-s+1)T(r,f) + S(r,f),
\end{aligned}$$

which is a contradiction.

Case 2). If $s = k+1$ and $z^{n-s} + \cdots + a_s = 0$ has at least one zero with multiplicity 2. From (3.20), we have

$$(n-k)T(r,f) \leqslant (n-s)T(r,f) + S(r,f),$$

a contradiction.

Case 3). If $s \leqslant k+1$ and $z^{n-s} + \cdots + a_s = 0$ has at least two zeros with multiplicity 2 or at least one zero with multiplicity 3, we also get a contradiction from (3.20).

4. Properly meromorphic solutions

In this section, we will consider the properly meromorphic solutions on non-linear differential-difference equations.

Wen etc. [19] proved that there is no properly meromorphic solutions with hyper-order less than one on (1.5) by considering the poles multiplicities provided that $n \geqslant 2$, which is also true for (1.6). In fact, we know that both (1.5) and (1.6) has no any properly meromorphic solutions by the follows statements. We assume that $f(z)$ is a properly meromorphic solution of (1.5) or (1.6) and $f(z)$ has a pole z_0, then $z_k = z_0 + kc$ are also the poles of $f(z)$, where k is any integer, thus $f(z)$ should have infinitely many poles. Let $\mathbb{E}$ be the set of all poles multiplicities of $f(z)$ and m

be the minimum of $\mathbb{E}$, where m is called the index of $f(z)$. Obviously, $f(z+c)$ has the same index m, but the index of $f(z)^n$ is mn, which is impossible for (1.5) or (1.6), when $n \geqslant 2$. Thus, $f(z)$ has no poles. Similarly, if $L(z,f)$ is a linear difference polynomial, then (3.1) has no properly meromorphic solutions.

Remark that if $L(z,f)$ includes the derivatives of $f(z)$ or the derivatives of $f(z+c)$, then (3.1) may have properly meromorphic solutions, which can be seen by the examples below.

Examples (1) Properly meromorphic function $f(z)=\frac{1}{1-e^z}$ solves the follows two equations

$$f(z)^2-e^{-z}f'(z+2\pi i)=0, \qquad f(z)^2-e^{-z}f'(z)=0.$$

(2) Properly meromorphic function $f(z)=\frac{1}{e^z+1}$ solves

$$f(z)^3-\frac{5}{2}f(z)^2+2f(z)+\frac{1}{2}e^zf''(z)=\frac{1}{2}.$$

and $f(z)=\frac{1-e^z}{2(e^z+1)}$ solves

$$f(z)^3-f(z)^2+\frac{1}{4}f(z)+\frac{1}{2}e^zf''(z)=0.$$

(3) Properly meromorphic solutions with infinite order of (3.1) also exist. For example, the function $f(z)=\frac{1}{e^{-e^{-z}}-1}$ solves

$$f(z)^2+f(z)+e^zf'(z)=0.$$

and the function $f(z)=\frac{3e^{-e^{-z}}-2}{e^{-e^{-z}}-1}$ solves

$$f(z)^2-5f(z)-e^zf'(z)=-6.$$

Remark that some functions are periodic functions in the above examples, so $f'(z)$ can be replaced with $f'(z+c)$ in the above equations for suitable constants c. An elementary calculation to find that the non-linear differential equation

$$f(z)^2+e^{Q(z)}f^{'}(z)=0 \tag{4.1}$$

has solutions $f(z)=\frac{1}{C+\int e^{-Q(z)}dz}$, C is a constant. In addition, the solutions of

$$f(z)^2+a_1f(z)+q(z)e^{Q(z)}f'(z)=0 \tag{4.2}$$

can be expressed by $f=\frac{1}{e^{\int\frac{a_1}{q(z)}e^{-Q(z)}dz}\left(\int\frac{e^{-Q(z)}}{q(z)}e^{\int\frac{-a_1}{q(z)}e^{-Q(z)}dz}dz+C\right)}$.

Question 1: How to classify the properly meromorphic solutions of

$$f^n+a_{n-1}f^{n-1}+\cdots+a_1f+q(z)e^{Q(z)}L(z,f)=P(z), \tag{4.3}$$

where $L(z,f)$ is a k-homogeneous differential-difference polynomial, $k\geqslant 1$ is an integer and $q(z)$, $P(z)$, $Q(z)$ are polynomials, $a_1,\cdots,a_{n-1}$ are constants.

In the paper, using the exponential polynomials, we consider the simple case of $n=2$, $L(z,f)=f'(z)$ and $P(z)\equiv 0$ in (4.3), that is

$$f^2+a_1f+q(z)e^{Q(z)}f'(z)=0, \tag{4.4}$$

where $Q(z)=b_qz^q+\cdots+b_0,\ b_q\neq 0,\ a_1$ is a constant and $q(z)$ is a non-zero polynomial.

It is easy to find that the meromorphic solutions of (4.4) have only finitely many zeros. For the simplified expressions, we can consider the meromorphic solutions f with the form

$$f(z)=\frac{1}{g(z)}=\frac{1}{G_0(z)+G_1(z)e^{\omega_1z^q}+\cdots+G_m(z)e^{\omega_mz^q}},$$

where $g(z)$ is an exponential polynomial and $G_j(z)(j=0,1,\cdots,m)$ are either exponential polynomials of degrees $<q$ or ordinary polynomials in z.

Theorem 4.1. *(i) If $a_1=0$, then (4.4) admits properly meromorphic solutions of the form $f=\frac{1}{g}=\frac{1}{d+Ae^{\omega_1z}}$, where d,A,ω_1 are constants.*

(ii) If $a_1\neq 0$, then (4.4) has no meromorphic solutions of the form $f=\frac{1}{g}$.

Proof of Theorem 4.1. (i) If $a_1=0$, substitute $f=\frac{1}{g}$ and

$$g(z)=G_0(z)+G_1(z)e^{\omega_1z^q}+\cdots+G_m(z)e^{\omega_mz^q}$$

into (4.4), then

$$1-q(z)e^{Q_0(z)}(G_0'(z)e^{b_qz^q}+G_{1,1}(z)e^{(\omega_1+b_q)z^q}+\cdots+G_{m,1}(z)e^{(\omega_m+b_q)z^q})=0, \tag{4.5}$$

where $Q_0(z)=Q(z)-b_qz^q$ is a polynomial of degree $\leqslant q-1$ and

$$G_{k,1}(z)=G_k'(z)+q\omega_kz^{q-1}G_k(z)\not\equiv 0 \tag{4.6}$$

for $k=1,\cdots,m$.

If $m\geqslant 2$, from Lemma 2.7 and (4.5), we get that at least one of $q(z)e^{Q_0(z)}G_{1,1}(z)$ and $q(z)e^{Q_0(z)}G_{m,1}(z)$ is equal to zero, thus $G_{1,1}(z)$ or $G_{m,1}(z)$ is equal to zero, which is impossible.

If $m=1$, then (4.5) reduces to

$$1-q(z)e^{Q_0(z)}\left(G_0'(z)e^{b_qz^q}+G_{1,1}(z)e^{(\omega_1+b_q)z^q}\right)=0. \tag{4.7}$$

Let $h_1(z)=q(z)e^{Q_0(z)}G_0'(z)e^{b_qz^q}$ and $h_2(z)=q(z)e^{Q_0(z)}G_{1,1}(z)e^{(\omega_1+b_q)z^q}$. Thus, $h_1(z)+h_2(z)=1$. Using the second main theorem of Nevanlinna theory for $h_1(z)$, we have $h_1(z)$ and $h_2(z)$ must be constants. Hence, $G_0'(z)=0$ and $\omega_1=-b_q$ by the expressions of $h_1(z)$ and $h_2(z)$. Let $g(z)=d+G_1(z)e^{\omega_1z^q}$. We proceed to prove that $q=1$. Otherwise, (4.7) can be written as

$$q(z)e^{Q(z)+\omega_1z^q}(G_1'(z)+G_1(z)q\omega_1z^{q-1})=1. \tag{4.8}$$

(4.8) means that $q(z)$ is a non-zero constant. Furthermore, $Q(z)+\omega_1 z^q$ and $G_1'(z)+G_1(z)q\omega_1 z^{q-1}$ must be constants, otherwise $G_1(z)$ is of order q. Hence, $q=1$ and $G_1(z)$ is a non-zero constant A. Thus, $g(z)=d+Ae^{\omega_1 z}$.

(ii) If $a_1\neq 0$, substitute $f=\frac{1}{g}$ into (4.4), we have

$$1+a_1(G_0(z)+G_1(z)e^{\omega_1 z^q}+\cdots+G_m(z)e^{\omega_m z^q})-q(z)e^{Q_0(z)}(G_0'(z)e^{b_q z^q}+ \tag{4.9} G_{1,1}(z)e^{(\omega_1+b_q)z^q}+\cdots+G_{m,1}(z)e^{(\omega_m+b_q)z^q})=0,$$

where $G_{k,1}(z)$ are the same as (4.6). If $G_0'(z)\neq 0$, then b_q, ω_1+b_q, $\cdots$, ω_m+b_q are $m+1$ distinct constants, hence $\{\omega_1,\cdots,\omega_m\}\neq\{b_q,\omega_1+b_q,\cdots,\omega_m+b_q\}$, then there exists $i\in\{0,1,2,\cdots,m\}$ such that $\omega_i+b_q\neq\omega_j$, $j=1,\cdots,m$ and $\omega_0=0$. By Lemma 2.7, we have $G_{i,1}(z)\equiv 0$, which is a contradiction. Thus, we have $G_0'(z)=0$ and $\{\omega_1,\cdots,\omega_m\}$=$\{\omega_1+b_q,\cdots,\omega_m+b_q\}$ for $m\geqslant 1$, which is also impossible unless $b_q=0$.

Remark 4.1. (1) The case (i) shows that all properly meromorphic solutions with the form $f(z)=\frac{1}{g}$ are of order 1. However, if $a_1=0$ and $q(z)$ is a rational function, then (4.4) has properly meromorphic solutions f with finite order $\sigma(f)>1$. For example, the function $f(z)=\frac{1}{e^{z^n}+1}$ solves

$$f(z)^2+\frac{1}{nz^{n-1}}e^{-z^n}f'(z)=0.$$

(2) If $P(z)\not\equiv 0$ in (4.3), the equation

$$f^2+a_1 f+q(z)e^{Q(z)}f'(z)=P(z) \tag{4.10}$$

may admit properly meromorphic solutions of the form $f=\frac{h}{g}$, where $h(z)$ and $g(z)$ are exponential polynomials. The functions $f_1(z)=\frac{1}{e^z-1}$ and $f_2(z)=\frac{1-e^z}{2e^z-1}$ solve

$$f(z)^2+2f(z)+e^z f'(z)=-1.$$

Hence, to classify the general ratios of exponential polynomials for non-linear differential-difference equation is deserved to considering.

(3) We have the basic discussions on $L(z,f)=f'(z)$ in (4.4). However, if $L(z,f)$ includes differential-difference polynomials, for example $L(z,f)=f'(z+c)$, the corresponding substitution will be more complicated than (4.5). In this case, it is not clear for the expressions of properly meromorphic solutions of (4.3).

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ОПЕРАТОРЫ $\mathcal{L}$-ВИНЕРА-ХОПФА В ВЕСОВЫХ ПРОСТРАНСТВАХ В СЛУЧАЕ БЕЗОТРАЖАТЕЛЬНОГО ПОТЕНЦИАЛА

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Аннотация. Заменой в определении оператора свертки преобразования Фурье спектральным преобразованием оператора Штурма-Лиувилля $\mathcal{L}$, порожденного безотражательным потенциалом, вводится понятие оператора $\mathcal{L}$-Винера-Хопфа в лебеговых пространствах с весом Макенхаупта. Получены критерии фредгольмовости и обратимости и формула для индекса в случае кусочно-непрерывного символа.

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Ключевые слова: безотражательный потенциал; оператор $\mathcal{L}$-Винера-Хопфа; оператор Фредгольма.

1. Введение

Заменой в классическом определении оператора свертки преобразования Фурье спектральным преобразованием самосопряженного оператора Штурма-Лиувилля в работе [1] введены понятия оператора $\mathcal{L}$-свертки и оператора $\mathcal{L}$-Винера-Хопфа, действующих в пространствах L_p. В случае когда потенциал соответствующего уравнения Штурма-Лиувилля является нулевым, эти операторы совпадают соответственно с оператором свертки и оператором Винера-Хопфа. В работах [2] – [5] при различных предположениях относительно символа, изучены свойства фредгольмовости и обратимости оператора $\mathcal{L}$-Винера-Хопфа в том случае, когда потенциал является безотражательным. Напомним, что оператор $A : X \to Y$, где X, Y – банаховы пространства, называется фредгольмовым, если его образ замкнут (т.е. $\operatorname{Im} A = \overline{\operatorname{Im} A}$), и конечномерны его ядро $\ker A := \{x \in X : Ax = 0\}$ и коядро $\operatorname{Coker} A := {}^{Y}/_{\operatorname{Im} A}$. Число $\operatorname{Ind} A := \dim\ker A - \dim\operatorname{Coker} A$ называют индексом оператора A, а множество $\{\lambda \in \mathbb{C} : T-\lambda I \text{ не фредгольмов}\}$ существенным спектром оператора A.

Пусть E либо $\mathbb{R}$, либо $\mathbb{R}_\pm := \{\pm x > 0 : x \in \mathbb{R}\}$, и $w : E \to [0,\infty]$ весовая функция (т.е. измеримая функция такая, что лебегова мера множества $w^{-1}\{0,\infty\}$ равна нулю), а $L_p(E,w)$, $1 < p < \infty$, лебегово пространство с нормой

$$\|f\|_{p,w} = \|fw\|_p = \left(\int_E |f(x)|^p \, w(x)^p \, dx \right)^{1/p} .$$

Через $A_p(\mathbb{R})$, $1 < p < \infty$, будем обозначать множество весов на $\mathbb{R}$, удовлетворяющих известному условию A_p:

$$\sup \left(\frac{1}{|\mathrm{I}|} \int_{\mathrm{I}} w(x)^p \, dx \right)^{1/p} \left(\frac{1}{|\mathrm{I}|} \int_{\mathrm{I}} w(x)^{-q} \, dx \right)^{1/q} < \infty,$$

где I пробегает все ограниченные интервалы вещественной прямой $\mathbb{R}$, $|\mathrm{I}|$ – длина интервала I и $1/p + 1/q = 1$.

Данная работа посвящена исследованию задачи фредгольмовости оператора $\mathcal{L}$-Винера-Хопфа в пространстве $L_p(\mathbb{R}_+, w) := L_p(\mathbb{R}_+, w|_{R_+})$ в случае когда $w \in A_p(\mathbb{R})$. Получены критерии фредгольмовости и формула для вычисления индекса в случае кусочно-непрерывного (непрерывного) символа оператора $\mathcal{L}$.

2. Спектральное преобразование. Данные рассеяния

Пусть $\mathcal{L}$ – самосопряженный в $L_2(\mathbb{R})$ оператор, порожденный дифференциальным выражением

$$(\ell y)(x) = -y''(x) + v(x)y(x), \qquad x \in \mathbb{R}, \tag{2.1}$$

где

$$\int_{-\infty}^{\infty} (1 + |x|) \, v(x) \, dx < \infty. \tag{2.2}$$

Важную роль (см., например, [6] – [8]) в спектральной теории оператора $\mathcal{L}$ и, в частности, в обратной задаче теории рассеяния на оси, уравнения

$$\ell y = \lambda^2 y, \tag{2.3}$$

играют решения Йоста, т.е. решения $e_+(x,\lambda)$ ($x \in \mathbb{R}$, $\operatorname{Im}\lambda \ge 0$) и $e_-(x,\lambda)$ ($x \in \mathbb{R}$, $\operatorname{Im}\lambda \le 0$) определяемые граничными условиями

$$\lim_{x\to\pm\infty} e^{-i\lambda x} e_\pm(x,\lambda) = 1, \quad \lim_{x\to\pm\infty} e^{-i\lambda x} e'_\pm(x,\lambda) = i\lambda.$$

Эти решения (см. [6] – [8]) допускают представление

$$e_{\pm}(x,\lambda) = (\mathrm{I} + \mathcal{K}_{\pm})(e^{i\lambda x}), \qquad \lambda \in \mathbb{R}, \tag{2.4}$$

где операторы преобразования $\mathrm{I} + \mathcal{K}_{\pm}$ действуют по формулам

$$((\mathrm{I} + \mathcal{K}_{+})y)(x) = y(x) + \int\limits_{x}^{\infty} K_{+}(x,t)\, y(t)\, dt$$

$$((\mathrm{I} + \mathcal{K}_{-})y)(x) = y(x) + \int\limits_{-\infty}^{x} K_{-}(x,t)\, y(t)\, dt \tag{2.5}$$

и ограничены соответственно в пространствах $L_p(\gamma,\infty)$ и $L_p(-\infty,\gamma)$, $1 \le p \le \infty$, при всех $\gamma \in \mathbb{R}$. Ядра $K_{+}(x,t)$, $-\infty < x \le t < \infty$ и $K_{-}(x,t)$, $-\infty < t \le x < \infty$ удовлетворяют известным уравнениям Гельфанда-Левитана-Марченко (см. [6] – [8]).

При вещественных значениях $\lambda \neq 0$ пары функций $e_{+}(x,\lambda)$, $e_{+}(x,-\lambda)$ и $e_{-}(x,\lambda)$, $e_{-}(x,-\lambda)$ образуют фундаментальные системы решений (2.3). В частности

$$e_{+}(x,\lambda) = b(\lambda)e_{-}(x,-\lambda) + b_0(\lambda)e_{-}(x,\lambda).$$

Функция $b_0(\lambda)$ аналитична в верхней полуплосткости $\operatorname{Im}\lambda > 0$ и имеет там лишь конечное число простых нулей $i\mu_k$ ($\mu_k > 0$, $k = 1, \dots N$) которые лежат на мнимой полуоси.

Дискретный спектр оператора $\mathcal{L}$ совпадает с множеством $\{(i\mu_1)^2, \dots, (i\mu_N)^2\}$.

Каждое собственное значение $\lambda_k = (i\mu_k)^2$ ($k = 1, \dots, N$) является простым и ему соответствуют правая собственная функция $e_{+}(x, i\mu_k)$ и линейно зависимая от нее левая собственная функция $e_{-}(x, -i\mu_k)$ (см. [6] – [8]). Функция $t(\lambda) = b_0^{-1}(\lambda)$ ($\lambda \in \mathbb{R}$) называется коэффициентом прохождения. Решения уравнения (2.3),

$$u_{\mp}(x,\lambda) := t(\lambda)\, e_{\pm}(x,\pm\lambda) \tag{2.6}$$

порождают интегралы

$$(U_{\mp}y)(\lambda) = \int\limits_{-\infty}^{\infty} u_{\mp}(x,\lambda)\, y(x)\, dx, \qquad \lambda \in \mathbb{R}, \tag{2.7}$$

которые сходятся по норме $L_2(\mathbb{R})$ и определяют ограниченные операторы $U_{\mp} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ (см. [6], а также [1]).

Далее, мы через $m(a)$ будем обозначать действующий в функциональных пространствах оператор умножения на функцию a $(m(a)y := ay)$, а через $J : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ оператор действующий по формуле $(Jy)(x) = y(-x)$.

Под спектральным преобразованием оператора $\mathcal{L}$ мы понимаем оператор

$$U := m(\chi_+)U_- + m(\chi_-)JU_+ : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \tag{2.8}$$

где χ_+ (χ_-) – характеристическая функция $\mathbb{R}_+$ $(\mathbb{R}_-)$. Оператор U удовлетворяет равенствам

$$U^*U = \mathrm{I} - P, \quad UU^* = \mathrm{I},$$

где I – единичный оператор, а P – ортогональный проектор в $L_2(\mathbb{R})$ на собственное подпространство соответствующее дискретному спектру оператора $\mathcal{L}$ (см. [6], [1]).

В случае $v = 0$, оператор U совпадает с преобразованием Фурье $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$:

$$(Fy)(\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{i\lambda x} y(x)dx.$$

Заметим также (см. [6] и [1]), что на всюду плотном в $L_2(\mathbb{R})$ множестве имеет место равенство

$$U\mathcal{L}U^* = m(\lambda^2).$$

Функции $r^-(\lambda) = b(\lambda)t(\lambda)$ и $r^+(\lambda) = -b(-\lambda)t(\lambda)$ называются соответственно левым и правым коэффициентами отражения.

Обратные величины норм собственных функций

$$m_k^{\pm} := \left(\int\limits_{-\infty}^{\infty} |e_{\pm}(x, \pm i\mu_k)|^2 \, dx \right)^{-1/2}$$

называют нормирующими множителями, а наборы величин $\{r^+(\lambda), i\mu_k, m_k^+; k = 1, \ldots, N\}$ и $\{r^-(\lambda), i\mu_k, m_k^-; k = 1, \ldots, N\}$ называют соответственно правым и левым данными рассеяния.

Обратная задача теории рассеяния уравнения (2.3) состоит в восстановлении потенциала по левым или правым данным рассеяния и нахождении необходимых и достаточных условий, которым должен удовлетворять взятый набор $\{r(\lambda), i\mu_k, m_k : k = 1, \ldots, N\}$, чтобы он являлся левыми либо правыми данными рассеяния уравнения (2.3) при некотором потенциале v удовлетворяющему условию (2.2).

3. Безотражательный потенциал

Как известно, (см. [6] – [8]) набор вида $\{0, i\mu_k, m_k : k = 1, \dots, N\}$, где μ_k, m_k положительные числа, причем μ_k различны друг от друга и всегда являются данными рассеяния. Потенциалы, имеющие данные рассеяния такого типа, называются *безотражательными* (поскольку в этом случае $r_\pm = 0$). Заметим (см. например [8]), что в этом случае коэффициент прохождения определяется по формуле

$$t(\lambda) = \prod_{k=1}^{N} \frac{\lambda + i\mu_k}{\lambda - i\mu_k}. \tag{3.1}$$

Чтобы наборы

$$\{0, i\mu_k, m_k^+ : k = 1, \dots, N\}, \qquad \{0, i\mu_k, m_k^- : k = 1, \dots, N\}, \tag{3.2}$$

были одновременно правыми и левыми данными рассеяния одного и того же уравнения (см. [6]), необходимо и достаточно, чтобы

$$\left(m_k^- m_k^+\right)^2 = -\frac{1}{\dot{b}_0(i\mu_k)^2} = 4\mu_k^2 \prod_{j\neq k} \left(\frac{\mu_k + \mu_j}{\mu_k - \mu_j}\right)^2, \ k = 1, \dots, N. \tag{3.3}$$

Заметим также, что безотражательные потенциалы играют важную роль в теории интегрирования нелинейных дифференциальных уравнений. С их помощью строятся точные, так называемые N-солитонные, решения уравнения Картевега-де Фриза (см., например, [9] – [11]).

Пусть данные рассеяния определены равенствами (3.2), (3.3). Обозначим $\psi_k^\pm(x) := m_k^\pm e^{\mp\mu_k x}$, $k = 1, \dots, N$, $x \in \mathbb{R}$. Легко видеть (см. [6] – [8]), что при этих данных рассеяния уравнения Гельфанда-Ливитана-Марченко имеют вид

$$\sum_{k=1}^{N} m_k^+ \psi_k^+(x+y) + K_+(x,y) + \sum_{k=1}^{N} \psi_k^+(y) \int_x^\infty K_+(x,\sigma)\psi_k^+(\sigma)\, d\sigma = 0, \tag{3.4}$$

$$\sum_{k=1}^{N} m_k^- \psi_k^-(x+y) + K_-(x,y) + \sum_{k=1}^{N} \psi_k^-(y) \int_x^\infty K_-(x,\sigma)\psi_k^-(\sigma)\, d\sigma = 0. \tag{3.5}$$

Следуя [8, 9], решения (3.4) и (3.5) будем искать в виде

$$K_\pm(x,y) = -\sum_{k=1}^{N} \varphi_k^\pm(x)\psi_k^+(y). \tag{3.6}$$

Подставляя определяемые по формулам (3.6) функции K_+ и K_- соответственно в (3.4) и (3.5), мы придем к линейным уравнениям относительно φ_k^+ и φ_k^-, $k =$

$1, \dots, N$:

$$(E_N + V_+(x))\,\varphi^+(x) = \psi^+(x) \tag{3.7}$$

$$(E_n + V_-(x))\,\varphi^-(x) = \psi^-(x), \tag{3.8}$$

где $\varphi^{\pm}(x) = \left(\varphi_1^{\pm}, \dots, \varphi_N^{\pm}\right)^T$, $\psi^{\pm}(x) = \left(\psi_1^{\pm}(x), \dots, \psi_n^{\pm}(x)\right)^T$, E_N – единичная матрица и

$$V_{\pm}(x) = \left(\frac{\psi_i^{\pm}(x)\psi_j^{\pm}(x)}{\mu_i + \mu_j}\right)_{i,j=1}^{N}. \tag{3.9}$$

Однозначность решения (3.7) доказана в [9]. Поступая аналогичным образом заметим, что квадратичная форма отвечающая матрице $E_N + V_-(x)$ имеет вид

$$\sum_{k=1}^{N} X_k^2 + \int_{-\infty}^{x} \left[\sum_{j=1}^{N} \psi_j^-(\sigma) X_j\right]^2 d\sigma.$$

Из положительной определенности квадратичной формы следует единственность решения (3.8). Таким образом, функции $K_+(x,y)$ и $K_-(x,y)$, определенные формулой (3.6), являются ядрами операторов преобразований. Из (2.4) следует, что решения Йоста определяются равенствами

$$e_{\pm}(x,\lambda) = e^{i\lambda x}\left(1 - \sum_{j=1}^{N} \frac{1}{\mu_j \mp i\lambda}\,\varphi_j^{\pm}(x)\psi_j^{\pm}(x)\right). \tag{3.10}$$

В частности, пользуясь (3.7) и (3.8) получим, что

$$e_{\pm}(x, \pm i\mu_k) = \frac{1}{m_k^{\pm}}\varphi_k^{\pm}(x),$$

т.е. $\varphi_k^+(x)$ и $\varphi_k^-(x)$ являются нормированными собственными функциями соответствующих собственному значению $\lambda_k = (i\mu_k)^2$.

В [9] по существу доказано, что

$$\lim_{x\to\infty} \varphi_j^+(x) e^{\mu_j} = m_j^+, \quad j = 1, \dots, N. \tag{3.11}$$

Поступая аналогичным образом, заметим, что в силу формулы Крамера

$$\varphi_j^-(x) = \frac{1}{\det(E_N + V_-(x))} \sum_{k=1}^{N} \psi_k^-(x) Q_{jk}(x),$$

где Q_{jk} – алгебраические дополнения к элементам j-го столбца матрицы $E_N + V_-(x)$. Учитывая, что $\lim\limits_{x\to-\infty} \det(E_N + V_-(x)) = 1$ и $\lim\limits_{x\to-\infty} Q_{jk}(x) = \delta_{jk}$ (δ_{jk} – символ Кронекера), получим, что

$$\lim_{x\to-\infty} \varphi_j^-(x) e^{-\mu_j x} = m_j^-. \tag{3.12}$$

В силу линейной зависимости функций φ_j^+ и φ_j^-, из (3.11) и (3.12) следует справедливость следующего утверждения.

Предложение 3.1. *В случае безотражательного потенциала нормированные собственные функции $\varphi_j^\pm$ удовлетворяют неравенствам*

$$\left|\varphi_j^\pm(x)\right| \le c_j e^{-\mu_j|x|}, \quad x \in \mathbb{R},\ j = 1, \dots, N. \tag{3.13}$$

Кроме того справедливы равенства (3.11), (3.12).

Заметим также, что безотражательные потенциалы допускают простое описание (см. [8]-[11]), а именно

$$v(x) = -2\frac{d^2}{dx^2}\ \ln\det\left(E_N + V_\pm(x)\right).$$

При $r^\pm = 0$ и $N = 0$, потенциал $v = 0$. По этой причине мы нулевой потенциал также будем считать безотражательным.

4. Операторы преобразования в весовых пространствах

С помощью непрерывной на $\mathbb{R}_+$ ($\mathbb{R}_-$) функции ϕ и удовлетворяющей там неравенству $|\phi(x)| < ce^{-\mu x}$, $x \in \mathbb{R}_+$ ($|\phi(x)| \le ce^{\mu x}$, $x \in \mathbb{R}_-$), где μ и c положительные постоянные, построим ограниченные операторы $N_{\phi,1}^+$, $N_{\phi,2}^+ : L_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+)$, $N_{\phi,1}^-$, $N_{\phi,2}^- : L_p(\mathbb{R}_-) \to L_p(\mathbb{R}_-)$ действующие по формулам

$$\left(N_{\phi,1}^+ y\right)(x) = \int_x^\infty \phi(\sigma)y(\sigma)\,d\sigma, \qquad \left(N_{\phi,2}^+ y\right)(x) = \int_0^x \phi(\sigma)y(\sigma)\,d\sigma,$$

$$\left(N_{\phi,1}^- y\right)(x) = \int_{-\infty}^x \phi(\sigma)y(\sigma)\,d\sigma, \qquad \left(N_{\phi,2}^- y\right)(x) = \int_x^0 \phi(\sigma)y(\sigma)\,d\sigma.$$

Из формул (3.6) следует, что в случае безотражательного потенциала, операторы преобразования действуют по формулам

$$\mathrm{I} + \mathcal{K}_\pm = \mathrm{I} - \sum_{k=1}^N m\left(\varphi_k^\pm\right) N^\pm_{\psi_k^\pm,1}. \tag{4.1}$$

Операторы $\mathrm{I} - \Gamma_\pm : L_p(\mathbb{R}_\pm \to L_p(\mathbb{R}_\pm))$, $1 < p < \infty$ определим по формулам

$$\mathrm{I} - \Gamma_\pm = \mathrm{I} - \sum_{k=1}^N m\left(\psi_k^\pm\right) N^\pm_{\varphi_k^\pm,2}. \tag{4.2}$$

Оператор $\mathrm{I} - \Gamma_\pm$ действующий в пространстве $L_q(\mathbb{R}_\pm)$ ($\frac{1}{p} + \frac{1}{q} = 1$) является сопряженным к оператору $\mathrm{I}+\mathcal{K}_\pm$ действующему в $L_p(\mathbb{R}_\pm)$. Пусть $w \in A_p(\mathbb{R})$, а операторы $\pi_\pm^0 : L_p(\mathbb{R}_\pm, w) \to L_p(\mathbb{R}, w)$, $\pi_\pm : L_p(\mathbb{R}, w) \to L_p(\mathbb{R}_\pm, w)$, где $L_p(\mathbb{R}_\pm, w) :=$

$L_p\left(\mathbb{R}_\pm, w|_{\mathbb{R}_\pm}\right)$, действуют по формулам

$$\left(\pi_+^0 y\right)(x) = \begin{cases} y(x) & x \in \mathbb{R}_+ \\ 0 & x \in \mathbb{R}_- \end{cases}, \qquad \left(\pi_-^0 y\right)(x) = \begin{cases} 0 & x \in \mathbb{R}_+ \\ y(x) & x \in \mathbb{R}_- \end{cases}.$$

$(\pi_\pm y)(x) = y(x)$, $x \in \mathbb{R}_\pm$.

Определим также операторы $\mathrm{I} + \mathcal{K}$, $\mathrm{I} - \Gamma : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, действующие по формулам: $\mathrm{I} + \mathcal{K} = \pi_+^0(\mathrm{I} + \mathcal{K}_+)\pi_+ + \pi_-^0(\mathrm{I} + \mathcal{K}_-)\pi_-$, $\mathrm{I} - \Gamma = \pi_+^0(\mathrm{I} - \Gamma_+)\pi_+ + \pi_-^0(\mathrm{I} - \Gamma_-)\pi_-$.

Операторы $\mathrm{I}+\mathcal{K}_\pm$, $\mathrm{I}-\Gamma_\pm$, обратимы в $L_p(\mathbb{R}_\pm)$, а операторы $\mathrm{I}+\mathcal{K}$, $\mathrm{I}-\Gamma$ обратимы в $L_p(\mathbb{R})$. Обратные этих операторов построены в [3].

Прежде чем установить аналогичные свойства этих операторов в пространствах $L_p(\mathbb{R}_\pm, w)$, $L_p(\mathbb{R}, w)$, при $w \in A_p(\mathbb{R})$, приведем одно простое свойство весовой функции $w \in A_p(\mathbb{R})$.

Пусть $S_\mathbb{R} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ преобразование Гильберта, т.е. сингулярный интегральный оператор с ядром Коши на оси:

$$(S_\mathbb{R} y)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int\limits_{|s-x|>\varepsilon} \frac{1}{s-x}\, y(s)\, ds, \qquad x \in \mathbb{R}.$$

Как известно оператор $S_\mathbb{R}$ ограничен в $L_2(\mathbb{R})$. Кроме того (см. [12, 13]) $w \in \mathcal{A}_p(\mathbb{R})$, $1 < p < \infty$, тогда и только тогда, когда $L_2(\mathbb{R}) \cap L_p(\mathbb{R}, w)$ плотно в $L_p(\mathbb{R}, w)$ и существует константа $C_{p,w}$ такая, что $\|S_\mathbb{R} y\|_{p,w} \le C_{p,w} \|y\|_{p,w}$ для всех $y \in L_2(\mathbb{R}) \cap L_p(\mathbb{R}, w)$ одновременно. Аналогично для единичной окружности $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ через $A_p(\mathbb{T})$ обозначим множество весовых функций $\rho : \mathbb{T} \to [0, \infty]$ удовлетворяющих условию

$$\sup_{\mathrm{I}} \left(\frac{1}{|\mathrm{I}|} \int\limits_{\mathbb{T}} \rho(\tau)^p \, |d\tau| \right)^{1/p} \left(\frac{1}{|\mathrm{I}|} \int\limits_{\mathbb{T}} \rho(\tau)^{-q} \, |d\tau| \right)^{1/q}$$

где I пробегает все дуги $\mathbb{T}$, а $|\mathrm{I}|$ – длина дуги I. По заданной на $\mathbb{R}$ весовой функцией w, построим, заданную на $\mathbb{T}$, весовую функцию ρ:

$$\rho(t) = w\left(\frac{i(1+t)}{1-t}\right) |1-t|^{1-2/p}, \qquad t \in \mathbb{T}.$$

Как известно (см. [13]) $w \in A_p(\mathbb{R})$, $1 < p < \infty$, тогда и только тогда, когда $\rho \in A_p(\mathbb{T})$. Из условия $\rho \in A_p(\mathbb{T})$, $1 < p < \infty$, следует, что $\rho \in L_p(\mathbb{T})$ и $\rho^{-1} \in L_q(\mathbb{T})$ ($\frac{1}{q} + \frac{1}{p} = 1$). Замена переменной $x = i(\sigma+1)/(1-\sigma)$ позволяет условие

$$\int\limits_{\mathbb{T}} \left(w\left(\frac{i(1+t)}{1-t} \right) \right)^p |1-t|^{p-2} \, |dt| < \infty$$

записать в виде

$$\frac{1}{2}\int\limits_{\mathbb{R}} w^p(x)\left|\frac{2}{x+i}\right|^p dx<\infty,$$

т.е. $w(x)\frac{1}{x+i}\in L_p(\mathbb{R})$.

Аналогично, из условия $\rho^{-1}\in L_q(\mathbb{T})$ следует, что $w^{-1}(x)\frac{1}{x+i}\in L_q(\mathbb{R})$. Таким образом, имеет место следующий факт.

Предложение 4.1. *Если $w\in A_p(\mathbb{R})$, то*

$$\frac{w(x)}{x+i}\in L_p(\mathbb{R}) \text{ и } \frac{w^{-1}(x)}{x+i}\in L_q(\mathbb{R}), \qquad \frac{1}{p}+\frac{1}{q}=1.$$

Справедливо следующее утверждение.

Лемма 4.1. *Пусть v – безотражательный потенцил и $w\in A_p(\mathbb{R})$, $1<p<\infty$. Тогда операторы $\mathrm{I}+\mathcal{K}_\pm$, $\mathrm{I}-\Gamma_\pm$ ограничены в пространствах $L_p(\mathbb{R}_\pm,w)$, а операторы $\mathrm{I}+\mathcal{K}$, $\mathrm{I}-\Gamma$ ограничены в пространствах $L_p(\mathbb{R},w)$. Кроме того эти операторы обратимы и справедливы равенства*

$$(\mathrm{I}+\mathcal{K}_\pm)^{-1}=\mathrm{I}+\mathcal{L}_\pm:=\mathrm{I}+\sum_{k=1}^{N} m\left(\psi_k^\pm\right)N^\pm_{\varphi_k^\pm,1} \tag{4.3}$$

$$(\mathrm{I}-\Gamma_\pm)^{-1}=\mathrm{I}+Q_\pm:=\mathrm{I}+\sum_{k=1}^{N} m\left(\varphi_k^\pm\right)N^\pm_{\psi_k^\pm,2} \tag{4.4}$$

$$(\mathrm{I}+\mathcal{K})^{-1}=\pi_+^0(\mathrm{I}+\mathcal{K}_+)^{-1}\pi_+ +\pi_-^0(\mathrm{I}+\mathcal{K}_-)^{-1}\pi_+ \tag{4.5}$$

$$(\mathrm{I}-\Gamma)^{-1}=\pi_+^0(\mathrm{I}-\Gamma_+)^{-1}\pi_+ +\pi_-^0(\mathrm{I}-\Gamma_-)^{-1}\pi_+\,. \tag{4.6}$$

Доказательство. Пользуясь ограниченностью на $\mathbb{R}_+$ функции $e^{-\mu x}|x+i|$ при $\mu>0$, легко видеть, что функция $\psi=e^{-\mu x}$ $(\mu\geq 0)$ принадлежит $L_p(\mathbb{R}_+,w)$. Действительно, в силу предложения 4.1

$$\|\psi\|_{p,w}=\left(\int\limits_0^\infty e^{-\mu px}\,|x+i|^p\left(\frac{w(x)}{|x+i|}\right)^p dx\right)^{1/p}\leq c\left\|\frac{w}{x+i}\right\|_p.$$

В частности $\varphi_k^+,\psi_k^+\in L_p(\mathbb{R}_+,w)$, $k=1,\ldots,N$. Пользуясь предложением 3.1 и неравенством Гёльдера, получим

$$\left|\left(m(\psi_k^+)N^+_{\varphi_k,1}y\right)(x)\right|\leq c_k\psi_k^+(x)\int\limits_x^\infty e^{-\mu_k\tau}|y(\tau)|\,d\tau\leq$$

$$\leq c_k\psi_k^+(x)\left(\int\limits_x^\infty e^{-\mu_k q\tau}\frac{(w(\tau))^{-q}}{(\tau+i)}\,d\tau\right)^{1/q}\|y\|_{p,w}\leq c_k\left\|\frac{w^{-1}}{x+i}\right\|_q\|y\|_{p,w}\cdot\psi_k^+(x),$$

То есть

$$\left\|m\left(\psi_k^+\right) N_{\varphi_k,1}^+ y\right\|_{p,w} \leq c_k \left\|\psi_k^+\right\|_{p,w} \left\|\frac{w^{-1}}{x+i}\right\|_q \|y\|_{p,w}.$$

Аналогично доказывается ограниченность операторов $m\left(\psi_k^-\right) N_{\varphi_k^-,1}^-$, $m\left(\psi_k^\pm\right) N_{\varphi_k^\pm,2}^\pm$, $m\left(\varphi_k^\pm\right) N_{\psi_k^\pm,1}^\pm$, $m\left(\psi_k^\pm\right) N_{\varphi_k^\pm,2}^\pm$. Отсюда следует, ограниченность операторов $\mathrm{I}+\mathcal{K}_\pm$, $\mathrm{I}-\Gamma_\pm$, $\mathrm{I}+\mathcal{L}_\pm$, $\mathrm{I}+Q_\pm$.

В [3] для $y \in L_p(\mathbb{R}_\pm, w) \cap L_2(\mathbb{R}_\pm)$ доказаны тождества

$$(\mathrm{I}+\mathcal{K}_\pm)(\mathrm{I}+\mathcal{L}_\pm) y = y, \qquad (\mathrm{I}+\mathcal{L}_\pm)(\mathrm{I}+\mathcal{K}_\pm) y = y$$
$$(\mathrm{I}-\Gamma_\pm)(\mathrm{I}+Q_\pm) y = y, \qquad (\mathrm{I}+Q_\pm)(\mathrm{I}-\Gamma_\pm) y = y$$

Из всюду плотности $L_p(\mathbb{R}_\pm, w) \cap L_p(\mathbb{R}_\pm)$ в $L_p(\mathbb{R}_\pm, w)$ и ограниченности соответствующих операторов в $L_p(\mathbb{R}_\pm, w)$ следуют равенства (4.3), (4.4). Ограниченность операторов $\mathrm{I}+\mathcal{K}$, $\mathrm{I}-\Gamma$ является следствием ограниченности операторов $\mathrm{I}+\mathcal{K}_\pm$, $\mathrm{I}-\Gamma_\pm$, а формулы (4.5), (4.6) очевидны. Лемма доказана. □

5. Операторы $\mathcal{L}$-Винера-Хопфа. U-мультипликаторы

Пусть v – безотражательный потенциал порожденный правыми данными рассеяния $\{0, i\mu_k, m_k^+ : k = 1, \ldots, N\}$ (соответственно левыми данными рассеяния $\{0, i\mu_k, m_k^- : k = 1, \ldots, N\}$), $w \in \mathcal{A}_p(\mathbb{R})$, а оператор U построен по формулам (2.6)-(2.8), (3.1), (3.10). В работе [3], для оператора U получены явные представления.

Функцию $a \in L_\infty(\mathbb{R})$ будем называть U-мультипликатором в $L_p(\mathbb{R}, w)$, если отображение $f \mapsto U^* m(a) U f$ отображает $L_2(\mathbb{R}) \cap L_p(\mathbb{R}, w)$ в себя и существует постоянная $c > 0$ такая, что одновременно для всех $f \in L_2(\mathbb{R}) \cap L_p(\mathbb{R}, w)$ имеет место неравенство

$$\|U^* m(a) U f\|_{p,w} \leq c \|f\|_{p,w}$$

Поскольку $L_2(\mathbb{R}) \cap L_p(\mathbb{R}, w)$ плотно в $L_p(\mathbb{R}, w)$, то сказанное означает, что оператор $U^* m(a) U$ допускает непрерывное продолжение до действующего на $L_p(\mathbb{R}, w)$ ограниченного оператора, который мы будем обозначать через $W_{\mathcal{L}}^0(a)$ и будем называть оператором $\mathcal{L}$-свертки на $L_p(\mathbb{R}, w)$ с символом a.

Множество U-мультипликаторов будем обозначать через $\mathcal{M}_{p,w,\mathcal{L}}$. Оператор $W_{\mathcal{L}}(a) := \pi_+ W_{\mathcal{L}}^0(a) \pi_+^0 : L_p(\mathbb{R}_+, w) \to L_p(\mathbb{R}_+, w)$, $1 < p < \infty$, будем называть оператором $\mathcal{L}$-Винера-Хопфа с символом a. Поскольку при $v = 0$, оператор U совпадает с преобразованием Фурье F, то операторы $W_{\mathcal{L}}^0(a)$ и $W_{\mathcal{L}}(a)$ в этом случае совпадают соответственно определенными в весовых пространствах оператором свертки и оператором Винера-Хопфа (см. [13]). По этой причине далее в обозначениях $\mathcal{M}_{p,w,\mathcal{L}}$, $W^0_{\mathcal{L}}(a)$, $W_{\mathcal{L}}(a)$ в случае $v = 0$, мы опускаем индекс $\mathcal{L}$ и будем пользоваться обозначениями $\mathcal{M}_{p,w}$, $W^0(a)$, $W(a)$ соответственно. Класс мульпликаторов Фурье $\mathcal{M}_{p,w}$ (см. [13]) является банаховой алгеброй с нормой

$$\|a\|_{\mathcal{M}_{p,w}} := \left\|W^0(a)\right\|_{B(L_p(\mathbb{R},w))}.$$

Через $PC := PC(\dot{\mathbb{R}})$ обозначим алгебру всех кусочно-непрерывных функций на $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Другими словами, функция a принадлежит PC тогда и только тогда, когда для любого $x_0 \in \dot{\mathbb{R}}$ существуют пределы $a(x_0 - 0) := \lim\limits_{x \to x_0 - 0} a(x)$, $a(x_0 + 0) := \lim\limits_{x \to x_0 + 0} a(x)$, причем

$$a(\infty - 0) := a(+\infty) = \lim_{x \to +\infty} a(x), \qquad a(\infty + 0) := a(-\infty) = \lim_{x \to -\infty} a(x).$$

Функции из PC имеющие ограниченную вариацию $V(a)$ принадлежат алгебре $\mathcal{M}_{p,w}$ (см.17.1 [13]). Следствием этого факта является следующее утверждение

Предложение 5.1. *Определенная формулой* (3.1) *коэффициент прохождения* $t(\lambda)$ *и ее сопряженная* $\bar{t}(\lambda) = t(-\lambda) = t^{-1}(\lambda)$ *принадлежат* $\mathcal{M}_{p,w}$.

Доказательство. Рассмотрим функции $\hbar_\mu(\lambda) := (\lambda - i\mu)^{-1}$ $(\mu \in \mathbb{R} \setminus \{0\})$. Функции $\operatorname{Re} \hbar_\mu(\lambda) = \lambda(\lambda^2 + \mu^2)^{-1}$, $\operatorname{Im} \hbar_\mu(\lambda) = \mu(\lambda^2 + \mu^2)^{-1}$ имеют ограниченную вариацию и поэтому $V(\hbar_\mu) \le V(\operatorname{Re} \hbar_\mu) + V(\operatorname{Im} \hbar_\mu) < \infty$. Доказательство предложения следует теперь из равенств

$$f_\mu(\lambda) := \frac{\lambda + i_\mu}{\lambda - i\mu} = 1 + 2i\mu \frac{1}{\lambda - i\mu}, \quad t(\lambda) = \prod_{k=1}^{N} f_{\mu_k}(\lambda), \quad \bar{t}(\lambda) = \prod_{k=1}^{N} f_{-\mu_k}(\lambda)$$

и того факта, что $\mathcal{M}_{p,w}$ является алгеброй. Предложение доказано. □

Докажем теперь следующее утверждение.

Теорема 5.1. *Пусть* $w \in \mathcal{A}_p(\mathbb{R})$, $a \in \mathcal{M}_{p,w}$. *Тогда* $a \in \mathcal{M}_{p,w,\mathcal{L}}$ *и в пространстве* $L_p(\mathbb{R}_+, w)$ *справедливо тождество*

$$W_{\mathcal{L}}(a) = (\mathrm{I} + \mathcal{K}_+) W(a) (\mathrm{I} - \Gamma_+). \tag{5.1}$$

Доказательство. Для $a \in L_\infty(\mathbb{R})$ в пространстве $L_2(\mathbb{R})$ (см. [3]) справедливо тождество

$$W^0_{\mathcal{L}}(a) = (\mathrm{I} + \mathcal{K})(m(\chi_+), m(\chi_-)) \begin{pmatrix} W^0(a) & W^0(\bar{t}a) \\ W^0(ta) & W^0(a) \end{pmatrix} \begin{pmatrix} m(\chi_+) \\ m(\chi_-) \end{pmatrix} (\mathrm{I} - \Gamma). \tag{5.2}$$

В силу предыдущего предложения, из $a \in \mathcal{M}_{p,w}$ следует, что ta и $\bar{t}a$ также принадлежат $\mathcal{M}_{p,w}$ и по этой причине правая часть тождества (5.2) переводит $L_p(\mathbb{R}, w) \cap L_2(\mathbb{R})$ в себя и допускает продолжения до ограниченного оператора в пространстве $L_p(\mathbb{R}, w)$. Таким образом $a \in \mathcal{M}_{p,w,\mathcal{L}}$ и тождество (5.2) справедливо и в пространстве $L_p(\mathbb{R}, w)$.

Учитывая тождества $(\mathrm{I}-\Gamma)\pi_+^0 = \pi_+^0(\mathrm{I}-\Gamma_+)$, $\pi_+(\mathrm{I}+\mathcal{K}) = (\mathrm{I}+\mathcal{K}_+)\pi_+$, $\pi_+ m(\chi_+) = \pi_+$, $\pi_+ m(\chi_-) = 0$, $m(\chi_+)\pi_+^0 = \pi_+^0$ получим тождество (5.1). □

6. Фредгольмовость оператора $\mathcal{L}$-Винера-Хопфа

Обозначим через $PC_{p,w}$ $(C_{p,w}(\dot{\mathbb{R}}))$ замыкание всех функций PC $(C(\dot{\mathbb{R}}))$ имеющих ограниченную вариацию, в банаховой алгебре $\mathcal{M}_{p,w}$ и пусть $C_{p,w}(\bar{\mathbb{R}}) := PC_{p,w} \cap C(\bar{\mathbb{R}})$, где $\bar{\mathbb{R}} = R \cup \{\pm\infty\}$ двухточечная компактификация $\mathbb{R}$. Теорема 5.1 позволяет изучать фредгольмовы свойства оператора $W_{\mathcal{L}}(a)$ в пространствах $L_p(\mathbb{R}_+, w)$ на основе известных соответствующих свойств оператора $W(a)$ в тех же пространствах (см.17.2 [13]).

Ниже мы считаем, что $w \in \mathcal{A}_p(\mathbb{R})$, а потенциал ν является безотражательным.

Теорема 6.1. *Пусть $a \in PC_{p,w}$. Если оператор $W_{\mathcal{L}}(a)$ фредгольмов в $L_p(\mathbb{R}_+, w)$ и $\operatorname{Ind} W_{\mathcal{L}}(a) = 0$, то оператор $W_{\mathcal{L}}(a)$ обратим в пространстве $L_p(\mathbb{R}_+, w)$.*

Пусть $\nu \in (0,1), z_1, z_2 \in \mathbb{C}$. Множество

$$\mathcal{A}(z_1, z_2, \nu) := \left\{ \frac{z_2 e^{2\pi(x+i\nu)} - z_1}{e^{2\pi(x+i\nu)} - 1}, x \in \mathbb{R} \right\} \cup \{z_1, z_2\}.$$

является дугой окружности соединяющей точки z_1 и z_2 и содержащая концевые точки z_1 и z_2. Множество $\mathcal{A}(z, z, \nu)$ вырождается в точку $\{z\}$. Множество $\mathcal{A}(z_1, z_2, 1/2)$ совпадает с отрезком соединяющий точки z_1 и z_2. В случае $\nu > 1/2$ из точек $\mathcal{A}(z_1, z_2, \nu)$ отличных от z_1 и z_2, отрезок $[z_1, z_2]$ виден под углом $2\pi(1-\nu)$ и при переходе от точки z_1 к точке z_2 отрезок остается справа. В случае $\nu < 1/2$, из отличных от z_1 и z_2 точек $\mathcal{A}(z_1, z_2, \nu)$ отрезок виден под углом $2\pi\nu$ и при переходе от точки z_1 к z_2 отрезок остается слева.

Для $0 < \nu_1 \le \nu_2 < 1$ множество

$$\mathcal{H}(z_1, z_2; \nu_1, \nu_2) = \bigcup_{\nu \in [\nu_1, \nu_2]} \mathcal{A}(z_1, z_2, \nu)$$

называется рогом между z_1 и z_2 определяемое числами ν_1, ν_2 (см. [13]).

Каждое из множеств (см. [13], теорема 16.17)

$$\mathrm{I}_0(p,w) := \left\{ \lambda \in \mathbb{R} : \left| \frac{\xi}{\xi+i} \right|^{\lambda} w(\xi) \in A_p(\mathbb{R}) \right\},$$
$$\mathrm{I}_\infty(p,w) := \left\{ \lambda \in \mathbb{R} : |\xi+i|^{-\lambda}\, w(\xi) \in A_p(\mathbb{R}) \right\}$$

является открытым интервалом длиной не превышающей единицу и содержащий $0 : \mathrm{I}_x(p,w) = (-\nu_x^-(p,w); 1-\nu_x^+(p,w))$ с $0 < \nu_x^- \le \nu_x^+ < 1$ и $x = 0$ либо $x = \infty$.

Теорема 6.2. *Пусть $a \in PC_{p,w}$. Тогда существенный спектр оператора $W_{\mathcal{L}}(a)$ в $L_p(\mathbf{R}_+, w)$ совпадает с множеством*

$$G_{p,w} := \bigcup_{\alpha\in\mathbb{R}} \mathcal{H}\left(a(x-0), a(x+0); \nu_\infty^-(p,w), \nu_\infty^+(p,\infty)\right)$$
$$\bigcup \mathcal{H}\left(a(+\infty), a(-\infty); \nu_0^-(p,w), \nu_0^+(p,w)\right)$$

Если $0 \notin G_{p,w}$, то индекс оператора $W_{\mathcal{L}}(a)$ в $L_p(\mathbf{R}_+, w)$ совпадает с количеством оборотов вокруг нуля точки при ее обходе естественным образом ориентированной кривой

$$\gamma_{p,w} := \bigcup_{\alpha\in\mathbb{R}} \mathcal{A}\left(a(x-0), a(x+0); \left(\nu_\infty^-(p,w) + \nu_\infty^+(p,w)\right)/2\right)$$
$$\bigcup \mathcal{A}\left(a(+\infty), a(-\infty); \left(\nu_0^-(p,w) + \nu_0^+(p,w)\right)/2\right)$$

Теорема становится более прозрачной в случае, когда количество разрывов функции a конечно. Мы остановимся на случае, когда a непрерывна на $\mathbb{R}$ и может иметь разрыв только в бесконечности. Под $\arg a$ отличной всюду на $\bar{\mathbb{R}}$ от нуля непрерывной функции $a \in C(\bar{\mathbb{R}})$ будем понимать произвольную непрерывную функцию на $\bar{\mathbb{R}}$ удовлетворяющую равенству $a = |a|e^{i\arg a}$.

Теорема 6.3. *Пусть $a \in C(\bar{\mathbb{R}})$. Тогда следующие утверждения эквивалентны:*

(i) *$W_{\mathcal{L}}(a)$ является фредгольмовым оператором в $L_p(\mathbf{R}_+, w)$;*

(ii) *$a(x) \ne 0$ для всех $x \in \bar{\mathbb{R}}$, а число $\nu + \frac{1}{2\pi} \arg(a^{-1}(+\infty)a(-\infty))$ не является целым ни при каком $\nu \in \left[\nu_0^-, \nu_0^+\right]$;*

(iii) *$a(x) \ne 0$ для всех $x \in \bar{\mathbb{R}}$ и $a^{-1}(+\infty)a(-\infty)$ не принадлежит множеству $\{re^{2\pi i\varphi}; r \in [0,\infty], \varphi \in [1-\nu_0^+, 1-\nu_0^-]\}$.*

В случае, когда $W_{\mathcal{L}}(a)$ является фредгольмовым в $L_p(\mathbf{R}_+, w)$, его индекс вычисляется по формуле

$$\operatorname{Ind} W_{\mathcal{L}}(a) = -\frac{1}{2\pi}\left(\arg\, a(+\infty) - \arg\, a(-\infty)\right) +$$

$$\frac{1}{2}\left(\nu_0^- + \nu_0^+\right) - \left\{\frac{1}{2}\left(\nu_0^- + \nu_0^+\right) + \frac{1}{2\pi}\arg(a^{-1}(+\infty)a(-\infty))\right\},$$

где $\{x\}$ *– дробная часть числа* x.

Если $a \in C_{p,w}(\dot{\mathbb{R}})$, *то оператор* $W_{\mathcal{L}}(a)$ *фредгольмов тогда и только тогда когда* $a(x) \neq 0$ *для всех* $x \in \dot{\mathbb{R}}$ *и в этом случае* $\operatorname{Ind} W_{\mathcal{L}}(a) = \frac{1}{2\pi}(\arg a(-\infty) - \arg a(+\infty))$.

Abstract. By replacement in the definition of the Convolution operators of Fourier transform by a spectral transform of a Sturm-Liouville operator $\mathcal{L}$ generation reflectionless potential the concept $\mathcal{L}$-Wiener-Hopf operator on Lebesque spaces with Muckenhoupt weights is introduced. In the case of piecewise continuous symbol the Fredholm criterion and formula for index are obtained.

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ON THE INVERSE LQG HOMING PROBLEM

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Abstract. The problem of minimizing or maximizing the time spent by a controlled diffusion process in a given interval is known as LQG homing. The optimal control, when it is possible to obtain an explicit solution to such a problem, is often expressed as special functions. Here, the inverse problem is considered: we determine, under certain assumptions, the processes for which the optimal control is a simple power function. Moreover, the problem is extended to the case of jump-diffusion processes.

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1. Introduction

Let $\{X_u(t), t \geq 0\}$ be the one-dimensional controlled diffusion process defined by the stochastic differential equation

$$\text{(1.1)} \qquad \mathrm{d}X_u(t) = f[X_u(t)]\,\mathrm{d}t + b[X_u(t)]\,u[X_u(t)]\,\mathrm{d}t + \sigma[X_u(t)]\,\mathrm{d}B(t),$$

where $b(\cdot)$ is not identical to zero, $\sigma(\cdot)$ is non-negative and $\{B(t), t \geq 0\}$ is a standard Brownian motion. The random variable

$$\text{(1.2)} \qquad T(x) := \inf\{t > 0 : X_u(t) \notin (a, b) \mid X_u(0) = x \in (a, b)\}$$

is called a *first-passage time* in probability. The problem of finding the control $u^*(x)$ that minimizes the expected value of the cost function

$$\text{(1.3)} \qquad J(x) := \int_0^{T(x)} \left\{ \tfrac{1}{2}\, q[X_u(t)]\, u^2[X_u(t)] + \lambda \right\} \mathrm{d}t + K[X_u[T(x)]],$$

where $q(\cdot)$ is positive in (a, b) and λ is a real parameter, is a particular *LQG homing* problem; see Whittle [8]. If the parameter λ is positive (respectively, negative), then the optimizer wants the controlled process $X_u(t)$ to leave the interval (a, b) as soon (resp., late) as possible, taking the quadratic control costs and termination cost $K(\cdot)$ into account. Notice that the optimal control problem considered is time-invariant. In the general formulation, $\{X_u(t), t \geq 0\}$ can be an n-dimensional controlled diffusion process, and all the functions can depend explicitly on t. The cost function

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can also take the risk-sensitivity of the optimizer into account; see Whittle [9] or Kuhn [2] and Makasu [7]. Moreover, Lefebvre and Moutassim [6] considered the case when the uncontrolled process is a Wiener process with random parameters.

In addition to being of theoretical interest, LQG homing problems have many applications in various areas: financial mathematics, reliability theory, hydrology, etc. Recently, the author (Lefebvre [3]) has considered this type of problem in the context of epidemiology. More precisely, he considered a stochastic version of the classic three-dimensional model for the spread of epidemics due to Kermack and McKendrick. The aim was to end the epidemic as soon as possible. In practice, no one knows how long an epidemic will last. Therefore, the final time in this optimal control problem is indeed a random variable. See also Ionescu *et al.* [1].

Whittle [8] has shown that, under some conditions, it is sometimes possible to express the optimal control $u^*(x)$ in terms of a mathematical expectation for the uncontrolled process $\{X_0(t), t \geq 0\}$ obtained by setting $u[X_u(t)] \equiv 0$ in Eq. (1.1). However, solving the purely probabilistic problem is generally quite difficult, especially in two or more dimensions.

When an explicit solution to an LQG homing problem can be found, the optimal control is often expressed in terms of special functions or integrals that can only be evaluated numerically; see, for instance, Lefebvre [4]. Here, we consider the inverse problem: we will try to determine what are the problems for which the optimal control $u^*(x)$ is a simple power function, namely a constant, a linear function of x or proportional to $1/x$. Moreover, we assume that the functions $b(\cdot)$ and $q(\cdot)$ are also power functions.

To solve an LQG homing problem, we can make use of dynamic programming: we define the value function

$$F(x) = \inf_{u[X_u(t)],\, 0 \leq t \leq T(x)} E\left[J(x)\right]. \tag{1.4}$$

We can show (see Whittle [8]) that the function F satisfies the *dynamic programming equation*

$$0 = \inf_{u(x)} \left\{ \frac{1}{2}\, q(x)\, u^2(x) + \lambda + f(x)\, F'(x) + b(x)\, u(x)\, F'(x) + \frac{\sigma^2(x)}{2}\, F''(x) \right\}. \tag{1.5}$$

We deduce from the above equation that the optimal control can be expressed as follows:

$$u^*(x) = -\frac{b(x)}{q(x)}\, F'(x). \tag{1.6}$$

Hence, substituting this expression for $u^*(x)$ into Eq. (1.5), we obtain that we must solve the non-linear second-order differential equation

$$0 = -\frac{1}{2}\frac{b^2(x)}{q(x)}\,[F'(x)]^2 + \lambda + f(x)\,F'(x) + \frac{\sigma^2(x)}{2}\,F''(x). \tag{1.7}$$

This equation is valid for $a < x < b$ and is subject to the boundary conditions

$$F(a) = K(a) \quad \text{and} \quad F(b) = K(b). \tag{1.8}$$

Notice that Eq. (1.7) is a Riccati equation for $G(x) := F'(x)$.

In the next section, we will assume that the optimal control $u^*(x)$ is a certain power function of x and we will try to determine the value of the functions $b(\cdot)$, $q(\cdot)$ and $K(\cdot)$ for which this power function is indeed the exact solution to the optimal control problem. Then, in Section 3, the inverse LQG homing problem will be extended to the case of jump-diffusion processes.

2. Inverse problem

The functions $b(\cdot)$ and $q(\cdot)$ in LQG homing problems are generally power functions. Actually, they are often assumed to be respectively a non-zero and a positive constant. Here, we assume that

$$b(x) = b_0\,x^m \quad \text{and} \quad q(x) = q_0\,x^n, \tag{2.1}$$

where $b_0 \neq 0$ and $q_0 > 0$. Moreover, $m, n \in \{0, 1, 2, \ldots\}$. If $a \geq 0$ in the interval (a, b), then n can be an odd integer; otherwise, it must be an even integer (including 0). In the case of the function $K(\cdot)$, it is often chosen to be identical to zero. In this paper, it can be any real function.

Case I. Assume first that the optimal control is a constant: $u^*(x) \equiv u_0$. An important special case is the one when $u^*(x) \equiv 0$. We then deduce from Eq. (1.6) that $F(x) \equiv F_0$. Therefore, this solution can only be the exact one if $\lambda = 0$. Moreover, we must have $K(a) = K(b) = F_0$. With these assumptions, it is actually obvious that the optimizer must not use any control, for any functions $b(\cdot)$, $q(\cdot)$, $f(\cdot)$ and $\sigma(\cdot)$.

Next, if $u^*(x) \equiv u_0 \neq 0$, we obtain that

$$F'(x) = -u_0\,\frac{q(x)}{b(x)} = -\frac{u_0\,q_0}{b_0}\,x^{n-m}. \tag{2.2}$$

Equation (1.7) becomes

$$0 = -\frac{1}{2}u_0^2\,q_0\,x^n + \lambda - f(x)\,\frac{u_0\,q_0}{b_0}\,x^{n-m} - \frac{\sigma^2(x)}{2}\,\frac{u_0\,q_0}{b_0}\,(n-m)\,x^{n-m-1}. \tag{2.3}$$

In the special case when $m = n$, $F'(x)$ is a constant and Eq. (2.3) is satisfied if and only if

$$f(x) = -\frac{1}{2} u_0 b_0 x^n + \lambda \frac{b_0}{u_0 q_0}. \tag{2.4}$$

Furthermore, the final cost must be given by

$$K(a) = -\frac{u_0 q_0 a}{b_0} + F_0 \quad \text{and} \quad K(b) = -\frac{u_0 q_0 b}{b_0} + F_0, \tag{2.5}$$

where F_0 is a real constant.

Now, there are some conditions on the functions $f(\cdot)$ and $\sigma(\cdot)$ that must be satisfied for the uncontrolled process $\{X_0(t), t \geq 0\}$ to be a diffusion process. For the applications, the most important cases are the ones when $f(x) = f_0 x^p$ and $\sigma^2(x) = \sigma_0^2 x^r$, where $p \in \{-1, 0, 1\}$, $r \in \{0, 1, 2\}$, f_0 is a real constant and σ_0 is a positive constant.

Proposition 2.1. *Assume that $m = n \in \{0, 1, 2, \ldots\}$. If the conditions in Eq. (2.4) and Eq. (2.5) are satisfied, where n is such that the uncontrolled stochastic process $\{X_0(t), t \geq 0\}$ with $f(x)$ defined in Eq. (2.4) is a diffusion process, then the optimal control $u^*(x)$ is a non-zero constant u_0.*

Remark. (i) Notice that there is no explicit condition on the function $\sigma(x)$. (ii) When $n = 0$, the function $f(x)$ is a constant. Then, if $\sigma(x)$ is also a constant, $\{X_0(t), t \geq 0\}$ is a Wiener process. If $n = 1$ and $\sigma(x)$ is a constant, then $\{X_0(t), t \geq 0\}$ could be an Ornstein-Uhlenbeck process (if $u_0 b_0$ is positive). Finally, if $\lambda = 0$, $n = 1$ and $\sigma^2(x) = \sigma_0^2 x^2$, then the uncontrolled process is a geometric Brownian motion. We see that the optimal control is not equal to zero, even if $\lambda = 0$. This is due to the fact that $K(a) \neq K(b)$. (iii) When $n = 1$, we have $q(x) = q_0 x$. Because the function $q(x)$ is assumed to be positive in the interval (a, b), we must impose the additional condition $a \geq 0$. (iv) The Wiener process (or Brownian motion) is the basic diffusion process and the Ornstein-Uhlenbeck process is widely used in physics and biology, in particular. Geometric (or exponential) Brownian motion is the fundamental diffusion process in financial mathematics.

There are of course many *mathematical* cases that can be considered. However, the most frequent ones for the function $b(x)$ (respectively, $q(x)$) are those when $m = 0$ and $m = 1$ (resp., $n = 0$ and $n = 2$). Since we now assume that $m \neq n$, there are three important cases to examine: $(m, n) = (0, 2), (1, 0)$ and $(1, 2)$.

Firstly, with $(m, n) = (0, 2)$, Eq. (2.3) reduces to

$$0 = -\frac{1}{2} u_0^2 q_0 x^2 + \lambda - f(x) \frac{u_0 q_0}{b_0} x^2 - \sigma^2(x) \frac{u_0 q_0}{b_0} x. \tag{2.6}$$

Let $\lambda = 0$. Then, we can choose $f(x) \equiv f_0$ and $\sigma^2(x) = \sigma_0^2 x$, where f_0 is such that

$$f_0 = -\frac{1}{2} b_0 u_0 - \sigma_0^2. \tag{2.7}$$

Moreover, we must have $a \geq 0$ and the function $K(x)$ must satisfy the conditions

$$K(a) = -\frac{u_0 q_0 a^3}{3 b_0} + F_0 \quad \text{and} \quad K(b) = -\frac{u_0 q_0 b^3}{3 b_0} + F_0, \tag{2.8}$$

where F_0 is a real constant.

Secondly, if we choose $(m, n) = (1, 0)$, Eq. (2.3) simplifies to

$$0 = -\frac{1}{2} u_0^2 q_0 + \lambda - f(x) \frac{u_0 q_0}{b_0} x^{-1} + \frac{\sigma^2(x)}{2} \frac{u_0 q_0}{b_0} x^{-2}. \tag{2.9}$$

The most interesting particular solution is when $\{X_0(t), t \geq 0\}$ is a geometric Brownian motion with $f(x) = f_0 x$ and $\sigma^2(x) = \sigma_0^2 x^2$. Then, the various parameters must be chosen so that

$$0 = -\frac{1}{2} u_0^2 q_0 + \lambda - f_0 \frac{u_0 q_0}{b_0} + \frac{\sigma_0^2}{2} \frac{u_0 q_0}{b_0}. \tag{2.10}$$

This time, λ could be any real number. Because the optimally controlled process $\{X_{u^*}(t), t \geq 0\}$ is also a geometric Brownian motion, with infinitesimal mean $(f_0 + b_0 u_0) x$, and geometric Brownian motions are strictly positive (or strictly negative), we should assume that $a > 0$. Moreover, the function $K(x)$ must satisfy the following conditions:

$$K(a) = -\frac{u_0 q_0 \ln(a)}{b_0} + F_0 \quad \text{and} \quad K(b) = -\frac{u_0 q_0 \ln(b)}{b_0} + F_0, \tag{2.11}$$

for a certain constant F_0.

Thirdly, when $(m, n) = (1, 2)$, we deduce from Eq. (2.3) that

$$0 = -\frac{1}{2} u_0^2 q_0 x^2 + \lambda - f(x) \frac{u_0 q_0}{b_0} x - \frac{\sigma^2(x)}{2} \frac{u_0 q_0}{b_0}. \tag{2.12}$$

There are two interesting particular solutions: as above, if $\{X_0(t), t \geq 0\}$ is a geometric Brownian motion with $f(x) = f_0 x$ and $\sigma^2(x) = \sigma_0^2 x^2$, and if $\lambda = 0$, we must have

$$f_0 = -\frac{1}{2} (b_0 u_0 + \sigma_0^2) \tag{2.13}$$

and $a > 0$. Furthermore, $\{X_0(t), t \geq 0\}$ could be an Ornstein-Uhlenbeck process with infinitesimal mean $f_0 x$ and infinitesimal variance σ_0^2, where

$$f_0 = -\frac{b_0 u_0}{2} \ (< 0) \quad \text{and} \quad \sigma_0^2 = \frac{2 \lambda b_0}{u_0 q_0} \ (> 0). \tag{2.14}$$

Finally, in both cases the conditions

$$K(a) = -\frac{u_0 q_0 a^2}{2 b_0} + F_0 \quad \text{and} \quad K(b) = -\frac{u_0 q_0 b^2}{2 b_0} + F_0, \tag{2.15}$$

where F_0 is a real constant, must be satisfied.

Case II. Assume now that the optimal control $u^*(x)$ is linear: $u^*(x) = u_0\, x$, where $u_0 \neq 0$. Equation (1.6) then implies that

$$F'(x) = -u_0\, x\, \frac{q(x)}{b(x)} = -\frac{u_0\, q_0}{b_0}\, x^{n-m+1}, \tag{2.16}$$

so that Eq. (1.7) takes the form

$$0 = -\frac{1}{2}\, u_0^2\, q_0\, x^{n+2} + \lambda - f(x)\, \frac{u_0\, q_0}{b_0}\, x^{n-m+1} - \frac{\sigma^2(x)}{2}\, \frac{u_0\, q_0}{b_0}\, (n-m+1)\, x^{n-m}. \tag{2.17}$$

Here the special case is when $m = n+1$; then, $F'(x)$ is a constant and we find that Eq. (2.17) is satisfied if and only if (iff) the function $f(x)$ is such that

$$f(x) = -\frac{1}{2}\, u_0\, b_0\, x^{n+2} + \lambda\, \frac{b_0}{u_0\, q_0}. \tag{2.18}$$

As in Case I, the function $K(x)$ must satisfy the conditions in Eq. (2.5).

Proposition 2.2. *Assume that $m = n+1 \in \{1, 2, \ldots\}$. If the function $f(x)$ can be expressed as in Eq. (2.18), where n is such that the uncontrolled stochastic process $\{X_0(t), t \geq 0\}$ is a diffusion process, and if the final cost satisfies both conditions in Eq. (2.5), then the optimal control $u^*(x)$ is linear: $u^*(x) = u_0\, x$, where $u_0 \neq 0$.*

Remark. Again, we can choose any admissible function $\sigma(x)$. The most interesting case is when $n = 0$ and $\sigma^2(x) = \sigma_0^2\, x^2$.

When $m = n$, Eq. (2.17) becomes

$$0 = -\frac{1}{2}\, u_0^2\, q_0\, x^{n+2} + \lambda - f(x)\, \frac{u_0\, q_0}{b_0}\, x - \frac{\sigma^2(x)}{2}\, \frac{u_0\, q_0}{b_0}. \tag{2.19}$$

With $n = 0$, there are two important solutions: firstly, we can have

$$f(x) = \frac{\lambda\, b_0}{u_0\, q_0\, x} \quad \text{and} \quad \sigma^2(x) = -b_0\, u_0\, x^2, \tag{2.20}$$

provided that $b_0\, u_0 < 0$. If $\lambda = 0$, the uncontrolled process is then a geometric Brownian motion. Secondly, we can also have

$$f(x) = -\frac{b_0\, u_0}{2}\, x \quad \text{and} \quad \sigma^2(x) \equiv \frac{2\,\lambda\, b_0}{u_0\, q_0}\ (>0). \tag{2.21}$$

This time, if $b_0\, u_0 > 0$, $\{X_0(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process. Furthermore, in both cases Eq. (2.15) must be satisfied.

To conclude this part, let us consider the case when $m = n-1 \in \{0, 1, \ldots\}$; we deduce from Eq. (2.17) that

$$0 = -\frac{1}{2}\, u_0^2\, q_0\, x^{n+2} + \lambda - f(x)\, \frac{u_0\, q_0}{b_0}\, x^2 - \sigma^2(x)\, \frac{u_0\, q_0}{b_0}\, x. \tag{2.22}$$

If $\lambda = 0$ and $n = 1$, we can take $f(x) = f_0\, x$ and $\sigma^2(x) = \sigma_0^2\, x^2$, provided that $a > 0$, the conditions in Eq. (2.8) are satisfied and

$$f_0 = -\frac{1}{2}\, b_0\, u_0 - \sigma_0^2. \tag{2.23}$$

Case III. Lastly, suppose that $a > 0$ and that the optimal control $u^*(x)$ is inversely proportional to x: $u^*(x) = u_0/x$, where $u_0 \neq 0$. We then deduce from Eq. (1.6) that

$$F'(x) = -\frac{u_0}{x}\,\frac{q(x)}{b(x)} = -\frac{u_0\,q_0}{b_0}\,x^{n-m-1}. \tag{2.24}$$

It follows that Eq. (1.7) becomes

$$0 = -\frac{1}{2}\,u_0^2\,q_0\,x^{n-2} + \lambda - f(x)\,\frac{u_0\,q_0}{b_0}\,x^{n-m-1} - \frac{\sigma^2(x)}{2}\,\frac{u_0\,q_0}{b_0}\,(n-m-1)\,x^{n-m-2}. \tag{2.25}$$

The function $F'(x)$ is a constant when $n - m = 1$. Equation (2.25) is then satisfied iff

$$f(x) = -\frac{1}{2}\,u_0\,b_0\,x^{n-2} + \lambda\,\frac{b_0}{u_0\,q_0}. \tag{2.26}$$

As in the previous cases, the termination cost function $K(x)$ must satisfy both conditions in Eq. (2.5).

Proposition 2.3. *Assume that $m = n - 1 \in \{0, 1, 2, \ldots\}$ and that the function $f(x)$ can be expressed as in Eq. (2.26), where $n \geq 1$ is such that the uncontrolled stochastic process $\{X_0(t), t \geq 0\}$ is a diffusion process. If the two conditions in Eq. (2.5), with $a > 0$, are satisfied, then the optimal control $u^*(x)$ is inversely proportional to x: $u^*(x) = u_0/x$, where $u_0 \neq 0$.*

Remark. As above, we can choose any admissible function $\sigma(x)$. When $n = 1$ and $\lambda = 0$, we have

$$f(x) = -\frac{u_0\,b_0}{2\,x}. \tag{2.27}$$

Then, if $\sigma^2(x) \equiv \sigma_0^2$, the uncontrolled process could be a Bessel process, which is another important diffusion process. Moreover, if $n = 2$,

$$f(x) \equiv f_0 = -\frac{1}{2}\,u_0\,b_0 + \lambda\,\frac{b_0}{u_0\,q_0}. \tag{2.28}$$

If we choose $\sigma^2(x) \equiv \sigma_0^2$, then $\{X_0(t), t \geq 0\}$ is a Wiener process.

Let us finally consider the particular case when $(m, n) = (0, 2)$. The conditions in Eq. (2.15) must then be satisfied. Moreover, Eq. (2.25) reduces to

$$0 = -\frac{1}{2}\,u_0^2\,q_0 + \lambda - f(x)\,\frac{u_0\,q_0}{b_0}\,x - \frac{\sigma^2(x)}{2}\,\frac{u_0\,q_0}{b_0}. \tag{2.29}$$

There are two interesting solutions: firstly, we can have $f(x) = f_0/x$ and $\sigma^2(x) \equiv \sigma_0^2$, with

$$f_0 = -\frac{1}{2}\left(u_0\,b_0 + \sigma_0^2\right) + \lambda\,\frac{b_0}{u_0\,q_0}. \tag{2.30}$$

Hence, $\{X_0(t), t \geq 0\}$ could be a Bessel process. Secondly, we can also have

$$\lambda = \frac{1}{2}\,u_0^2\,q_0 \quad \text{and} \quad f(x) = -\frac{\sigma^2(x)}{2\,x}. \tag{2.31}$$

An important case is the one for which $\sigma^2(x) = \sigma_0^2 x^2$, so that $f(x) = -\sigma_0^2 x/2$. The uncontrolled process is then a geometric Brownian motion. If $\sigma^2(x) \equiv \sigma_0^2$, then $\{X_0(t), t \geq 0\}$ could again be a Bessel process.

3. Jump-diffusion processes

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate α, and $\{Y_i, i = 1, 2, \ldots\}$ be independent and identically distributed (i.i.d.) random variables having the common probability density function $f_Y(y)$. In this section, we will extend the inverse LQG homing problem to the case when $\{X_u(t), t \geq 0\}$ is a controlled jump-diffusion process defined by

$$\begin{aligned} X_u(t) &= X_u(0) + \int_0^t \{f[X(s)] + b[X_u(s)]\, u[X_u(s)]\}\, \mathrm{d}s \\ &\quad + \int_0^t \sigma[X_u(s)]\, \mathrm{d}B(s) + \sum_{i=1}^{N(t)} Y_i. \end{aligned} \tag{3.1}$$

The stochastic processes $\{N(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ are assumed to be independent. Jump-diffusion processes are widely used in financial mathematics, among other fields.

The ordinary differential equation satisfied by the value function $F(x)$ becomes an integro-differential equation (see Lefebvre [5]):

$$\begin{aligned} 0 = -\frac{1}{2}\frac{b^2(x)}{q(x)}\,[F'(x)]^2 + \lambda + f(x)\,F'(x) + \frac{\sigma^2(x)}{2}\,F''(x) \\ + \ \alpha\left\{\int_{-\infty}^{\infty} F(x+y)\, f_Y(y)\,\mathrm{d}y - F(x)\right\}. \end{aligned} \tag{3.2}$$

Moreover, because there can now be an overshoot, the boundary conditions become

$$F(x) = K(x) \quad \text{if } x \notin (a, b). \tag{3.3}$$

In the case when the jump size is a constant ϵ, so that $f_Y(y)$ becomes the Dirac delta function $\delta(y-\epsilon)$, the above integro-differential equation is reduced to a differential-difference equation:

$$0 = -\frac{1}{2}\frac{b^2(x)}{q(x)}\,[F'(x)]^2 + \lambda + f(x)\,F'(x) + \frac{\sigma^2(x)}{2}\,F''(x) + \alpha\,[F(x+\epsilon) - F(x)]\,. \tag{3.4}$$

Although Eq. (3.2) is obviously difficult to solve explicitly, by choosing the functions f_Y and K appropriately, the variety of problems for which the optimal control $u^*(x)$ is a power of x is very large. We will present below various examples of such problems. The same cases for $u^*(x)$ as in the preceding section will be considered. In financial mathematics, an example of an LQG homing problem might consist in finding the optimal investment policy when the investor decides to sell his/her shares of a given company the first time they reach a certain level, which is a random time. An optimal solution that is a simple power function is very easy to implement.

Case I. If $u^*(x) \equiv 0$, so that $F(x) \equiv F_0$, we saw in Section 2 that we must then have $\lambda = 0$ and $K(a) = K(b) = F_0$, so that the value of the optimal control was obvious. However, in the case of jump-diffusion processes, we can have $u^*(x) \equiv 0$ in non-trivial problems. Indeed, assume that $[a, b] = [0, 1]$ and that $Y \sim \mathrm{U}[-2, 2]$; that is, Y is uniformly distributed on the interval $[-2, 2]$. Then, we have $X[T(x)] \in (-2, 0]$ or $[1, 3)$. Let us define

$$I(x) = \int_{-\infty}^{\infty} F(x+y)\, f_Y(y)\, \mathrm{d}y. \tag{3.5}$$

We may write that

$$\begin{aligned} I(x) &= \frac{1}{4}\left\{\int_{-2}^{-x} K(x+y)\,\mathrm{d}y + \int_{-x}^{1-x} F(x+y)\,\mathrm{d}y + \int_{1-x}^{2} K(x+y)\,\mathrm{d}y\right\} \\ &= \frac{1}{4}\left\{\int_{-2}^{-x} K(x+y)\,\mathrm{d}y + F_0 + \int_{1-x}^{2} K(x+y)\,\mathrm{d}y\right\}. \end{aligned} \tag{3.6}$$

Let $K(0) = K(1) = F_0$ (as required), but $K(x) \equiv F_1$ if $x \in (-2, 0)$ or $x \in (1, 3)$. We have $I(x) = \frac{1}{4}\,(3F_1 + F_0)$. We therefore may state that Eq. (3.2) is satisfied if and only if

$$\lambda + \frac{3\alpha}{4}\,(F_1 - F_0) = 0. \tag{3.7}$$

Thus, when $F_1 \neq F_0$, so that $\lambda \neq 0$ as well, the optimal strategy is nevertheless to use no control at all. This example can obviously be generalized.

Remark. The function $K(x)$ is not necessarily continuous. In fact, it is natural to have a possibly different final cost when there is an overshoot.

Next, in the case when $u^*(x) \equiv u_0 \neq 0$, the value function must be of the form

$$F(x) = \kappa\,\frac{x^{n-m+1}}{n-m+1} + F_0, \tag{3.8}$$

where

$$\kappa := -\frac{u_0\, q_0}{b_0}. \tag{3.9}$$

For the sake of brevity and simplicity, we will assume that $n = m$, so that $F(x) = \kappa x + F_0$. We take again $[a, b] = [0, 1]$, and we choose $K(x) = F(x)$, for $x \notin (0, 1)$. Then, if $Y \sim \mathrm{U}[-2, 2]$ (as above), we calculate

$$I(x) = \frac{1}{4}\int_{-2}^{2} [\kappa\,(x+y) + F_0]\,\mathrm{d}y = \kappa x + F_0, \tag{3.10}$$

so that we return to the case when there are no jumps, that is, $\alpha = 0$. Instead, let us take $Y \sim \mathrm{U}(0,1)$, which implies that there are only positive jumps. With this choice, we have

$$I(x) = \int_0^1 [\kappa(x+y) + F_0]\,\mathrm{d}y = \kappa\left(x + \frac{1}{2}\right) + F_0 = F(x) + \frac{\kappa}{2}. \tag{3.11}$$

It follows that Eq. (3.2) reduces to

$$0 = -\frac{1}{2}\frac{b_0^2}{q_0}\kappa^2 + \lambda + f(x)\kappa + \frac{\alpha\kappa}{2} = \lambda + \kappa\left(\frac{1}{2}b_0 u_0 + f(x) + \frac{\alpha}{2}\right). \tag{3.12}$$

Therefore, $f(x)$ must be a constant f_0 such that the above equation is satisfied, and we can choose any admissible infinitesimal variance $\sigma^2(x)$. In particular, the continuous part of the process $\{X_u(t), t \geq 0\}$ could be a controlled Brownian motion.

Case II. We make the following assumptions: the optimal control $u^*(x)$ is of the form $u^*(x) = u_0 x$, where $u_0 \neq 0$, the interval $[a, b]$ is $[0, 1]$ and $m = n = 0$. We have

$$F(x) = \kappa\frac{x^2}{2} + F_0 \quad \text{for } x \in (0,1). \tag{3.13}$$

As above, we choose $K(x) = F(x)$ for $x \notin (0,1)$. With $Y \sim \mathrm{U}(-2,2)$, we obtain that

$$I(x) = \frac{1}{4}\int_{-2}^{2}\left[\kappa\frac{(x+y)^2}{2} + F_0\right]\mathrm{d}y = F(x) + \frac{2\kappa}{3}. \tag{3.14}$$

Then, Eq. (3.2) becomes

$$0 = -\frac{1}{2}q_0 u_0^2 x^2 + \lambda + f(x)\kappa x + \frac{\kappa}{2}\sigma^2(x) + \frac{2\alpha\kappa}{3}. \tag{3.15}$$

There are numerous important processes for which the above equation holds, including the cases when the continuous part of $\{X_0(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process, or a geometric Brownian motion.

Case III. Suppose that $[a, b] = [1, 2]$, $m = 0$, $n = 1$ and $u^*(x) = u_0/x$, where $u_0 \neq 0$. The value function $F(x)$ becomes

$$F(x) = \kappa x + F_0. \tag{3.16}$$

With $K(x) = F(x)$ for $x \notin (1,2)$ and $Y \sim \mathrm{U}(0,1)$, we have

$$I(x) = \int_0^1 [\kappa(x+y) + F_0]\,\mathrm{d}y = F(x) + \frac{\kappa}{2}. \tag{3.17}$$

Hence, Eq. (3.2) is

$$0 = -\frac{q_0 u_0^2}{2x} + \lambda + f(x)\kappa + \frac{\alpha\kappa}{2}. \tag{3.18}$$

The continuous part of $\{X_0(t), t \geq 0\}$ could be a Bessel process.

4. Conclusion

In this paper, we obtained various explicit and exact solutions to LQG homing problems for important one-dimensional diffusion processes by considering the inverse problem. Instead of trying to find the solution to the appropriate non-linear second-order differential equation satisfied by the value function, from which the optimal control follows at once, we looked for problems for which the optimal control $u^*(x)$ was either a constant, a linear function of x or inversely proportional to x. We saw that there are indeed interesting problems for which the exact solution is simple.

We could have considered other cases, but the aim was to present solutions to realistic problems involving important diffusion processes, such as the Wiener process and geometric Brownian motion, rather than purely mathematical examples. Moreover, we could of course consider other particular forms for the optimal control; for instance, the case when $u^*(x)$ is a quadratic function of x is of interest.

Finally, in Section 3 we presented an extension of the inverse LQG homing problem to the important case of jump-diffusion processes. Although the equation satisfied by the value function is much more complicated, we saw that it is possible to find many interesting examples for which the optimal control u_0^* is a constant or a power of x.

As a sequel to this paper, we could consider multidimensional LQG homing problems, either for diffusion or jump-diffusion processes. There are few such problems that have been solved explicitly and exactly so far in two or more dimensions, because the equation satisfied by the value function is then a non-linear partial differential (or integro-differential) equation. Therefore it would indeed be interesting to find important problems that actually have simple solutions.

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FEKETE-SZEGÖ INEQUALITIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS RELATED WITH NEPHROID DOMAIN

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Abstract. The purpose of this paper is to consider coefficient estimates in a class of functions $\mathscr{M}_{\alpha,\,\lambda}(q)$ consisting of analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ subordinating with nephroid domain, to derive certain coefficient estimates a_2, a_3 and Fekete-Szegö inequality for $f \in \mathscr{M}_{\alpha,\,\lambda}(q)$. A similar result have been done for the function f^{-1}. Further application of our results to certain functions defined by convolution products with a normalized analytic function is given, and in particular we obtain Fekete-Szegö inequalities for certain subclasses of functions defined through neutrosophic Poisson distribution.

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Keywords: analytic function; starlike function; convex function; subordination; Fekete-Szegö inequality; neutrosophic Poisson distribution; Hadamard product.

1. Introduction

Let $\mathscr{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\Delta := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent functions. A function $f \in \mathscr{S}$ is said to be *starlike* in Δ if and only if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > 0, \qquad (z \in \Delta)$$

and on the other hand,a function $f \in \mathscr{S}$ is said to be *convex* in Δ if and only if

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0, \quad (z \in \Delta)$$

denoted by $\mathscr{S}^*$ and $\mathscr{C}$ respectively.

Let f_1 and f_2 be functions analytic in Δ. Then we say that the function f_1 is subordinate to f_2 if there exists a Schwarz function $w(z)$, analytic in Δ with

$w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$, such that $f_1(z) = f_2(w(z))$ $(z \in \Delta)$. We denote this subordination by

$$f_1 \prec f_2 \quad \text{or} \quad f_1(z) \prec f_2(z) \quad (z \in \Delta).$$

In particular, if the function f_2 is univalent in Δ, the above subordination is equivalent to $f_1(0) = f_2(0)$ *and* $f(\Delta) \subset f_2(\Delta)$. The function $q(z) = 1 + z - \frac{z^3}{3}$ maps Δ onto the region bounded by the nephroid

$$\left((u-1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4v^2}{3} = 0,$$

which is symmetric about the real axis and lies completely inside the right-half plane $u > 0$. Geometrically, a nephroid is the locus of a point on the circumference of a circle of radius ρ traversing positively the outside of a fixed circle of radius 2ρ. It is an algebraic curve of degree six and is an epicycloid having two cusps. The plane curve nephroid was studied by Huygens and Tschirnhausen around 1679 in connection with the theory of caustics, a method of deriving a new curve based on a given curve and a point. In 1692, J. Bernoulli showed that the nephroid is the catacaustic (envelope of rays emanating from a specified point) of a cardioid for a luminous cusp. However, the name nephroid, which means kidney shaped, was first used by the English mathematician Richard A. Proctor in 1878 in his book "The Geometry of Cycloids". (For more details see [20] and references cited therein)

Definition 1.1. [20] Let $\mathscr{S}^*(q)$ denote the class of analytic functions f in the unit disc Δ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec 1 + z - \frac{z^3}{3} =: q(z), \quad z \in \Delta. \tag{1.2}$$

and $\mathscr{C}(q)$ if

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + z - \frac{z^3}{3} =: q(z), \quad z \in \Delta. \tag{1.3}$$

Further they proved by considering, $q(z)$ as a holomorphic solution of the differential equation

$$\frac{zq'(z)}{q(z)} = 1 + z - \frac{z^3}{3}, \quad z \in \Delta,\ q(0) = 0,\ q'(0) = 1,$$

i.e.

$$\Omega_n(z) = z\exp\left(\int_0^z \frac{q(\zeta^{n-1}) - 1}{\zeta} d\zeta\right) = z + \frac{z^n}{n-1} + \frac{z^{2n-1}}{2(n-1)^2} + \cdots, \quad z \in \Delta \tag{1.4}$$

plays the extremal role of the class $\mathscr{S}_q^*$ as noted by Wani and Swaminathan [20]. Also

$$\Upsilon_n(z) = \exp\left(\int_0^z \frac{q(\zeta^{n-1}) - 1}{\zeta} d\zeta\right) = z + \frac{z^n}{n(n-1)} + \frac{z^{2n-1}}{2(2n-1)(n-1)^2} + \cdots, \tag{1.5}$$

$z \in \Delta$, plays the extremal role of the class $\mathscr{C}_q$ as noted by Wani and Swaminathan [20]. It may be noted from (1.3) of Definition1.1 that the set $q(\Delta)$ lies in the right half-plane and it is not a starlike domain with respect to the origin.

Recently, Raina and Sokol [15] have studied and obtained some coefficient inequalities for the class $\mathscr{S}^\star(z+\sqrt{1+z^2})$ and these results are further improved by Sokol and Thomas [19] further the Fekete-Szegö inequality for functions in the class $\mathscr{C}(q)$ were obtained and in view of the Alexander result between the class $\mathscr{S}^*(z+\sqrt{1+z^2})$ and $\mathscr{C}(z+\sqrt{1+z^2})$, the Fekete-Szegö inequality for functions in $\mathscr{S}^*(z+\sqrt{1+z^2})$ were also obtained. For a brief history of Fekete-Szegö problem for the class of starlike, convex and various other subclasses of analytic functions, we refer the interested reader to [18]. Let $\alpha \geq 0,\ \lambda \geq 0$ and $0 \leq \rho < 1$ and $f \in \mathscr{A}$. We say that $f \in M(\alpha, \lambda, \rho)$ if it satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} > \rho.$$

The class $M(\alpha, \lambda, \rho)$ was introduced by Guo and Liu [4].

Motivated essentially by the aforementioned works, (see [15, 17] and [1]) in this paper we define the following class $\mathscr{M}_{\alpha,\lambda}(q)$ given in Definition1.2. First,we shall find estimations of first few coefficients of functions f of the form (1.1) belonging to $\mathscr{M}_{\alpha,\lambda}(q)$ and we prove the Fekete-Szegö inequality $f \in \mathscr{M}_{\alpha,\lambda}(q)$ and also for $f^{-1} \in \mathscr{M}_{\alpha,\lambda}(q)$. Also we give applications of our results to certain functions defined through Poisson distribution .

Now, we define the following class $\mathscr{M}_{\alpha,\lambda}(q)$:

Definition 1.2. For $\alpha \geq 0$, $\lambda \geq 0$ a function $f \in \mathscr{A}$ is in the class $\mathscr{M}_{\alpha,\lambda}(q)$ if

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\}$$

$$\prec \quad 1 + z - \frac{z^3}{3} = q(z);\ z = re^{i\theta} \in \Delta. \tag{1.6}$$

Note that by specializing the parameter we get the following subclasses based on nephroid domain (see [20]).

- $\mathscr{M}_{0,0}(q) \equiv \mathscr{S}^*(q) = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec q(z) = 1 + z - \frac{z^3}{3}, z \in \Delta\right\}$
- $\mathscr{M}_{0,1}(q) \equiv \mathscr{C}(q) = \left\{f \in \mathcal{A} : \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q(z) = 1 + z - \frac{z^3}{3}, z \in \Delta\right\}$
- $\mathscr{M}_{0,\lambda}(q) \equiv \mathscr{M}_\lambda(q)$
  $= \{f \in \mathcal{A} : (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q(z) = 1 + z - \frac{z^3}{3}, z \in \Delta\}$
- $\mathscr{M}_{\alpha,0}(q) \equiv \mathscr{B}^\alpha(q) = \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} \prec q(z) = 1 + z - \frac{z^3}{3}, z \in \Delta\}$ . .

2. A Coefficient Estimate

To prove our main result, we need the following:

Lemma 2.1. [8] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2, & \text{if} \quad v \leq 0, \\ 2, & \text{if} \quad 0 \leq v \leq 1, \\ 4v - 2, & \text{if} \quad v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $\dfrac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_1(z)$ is $\dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \quad (0 \leq \eta \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).$$

We also need the following:

Lemma 2.2. [3] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|\, c_n \,| \leq 2 \;\; for\ all\ n \geq 1 \qquad \text{and} \qquad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

The class of all such functions with positive real part are denoted by $\mathscr{P}$.

Lemma 2.3. [7] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leq 2\max(1, |2v - 1|).$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 2.4. [6] *Let* $P(z) = 1+c_1z+c_2z^2+c_3z^3+.....$ *be in* $\mathcal{P}$ *then for any complex number* μ,

$$\left|c_2 - \mu\frac{c_1^2}{2}\right| \leq \max\{2, 2|\mu-1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2; \\ 2|\mu-1|, & elsewhere. \end{cases}$$

The result is sharp for the functions defined by $P(z) = \frac{1+z^2}{1-z^2}$ *or* $P(z) = \frac{1+z}{1-z}$.

Theorem 2.1. *Let* $\alpha \geq 0$ *and* $\lambda \geq 0$. *If* $f(z)$ *given by* (1.1) *belongs to* $\mathscr{M}_{\alpha,\lambda}(q)$, *then*

$$\begin{aligned} |a_2| &\leq \frac{1}{(1+\alpha)(1+\lambda)}, \\ |a_3| &\leq \frac{1}{(\alpha+2)(1+2\lambda)}\max\{1, \left|\left(\frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2((1+\alpha)(1+\lambda))^2}\right)\right|\}. \end{aligned}$$

Proof. If $f \in \mathscr{M}_{\alpha,\lambda}(q)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\}$$

$$= q(w(z)) = 1 + w(z) - \frac{(w(z))^3}{3}. \tag{2.1}$$

Define the function $P(z)$ by

$$P(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \cdots$$

it is easy to see that

$$w(z) = \frac{P(z)-1}{P(z)+1} = \frac{1}{2}\left[c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right]. \tag{2.2}$$

Since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Let us define the function $p(z)$ by

$$\begin{aligned} p(z) :&= \left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} \\ &= 1 + b_1z + b_2z^2 + \cdots. \end{aligned} \tag{2.3}$$

In view of the equations (2.1), (2.2), (2.3), we have

$$p(z) = q\left(\frac{P(z)-1}{P(z)+1}\right). \tag{2.4}$$

Hence

(2.5)

$$\begin{aligned} 1 + w(z) - \frac{(w(z))^3}{3} &= 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right)z^3 - \frac{{c_1}^3}{24}z^3\cdots \\ &= 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{{c_1}^3}{12}\right)z^3 + \cdots, \quad z \in \mathbb{D}. \end{aligned}$$

Using (2.2) in (2.4), we get,

$$b_1 = \frac{c_1}{2} \qquad \text{and} \qquad b_2 = \frac{c_2}{2} - \frac{c_1^2}{4}.$$

A computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3a_2)z^3 + \cdots.$$

Similarly we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + \cdots.$$

An easy computation shows that

$$\begin{aligned} &\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} \\ &= 1 + (1+\alpha)(1+\lambda)a_2 z + (\alpha+2)(1+2\lambda)a_3 z^2 \\ &+ \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right)a_2^2 z^2 + \cdots. \end{aligned}$$

In view of the equation (2.3), we see that

$$b_1 = (1+\alpha)(1+\lambda)a_2 \tag{2.6}$$

$$b_2 = (\alpha+2)(1+2\lambda)a_3 + \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right)a_2^2 \tag{2.7}$$

or equivalently, we have

$$a_2 = \frac{c_1}{2(1+\alpha)(1+\lambda)}, \tag{2.8}$$

$$\begin{aligned} a_3 &= \frac{1}{(\alpha+2)(1+2\lambda)}\left(\frac{c_2}{2} - \frac{c_1^2}{4}\left[1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right]\right), \\ &= \frac{1}{2(\alpha+2)(1+2\lambda)}\left(c_2 - \frac{c_1^2}{2}\left[1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right]\right) \qquad (2.9) \\ &= \frac{1}{2(\alpha+2)(1+2\lambda)}\left(c_2 - vc_1^2\right) \end{aligned}$$

where

$$v = \frac{1}{2}\left(1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right). \tag{2.10}$$

Therefore, we have

$$|a_2| \le \frac{1}{(1+\alpha)(1+\lambda)}$$

and by using the estimate

$$|c_2 - vc_1^2| \le 2\max(1, |2v-1|)$$

given in Lemma 2.3,we have

$$\begin{aligned}|a_3| &\leq \frac{1}{(\alpha+2)(1+2\lambda)}\max\{1,\left|2\times\frac{1}{2}\left(1+\frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right)-1\right|\}\\ &= \frac{1}{(\alpha+2)(1+2\lambda)}\max\{1,\left|\left(\frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right)\right|\}\end{aligned}$$

Remark 2.1. Let $\alpha=0$ and $\lambda\geq 0$. If $f(z)$ given by (1.1) belongs to $\mathscr{M}_\lambda(q)$, then

$$\begin{aligned}|a_2| &\leq \frac{1}{1+\lambda},\\ |a_3| &\leq \frac{1}{2(1+2\lambda)}\max\{1,\left|\frac{3\lambda+1}{2(1+\lambda)^2}\right|\}=\frac{3\lambda+1}{4(1+2\lambda)(1+\lambda)^2}.\end{aligned}$$

Remark 2.2. Let $\lambda=0$. If $f(z)$ given by (1.1) belongs to $\mathscr{B}^\alpha(q)$, then

$$|a_2|\leq\frac{1}{1+\alpha},\qquad \textit{and}\qquad |a_3|\leq\frac{1}{\alpha+2}\max\{1,\left|\left(\frac{\alpha^2+\alpha--2}{2\left(1+\alpha\right)^2}\right)\right|\}.$$

Remark 2.3. (see [20]) Let $\alpha=0$ and $\lambda=0$. If $f(z)$ given by (1.1) belongs to $\mathscr{S}^*(q)$, then

$$|a_2|\leq 1,\qquad \textit{and}\qquad |a_3|\leq\frac{1}{2}\max\{1,\left|\frac{1}{2}\right|\}=\frac{1}{2}.$$

Remark 2.4. (see [20]) Let $\alpha=0$ *and* $\lambda=1$. If $f(z)$ given by (1.1) belongs to $\mathscr{C}(q)$, then

$$|a_2|\leq\frac{1}{2},\qquad \textit{and}\qquad |a_3|\leq\frac{1}{6}\max\{1,\left|\frac{1}{2}\right|\}=\frac{1}{6}.$$

Theorem 2.2. *Let* $0\leq\mu\leq 1,\ \alpha\geq 0$ *and* $\lambda\geq 0$. *If* $f(z)$ *given by* (1.1) *belongs to* $\mathscr{M}_{\alpha,\lambda}(q)$, *then*

$$|a_3-\mu a_2^2| \leq \begin{cases}\dfrac{1}{2\xi}\left(-\dfrac{\gamma}{\tau^2}\right), & \text{if}\quad \mu\leq\sigma_1,\\[2ex] \dfrac{1}{\xi}, & \text{if}\quad \sigma_1\leq\mu\leq\sigma_2,\\[2ex] \dfrac{1}{2\xi}\left(\dfrac{\gamma}{\tau^2}\right), & \text{if}\quad \mu\geq\sigma_2,\end{cases}$$

where, for convenience,

$$\sigma_1=\frac{-2\tau^2+2(\alpha+3)\lambda-\rho}{2\xi};\sigma_2=\frac{2\tau^2+2(\alpha+3)\lambda-\rho}{2\xi};\sigma_3=\frac{2(\alpha+3)\lambda-\rho}{2\xi},$$

$$\gamma:=\rho-2(\alpha+3)\lambda+2\mu\xi, \tag{2.11}$$

$$\rho:=\alpha^2+\alpha-2, \tag{2.12}$$

$$\xi:=(\alpha+2)(1+2\lambda), \tag{2.13}$$

and

$$\tau := (1+\alpha)(1+\lambda). \tag{2.14}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{\xi}\left(1 + \frac{\gamma}{2\tau^2}\right)|a_2|^2 \leq \frac{1}{\xi}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{\xi}\left(1 - \frac{\gamma}{2\tau^2}\right)|a_2|^2 \leq \frac{1}{\xi}.$$

These results are sharp.

Proof. Now by making use of (2.8) and (2.9) , we get

$$\begin{aligned} & a_3 - \mu a_2^2 = \\ = \; & \frac{1}{(\alpha+2)(1+2\lambda)}\left(\frac{c_2}{2} - \frac{c_1^2}{4} - \left(\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{8\left((1+\alpha)(1+\lambda)\right)^2}\right)c_1^2\right) \\ = \; & \frac{1}{2(\alpha+2)(1+2\lambda)}\left(c_2 - \frac{c_1^2}{2}\left(1 + \frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{2\left((1+\alpha)(1+\lambda)\right)^2}\right)\right) \end{aligned}$$

where

$$v := \frac{1}{2}\left(1 + \frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{2\left((1+\alpha)(1+\lambda)\right)^2}\right).$$

That is simply

$$v := \frac{1}{2}\left(1 + \frac{\rho - 2(\alpha+3)\lambda + 2\mu\xi}{2\tau^2}\right) = \frac{1}{2}\left(1 + \frac{\gamma}{2\tau^2}\right).$$

The assertion of Theorem 2.2 now follows by an application of Lemma 2.1.

To show that the bounds are sharp, we define the functions the functions F_η and G_η $(0 \leq \eta \leq 1)$, respectively, with $F_\eta(0) = 0 = F_\eta'(0) - 1$ and $G_\eta(0) = 0 = G_\eta'(0) - 1$ by

$$\begin{aligned} & \frac{z(F_\eta)'(z)}{F_\eta(z)}\left(\frac{F_\eta(z)}{z}\right)^\alpha \\ & +\lambda\left[1 + \frac{z(F_\eta)''(z)}{(F_\eta)'(z)} - \frac{z(F_\eta)'(z)}{F_\eta(z)} + \alpha\left(\frac{z(F_\eta)'(z)}{F_\eta(z)} - 1\right)\right] = q\left(\frac{z(z+\eta)}{1+\eta z}\right), \end{aligned}$$

and

$$\begin{aligned} & \frac{z(G_\eta)'(z)}{G_\eta(z)}\left(\frac{G_\eta(z)}{z}\right)^\alpha \\ & +\lambda\left[1 + \frac{z(G_\eta)''(z)}{(G_\eta)'(z)} - \frac{z(G_\eta)'(z)}{G_\eta(z)} + \alpha\left(\frac{z(G_\eta)'(z)}{G_\eta(z)} - 1\right)\right] = q\left(-\frac{z(z+\eta)}{1+\eta z}\right), \end{aligned}$$

respectively. Clearly the functions $K_q := q(z), F_\eta, G_\eta$ are members of $\mathscr{M}_{\alpha,\lambda}(q)$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_q or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_q = q(z^2)$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_η or one of its rotations. $\square$

By making use of Lemma 2.3, we immediately obtain the following:

Theorem 2.3. *Let $0 \le \alpha \le 1$, and $0 \le \lambda \le 1$. If $f \in \mathscr{M}_{\alpha,\lambda}(q)$, then for complex μ, we have*

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ \le\ & \frac{1}{(\alpha+2)(1+2\lambda)} \max\left\{1, \left|\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{2((1+\alpha)(1+\lambda))^2}\right|\right\} \\ =\ & \frac{1}{\xi}\max\left\{1, \left|\frac{\rho-2(\alpha+3)\lambda+2\mu\xi}{2\tau^2}\right|\right\}, \end{aligned}$$

where ρ, ξ, τ are as defined in (2.12), (2.13) *and* (2.14). *The result is sharp.*

Remark 2.5. (1) For the choice $\alpha = 0$, and $\lambda = 1$, Theorem 2.3, coincides with the result obtained for the class $f \in \mathscr{C}(q)$ as

$$|a_3 - \mu a_2^2| \le \frac{1}{6}\max\left\{1, \left|\frac{3\mu}{2}-1\right|\right\}.$$

(2) For the choices $\alpha = 0$, and $\lambda = 0$, Theorem 2.3 reduces to the result for the class $f \in \mathscr{S}^*(q)$ (see [20]) as

$$|a_3 - \mu a_2^2| \le \frac{1}{2}\max\{1, |2\mu-1|\}.$$

(3) For the choice of $\alpha = 0$, Theorem 2.3, reduces the result for the class $f \in \mathscr{M}_\lambda(q)$ as

$$|a_3 - \mu a_2^2| \le \frac{1}{1+2\lambda}\max\left\{1, \left|\frac{-2-6\lambda+4\mu(1+2\lambda)}{2(1+\lambda)^2}\right|\right\}.$$

(4) For the choice of $\lambda = 0$, Theorem 2.3, reduces the result for $f \in \mathscr{B}^\alpha(q)$

$$|a_3 - \mu a_2^2| \le \frac{2}{\alpha+2}\max\left\{1, \left|\frac{\alpha^2+\alpha-2+2\mu(\alpha+2)}{2(1+\alpha)^2}\right|\right\}.$$

3. Coefficient inequalities for the function f^{-1}

Theorem 3.1. *If $f \in \mathscr{M}_{\alpha,\lambda}(q)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of f with $|w| < r_0$ where r_0 is greater than the radius of the Koebe domain of the class $f \in \mathscr{M}_{\alpha,\lambda}(q)$, then for any complex number μ, we have*

$$| d_3 - \mu d_2^2 | \le \frac{1}{\xi} max\left\{1, \left|\frac{2\tau^2+\rho-2(\alpha+3)\lambda+(4+2\mu)\xi}{\tau^2}-1\right|\right\}, \tag{3.1}$$

where ρ, ξ, τ are as defined in (2.12), (2.13) *and* (2.14).

Proof. As

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{3.2}$$

is the inverse function of f, it can be seen that

$$f^{-1}(f(z)) = f\{f^{-1}(z)\} = z. \tag{3.3}$$

From equations (1.1) and (3.3), it can be reduced to

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z. \tag{3.4}$$

From (3.3) and (3.4), one can obtain

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2 d_2 + d_3)z^3 + ... = z. \tag{3.5}$$

By comparing the coefficients of z and z^2 from relation (3.5), it can be seen that

$$d_2 = -a_2, \qquad d_3 = 2a_2^2 - a_3. \tag{3.6}$$

From relations (2.8), (2.9), and (3.6)

$$d_2 = -\frac{c_1}{2(1+\alpha)(1+\lambda)}; \tag{3.7}$$

$$\begin{aligned} d_3 &= \frac{1}{2(\alpha+2)(1+2\lambda)} \\ &\times \left(c_2 - \frac{2\left((1+\alpha)(1+\lambda)\right)^2 + 4(\alpha+2)(1+2\lambda) + \alpha^2 + \alpha - 2(\alpha+3)\lambda - 2}{4\left((1+\alpha)(1+\lambda)\right)^2} c_1^2 \right); \\ &= \frac{1}{2\xi} \left(c_2 - \frac{2\tau^2 + 4\xi + \rho - 2(\alpha+3)\lambda}{2\tau^2} c_1^2 \right); \end{aligned} \tag{3.8}$$

and ρ, ξ, τ are as defined in (2.12), (2.13) and (2.14). For any complex number μ, consider

$$d_3 - \mu d_2^2 = \frac{1}{2\xi}\left(c_2 - \frac{2\tau^2 + \rho - 2(\alpha+3)\lambda + (4+2\mu)\xi}{2\tau^2} c_1^2 \right). \tag{3.9}$$

Taking modulus on both sides and by applying Lemma 2.3 on the right hand side of (3.9), one can obtain the result as in (3.1). Hence this completes the proof. □

Remark 3.1. Suitably specializing the parameters in Theorem 3.1 one can easily state above result for the function classes $\mathscr{M}_{0,\lambda}(q) \equiv \mathscr{M}_{\lambda}(q)$; $\mathscr{M}_{\alpha,0}(q) \equiv \mathscr{B}^{\alpha}(q)$; $\mathscr{M}_{0,0}(q) \equiv \mathscr{S}^*(q)$ and $\mathscr{M}_{0,1}(q) \equiv \mathscr{C}(q)$.

4. Application to functions defined by neutrosophic Poisson distribution

By letting $\wp_N(z)$ as the neutrosophic Poisson distribution series we study the following results (for detaiss see[12, 14]) . As is well known that the classical probability distributions only deals with specified data and specified parameter values, while neutrosophic probability distribution gives a more general and clear ones. In fact, Neutrosophic Poisson distribution of a discrete variable X is a classical Poisson distribution of x with the imprecise parameter value. A variable X is said to have neutrosophic Poisson distribution if its probability with the value $k \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$ is

$$NP(x = k) = \frac{(m_N)^k}{k!} e^{-m_N}, k = 0.1, 2, 3 \cdots .$$

where the distribution parameter m_N is the expected value and the variance, that is to say, $NE(x) = NV(x) = m_N$ for the neutrosophic statistical number $N = d + I$ (refer to [5] and also see [14]the references cited). Define a power series whose coefficients are probabilities of neutrosophic Poisson distribution by

$$\Phi(m_N, z) = z + \sum_{n=2}^{\infty} \frac{(m_N)^{n-1}}{(n-1)!} e^{-m_N} z^n, \quad z \in \mathbb{D}.$$

For $f \in \mathcal{A}$, we take the convolution operator $*$ and introduce the linear operator $\Lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} \Lambda f(z) &= \Phi(m_N, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(m_N)^{n-1}}{(n-1)!} e^{-m_N} a_n z^n \\ &= z + \sum_{n=m+1}^{\infty} \Psi(m_N, n) a_n z^n, \end{aligned} \tag{4.1}$$

where

$$\Psi_n := \Psi(m_N, n) = \frac{(m_N)^{n-1}}{(n-1)!} e^{-m_N}.$$

Specially

$$\Psi_2 := m_N e^{-m_N}, \quad \Psi_3 := \frac{(m_N)^2}{2} e^{-m_N}. \tag{4.2}$$

For the application of the results given in the previous section, we define the class $\mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$, in the following way:

$$\mathscr{M}^{\varphi}_{\alpha,\lambda}(q) := \{f \in \mathscr{A} \quad \text{and} \quad (f * \varphi) \in \mathscr{M}_{\alpha,\lambda}(q)\}$$

where

$$\varphi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n, \quad (\varphi_n > 0); \quad (f * \varphi) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n$$

and $\mathscr{M}_{\alpha,\lambda}(q)$ is given by Definition 1.2 and $*$ denote the convolution or Hadamard product of two series. We define the class $\mathscr{M}^{m}_{\alpha,\lambda}(q)$ in the following way:

$$\mathscr{M}^{m}_{\alpha,\lambda}(q) := \{f \in \mathscr{A} \quad \text{and} \quad \Lambda f \in \mathscr{M}_{\alpha,\lambda}(q)\}$$

where $\mathscr{M}_{\alpha,\lambda}(q)$ is given by Definition 1.2.

In following theorem,we obtain the coefficient estimate for functions in the class $\mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$, from the corresponding estimate for functions in the class $\mathscr{M}_{\alpha,\lambda}(q)$.

Applying Theorem 2.2 for the function $(f*\varphi)(z) = z + \varphi_2 a_2 z^2 + \varphi_3 a_3 z^3 + \cdots$, we get the next Theorems 4.1 and 4.2 after an obvious change of parameter μ.

Theorem 4.1. *Let $0 \leq \alpha \leq 1$, and $0 \leq \lambda \leq 1$. If $f \in \mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$, then for complex μ, we have*

$$|a_3 - \mu a_2^2| = \frac{1}{(\alpha+2)(1+2\lambda)\varphi_3} max\left\{1, \left|\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda}{2((1+\alpha)(1+\lambda))^2} + \frac{\mu(\alpha+2)(1+2\lambda)\varphi_3}{((1+\alpha)(1+\lambda)\varphi_2)^2}\right|\right\}.$$

Theorem 4.2. *Let $0 \leq \mu \leq 1$, $\alpha \geq 0$, $\lambda \geq 0$ and $\varphi_n > 0$. If $f(z)$ given by (1.1) belongs to $\mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{1}{2\xi\varphi_3}\left(-\dfrac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{1}{\xi\varphi_3}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{1}{2\xi\varphi_3}\left(\dfrac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

where, for convenience, $\gamma_2 := \rho - 2(\alpha+3)\lambda + 2\mu\xi\frac{\varphi_3}{\varphi_2^2}$,

$$\sigma_1 := \frac{\varphi_2^2}{\varphi_3}\left[\frac{2(\alpha+3)\lambda - \rho - 2\tau^2}{2\xi}\right], \sigma_2 = \frac{\varphi_2^2}{\varphi_3}\left[\frac{2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}\right]$$

and ρ, ξ, τ are as defined in(2.12)(2.13) *and* (2.14).

Now ,we obtain the coefficient estimate for $f \in \mathscr{M}^{m}_{\alpha,\lambda}(q)$, from the corresponding estimate for $f \in \mathscr{M}_{\alpha,\lambda}(q)$. Applying Theorem 2.2 for the function $\Lambda f = z + \Psi_2 a_2 z^2 + \Psi_3 a_3 z^3 + \cdots$, we get the following Theorems 4.3 and 4.4 after an obvious change of the parameter μ as in above theorems.

For Ψ_2 and Ψ_3 given by (4.2) Theorems 4.1 and 4.2 reduces to the following:

Theorem 4.3. *Let $0 \leq \alpha \leq 1$, and $0 \leq \lambda \leq 1$. If $f \in \mathscr{M}^{m}_{\alpha,\lambda}(q)$, then for complex μ, we have*

$$|a_3 - \mu a_2^2| = \frac{2}{(\alpha+2)(1+2\lambda)m_N^2 e^{-m_N}} \times$$
$$\times \max\left\{1, \left|\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda}{2((1+\alpha)(1+\lambda))^2} + \frac{\mu(\alpha+2)(1+2\lambda)}{2((1+\alpha)(1+\lambda))^2 e^{-m_N}}\right|\right\}.$$

Theorem 4.4. *Let $0 \leq \mu \leq 1$, $\alpha \geq 0$, $\lambda \geq 0$ and $\psi_n > 0$. If $f(z)$ given by (1.1) belongs to $\mathscr{M}^{m}_{\alpha,\lambda}(q)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{1}{\xi m_N^2 e^{-m_N}}\left(-\dfrac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{2}{\xi m_N^2 e^{-m_N}}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{1}{\xi m_N^2 e^{-m_N}}\left(\dfrac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

where $\gamma_2 := \rho - 2(\alpha+3)\lambda + \frac{\mu\xi}{e^{-m_N}}$, *for convenience we write*

$$\sigma_1 \;:=\; e^{-m_N}\left[\frac{2\tau^2+2(\alpha+3)\lambda-\rho}{2\xi}\right], \sigma_2 = e^{-m_N}\left[\frac{2\tau^2+2(\alpha+3)\lambda-\rho}{2\xi}\right]$$

and ρ, ξ, τ *are as defined in* (2.12), (2.13) *and* (2.14).

A variable $\mathcal{X}$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \cdots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!}$, ...respectively, where m is called the parameter. Thus

$$P(\mathcal{X} = r) = \frac{m^r e^{-m}}{r!}, \; r = 0, 1, 2, 3, \cdots.$$

In [13], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad z \in \Delta,$$

where $m > 0$. By ratio test the radius of convergence of above series is infinity. Using the Hadamard product, Porwal[13] (see also, [1, 9, 10] introduced a new linear operator $\mathcal{I}^m(z) : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{I}^m f = \mathcal{K}(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, = z + \sum_{n=2}^{\infty} \psi_n(m) a_n z^n, \qquad z \in \Delta,$$

where Since, $\mathcal{I}^m f = z + \sum_{n=2}^{\infty} \psi_m a_n z^n$, *where* $\psi_n = \frac{m^{n-1}}{(n-1)!} e^{-m}$, we have

$$\psi_2 = me^{-m} \quad \textit{and} \quad \psi_3 = \frac{m^2}{2} e^{-m}. \tag{4.3}$$

Remark 4.1. Suitably specializing the parameters in Theorems 4.3 and 4.4 one can easily state the results for the function classes associated with neutrosophic Poisson distribution and Poisson distribution as listed below:

(1) $\mathscr{M}^m_{0,\lambda}(q) \equiv \mathscr{M}^m_{\lambda}(q)$
(2) $\mathscr{M}^m_{0,0}(q) \equiv \mathscr{S}^*_m(q)$
(3) $\mathscr{M}_{\alpha,0}(q) \equiv \mathscr{B}^{\alpha}(q)$ and
(4) $\mathscr{M}^m_{0,1}(q) \equiv \mathscr{C}^m(q)$

which are new and not been studied sofar.

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SMOOTH FUNCTIONS AND GENERAL FOURIER COEFFICIENTS

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Abstract. As is known, the Fourier series of differentiable functions for classical orthonormal systems (trigonometric, Haar, Walsh, ...) are absolutely convergent. However, for general orthonormal systems (ONS) this fact does not hold. In the present paper, we consider some specific properties of special series of Fourier coefficients of differentiable functions with respect to the general ONS. The obtained results demonstrate that the properties of the general ONS and of the subsequence of this system are essentially different. Here we have shown that the received results are best possible.

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1. Auxiliary notations and theorems

Let (φ_n) be an ONS on $[0;1]$. Suppose that $f \in L_2$, then the Fourier coefficients of the function f are defined as follows:

$$C_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx, \quad n = 1, 2, \dots . \tag{1.1}$$

We denote

$$D_n(a) = \max_{1\le i\le n} \left| \int_0^{\frac{i}{n}} B_n(a, x)\,dx \right|, \tag{1.2}$$

where

$$B_n(a; x) = \sum_{k=1}^{n} a_k k^{\gamma} \int_0^x \varphi_k(u)\,du.$$

Also,

$$E_n(a, x) = \sum_{k=1}^{n} a_k k^{\gamma} \varphi_k(x)$$

and (a_k) is some sequence of real numbers.

Assume that $(0 < \gamma < 1)$

$$H_n(a) = \left(\sum_{k=1}^{n} a_k^2 k^{\gamma} \right)^{\frac{1}{2}}.$$

The bounded sequence we denote by (r_n), $r_n = O(1)$.

Lemma 1.1. *For any* $(a_n) \in \ell_2$ *and for* $h(x) = 1$ $x \in [0,1]$ *there holds*

$$\left| \int_0^1 E_n(a,x)\,dx \right| = O(1)\, H_n(a) H_n(C(h)),$$

where $H_n(C(h)) = H_n(a)$ *and* $C_k(h) = \int_0^1 h(x)\varphi_k(x)\,dx = a_k$ $(a = (a_k)$ *and* $C = (C_k(f))$.

Proof. According to the Cauchy inequality, we have (see (1.1))

$$\begin{aligned}\left| \int_0^1 E_n(a,x)\,dx \right| &= \left| \sum_{k=1}^n a_k k^\gamma \int_0^1 h(u)\varphi_k(u)\,du \right| \\ &\le \left(\sum_{k=1}^n a_k^2 k^\gamma \right)^{\frac{1}{2}} \left(\sum_{k=1}^n C_k^2(h) k^\gamma \right)^{\frac{1}{2}} = O(1) H_n(a) H_n(C(h)).\end{aligned}$$

Lemma 1.1 is proved. □

Lemma 1.2. *Let* $g(x) = x$ *for* $x \in [0,1]$ *and* $(a_n) \in \ell_2$, *then*

$$\left| \int_0^1 B_n(a,x)\,dx \right| = O(1) H_n(a) \big(H_n(C(h)) + H_n(C(g)) \big).$$

Proof. Integrating by parts we obtain

$$\begin{aligned}\int_0^1 B_n(a,x)\,dx &= \int_0^1 E_n(a,x)\,dx - \sum_{k=1}^n a_k k^\gamma \int_0^1 x\varphi_k(x)\,dx \\ &= \int_0^1 E_n(a,x)\,dx - \sum_{k=1}^n a_k k^\gamma C_k(g).\end{aligned}$$

Hence, using Hölder's inequality, from Lemma 1.1 it follows

$$\left| \int_0^1 B_n(a,x)\,dx \right| = O(1) H_n(a) H_n(C(h)) + O(1) H_n(a) H_n(C(g)).$$

Lemma 1.2 is proved. □

Lemma 1.3. *If* $(a_k) \in \ell_2$, $\gamma < 1$, *then for any* $i = 1, 2, \ldots, n$

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} |B_n(a,x)|\,dx = O(1).$$

Proof. By the Cauchy inequality

$$\begin{aligned}\left| \int_{\frac{i-1}{n}}^{\frac{i}{n}} B_n(a,x)\,dx \right| &= \left| \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sum_{k=1}^n a_k k^\gamma \int_0^x \varphi_k(u)\,du \right| \\ &\le \frac{1}{\sqrt{n}} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\sum_{k=1}^n a_k k^\gamma \int_0^x \varphi_k(u)\,du \right)^2 \right)^{\frac{1}{2}} \\ &\le \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n a_k^2 k^{2\gamma} \right)^{\frac{1}{2}} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \sum_{k=1}^n \left(\int_0^x \varphi_k(u)\,du \right)^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Then, according to the Bessel inequality, we have

$$\sum_{k=1}^{\infty}\left(\int_0^x \varphi_k(u)\,du\right)^2 \le 1.$$

Since $\gamma < 1$ and $\frac{1}{n}n^\gamma < 1$, we get

$$\left|\int_{\frac{i-1}{n}}^{\frac{i}{n}} B_n(a,x)\,dx\right| \le \frac{1}{\sqrt{n}}n^\gamma\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}\frac{1}{\sqrt{n}} = O(1).$$

Lemma 1.3 is proved. □

Theorem 1.1 (see [1])**.** *Let $f, F \in L_2$. Then*

$$\int_0^1 f(x)F(x)\,dx = n\sum_{i=1}^{n-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\Big(f(x) - f\Big(x+\frac{1}{n}\Big)\Big)\,dx\int_0^{\frac{i}{n}} F(x)\,dx \tag{1.3}$$

$$+\, n\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{i-1}{n}}^{\frac{i}{n}}(f(x)-f(t))dt\,F(x)\,dx + n\int_{1-\frac{1}{n}}^{1} f(x)\,dx\int_0^1 F(x)\,dx.$$

By V we denote the class of functions of bounded variation and by $V(f)$ the finite variation of function f on $[0,1]$. Let C_V be a class of functions f for which $f'(x) = \frac{d}{dx}f(x) \in V$.

By A we denote the class of absolutely continuous functions. A is a Banach space with the norm

$$\|f\|_A = \|f\|_C + \int_0^1 |f'(x)|\,dx.$$

2. The main problem

Suppose that $f \in C_V$ is an arbitrary function and (φ_n) is trigonometric [2, Ch. 4], Haar [3, Ch. 1] or Walsh [3, Ch. 1] system, then it is evident that if $0 < \gamma < 1$,

$$\sum_{k=1}^{\infty} C_k^2(f)k^\gamma = O(1)\sum_{k=1}^{\infty} k^{-2}k^\gamma < +\infty.$$

There arises the question: is the series

$$\sum_{k=1}^{\infty} C_k^2(f)k^\gamma$$

convergent for any $f \in C_V$ and for arbitrary ONS when $0 < \gamma < 1$?

It is known (see [4]) that if $f \in L_2$ is an arbitrary function ($f \nsim 0$) and $(a_k) \in \ell_2$ is an arbitrary sequence of numbers, then there exists an ONS (φ_n) such that

$$C_n(f) = da_n, \;\; n = 1,2,\dots \;\; (d \ne 0 \text{ depends only on } f \text{ and } (a_n)).$$

Assume that $g(x) = 1$ for $x \in [0;1]$ and let

$$a_n = \frac{1}{\sqrt{n}\log(n+1)}\,.$$

Then, since $(a_n) \in \ell_2$ as it was noted above, there exists an ONS (φ_n) such that

$$C_n(g) = da_n, \quad n = 1, 2, \ldots .$$

Hence

$$\sum_{k=1}^{\infty} C_k^2(g) k^{\gamma} = d^2 \sum_{k=1}^{\infty} \frac{1}{k \log^2(k+1)} k^{\gamma} = +\infty,$$

though in this case $g \in C_V$.

The similar problems are considered in the papers [5]-[8].

3. The main results

Theorem 3.1. *Let (φ_n) be an ONS on $[0;1]$ such that $H_n(C(h)) = O(1)$ and $H_n(C(g)) = O(1)$ (see Lemmas 1.1 and 1.2). If for arbitrary $(a_n) \in \ell_2$ (see (1.2))*

$$D_n(a) = O(1) H_n(a), \tag{3.1}$$

then for any $f \in C_V$, $0 < \gamma < 1$, there holds

$$\sum_{k=1}^{n} C_k^2(f) k^{\gamma} < +\infty.$$

Proof. For arbitrary $f \in L_2(0,1)$,

$$\sum_{k=1}^{n} C_k^2(f) k^{\gamma} = \sum_{k=1}^{n} C_k(f) k^{\gamma} \int_0^1 f(x) \varphi_k(x)\, dx = \int_0^1 f(x) \sum_{k=1}^{n} C_k(f) k^{\gamma} \varphi_k(x)\, dx = \int_0^1 f(x) E_n(C,x)\, dx, \tag{3.2}$$

where $E_n(C,x) = E_n(a,x)$ when $C_k(f) = a_k$, $k = 1, 2, \ldots$.

In (1.3) we substitute $F(x) = B_n(C;x)$ and $f(x) = f'(x)$:

$$\begin{aligned} \int_0^1 f'(x) B_n(C,x)\, dx &= n \sum_{i=1}^{n-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(f'(x) - f'\Big(x + \frac{1}{n}\Big)\Big)\, dx \int_0^{\frac{i}{n}} B_n(C,x)\, dx \\ &+ n \sum_{i=1}^{n-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \big(f'(x) - f'(t)\big) dt\, B_n(C,x)\, dx \\ &+ n \int_{1-\frac{1}{n}}^{1} f'(x)\, dx \int_0^1 B_n(C,x)\, dx = P_1 + P_2 + P_3. \end{aligned} \tag{3.3}$$

By conditions (3.1) and $f \in C_V$ we get $(\Delta_{in} = [\frac{i-1}{n}, \frac{i}{n}])$

$$\begin{aligned} |P_1| &= nO\Big(\frac{1}{n}\Big) \sum_{i=1}^{n-1} \sup_{x \in \Delta_{in}} \Big| f'(x) - f'\Big(x + \frac{1}{n}\Big)\Big| \Big| \int_0^{\frac{i}{n}} B_n(C,x)\, dx \Big| \\ &= O(1) V(f') D_n(C) = O(1) H_n(C). \end{aligned} \tag{3.4}$$

According to Lemma 1.3 ($0<\gamma<1$),

(3.5)
$$|P_2| = nO\Big(\frac{1}{n}\Big)\sum_{i=1}^{n}\max_{x,t\in\Delta_{in}}|f'(x)-f'(t)|\int_{\frac{i-1}{n}}^{\frac{i}{n}}|B_n(C,x)|\,dx = O(1)V(f') = O(1).$$

Next, Lemma 1.2 and conditions of Theorem 3.1 imply

(3.6)
$$|P_3| = \Big|\int_0^1 B_n(a,x)\,dx\Big| = O(1)H_n(C)\big(H_n(C(h))+H_n(C(g))\big) = O(1)H_n(C) = O(1).$$

Taking into account (3.4), (3.5) and (3.6) in (3.3) we get

$$\Big|\int_0^1 f'(x)B_n(C,x)\,dx\Big| = O(1)H_n(C)+O(1). \tag{3.7}$$

Using (3.2) and integration by parts we have

(3.8)
$$\sum_{k=1}^{n}C_k^2(f)k^\gamma = \int_0^1 f(x)E_n(C,x)\,dx = f(1)\int_0^1 E_n(C,x)\,dx - \int_0^1 f'(x)B_n(C,x)\,dx.$$

It can be easily verified that (see (3.8), (3.7) and Lemma 1.1)

$$\sum_{k=1}^{n}C_k^2(f)k^\gamma = O(1)H_n(C)H_n(C(h)) + O(1)H_n(C)+O(1)$$
$$= O(1)+O(1)\Big(\sum_{k=1}^{n}C_k^2(f)k^\gamma\Big)^{\frac12}.$$

So

$$\Big(\sum_{k=1}^{n}C_k^2(f)k^\gamma\Big)^{\frac12} = O(1).$$

Finally, for any $f\in C_V$,

$$\sum_{k=1}^{\infty}C_k^2(f)k^\gamma < +\infty.$$

Theorem 3.1 is proved. □

Theorem 3.2. *Let (φ_n) be an ONS on $[0;1]$. If for some $(b_n)\in\ell_2$*

$$\limsup_{n\to\infty}\frac{1}{H_n(b)}\,|D_n(b)| = +\infty, \tag{3.9}$$

then there exists a function $s\in C_V$ such that

$$\sum_{n=1}^{\infty}C_n^2(s)n^\gamma = +\infty.$$

Proof. First, we suppose that

$$\lim_{n\to\infty}H_n(C(h)) = +\infty \quad\text{or}\quad \lim_{n\to\infty}H_n(C(g)) = +\infty.$$

Since $h(x)=1$ and $g(x)=x$, when $x\in[0,1]$, we conclude that

$$\lim_{n\to\infty}\sum_{k=1}^{n} C_k^2(h)k^\gamma=+\infty \quad \text{or} \quad \lim_{n\to\infty}\sum_{k=1}^{n} C_k^2(g)k^\gamma=+\infty.$$

In such a case Theorem 3.2 is proved.

Now we assume that

$$H_n(C(h))=O(1) \quad \text{and} \quad H_n(C(g))=O(1). \tag{3.10}$$

We have

$$D_n(a)=\max_{1\le i\le n}\left|\int_0^{\frac{i}{n}} B_n(a,x)\,dx\right|=\left|\int_0^{\frac{i_n}{n}} B_n(a,x)\,dx\right|, \text{ where } 1\le i_n<n.$$

Here we must note that if $i_n=n$ and

$$\limsup_{n\to\infty}\frac{1}{H_n(b)}\left|\int_0^1 B_n(a,x)\,dx\right|=+\infty,$$

then according to Lemma 1.3

$$\left|\int_{1-\frac{1}{n}}^1 B_n(a,x)\,dx\right|=O(1)$$

and

$$\limsup_{n\to\infty}\frac{1}{H_n(b)}\left|\int_0^{1-\frac{1}{n}} B_n(a,x)\,dx\right|$$
$$=\limsup_{n\to\infty}\frac{1}{H_n(b)}\left|\int_0^1 B_n(a,x)\,dx\right|-\limsup_{n\to\infty}\frac{1}{H_n(b)}\left|\int_{1-\frac{1}{n}}^1 B_n(a,x)\,dx\right|=+\infty.$$

We define the sequence of functions (f_n) as follows:

$$f_n(x)=\begin{cases}0 & \text{when } x\in[0,\frac{i_n-2}{n}],\\ 1 & \text{when } x\in[\frac{i_n}{n},1],\\ \frac{nx-i_n+2}{2} & \text{when } x\in[\frac{i_n-2}{n},\frac{i_n}{n}].\end{cases} \tag{3.11}$$

In (3.3) we substitute $f'=f_n$, then

(3.12)

$$\int_0^1 f_n(x)B_n(b,x)\,dx=n\sum_{i=1}^{n-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\Big(f_n(x)-f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx\int_0^{\frac{i}{n}} B_n(b,x)\,dx$$
$$+n\sum_{i=1}^{n-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{i-1}{n}}^{\frac{i}{n}}(f_n(x)-f_n(t))dtB_n(b,x)\,dx$$
$$+n\int_{1-\frac{1}{n}}^1 f_n(x)\,dx\int_0^1 B_n(b,x)\,dx=S_1+S_2+S_3.$$

By (3.11), since $|f_n(x)-f_n(t)|\le 1$ when $x,t\in[0,1]$ and $f_n(x)-f_n(t)=0$ when $x,t\in[0,\frac{i_n-2}{n}]$ or $x,t\in[\frac{i_n}{n},1]$, using Lemma 1.1, we receive

$$|S_2|\le n\frac{1}{n}\int_{\frac{i_n-2}{n}}^{\frac{i_n}{n}}|B_n(b,x)|\,dx=O(1). \tag{3.13}$$

Next, taking into account Lemma 1.2 and (3.10), we get

$$\left|\int_0^1 B_n(b,x)\,dx\right| = O(1)H_n(b)H_n(C(h)) + O(1)H_n(b)H_n(C(g)) = O(1)H_n(b).$$

Hence it follows that

$$|S_3| \le n\int_{1-\frac{1}{n}}^1 |f_n(x)|\,dx \left|\int_0^1 B_n(b,x)\,dx\right| = O(1)H_n(b). \tag{3.14}$$

Taking into consideration (3.11) we have

a) $\displaystyle\int_{\frac{i_n-3}{n}}^{\frac{i_n-2}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx = -\int_{\frac{i_n-2}{n}}^{\frac{i_n-1}{n}} \frac{nx-i_n+2}{2}\,dx = -\frac{1}{4n}\,;$

b) $\displaystyle\int_{\frac{i_n-2}{n}}^{\frac{i_n-1}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx = -\frac{1}{2n};$

c) $\displaystyle\int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx = \int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} \frac{nx-i_n+2}{2}\,dx - \frac{1}{n}$

$\displaystyle\qquad = \frac{3}{4n} - \frac{1}{n} = -\frac{1}{4n}\,;$

d) $\displaystyle\int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx = 0$ when $i \le i_n-3$ or $i \ge i_n+1$.

Therefore, due to a)–d) we get

$$\begin{aligned}
|S_1| &= n\left|\sum_{i=i_n-2}^{i_n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(f_n(x) - f_n\Big(x+\frac{1}{n}\Big)\Big)\,dx \int_0^{\frac{i}{n}} B_n(b,x)\,dx\right| \\
&= n\left|\frac{1}{4n}\int_0^{\frac{i_n-2}{n}} B_n(b,x)\,dx + \frac{1}{2n}\int_0^{\frac{i_n-1}{n}} B_n(b,x)\,dx + \frac{1}{4n}\int_0^{\frac{i_n}{n}} B_n(b,x)\,dx\right| \\
&\ge \left|\int_0^{\frac{i_n}{n}} B_n(b,x)\,dx\right| - \frac{1}{4}\int_{\frac{i_n-2}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx - \frac{1}{2}\int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx.
\end{aligned}$$

Since (see Lemma 1.3)

$$\int_{\frac{i_n-2}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx = O(1) \quad\text{and}\quad \int_{\frac{i_n-1}{n}}^{\frac{i_n}{n}} |B_n(b,x)|\,dx = O(1),$$

we have

$$|S_1| \ge D_n(b) - O(1). \tag{3.15}$$

Hence from (3.12), because of (3.13), (3.14) and (3.15), it follows

$$\left|\int_0^1 f_n(x)B_n(b,x)\,dx\right| \ge D_n(b) - O(1).$$

From here and (3.9),

$$\limsup_{n\to\infty} \frac{1}{H_n(b)}\left|\int_0^1 f_n(x)B_n(b,x)\,dx\right| = +\infty. \tag{3.16}$$

It can be easily verified that

$$\Delta_n(f) = \frac{1}{H_n(b)} \int_0^1 f(x) B_n(b,x)\, dx, \quad n = 1, 2, \dots ,$$

is a sequence of linear and bounded functionals on A.

On the other hand,

$$||f_n||_A = ||f_n||_C + \int_0^1 |f_n'(x)|\, dx \le 2. \tag{3.17}$$

Since (3.16)

$$\limsup_{n\to\infty} |\Delta_n(f_n)| = +\infty$$

and (3.17), by virtue of Banach–Steinhaus Theorem there exists a function $u \in A$ such that

$$\limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \int_0^1 u(x) B_n(b,x)\, dx \right| = +\infty. \tag{3.18}$$

We assume that

$$s(x) = \int_0^x u(t) dt.$$

It can be easily verified (see (3.8)) that

$$\sum_{k=1}^n C_k(s) b_k k^\gamma = \int_0^1 s(x) \sum_{k=1}^n b_k k^\gamma \varphi_k(x)\, dx = \int_0^1 s(x) E_n(b,x)\, dx$$
$$= s(1) \int_0^1 E_n(b,x)\, dx - \int_0^1 s'(x) B_n(b,x)\, dx.$$

From here, since $s'(x) = u(x)$, by virtue of Lemma 1.1 and (3.10) (see (3.18)), we get

$$\limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \sum_{k=1}^n C_k(s) b_k k^\gamma \right| \ge \limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \int_0^1 u(x) B_n(b,x)\, dx \right| - \limsup_{n\to\infty} \frac{|s(1)|}{H_n(b)} \left| \int_0^1 E_n(b,x)\, dx \right| = +\infty. \tag{3.19}$$

Now using the Cauchy inequality,

$$\left| \sum_{k=1}^n b_k k^\gamma C_k(s) \right| \le \left(\sum_{k=1}^n b_k^2 k^\gamma \right)^{\frac{1}{2}} \left(\sum_{k=1}^n C_k^2(s) k^\gamma \right)^{\frac{1}{2}} = H_n(b) \left(\sum_{k=1}^n C_k^2(s) k^\gamma \right)^{\frac{1}{2}}.$$

Finally, due to (3.19) we get

$$\lim_{n\to\infty} \left(\sum_{k=1}^n C_k^2(s) k^\gamma \right)^{\frac{1}{2}} = \limsup_{n\to\infty} \frac{1}{H_n(b)} \left| \sum_{k=1}^n b_k k^\gamma C_k(s) \right| = +\infty.$$

Since $s' \in A$, Theorem 3.2 is proved. □

Theorem 3.3. *From any ONS one can insolate a subsequence* (φ_{n_k}) *such that for any function* $f \in C_V$,

$$\sum_{k=1}^{\infty} C_{n_k}^2(f) k^{\gamma} < +\infty,$$

where $C_{n_k}(f) = \int_0^1 f(x)\varphi_{n_k}(x)\,dx$ *and* $0 < \gamma < 1$.

Proof. Let the ONS (φ_n) be a complete system on $[0,1]$. Then, by the Parceval equality, for any $x \in [0,1]$ we have

$$\sum_{n=1}^{\infty} \Big(\int_0^x \varphi_n(u)\,du \Big)^2 = x \quad \text{and} \quad \sum_{n=1}^{\infty} \Big(\int_0^1 x\varphi_n(u)\,du \Big)^2 = \frac{1}{2}.$$

According to the Dini Theorem there exists a sequence of natural numbers (n_k) such that

$$\sum_{s=n_k}^{\infty} \Big(\int_0^x \varphi_s(u)\,du \Big)^2 < \frac{1}{k^2} \quad \text{and} \quad \sum_{s=n_k}^{\infty} \Big(\int_0^1 x\varphi_s(u)\,du \Big)^2 < \frac{1}{k^2}$$

uniformly with respect to $x \in [0,1]$. From here, uniformly with respect to $x \in [0,1]$, we obtain

$$\Big| \int_0^x \varphi_{n_k}(u)\,du \Big| < \frac{1}{k} \quad \text{and} \quad \Big| \int_0^1 x\varphi_{n_k}(u)\,du \Big| < \frac{1}{k}, \quad k = 1, 2, \ldots. \tag{3.20}$$

In our case let

$$B_m(a,x) = \sum_{k=1}^{m} a_k k^{\gamma} \int_0^x \varphi_{n_k}(t) dt \quad \text{and} \quad H_m(a) = \big(\sum_{k=1}^{m} a_k^2 k^{\gamma} \big)^{\frac{1}{2}}.$$

Next, for arbitrary $(a_n) \in \ell_2$ and $0 < \gamma < 1$ we get (see (3.2) and (3.20))

$$\begin{aligned} D_m(a) &= \max_{1 \le i \le m} \Big| \int_0^{\frac{i}{m}} B_m(a,x)\,dx \Big| = \Big(\int_0^1 B_m^2(a,x)\,dx \Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{k=1}^{m} a_k^2 k^{\gamma} \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{m} k^{\gamma} \Big(\int_0^x \varphi_{n_k}(u)\,du \Big)^2 \Big)^{\frac{1}{2}} \\ &= H_m(a) \Big(\sum_{k=1}^{m} k^{\gamma} k^{-2} \Big)^{\frac{1}{2}} = O(1) H_m(a). \end{aligned}$$

Thus

$$D_m(a) = O(1) H_m(a). \tag{3.21}$$

In addition (see (3.20)),

$$\begin{aligned} H_m(C(h)) = \Big(\sum_{k=1}^{m} C_{n_k}^2(h) k^{\gamma} \Big)^{\frac{1}{2}} &= \Big(\sum_{k=1}^{m} k^{\gamma} \Big(\int_0^1 \varphi_{n_k}(x)\,dx \Big)^2 \Big)^{\frac{1}{2}} \\ &= \Big(\sum_{k=1}^{m} k^{\gamma} k^{-2} \Big)^{\frac{1}{2}} = O(1) \end{aligned}$$

and

$$H_m(C(g)) = \Big(\sum_{k=1}^{m} C_{n_k}^2(g)k^\gamma\Big)^{\frac{1}{2}} = \Big(\sum_{k=1}^{m} k^\gamma\Big(\int_0^1 x\varphi_{n_k}(x)\,dx\Big)^2\Big)^{\frac{1}{2}}$$
$$= \Big(\sum_{k=1}^{m} k^\gamma k^{-2}\Big)^{\frac{1}{2}} = O(1).$$

According to (3.21) and Theorem 3.1, for any $f \in C_V$ the series $\sum_{k=1}^{\infty} C_k^2(f)k^\gamma$ is convergent. □

4. Problems of efficiency

Theorem 4.1. *Let (φ_n) be an ONS and*

$$\int_0^x \varphi_n(u)\,du = O(1)\,\frac{1}{n}$$

uniformly with respect to $x \in [0,1]$. Then for arbitrary $(a_n) \in \ell_2$,

$$D_n(a) = O(1)H_n(a). \tag{4.1}$$

Proof. In our case

$$D_n(a) = \max_{1\le i\le n}\Big|\int_0^{\frac{i}{n}} B_n(a,x)\,dx\Big| = \max_{1\le i\le n}\Big|\sum_{k=1}^{n} a_k k^\gamma \int_0^{\frac{i}{n}}\int_0^x \varphi_k(u)\,du\,dx\Big|$$
$$= O(1)\sum_{k=1}^{n}\frac{1}{k}|a_k|k^\gamma = O(1)\Big(\sum_{k=1}^{n} a_k^2 k^\gamma\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n} k^{-2+\gamma}\Big)^{\frac{1}{2}} = O(1)H_n(a).$$

So, the trigonometric ($\sqrt{2}\cos 2\pi nx$, $\sqrt{2}\sin 2\pi nx$) and Walsh systems satisfy condition (4.1). □

Theorem 4.2. *If (X_n) is the Haar system, then for an arbitrary $(a_n) \in \ell_2$,*

$$D_n(a) = O(1)H_n(a).$$

Proof. The definition of the Haar system implies (see [3, Ch. 1])

$$\Big|\int_0^x \sum_{k=2^m+1}^{2^{m+1}} a_k k^\gamma X_k(u)\,du\Big| \le 2^{-\frac{m}{2}}|a_{k(m)}|k^\gamma(m),$$

where $2^m < k(m) \le 2^{m+1}$.

Without loss of generality, we suppose

$$B_n(a;x) = \sum_{k=2}^{n} a_k k^\gamma \int_0^x \varphi_k(u)\,du.$$

From here, if $n = 2^q$, for an arbitrary $(a_n) \in \ell_2$ $(0 < \gamma < 1)$ we have

$$D_n(a) = \max_{1 \le i \le n} \left| \int_0^{\frac{i}{n}} B_n(a,x)\,dx \right|$$
$$= \max_{1 \le i \le n} \left| \sum_{m=0}^{q-1} \int_0^{\frac{i}{2^q}} \sum_{k=2^m+1}^{2^{m+1}} \int_0^x X_k(u)\,du\; k^\gamma a_k\,dx \right|$$
$$= O(1) \sum_{m=0}^{q-1} 2^{-\frac{m}{2}} k^\gamma(m) |a_{k(m)}|$$
$$= O(1) \left(\sum_{m=0}^{q-1} \sum_{k=2^m+1}^{2^{m+1}} a_k^2 k^\gamma \right)^{\frac{1}{2}} \left(\sum_{m=0}^{q} 2^{-m} 2^{\gamma m} \right)^{\frac{1}{2}} = O(1) H_n(a).$$

It is easy to prove that when $n = 2^q + l$, $1 \le l \le 2^q$, the condition $D_n(a) = H_n(a)$ is valid. □

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Contents