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ON COMPLEX SCHRÖDINGER TYPE EQUATIONS WITH SOLUTIONS IN A GIVEN DOMAIN

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Abstract. Value distribution, particularly the numbers of a-points, weren't studied for meromorphic functions in a given domain which are solutions of some complex differential equations. In fact we have here a "virgin land". A new program of investigations of similar solutions in a given domain was initiated quite recently. In this program some geometric methods were offered to study some standard problems as well as some new type problems related to Gamma-lines and Blaschke characteristic for a-points of the solutions of different equations. In this paper we apply these methods to get bounds for length of Gamma-lines and Blaschke characteristic for a-points for solutions of equations $w'' = gw^{\mu}$ considered in a given domain.

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Keywords: Schrödinger type equations; solutions of complex equations in a domain; Gamma-lines.

1. Introduction

There is a huge number of investigations in complex differential equations (CDE) when the solutions are meromorphic in the complex plane or in the unit disk. The main attention was paid to the value distribution type phenomena of the solutions, particularly to the zeros (more generally to the a-points) of these solutions. Meantime we have very few studies of meromorphic solutions in a given domain, particularly zeros of similar solutions weren't touched at all. In fact our present situation with the solutions in a given domain is similar to that in the beginning of 20th century when studies of the growth of solutions in the complex plane were started.

Recently a new program of investigations of CDE-s with solutions in a given domain was initiated in [4], where different characteristics of solutions were studied for different CDE-s. In this paper we consider two characteristics for the solutions in a given domain of equations $w'' = gw^{\mu}$, where μ is a positive integer number.

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2. On a-points of solutions of $w'' = gw^\mu$

Denote $D_1 = \{z : |z| < 1\}$. Let $w(z)$ be a meromorphic function in D_1. Denote a-points of w by $z_i(a) \in D_1$. The Blaschke sum of zeros of w, i.e. $\sum_i (1 - |z_i(0)|)$, was widely used in the study of meromorphic functions in D_1, particularly in CDE-s with solutions in the unit disk. For a given analytic function in D_1, Pommerenke considered in [10] (1982) the equation $w'' = gw$ (one dimensional complex Schrödinger equation) with solutions w in D_1 and proved for the zeros $z_i(0)$ of w: assumption $\int\int_{D_1} |g(z)|^{1/2} d\sigma < \infty$ implies $\sum_i (1 - |z_i(0)|) < \infty$. A new stage of studies of this equation related to interrelations of g and Blaschke sum for D_1 was started recently by Heittokangas [6] (2005); for further developments see his survey in the book [8].

As we mentioned above our aim is to study CDE-s with solutions in domains D.

Assume that D is a simply connected domain with smooth boundary ∂D of finite length $l(D)$ and area $S(D)$.

We study the following more general equation

$$w'' = gw^\mu, \tag{S$^\mu$}$$

where μ is a positive integer number and $g(z)$ is a regular function in $\bar{D} = D \cup \partial D$.

As a characteristic of a-points we consider the following *Blaschke sum of a-points for a given domain D* (considered first in [2, Chapter 1]) which we define as $\mathcal{N}(D, a, w) := \sum_i \mathrm{Dist}(z_i(a), \partial D)$, where $\mathrm{Dist}(x, y)$ stands obviously for the distance between x and y. Notice that in the case when D is the disk D_1 we have $\mathrm{Dist}(z_i(0), \partial D) = 1 - |z_i(0)|$; respectively the Blaschke sum for D becomes usual Blaschke sum for D_1.

For a regular function w in $\bar{D}$ we denote $M(w) := \max_{z \in \partial D} |w(z)|$ [2] and $m(w') := \min_{z \in \bar{D}} |w'(z)|$.

Theorem 2.1. *For an arbitrary regular in $\bar{D}$ solution $w(z)$ of equation (S^μ) and any complex value $a \neq 0$ we have*

$$\mathcal{N}(D, a, w) \le K_{11} M^\mu(w) + K_{12} m(w') + K_{13}, \tag{2.1}$$

where K_{11}, K_{12}, K_{13} are independent of w.

Some comments. Notice that if we know the magnitude $w'(z_0)$ at any point $z_0 \in \bar{D}$ we can substitute $m(w')$ in (2.1) by $|w'(z_0)|$. The coefficients depend on the

[2] Here we may remember that in numerous studies concerning regular functions w in the disks $D(r) := \{z : |z| < r\}$ (instead of the domains D) the magnitude $M(w)$ plays a role of a characteristic. The same is true also for entire functions; in this case we deal usually with $\ln M(w) := \ln \max_{z \in \partial D(r)} |w(z)|$.

equation, the value a and the domain D. They are determined in the simple terms:

$$K_{11} = \frac{3\pi + 3\mu}{4|a|} M(g) l(D) S(D), \quad K_{12} = \frac{\pi + \mu}{2|a|} S(D),$$

and

$$K_{13} = \frac{1}{4} \iint\limits_{D} \left| \frac{g'(z)}{g(z)} \right| d\sigma + \frac{\pi + 2}{8} l(D).$$

Thus, K_{11}, K_{12}, K_{13} are finite when the last double integral is finite so that (2.1) yields, in this case, simply determined bounds for $\mathcal{N}(D, a, w)$.

Finally, we notice that in the case when $g(z)$ is a polynomial of degree n the upper bounds of the double integral can be easily given by n and $S(D)$.

3. Gamma-lines of solutions of $w'' = gw^{\mu}$

Gamma-lines, motivation of their studies and the preceding results. Let $w(z) := u + iv := \mathrm{Re} w + i \, \mathrm{Im} w$ be a meromorphic function in D. Consider level sets of $u - A$, $-\infty < A < +\infty$, that is solutions $u(x, y) = A$ (or $\mathrm{Re} w(z) = A$). (By the definition, level sets of real functions $u(x, y)$ are solutions of $u(x, y) = 0$). In turn level sets are particular cases of Gamma-lines of w which are those curves in D whose w-images belong to a given curve. For instance, when Γ is the real axis, Gamma-lines become level sets of function $u(x, y)$, i.e. solutions of $u(x, y) = 0$,

One can notice a striking similarity between the a-points (which are the solutions $w(z) = a$) and the level sets (which are solutions of $u(x, y) = A$). On the other hand, level sets of $u - A$ admit a lot of interpretations (streaming line, potential line, isobar, isoterm) in different applied fields of engineering, physics, environmental and other problems. Due to the above arguments (similarity with a-points and applicability), it is pertinent to study largely level sets for different classes of meromorphic functions particularly for the solutions w of different classes of complex differential equations.

We denote the length of Gamma-lines of w lying in D by $L(D, \Gamma, w)$. These lengths were widely studied in [2] for large classes of smooth Jordan curves Γ (bounded or unbounded) in the complex plane. The only restriction for Γ is that $\nu(\Gamma) = \mathrm{Var}_{z \in \Gamma} \alpha_{\Gamma}(z) < \infty$, where Var means variation, $\alpha_{\Gamma}(z)$ is the angle between the tangent to Γ at $z \in \Gamma$ and the real axis.

As to Gamma-lines for solutions of equation, they were considered first recently in [1] for solutions in D of equation $w'' = gw$, where estimates of $L(D, \Gamma, w)$ were given in terms of Ahlfors-Shimizu classical characteristic.

In this section, we give upper bounds of $L(D,\Gamma,w)$ for solution w of (S^μ). The bounds will be given in terms of $M(w)$, which in application mean often some important physical concepts.

Theorem 3.1. *Let $w(z)$ be a regular function in $\bar{D}$ which is a solution of equation (S^μ) and Γ a smooth Jordan curve with $\nu(\Gamma) < \infty$ which does not pass through zero. Then*

$$L(D,\Gamma,w) \le K_{21}M^\mu(w) + K_{22}m(w') + K_{23}, \tag{3.1}$$

where K_{21}, K_{22}, K_{23} are independent of w.

The coefficients depend on the equation, the curve Γ and the domain D. They are determined in the simple terms:

$$K_{21} = K(\Gamma)\frac{3\pi+3\mu}{|a_\Gamma|}M(g)l(D)S(D), \quad K_{22} = K(\Gamma)\frac{2\pi+2\mu}{|a_\Gamma|}S(D),$$

where $K(\Gamma) = 3(\nu(\Gamma)+1)$, a_Γ is the closest to the zero point belonging to Γ [3] and

$$K_{23} = K(\Gamma)\iint\limits_D \left|\frac{g'(z)}{g(z)}\right| d\sigma + K(\Gamma)\frac{\pi+2}{2}l(D).$$

Theorem 3.2. *Assuming in Theorem 3.1 that Γ is a straight line which does not pass through zero, we have*

$$L(D,\Gamma,w) \le K_{31}M^\mu(w) + K_{32}m(w') + K_{33}, \tag{3.2}$$

where K_{31}, K_{32}, K_{33} are independent of w.

Assuming that a is the closet to zero point on Γ we have

$$K_{31} = \frac{3\pi+3\mu}{2|a|}M(g)l(D)S(D), \quad K_{32} = \frac{\pi+\mu}{|a|}S(D),$$

and

$$K_{33} = \frac{1}{2}\iint\limits_D \left|\frac{g'(z)}{g(z)}\right| d\sigma + \frac{\pi+2}{4}l(D).$$

4. Proofs

Proof of Theorem 3.1. We need the following "basic identity for Gamma-lines" (see [2, item 1.1.3, identity (1.1.6)]). We state it as

Lemma 4.1. *For any regular function w in D we have*

$$\int_0^\infty L(D,\Gamma(R),w)dR = \iint\limits_D |w'|\, d\sigma,$$

where $\Gamma(R)$ is the circumference $\{w: \ |w| = R\}$.

[3] If we have more than one similar point we take arbitrary of them.

For a given $a \in \mathbb{C}$, $a \neq 0$, we denote $D(|a|/2, 3|a|/4) := \{z : |a|/2 < |w(z)| < 3|a|/4\}$. This set consists of some connected components which are simply connected or multiply connected components. Dividing multiply connected components into some simply connected ones we can consider $D(|a|/2, 3|a|/4)$ as a union of simply connected domains $D_\lambda(|a|/2, 3|a|/4)$, where λ is a counting index of these domains. Applying Lemma 4.1 in each $D_\lambda(|a|/2, 3|a|/4)$ and then summing up for all indexes λ we obtain

$$\int_{|a|/2}^{3|a|/4} L(D, \Gamma(R), w)dR = \iint\limits_{D(|a|/2,3|a|/4)} |w'| \, d\sigma.$$

Due to the mean value theorem we conclude that there is $R^* \in (|a|/2, 3|a|/4)$ such that

$$L(D, \Gamma(R^*), w) = \frac{4}{|a|} \iint\limits_{D(|a|/2,3|a|/4)} |w'| \, d\sigma. \tag{4.1}$$

Denote $D(|w| > c) = \{z : |w(z)| > c > 0\}$. The set $D(|w| > c)$ may consists of one or more domains $D_\eta(|w| > c)$; clearly they can be as simply connected as well as multiply connected. By $\partial D_\eta(|w| > c)$ we denote the union of all boundary components of $D_\eta(|w| > c)$. Notice that the boundary $\partial D_\eta(|w| > R^*)$ should have a (non empty) common part $\partial D_\eta(|w| > R^*) \cap \partial D$ with ∂D. (Indeed, assume contrary, that $\partial D_\eta(|w| > R^*)$ lies fully inside D. Then w should have a pole inside D which contradicts our assumption that w is regular in $\bar{D}$). Observing that the different common parts (taken for different η) do not overlap we obtain

$$\sum_\eta l\left(\partial D_\eta(|w| > R^*)\right) \leq L(D, \Gamma(R^*), w) + l(\partial D). \tag{4.2}$$

We need also the following "principle of logarithmic derivatives", which was established recently [3] by making use of Gamma-lines technic.

Lemma 4.2. *Let d be a bounded domain with piecewise smooth boundary (d can be also multiply connected); we assume that the intersection of d with any straight line consists of finite number of intervals. Then for any meromorphic function f in the closure of d and any integer $k \geq 1$ we have*

$$\iint\limits_d \left|\frac{f'(z)}{f(z)}\right| d\sigma \leq \iint\limits_d \left|\frac{f^{(k+1)}(z)}{f^{(k)}(z)}\right| d\sigma + \frac{k\pi}{2} l(\partial d). \tag{4.3}$$

Comment 1. In [3] we assumed that the intersection of d with any straight line consists of finite number of intervals. This restriction on "intersection" was putted just for simplicity of the proof. To avoid this it is enough to consider a domain d^* ("very close" to d) which satisfies this restriction. Then we can apply (4.3) to d^* and make limit transfer to d. We will come to the above wording of Lemma 4.1.

Assume now that f is our regular function w in $\bar{D}$ and d is one of the domains $D_\eta(|w| > R^*)$. Notice that the part of the boundary $\partial D_\eta(|w| > R^*)$ lying in D consists of piecewise analytic curves with a finite number of possible turning points where $w' = 0$. This implies that the boundary of each $\partial D_\eta(|w| > R^*)$ is piecewise smooth so that we can apply (4.3). Applying it for the derivative w' in a given domain $D_\eta(|w| > R^*)$ with $k \geq 2$ we have

(4.4)
$$\iint\limits_{D_\eta(|w|>R^*)} \left|\frac{w''(z)}{w'(z)}\right| d\sigma \leq \iint\limits_{D_\eta(|w|>R^*)} \left|\frac{w^{(k+1)}(z)}{w^{(k)}(z)}\right| d\sigma + \frac{\pi}{2}(k-1) l\left(\partial D_\eta(|w| > R^*)\right).$$

Further, we need the following "tangent variation principle" (see [2, item 1.2.2 inequalities 1.2.8 and 1.2.9]).

Lemma 4.3. *For any meromorphic function $f(z)$ in $\bar{D}$ and any smooth Jordan curve Γ (bounded or unbounded) with $\nu(\Gamma) < \infty$ we have*

$$L(D, \Gamma, f) \leq K(\Gamma) \left\{ \iint\limits_{D} \left|\frac{f''(z)}{f'(z)}\right| d\sigma + l(\partial D) \right\}, \tag{4.5}$$

where $K(\Gamma) = 3(\nu(\Gamma) + 1)$.

Comment 2. In particular case when Γ is a straight line, the above formula can be improved. Due to Theorem 1 in [5] we have in this case

$$L(D, \Gamma, f) \leq \frac{1}{2} \iint\limits_{D} \left|\frac{f''(z)}{f'(z)}\right| d\sigma + \frac{1}{2} l(\partial D). \tag{4.6}$$

Applying (4.4) to the regular function w in any of the domains $D_\eta(|w| > R^*)$ and combining with (4.4) we obtain: for any smooth Jordan curve Γ with $\nu(\Gamma) < \infty$,

$$L(D_\eta(|w| > R^*), \Gamma, w) \leq$$
$$K(\Gamma) \left\{ \iint\limits_{D_\eta(|w|>R^*)} \left|\frac{w^{(k+1)}(z)}{w^{(k)}(z)}\right| d\sigma + \frac{\pi}{2}(k-1)\, l\left(\partial D_\eta(|w| > R^*)\right) + l(D) \right\}.$$

Summing up this inequality by η we get the following formula for $D(|w| > R^*)$:

$$L(D(|w| > R^*), \Gamma, w) \leq$$
$$K(\Gamma) \left\{ \iint\limits_{D(|w|>R^*)} \left|\frac{w^{(k+1)}(z)}{w^{(k)}(z)}\right| d\sigma + \frac{\pi}{2}(k-1)\, l\left(\partial D(|w| > R^*)\right) + l(D) \right\},$$

where

$$l\left(\partial D(|w| > R^*)\right) = \sum_\eta l\left(\partial D_\eta(|w| > R^*)\right).$$

Applying (4.2) to the last inequality we obtain

$$L(D(|w|>R^*),\Gamma,w)\le K(\Gamma)\times \tag{4.7}$$

$$\left\{\iint\limits_{D(|w|>R^*)}\left|\frac{w^{(k+1)}(z)}{w^{(k)}(z)}\right|d\sigma+\frac{\pi(k-1)}{2}L(D,\Gamma(R^*),w)+\left(\frac{\pi\,(k-1)}{2}+1\right)l(D)\right\}.$$

Comment 3. For a straight line Γ we can apply (4.6) instead of (4.5). Respectively instead of (4.7) we get

$$L(D(|w|>R^*),\Gamma,w)\le\frac{1}{2}\iint\limits_{D(|w|>R^*)}\left|\frac{w^{(k+1)}(z)}{w^{(k)}(z)}\right|d\sigma+$$

$$\frac{\pi}{4}\,(k-1)\,L(D,\Gamma(R^*),w)+\frac{1}{2}\left(\frac{\pi\,(k-1)}{2}+1\right)l(D). \tag{4.8}$$

Now we consider a curve Γ in Theorem 3.1 which does not pass through zero. Assume a_Γ is the point on Γ which is the closest to the point 0; if we have more than one similar points we take arbitrary one of them. With this value a_Γ we define as above corresponding value $R^*_\Gamma\in(|a_\Gamma|/2,3|a_\Gamma|/4)$ and notice that the curve Γ (which we consider in w-plane) lies fully in the set $D(|w|>R^*_\Gamma)$. Respectively Gamma-lines of this Γ lie fully in the set $D\,(|w|>R^*_\Gamma)$ so that we have $L(D(|w|>R^*_\Gamma),\Gamma,w)=L(D,\Gamma,w)$ and (4.7) yields

$$L(D,\Gamma,w)\le K(\Gamma)\times \tag{4.9}$$

$$\left\{\iint\limits_{D(|w|>R^*_\Gamma)}\left|\frac{w^{(k+1)}(z)}{w^{(k)}(z)}\right|d\sigma+\frac{\pi}{2}\,(k-1)\,L(D,\Gamma(R^*_\Gamma),w)+\left(\frac{\pi\,(k-1)}{2}+1\right)l(D)\right\}.$$

Now we apply the last inequality to our solution $w(z)$ of equation (S^μ) for $\mu=2$, we have for any $z\in\bar{D}$

$$\left|\frac{w'''(z)}{w''(z)}\right|=\left|\frac{g'(z)\,(w(z))^\mu+\mu g(z)\,(w(z))^{\mu-1}\,w'(z)}{g(z)\,(w(z))^\mu}\right|\le\left|\frac{g'(z)}{g(z)}\right|+\mu\left|\frac{w'(z)}{w(z)}\right|.$$

Thus, due to definition of R^*_Γ, for any $z\in D(|w|>R^*_\Gamma)$ we have $|w(z)|>|a_\Gamma|/2$, consequently

$$\left|\frac{w'''(z)}{w''(z)}\right|\le\left|\frac{g'(z)}{g(z)}\right|+\frac{2\mu}{|a_\Gamma|}\,|w'(z)|$$

and taking into account that $D(|w|>R^*_\Gamma)\subset D$ we get

$$\iint\limits_{D(|w|>R^*_\Gamma)}\left|\frac{w'''(z)}{w''(z)}\right|d\sigma\le$$

$$\iint\limits_{D(|w|>R^*_\Gamma)}\left\{\left|\frac{g'(z)}{g(z)}\right|+\frac{2\mu}{|a_\Gamma|}\,|w'(z)|\right\}d\sigma\le\iint\limits_{D}\left|\frac{g'(z)}{g(z)}\right|d\sigma+\frac{2\mu}{|a_\Gamma|}\iint\limits_{D}|w'(z)|\,d\sigma.$$

Due to (4.1) we also have

$$L(D,\Gamma(R^*_\Gamma),w) \le \frac{4}{|a_\Gamma|}\iint\limits_D |w'|\,d\sigma$$

so that applying the last two inequalities to (4.9) applied for $\mu = 2$ we obtain

$$L(D,\Gamma,w) \le$$

$$K(\Gamma)\left\{\iint\limits_D \left|\frac{g'(z)}{g(z)}\right| d\sigma + \frac{2\pi+2\mu}{|a_\Gamma|}\iint\limits_D |w'(z)|\,d\sigma + \left(\frac{\pi}{2}+1\right) l(D)\right\}. \tag{4.10}$$

Since w and g are regular functions and μ is an integer we conclude that gw^μ is a regular function so that taking into account that $w'' = gw^\mu$ we have for an arbitrary $z_0 \in \bar{D}$

$$w'(z) - w'(z_0) = \int\limits_{z_0}^{z} w''(Z)dZ = \int\limits_{z_0}^{z} g(Z)\,(w(Z))^\mu\,dZ.$$

Consequently we have $|w'(z)| \le M(g)M^\mu(w)l_D(z,z_0) + |w'(z_0)|$, where $l_D(z,z_0)$ is the length of a curve, say γ, which lies in $\bar{D}$ and connects z and z_0. We always can connect z with a point $z^* \in \partial D$ and z_0 with a point $z_0^* \in \partial D$ by some curves with the lengths $l(D)/2$ and then can connect the points z^* and z_0^* by a part of the boundary ∂D of the length $l(D)/2$. Thus we always can take γ such that $l_D(z,z_0) \le 3l(D)/2$. Also we can take z_0 such that $|w'(z_0)|$ reaches its minimum in $\bar{D}$ (that is $|w'(z_0)| := m(w') := \min_{z\in\bar{D}} |w'(z)|$). With similar notations we obtain

$$\iint\limits_D |w'(z)|d\sigma \le \frac{3}{2}M(g)M^\mu(w)l(D)S(D) + m(w')S(D).$$

Consequently (4.10) implies

$$L(D,\Gamma,w) \le K(\Gamma)\frac{3\pi+3\mu}{|a_\Gamma|}M(g)M^\mu(w)l(D)S(D) +$$

$$K(\Gamma)\frac{2\pi+2\mu}{|a_\Gamma|}m(w')S(D) + K(\Gamma)\iint\limits_D \left|\frac{g'(z)}{g(z)}\right| d\sigma +$$

$$K(\Gamma)\frac{\pi+2}{2}l(D) = K_{21}M^\mu(w) + K_{22}m(w') + K_{23}, \tag{4.11}$$

with K_{21}, K_{22}, K_{23} given after Theorem 3.1. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. This theorem is a particular case of Theorem 3.1 where we deal with a straight line Γ. Due to Comment 3, we see that the constant $K(\Gamma)$ in (4.7) is replaced by $1/2$ for the straight line; respectively we should apply (4.8) (instead of (4.7)) in the above proofs. Applying (4.8) we obtain (4.9), (4.10) and

(4.11) with $K(\Gamma)$ replaced by $1/2$. Respectively we get the proof of Theorem 3.2 with the coefficients K_{31}, K_{32} and K_{33} given after Theorem 3.2.

Proof of Theorem 2.1. The next inequality giving interrelations between Blaschke characteristic and Gamma-lines was proved in [2, item 1.5], (see also [4, item 7.1])): *for any regular function w in D and any smooth Jordan curve Γ connecting a with ∞ we have* $\mathcal{N}(D,a,w) \leq L(D,\Gamma,w)$. Since any straight line passing through a contain two parts connecting a with ∞ we have for any straight line Γ

$$\mathcal{N}(D,a,w) \leq \frac{1}{2}L(D,\Gamma,w).$$

Due to Theorem 3.2 we have upper bounds $L(D,\Gamma,w)$ for any straight line Γ, which does not pass through zero. Respectively, Theorem 3.2 and the previous inequality give the following upper bounds for $\mathcal{N}(D,a,w)$:

$$\mathcal{N}(D,a,w) \leq \frac{1}{2}L(D,\Gamma,w) \leq \frac{1}{2}\left[K_{31}M^{\mu}(w) + K_{32}m(w') + K_{33}\right].$$

Denoting $K_{11} = \frac{1}{2}K_{31}$, $K_{12} = \frac{1}{2}K_{32}$ and $K_{13} = \frac{1}{2}K_{33}$ we obtain Theorem 2.1.

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STATISTICAL ESTIMATION FOR STATIONARY MODELS WITH TAPERED DATA

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Abstract. In this paper, we survey some recent results on parametric and nonparametric statistical estimation about the spectrum of stationary models with tapered data, as well as, a question concerning robustness of inferences, carried out on a linear stationary process contaminated by a small trend. We also discuss some questions concerning tapered Toeplitz matrices and operators, central limit theorems for tapered Toeplitz type quadratic functionals, and tapered Fejér-type singular integrals. These are the main tools for obtaining the corresponding results, and also are of interest in themselves. The processes considered will be discrete-time and continuous-time Gaussian, linear or Lévy-driven linear processes with memory.

MSC2010 numbers: 62F10; 62F12; 62G05; 60G10.

Keywords: tapered data; stationary processes; spectral density; parametric and nonparametric estimation; robustness.

1. Introduction

Let $\{X(t),\ t \in \mathbb{U}\}$ be a centered real-valued stationary process with spectral density $f(\lambda)$, $\lambda \in \Lambda$, and covariance function $r(t)$, $t \in \mathbb{U}$. We consider simultaneously the

continuous-time (c.t.) case, where $\mathbb{U} = \mathbb{R} := (-\infty, \infty)$, and the discrete-time (d.t.) case, where $\mathbb{U} = \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. The domain Λ of the frequency variable λ is $\Lambda = \mathbb{R}$ in the c.t. case, and $\Lambda := [-\pi.\pi]$ in the d.t. case.

We want to make statistical inferences (parametric and nonparametric estimation) about the spectrum of $X(t)$. In the classical setting, the inferences are based on an observed finite realization $\mathbf{X}_T$ of the process $X(t)$: $\mathbf{X}_T := \{X(t),\, t \in D_T\}$, where $D_T := [0, T]$ in the c.t. case and $D_T := \{1, \ldots, T\}$ in the d.t. case.

A sufficiently developed inferential theory is now available for stationary models based on the standard (non-tapered) data $\mathbf{X}_T$. We cite merely the following references Avram et al. [3], Casas and Gao [8], Dahlhaus [12], Dahlhaus and Wefelmeyer [14], Dzhaparidze [15], Dzhaparidze and Yaglom [16], Fox and Taqqu [17], Gao [18], Gao et al. [19], Ginovyan [20, 21, 24, 25], Giraitis et al. [37], Giraitis and Surgailis [38], Guyon [40], Has'minskii and Ibragimov [41], Heyde and Dai [42], Ibragimov [43, 44], Ibragimov and Khas'minskii [45], Leonenko and Sakhno [47], Taniguchi

[49], Taniguchi and Kakizawa [50], Tsai and Chan [52], Walker [53], Whittle [54], where can also be found additional references.

In the statistical analysis of stationary processes, however, the data are frequently tapered before calculating the statistic of interest, and the statistical inference procedure, instead of the original data $\mathbf{X}_T$, is based on the *tapered data*: $\mathbf{X}_T^h := \{h_T(t)X(t),\, t \in D_T\}\}$, where $h_T(t) := h(t/T)$ with $h(t)$, $t \in \mathbb{R}$ being a *taper function*.

The use of data tapers in nonparametric time series was suggested by Tukey [51]. The benefits of tapering the data have been widely reported in the literature (see, e.g., Brillinger [6], Dahlhaus [10, 11], Dahlhaus and Künsch [13], Guyon [40], and references therein). For example, data-tapers are introduced to reduce the so-called 'leakage effects', that is, to obtain better estimation of the spectrum of the model in the case where it contains high peaks. Other application of data-tapers is in situations in which some of the data values are missing. Also, the use of tapers leads to bias reduction, which is especially important when dealing with spatial data. In this case, the tapers can be used to fight the so-called 'edge effects'.

In this paper, we survey some recent results on parametric and nonparametric statistical estimation about the spectrum of stationary models with tapered data, as well as, a question concerning robustness of inferences, carried out on a linear stationary process contaminated by a small trend. We also discuss some questions concerning tapered Toeplitz matrices and operators, central limit theorems for tapered Toeplitz type quadratic functionals, and tapered Fejér-type kernels and singular integrals. These are the main tools for obtaining the corresponding results, and also are of interest in themselves. The processes considered will be discrete-time and continuous-time Gaussian, linear or Lévy-driven linear processes with memory.

The rest of the paper is structured as follows. In Section 2 we specify the model of interest - a stationary process, recall some key notions and results from the theory of stationary processes, and introduce the data tapers and tapered periodogram. In Section 3 we discuss the nonparametric estimation problem. We analyze the asymptotic properties, involving asymptotic unbiasedness, bias rate convergence, consistency, a central limit theorem and asymptotic normality of the empirical spectral functionals. In Section 4 we discuss the parametric estimation problem. We present sufficient conditions for consistency and asymptotic normality of minimum contrast estimator based on the Whittle contrast functional for stationary linear models with tapered data. A question concerning robustness of inferences, carried out on a linear stationary process contaminated by a small trend is discussed in Section 5. In Section 6 we briefly discuss the methods and tools, used to prove the results stated in Sections 3–5.

2. Preliminaries

In this section we specify the model of interest - a stationary process, recall some key notions and results from the theory of stationary processes, and introduce the data tapers and tapered periodogram.

2.1. **The model.** *Second-order (wide-sense) stationary process.* Let $\{X(u),\ u \in \mathbb{U}\}$ be a centered real-valued second-order (wide-sense) stationary process defined on a probability space $(\Omega, \mathcal{F}, P)$ with covariance function $r(t)$, that is, $\mathbb{E}[X(u)] = 0$, $r(u) = \mathbb{E}[X(t+u)X(t)]$, $u, t \in \mathbb{U}$, where $\mathbb{E}[\cdot]$ stands for the expectation operator with respect to measure P. We consider simultaneously the c.t. case, where $\mathbb{U} = \mathbb{R} := (-\infty, \infty)$, and the d.t. case, where $\mathbb{U} = \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. We assume that $X(u)$ is a *non-degenerate process*, that is, $\mathrm{Var}[X(u)] = \mathbb{E}|X(u)|^2 = r(0) > 0$. (Without loss of generality, we assume that $r(0) = 1$). In the c.t. case the process $X(u)$ is also assumed mean-square continuous, that is, $\mathbb{E}[X(t) - X(s)]^2 \to 0$ as $t \to s$.

By the Herglotz theorem in the d.t. case, and the Bochner-Khintchine theorem in the c.t. case (see, e.g., Cramér and Leadbetter [9]), there is a finite measure μ on $(\Lambda, \mathfrak{B}(\Lambda))$, where $\Lambda = \mathbb{R}$ in the c.t. case, and $\Lambda = [-\pi.\pi]$ in the d.t. case, and $\mathfrak{B}(\Lambda)$ is the Borel σ-algebra on Λ, such that for any $u \in \mathbb{U}$ the covariance function $r(u)$ admits the following *spectral representation*:

$$r(u) = \int_\Lambda \exp\{i\lambda u\} d\mu(\lambda), \quad u \in \mathbb{U}. \tag{2.1}$$

The measure μ in (2.1) is called the *spectral measure* of the process $X(u)$. The function $F(\lambda) := \mu[-\pi, \lambda]$ in the d.t. case and $F(\lambda) := \mu[-\infty, \lambda]$ in the c.t. case, is called the *spectral function* of the process $X(t)$. If $F(\lambda)$ is absolutely continuous (with respect to Lebesgue measure), then the function $f(\lambda) := dF(\lambda)/d\lambda$ is called the *spectral density* of the process $X(t)$. Notice that if the spectral density $f(\lambda)$ exists, then $f(\lambda) \geq 0$, $f(\lambda) \in L^1(\Lambda)$, and (2.1) becomes

$$r(u) = \int_\Lambda \exp\{i\lambda u\} f(\lambda) d\lambda, \quad u \in \mathbb{U}. \tag{2.2}$$

Thus, the covariance function $r(u)$ and the spectral function $F(\lambda)$ (resp. the spectral density $f(\lambda)$) are equivalent specifications of the second order properties for a stationary process $X(u)$.

Linear processes. Existence of spectral density functions. We consider here stationary processes possessing spectral densities. For the following results we refer to Ibragimov and Linnik [46].

(a) The spectral function $F(\lambda)$ of a d.t. stationary process $\{X(u),\ u \in \mathbb{Z}\}$ is absolutely continuous (with respect to the Lebesgue measure) if and only if it can be represented as an infinite moving average:

$$X(u) = \sum_{k=-\infty}^{\infty} a(u-k)\xi(k), \qquad \sum_{k=-\infty}^{\infty} |a(k)|^2 < \infty, \tag{2.3}$$

where $\{\xi(k), k \in \mathbb{Z}\} \sim$ WN(0,1) is a standard white-noise, that is, a sequence of orthonormal random variables.

(b) The covariance function $r(u)$ and the spectral density $f(\lambda)$ of $X(u)$ are given by formulas:

$$r(u) = \sum_{k=-\infty}^{\infty} a(u+k)a(k), \quad f(\lambda) = \frac{1}{2\pi}\left|\sum_{k=-\infty}^{\infty} a(k)e^{-ik\lambda}\right|^2, \quad \lambda \in [-\pi, \pi]. \tag{2.4}$$

Similar results hold for c.t. processes. Indeed, the following holds.

(a) The spectral function $F(\lambda)$ of a c.t. stationary process $\{X(u),\, u \in \mathbb{R}\}$ is absolutely continuous (with respect to Lebesgue measure) if and only if it can be represented as an infinite continuous moving average:

$$X(u) = \int_{\mathbb{R}} a(u-t)d\xi(t),, \qquad \int_{\mathbb{R}} |a(t)|^2 dt < \infty, \tag{2.5}$$

where $\{\xi(t), t \in \mathbb{R}\}$ is a process with orthogonal increments and $\mathbb{E}|d\,\xi(t)|^2 = dt$.

(b) The covariance function $r(u)$ and the spectral density $f(\lambda)$ of $X(u)$ are given by formulas:

$$r(u) = \int_{\mathbb{R}} a(u+x)a(x)dx, \quad f(\lambda) = \frac{1}{2\pi}\left|\int_{\mathbb{R}} e^{-i\lambda t}a(t)dt\right|^2, \quad \lambda \in \mathbb{R}. \tag{2.6}$$

The function $a(\cdot)$ in representations (2.3) and (2.5) plays the role of a *time-invariant filter*, and the linear processes defined by (2.3) and (2.5) can be viewed as the output of a linear filter $a(\cdot)$ applied to the process $\xi(t)$, called the innovation or driving process of $X(t)$.

Processes of the form (2.3) and (2.5) appear in many fields of science (economics, finance, physics, etc.), and cover large classes of popular models in time series modeling. For instance, the classical autoregressive moving average models and their continuous counterparts the c.t. autoregressive moving average models are of the form (2.3) and (2.5), respectively, and play a central role in the representations of stationary time series (see, e.g., Brockwell and Davis [7]).

Lévy-driven linear process. We first recall that a Lévy process, $\{\xi(t),\ t \in \mathbb{R}\}$ is a process with independent and stationary increments, continuous in probability, with sample-paths which are right-continuous with left limits (càdlàg) and $\xi(0) = \xi(0-) = 0$. The Wiener process $\{B(t),\ t \geq 0\}$ and the centered Poisson process $\{N(t) - \mathbb{E}N(t),\ t \geq 0\}$ are typical examples of centered Lévy processes. A Lévy-driven linear process $\{X(t),\ t \in \mathbb{R}\}$ is a real-valued c.t. stationary process defined by

(2.5), where $\xi(t)$ is a Lévy process satisfying the conditions: $\mathbb{E}\xi(t) = 0$, $\mathbb{E}\xi^2(1) = 1$ and $\mathbb{E}\xi^4(1) < \infty$. In the case where $\xi(t) = B(t)$, $X(t)$ is a Gaussian process (see Bai et al. [4]):

Dependence (memory) structure of the model. In the frequency domain setting, the statistical and spectral analysis of stationary processes requires *two types of conditions* on the spectral density $f(\lambda)$. The first type controls the *singularities* of $f(\lambda)$, and involves the *dependence (or memory) structure* of the process, while the second type – controls the *smoothness* of $f(\lambda)$. The memory structure of a stationary process is essentially a measure of the dependence between all the variables in the process, considering the effect of all correlations simultaneously. Traditionally memory structure has been defined in the time domain in terms of decay rates of the autocorrelations, or in the frequency domain in terms of rates of explosion of low frequency spectra (see, e.g., Beran et al. [5], Giraitis et al. [37], Guégan [39]). It is convenient to characterize the memory structure in terms of the spectral density function. We will distinguish the following types of stationary models:

(a) short memory (or short-range dependent),

(b) long memory (or long-range dependent),

(c) intermediate memory (or anti-persistent).

Short-memory models. Much of statistical inference is concerned with *short-memory* stationary models, where the spectral density $f(\lambda)$ of the model is bounded away from zero and infinity, that is, there are constants C_1 and C_2 such that $0 < C_1 \le f(\lambda) \le C_2 < \infty$.

A typical d.t. short memory model example is the stationary Autoregressive Moving Average (ARMA)(p, q) process $X(t)$ defined to be a stationary solution of the difference equation:

$$\psi_p(B)X(t) = \theta_q(B)\varepsilon(t), \quad t \in \mathbb{Z},$$

where ψ_p and θ_q are polynomials of degrees p and q, respectively, B is the backshift operator defined by $BX(t) = X(t-1)$, and $\{\varepsilon(t), t \in \mathbb{Z}\}$ is a d.t. white noise, that is, a sequence of zero-mean, uncorrelated random variables with variance σ^2. The spectral density $f(\lambda)$ of (ARMA)(p, q) process is a rational function (see, e.g., Brockwell and Davis [7], Section 3.1):

$$f(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{|\theta_q(e^{-i\lambda})|^2}{|\psi_p(e^{-i\lambda})|^2}. \tag{2.7}$$

A typical c.t. short-memory model example is the stationary c.t. ARMA(p, q) processes, denoted by CARMA(p, q). The spectral density function $f(\lambda)$ of a

CARMA(p, q) process $X(t)$ is given by the following formula (see, e.g., Tsai and Chan [52]):

$$f(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{|\beta_q(i\lambda)|^2}{|\alpha_p(i\lambda)|^2}, \tag{2.8}$$

where $\alpha_p(z)$ and $\beta_q(z)$ are polynomials of degrees p and q, respectively.

Discrete-time long-memory and anti-persistent models. Data in many fields of science (economics, finance, hydrology, etc.), however, is well modeled by stationary processes whose spectral densities are *unbounded* or *vanishing* at some fixed points (see, e.g., Beran et al. [5], Guégan [39], and references therein). A *long-memory* model is defined to be a stationary process with *unbounded* spectral density, and an *anti-persistent* model – a stationary process with *vanishing* (at some fixed points) spectral density.

In the discrete context, a basic model that displays long-memory or is anti-persistent is the Autoregressive Fractionally Integrated Moving Average (ARFIMA) $(p, d, q))$ process $X(t)$ defined to be a stationary solution of the difference equation:

$$\psi_p(B)(1 - B)^d X(t) = \theta_q(B)\varepsilon(t), \quad d < 1/2,$$

where B is the backshift operator, $\varepsilon(t)$ is a d.t. white noise, and ψ_p and θ_q are polynomials of degrees p and q, respectively. The spectral density $f_X(\lambda)$ of $X(t)$ is given by

$$f_X(\lambda) = |1 - e^{-i\lambda}|^{-2d} f(\lambda) = (2\sin(\lambda/2))^{-2d} f(\lambda), \quad d < 1/2, \tag{2.9}$$

where $f(\lambda)$ is the spectral density of an ARMA(p, q) process, given by (2.7). Observe that for $0 < d < 1/2$ the model $X(t)$ specified by the spectral density (2.9) displays long-memory, for $d < 0$ – intermediate-memory, and for $d = 0$ – short-memory. For $d \geq 1/2$ the function $f_X(\lambda)$ in (2.9) is not integrable, and thus it cannot represent a spectral density of a stationary process.

Continuous-time long-memory and anti-persistent models. In the continuous context, a basic process which has commonly been used to model long-range dependence is the *fractional Brownian motion* (fBm) $\{B_H(t), t \in \mathbb{R}\}$ with Hurst index H, $0 < H < 1$, defined to be a centered Gaussian H-self-similar process having stationary increments. The fBm B_H can be regarded as a Gaussian process having a 'spectral density':

$$f(\lambda) = c|\lambda|^{-(2H+1)}, \quad c > 0, \quad 0 < H < 1, \quad \lambda \in \mathbb{R}. \tag{2.10}$$

The form (2.10) can be understood in a generalized sense (see, e.g., Yaglom [55]), since the fBm B_H is a nonstationary process.

A proper stationary model in lieu of fBm is the *fractional Riesz-Bessel motion* (fRBm), introduced in Anh et al. [1], and defined as a c.t. Gaussian process $X(t)$

with spectral density

$$f(\lambda) = c\,|\lambda|^{-2\alpha}(1+\lambda^2)^{-\beta}, \quad \lambda \in \mathbb{R},\, 0<c<\infty,\, 0<\alpha<1,\, \beta>0. \tag{2.11}$$

The exponent α determines the long-range dependence, while the exponent β indicates the second-order intermittency of the process (see, e.g., Anh et al. [2] and Gao et al. [19]).

Notice that the process $X(t)$, specified by the spectral density (2.11), is stationary if $0<\alpha<1/2$ and is non-stationary with stationary increments if $1/2 \le \alpha < 1$.

Comparing (2.10) and (2.11), we observe that the spectral density of fBm is the limiting case as $\beta \to 0$ that of fRBm with Hurst index $H = \alpha - 1/2$.

Another important c.t. long-memory model is the CARFIMA(p,H,q) process. The spectral density $f(\lambda)$ of a CARFIMA(p,H,q) process is given by formula (see, e.g., Tsai and Chan [52]):

$$f(\lambda) = \frac{\sigma^2}{2\pi}\Gamma(2H+1)\sin(\pi H)|\lambda|^{1-2H}\frac{|\beta_q(i\lambda)|^2}{|\alpha_p(i\lambda)|^2}, \tag{2.12}$$

where $\alpha_p(z)$ and $\beta_q(z)$ are polynomials of degrees p and q, respectively. Notice that for $H=1/2$, the spectral density given by (2.12) becomes that of the short-memory CARMA(p,q) process, given by (2.8).

2.2. **Data tapers and tapered periodogram.** Our inference procedures will be based on the tapered data $\mathbf{X}_T^h$:

$$\mathbf{X}_T^h := \begin{cases} \{h_T(t)X(t),\, t=1,\ldots,T\} & \text{in the d.t. case,} \\ \{h_T(t)X(t),\, 0\le t\le T\} & \text{in the c.t. case,} \end{cases} \tag{2.13}$$

where

$$h_T(t) := h(t/T) \tag{2.14}$$

with $h(t)$, $t\in\mathbb{R}$ being a *taper function*.

Throughout the paper, we will assume that the taper function $h(\cdot)$ satisfies the following assumption.

Assumption 2.1. The taper $h:\mathbb{R}\to\mathbb{R}$ is a continuous nonnegative function of bounded variation and of bounded support $[0,1]$, such that $H_k \neq 0$, where

$$H_k := \int_0^1 h^k(t)dt, \quad k\in\mathbb{N} := \{1,2,\ldots\}. \tag{2.15}$$

Note. The case $h(t) = \mathbb{I}_{[0,1]}(t)$, where $\mathbb{I}_{[0,1]}(\cdot)$ denotes the indicator of the segment $[0,1]$, will be referred to as the *non-tapered* case.

Remark 2.1. For the d.t. case, an example of a taper function $h(t)$ satisfying Assumption 2.1 is the Tukey-Hanning taper function $h(t) = 0.5(1-\cos(\pi t))$ for $t\in[0,1]$. For the c.t. case, a simple example of a taper function $h(t)$ satisfying Assumption 2.1 is the function $h(t) = 1-t$ for $t\in[0,1]$.

Denote by $H_{k,T}(\lambda)$ the *tapered Dirichlet type kernel*, defined by

$$H_{k,T}(\lambda) := \begin{cases} \sum_{t=1}^{T} h_T^k(t)e^{-i\lambda t} & \text{in the d.t. case,} \\ \int_0^T h_T^k(t)e^{-i\lambda t}dt & \text{in the c.t. case.} \end{cases} \tag{2.16}$$

Define the finite Fourier transform of the tapered data (2.13):

$$d_T^h(\lambda) := \begin{cases} \sum_{t=1}^{T} h_T(t)X(t)e^{-i\lambda t} & \text{in the d.t. case,} \\ \int_0^T h_T(t)X(t)e^{-i\lambda t}dt & \text{in the c.t. case.} \end{cases} \tag{2.17}$$

and the tapered periodogram $I_T^h(\lambda)$ of the process $X(t)$:

$$I_T^h(\lambda) := \frac{1}{C_T}\, d_T^h(\lambda)d_T^h(-\lambda) = \tag{2.18}$$

$$= \begin{cases} \frac{1}{C_T}\left|\sum_{t=1}^{T} h_T(t)X(t)e^{-i\lambda t}\right|^2 & \text{in the d.t. case,} \\ \frac{1}{C_T}\left|\int_0^T h_T(t)X(t)e^{-i\lambda t}dt\right|^2 & \text{in the c.t. case.} \end{cases}$$

where

$$C_T := 2\pi H_{2,T}(0) \neq 0. \tag{2.19}$$

Notice that for non-tapered case ($h(t) = \mathbb{I}_{[0,1]}(t)$), we have $C_T = 2\pi T$.

3. Nonparametric estimation problem

Suppose we observe a finite realization $\mathbf{X}_T := \{X(u), 0 \leq u \leq T$ (or $u = 1, \dots, T$ in the d.t. case)$\}$ of a centered stationary process $X(u)$ with an *unknown* spectral density function $f(\lambda)$, $\lambda \in \Lambda$. We assume that $f(\lambda)$ belongs to a given (infinite-dimensional) class $\mathcal{F} \subset L^p := L^p(\Lambda)$ $(p \geq 1)$ of spectral densities possessing some specified smoothness properties. The problem is to estimate the value $J(f)$ of a given functional $J(\cdot)$ at an *unknown* 'point' $f \in \mathcal{F}$ on the basis of an observation $\mathbf{X}_T$, and investigate the asymptotic (as $T \to \infty$) properties of the suggested estimators, depending on the dependence structure of the model $X(u)$ and the smoothness structure of the 'parametric' set $\mathcal{F} \subset L^p(\Lambda)$ $(p \geq 1)$.

Linear and non-linear functionals of the periodogram play a key role in the parametric estimation of the spectrum of stationary processes, when using the minimum contrast estimation method with various contrast functionals (see, e.g., Dzhaparidze [15], Guyon [40], Leonenko and Sakhno [47], Taniguchi and Kakizawa [50], and references therein). In this section, we review the asymptotic properties, involving asymptotic unbiasedness, bias rate convergence, consistency, a central limit theorem and asymptotic normality of the empirical spectral functionals based on the tapered data. Some of these properties were discussed and proved in Ginovyan and Sahakyan [34, 35]. For non-tapered case, these properties were established in the papers Ginovyan [22, 25]. The results stated in this section are used to prove consistency and asymptotic normality of the minimum contrast estimator based on the Whittle contrast functional for stationary linear models with tapered data (see Section 4). Here we follow the papers Ginovyan [23, 25, 26], and Ginovyan and Sahakyan [34, 35].

3.1. **Estimation of linear spectral functionals.** We are interested in the nonparametric estimation problem, based on the tapered data (2.13), of the following linear spectral functional:

$$J = J(f,g) := \int_\Lambda f(\lambda)g(\lambda)d\lambda, \tag{3.1}$$

where $g(\lambda) \in L^q(\Lambda),\ 1/p + 1/q = 1$.

As an estimator J_T^h for functional $J(f)$, given by (3.1), based on the tapered data (2.13), we consider the averaged tapered periodogram (or a simple 'plug-in' statistic), defined by

$$J_T^h \quad = \quad J(I_T^h) := \int_\Lambda I_T^h(\lambda)g(\lambda)d\lambda, \tag{3.2}$$

where $I_T^h(\lambda)$ is the tapered periodogram of the process $X(t)$ given by (2.18). Denote

$$Q_T^h := \begin{cases} \sum_{t=1}^T \sum_{s=1}^T \widehat{g}(t-s)h_T(t)h_T(s)X(t)X(s) & \text{in the d.t. case,} \\ \int_0^T \int_0^T \widehat{g}(t-s)h_T(t)h_T(s)X(t)X(s)\,dt\,ds & \text{in the c.t. case,} \end{cases} \tag{3.3}$$

where $\widehat{g}(t)$ is the Fourier transform of function $g(\lambda)$:

$$\widehat{g}(t) := \int_\Lambda e^{i\lambda t} g(\lambda)d\lambda, \quad t \in \Lambda. \tag{3.4}$$

In view of (2.18) and (3.2) – (3.4) we have

$$J_T^h = C_T^{-1} Q_T^h, \tag{3.5}$$

where C_T is as in (2.19). We will refer to $g(\lambda)$ and to its Fourier transform $\widehat{g}(t)$ as a *generating function* and *generating kernel* for the functional J_T^h, respectively.

Thus, to study the asymptotic properties of the estimator J_T^h, we have to study the asymptotic distribution (as $T \to \infty$) of the tapered Toeplitz type quadratic functional Q_T^h given by (3.3) (for details see Section 6.2).

3.2. **Asymptotic unbiasedness.** We begin with the following assumption.

Assumption 3.1. The function

$$\Psi(u) = \int_\Lambda f(v)g(u+v)\,dv \tag{3.6}$$

belongs to $L^1(\Lambda) \cap L^2(\Lambda)$ and is continuous at $u = 0$.

Theorem 3.1. *Let the functionals* $J := J(f, g)$ *and* $J_T^h := J(I_T^h, g)$ *be defined by (3.1) and (3.2), respectively. Then under Assumptions 2.1 and 3.1 the statistic* J_T^h *is an asymptotically unbiased estimator for* $J(f)$*, that is, the following relation holds:*

$$\lim_{T\to\infty} [E(J_T^h) - J] = 0. \tag{3.7}$$

Remark 3.1. Using Hölder inequality, it can easily be shown that if $f \in L^1(\Lambda) \cap L^{p_1}(\Lambda)$ and $g \in L^1(\Lambda) \cap L^{p_2}(\Lambda)$ with $1 \le p_1, p_2 \le \infty$, $1/p_1 + 1/p_2 \le 1$, then the relation (3.7) is satisfied.

Under additional smoothness conditions on functions $f(\lambda)$ and $g(\lambda)$ we can estimate the rate of convergence in (3.7). To state the corresponding result, we first introduce some notation and assumptions.

Given numbers $p \ge 1$, $0 < \alpha < 1$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers, we set $\beta = \alpha + r$ and denote by $H_p(\beta)$ the L^p-Hölder class, that is, the class of those functions $\psi(\lambda) \in L^p(\Lambda)$, which have r-th derivatives in $L^p(\Lambda)$ and with some positive constant C satisfy

$$||\psi^{(r)}(\cdot + h) - \psi^{(r)}(\cdot)||_p \le C|h|^\alpha.$$

Assumption 3.2. We say that a pair of integrable functions $(f(\lambda), g(\lambda))$, $\lambda \in \Lambda$, satisfies condition $(\mathcal{H})$, and write $(f, g) \in (\mathcal{H})$, if $f \in H_p(\beta_1)$ for $\beta_1 > 0$, $p > 1$ and $g \in H_q(\beta_2)$ for $\beta_2 > 0$, $q > 1$ with $1/p + 1/q = 1$, and one of the conditions a) – d) is satisfied:

a) $\beta_1 > 1/p$, $\beta_2 > 1/q$,

b) $\beta_1 \le 1/p$, $\beta_2 \le 1/q$ and $\beta_1 + \beta_2 > 1/2$,

c) $\beta_1 > 1/p$, $1/q - 1/2 < \beta_2 \le 1/q$,

d) $\beta_2 > 1/q$, $1/p - 1/2 < \beta_1 \le 1/p$.

Remark 3.2. In Ginovian [22] it was proved that if $(f, g) \in (\mathcal{H})$, then there exist numbers p_1 $(p_1 > p)$ and q_1 $(q_1 > q)$, such that $H_p(\beta_1) \subset L_{p_1}$, $H_q(\beta_2) \subset L_{q_1}$ and $1/p_1 + 1/q_1 \le 1/2$.

Assumption 3.3. The spectral density f and the generating function g are such that $f, g \in L^1(\Lambda) \cap L^2(\Lambda)$ and g is of bounded variation.

The following theorem controls the bias $E(J_T^h) - J$ and provides sufficient conditions assuring the proper rate of convergence of bias to zero, necessary for asymptotic normality of the estimator J_T^h. Specifically, we have the following result.

Theorem 3.2. *Let the functionals* $J := J(f, g)$ *and* $J_T^h := J(I_T^h, g)$ *be defined by (3.1) and (3.2), respectively. Then under Assumptions 2.1 and 3.2 (or 3.3), the* *following asymptotic relation holds:*

$$T^{1/2}\left[\mathbb{E}(J_T^h) - J\right] \to 0 \quad \text{as} \quad T \to \infty. \tag{3.8}$$

Remark 3.3. We call an estimator J_T^h of J *asymptotically unbiased of the order of* T^β, $\beta > 0$ if $\lim_{T\to\infty} T^\beta[E(J_T^h) - J] = 0$. Thus, Theorem 3.2 states that the statistic J_T^h is an asymptotically unbiased estimator for J of the order of $T^{1/2}$.

3.3. **Consistency.** Recall that an estimator J_T^h of J is said to be (a) consistent if $J_T^h \to J$ in probability as $T \to \infty$, (b) mean square consistent if $\mathbb{E}(J_T^h - J)^2 \to 0$ as $T \to \infty$, (c) $\sqrt{T}$-consistent in the mean square sense if $\mathbb{E}\left([\sqrt{T}(J_T^h - J)]^2\right) = O(1)$ as $T \to \infty$,

To state the corresponding results we first introduce the following assumption.

Assumption 3.4. The filter $a(\cdot)$ and the generating kernel $\widehat{g}(\cdot)$ are such that

$$a(\cdot) \in L^p(\Lambda) \cap L^2(\Lambda), \quad \widehat{g}(\cdot) \in L^q(\Lambda) \quad \text{with} \quad 1 \le p, q \le 2, \quad 2/p + 1/q \ge 5/2.$$

We begin with a result on the asymptotic behavior of the variance $\mathrm{Var}(J_T^h) = \mathbb{E}(J_T^h - \mathbb{E}(J_T^h))^2$. The proof of the next theorem can be found in Ginovyan and Sahakyan [34].

Theorem 3.3. *Let the functionals* $J := J(f, g)$ *and* $J_T^h := J(I_T^h, g)$ *be defined by (3.1) and (3.2), respectively. Then under Assumptions 2.1 and 3.4 the following asymptotic relation holds:*

$$\lim_{T\to\infty} T\mathrm{Var}(J_T^h) = \sigma_h^2(J), \tag{3.9}$$

where

$$\sigma_h^2(J) := 4\pi e(h) \int_\Lambda f^2(\lambda) g^2(\lambda) d\lambda + \kappa_4 e(h) \left[\int_\Lambda f(\lambda) g(\lambda) d\lambda\right]^2. \tag{3.10}$$

Here κ_4 *is the fourth cumulant of* $\xi(1)$, *and*

$$e(h) := \frac{H_4}{H_2^2} = \int_0^1 h^4(t)dt \left(\int_0^1 h^2(t)dt\right)^{-2}. \tag{3.11}$$

From Theorems 3.1–3.3 we infer the following result.

Theorem 3.4. *The following assertions hold.*

(a) *Under Assumptions 2.1, 3.1 and 3.4 the statistic* J_T^h *is a mean square consistent estimator for* J.
(b) *Under Assumptions 2.1, 3.2 (or 3.3) and 3.4 the statistic* J_T^h *is a* $\sqrt{T}$-*consistent in the mean square sense estimator for* J.

3.4. **Asymptotic normality.** The next result contains sufficient conditions for functional J_T^h to obey the central limit theorem (CLT), and was proved in Ginovyan and Sahakyan [34].

Theorem 3.5 (CLT). *Let $J := J(f,g)$ and $J_T^h := J(I_T^h, g)$ be defined by (3.1) and (3.2), respectively. Then under Assumptions 2.1 and 3.4 the functional J_T^h obeys the central limit theorem. More precisely, we have*

$$T^{1/2}\left[J_T^h - \mathbb{E}(J_T^h)\right] \xrightarrow{d} \eta \quad \text{as} \quad T \to \infty, \tag{3.12}$$

where the symbol $\xrightarrow{d}$ stands for convergence in distribution, and η is a normally distributed random variable with mean zero and variance $\sigma_h^2(J)$ given by (3.10) *and* (3.11).

Taking into account the equality

$$T^{1/2}\left[J_T^h - J\right] = T^{1/2}\left[\mathbb{E}(J_T^h) - J\right] + T^{1/2}\left[J_T^h - \mathbb{E}(J_T^h)\right], \tag{3.13}$$

as an immediate consequence of Theorems 3.2 and 3.5, we obtain the next result that contains sufficient conditions for a simple 'plug-in' statistic $J(I_T^h)$ to be an asymptotically normal estimator for a linear spectral functional J.

Theorem 3.6. *Let the functionals $J := J(f,g)$ and $J_T^h := J(I_T^h, g)$ be defined by (3.1) and (3.2), respectively. Then under Assumptions 2.1, 3.2 (or 3.3) and 3.4 the statistic J_T^h is an asymptotically normal estimator for functional J. More precisely, we have*

$$T^{1/2}\left[J_T^h - J\right] \xrightarrow{d} \eta \quad \text{as} \quad T \to \infty, \tag{3.14}$$

where η is as in Theorem 3.5, that is, η is a normally distributed random variable with mean zero and variance $\sigma_h^2(J)$ given by (3.10) *and* (3.11).

Remark 3.4. Notice that if the underlying process $X(u)$ is Gaussian, then in formula (3.10) we have only the first term. Using the results from Ginovyan [22] and Ginovyan and Sahakyan [29, 30], it can be shown that in this case Theorem 3.6 is true under Assumptions 2.1 and 3.4.

4. Parametric estimation problem

We assume here that the spectral density $f(\lambda)$ belongs to a given parametric family of spectral densities $\mathcal{F} := \{f(\lambda,\theta) : \theta \in \Theta\}$, where $\theta := (\theta_1, \dots, \theta_p)$ is an unknown parameter and Θ is a subset in the Euclidean space $\mathbb{R}^p$. The problem of interest is to estimate the unknown parameter θ on the basis of the tapered data (2.13), and investigate the asymptotic (as $T \to \infty$) properties of the suggested estimators, depending on the dependence (memory) structure of the model $X(t)$ and the smoothness of its spectral density f.

There are different methods of estimation: maximum likelihood, Whittle, minimum contrast, etc. Here we focus on the Whittle method.

4.1. **The Whittle estimation procedure.** The Whittle estimation procedure, originally devised for d.t. short memory stationary processes, is based on the smoothed periodogram analysis on a frequency domain, involving approximation of the likelihood function and asymptotic properties of empirical spectral functionals (see Whittle [54]). The Whittle estimation method since its discovery has played a major role in the asymptotic theory of parametric estimation in the frequency domain, and was the focus of interest of many statisticians. Their aim was to weaken the conditions needed to guarantee the validity of the Whittle approximation for d.t. short memory models, to find analogues for long and intermediate memory models, to find conditions under which the Whittle estimator is asymptotically equivalent to the exact maximum likelihood estimator, and to extend the procedure to the c.t. models and random fields.

For the d.t. case, it was shown that for Gaussian and linear stationary models the Whittle approach leads to consistent and asymptotically normal estimators under short, intermediate and long memory assumptions. Moreover, it was shown that in the Gaussian case the Whittle estimator is also asymptotically efficient in the sense of Fisher (see, e. g., Dahlhaus [12], Dzhaparidze [15], Fox and Taqqu [17], Giraitis and Surgailis [38], Guyon [40], Taniguchi and Kakizawa [50], Walker [53], and references therein).

For c.t. models, the Whittle estimation procedure has been considered, for example, in Avram et al. [3], Casas and Gao [8], Dzhaparidze and Yaglom [16], Gao [18], Gao et al. [19], Leonenko and Sakhno [47], Tsai and Chan [52], where can also be found additional references. In this case, it was proved that the Whittle estimator is consistent and asymptotically normal.

The Whittle estimation procedure based on the d.t. tapered data has been studied in Dahlhaus [10], Dahlhaus and Künsch [13], Guyon [40], Ludeña and Lavielle [48]. In the case where the underlying model is a Lévy-driven c.t. linear process with possibly unbounded or vanishing spectral density function, consistency and asymptotic normality of the Whittle estimator was established in Ginovyan [27].

To explain the idea behind the Whittle estimation procedure, assume for simplicity that the underlying process $X(t)$ is a d.t. Gaussian process, and we want to estimate the parameter θ based on the sample $X_T := \{X(t), t = 1, \dots, T\}$. A natural approach is to find the maximum likelihood estimator (MLE) $\widehat{\theta}_{T,MLE}$ of θ, that is, to maximize the likelihood function, or to minimize the $-1/T\times$log-likelihood function $L_T(\theta)$, which in this case takes the form:

$$L_T(\theta) := \frac{1}{2}\ln 2\pi + \frac{1}{2T}\ln\det B_T(f_\theta) + \frac{1}{2T}X_T'[B_T(f_\theta)]^{-1}X_T,$$

where $B_T(f_\theta)$ is the Toeplitz matrix generated by f_θ. Unfortunately, the above function is difficult to handle, and no explicit expression for the estimator $\widehat{\theta}_{T,MLE}$ is known (even in the case of simple models). An approach, suggested by P. Whittle, called the Whittle estimation procedure, is to approximate the term $\ln\det B_T(f_\theta)$ by $\frac{T}{2}\int_{-\pi}^{\pi}\ln f_\theta(\lambda)d\lambda$ and the inverse matrix $[B_T(f_\theta)]^{-1}$ by the Toeplitz matrix $B_T(1/f_\theta)$. This leads to the following approximation of the log-likelihood function $L_T(\theta)$, introduced by Whittle [54], and called Whittle functional:

$$L_{T,W}(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[\ln f_\theta(\lambda) + \frac{I_T(\lambda)}{f_\theta(\lambda)}\right]d\lambda,$$

where $I_T(\lambda)$ is the ordinary periodogram of the process $X(t)$.

Now minimizing the Whittle functional $L_{T,W}(\theta)$ with respect to θ, we get the Whittle estimator $\widehat{\theta}_T$ for θ. It can be shown that if

$$T^{1/2}(L_T(\theta) - L_{T,W}(\theta) \to 0 \quad \text{as} \quad n\to\infty \quad \text{in probability},$$

then the MLE $\widehat{\theta}_{T,MLE}$ and the Whittle estimator $\widehat{\theta}_T$ are asymptotically equivalent in the sense that $\widehat{\theta}_T$ also is consistent, asymptotically normal and asymptotically Fisher-efficient (see, e.g., Dzhaparidze and Yaglom [16]).

In the continuous context, the Whittle procedure of estimation of a spectral parameter θ based on the sample $X_T := \{X(t),\, 0\le t\le T\}$ is to choose the estimator $\widehat{\theta}_T$ to minimize the weighted Whittle functional:

$$U_T(\theta) := \frac{1}{4\pi}\int_{\mathbb{R}}\left[\ln f(\lambda,\theta) + \frac{I_T(\lambda)}{f(\lambda,\theta)}\right]\cdot w(\lambda)\,d\lambda, \tag{4.1}$$

where $I_T(\lambda)$ is the continuous periodogram of $X(t)$, and $w(\lambda)$ is a weight function ($w(-\lambda) = w(\lambda)$, $w(\lambda)\ge 0$, $w(\lambda)\in L^1(\mathbb{R})$) for which the integral in (4.1) is well defined. An example of common used weight function is $w(\lambda) = 1/(1+\lambda^2)$.

The Whittle procedure of estimation of a spectral parameter θ based on the tapered sample (2.13) is to choose the estimator $\widehat{\theta}_{T,h}$ to minimize the weighted tapered Whittle functional:

$$U_T^h(\theta) := \frac{1}{4\pi}\int_{\Lambda}\left[\log f(\lambda,\theta) + \frac{I_T^h(\lambda)}{f(\lambda,\theta)}\right]\cdot w(\lambda)\,d\lambda, \tag{4.2}$$

where $I_T^h(\lambda)$ is the tapered periodogram of $X(t)$, given by (2.18), and $w(\lambda)$ is a weight function for which the integral in (4.2) is well defined. Thus, the Whittle estimator $\widehat{\theta}_{T,h}$ of θ based on the tapered sample (2.13) is defined by

$$\widehat{\theta}_{T,h} := \operatorname*{Arg\,min}_{\theta\in\Theta} U_T^h(\theta). \tag{4.3}$$

4.2. **Asymptotic properties of the Whittle estimator.** To state results involving properties of the Whittle estimator, we first introduce the following set of assumptions.

Assumption 4.1. The true value θ_0 of the parameter θ belongs to a compact set Θ, which is contained in an open set S in the p-dimensional Euclidean space $\mathbb{R}^p$, and $f(\lambda,\theta_1) \neq f(\lambda,\theta_2)$ whenever $\theta_1 \neq \theta_2$ almost everywhere in Λ with respect to the Lebesgue measure.

Assumption 4.2. The functions $f(\lambda,\theta)$, $f^{-1}(\lambda,\theta)$ and $\frac{\partial}{\partial\theta_k} f^{-1}(\lambda,\theta)$, $k=1,\dots,p$, are continuous in (λ,θ).

Assumption 4.3. The functions $f := f(\lambda,\theta)$ and $g := w(\lambda)\frac{\partial}{\partial\theta_k} f^{-1}(\lambda,\theta)$ satisfy Assumptions 3.3 or 3.4 for all $k=1,\dots,p$ and $\theta\in\Theta$.

Assumption 4.4. The functions $a := a(\lambda,\theta)$ and $b := \widehat{g}$, where g is as in Assumption 4.3, satisfy Assumption 3.1.

Assumption 4.5. The functions $\frac{\partial^2}{\partial\theta_k\partial\theta_j} f^{-1}(\lambda,\theta)$ and $\frac{\partial^3}{\partial\theta_k\partial\theta_j\partial\theta_j} f^{-1}(\lambda,\theta)$, $k,j,l = 1,\dots,p$, are continuous in (λ,θ) for $\lambda\in\Lambda$, $\theta\in N_\delta(\theta_0)$, where $N_\delta(\theta_0) := \{\theta : |\theta-\theta_0|<\delta\}$ is some neighborhood of θ_0.

Assumption 4.6. The matrices

$$W(\theta) := \|w_{ij}(\theta)\|,\ A(\theta) := \|a_{ij}(\theta)\|,\ B(\theta) := \|b_{ij}(\theta)\|,\ i,j=1,\dots,p \tag{4.4}$$

are positive definite, where

$$w_{ij}(\theta) = \frac{1}{4\pi}\int_\Lambda \frac{\partial}{\partial\theta_i}\ln f(\lambda,\theta)\frac{\partial}{\partial\theta_j}\ln f(\lambda,\theta) w(\lambda)d\lambda, \tag{4.5}$$

$$a_{ij}(\theta) = \frac{1}{4\pi}\int_\Lambda \frac{\partial}{\partial\theta_i}\ln f(\lambda,\theta)\frac{\partial}{\partial\theta_j}\ln f(\lambda,\theta) w^2(\lambda)d\lambda, \tag{4.6}$$

$$b_{ij}(\theta) = \frac{\kappa_4}{16\pi^2}\int_\Lambda \frac{\partial}{\partial\theta_i}\ln f(\lambda,\theta)w(\lambda)d\lambda\int_{\mathbb{R}} \frac{\partial}{\partial\theta_j}\ln f(\lambda,\theta) w(\lambda)d\lambda, \tag{4.7}$$

and κ_4 is the fourth cumulant of $\xi(1)$.

The next theorem contains sufficient conditions for Whittle estimator to be consistent (see Ginovyan [27]).

Theorem 4.1. *Let $\widehat{\theta}_{T,h}$ be the Whittle estimator defined by* (4.3) *and let θ_0 be the true value of parameter θ. Then, under Assumptions 4.1–4.4 and 2.1, the statistic $\widehat{\theta}_{T,h}$ is a consistent estimator for θ, that is, $\widehat{\theta}_{T,h}\to\theta_0$ in probability as $T\to\infty$.*

Having established the consistency of the Whittle estimator $\widehat{\theta}_{T,h}$, we can go on to obtain the limiting distribution of $T^{1/2}\left(\widehat{\theta}_{T,h}-\theta_0\right)$ in the usual way by applying the Taylor's formula, the mean value theorem, and Slutsky's arguments. Specifically we have the following result, showing that under the above assumptions, the Whittle estimator $\widehat{\theta}_{T,h}$ is asymptotically normal (see Ginovyan [27]).

Theorem 4.2. *Suppose that Assumptions 4.1–4.6 and 2.1 are satisfied. Then the Whittle estimator* $\widehat{\theta}_{T,h}$ *of an unknown spectral parameter* θ *based on the tapered data* (2.13) *is asymptotically normal. More precisely, we have*

$$T^{1/2}\left(\widehat{\theta}_{T,h}-\theta_0\right)\xrightarrow{d} N_p\left(0, e(h)\Gamma(\theta_0)\right) \quad \text{as} \quad T\to\infty, \tag{4.8}$$

where $N_p(\cdot,\cdot)$ *denotes the p-dimensional normal law,* $\xrightarrow{d}$ *stands for convergence in distribution,*

$$\Gamma(\theta_0) = W^{-1}(\theta_0)\left(A(\theta_0)+B(\theta_0)\right)W^{-1}(\theta_0), \tag{4.9}$$

where the matrices W*,* A *and* B *are defined in (4.4)-(4.7), and the tapering factor* $e(h)$ *is given by formula* (3.11)*.*

Remark 4.1. In the d.t. case as a weight function we take $w(\lambda)\equiv 1$, and the matrices $A(\theta_0)$ and $W(\theta_0)$ coincide (see (4.4) – (4.6)). So, in this case, formula (4.9) becomes $\Gamma(\theta_0) = W^{-1}(\theta_0)\left(W(\theta_0)+B(\theta_0)\right)W^{-1}(\theta_0)$. If, in addition, the underlying process is Gaussian ($\kappa_4 = 0$, and hence $B(\theta_0) = 0$), and the taper h is chosen so that the tapering factor $e(h)$ is equal to one, then we have $\Gamma(\theta_0) = W^{-1}(\theta_0)$, that is, the Whittle estimator $\widehat{\theta}_{T,h}$ is Fisher-efficient.

5. Robustness to small trends of estimation

In time series analysis, much of statistical inferences about unknown spectral parameters or spectral functionals are concerned with the stationary models, in which case it is assumed that the models are centered, or have constant means. In this section, we are concerned with the robustness of inferences, carried out on a stationary models, possibly exhibiting long memory, contaminated by a small trend. Specifically, let $\{X(t),\, t\in\mathbb{U}\}$ be a centered stationary process possessing a spectral density $f_X(\lambda)$, $\lambda\in\Lambda$. Assuming that either f_X is known with the exception of a vector parameter $\theta\in\Theta\subset\mathbb{R}^p$, or f_X is completely unknown and belongs to a given class $\mathcal{F}$, we want to make inferences about θ or the value $J(f_X)$ of a given functional $J(\cdot)$ at an unknown point $f_X\in\mathcal{F}$ in the case where the actual observed data are in the contaminated form:

$$Y(t) = X(t) + M(t), \quad t\in D_T, \tag{5.1}$$

where $M(t)$ is a deterministic trend, and $D_T = [0,T]$ in the c.t. case and $D_T = \{1,\ldots,T\}$ in the d.t. case.

The process $X(t)$ is what we believe is being observed but in reality the data are in the contaminated form $Y(t)$. In this case standard inferences can be carried on the basis of the stationary model $X(t)$, and we are interested in question whether the conclusions are robust against this kind of departure from the stationarity.

In the non-tapered case, this problem for d.t. models was considered in Heyde and Dai [42] (see also Taniguchi and Kakizawa [50], Theorems 6.4.1 and 6.4.2). For c.t. models it was studied in Ginovyan and Sahakyan [33].

The results stated below show that if the trend $M(t)$ is 'small', then the asymptotic properties of estimators of the parameter θ and the functional $J(f)$, stated in Sections 3 and 4 for a stationary model $X(t)$, remain valid for the contaminated model $Y(t)$, that is, both the parametric and nonparametric estimating procedures are robust against replacing the stationary model $X(t)$ by the non-stationary $Y(t)$. To this end, similar to the non-tapered case, we first establish an asymptotic relation between stationary and contaminated tapered periodograms. For simplicity, the results that follow we prove in the c.t. case, the proofs in the d.t. case are similar.

5.1. **A relation between stationary and contaminated tapered periodograms.** The next result shows that a small trend of the form $|M(t)| \le C|t|^{-\beta}$, $\beta > 1/4$, does not effect the asymptotic properties of the empirical spectral linear functionals of a tapered periodogram. Note that this result is of general nature, and do not require from the model $X(t)$ to be linear.

Theorem 5.1. *Let $\{X(t),\, t \in \mathbb{U}\}$ be a stationary mean zero process, $\{M(t),\, t \in \mathbb{U}\}$ be a deterministic trend, $Y(t) = X(t) + M(t)$, and let $I^h_{TX}(\lambda)$ and $I^h_{TY}(\lambda)$ be the tapered periodograms of $X(t)$ and $Y(t)$, respectively. Let $g(\lambda)$, $\lambda \in \Lambda$ be an even integrable function. If the trend $M(t)$ and the Fourier transform $a(t) := \widehat{g}(t)$ of $g(\lambda)$ are such that $M(t)$ is locally integrable on $\mathbb{R}$ and*

$$|M(t)| \le C|t|^{-\beta}, \qquad |a(t)| \le C|t|^{-\gamma}, \quad t \in \Lambda, \quad 2\beta + \gamma > 3/2, \tag{5.2}$$

with some constants $C > 0$, $\gamma > 0$ and $\beta > 1/4$, then

$$T^{1/2} \int_\Lambda g(\lambda) \left[I^h_{TY}(\lambda) - I^h_{TX}(\lambda) \right] d\lambda \xrightarrow{P} 0 \quad \text{as} \quad T \to \infty, \tag{5.3}$$

where $\xrightarrow{P}$ stands for convergence in probability, provided that one of the following conditions holds:

(i) *the process $X(t)$ has short or intermediate memory, that is, the covariance function $r(t) := r_X(t)$ of $X(t)$ satisfies $r \in L^1(\Lambda)$, and $\beta + \gamma > 1$,*

(ii) *the process $X(t)$ has long memory with covariance function $r(t)$ satisfying*

$$|r(t)| \le C|t|^{-\alpha}, \quad t \in \Lambda, \quad \alpha + \gamma \ge 3/2 \tag{5.4}$$

with some constants $C > 0$, $0 < \alpha \le 1$, and $\alpha + 2\beta > 1$ if $\beta < 1 < \gamma$.

Proof. In view of (2.18) and (5.1) we can write

$$
\begin{aligned}
& I^h_{T,X}(\lambda) - I^h_{T,Y}(\lambda) = \\
= \; & \frac{1}{C_T}\left(\left|\int_0^T e^{-i\lambda t} h_T(t) X(t)\,dt\right|^2 - \left|\int_0^T e^{-i\lambda t} h_T(t) Y(t)\,dt\right|^2\right) \\
= \; & \frac{1}{C_T}\left(\left|\int_0^T e^{-i\lambda t} h_T(t)[Y(t)+M(t)]\,dt\right|^2 - \left|\int_0^T e^{-i\lambda t} h_T(t) Y(t)\,dt\right|^2\right) \\
= \; & \frac{1}{C_T}\int_0^T\int_0^T e^{i\lambda(t-s)} h_T(t)h_T(s)\left[Y(t)M(s)+Y(s)M(t)+M(t)M(s)\right]\,dtds
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} g(\lambda,\theta)\left[I^h_{T,X}(\lambda) - I^h_{T,Y}(\lambda)\right] d\lambda \\
& = \frac{1}{H_2 T}\int_0^T\int_0^T [Y(t)M(s)+Y(s)M(t)+M(t)M(s)]\, h_T(t)h_T(s)a(t-s)\,dtds
\end{aligned}
$$

$$
(5.5) \le \frac{C}{T}\int_0^T\int_0^T |Y(t)M(s)+Y(s)M(t)+M(t)M(s)|\,|a(t-s)|\,dtds,
$$

since the function h is bounded on $\mathbb{R}$ by Assumption 2.1.

Thus, to complete the proof it is enough to observe that under the conditions of the theorem we have (see Ginovyan and Sahakyan [33], relations (6.11) and (6.12)):

$$
T^{-1/2}\int_0^T\int_0^T M(t)|M(s)a(t-s)|\,dtds \to 0 \quad \text{as} \quad T\to\infty
$$

and

$$
T^{-1/2}\int_0^T\int_0^T |Y(t)M(s)a(t-s)|\,dtds \xrightarrow{P} 0 \quad \text{as} \quad T\to\infty. \qquad \square
$$

Remark 5.1. It is easy to check that the statement of Theorem 5.1 holds, in particular, if the parameters α, β and γ satisfy the following conditions:

in the case *(i)*: $\beta > 1/2,\ \gamma \ge 1/2$,

in the case *(ii)*: $\alpha \ge 3/4,\ \beta > 3/8,\ \gamma \ge 3/4$.

Remark 5.2. In the non-tapered d.t. case, Theorem 5.1 (with additional conditions $\gamma = 1$ in the case *(i)*, and $\gamma > 1,\ \alpha < 1/2$ in the case *(ii)*), was proved by Heyde and Dai [42] (see also Taniguchi and Kakizawa [50], Theorems 6.4.1 and 6.4.2).

5.2. **Robustness to small trends of nonparametric estimation.** The next result shows that a small trend of the form $|M(t)| \le C|t|^{-\beta}$ does not effect the asymptotic properties of the estimator of a linear spectral functional $J(f)$, that is, the nonparametric estimation procedure is robust to the presence of a small trend in the model.

Theorem 5.2. *Suppose that the assumptions of Theorems 3.6 and 5.1 are fulfilled. Then the statistic* $J(I^h_{TY})$ *is consistent and asymptotically normal estimator for functional* $J(f)$ *with asymptotic variance* $\sigma^2_h(J)$ *given by* (3.10) *and* (3.11)*, that is, the asymptotic relation (3.14) is satisfied with* $I^h_{TX}(\lambda)$ *replaced by the contaminated periodogram* $I^h_{TY}(\lambda)$*:*

$$T^{1/2}\left[J(I^h_{TY}) - J(f)\right] \xrightarrow{d} \eta \quad \text{as} \quad T \to \infty, \tag{5.6}$$

where η *is* $N(0, \sigma^2_h(J))$ *with* $\sigma^2_h(J)$ *given by* (3.10) *and* (3.11).

Proof of Theorem 5.2. In view of (3.1) and (3.2) we can write

$$\begin{aligned}
T^{1/2}\left[J(I^h_{TY}) - J(f)\right] &= T^{1/2}\left[\int_{\mathbb{R}} I^h_{TY}(\lambda)g(\lambda)d\lambda - \int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda\right] \\
&= T^{1/2}\left[\int_{\mathbb{R}} I^h_{TY}(\lambda)g(\lambda)d\lambda - \int_{\mathbb{R}} I^h_{TX}(\lambda)g(\lambda)d\lambda\right] \\
&+ T^{1/2}\left[\int_{\mathbb{R}} I^h_{TX}(\lambda)g(\lambda)d\lambda - \int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda\right] \\
&= T^{1/2}\int_{\mathbb{R}} g(\lambda)\left[I^h_{TY}(\lambda) - I^h_{TX}(\lambda)\right]d\lambda + T^{1/2}\left[J(I^h_{TX}) - J(f)\right]. \qquad (5.7)
\end{aligned}$$

By Theorem 5.1, the first term on the right-hand side of (5.7) goes to zero in probability as $T \to \infty$, while by Theorem 3.6, the second term on the right-hand side of (5.7) goes in distribution to η, and the result follows. □

5.3. **Robustness to small trends of parametric estimation.** The next result shows that a small trend of the form $|M(t)| \le C|t|^{-\beta}$, $\beta > 1/4$, does not effect the asymptotic properties of the Whittle estimator of an unknown spectral parameter θ, that is, the Whittle parametric estimation procedure based on the tapered sample (2.13) is robust to the presence of a small trend in the model.

Theorem 5.3. *Suppose that the assumptions of Theorem 5.1 with* $g = f^{-1}(\lambda, \theta) \cdot w(\lambda)$ *are satisfied. Then under the conditions of Theorems 4.2 the Whittle estimator* $\widehat{\theta}_{TY,h}$*, constructed on the basis of the contaminated tapered periodogram* $I^h_{T,Y}(\lambda)$*, is consistent and asymptotically normal estimator for an unknown spectral parameter* θ*, that is, the asymptotic relation (4.8) is satisfied with* $I^h_{TX}(\lambda)$ *replaced by the contaminated periodogram* $I^h_{TY}(\lambda)$*:*

$$T^{1/2}\left(\widehat{\theta}_{TY,h} - \theta_0\right) \xrightarrow{d} N_p\left(0, e(h)\Gamma(\theta_0)\right) \quad \text{as} \quad T \to \infty, \tag{5.8}$$

where the matrix $\Gamma(\theta_0)$ *is defined in (4.9).*

Proof of Theorem 5.3. By Taylor's formula for $\frac{\partial}{\partial\theta}U^h_{TX}\left(\widehat{\theta}_{TX,h}\right)$, where $U^h_{TX}(\cdot)$ is the tapered Whittle functional defined by (4.2) and $\widehat{\theta}_{TX,h}$ is the Whittle estimator constructed on the basis of observation $\mathbf{X}_T = \{X(t),\, 0 \le t \le T\}$, for $|\widehat{\theta}^*_T - \theta_0| < |\widehat{\theta}_{TX,h} - \theta_0|$ and for sufficiently large T, we can write

$$(5.9) \quad T^{1/2}\left[\widehat{\theta}_{TX,h} - \theta_0\right] = -T^{1/2}\left[\frac{\partial^2}{\partial\theta\partial\theta'}U^h_{TX}(\theta^*_T)\right]^{-1}\left[\frac{\partial}{\partial\theta}U^h_{TX}(\theta_0)\right] + o_P(1).$$

Next, by Theorem 5.1, we have

$$(5.10) \qquad U^h_{TY}(\theta_T) = U^h_{TX}(\theta_T) + o_P(1).$$

Again using Taylor's formula for $\frac{\partial}{\partial\theta}U^h_{TY}\left(\widehat{\theta}_{TY,h}\right)$, where now $U^h_{TY}(\cdot)$ and $\widehat{\theta}_{TY,h}$ are respectively the Whittle functional and the Whittle estimator, constructed on the basis of the contaminated observation $\mathbf{Y}_T = \{Y(t),\, 0 \le t \le T\}$, and taking into account the relations (5.9) and (5.10), we can infer that

$$T^{1/2}\left[\widehat{\theta}_{TY,h} - \theta_0\right] = T^{1/2}\left[\widehat{\theta}_{TX,h} - \theta_0\right] + o_P(1),$$

showing that the estimator $\widehat{\theta}_{TY,h}$ possesses the same asymptotic properties as $\widehat{\theta}_{TX,h}$.

Hence the result follows from Theorems 4.2. □

Remark 5.3. In the non-tapered case, Theorems 5.1 – 5.3 were proved in Ginovyan and Sahakyan [33].

6. Methods and tools

In this section we briefly discuss the methods and tools, used to prove the results stated in Sections 3–5.

6.1. Approximation of traces of products of Toeplitz matrices and operators.

The trace approximation problem for truncated Toeplitz operators and matrices has been discussed in detail in the survey paper Ginovyan et al. [36] in the non-tapered case. Here we present some important results in the tapered case, which were used to prove the results stated in Sections 3–5.

Let $\psi(\lambda)$ be an integrable real symmetric function defined on $[-\pi,\pi]$, and let $h(t)$, $t \in [0,1]$ be a taper function. For $T = 1, 2, \ldots$, the $(T \times T)$*-truncated tapered Toeplitz matrix* generated by ψ and h, denoted by $B^h_T(\psi)$, is defined by the following equation:

$$(6.1) \qquad B^h_T(\psi) := \|\widehat{\psi}(t-s)h_T(t)h_T(s)\|_{t,s=1,2\ldots,T},$$

where $\widehat{\psi}(t)$ $(t \in \mathbb{Z})$ are the Fourier coefficients of ψ.

Given a real number $T > 0$ and an integrable real symmetric function $\psi(\lambda)$ defined on $\mathbb{R}$, the T*-truncated tapered Toeplitz operator* (also called *tapered Wiener-Hopf operator*) generated by ψ and a taper function h, denoted by $W^h_T(\psi)$ is defined as follows:

$$(6.2) \qquad [W^h_T(\psi)u](t) = \int_0^T \hat{\psi}(t-s)u(s)h_T(s)ds, \quad u(s) \in L^2([0,T];h_T),$$

where $\hat{\psi}(\cdot)$ is the Fourier transform of $\psi(\cdot)$, and $L^2([0,T];h_T)$ denotes the weighted L^2-space with respect to the measure $h_T(t)dt$.

Let h be a taper function satisfying Assumption 2.1, and let $A_T^h(\psi)$ be either the $T\times T$ tapered Toeplitz matrix $B_T^h(\psi)$, or the T-truncated tapered Toeplitz operator $W_T^h(\psi)$ generated by a function ψ (see (6.1) and (6.2)).

Observe that, in view of (2.15), (2.19), (6.1) and (6.2), we have

$$\frac{1}{T}\operatorname{tr}\left[A_T^h(\psi)\right] = \frac{1}{T}\cdot\widehat{\psi}(0)\cdot\int_0^T h_T^2(t)dt = 2\pi H_2\int_\Lambda \psi(\lambda)d\lambda. \tag{6.3}$$

What happens to the relation (6.3) when $A_T^h(\psi)$ is replaced by a product of Toeplitz matrices (or operators)? Observe that the product of Toeplitz matrices (resp. operators) is not a Toeplitz matrix (resp. operator).

The idea is to approximate the trace of the product of Toeplitz matrices (resp. operators) by the trace of a Toeplitz matrix (resp. operator) generated by the product of the generating functions. More precisely, let $\{\psi_1,\psi_2,\dots,\psi_m\}$ be a collection of integrable real symmetric functions defined on Λ. Let $A_T^h(\psi_i)$ be either the $T\times T$ tapered Toeplitz matrix $B_T^h(\psi_i)$, or the T-truncated tapered Toeplitz operator $W_T^h(\psi_i)$ generated by a function ψ_i and a taper function h. Define

$$S_{A,\mathcal{H},h}(T) := \frac{1}{T}\operatorname{tr}\left[\prod_{i=1}^m A_T^h(\psi_i)\right], \quad M_{\Lambda,\mathcal{H},h} := (2\pi)^{m-1}H_m\int_\Lambda\left[\prod_{i=1}^m \psi_i(\lambda)\right]d\lambda,$$

where H_m is as in (2.15), and let

$$\Delta(T) := \Delta_{A,\Lambda,\mathcal{H},h}(T) = |S_{A,\mathcal{H},h}(T) - M_{\Lambda,\mathcal{H},h}|. \tag{6.4}$$

Proposition 6.1. *Let $\Delta(T)$ be as in (6.4). Each of the following conditions is sufficient for*

$$\Delta(T) = o(1) \quad \text{as} \quad T\to\infty. \tag{6.5}$$

- **(C1)** $\psi_i \in L^1(\Lambda)\cap L^{p_i}(\Lambda)$, $p_i>1$, $i=1,2,\dots,m$, *with* $1/p_1+\cdots+1/p_m\le 1$.
- **(C2)** *The function* $\varphi(\mathbf{u})$ *defined by*

$$\varphi(\mathbf{u}) := \int_\Lambda \psi_1(\lambda)\psi_2(\lambda-u_1)\psi_3(\lambda-u_2)\cdots\psi_m(\lambda-u_{m-1})\,d\lambda, \tag{6.6}$$

where $\mathbf{u}=(u_1,u_2,\dots,u_{m-1})\in\Lambda^{m-1}$, *belongs to* $L^{m-2}(\Lambda^{m-1})$ *and is continuous at* $\mathbf{0}=(0,0,\dots,0)\in\Lambda^{m-1}$.

Remark 6.1. In the non-tapered case, Proposition 6.1 was proved in Ginovyan et al. [36], while in the tapered case, it was proved in Ginovyan [28]. Proposition 6.1 was used to prove Theorems 3.5, 3.6, and 4.2. More results concerning the trace approximation problem for truncated Toeplitz operators and matrices can be found in Ginovyan and Sahakyan [31, 32], and in Ginovyan et al. [36].

6.2. **Central limit theorems for tapered quadratic functionals.** In this subsection we state central limit theorems for tapered quadratic functional Q_T^h given by (3.3), which were used to prove the results stated in Sections 3–5.

Let $A_T^h(f)$ be either the $T\times T$ tapered Toeplitz matrix $B_T^h(f)$, or the T-truncated tapered Toeplitz operator $W_T^h(f)$ generated by the spectral density f and taper h, and let $A_T^h(g)$ denote either the $T\times T$ tapered Toeplitz matrix, or the T-truncated tapered Toeplitz operator generated by the functions g and h (for definitions see formulas (6.1) and (6.2)). Similar to the non-tapered case, we have the following results (cf. Ginovyan et al. [36], Ibragimov [43]).

1. The quadratic functional Q_T^h in (3.3) has the same distribution as the sum $\sum_{j=1}^\infty \lambda_{j,T}\xi_j^2$, where $\{\xi_j, j\geq 1\}$ are independent $N(0,1)$ Gaussian random variables and $\{\lambda_{j,T}, j\geq 1\}$ are the eigenvalues of the operator $A_T^h(f)\,A_T^h(g)$.
2. The characteristic function $\varphi(t)$ of Q_T^h is given by formula: $\varphi(t)=\prod_{j=1}^\infty|1-2it\lambda_{j,T}|^{-1/2}$.
3. The k–th order cumulant $\chi_k(Q_T^h)$ of Q_T^h is given by formula:

$$\chi_k(Q_T^h)=2^{k-1}(k-1)!\sum_{j=1}^\infty \lambda_{j,T}^k=2^{k-1}(k-1)!\,\mathrm{tr}\,[A_T^h(f)\,A_T^h(g)]^k. \tag{6.7}$$

Thus, to describe the asymptotic distribution of the quadratic functional Q_T^h, we have to control the traces and eigenvalues of the products of truncated tapered Toeplitz operators and matrices.

CLT for Gaussian models. We assume that the model process $X(t)$ is Gaussian, and with no loss of generality, that $g\geq 0$. We will use the following notation. By $\widetilde{Q}_T^h$ we denote the standard normalized quadratic functional:

$$\widetilde{Q}_T^h=T^{-1/2}\left(Q_T^h-\mathbb{E}[Q_T^h]\right). \tag{6.8}$$

Also, we set

$$\sigma_h^2:=16\pi^3H_4\int_\Lambda f^2(\lambda)g^2(\lambda)\,d\lambda, \tag{6.9}$$

where H_4 is as in (2.15). The notation

$$\widetilde{Q}_T^h\xrightarrow{d}\eta\sim N(0,\sigma_h^2)\quad\text{as}\quad T\to\infty \tag{6.10}$$

will mean that the distribution of the random variable $\widetilde{Q}_T^h$ tends (as $T\to\infty$) to the centered normal distribution with variance σ_h^2.

The following theorems were proved in Ginovyan and Sahakyan [35].

Theorem 6.1. *Each of the following conditions is sufficient for the quadratic form Q_T^h to obey the CLT, that is, for $\widetilde{Q}_T^h\xrightarrow{d}\eta\sim N(0,\sigma_h^2)$ as $T\to\infty$ with σ_h^2 is as in* (6.9).

(T1) *$f \cdot g \in L^1(\Lambda) \cap L^2(\Lambda)$, the taper function h satisfies Assumption 2.1, and for $T \to \infty$*

$$\chi_2(\widetilde{Q}_T^h) = \frac{2}{T} tr\big[A_T^h(f) A_T^h(g)\big]^2 \longrightarrow \sigma_h^2. \tag{6.11}$$

(T2) *The function*

$$\varphi(x_1, x_2, x_3) = \int_\Lambda f(u) g(u - x_1) f(u - x_2) g(u - x_3)\, du \tag{6.12}$$

belongs to $L^2(\Lambda^3)$ and is continuous at $(0, 0, 0)$, and the taper function h satisfies Assumption 2.1.

(T3) *$f(\lambda) \in L^p(\Lambda)$ $(p \geq 1)$ and $g(\lambda) \in L^q(\Lambda)$ $(q \geq 1)$ with $1/p + 1/q \leq 1/2$, and the taper function h satisfies Assumption 2.1.*

To state the next theorem, we recall the class $SV_0(\mathbb{R})$ of slowly varying functions at zero $u(\lambda)$, $\lambda \in \mathbb{R}$, satisfying the following conditions: for some $a > 0$, $u(\lambda)$ is bounded on $[-a, a]$, $\lim_{\lambda \to 0} u(\lambda) = 0$, $u(\lambda) = u(-\lambda)$ and $0 < u(\lambda) < u(\mu)$ for $0 < \lambda < \mu < a$.

Theorem 6.2. *Assume that the functions f and g are integrable on $\mathbb{R}$ and bounded outside any neighborhood of the origin, and satisfy for some $a > 0$*

$$f(\lambda) \leq |\lambda|^{-\alpha} L_1(\lambda), \quad |g(\lambda)| \leq |\lambda|^{-\beta} L_2(\lambda), \quad \lambda \in [-a, a], \tag{6.13}$$

for some $\alpha < 1$, $\beta < 1$ with $\alpha + \beta \leq 1/2$, where $L_1(x)$ and $L_2(x)$ are slowly varying functions at zero satisfying

$$L_i \in SV_0(\mathbb{R}), \quad \lambda^{-(\alpha+\beta)} L_i(\lambda) \in L^2[-a, a], \ i = 1, 2. \tag{6.14}$$

Also, let the taper function h satisfy Assumption 2.1. Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0, \sigma_h^2)$ as $T \to \infty$.

The condition $\alpha < 1$, $\beta < 1$ in Theorem 6.2 ensure that the Fourier transforms of f and g are well defined. For $\alpha > 0$ the process $X(t)$ may exhibit long-range dependence. We also allow here $\alpha+\beta$ to assume the critical value $1/2$. The assumptions $f \cdot g \in L^1(\Lambda)$, $f, g \in L^\infty(\Lambda \setminus [-a, a])$ and (6.14) imply that $f \cdot g \in L^2(\Lambda)$, so that the variance σ_h^2 in (6.9) is finite.

CLT for Lévy-driven stationary linear models. Now we assume that the underlying model $X(t)$ is a Lévy-driven stationary linear process defined by (2.5), where $a(\cdot)$ is a filter from $L^2(\mathbb{R})$, and $\xi(t)$ is a Lévy process satisfying the conditions: $\mathbb{E}\xi(t) = 0$, $\mathbb{E}\xi^2(1) = 1$ and $\mathbb{E}\xi^4(1) < \infty$.

The central limit theorem that follows was proved in Ginovyan and Sahakyan [34].

Theorem 6.3. *Assume that the filter $a(\cdot)$ and the generating kernel $\widehat{g}(\cdot)$ are such that*

$$a(\cdot) \in L^p(\mathbb{R}) \cap L^2(\mathbb{R}), \quad \widehat{g}(\cdot) \in L^q(\mathbb{R}), \quad 1 \le p, q \le 2, \quad 2/p + 1/q \ge 5/2, \tag{6.15}$$

and the taper h satisfies Assumption 2.1. Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0, \sigma^2_{L,h})$ as $T \to \infty$, where

$$\sigma^2_{L,h} = 16\pi^3 H_4 \int_{\mathbb{R}} f^2(\lambda) g^2(\lambda) d\lambda + \kappa_4 4\pi^2 H_4 \left[\int_{\mathbb{R}} f(\lambda) g(\lambda) d\lambda\right]^2, \tag{6.16}$$

where H_4 is as in (2.15)*.*

Remark 6.2. Notice that if the underlying process $X(t)$ is Gaussian, then in formula (6.16) we have only the first term and so $\sigma^2_{L,h} = \sigma^2_h$ (see (6.9)), because in this case $\kappa_4 = 0$. On the other hand, the condition (6.15) is more restrictive than the conditions in Theorems 6.1 and 6.2. Thus, for Gaussian processes Theorems 6.1 and 6.2 improve Theorem 6.3. For non-tapered case Theorem 6.3 was proved in Bai et al. [4].

6.3. **Fejér-type kernels and singular integrals.** We define Fejér-type tapered kernels and singular integrals, and state some of their properties.

For a number k ($k = 2, 3, \ldots$) and a taper function h satisfying Assumption 2.1 consider the following Fejér-type tapered kernel function:

$$F^h_{k,T}(\mathbf{u}) := \frac{H_T(\mathbf{u})}{(2\pi)^{k-1} H_{k,T}(0)}, \quad \mathbf{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{R}^{k-1}, \tag{6.17}$$

where

$$H_T(\mathbf{u}) := H_{1,T}(u_1) \cdots H_{1,T}(u_{k-1}) H_{1,T}\left(-\sum_{j=1}^{k-1} u_j\right), \tag{6.18}$$

and the function $H_{k,T}(\cdot)$ is defined by (2.16) with $H_{k,T}(0) = T \cdot H_k \neq 0$ (see (2.15)).

The next result shows that, similar to the classical Fejér kernel, the tapered kernel $F^h_{k,T}(\mathbf{u})$ is an approximation identity (see Ginovyan and Sahakyan [34], Lemma 3.4).

Proposition 6.2. *For any $k = 2, 3, \ldots$ and a taper function h satisfying Assumption 2.1 the kernel $F^h_{k,T}(\mathbf{u})$, $\mathbf{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{R}^{k-1}$, possesses the following properties:*

a) $\sup_{T>0} \int_{\mathbb{R}^{k-1}} \left|F^h_{k,T}(\mathbf{u})\right| d\mathbf{u} = C_1 < \infty$;

b) $\int_{\mathbb{R}^{k-1}} F^h_{k,T}(\mathbf{u})\, d\mathbf{u} = 1$;

c) $\lim_{T\to\infty} \int_{\mathbb{E}^c_\delta} \left|F^h_{k,T}(\mathbf{u})\right| d\mathbf{u} = 0$ *for any* $\delta > 0$;

d) *If $k > 2$ for any $\delta > 0$ there exists a constant $M_\delta > 0$ such that $\left\|F^h_{k,T}\right\|_{L^{p_k}(\mathbb{E}^c_\delta)} \le M_\delta$ for $T > 0$, where $p_k = \frac{k-2}{k-3}$ for $k > 3$, $p_3 = \infty$, $\mathbb{E}^c_\delta = \mathbb{R}^{k-1} \setminus \mathbb{E}_\delta$, and*

$$\mathbb{E}_\delta = \{\mathbf{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{R}^{k-1} : |u_i| \le \delta,\ i = 1, \ldots, k-1\}.$$

e) *If the function* $Q \in L^1(\mathbb{R}^{k-1}) \bigcap L^{k-2}(\mathbb{R}^{k-1})$ *and is continuous at* $\mathbf{v} = (v_1, \ldots, v_{k-1})$ *(L^0 is the space of measurable functions), then*

$$\lim_{T\to\infty} \int_{\mathbb{R}^{k-1}} Q(\mathbf{u}+\mathbf{v}) F^h_{k,T}(\mathbf{u}) d\mathbf{u} = Q(\mathbf{v}). \tag{6.19}$$

Denote

$$\Delta^h_{2,T} := \int_{\mathbb{R}^2} f(\lambda) g(\lambda+\mu) F^h_{2,T}(\mu) d\lambda d\mu - \int_{\mathbb{R}} f(\lambda) g(\lambda) d\lambda, \tag{6.20}$$

where $F^h_{2,T}(\mu)$ is given by (6.17) and (6.18).

The next two propositions give information on the rate of convergence to zero of $\Delta^h_{2,T}$ as $T \to \infty$ (see Ginovyan and Sahakyan [34], Lemmas 4.1 and 4.2).

Proposition 6.3. *Assume that Assumptions 2.1 and 3.3 are satisfied. Then the following asymptotic relation holds:*

$$\Delta^h_{2,T} = o\left(T^{-1/2}\right) \quad \text{as} \quad T \to \infty. \tag{6.21}$$

Proposition 6.4. *Assume that Assumptions 2.1 and 3.2 are satisfied. Then the following inequality holds:*

$$|\Delta^h_{2,T}| \le C_h \begin{cases} T^{-(\beta_1+\beta_2)}, & \text{if } \beta_1+\beta_2 < 1 \\ T^{-1} \ln T, & \text{if } \beta_1+\beta_2 = 1 \\ T^{-1}, & \text{if } \beta_1+\beta_2 > 1, \end{cases} \qquad T > 0, \tag{6.22}$$

where C_h is a constant depending on h.

Notice that for non-tapered case ($h(t) = \mathbb{I}_{[0,1]}(t)$), Propositions 6.3 and 6.4 were proved in Ginovyan and Sahakyan [30] (see also Ginovyan and Sahakyan [31, 32]). In the d.t. tapered case, Proposition 6.3 under different conditions was proved in Dahlhaus [10].

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MULTIPLICITY OF SOLUTIONS FOR A FRACTIONAL LAPLACIAN EQUATION INVOLVING A PERTURBATION

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Abstract. A fractional Laplacian equation involving a perturbation is investigated. Under certain conditions, we obtain at least two solutions to this equation.

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1. Introduction

Fractional Laplacian equations have been applied to many subjects, such as, anomalous diffusion, elliptic problems with measure data, gradient potential theory, minimal surfaces, non-uniformly elliptic problems, optimization, phase transitions, quasigeostrophic flows, singular set of minima of variational functionals, and water waves (see [2]-[11] and the references therein). Fractional Brezis-Nirenberg problems had been investigated by many researchers (such as [2, 10]).

$$\begin{cases} (-\Delta)^s u + \lambda u = |u|^{2_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < s < 1, N > 2s$, $2_s^* := \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, Ω is an open bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and the fractional Laplacian is defined by

$$-(-\Delta)^s u(x) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{N+2s}} \mathrm{d}\zeta \right)^{-1}. \tag{1.1}$$

Define Hilbert space $D^s(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{D^s}$ induced by the following scalar product

$$\langle u, v \rangle_{D^s} := \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

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If Ω is an open bounded Lipschitz domain, then $D^s(\Omega)$ coincides with the Sobolev space

$$X_0 := \{f \in X : f = 0 \text{ a.e. in } \Omega^c\},$$

where X is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function f in X belongs to $L^2(\Omega)$ and the map $(x,y) \mapsto \big(f(x) - f(y)\big)|x-y|^{-\frac{N}{2}+s}$ is in $L^2\left(\mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c),\ \mathrm{d}x\mathrm{d}y\right)$, and Ω^c is the complement of Ω in $\mathbb{R}^N$. Consider fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^{2N}) \right\},$$

equiped the Gagliardo seminorm

$$[u]^2_{H^s(\mathbb{R}^N)} := \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

The fractional Laplacian operator can be defined by

$$\begin{aligned}
(-\Delta)^s u(x) &= C_{N,s}\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \\
&= C_{N,s} \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \\
&= -\frac{1}{2} C_{N,s} \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \mathrm{d}y,
\end{aligned}$$

where $C_{N,s}$ is given by (1.1) and P.V. is the principle value defined by the latter formula. Define the fractional Sobolev space

$$H^s(\Omega) := \left\{ x \in H^s(\mathbb{R}^N) : u = 0 \ \text{ a.e. in } \Omega^c \right\},$$

equipped with the seminorm

$$\|u\|_{H^s(\Omega)} := \left(\lambda \int_\Omega |u|^2 \mathrm{d}x + \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

which was introduced in [10]. From $u = 0$ a.e. in Ω^c, it is easy to see that

$$|u|_2^2 := \int_\Omega |u|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x,$$

$$\int_{\mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

Hence, we just denote $\|u\|_{H^s(\Omega)}$ by

$$\|u\|_{H^s} := \left(\lambda \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x + \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

It follows from Lemma 7 in [8] that $\big(H^s(\Omega), \|\cdot\|_{H^s}\big)$ is a Hilbert space.

In present paper, we study the following fractional Laplacian equation involving a perturbation

$$\begin{cases} (-\Delta)^s u + \lambda u = |u|^{p-2}u + h(x) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \Omega^c, \end{cases} \tag{1.2}$$

where $0 < s < 1$, λ is a real parameter, $p \in (2, 2_s^*)$, $h \in L^2(\Omega)$, and $\Omega \subset \mathbb{R}^N$ is an open bounded Lipschitz domain. Via classic methods (see [1] for example), we obtain multiplicity of solutions for fractional Laplacian equation (1.2). The solutions of equation (1.2) coincide with the critical points of the following energy functional

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \frac{\lambda}{2}\int_\Omega |u|^2 \mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \mathrm{d}x - \int_\Omega hu\mathrm{d}x \\ &= \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p}|u|_p^p - \int_\Omega hu\mathrm{d}x, \quad \forall u \in H^s(\Omega). \end{aligned}$$

If $h \equiv 0$, then equation (1.2) becomes

$$\begin{cases} (-\Delta)^s + \lambda u = |u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \Omega^c. \end{cases} \tag{1.3}$$

Define the energy functional of equation (1.3) and corresponding Nehari manifold as follows:

$$I(u) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p}|u|_p^p, \quad \forall u \in H^s(\Omega),$$

and

$$\mathcal{N} = \{u \in H^s(\Omega) : u \neq 0, I'(u)u = 0\} = \left\{u \in H^s(\Omega) : u \neq 0, \|u\|_{H^s}^2 = |u|_p^p\right\}.$$

Our main result reads as follows.

Theorem 1.1. *There exists $\epsilon > 0$ such that for every $h \in L^2(\Omega)$ with $|h|_2 \leqslant \epsilon$, equation* (1.2) *has at least two solutions.*

2. The proof of Theorem 1.1

We need the following fractional Sobolev embedding results, which was proved in [8].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz domain. Then $H^s(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, 2_s^*]$, and $H^s(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for every $q \in [1, 2_s^*)$.*

From Lemma 2.1, we can define a constant S_p.

$$S_p := \inf\{C > 0 : |u|_p \leqslant C\|u\|_{H^s}, \forall u \in H^s(\Omega)\}.$$

Next, we give some numbers which will be used in the proof.

$$a_1 = \left(\frac{1}{(p-1)S_p^p}\right)^{\frac{1}{p-2}}, a_2 = \left(\frac{1}{2}a_1^2\right)^{\frac{1}{p}}, a_3 = \frac{1}{2}\min\left\{a_1, \frac{a_2}{S_p}\right\}.$$

It is easy to find that $a_3 < a_1$.

Lemma 2.2. *There exists $\epsilon_1 > 0$ such that for every $h \in L^2(\Omega)$ with $|h|_2 \leqslant \epsilon_1$ and for every $u \in H^s(\Omega)$, if*

$$\|u\|_{H^s}^2 = \int_\Omega |u|^p dx + \int_\Omega hu dx = |u|_p^p + \int_\Omega hu dx, \tag{2.1}$$

then either $\|u\|_{H^s} > a_1$ and $|u|_p \geqslant a_2$ or $\|u\|_{H^s} < a_3$.

Proof. It follows from (2.1) that

$$\|u\|_{H^s}^2 \leqslant S_p^p \|u\|^p + |h|_2 |u|_2.$$

By Lemma 2.1, we get $|u|_2 \leqslant C_1 \|u\|_{H^s}$. Then, $\|u\|_{H^s}^2 \leqslant S_p^p \|u\|^p + C_1 |h|_2 \|u\|_{H^s}$. If $u \neq 0$ in $H^s(\Omega)$, then

$$\|u\|_{H^s} - S_p^p \|u\|^{p-1} - C_1 |h|_2 \leqslant 0.$$

For calculation convenience, we define function ϕ: $[0, +\infty) \to \mathbb{R}$ by

$$\phi(t) = t - S_p^p t^{p-1} - C_1 |h|_2.$$

Since $\phi'(t) = 1 - (p-1) S_p^p t^{p-2}$, we get the maxinmum point of ϕ as $a_1 = \left((p-1) S_p^p\right)^{-\frac{1}{p-2}}$. It is easy to see that ϕ is strictly increasing on $(0, a_1)$, strictly decreasing on $(a_1, +\infty)$ and $\phi(0) < 0$, $\lim_{t \to +\infty} \phi(t) = -\infty$.

In order to observe the characteristics of the function ϕ, we calculate the maximum value of ϕ,

$$\begin{aligned}
\phi(a_1) &= \left(\frac{1}{(p-1) S_p^p}\right)^{\frac{1}{p-2}} - S_p^p \left(\frac{1}{(p-1) S_p^p}\right)^{\frac{p-1}{p-2}} - C|h|_2 \\
&= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{S_p^p}\right)^{\frac{1}{p-2}} - \left(\frac{1}{p-1}\right)^{1+\frac{1}{p-2}} \left(\frac{1}{S_p^p}\right)^{\frac{1}{p-2}} - C|h|_2 \\
&= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{S_p^p}\right)^{\frac{1}{p-2}} \left(1 - \frac{1}{p-1}\right) - C|h|_2 \\
&= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{S_p^p}\right)^{\frac{1}{p-2}} \frac{p-2}{p-1} - C|h|_2 =: \alpha_1 - C|h|_2,
\end{aligned}$$

and if we take $|h|_2 \leqslant \frac{\alpha_1}{2C}$, then

$$\phi(a_1) \geqslant \alpha_1 - C\frac{\alpha_1}{2C} = \alpha_1 - \frac{\alpha_1}{2} = \frac{\alpha_1}{2} > 0,$$

which means the function ϕ has two zeros t_1, t_2 and $t_1 < a_1 < t_2$. Then $\phi(t) > 0$ for all $t \in (t_1, t_2)$, while $\phi(t) < 0$ for all $t \in [0, t_1) \cup (t_2, +\infty)$. Substituting t_1 into the function ϕ, we get that

$$C|h|_2 = t_1 - S_p^p t_1^{p-1} = t_1 \left(1 - S_p^p t_1^{p-2}\right).$$

Since $t_1 < a_1$, we have

$$C|h|_2 \geqslant t_1 \left(1 - S_p^p a_1^{p-2}\right) = t_1 \left(1 - \frac{1}{p-1}\right) = t_1 \frac{p-2}{p-1},$$

i.e., $t_1 \leqslant \frac{p-1}{p-2} C|h|_2$. If we take

$$|h|_2 < \frac{p-1}{p-1} \frac{a_3}{C},$$

then

$$t_1 < \frac{p-1}{p-2} \frac{p-2}{p-1} \frac{a_3}{C} = a_3.$$

In summary, for

$$|h|_2 < \min \left\{ \frac{p-2}{p-1} \frac{a_3}{C}, \frac{\alpha_1}{2C} \right\},$$

we get $\phi \leqslant 0$ implies $t < a_3$ or $t > a_1$. If (2.1) hold and $\|u\|_{H^s} > a_1$, we get

$$|u|_p^p = \|u\|_{H^s}^2 - \int_\Omega hu\mathrm{d}x \geqslant a_1^2 - |h|_2 |u|_2 \geqslant a_1^2 - a|h|_2 |u|_p,$$

where $a = |\Omega|^{\frac{p-2}{2p}}$. Namely

$$|u|_p^p + a|h|_2|u|_p - a_1^2 \geqslant 0. \tag{2.2}$$

Regarding $|u|_p$ as a variable, we get a function $\gamma : [0, +\infty) \to \mathbb{R}$, defined by

$$\gamma(t) = t^p + a|h|_2 t - a_1^2.$$

Since $\gamma'(t) = pt^{p-1} + a|h|_2 > 0$, for all $t > 0$, γ is strictly increasing. Therefore, if

$$|h|_2 < \frac{a_1^2}{2aa_2},$$

then

$$\begin{aligned} \gamma(a_2) &= a_2^p + a|h|_2 a_2 - a_1^2 = \frac{1}{2} a_1^2 + a|h|_2 a_2 - a_1^2 \\ &= a|h|_2 a_2 - \frac{1}{2} a_1^2 < a \frac{a_1^2}{2aa_2} a_2 - \frac{1}{2} a_1^2 = 0. \end{aligned}$$

We see that $\gamma(t) < 0$ for $t \in [0, a_2]$. By (2.2) we derive that $|u|_p \geqslant a_2$.

Summing up, if we choose

$$\epsilon_1 = \min \left\{ \frac{p-2}{p-1} \frac{a_3}{C}, \frac{\alpha_1}{2C}, \frac{a_1^2}{2aa_2} \right\},$$

then Lemma 2.2 holds. □

In the sequel, we always assume $|h|_2 < \epsilon_1$. Now define

$$\begin{aligned} \mathcal{N}_h &:= \{u \in H^s(\Omega) : J'(u)u = 0, \|u\|_{H^s} > a_1\} \\ &= \left\{ u \in H^s(\Omega) : \|u\|_{H^s}^2 = |u|_p^p + \int_\Omega hu\mathrm{d}x, \|u\|_{H^s} > a_1 \right\}, \end{aligned}$$

and $m_h = \inf_{u \in \mathcal{N}_h} J(u)$. Notice that $\mathcal{N}_h$ is a subset of Nehari mainfold and for $u \in \mathcal{N}_h$, we have

$$J(u) = \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|_{H^s}^2 - \left(1 - \frac{1}{p} \right) \int_\Omega hu\mathrm{d}x.$$

Now, we prove that $\mathcal{N}_h$ is not empty.

Lemma 2.3. *There exists $\epsilon_2 \in (0, \epsilon_1]$ such that for every $h \in L^2(\Omega)$ with $|h|_2 \leqslant \epsilon_2$, there results $\mathcal{N}_h \neq 0$.*

Proof. Consider function

$$t \mapsto J'(tu)\,tu = t^2\|u\|_{H^s}^2 - t^p \int_\Omega |u|^p \mathrm{d}x - t \int_\Omega hu\mathrm{d}x$$
$$= t\left[t\|u\|^2 - t^{p-1}|u|_p^p - \int_\Omega hu\mathrm{d}x\right],$$

where $u \in H^s(\Omega) \setminus \{0\}$, $t \in (0, +\infty)$. Since $t > 0$, we only consider the following function

$$\gamma(t) = t\|u\|_{H^s}^2 - t^{p-1}|u|_p^p - \int_\Omega hu\mathrm{d}x,$$

since $p \in (2, 2_s^*)$, the function γ has a global maximum. Solving

$$\gamma'(t) = \|u\|_{H^s}^2 - (p-1)\,t^{p-2}|u|_p^p = 0,$$

we have the function γ has a global maximum at

$$t' = \left(\frac{\|u\|_{H^s}^2}{(p-1)|u|_p^p}\right)^{\frac{1}{p-2}},$$

and

$$\gamma(t') = \frac{\|u\|_{H^s}^{\frac{2(p-1)}{p-2}}}{|u|_p^{\frac{p}{p-2}}} \frac{1}{(p-1)^{\frac{1}{p-2}}} - \int_\Omega hu\mathrm{d}x =: \frac{\|u\|_{H^s}^{\frac{2(p-1)}{p-2}}}{|u|_p^{\frac{p}{p-2}}}\alpha - \int_\Omega hu\mathrm{d}x$$
$$\geqslant \frac{\|u\|_{H^s}^{\frac{2(p-1)}{p-2}}}{\|u\|_p^{\frac{p}{p-2}} S_p^{\frac{p}{p-2}}}\alpha - \int_\Omega hu\mathrm{d}x \geqslant \|u\|_{H^s} \frac{1}{S_p^{\frac{p}{p-2}}}\alpha - C|h|_2\|u\|$$
$$= \|u\|_{H^s}\left(\frac{\alpha}{S_p^{\frac{p}{p-2}}} - C|h|_2\right).$$

Thus, if

$$|h|_2 \leqslant \frac{\alpha}{2CS_p^{\frac{p}{p-2}}},$$

there results $\gamma(t') > 0$. Moreover, $\gamma(t)$ is strictly increasing in $(0, t')$, strictly decreasing in $(t', +\infty)$ and $\lim_{t\to+\infty} \gamma(t) = -\infty$. Then the function γ has at least one zero $t_1 \in (t', +\infty)$. Then there exists $v = t_1 u$ satisfies (2.1). Next, we verify that v satisfies $\|v\|_{H^s} > a_1$, we get $v \in \mathcal{N}_h$. Since

$$\|v\|_{H^s} = \|t_1 u\|_{H^s} = t_1\|u\|_{H^s} > t'\|u\|_{H^s} = \left(\frac{\|u\|_{H^s}^2}{(p-1)\,|u|_p^p}\right)^{\frac{1}{p-2}}$$
$$= \|u\|_{H^s}^{\frac{p}{p-2}} \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{|u|_p^p}\right)^{\frac{1}{p-2}}$$
$$\geqslant (|u|_p)^{\frac{p}{p-2}} \left(\frac{1}{S_p}\right) \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{|u|_p}\right)^{\frac{p}{p-2}}$$
$$= \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} \left(\frac{1}{S_p}\right)^{\frac{p}{p-2}} = a_1,$$

the proof is completed with $\epsilon_2 = \min\left\{\epsilon_1, \frac{\alpha}{2CS_p^{\frac{p}{p-2}}}\right\}$. □

We now show that m_h are uniformly bounded from above and below by three Lemmas below.

Lemma 2.4. *Let $\epsilon_3 = \min\{1, \epsilon_2\}$. Then there exists $C > 0$ such that for every $|h|_2 < \epsilon_3$, there results $m_h \leqslant C$.*

Proof. Denote u_0 and m_0 as the solution and the level of the solution of equation (1.2), that is, $u_0 \in \mathcal{N}$, $I(u_0) = \min_{u\in\mathcal{N}} I(u) = m_0$. Due to Lemma 2.3, letting $|h|_2 < \epsilon_3$, there exists $t_h > 0$ such that $t_h u_0 \in \mathcal{N}_h$. Then

$$\|t_h u_0\|_{H^s}^2 = \int_\Omega |t_h u_0|^p \mathrm{d}x + \int_\Omega h u \mathrm{d}x. \tag{2.3}$$

Noticing $u_0 \in \mathcal{N}$, i.e., $\|u_0\|_{H^s}^2 = |u_0|_p^p$, (2.3) is equivalent to

$$\left(t_h^2 - t_h^p\right)\|u_0\|_{H^s}^2 = t_h \int_\Omega h u_0 \mathrm{d}x,$$

namely,

$$\left(t_h - t_h^{p-1}\right)\|u_0\|_{H^s}^2 = \int_\Omega h u_0 \mathrm{d}x,$$

which implies that

$$\left(t_h - t_h^{p-1}\right)\|u_0\|_{H^s}^2 \geqslant -C_1 |h|_2 \|u_0\|_{H^s},$$

that is

$$t_h - t_h^{p-1} \geqslant -\frac{C_1|h|_2}{\|u_0\|_{H^s}} \geqslant -\frac{C_1}{\|u_0\|_{H^s}}. \tag{2.4}$$

Consider function $\phi : t \mapsto t - t^{p-1}$. Since $\lim_{t\to+\infty} \phi(t) = -\infty$, there exists $C_2 > 0$ there $t_h \leqslant C_2$, and then

$$\begin{aligned} m_h \leqslant J(t_h u_0) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|t_h u_0\|_{H^s}^2 - \left(1 - \frac{1}{p}\right)\int_\Omega h u \mathrm{d}x \\ &\leqslant \left(\frac{1}{2} - \frac{1}{p}\right) C_2^2 \|u_0\|_{H^s}^2 + \left(1 - \frac{1}{p}\right) C_2 C_1 |h|_2 \|u_0\|_{H^s} \\ &\leqslant \left(\frac{1}{2} - \frac{1}{p}\right) C_2^2 \|u_0\|_{H^s}^2 + \left(1 - \frac{1}{p}\right) C_2 C_1 \|u_0\|_{H^s} =: C. \end{aligned}$$

□

To prove that m_h are uniform bound from below, we need a related Lemma.

Lemma 2.5. *For h that satisfies the condition in Lemma 2.4, there exists a normal number C_3 and a minimizing sequence $\{u_k\}_k$ for m_h such that $\|u_k\|_{H^s} \leqslant C_3$, and $|u_k|_p \leqslant S_p C_3$ for all k .*

Proof. Let $\{v_k\}_k$ be a minimizing sequence for m_h, i.e., $v_k \in \mathcal{N}_h$ and $J(v_k) \to m_h$ since $m_h \leqslant C$, there exists k' such that for every $k \geqslant k'$, $J(v_k) \leqslant 2C$. Then

$$\begin{aligned}2C \geqslant J(v_k) &= \left(\frac{1}{2} - \frac{1}{p}\right) \|v_k\|_{H^s}^2 - \left(1 - \frac{1}{p}\right) \int_\Omega h v_k \mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right) \|v_k\|_{H^s}^2 - \left(1 - \frac{1}{p}\right) C_1 |h|_2 \|v_k\|_{H^s} =: a\|v_k\|_{H^s}^2 - b\|v_k\|_{H^s}.\end{aligned}$$

We get

$$\frac{b + \sqrt{b^2 + 8ac}}{2a} =: C_3,$$

and $|v_k|_p \leqslant S_p \|v_k\|_{H^s} = S_p C_3$, where $u_k = v_{k'+k}$. □

The preparation work has been completed. Now we prove the boundness from below.

Lemma 2.6. *There exists $\epsilon_4 \in (0, \epsilon_3]$ such that if $|h|_2 < \epsilon_4$, then $m_h \geqslant \frac{1}{2} m_0 > 0$.*

Proof. We consider $\{u_k\}_k$ obtained in Lemma 2.5. Let t_k be such that $t_k u_k \in \mathcal{N}$, which is equivalent to

$$\|t_k u_k\|_{H^s}^2 = \int_\Omega |t_k u_k|^p \mathrm{d}x,$$

namely,

$$t_k^2 \|u_k\|_{H^s}^2 = t_k^p \int_\Omega |u_k|^p \mathrm{d}x,$$

i.e. ,

$$t_k = \left(\frac{\|u_k\|_{H^s}^2}{|u_k|_p^p}\right)^{\frac{1}{p-2}}.$$

Since $u_k \in \mathcal{N}_h$, we have

$$\|u_k\|_{H^s}^2 = |u_k|_p^p + \int_\Omega h u_k \mathrm{d}x.$$

Then

$$t_k = \left(\frac{|u_k|_p^p + \int_\Omega h u_k \mathrm{d}x}{|u_k|_p^p}\right)^{\frac{1}{p-2}} = \left(1 + \frac{\int_\Omega h u_k \mathrm{d}x}{|u_k|_p^p}\right)^{\frac{1}{p-2}},$$

and

$$\begin{aligned} m_0 \leqslant I(t_k u_k) &= \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2 \|u_k\|_{H^s}^2 \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2 \|u_k\|_{H^s}^2 - \left(1 - \frac{1}{p}\right) t_k^2 \int_\Omega h u_k \mathrm{d}x + \left(1 - \frac{1}{p}\right) t_k^2 \int_\Omega h u_k \mathrm{d}x \qquad (2.5) \\ &= t_k^2 J(u_k) + \left(1 - \frac{1}{p}\right) t_k^2 \int_\Omega h u_k \mathrm{d}x.\end{aligned}$$

By Lemma 2.2 and Lemma 2.5, we have

$$t_k = \left(1 + \frac{\int_\Omega h u_k \mathrm{d}x}{|u_k|_p^p}\right)^{\frac{1}{p-2}} \leqslant \left(1 + \frac{C_1 |h|_2 \|u_k\|_{H^s}}{|u_k|_p^p}\right)^{\frac{1}{p-2}} \leqslant \left(1 + \frac{C_1 C_3 |h|_2}{a_2^p}\right)^{\frac{1}{p-2}}.$$

If

$$|h|_2 \leqslant \frac{a_2^p}{C_1C_3}\left[\left(\frac{3}{4}\right)^{\frac{p-2}{2}}-1\right],$$

then

$$t_k \leqslant \left(\frac{C_1C_3}{a_2^p}\frac{a_2^p}{C_1C_3}\left[\left(\frac{3}{4}\right)^{\frac{p-2}{p}}-1\right]\right)^{\frac{1}{p-2}}=\left(\frac{3}{4}\right)^{\frac{1}{2}}.$$

Now we consider (2.5)

$$\left|(1-\frac{1}{p})t_k^2\int_\Omega hu_k\mathrm{d}x\right| \leqslant \left(1-\frac{1}{p}\right)t_k^2|h|_2C_1\|u_k\|_{H^s} \leqslant \frac{3}{4}|h|_2\left(1-\frac{1}{p}C_1C_3\right).$$

If we take

$$|h|_2 < \frac{4m_0}{9\left(1-\frac{1}{p}\right)C_1C_3},$$

then

$$\left|(1-\frac{1}{p})t_k^2\int_\Omega hu_k\mathrm{d}x\right| \leqslant \frac{m_0}{3}.$$

Then we can write $m_0 \leqslant t_k^2J(u_k)+\frac{m_0}{3}$, i.e., $t_k^2J(u_k)\geqslant \frac{2}{3}m_0$. Since

$$J(u_k)>0,\ t_k^2\leqslant \frac{4}{3},$$

we get that $\frac{2}{3}m_0 \leqslant t_k^2J(u_k)\leqslant \frac{4}{3}J(u_k)$, i.e.,

$$\frac{1}{2}m_0 \leqslant J(u_k), as\ k\to\infty, \tag{2.6}$$

which implies that $\frac{1}{2}m_0 \leqslant m_h$. If we choose

$$\epsilon_4=\min\left\{\epsilon_3, \frac{a_2^p}{C_1C_3}\left[\left(\frac{4}{3}\right)^{\frac{p-2}{2}}-1\right], \frac{m_0}{a\left(1-\frac{1}{p}\right)C_1C_3}\right\},$$

then Lemma 2.6 holds. □

The next thing to prove is an important part of the theorem, namely the minimum of J on $\mathcal{N}_h$ is attained.

Lemma 2.7. *There exists $\epsilon_5 \in (0,\epsilon_4]$ such that for every $|h|_2<\epsilon_5$, m_h is attained by some $u\in\mathcal{N}_h$.*

Proof. We consider $\{u_k\}_k$ obtained in Lemma 2.5 and $|h|_2<\epsilon_4$. Since Ω is bounded, there exists $u\in H^s(\Omega)$ such that $u_k \rightharpoonup u$ in $H^s(\Omega)$. By Lemma 2.1, we have $u_k\to u$ in $L^p(\Omega)$ and in $L^2(\Omega)$. Then we derive that

$$J(u)\leqslant \liminf_k J(u_k)=m_h, \tag{2.7}$$

and

$$\|u\|_{H^s}^2 \leqslant |u|_p^p+\int_\Omega hu\mathrm{d}x. \tag{2.8}$$

Consider the case of equal sign in (2.8). From the Lemma 2.1, we have if (2.7) hold, then either $\|u\|_{H^s} > a_1$ or $\|u\|_{H^s} < a_3$. If $\|u\|_{H^s} > a_1$, then $u \in \mathcal{N}_h$ and (2.7) implies that u is the minimum we are looking for. If $\|u\|_{H^s} < a_3$, then

$$|u|_p \leqslant S_p \|u\|_{H^s} \leqslant S_p a_3 < S_p \frac{a_2}{S_p} = a_2,$$

which is a controdiction with $|u|_p \geqslant a_2$ from Lemma 2.2. Next consider the case of strict inequality in (2.8), namly,

$$\|u\|_{H^s}^2 < |u|_p^p + \int_\Omega hu\mathrm{d}x. \tag{2.9}$$

If we can show that (2.9) dose not hold, then (2.8) only holds when the equal sign is taken. At this time, according to the previous proof, u is the minimum we are looking for, and the proof of Lemma 2.7 is completed. So we only need to show that (2.9) can not hold. By (2.9), there exists $t^* > 0$ such that $t^* u \in \mathcal{N}_h$ and $t^* > t'$ according to (2.8), we have

$$\begin{aligned} t' &\leqslant \left(\frac{|u|_p^p + \int_\Omega hu\mathrm{d}x}{(p-1)\,|u|_p^p} \right)^{\frac{1}{p-2}} = \left(\frac{1}{p-1} + \frac{\int_\Omega hu\mathrm{d}x}{(p-1)\,|u|_p^p} \right)^{\frac{1}{p-2}} \\ &\leqslant \left(\frac{1}{p-1} + \frac{|h|_2 C_1 \|u\|_{H^s}}{(p-1)\,|u|_p^p} \right)^{\frac{1}{p-2}} \leqslant \left(\frac{1}{p-1} + \frac{|h|_2 C_1 C_3}{(p-1)\,a_2^p} \right)^{\frac{1}{p-2}}. \end{aligned}$$

If we choose

$$\epsilon_5 = \min\left\{ \frac{(p-2)\,(p-1)\,a_2^p}{2C_1C_3}, \epsilon_4 \right\},$$

then $t' \leqslant 1$.

For the function γ in Lemma 2.5, since $t^* u \in \mathcal{N}_h$, we have $\gamma(t^*) = 0$ and the ineqality (2.9) is equivalent to $\gamma(1) < 0$. Since $t' < 1$ and $t' < t^*$, we see that $t^* < 1$. According to the definition of m_h, we derive that

$$\begin{aligned} m_h \leqslant J(t^* u) &= (t^*)^2 \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|_{H^s}^2 - t^* \left(1 - \frac{1}{p} \right) \int_\Omega hu\mathrm{d}x \\ &\leqslant (t^*)^2 \liminf_k \left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|_{H^s}^2 - t^* \lim_k \left(1 - \frac{1}{p} \right) \int_\Omega hu\mathrm{d}x \\ &\leqslant (t^*) \liminf_k \left[\left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|_{H^s}^2 - \left(1 - \frac{1}{p} \right) \int_\Omega hu\mathrm{d}x \right] \\ &= t^* \liminf_k J(u_k) = t^* m_h < m_h. \end{aligned}$$

Observing the first and last two terms of the above inequality, we obtain that $m_h < m_h$, which is impossible, so the inequality (2.9) does not hold. □

Now we prove that u is the critical point of the functional J.

Lemma 2.8. *There exists $\epsilon_6 \in (0, \epsilon_5)$ such that if $|h|_2 < \epsilon_6$, then u satisfies $J'(u)\,v = 0$ for all $v \in H^s(\Omega)$.*

Proof. Fix $v \in H^s(\Omega)$ and consider function $\phi : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ defined by

$$\phi(s,t) := t^2 \|u + sv\|_{H^s}^2 - t^p |u + sv|^p - t \int_\Omega h(u + sv)\,\mathrm{d}x.$$

Since $u \in \mathcal{N}_h$, we have $\phi(0,1) = 0$. So ϕ is a first-order continuous function and

$$\frac{\partial \phi}{\partial t}(0,1) = 2\|u\|_{H^s}^2 - p|u|_p^p - \int_\Omega hu\mathrm{d}x = (2-p)\|u\|^2 + (p-1)\int_\Omega hu\mathrm{d}x.$$

Letting $\frac{\partial \phi}{\partial t}(0,1) = 0$, then

$$\|u\|_{H^s}^2 = \frac{p-1}{p-2}\int_\Omega hu\mathrm{d}x \leqslant \frac{p-1}{p-2}|h|_2 C_1 \|u\|_{H^s},$$

i.e.,

$$\|u\|_{H^s} \leqslant \frac{p-1}{p-2}|h|_2 C_1.$$

If we take

$$|h|_2 < \frac{p-2}{C_1(p-1)} a_1,$$

then

$$\|u\|_{H^s} < \frac{p-1}{p-2}\frac{p-2}{C_1(p-1)} a_1 C_1 = a_1,$$

which contradicts $u \in \mathcal{N}_h$. So for such choices of h, there must be $\frac{\partial \phi}{\partial t}(0,1) \neq 0$. By the Implicit Function Theorem, there exist a number $\delta > 0$ and a C^1 function $t(s) : (-\delta, \delta) \to \mathbb{R}$ such that $\phi(s, t(s)) = 0$ for every $s \in (-\delta, \delta)$ and $t(0) = 1$. Since $\|u\|_{H^s} > a_1$, we can also take δ small enough such that $t(s)(u + sv) > a_1$. We now study the behaivior of the function $\gamma(s) = J(t(s)(u + sv))$. It can be obtained that γ is differentiable and has a local minimum at $s = 0$. Since $u \in \mathcal{N}_h$, we have

$$0 = \gamma'(0) = J'(u)[t'(0)u + t(0)v] = t'(0)J'(u)u + J'(u)v = J'(u)v,$$

which implies that when $\epsilon_6 < \min\left\{\epsilon_5, \frac{(p-2)a_1}{C_1(p-1)}\right\}$, the minimum u satisfies $J'(u)v = 0$ for all $v \in H^s(\Omega)$. $\square$

So far, we have found a solution to equation (1.2). Next, we show that equation (1.2) has other solution.

Lemma 2.9. *For every $\epsilon > 0$, there exists $\delta > 0$ such that if $|h|_2 < \delta$, equation (1.2) admits a solution u_h satisfying $\|u_h\|_{H^s} < \epsilon$.*

Proof. Recalling $I(u) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p}|u|_p^p$, since

$$S_p = \inf\left\{C > 0 : |u|_p \leqslant C\|u\|_{H^s}, \forall u \in H^s(\Omega)\right\},$$

we have

$$I(u) \geqslant \frac{1}{2}\|u\|_{H^s}^2 - \frac{S_p^p}{p}\|u\|_{H^s}^p.$$

The function

$$\phi(t) := \frac{1}{2}t^2 - \frac{S_p^p}{p}t^p$$

is continuous, strictly increasing in a right neighborhood of 0, and $\phi(0) = 0$. There exists $\epsilon' \leqslant \epsilon$ such that for all $t \in (0, \epsilon')$, we have $\phi(t) > 0$. Then for any $\eta \in (0, \epsilon')$, we have $I(u) \geqslant \phi(\eta) > 0$ for $\|u\|_{H^s} = \eta$. We also have

$$J(u) = I(u) - \int_\Omega hu\mathrm{d}x \geqslant \phi(\eta) - |h|_2 C_1 \eta.$$

Choosing $\delta = \frac{\phi(\eta)}{2C_1\eta}$ and $|h|_2 < \delta$, we derive that $J(u) \geqslant \frac{\phi(\eta)}{2} > 0$ for $\|u\|_{H^s} = \eta$.

Define

$$B_\eta = \{u \in H^s(\Omega) : \|u\|_{H^s} \leqslant \eta\},$$

and $n_\eta = \inf_{u \in B_\eta} J(u)$. Obviously, $-\infty < n_\eta \leqslant J(0) = 0$. Then we may proved that n_η is achieved by some $u_h \in B_\eta$. Since $J(u_h) = n_\eta \leqslant 0$, it can not be $\|u_h\|_{H^s} = \eta$, which means u_h lies in the interior of the ball B_η and u_h is a local minimum for J, moreover, u_h is a solution of equation (1.2). □

Proof of Theorem 1.1. By Lemma 2.9, choosing $\epsilon = a_1$, we can fix $\delta > 0$ such that for every $|h|_2 < \delta$ there exists a solution u_h of equation (1.2) with $\|u_h\|_{H^s} < a_1$.

If we take $|h|_2 < \varepsilon := \min\{\epsilon_6, \delta\}$, then, by Lemma 2.8, we obtain a different solution u to equation (1.2), satisfying $\|u\|_{H^s} > a_1$. □

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ОБ ОДНОМ КЛАССЕ МНОГОЧЛЕНОВ, ГИПЕРБОЛИЧЕСКИХ С ВЕСОМ

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Аннотация. Доказывается, что если $\lambda \in \mathbb{R}^n$, $1 < \lambda_n \leq \lambda_{n-1} \leq ... \leq \lambda_1$, $\Re := \{\nu \in \mathbb{R}^n_+, (\lambda,\nu) \leq 1\}$, $\mathcal{M} := \{\nu \in \mathbb{R}^n_+, \sum_{j=1}^{n-1} \lambda_j \nu_j + \lambda_{n-1}\nu_n \leq 1\}$ и многочлен $P(\xi) = P(\xi_1, ..., \xi_n)$ является $\Re$- гиперболическим относительно вектора $\eta = (\eta_1, ..., \eta_n) \in R^n$, $\eta_n \neq 0$, то он $\mathcal{M}$-гиперболичен относительно η.

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Ключевые слова: многогранник Ньютона; (слабо) гиперболический многочлен (оператор); вес гиперболичности.

1. Введение. Постановка вопроса

Будем пользоваться следующими стандартными обозначениями: $\mathbb{N}-$ множество натуральных чисел, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_0^n = \mathbb{N}_0 \times \cdots \times \mathbb{N}_0-$ множество всех $n-$мерных мультииндексов, те. векторов $\alpha = (\alpha_1, ..., \alpha_n)$, где $\alpha_j \in \mathbb{N}_0$ $(j = 1, ..., n)$, $\mathbb{R}^n(\mathbb{E}^n)-$ $n-$мерное вещественное евклидово пространство точек $\xi = (\xi_1, ..., \xi_n)$ (соответственно точек $(x = (x_1, ..., x_n))$. $\mathbb{R}^n_+ := \{\xi \in \mathbb{R}^n, \xi_j \geq 0 \ (j = 1, ..., n)\}$, $\mathbb{R}^n_0 := \{\xi \in \mathbb{R}^n, \xi_1 ... \xi_n \neq 0 \ \}$.

Для $\xi, \eta \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, $\nu \in \mathbb{R}^n_+$ и $t \in \mathbb{R}$ обозначим $(\xi, \eta) = \xi_1\eta_1 + ... + \xi_n\eta_n$, $t\,\xi = (t\,\xi_1, ..., t\,\xi_n)$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2}$, $|\,\nu| = \nu_1 + \cdots + \nu_n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, где $D_j = \partial/\partial\xi_j$ $(j = 1, \cdots, n)$, $|\xi^\nu| = |\xi_1|^{\nu_1} ... |\xi_n|^{\nu_n}$, $G(\nu) := \{0 \neq \mu \in \mathbb{R}^n_0, \, \mu \neq \nu, \, \mu_j = \nu_j$ либо $\mu_j = 0, \ (1 \leq j \leq n).\}$

Пусть $P(\xi) = \sum_\alpha \gamma_\alpha \, \xi^\alpha$ многочлен с постоянными коэффициентами, где сумма распространяется по канечному набору мультииндексов $(P) := \{\alpha \in \mathbb{N}_0^n, \, \gamma_\alpha \neq 0\}$.

[1]Работа выполнена в рамках Программы Развития РАУ по НИР

Обозначим $m := \max_{\alpha \in (P)} |\alpha|$ и представим многочлен P в виде суммы однородных многочленов

$$P(\xi) = \sum_{j=0}^{m} P_j(\xi) = \sum_{j=0}^{m} \sum_{|\alpha|=j} \gamma_\alpha \, \xi^\alpha. \tag{1.1}$$

Определение 1.1. *(см. [1] или [2] определение 12.3.3) Пусть* $0 \neq \eta \in \mathbb{R}^n$ $\xi \in R^n$. *Многочлен* P *называется гиперболическим по Гордингу относительно вектора* η, *если* $P_m(\eta) \neq 0$ *и существует число* $\tau_0 > 0$, *такое, что* $P(\xi + i\tau\,\eta) \neq 0$ *для всех* $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{C}$, $|Re\tau| \geq \tau_0$.

Определение 1.2. *(см [3] или [4]) функцию* g, *определенную на* $\mathbb{R}^n$, *назовем весом гиперболичности, если 1)* $\inf_{\xi \in \mathbb{R}^n} g(\xi) > 0$, *2) существуют числа* $a \in [0,1)$ *и* $c > 0$ *такие, что* $g(\xi + \eta) \leq c\,[g(\xi) + |\eta|^a]$ $\forall \xi, \eta \in \mathbb{R}^n$. *Очевидно, что для веса гиперболичности* g, $\lim_{\xi \to \infty} g(\xi)/|\xi| = 0$.

Определение 1.3. *(см [3] или [4]) Пусть функция* g *является весом гиперболичности и* $0 \neq \eta \in \mathbb{R}^n$. *Скажем, что многочлен* P *является* $g-$*гиперболическим относительно вектора* η, *если* $P_m(\eta) \neq 0$ *и существует число* $c > 0$ *такое, что* $P(\xi + i\tau\,\eta) \neq 0$ *для любой пары* $(\xi, \tau) : \xi \in \mathbb{R}^n$, $\tau \in \mathbb{C}$, $|Re\tau| \geq c\,g(\xi)$.

Пусть $\mathcal{A} := \{\nu^j \in \mathbb{R}^n_+; j = 1, 2, ..., M\}$ (конечный) набор точек. Наименьшую выпуклую оболочку набора $\mathcal{A}$ (которая является многогранником) назовем многогранником Ньютона (далее М.Н) множества $\mathcal{A}$ (см., например, [5] - [7]) и обозначим через $\Re(\mathcal{A})$. Многогранник $\Re \subset \mathbb{R}^n_+$ называется вполне правильным, если 1) $\Re$ имеет вершину в начале координат $\mathbb{R}^n_+$, 2) отличную от начала координат вершину на каждой оси координат $\mathbb{R}^n_+$, и 3) все координаты внешних (относительно $\Re$) нормалей $(n-1)$ -мерных некоординатных граней (далее $\Re-$нормаль) положительны. Для в.п. многогранника $\Re$ введем обозначения

$\Re^0-$ множество его вершин,

$\Re^0_j = \{\nu = (\nu_1, ..., \nu_n) \in \Re^0, \nu_j \neq 0\}$ $j = 1, ..., n$,

$\Lambda(\Re)-$множество $\Re-$нормалей $(n-1)$ -мерных некоординатных граней, нормированных так, что $\sup_{\nu \in \Re}(\lambda, \nu) = 1$,

$\rho(\Re) := \max_{\nu \in \Re} |\nu| = \max_{\nu \in \Re^0} |\nu|$,

$d_j(\Re) := \max_{\lambda \in \Lambda(\Re)} (1/\lambda_j)$ $(j = 1, ..., n)$; $d(\Re) := \max_{1 \leq j \leq n} d_j(\Re)$,

$\partial\Re-$ множество точек $\nu \in \Re$ для которых существует вектор $\lambda \in \Lambda(\Re)$ такой, что $(\lambda, \nu) = 1$,

$\kappa_{\Re}(\nu) := \min_{\lambda\in\Lambda(\Re)}(1/(\lambda,\nu))$,

$\Re^*-$ многогранник Ньютона набора точек $\{(0,...,0,d_j(\Re),0,...,0)\ (j=1,...,n)\}$.

Легко убедиться, что для любого вполне правильного многогранника $\Re$ 1) многогранник $\Re^*$ является вполне правильным и $card\,\Lambda(\Re)=1$, 2) $\Re\subset\Re^*$, 3) $\Re=\Re^*$, тогда и только тогда, когда $card\Lambda(\Re)=1$ и 4) $\rho(\Re^*)=d(\Re^*)=d(\Re)$.

Из определения числа $\kappa_{\Re}(\nu)$, множества $G(\nu)$ и многогранника $\Re^*$ непосредственно следует

Предложение 1.1. *Пусть $\Re(\mathcal{A})$ в.п многогранник. Тогда*

1) $\mu/\kappa_{\Re}(\nu)\in\Re$ для любых $\nu\in\partial\Re$ и $\mu\in G(\nu)$

2) $\{\nu\in\mathbb{R}^n_+,(\lambda,\nu)\le 1\}\subset\Re^*$ для любого $\lambda\in\Lambda(\Re)$.

Через $\mathcal{B}_n$ обозначим множество $n-$мерных вполне правильных многогранников $\Re\subset\mathbb{R}^n_+$ для которых $d(\Re)<1$, а для вполне правильного многогранника $\Re\in\mathcal{B}_n$ положим $h_{\Re}(\xi):=\sum_{\nu\in\Re^0}|\xi^\nu|$. $h_{\Re}-$гиперболический многочлен назовем $\Re-$гиперболическим. При $s>1$ и $\Re=\{\nu\in\mathbb{R}^n_+:|\nu|\le 1/s\}$ $\Re-$гиперболический многочлен называется $s-$гиперболическим (см [8] - [9] и [10]).

Известно 1) (см [2], Следствие 12.5.7), что если многочлен P гиперболичен по Гордингу относительно вектора η, то для произвольных $f\in C^\infty(H)$ и $\varphi_j\in C^\infty(\partial H)$ $j=0,...,m-1$ решение следующей задачи Коши

$$P(D)u=f;\ <D,\eta>^k u\,|_{\partial H}=\varphi_j\ j=0,...,m-1$$

принадлежит $C^\infty(H)$, где $H:=\{x\in\mathbb{R}^n,(x,\eta)\ge 0\}$,

2) (см [3] или [4]) Пусть $\mathcal{M}\in\mathcal{B}_{n-1}$ многогранник такой, что $\mathcal{M}\Subset\{\nu'\in\mathbb{R}^{n-1},(0,\nu')\in\Re\}$, $G^{\mathcal{M}}-$ мультианизотропное пространство Жевре, $\tilde{\mathcal{M}}-$ многогранник Ньютона набора $\{(0,\nu'),\ \nu'\in\mathcal{M}^0\}\cup\{(d(\mathcal{M}),0')\}$, $\mathring{G}^{\mathcal{M}}:=G^{\mathcal{M}}\cap C_0^\infty$. Если многочлен P является $\Re-$гиперболическим относительно вектора $\eta=(1,0,...,0)$, то для произвольных $\varphi_j\in\mathring{G}^{\mathcal{M}}$ $j=0,...,m-1$ следующая задача Коши

$$P(D)u=0,x_1>0;\ D_1^j u\,|_{x_1=0}=\varphi_j\ j=0,...,m-1$$

имеет единственное решение $u\in G^{\tilde{\mathcal{M}}}$.

Аналогичные результаты для $s-$гиперболических операторов получены в [8].

Известно, (см., например, [8], [4], [9]), что поведение корня $t=t(\xi)$ гиперболического многочлена $P(\xi+t\,\eta)$ при возрастании $|\xi|$, непосредственно влияет на корректности постановки задачи Коши в пространствах Жевре. При этом, оказывается, что чем меньше скорость возрастания $t(\xi)$ при возрастании $|\xi|$, тем в более широких пространствах Жевре задача Коши для соответствующих операторов $P(D)$ поставлена корректно. Подобное явление происходит и при изучении вопроса о существования фундаментальных решений в функциональных пространствах, порожденных соответствующими пространствами Жевре, т.е. чем меньше скорость возрастания $t(\xi)$ при возрастании $|\xi|$, тем в более гладких по Жевре пространствах для операторов $P(D)$ имеется фундаментальное решение.

В настоящей работе, предполагая определенное априорное поведене корня $t = t(\xi)$ гиперболического многочлена $P(\xi + t\,\eta)$ при возрастании $|\xi|$, доказывается, что в некоторых случаях эта функция может иметь меньшую скорость возрастания.

2. Некоторые свойства многогранников Ньютона

Лемма 2.1. *Пусть* $\Re \in \mathcal{B}_n$ *Тогда для произвольных* $\nu \in \partial\Re$ *и* $\mu \in G(\nu)$ *существует вектор* $\lambda \in \Lambda(\Re)$ *такой, что*
$\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1}\,(\lambda, \nu-\mu) \geq 1,$ *т.е.* $\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1}(\nu-\mu) \notin \Re \setminus \partial\Re.$

Доказательство. Из определения множества $\partial\Re$, и в силу условия леммы следует, что существует вектор $\lambda^0 \in \Lambda(\Re)$ для которого $(\lambda^0, \nu) = 1$. Так как

$$\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1} = \max_{\lambda\in\Lambda(\Re)} \frac{1}{1-(\lambda,\mu)},$$

то $\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1}\,(\lambda^0, \nu-\mu) = \big[\max\limits_{\lambda\in\Lambda(\Re)} \frac{1}{1-(\lambda,\mu)}\big]\ (\lambda^0,\nu-\mu) \geq \frac{(\lambda^0,\nu-\mu)}{1-(\lambda^0,\nu)} \geq 1.$ □

Лемма 2.2. *Пусть* $\Re \in \mathcal{B}_n$. *Тогда* $\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1}\,(\nu-\mu) \in \Re^*$ *для произвольных* $\nu \in \partial\Re$ *и* $\mu \in G(\nu)$.

Доказательство. В силу пункта 2) Предложения 1.1 достаточно показать, что существует вектор $\lambda^0 \in \Lambda(\Re)$ такой, что $\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1}\,(\lambda^0, \nu-\mu) \leq 1$. Так как $\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1} = \max\limits_{\lambda\in\Lambda(\Re)} \frac{1}{1-(\lambda,\mu)}$, то существует вектор $\lambda^0 \in \Lambda(\Re)$ для которого $\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1} = \frac{1}{1-(\lambda^0,\mu)}$. Отсюда, в силу определения множества $\Lambda(\Re)$ имеем

$$\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1}(\lambda^0, \nu-\mu) = \frac{(\lambda^0,\nu-\mu)}{1-(\lambda^0,\mu)} \leq 1.$$

□

Лемма 2.3. *Пусть* $\Re \in \mathcal{B}_n$, $1 \leq j \leq n$ *и* $l(j) := (0,...,l_j,0,...,0)$ *вершина* $\Re$ *лежащая на оси* ξ_j. *Если* $(\lambda, l(1)) = 1$ *для любого* $\lambda \in \Lambda(\Re)$, *то* $\Re^0_1 \setminus \{l(1)\} = \emptyset$.

Доказательство. Пусть, например, $j = 1$. Предположим обратное, что при условиях леммы $\Re_1^0 \setminus \{l(1)\} \neq \emptyset$ и $\mu \in \Re_1^0 \setminus \{l(1)\}$. Пусть, рады определенности (быть может после перенумерации индексов $j : 2 \leq j \leq n$), $\mu = (\mu_1, \mu_2, ..., \mu_r, 0, ..., 0)$, $2 \leq r \leq n$, где $\mu_1 ... \mu_r \neq 0$.

Для натурального числа $k : k \leq n$ и точки $\xi \in \mathbb{R}^n$ обозначим $\xi^{(k)} := (\xi_1, ..., \xi_k)$ и пусть $\mathfrak{M} \subset \mathbb{R}_+^r$ М.Н. набора $\{\nu^{(r)} \in R_+^r; \nu = (\nu^{(r)}, 0, ..., 0) \in \Re^0\}$. Так как многогранник $\Re$ является вполне правильным, то $\mathfrak{M} = \{\nu^{(r)} \in R_+^r; (\lambda^{(r)}, \nu^{(r)}) \leq 1 \ \forall \lambda \in \Lambda(\Re)\}$.

Так как $\mu^{(r)}$ вершина многогранника $\mathfrak{M}$, то существуют r штук линейно независимых векторов $\{\lambda^{(r)}(j)\}$ $(\lambda(j) \in \Lambda(\Re)\ j = 1, ..., r)$ такие, что $(\lambda^{(r)}(j), \mu^{(r)}) = 1$ $(j = 1, ..., r)$. С другой стороны, в силу условия леммы

$$(\lambda^{(r)}(j), l^{(r)}(1)) = (\lambda(j), l(1)) = 1 \ (j = 1, ..., r).$$

Так как множество векторов $\{\lambda^{(r)}(j)\}$ линейно независимы, то отсюда получаем, что $\mu^{(r)} = l^{(r)}(1)$, следовательно (в силу определения $l(1)$ и μ) $\mu = l(1)$, что противоречит условиям $\mu_j \neq 0$ $(j = 1, ..., r, \ r \geq 2)$ и доказывает лемму. □

Замечание 2.1. *Легко убедиться, что для любого вполне правильного многогранника* $\Re$ $l(j) = (0, ..., 0, \min\limits_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j}, 0, ..., 0)$ *вершина многогранника* $\Re$ *лежащая на оси* ξ_j $(j = 1, ..., n)$.

Лемма 2.4. *Пусть* $\Re \in \mathcal{B}_n$, $1 \leq j \leq n$ *и* $l(j) := (0, ..., l_j, 0, ..., 0)$ *вершина* $\Re$ *лежащая на оси* ξ_j. *Тогда* $\Re_j^0 \setminus \{l(j)\} \neq \emptyset$ *в том и только в том случае, когда* $d_j(\Re) > l_j$, *или, что то же самое,*

$$\min_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j} < \max_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j}. \tag{2.1}$$

Доказательство. Ради удобства записи, доказательство проведем, например, для $j = 1$.

Достаточность. Пусть, наоборот, $\Re_1^0 \setminus \{l(1)\} \neq \emptyset$, но $\min\limits_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_1} = \max\limits_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_1}$. Так как $(\lambda^0, l(1)) = 1$ для некоторого $\lambda^0 \in \Lambda(\Re)$ и $l(1) = (l_1, 0, ..., 0)$, то $(\lambda, l(1)) = 1$ для любого $\lambda \in \Lambda(\Re)$. Отсюда, в силу Леммы 2.3 получаем, что $\Re_1^0 \setminus \{l(1)\} = \emptyset$. Получили противоречие, которое доказывает достаточность.

Необходимость. Пусть выполняется (2.1). Покажем, что $\Re_1^0 \setminus \{l(1)\} \neq \emptyset$. Выберем векторы λ^0 и λ^1 из $\Lambda(\Re)$ так, чтобы $\lambda_1^0 = \min\limits_{\lambda \in \Lambda(\Re)} \lambda_1$ и $\lambda_1^1 = \max\limits_{\lambda \in \Lambda(\Re)} \lambda_1$. Тогда $(\lambda^0, l(1)) = \lambda_1^0\, l_1 < \lambda_1^1\, l(1) = (\lambda^1, l(1))$.

Так как λ^0 нормаль к некоторой $(n-1)-$мерной некоординатной грани $\Re$, то существует вершина $e \in \Re^0$ такая, что $e_1 \neq 0$ и $(\lambda^0, e) = 1$. Так как $(\lambda^0, l(1)) < 1$, то отсюда следует, что $e \neq l(1)$, т.е. $e \notin \Re^0_1 \setminus \{l(1)\}$. □

Предложение 2.1. *Пусть для* $\Re \in \mathcal{B}_n$ *(после возможной перенумерации индексов) существуют номер* $r \geq 2$ *и вершина* $e = (e_1, ..., e_r, 0, ..., 0) \in \Re^0$, *для которых* $e_1 ... e_r \neq 0$. *Тогда* $d_j(\Re) > l_j$, $j = 1, ..., r$, *где* $l(j) := (0, ..., l_j, 0, ..., 0)$ *вершина* $\Re$ *лежащая на оси* ξ_j, $j = 1, ..., r$.

Доказательство. Предположим обратное, что, например, $d_1(\Re) = l_1$. В силу Леммы 2.4 имеем $\Re^0_1 \setminus \{l(1)\} = \emptyset$. Так как $e \neq l(1)$, то это противоречит условию $e \in \Re^0$ и доказывает предложение. □

Лемма 2.5. *Пусть* $\Re \in \mathcal{B}_n$, $1 \leq j \leq n$ *и* $l(j) := (0, ..., l_j, 0, ..., 0)$ *вершина* $\Re$ *лежащая на оси* ξ_j. *Если* $\Re^0_j \setminus \{l(j)\} \neq \emptyset$ *то существует вершина* $e \in \Re^0_j \setminus \{l(j)\}$ *такая, что* $e_j \, \frac{t_j}{(t_j-1)} = d_j(\Re)$ $(= \max\limits_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j})$, *где* $t_j = t_j(e) = \kappa_{\Re}(e_1, ..., e_{j-1}, 0, e_{j+1}, ..., e_n)$ $= \min\limits_{\lambda \in \Lambda(\Re)} \frac{1}{(\lambda, e) - (\lambda_j, e_j))}$.

Доказательство. Из условия $\Re^0_j \setminus \{l(j)\} \neq \emptyset$ настоящей леммы, из Леммы 2.4 и Замечания 2.1 имеем $d_j(\Re) = \max\limits_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j} > \min\limits_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j} = l_j$.

Выберем вектор $\lambda^0 \in \Lambda(\Re)$ так, чтобы $\frac{1}{\lambda^0_j} = \max\limits_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j}$. Тогда $d_j(\Re) = \frac{1}{\lambda^0_j}$ и $\frac{1}{\lambda^0_j} > \min\limits_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j}$. Так как $l_j < d_j(\Re)$, то $(\lambda^0, l(j)) < 1$. С другой стороны, так как λ^0 нормаль к некоторой $(n-1)-$мерной некоординатной грани $\Re$, то существует вершина $e \in \Re^0_j$ для которой $(\lambda^0, e) = 1$. Следовательно $e \neq l(j)$. Покажем, что

$$d_j(\Re) = e_j \max_{\lambda \in \Lambda(\Re)} \frac{1}{1 - (\lambda, e) + \lambda_j \, e_j}. \tag{2.2}$$

В самом деле, так как $d_j(\Re) = \frac{1}{\lambda^0_j}$, то в силу определения $\Lambda(\Re)$ имеем

$$d_j(\Re) = \frac{e_j}{e_j \, \lambda^0_j} \leq e_j \max_{\lambda \in \Lambda(\Re)} \frac{1}{1 - (\lambda, e) + \lambda_j \, e_j} \leq e_j \max_{\lambda \in \Lambda(\Re)} \frac{1}{\lambda_j \, e_j} = \frac{e_j}{\lambda^0_j \, e_j} = d_j(\Re),$$

откуда получаем равенство (2.2).

Так как $\frac{t_j}{(t_j-1)} = \max\limits_{\lambda \in \Lambda(\Re)} \frac{1}{1-(\lambda,e)+\lambda_j \, e_j}$, то из равенства (2.2) непосредственно получаем утверждение леммы. □

Следующее предложение непосредственно получается с применением неравенства Гелдера (см. также [6] или [7])

Предложение 2.2. *Пусть* $\Re \in \mathcal{B}_n$ *в.п многогранник,* $\mu \in \Re$, *и функция* $h_\Re(\xi) := \sum_{\nu \in \Re^0} |\xi^\nu|$ *определяется как выше. Тогда* $|\xi^\mu| \le c\, h_\Re(\xi) \ \ \forall\, \xi \in \mathbb{R}^n$ *с некоторой постоянной* $c > 0$.

Теорема 2.1. *Для произвольного* $\Re \in \mathcal{B}_n$ *существует постоянная* $c_1 > 0$ *такая, что*

$$h_\Re(\xi + \eta) \le c_1 \left[h_\Re(\xi) + h_{\Re^*}(\eta)\right] \ \forall\, \xi, \eta \in \mathbb{R}^n.$$

Доказательство. Так как $\kappa_\Re(\mu) > 1$ для любых $\mu \in G(\nu)$ и $\nu \in \Re^0$, то в силу неравенства Гёльдера имеем с некоторой постоянной $c_2 > 0$ для всех $\xi, \eta \in \mathbb{R}^n$

$$|(\xi+\eta)^\nu| = \prod_{j=1}^{n} |\xi_j + \eta_j|^{\nu_j} \le c_2 \prod_{j=1}^{n} (|\xi_j|^{\nu_j} + |\eta_j|^{\nu_j})$$

$$\le c_2[|\xi|^\nu + |\eta|^\nu) + \sum_{\mu \in G(\nu)} |\xi^\nu|\, |\eta^{\nu-\mu}| \le c_2 \left[(|\xi|^\nu + |\eta|^\nu)\right.$$

$$+ \sum_{\mu \in G(\nu)} [\frac{|\xi^\mu|^{\kappa_\Re(\mu)}}{\kappa_\Re(\mu)} + \frac{\kappa_\Re(\mu) - 1}{\kappa_\Re(\mu)}\, |\eta^{\nu-\mu}|^{\frac{\kappa_\Re(\mu)}{\kappa_\Re(\mu)-1}}].$$

Отсюда, в силу пункта 1) Предложения 1.1, Леммы 2.2 и Предложения 2.2, с некоторой постоянной $c_3 > 0$ имеем для всех $\xi, \eta \in \mathbb{R}^n$

$$|(\xi+\eta)^\nu| \le c_3 \left[h_\Re(\xi) + h_\Re(\eta) + h_{\Re^*}(\eta)\right]].$$

Так как $\Re \subset \Re^*$ то отсюда с некоторой постоянной $c_4 > 0$ имеем

$$|(\xi+\eta)^\nu| \le c_4 \left[h_\Re(\xi) + h_{\Re^*}(\eta)\right] \ \forall \xi, \eta \in \mathbb{R}^n.$$

Откуда, в силу определения функции $h_\Re$ и конечности множества $\Re^0$, получаем утверждение Теоремы 2.1. □

Теорема 2.2. *Пусть* $\Re \in \mathcal{B}_n$, *а вполне правильный многогранник* $\mathcal{M} \subset \mathbb{R}^n_+$ *такой, что* $\Re^* \not\subset \mathcal{M}$. *Тогда*

$$\sup_{\xi,\eta \in \mathbb{R}^n} \frac{h_\Re(\xi+\eta)}{h_\Re(\xi) + h_\mathcal{M}(\eta)} = \infty.$$

Доказательство. Исходя из определения многогранника $\Re^*$ и условия $\Re^* \not\subset \mathcal{M}$ теоремы, легко убедиться, что существует индекс $j : 1 \le j \le n$ (пусть для определенности $j = 1$) такой, что $l_1(\mathcal{M}) < d_1(\Re^*)$, где $l(1, \mathcal{M}) := (l_1(\mathcal{M}), 0..., 0)$, $d(1, \Re^*) := (d_1(\Re^*), 0, ..., 0) = (d_1(\Re), 0, ..., 0)$ вершины лежащие на оси ξ_1 соответственно многогранников $\mathcal{M}$ и $\Re^*$. Так как $d_1(\Re) \ge l_1(\Re)$, то возможны следующие два случая: 1) $d_1(\Re) > l_1(\Re)$ и 2) $d_1(\Re) = l_1(\Re)$.

В случае 1) в силу лемм 2.5 и 2.4 существует вершина $e \in \Re_1^0 \setminus \{l(1,\Re)\}$ такая, что $e_1 \frac{t_1(e)}{t_1(e)-1} = d_1(\Re)$, где $t_1(e) = \kappa_{\Re}(0, e_2, ..., e_n) = \min\limits_{\lambda \in \Lambda(\Re)} \frac{1}{(\lambda, e) - \lambda_1 e_1}$.

Выберем вектор $\lambda^0 \in \Lambda(\Re)$ так, чтобы $t_1(e) = \frac{1}{(\lambda^0, e) - \lambda_1 e_1}$. Тогда в силу Леммы 2.5 (см. доказательство этой леммы) $(\lambda^0, e) = 1$ и $\lambda_1^0 d_1(\Re) = 1$.

Рассмотрим следующие возможные подслучаи случая 1): 1.1) $l_1(1, \mathcal{M}) \geq l_1(\Re)$ и 1.2) $l_1(1, \mathcal{M}) < l_1(\Re)$.

Так как $\Re$ вполне правильный многогранник, то из условия $e \in \Re_1^0 \setminus \{l(1,\Re)\}$ имеем, что $e_1 < l_1(\Re)$. Следовательно в подслучае 1.1) $e_1 < l_1(\mathcal{M})$ и

$$\frac{1}{l_1(\mathcal{M})} < \frac{\sum_{j=2}^n \lambda_j^0 e_j}{l_1(\mathcal{M}) - e_1} = \frac{((\lambda^0, e)) - \lambda_1^0 e_1}{l_1(\mathcal{M}) - e_1} = \frac{1 - \lambda_1^0 e_1}{l_1(\mathcal{M}) - e_1}.$$

Пусть $a \in (1/l_1(\mathcal{M}), (1 - \lambda_1^0 e_1))/(l_1(\mathcal{M}) - e_1)$, $\xi^s := (0, s^{\lambda_2^0}, ..., s^{\lambda_n^0})$, $\eta^s := (s^a, 0, ..., 0)$ $(s = 1, 2, ...)$. Тогда имеем с некоторой постоянной $c_5 > 0$ при всех s = 1,2, ...

$$h_{\Re}(\xi^s + \eta^s) \geq |(\xi^s + \eta^s)^e| = |\eta_1^s|^{e_1} |\xi_2^s|^{e_2} ... |\xi_n^s|^{e_n} = s^{a e_1 + \sum_{j=2}^n \lambda_j^0 e_j} = s^{a e_1 + 1 - \lambda_1^0 e_1},$$

$$h_{\Re}(\xi^s) = \sum_{\nu \in \Re^0} |(\xi^s)^\nu| = \sum_{\nu \in \Re^0, \nu_1 \neq 0} s^{(\lambda^0, \nu)} \leq c_5 s, \quad h_{\mathcal{M}}(\eta^s) = |\eta_1^s|^{l_1(\mathcal{M})} = s^{a l_1(\mathcal{M})}.$$

Так как, по определению числа a, $ae_1 + 1 - \lambda_1^0 e_1 > a\, l_1(\mathcal{M}) > 1$, то отсюда получаем утверждение теоремы в подслучае 1.1).

В подслучае 1.2) положим $\xi^s = 0$, $\eta^s = (s, 0, ..., 0)$ $s = 1, 2, ...$. Тогда $h_{\Re}(\xi^s + \eta^s) = h_{\Re}(\eta^s) = 1 + s^{l_1(\Re)}$, $h_{\Re}(\xi^s) = 1$, $h_{\mathcal{M}}(\eta^s) = 1 + s^{l_1(\mathcal{M})}$ $s = 1, 2, ...$. Отсюда следует утверждение теоремы и в подслучае 1.2).

Утверждение теоремы в случае 2) доказывается аналогично подслучаю 1.2). Надо только иметь в виду, что, так как в этом случае $l_1(\mathcal{M}) < d_1(\Re)(= l_1(\mathcal{M})$, то $l_1(\mathcal{M}) < l_1(\Re)$. □

Из Теоремы 2.2 непосредственно следует

Следствие 2.1. *Пусть $\Re \in \mathcal{B}_n$, $l_1 < d_1(\Re)$, где $l(1) = (l_1, 0, ..., 0)$ вершина $\Re$ лежащая на оси ξ_1. Если $\sup\limits_{t \in \mathbb{R}} \frac{h_{\Re}(\xi + (t, 0, ..., 0))}{h_{\Re}(\xi) + t^l} < \infty$ для некоторого $l > 0$ и для всеь $\xi \in \mathbb{R}^n$, то $l \geq d_1(\Re)$.*

Теорема 2.3. *Пусть $\Re \subset \mathbb{R}_+^n$ вполне правильный многогранник. Функция $h_{\Re}$ является весом гиперболичности тогда и только тогда, когда $\Re \in \mathcal{B}_n$.*

Доказательство. **Достаточность.** Пусть $\Re \in \mathcal{B}_n$ вполне правильный многогранник. Так ка $0 \in \Re^0$, то $h_\Re(\xi) \geq 1$ $\forall \xi \in \mathbb{R}^n$. Из условия $\Re \in \mathcal{B}_n$ следует существование постоянной $c_6 > 0$ такой, что $h_\Re(\xi+\eta) \leq c_6\, [h_\Re(\xi) + h_{\Re^*}(\eta)]$ $\forall \xi, \eta \in \mathbb{R}^n$.

Так как $\rho(\Re^*) = d(\Re)$, то отсюда и из Предложения 2.2 получаем, что с некоторой постоянной $c_6^{'} > 0$ справедлива оценка $h_\Re(\xi+\eta) \leq c_6^{'}\, [h_\Re(\xi) + |\eta|^{d(\Re)}]$ $\forall \xi, \eta \in \mathbb{R}^n$.

Так как $d(\Re) < 1$ в силу определения $\mathcal{B}_n$, то отсюда непосредственно следует, что функция $h_\Re$ является весом гиперболичности.

Необходимость. Пусть функция $h_\Re$ является весом гиперболичности, т.е. с некоторыми постоянными $a \in [0,1)$ и $c_7 > 0$

$$h_\Re(\xi+\eta) \leq c_7\, [h_\Re(\xi) + |\eta|^a] \;\; \forall \xi, \eta \in \mathbb{R}^n. \tag{2.3}$$

Покажем, что из оценки (2.3) следует, что $d(\Re) < 1$, т.е. $\Re \in \mathcal{B}_n$. Предположим обратное, что $d(\Re) \geq 1$. Пусть $\delta > d(\Re)$, $\mathcal{M} := \frac{1}{\delta}\Re = \{\mu : \delta\, \mu \in \Re\}$. Очевидно $d(\mathcal{M}) = d(\Re)/\delta < 1$. Следовательно $\mathcal{M} \in \mathcal{B}_n$ и с некоторой постоянной $c_8 > 0$

$$c_8^{-1} h_\mathcal{M}^\delta(\xi) \leq h_\Re(\xi) \leq c_8\, h_\mathcal{M}^\delta(\xi) \;\; \forall \xi \in \mathbb{R}^n. \tag{2.4}$$

Рассмотрим следующие возможные случаи

1) для некоторого индекса $j : 1 \leq j \leq n$ (пусть, ради определенности для j = 1) $l_1(\mathcal{M}) < d_1(\mathcal{M})$

2) $l_j(\mathcal{M}) = d_j(\mathcal{M})$ при всех $j : 1 \leq j \leq n$, где $l(j, \mathcal{M}) := (0,...,0,l_j,0,...,0)$ вершина $\mathcal{M}$, лежащая на оси ξ_j $(j = 1,...,n)$.

Пусть в случае 1) $l \in (l_1(\mathcal{M}), d_1(\mathcal{M}))$ некоторое число. Согласно Следствия 2.1 для любого $s = 1,2,...$ существует точка $\xi^s \in \mathbb{R}^n$ и число t_s такие, что

$$h_\mathcal{M}(\xi^s + (t_s,0,...,0)) \geq s\, [h_\mathcal{M}(\xi^s] + |t_s|^l \;\; (s = 1,2,...)\cdot \tag{2.5}$$

Из оценки (2.4), на основании Теоремы 2.1 имеем, что $t_s \to \infty$ при $s \to \infty$. Тогда из той же оценки (2.4) и левой части оценки (2.3) получаем $c_8^{-1}\, [s\, (h_\mathcal{M}(\xi^s) + |t_s|^l)]^\delta$ $\leq h_\Re(\xi^s + (t_s,0,...,0))$ $s = 1,2,...\cdot$ Отсюда, в силу оценки (2.3), с некоторой постоянной $c_9 > 0$ имеем для всех $s = 1,2,...$ $s^\delta\, [h_\mathcal{M}^\delta(\xi^s) + |t_s|^{l\,\delta}] \leq c_9\, [h_\Re(\xi^s) + |t_s|^a]$. Откуда, в свою очеред, в силу правой части неравенства (2.4), получаем, что

$$c_8^{-1}\, [s^\delta\, h_\Re(\xi^s) + |t_s|^{l\,\delta}] \leq c_9\, [h_\Re(\xi^s) + |t_s|^a\,] \;\; s = 1,2,...\cdot \tag{2.6}$$

Из этой оценки следует, что для достаточно больших s $s^{\delta}\,|t_s|^{l\,\delta} \le c_9\,|t_s|^{a}$ и так как $t_s \to \infty$ при $s \to \infty$, то отсюда получаем, что $l\,\delta < a$. Отсюда, в силу произвольности числа $l \in (l_1(\mathcal{M}), d_1(\mathcal{M})$, получаем, что $\delta\, d_1(\mathcal{M}) \le a$, следовательно $d_1(\Re) \le a$. Так как $a \in [0,1)$, то этим необходимая часть теоремы в случае 1) доказана.

Рассмотрим случай 2). В этом случае в силу Леммы 2.4 получаем, что множество $\Lambda(\Re)$ состоит из одного элемента $\Lambda(\Re) = \{(\frac{1}{d_1(\Re)}, ..., \frac{1}{d_n(\Re)})\}$ и поэтому $h_{\Re}(\xi) = 1 + \sum_{j=1}^{n} |\xi_j|^{d_j(\Re)}$. Тогда, применяя оценку (2.3) при $\xi = 0, \eta \in \mathbb{R}^n$, получим

$$1 + \sum_{j=1}^{n} |\eta_j|^{d_j(\Re)} = h_{\Re}(0+\eta) \le c_7\,[h_{\Re}(0) + |\eta|^a] = c_7\,(1 + |\eta|^a) \quad \forall \eta \in \mathbb{R}^n.$$

Отсюда непосредственно получаем, что $d(\Re) = \max_{1\le j\le n} d_j(\Re) \le a$. Так как $a \in [0,1)$, то это означает, что $\Re \in \mathcal{B}_n$. Этим часть теоремы, относящейся к необходимости в случае 2) также доказана. Теорема 2.3 доказана. □

Замечание 2.2. *Очевидно что, если $\Re \subset \mathbb{R}^n_+$ вполне правильный многогранник, для которого выполняется оценка (2.3), то $a > 0$.*

Лемма 2.6. *Пусть $\Re \in \mathcal{B}_n$, а $l(n) := (0, ..., 0, l_n)$ вершина $\Re$, лежащая на оси ξ_n. Если $l(n) \ge d(\mathcal{M})$, где $\mathcal{M}$ многогранник Ньютона набора $\{\nu' := (\nu_1, ..., \nu_{n-1}) \in \mathbb{R}^{n-1}_+, (\nu', 0) \in \Re^0\}$, то с некоторой постоянной $c_{10} \in (0,1)$ и для всех $\eta = (\eta_1, ..., \eta_n), \eta_n \ne 0$ имеем*

$$c_{10}\; h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta') = c_{10}\; h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) \le \min_{t\in\mathbb{R}} h_{\Re}(\xi - t\,\eta)$$

$$\le h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) = h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta', 0) \;\; \forall \xi \in \mathbb{R}^n. \tag{2.7}$$

Доказательство. Так как $\xi - \frac{\xi_n}{\eta_n}\,\eta = (\xi' - \frac{\xi_n}{\eta_n}\,\eta', 0)$ при $\xi \in \mathbb{R}^n$, то в силу определения многогранника $\mathcal{M}$ имеем $h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) = h_{\Re}(\xi' - \frac{\xi_n}{\eta_n}\,\eta', 0) = h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta')$ $\forall \xi \in \mathbb{R}^n$, откуда непосредственно получаем правую часть оценки (2.7).

Дакажем левую часть оценки (2.7). В силу Теоремы 2.1 с некоторой постоянной $c_{11} > 0$ и для любого $t \in \mathbb{R}$ имеем

$$h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta') = h_{\mathcal{M}}(\xi' - t\,\eta' + (t\,\eta_n - \xi_n)\frac{\eta'}{\eta_n})$$

$$\le\; c_{11}\,[h_{\mathcal{M}}(\xi' - t\,\eta') + h_{\mathcal{M}^*}((t\,\eta_n - \xi_n)\frac{\eta'}{\eta_n})].$$

Так как $\rho(\mathcal{M}^*) = d(\mathcal{M})$, то для всех $\xi \in \mathbb{R}^n$ и $t \in \mathbb{R}$ получаем

$$h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta') \le c_{11}\,[h_{\mathcal{M}}(\xi' - t\,\eta') + |\xi_n - t\,\eta_n|^{\rho(\mathcal{M}^*)}] = c_{11}\,[h_{\mathcal{M}}(\xi' - t\,\eta') + |\xi_n - t\,\eta_n|^{d(\mathcal{M})}].$$

В силу условия леммы $l_n \ge d(\mathcal{M})$ и $\{(\nu', 0) : \nu' \in \mathcal{M}\} \subset \Re$, поэтому, на основании Предложения 2.2, с некоторой постоянной $c_{12} > 0$ для всех $\xi \in \mathbb{R}^n$ и $t \in \mathbb{R}$ отсюда получаем

$$h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta') \le c_{12}\,[h_{\mathcal{M}}(\xi' - t\,\eta') + |\xi_n - t\,\eta_n|^{l_n}] \le c_{12}\,h_{\Re}(\xi - t\,\eta),$$

откуда, в силу произвольности числа $t \in \mathbb{R}$, получаем левую часть оценки (2.7). Лемма 2.6 доказана. □

Замечание 2.3. *Легко убедиться, что для любого* $\Re \in \mathcal{B}_n$, *и* $\eta \in \mathbb{R}^n$ *функция* $h_{\Re,\eta} := \min_{t\in\mathbb{R}} h_{\Re}(\xi - t\,\eta)$ *также является весом гиперболичности.*

Следствие 2.2. Пусть $0 < l_1 \le l_2 \le ... \le l_n < 1$ *а* $\Re$ *многогранник Ньютона набора* $\{l(j) := (0, ..., l_j, 0, ..., 0)\}_{j=1}^n$. *Тогда для любой точки* $\eta \in \mathbb{R}^n$, $\eta_n \ne 0$ *с некоторой постоянной* $c'_{12} \in (0, 1)$ *и для всех* $\xi \in \mathbb{R}^n$

$$c'_{12}\,[1 + \sum_{j=1}^{n} |\xi_j - \frac{\xi_n}{\eta_n}\,\eta_j|^{l^j}] \le h_{\Re,\eta}(\xi) \le [1 + \sum_{j=1}^{n} |\xi_j - \frac{\xi_n}{\eta_n}\,\eta_j|^{l^j}].$$

Доказательство. немедленно следует из Леммы 2.6, так как при условиях следствия $\Re \in \mathcal{B}_n$. □

Ниже мы будем пользоваться также следующим очевидным предложением

Предложение 2.3. *Пусть* $a \ne b$ *и* $\delta > 0$. *Тогда с некоторой постоянной* $c_{13} > 0$ *и при всех* $t, \vartheta \in \mathbb{R}$

$$c_{13}^{-1}\,(|t|^\delta + |\vartheta|^\delta) \le |t - a\,\vartheta|^\delta + |t - b\,\vartheta|^\delta \le c_{13}\,(|t|^\delta + |\vartheta|^\delta).$$

Лемма 2.7. *Пусть при условиях Леммы 2.6* $\eta^j \in \mathbb{R}^n$, $\eta^j_n \ne 0$ $(j = 1, ..., n)$ *линейно независимые векторы. Тогда с некоторой постоянной* $c_{14} > 0$ *и при всех* $\xi \in \mathbb{R}^n$

$$(2.8)\qquad c_{14}^{-1}\,[1 + \sum_{j=1}^{n-1} |\xi_j|^{l_j} + |\xi_n|^{l_{n-1}}] \le \sum_{j=1}^{n} h_{\Re,\eta^j}(\xi) \le c_{14}\,[1 + \sum_{j=1}^{n-1} |\xi_j|^{l_j} + |\xi_n|^{l_{n-1}}].$$

Доказательство. Так как векторы $\{\eta^j\}_{j=1}^n$ линейно независимы, то для любого $k : 1 \le k \le n-1$ существуют индексы $j_1, j_2 :\ j_1 \ne j_2$, $1 \le j_1, j_2 \le n$ такие, что $\eta_k^{j_1}/\eta_n^{j_1} \neq \eta_k^{j_2}/\eta_n^{j_2}$, то в силу предложения 2.3 с некоторой постоянной $c_{14} > 0$ имеем

$$c_{14}^{-1}\,(|\xi_k|^{l_k} + |\xi_n|^{l_k}) \leq |\xi_k - (\eta_k^{j_1}/\eta_n^{j_1})\,\xi_n|^{l_k} + |\xi_k - (\eta_k^{j_2}/\eta_n^{j_2})\,\xi_n|^{l_k}$$

$$\leq c_{14}\,(|\xi_k|^{l_k} + |\xi_n|^{l_k}) \;\; \forall \xi_k, \xi_n \in \mathbb{R}.$$

Так как $\max\limits_{1\leq j\leq n-1} l_j = l_{n-1}$, то отсюда непосредственно получается оценка (2.8), что доказывает лемму. □

Лемма 2.8. *Пусть* $\Re \in \mathcal{B}_n$, $\eta \in \mathbb{R}^n$, $\eta_n \neq 0$, *а* $\tilde{\Re} \subset \mathbb{R}_+^n$ *многогранник Ньютона набора* $\{\nu = (\nu_1, ..., \nu_{n-1}, 0) \in \Re^0\} \bigcup \{(0', d(\mathcal{M}))\}$, *где* $\mathcal{M} \subset \mathbb{R}_+^{n-1}$ *многогранник Ньютона набора* $\{\nu' = (\nu_1, ..., \nu_{n-1}) \in \mathbb{R}_+^{n-1}, (\nu', 0) \in \Re^0\}$. *Тогда с некоторимы положительнимы постояннимы* c_{15} *и* c_{16} *выполняются неравенства*

$$h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) \leq c_{15}\; h_{\tilde{\Re}}(\xi) \;\; \forall \xi \in \mathbb{R}^n, \tag{2.9}$$

$$h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) \leq c_{16}\; h_{\tilde{\Re},\eta}(\xi) \;\; \forall \xi \in \mathbb{R}^n. \tag{2.10}$$

Доказательство. Так как $(\xi - (\eta/\eta_n)\,\xi_n) = (\xi' - (\eta'/\eta_n)\,\xi_n, 0)$, то в силу Теоремы 2.1 с некоторой постоянной $c'_{15} > 0$ имеем

$$h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) = h_{\Re}(\xi' - \frac{\xi_n}{\eta_n}\,\eta', 0) = h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta')$$

$$\leq c_{15}\,[h_{\mathcal{M}}(\xi') + h_{\mathcal{M}^*}(\frac{\xi_n}{\eta_n}\,\eta')] \;\; \forall \xi \in \mathbb{R}^n.$$

С другой стороны, так как $\rho(\mathcal{M}^*) = d(\mathcal{M})$, то в силу определения многогранника $\tilde{\Re}$ и на основании Предложения 2.2, с некоторой постоянной $c''_{15} > 0$, отсюда получаем

$$h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) \leq c_{16}\, h_{\mathcal{M}}(\xi') + |\xi_n|^{d(\mathcal{M})}] \leq c_{16}\, h_{\tilde{\Re}}(\xi) \;\; \forall \xi \in \mathbb{R}^n.$$

Этим оценка (2.9) доказана. Докажем оценку (2.10).

Так как $(0', d(\mathcal{M}))$ вершина многогранника $\tilde{\Re}$ лежащая на оси ξ_n и, очевидно, что $\tilde{\Re} \in \mathcal{B}_n$, то в силу Леммы 2.6 с некоторой постоянной $c_{17} > 0$ при всех $\xi \in \mathbb{R}^n$ имеем

$$c_{17}^{-1}\, h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta') \leq h_{\tilde{\Re},\eta}(\xi) \leq c_{17}\, h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta').$$

Так как $h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) = h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta')$ при всех $\xi \in \mathbb{R}^n$, то отсюда непосредственно получаем оценку (2.10). Лемма 2.8 доказана. □

Из Леммы 1 работы [6] следует следующее

Предложение 2.4. *Пусть* $\Re, \mathcal{M} \subset \mathbb{R}^n_+$ *(выпуклые) мнгогранники. Тогда* $\Re \subset \mathcal{M}$ *в том и только в том случае, когда* $h_{\Re}(\xi) \le h_{\mathcal{M}}(\xi)$ *для всех* $\xi \in \mathbb{R}^n$.

Лемма 2.9. *Пусть* $\Re \in \mathcal{B}_n$, $\eta \in \mathbb{R}^{n-1}$, $\eta_n \neq 0$, *а* $\mathcal{M} \subset \mathbb{R}^{n-1}_+$ *многогранник Ньютона набора* $\{\nu' = (\nu_1, ..., \nu_{n-1}) \in \mathbb{R}^{n-1}_+,\ (\nu', 0) \in \Re^0\}$. *Тогда*

1) если для вполне правильного многогранника $\tilde{\Re} \subset \mathbb{R}^n_+$ *выполняется неравенство (ниже* $\mathcal{D} := \{j : 1 \le j \le n-1,\ \eta_j \neq 0.\}$*)*

$$h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) \le c'_{17}\, h_{\tilde{\Re}}(\xi)\ \ \forall \xi \in \mathbb{R}^n \tag{2.11}$$

с некоторой постоянной $c'_{17} > 0$ *, то*

$$\{\nu = (\nu', 0) \in \Re^0\} \cup \{(0', \max_{(\nu',0)\in\Re^0} \sum_{j\in\mathcal{D}} \nu_j)\} \subset \tilde{\Re}.$$

2) Пусть $\delta \in \mathbb{R}$. *Для того, чтобы с некоторой постоянной* $c_{18} > 0$ *выполнялось неравенство*

$$h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) \le c_{18}\,[h_{\mathcal{M}}(\xi') + |\xi_n|^{\delta}]\ \ \forall \xi \in \mathbb{R}^n, \tag{2.12}$$

где $\xi' = (\xi_1, ..., \xi_{n-1})$, *необходимо и достаточно условие* $\delta \ge \max_{j\in\mathcal{D}} d_j(\mathcal{M})$ *при* $\eta' \neq 0$.

Замечание 2.4. *Отметим, что условие леммы означает, что минимальный вполне правильный многогранник* $\tilde{\Re}$ *для которого выполняется неравенство (2.11) является многогранник Ньютона набора* $\{\nu = (\nu_1, ..., \nu_{n-1}, 0) \in \Re^0\} \cup \{(0', \max_{j\in\mathcal{D}} d_j(\mathcal{M}))\}$.

Доказательство леммы. Так как, в силу определения многогранника $\mathcal{M}$, $h_{\Re}(\xi - \frac{\xi_n}{\eta_n}\,\eta) = h_{\Re}(\xi' - \frac{\xi_n}{\eta_n}\,\eta', 0) = h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta')$, то на основания оценки (2.11) имеем

$$h_{\mathcal{M}}(\xi - \frac{\xi_n}{\eta_n}\,\eta) \le c'_{17}\, h_{\tilde{\Re}}(\xi)\ \ \forall \xi \in \mathbb{R}^n. \tag{2.11$'$}$$

Отсюда, в силу Предложения 2.4 и определения функции $h_{\tilde{\Re}}$, получаем, что $\mathcal{M} \subset \{\nu' = (\nu_1, ..., \nu_{n-1}) \in \mathbb{R}^{n-1}_+, (\nu', 0) \in \tilde{\Re}\}$. Следовательно $\{\nu = (\nu', 0) \in \mathcal{M}^0\}$ $\subset \{\nu = (\nu_1, ..., \nu_{n-1}, 0) \in \tilde{\Re}\} \subset \tilde{\Re}$.

Так как $\{\nu = (\nu', 0) \in \mathcal{M}^0\} = \{\nu = (\nu', 0) \in \Re^0\}$, то отсюда получаем, что $\{\nu = (\nu', 0) \in \Re^0\} \subset \tilde{\Re}$.

Теперь покажем, что $(0', \max_{\nu\in\mathcal{M}^0} \sum_{j\in\mathcal{D}} \nu^j) \in \tilde{\Re}$ при $\eta' \neq 0$. Пусть $\xi_j = 0$ при $j \in \mathcal{D}$, $\xi_j = 1$ при $j \notin \mathcal{D}$ и $\xi_n \in \mathbb{R}^1$. Тогда с некоторимы положительнимы постояннимы $c_{18}^{'}$ и c_{19} имеем для всех $\xi \in \mathbb{R}^n$

$$h_{\mathcal{M}}(\xi - \frac{\xi_n}{\eta_n}\,\eta) = \sum_{\nu' \in \mathcal{M}^0} |\frac{\xi_n}{\eta_n}\,\eta'|^{\nu'} = \sum_{\nu' \in \mathcal{M}^0} \prod_{j \in \mathcal{D}} |\frac{\xi_n}{\eta_n}\,\eta_j|^{\nu_j'}$$

$$\geq c_{18}^{'} \sum_{\nu' \in \mathcal{M}^0} |\xi_n|^{\sum\limits_{j \in \mathcal{D}} \nu_j} \geq c_{19}\,[1 + |\xi_n|^{\max\limits_{\nu' \in \mathcal{M}^0} \sum\limits_{j \in \mathcal{D}} \nu_j'}].$$

Так как $\tilde{\mathfrak{R}}$ вполне правильный многогранник, то с некоторой постоянной $c_{20} > 0$ и $\xi_j = 0$ при $j \in \mathcal{D}$, $\xi_j = 1$ при $j \in \mathcal{D}$, $1 \leq j \leq n-1$, и любого $\xi_n \in \mathbb{R}^1$ имеем $h_{\tilde{\mathfrak{R}}}(\xi) \leq c_{20}\,(1 + |\xi_n|^{\tilde{l}_n})$, где $\tilde{l}(n) := (0, ..., 0, \tilde{l}_n)$ вершина $\tilde{\mathfrak{R}}$ лежащая на оси ξ_n.

Из последных двух оценок, в силу неравенства (2.11') получаем, что $\max_{\nu' \in \mathcal{M}^0}(\sum_{j \in \mathcal{D}} \nu_j^{'}) \leq \tilde{l}_n$, следовательно $(0', \max_{\nu \in \mathfrak{R}^0} \sum_{j \in \mathcal{D}} \nu_j) \in \tilde{\mathfrak{R}}$, что доказывает утверждение пункта 1) леммы.

Достаточность пункта 2). Пусть $\delta \in \mathbb{R}$ любое при $\eta' = 0$ и $\delta \geq \max_{j \in \mathcal{D}} d_j(\mathcal{M})$ при $\eta' \neq 0$. Если $\eta' = 0$, то выполнение оценки (2.12) для любого $\delta \in \mathbb{R}$ непосредственно следует из того, что при для всех $\xi \in \mathbb{R}^n$

$$h_{\mathfrak{R}}(\xi - \frac{\xi_n}{\eta_n}\,\eta) = h_{\mathfrak{R}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta') = h_{\mathfrak{R}}(\xi', 0) = h_{\mathcal{M}}(\xi').$$

Пусть $\eta' \neq 0$. Тогда, в силу Теоремы 2.1 и определения многогранника $\mathcal{M}^*$, с некоторой постоянной $c_{21} > 0$ и при всех $\xi \in \mathbb{R}^n$ имеем

$$h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta') \leq c_{21}\,[h_{\mathcal{M}}(\xi') + [h_{\mathcal{M}^*}((\eta'/\eta_n)\,\xi_n)] = c_{21}\,[h_{\mathcal{M}}(\xi')$$

$$+1 + \sum_{j=1}^{n-1} |(\xi_n/\eta_n)\,\eta|^{d_j(\mathcal{M})}] = c_{21}\,[h_{\mathcal{M}}(\xi') + 1 + \sum_{j \in \mathcal{D}} |(\xi_n/\eta_n)\,\eta|^{d_j(\mathcal{M})}].$$

Так как $h_{\mathfrak{R}}(\xi - \frac{\xi_n}{\eta_n}\,\eta) = h_{\mathcal{M}}(\xi' - \frac{\xi_n}{\eta_n}\,\eta')$ для всех $\xi \in \mathbb{R}^n$, то отсюда получаем оценку (2.12) при $\delta \geq \max_{j \in \mathcal{D}} d_j(\mathcal{M})$.

Необходимость пункта 2). Предположим, для определенности, что $\mathcal{D} = \{j\}_{j=1}^r$ $1 \leq r \leq n-1$. Покажем, что при выполнении оценки (2.12) $\delta \geq \max_{j \in \mathcal{D}} d_j(\mathcal{M})$.

Сначала покажем, что $\delta \geq \max_{1 \leq j \leq r} l_j(\mathcal{M})$, где $l(j, \mathcal{M}) := (0, ..., 0, l_j(\mathcal{M}), 0, ..., 0)$ вершина $\mathcal{M}$, лежащая на оси ξ_j.

Из оценки (2.12) при $\xi' = 0$, $\xi_n \in \mathbb{R}$ имеем

$$1 + \sum_{j=1}^{r} |(\xi_n/\eta_n)\,\eta_j|^{l_j(\mathcal{M})} = h_{\mathcal{M}}((\xi_n/\eta_n)\,\eta') = h_{\mathcal{M}}(\xi' - (\xi_n/\eta_n)\,\eta')$$

$$= h_{\mathfrak{R}}(\xi - \frac{\xi_n}{\eta_n}\,\eta) \leq c_{18}\,[h_{\mathcal{M}}(\xi') + |\xi_n|^{\delta}] = c_{18}\,[h_{\mathcal{M}}(0') + |\xi_n|^{\delta}] = c_{18}\,(1 + |\xi_n|^{\delta}),$$

откуда непосредственно получаем, что $\delta \geq \max_{1 \leq j \leq r} l_j(\mathcal{M})$.

Теперь покажем, что $\delta \geq \max_{1\leq j\leq r} d_j(\mathcal{M})$. Предположим обратное, что $\delta < \max_{1\leq j\leq r} d_j(\mathcal{M})$. Тогда, по уже доказанной части получим, что

$$\max_{1\leq j\leq r} l_j(\mathcal{M}) \leq \delta < \max_{1\leq j\leq r} d_j(\mathcal{M}). \tag{2.13}$$

Пусть, для определенности, $d_1(\mathcal{M}) = \max_{1\leq j\leq r} d_j(\mathcal{M})$. Так как из (2.13) следует, что $l_1(\mathcal{M}) < d_1(\mathcal{M})$, то проводя рассуждения, аналогичные рассуждениям, проводимым при доказательстве Теоремы 1.2, для $\lambda^{0,1} = (\lambda_1^0, ..., \lambda_{n-1}^0)$, где $\lambda^0 = (\lambda_1^0, ..., \lambda_{n-1}^0, \lambda_n^0) \in \Lambda(\Re)$, $\lambda_1^0 = \min_{\lambda\in\Lambda(\Re)} \lambda_1$ получим, что $\lambda_1^0\, d_1(\mathcal{M}) = 1$, $\lambda_1^0\, l_1(\mathcal{M}) < 1$, $\lambda_j^0\, l_j(\mathcal{M}) \leq 1$ $(j = 2, ..., r)$, при этом существует вектор $\mu' = (\mu_1, ..., \mu_{n-1}) \in \mathcal{M}_1^0 \setminus \{l(1, \mathcal{M})\}$ такой, что $(\lambda^{0,1}, \mu') = 1$ и $\mu_1\, t_1(\mu')/(t_1(\mu') - 1) = d_1(\mathcal{M})$ (определение числа $t_1(\mu)$ см. в Лемме 2.5).

Пусть $l \in (\delta, d_1(\mathcal{M}))$ любое фиксированное число. Отметим, что так как 1)$l - l_1(\mathcal{M}) > \mu_1$, 2) многогранник $\mathcal{M}$ является вполне правильным и 3) $\lambda_1^0\, l < \lambda_1^0\, d_1\,(\mathcal{M})$, то $1/l < (1-\lambda_1^0\,\mu_1)/(l-\mu_1))$. Положим $\xi_1^s = 0$, $\xi_j^s = (sgn(\eta_j/\eta_n))\, s^{\lambda_j^0}$ $(j = 2, ..., r)$, $\xi_j^s = s^{\lambda_j^0}$ $(j = r+1, ..., n-1)$, и $\xi_n^s = s^a$ $(s = 1, 2, ...)$, где $a \in (1/l, (1-\lambda_1^0\,\mu_1)/(l-\mu_1))$.

В силу определения последовательности $\{\xi^s\}$ с некоторыми положительными постоянными c_{22}, c_{23} имеем

$$h_{\mathcal{M}}((\xi')^s - (\frac{\xi_n}{\eta_n}\,\eta')) \geq 1 + |((\xi')^s - (\frac{\xi_n}{\eta_n}\,\eta'))^{\mu}|$$

$$= 1 + |\xi_n^s|^{\mu_1} \prod_{j=2}^{r} |\xi_j^s - (\frac{\xi_n^s}{\eta_n}\,\eta')|^{\mu_j} \prod_{j=r+1}^{n-1} |\xi_j^s|^{\mu_j}$$

$$\geq c_{22}\,[1 + |\xi_n^s|^{\mu_1} \prod_{j=2}^{r} (|\xi_j^s|^{\mu_j} + |\xi_n^s|^{\mu_j}) \prod_{j=r+1}^{n-1} |\xi_j^s|^{\mu_j}]$$

$$\geq c_{23}\,[1 + |\xi_n^s|^{\mu_1} \prod_{j=2}^{n-1} |\xi_j^s|^{\mu_j}] = c_{23}\,[1 + s^{a\,\mu_1 + \sum_{j=2}^{n-1} \lambda_j^0\,\mu_j}]$$

$$= c_{23}\,(1 + s^{a\,\mu_1 + 1 - \lambda_1^0\,\mu_1})\;\; s = 1, 2, ...\cdot$$

Отсюда и из неравенства (2.12) имеем с некоторой постоянной $c_{24} > 0$ и для всех s = 1,2, ...

$$1 + s^{a\,\mu_1 + 1 - \lambda_1^0\,\mu_1} \leq c_{24}\,[h_{\mathcal{M}}((\xi')^s) + |\xi_n^s|^{\delta}].$$

Так как $((\lambda^0)', \nu') \leq 1$ для любого $\nu' \in \mathcal{M}$, то отсюда и из определения последовательности $\{\xi^s\}$, с некоторой постоянной $c_{25} > 0$ получаем

$$1 + s^{a\,\mu_1 + 1 - \lambda_1^0\,\mu_1} \leq c_{25}\,(1 + s + s^{a\,\delta})\;\; s = 1, 2, ...\cdot$$

Полученное неравенство противоречит тому, что из условий $l \in (\delta, d_1(\mathcal{M}))$, $a \in (1/l, (1-\lambda_1^0\,\mu_1)/(l-\mu_1))$ следует, что $a\,\mu_1 + 1 - \lambda_1^0\,\mu_1 > a\,\delta > 1$. Таким образом необходимость пункта 2) леммы и, тем самым, Лемма 2.9 доказаны. $\square$

Ниже мы будем рассматриват случай $n = 3$. Дело в том, что в трехмерном случае многогранник $\tilde{\mathfrak{R}}$ имеет удобный для изучения структуру. Именно, известно, что в двумерном случае $\mathfrak{R}-$ гиперболический многочлен является $s-$гиперболическим, а в трехмерном случае, если $l^1 = (l_1, 0, 0)$, $l^2 = (0, l_2, 0)$, $l^3 = (0, 0, l_3)$, вершины многогранника $\tilde{\mathfrak{R}}$, при этом $l_1 = l_3$, то $\mathfrak{R}-$ гиперболический многочлен является $s-$гиперболическим при $s = 1/l_3$ (см. [14]).

Лемма 2.10. *Пусть* $\mathcal{M} \in \mathcal{B}_2$, $l_3 \in (0,1)$, $\mathfrak{R}-$ *многогранник Ньютона набора* $\{(\nu, 0), \nu \in \mathcal{M}^0\} \cup \{(0, 0, l_3)\}$, $l(1) := (l_1, 0)$ *и* $l(2) := (0, l_2)$ $(l_1 \le l_2)$ *вершины многогранника* $\mathcal{M}$, *лежащие на координатных осях* ξ_1 *и* ξ_2 *соответсрвенно. Тогда*

1) если $l_3 \le l_2$, *то для любой точки* $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$; $(\eta_2 \neq 0)$ *справедливо неравенство*

$$c_{26}^{-1}\,[1 + |\xi_1 - (\eta_1/\eta_2)\,\xi_2|^{l_1} + |\xi_3 - (\eta_3/\eta_2)\,\xi_2|^{l_3}] \le h_{\mathfrak{R},\eta}(\xi)$$

$$\le c_{26}\,[1 + |\xi_1 - (\eta_1/\eta_2)\,\xi_2|^{l_1} + |\xi_3 - (\eta_3/\eta_2)\,\xi_2|^{l_3}] \;\; \forall \xi \in \mathbb{R}^3.$$

с некоторой постоянной $c_{26} > 0$

2) если $\eta^j = (\eta_1^j, \eta_2^j, \eta_3^j)$, $\eta_2^j \neq 0$ $(j = 1, 2, 3)$ *линейно независимые векторы, то для всех* $\xi \in \mathbb{R}^3$ *справедливо неравенство*

$$c_{27}^{-1}\,[1 + |\xi_1|^{l_1} + |\xi_2|^{l_3} + |\xi_3|^{l_3}] \le \sum_{j=1}^{3} h_{\mathfrak{R},\eta^j}(\xi) \le c_{27}\,[1 + |\xi_1|^{l_1} + |\xi_2|^{l_3} + |\xi_3|^{l_3}]$$

с некоторой постоянной $c_{27} > 0$.

Доказательство. Пусть $\tilde{\mathcal{M}} := \{\nu = (\nu_1, \nu_3) \in \mathbb{R}^2_+, (\nu_1, 0, \nu_3) \in \mathfrak{R}\}$. Очевидно $\tilde{\mathcal{M}}$ является многогранником Ньютона набора $\{(l_1, 0), (0, l_3)\}$. Так как $d(\tilde{\mathcal{M}}) = max\{l_1, l_3\} = \rho(\tilde{\mathcal{M}})$, то в силу условия леммы $l_2 \ge d(\tilde{\mathcal{M}})$. Следовательно, в силу Леммы 2.6, с некоторой постоянной $c_{28} \in (0, 1)$ имеем

$$c_{28}\; h_{\tilde{\mathcal{M}}}(\xi_1 - (\eta_1/\eta_2)\,\xi_2,\; \xi_3 - (\eta_3/\eta_2)\;\xi_2) \le h_{\mathfrak{R},\eta}(\xi)$$

$$\le h_{\tilde{\mathcal{M}}}(\xi_1 - (\eta_1/\eta_2)\,\xi_2,\; \xi_3 - (\eta_3/\eta_2)\,\xi_2) \;\; \forall \xi \in \mathbb{R}^3.$$

Отсюда, в силу определения многогранника $\tilde{\mathcal{M}}$, непосредственно получаем утверждение первого пункта леммы.

Доказательство второго пункта проводится аналогично доказательству Леммы 2.7 с применением первого пункта настоящей леммы. □

3. Основные результаты

Теорема 3.1. *Пусть* $\lambda = (\lambda_1, ..., \lambda_n)$, $1 < \lambda_n \leq \lambda_{n-1} \leq ... \leq \lambda_1$, $\Re = \{\nu \in \mathbb{R}^n_+, (\lambda, \nu) \leq 1\}$ *и* $\mathcal{M} := \{\nu \in \mathbb{R}^n_+, \sum_{j=1}^{n-1} \lambda_j \nu_j + \lambda_{n-1} \nu_n \leq 1\}$ *и многочлен* P $\Re-$ *гиперболичен относительно вектора* $\eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n$, $\eta_n \neq 0$, *то* P *также* $\mathcal{M}-$*гиперболичен относительно* η.

Доказательство. Из определения $\Re-$ гиперболичности P следует, что $P_m(\eta) \neq 0$ и существует постоянная $c_1 > 0$ такая, что

$$P(\xi + i\,\eta) \neq 0 \;\; \forall \xi \in \mathbb{R}^n, \tau \in \mathbb{C}, |Re\tau| \geq c_1\, h_{\Re}(\xi), \tag{3.1}$$

где $h_{\Re}(\xi) := 1 + \sum_{j=1}^{n} |\xi_j|^{1/\lambda_j}$.

В силу Теоремы 2.2 работы [14] соотношение (3.1) эквивалентно следующему: существует постоянная $c_2 > 0$ такая, что

$$P(\xi + i\,\tau\,\eta) \neq 0 \;\; \forall \xi \in \mathbb{R}^n, \tau \in \mathbb{C}, |Re\tau| \geq c_2\, h_{\Re,\eta}(\xi), \tag{3.2}$$

где $h_{\Re,\eta}(\xi) := \inf_{t \in \mathbb{R}} h_{\Re}(\xi - t\,\eta)$. Так как в силу Леммы 2.7 $h_{\Re,\eta}(\xi) \leq c_3\, h_{\mathcal{M}}(\xi)$ $\forall \xi \in \mathbb{R}^n$ с некоторой постоянной $c_3 > 0$, то отсюда и из неравенства (3.2) имеем с некоторой постоянной $c_4 > 0$,

$$P(\xi + i\,\tau\,\eta) \neq 0 \;\; \forall \xi \in \mathbb{R}^n, \tau \in \mathbb{C}, |Re\tau| \geq c_4\, h_{\mathcal{M}}(\xi). \tag{3.3}$$

Так как $P_m(\eta) \neq 0$, то из соотношения (1.3) получаем, чтто P $\mathcal{M}-$ гиперболичен относительно вектора η. Теорема 3.1 доказана. □

Теорема 3.2. *Пусть однородный многочлен* P_m *гиперболичен относительно вектора* $\eta = (\eta_1, ..., \eta_n)$, $\eta_n \neq 0$ *и* $Q \prec^{h_{\Re,\eta}} P_m$, $ord\, Q < m$, *где* $\Re$ *многогранник из Теоремы 3.1. Тогда существует окрестность* $U(\eta)$ *точки* η *такая, что многочлен* $P_m + Q$ $\mathcal{M}-$ *гиперболичен относительно любого вектора из* $U(\eta)$.

Доказательство. Так как в силу Леммы 2.7, $h_{\Re,\eta}(\xi) \leq c_3\, h_{\mathcal{M}}(\xi)$ $\forall \xi \in \mathbb{R}^n$, то в силу Леммы 3.1 работы [13] $Q \prec^{h_{\mathcal{M}}} P_m$. С другой стороны, так как многочлен P_m гиперболичен относительно любого вектора из некоторой окрестности $U(\eta)$ точки η (см.[2], Теорема 12.4.4), то отсюда в силу Теоремы 3.3 работы [13] получаем, что $P_m + Q$ $\mathcal{M}-$ гиперболичен относительно любого вектора из $U(\eta)$, что доказывает теорему. □

Теорема 3.3. *Пусть* $\Re \in \mathbb{B}_n$, $\mathcal{M} := \{\nu' = (\nu_1, ..., \nu_{n-1}) \in \mathbb{R}_+^{n-1}, (\nu', 0) \in \Re\}$ *и* $l_n \geq d(\mathcal{M})$, *где* $l(n) := (0, ..., 0, l_n)$ *вершина* $\Re$, *лежащая на оси* ξ_n. *Если многочлен* P $\Re-$ *гиперболичен относительно вектора* $\eta = (\eta_1, ..., \eta_n), \eta_n \neq 0$, *то* P *также* $\tilde{\Re}-$ *гиперболичен относительно того же вектора* η, *где* $\tilde{\Re}-$ *многогранник Ньютона набора* $\{(\nu', 0) : \nu' \in \mathcal{M}^0\} \cup \{0, d(\mathbb{M})\}$.

Доказательство. Так как, при условиях теоремы, в силу Леммы 2.8 $h_{\Re,\eta}(\xi) \leq c_5\, h_{\tilde{\Re}}$ $\forall \xi \in \mathbb{R}^n$ с некоторой постоянной $c_5 > 0$, то доказательство теоремы проводится буквальным повторением доказательства теоремы 3.1. □

Теорема 3.4. *Пусть* $\Re \in \mathbb{B}_2$, $l_3 \in (0,1)$, $\Re$ *многогранник Ньютона набора* $\{(\nu, 0) : \nu \in \mathcal{M}^0\} \cup \{(0, 0, l_3)\}, l(1) = (l_1, 0), l(2) = (0, l_2)$ *вершины многогранника* $\mathcal{M}$, *лежащие на осях* ξ_1 *и* ξ_2 *соответственно,* $l_3 \leq l_2$. *Если многочлен* P $\Re-$ *гиперболичен относительно вектора* $\eta = (\eta_1, \eta_2, \eta_3)$, $\eta_2 \neq 0$, *то* P *также* $\tilde{\Re} := \{\nu : \frac{\nu_1}{l_1} + \frac{\nu_2}{l_3} + \frac{\nu_3}{l_3} \leq 1\}-$ *гиперболичен относительно того же вектора* η.

Доказательство. Так как при условиях теоремы, в силу Леммы 2.10 $h_{\Re,\eta}(\xi) \leq c_6\, h_{\tilde{\Re}}$ $\forall \xi \in \mathbb{R}^n$ с некоторой постоянной $c_6 > 0$, то доказательство теоремы проводится буквальным повторением доказательства теоремы 3.1. □

Теорема 3.5. *Пусть, при условиях Теоремы 3.4, однородный многочлен* P_m *гиперболичен относительно вектора* $\eta = (\eta_1, \eta_2, \eta_3)$, $\eta_2 \neq 0$ *и* $Q \prec^{h_{\Re,\eta}} P_m$, $ord\, Q < m$. *Тогда существует окрестность* $U(\eta)$ *точки* η *такая, что многочлен* $P_m + Q$ $h_{\tilde{\Re}}-$ *гиперболичен относительно любого вектора из* $U(\eta)$.

Доказательство. проводится буквальным повторением доказательства теоремы 3.2, лиш с той разницей, что вместо Леммы 2.7, здесь надо ползоватся Леммой 2.10. □

Abstract. It is proved that if $\lambda \in \mathbb{R}^n$, $1 < \lambda_n \leq \lambda_{n-1} \leq ... \leq \lambda_1$, $\Re := \{\nu \in \mathbb{R}_+^n, (\lambda, \nu) \leq 1\}$, $\mathcal{M} := \{\nu \in R_+^n, \sum_{j=1}^{n-1} \lambda_j \nu_j + \lambda_{n-1} \nu_n \leq 1\}$ and the polynomial $P(\xi) = P(\xi_1, ..., \xi_n)$ is $\eta-$hyperbolic with respect to the vector $\eta \in \mathbb{R}^n$, $\eta_n \neq 0$, then it is also $\mathcal{M}-$hyperbolic and $\tilde{\Re}-$hyperbolic with respect to η.

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A CHARACTERIZATION OF WEIGHTED CENTRAL CAMPANATO SPACES

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Abstract. In this paper, we introduce the weighted central Campanato spaces $\dot{C}^{p,\lambda}(\omega)$ and characterize $\dot{C}^{p,\lambda}(\omega)$ by the boundedness of the commutators $[b, H]$ and $[b, H^*]$ from weighted central Morrey spaces to weighted central Morrey spaces for $\omega \in A_1$, where the commutators are generated by n-dimensional Hardy operators and symbol b. In particular, the Weighted Lipschitz estimates for the Commutators of Hardy operators are obtained if $0 < \lambda < 1/n$.

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Keywords: weighted central Campanato spaces; Hardy operators; commutator; Morrey spaces.

1. Introduction

In 1963, Campanato space $C^{p,\lambda}(\mathbb{R}^n)$ was first introduced by Campanato [1] in order to study elliptic regularity in the context of the heat equation. Let $-1/p < \lambda < 1/n$ and $1 \le p < \infty$, a locally integrable function f is said to belong to the Campanato space $C^{p,\lambda}(\mathbb{R}^n)$, if

$$\|f\|_{C^{p,\lambda}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{|B|^{1+\lambda p}} \int_B |f(x) - f_B|^p dx \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, $f_B = \frac{1}{|B|}\int_B f(y)dy$, where $|B|$ is the Lebesgue measure of B. If the supremum is taken over all balls $B(0, r)$, it is the central Campanato space $\dot{C}^{p,\lambda}(\mathbb{R}^n)$. The excellent structures of Campanato spaces render them useful in the studies of the regularity theory of PDEs, which allows us to give an integral characterization of the spaces of Hölder continuous functions. This leads to a generalization of the classical Sobolev embedding theorem [2, 3, 4, 5, 6]. It is well known that $C^{1,\frac{1}{n}(\frac{1}{p}-1)}$ is the dual space of the Hardy space H^p when $0 < p < 1$ [7]. Especially, $C^{1,0} = BMO(\mathbb{R}^n)$.

Many authors have focused on the researches of commutators for which the symbol functions belong to BMO spaces and Lipschitz spaces which are the special

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cases of Campanato spaces. More precisely,

$$C^{p,\lambda}(\mathbb{R}^n) = \begin{cases} BMO(\mathbb{R}^n) & \lambda = 0, \\ Lip_\beta(\mathbb{R}^n) & 0 < \lambda < 1/n. \end{cases}$$

Recently, there are lots of studies concerning Campanato spaces and central Campanato spaces. In 2013, Shi and Lu [8, 9] characterized the space $C^{p,\lambda}$ via the boundedness of fractional integral and Calderón-Zygmund singular integral operator on Morrey spaces. In [10], Zhao and Lu gave some creative characterizations of central Campanato spaces via the boundedness of commutators associated with the Hardy operators for $\lambda > 0$. In 2015, Shi got another characterization via the boundedness of commutators associated with the Hardy operators for $-1/p < \lambda < 0$ [11].

As is well known, Lipschitz spaces and Campanato spaces have equivalent norms if $1 \le p < \infty$. In 2018, Wang and Zhou [12] proved that they are still equivalent to $0 < p < 1$. In the weighted setting, J. García-Cuerva [13] proved the equivalence of weighted Lipschitz spaces and weighted Campanato spaces, which is stated as follows:

$$\begin{aligned} \|f\|_{Lip_{\beta,\omega}} &\approx \sup_Q \frac{1}{\omega(Q)^{\frac{\beta}{n}}} \left(\frac{1}{\omega(Q)} \int_Q |f(x) - f_Q|^p \omega(x)^{1-p} dx \right)^{\frac{1}{p}} \\ &\approx \sup_Q \frac{1}{\omega(Q)^{1+\frac{\beta}{n}}} \int_Q |f(x) - f_Q| dx \end{aligned}$$

if $1 \le p \le \infty$, $0 < \beta < 1$ and $\omega \in A_1$. Moreover, Hu and Zhou [14] extended its equivalence to $0 < p < 1$.

Inspired by the above works, in this paper, we introduce the weighted central Campanato spaces and characterize the weighted central Campanato spaces via the boundedness of commutators associated with the Hardy operators. In particular, we obtain characterizations of weighted central BMO spaces if $\lambda = 0$, the weighted Lipschitz estimates for the commutators of Hardy operators are derived according to the equivalence of weighted Lipschitz spaces and weighted Campanato spaces if $0 < \lambda < 1/n$.

2. Some preliminaries and notations

Most the notations we use are standard. $B(x,r)$ denotes the ball centered at x with radius r. For any $a > 0$, $aB(x,r) = B(x,ar)$. For a locally integrable function f, $f_B = \frac{1}{|B|} \int_B f(x)dx$, the Lebesgue measure of B by $|B|$. Also, ω is a nonnegative locally integrable function i.e. $\omega(E) = \int_E \omega(x)dx$, p' is the conjugate of p satisfying $1/p + 1/p' = 1$. C always stands for a constant independent of the main parameters and not necessarily the same at each occurrence.

In 1995, Christ and Grafakos [15] gave the definitions of the n-dimensional Hardy operator and its adjoint operator,

$$Hf(x) = \frac{1}{|x|^n}\int_{|y|<|x|} f(y)dy, \quad H^*f(x) = \int_{|y|\geq|x|} \frac{f(y)}{|y|^n}dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

H and H^* satisfy

$$\int_{R^n} g(x)Hf(x)dx = \int_{R^n} f(x)H^*g(x)dx.$$

Let b be a measurable locally integrable function and T be a linear operator. Then the commutator $[b,T]$ is defined by

$$[b,T]f = bTf - T(bf).$$

In [16], R. Coifman, R. Rochberg and G. Weiss proved that the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ if $b \in BMO(\mathbb{R}^n)$ and $1 \leq p < \infty$, where T is a Calderón-Zygmund singular integral operator.

In this article, the commutators of H and H^* are defined by

$$H_b f = [b,H]f = bHf - H(bf), \quad H_b^* f = [b,H^*]f = bH^*f - H^*(bf).$$

This article will prove that $H_b f$ and $H_b^* f$ are bounded from weighted central Morrey spaces to weighted central Morrey spaces if and only if b belongs to the weighted central Campanato spaces.

In the following, we give the definitions of weighted central Campanato spaces and weighted Morrey spaces.

Definition 2.1 Let ω be a nonnegative locally integrable function, a function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to belong to the weighted central Campanato space $\dot{C}^{p,\lambda}(\omega)(\mathbb{R}^n)$ for $-1/p < \lambda < 1/n$ and $1 \leq p < \infty$, if

$$\|f\|_{\dot{C}^{p,\lambda}(\omega)} = \sup_{r>0}\left(\frac{1}{\omega(B(0,r))^{1+\lambda p}}\int_{B(0,r)} |f(x) - f_{B(0,r)}|^p \omega(x)^{1-p}dx\right)^{\frac{1}{p}} < \infty.$$

If $\omega = 1$, $\dot{C}^{p,\lambda}(\omega)(\mathbb{R}^n) = \dot{C}^{p,\lambda}(\mathbb{R}^n)$. If the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $\omega = 1$, $\dot{C}^{p,\lambda}(\omega)(\mathbb{R}^n) = C^{p,\lambda}(\mathbb{R}^n)$, if $\lambda = 0$, it is the weighted central BMO space $CMO^p(\omega)$.

Definition 2.2 Let $1 \leq p < \infty$, ω is a nonnegative locally integrable function, a function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to belong to the weighted central BMO space $CMO^p(\omega)$ if

$$\|f\|_{CMO^p(\omega)} = \sup_{r>0}\left(\frac{1}{\omega(B(0,r))}\int_{B(0,r)} |f(x) - f_{B(0,r)}|^p \omega(x)^{1-p}dx\right)^{\frac{1}{p}} < \infty.$$

Obviously, $CMO^p(\omega) \subseteq CMO^q(\omega)$ if $1 \leq q < p < \infty$. When $\omega = 1$, $CMO^p(\omega) = CMO^p(\mathbb{R}^n)$. In particular, $BMO(\mathbb{R}^n) \subset CMO^p(\mathbb{R}^n)$ if $1 \leq p < \infty$, $CMO^p(\mathbb{R}^n) \subseteq CMO^q(\mathbb{R}^n)(1 \leq q < p < \infty)$. There is no analysis of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for $CMO^p(\mathbb{R}^n)$, so $CMO^p(\mathbb{R}^n)$ and $CMO(\mathbb{R}^n)$ are not equivalent.

Definition 2.3 [17] Let $1 \le p < \infty$, $-1/p < \lambda < 0$, ω_1, ω_2 are nonnegative locally integrable functions, a function $f \in L^p_{loc,\omega_2}(\mathbb{R}^n)$ is said to belong to the weighted Morrey space $M^{p,\lambda}(\omega_1,\omega_2)$ if

$$\|f\|_{M^{p,\lambda}(\omega_1,\omega_2)} = \sup_B \left(\frac{1}{\omega_1(B)^{1+\lambda p}} \int_B |f(x)|^p \omega_2(x) dx \right)^{1/p} < \infty.$$

If $\omega_1 = \omega_2 = 1$, $M^{p,\lambda}(\omega_1,\omega_2)(\mathbb{R}^n)$ is the classical Morrey space $M^{p,\lambda}(\mathbb{R}^n)$. In particular, Sakamoto and Yabuta [18] pointed out that $C^{p,\lambda}(\mathbb{R}^n)$ is equivalent to $M^{p,\lambda}(\mathbb{R}^n)$ when $1 \le p < \infty$ and $-1/p < \lambda < 0$. But Lin [19] gave a counterexample to verify that $M^{p,\lambda}(\mathbb{R}^n) \subseteq C^{p,\lambda}(\mathbb{R}^n)$ when $1 \le p < \infty$ and $-1/p < \lambda < 0$.

In order to characterize the weighted central Campanato spaces, we give the following definition of the weighted central Morrey space $\dot{M}^{p,\lambda}(\omega_1,\omega_2)$.

Definition 2.4 Let $1 \le p < \infty$, $-1/p < \lambda < 0$, ω_1, ω_2 are nonnegative locally integrable functions, a function $f \in L^p_{loc,\omega_2}(\mathbb{R}^n)$ is said to belong to the weighted central Morrey space $\dot{M}^{p,\lambda}(\omega_1,\omega_2)$ if

$$\|f\|_{\dot{M}^{p,\lambda}(\omega_1,\omega_2)} = \sup_{B(0,r)} \left(\frac{1}{\omega_1(B(0,r))^{1+\lambda p}} \int_{B(0,r)} |f(x)|^p \omega_2(x) dx \right)^{1/p} < \infty.$$

If $\omega_1 = \omega_2 = 1$, it is the central Morrey space $\dot{M}^{p,\lambda}(\mathbb{R}^n)$.

Definition 2.5 Let $1 < p < \infty$, we say $\omega \in A_p$ if

$$\sup_B \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty.$$

For the case $p = 1$, we say $\omega \in A_1$ if

$$\frac{1}{|B|} \int_B \omega(x) dx \le C ess \inf_{x \in B} \omega(x)$$

for every ball $B \subset \mathbb{R}^n$. A weight function $\omega \in A_\infty$ if it satisfies the A_p condition for some $1 \le p < \infty$.

Lemma 2.6 [20]. Let $\omega \in A_1$, then there are constants C_1, C_2 and $0 < \delta < 1$ for any measurable subset $E \subset B$,

$$C_1 \frac{|E|}{|B|} \le \frac{\omega(E)}{\omega(B)} \le C_2 \left(\frac{|E|}{|B|} \right)^\delta. \tag{2.1}$$

Lemma 2.7 [21]. Let $\omega \in A_1$, then for $1 < p < \infty$,

$$\int_B \omega(x)^{1-p'} dx \le C|B|^{p'} \omega(B)^{1-p'}, \tag{2.2}$$

where $1/p + 1/p' = 1$.

Proof: Since $A_1 \subset A_p$, ω satisfies the condition of the weight A_p. The above lemma can be obtained by simple calculation.

Lemma 2.8 [22]. The function class is called the reverse Hölder class if a function f satisfies the following condition

$$\sup_{x\in B}|f(x)-f_B| \le \frac{C}{|B|}\int_B |f(x)-f_B|dx. \tag{2.3}$$

Reverse Hölder class contains many kinds of functions, such as polynomial functions [23]. For more theories about reverse Hölder class, see [24].

3. A characterization of weighted central Campanato spaces

The main theorems are as follows.

Theorem 3.1. Let $\omega \in A_1(\mathbb{R}^n)$, $1<p<\infty$, $-1/p<\lambda<0$, $-1/p_i<\lambda_i<0(i=1,2)$, $\lambda=\lambda_1+\lambda_2$, $1/p=1/p_1+1/p_2$, b satisfies (2.3), then the following statements are equivalent:

(i) $b\in \dot{C}^{p_1,\lambda_1}(\omega)$;

(ii)$[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p_2,\lambda_2}(\omega,\omega)$ to $\dot{M}^{p,\lambda}(\omega,\omega^{1-p})$.

Theorem 3.2. Let $\omega\in A_1(\mathbb{R}^n)$, $1<p<\infty$, $1/p+1/p'=1$, $-\min\{1/(2p),1/(2p')\}<\lambda<0$, then the following statements are equivalent:

(i) $b\in \dot{C}^{\max(p,p'),\lambda}(\omega)$;

(ii) $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p,\lambda}(\omega,\omega)$ to $\dot{M}^{p,2\lambda}(\omega,\omega^{1-p})$. In addition, $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p',\lambda}(\omega,\omega)$ to $\dot{M}^{p',2\lambda}(\omega,\omega^{1-p'})$.

Theorem 3.3. Let $\omega\in A_1(\mathbb{R}^n)$, $1<p<\infty$, $1/p+1/p'=1$, $-\min\{1/(p),1/(p')\}<\lambda<0$, then the following statements are equivalent:

(i) $b\in CMO^{\max(p,p')}(\omega)$;

(ii) $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p,\lambda}(\omega,\omega)$ to $\dot{M}^{p,2\lambda}(\omega,\omega^{1-p})$. In addition, $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p',\lambda}(\omega,\omega)$ to $\dot{M}^{p',2\lambda}(\omega,\omega^{1-p'})$.

Proof of Theorem 3.1:

$(i)\Rightarrow(ii)$, for simplicity, we write $B_k=\{x\in\mathbb{R}^n: |x|\le 2^k\}$, $C_k=B_k\backslash B_{k-1}$ for $k\in\mathbb{Z}$. For a fixed ball $B=B(0,r)\subset\mathbb{R}^n$, let $B(0,r)=B_{k_0}$ with $k_0\in\mathbb{Z}$, we just need to prove that

$$\left(\frac{1}{\omega(B_{k_0})^{1+\lambda p}}\int_{B_{k_0}}|H_b f(x)|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}} \le C\|f\|_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \tag{3.1}$$

and

$$\left(\frac{1}{\omega(B_{k_0})^{1+\lambda p}}\int_{B_{k_0}}|H_b^* f(x)|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}} \le C\|f\|_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}. \tag{3.2}$$

On the one hand,

$$\int_{B_{k_0}} |H_b f(x)|^p \omega(x)^{1-p} dx = \int_{B_{k_0}} \left| \frac{1}{|x|^n} \int_{|y|<|x|} \big(b(x)-b(y)\big) f(y) dy \right|^p \omega(x)^{1-p} dx$$

$$\leq C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \int_{B_k} |b(x)-b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$+ C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \int_{B_k} |b(y)-b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx = I + II.$$

Applying Hölder's inequality, (2.1) and (2.2), we have the following estimates,

$$I = C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \sum_{i=-\infty}^{k} \int_{C_i} |b(x)-b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$\leq C \sum_{k=-\infty}^{k_0} 2^{-kpn} \int_{C_k} |b(x)-b_{B_k}|^p \omega(x)^{1-p} dx \left| \sum_{i=-\infty}^{k} \int_{C_i} |f(y)| dy \right|^p$$

$$= C \sum_{k=-\infty}^{k_0} 2^{-kpn} \int_{C_k} |b(x)-b_{B_k}|^p \omega(x)^{\frac{(1-p_1)p}{p_1}} \omega(x)^{\frac{p_1-p}{p_1}} dx \left| \sum_{i=-\infty}^{k} \int_{C_i} |f(y)| \omega(y)^{\frac{1}{p_2}} \omega(y)^{-\frac{1}{p_2}} dy \right|^p$$

$$\leq C \sum_{k=-\infty}^{k_0} 2^{-kpn} \left(\int_{B_k} |b(x)-b_{B_k}|^{p_1} \omega(x)^{1-p_1} dx \right)^{\frac{p}{p_1}} \omega(B_k)^{1-\frac{p}{p_1}}$$

$$\times \left| \sum_{i=-\infty}^{k} \left(\int_{C_i} |f(y)|^{p_2} \omega(y) dy \right)^{\frac{1}{p_2}} \left(\int_{C_i} \omega(y)^{1-p_2'} dy \right)^{\frac{1}{p_2'}} \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \sum_{k=-\infty}^{k_0} 2^{-kpn} \omega(B_k)^{1+\lambda_1 p} \left| \sum_{i=-\infty}^{k} |B_i| \omega(B_i)^{\lambda_2} \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+\lambda p} \left| \sum_{i=-\infty}^{k} 2^{(i-k)n(1+\lambda_2)} \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p} \sum_{k=-\infty}^{k_0} 2^{(k-k_0)n\delta(1+\lambda p)}$$

$$\leq C \|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p}.$$

Due to $1/p = 1/p_1 + 1/p_2$, using Hölder's inequality, (2.1) and (2.2), we can get

$$II = C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \sum_{i=-\infty}^{k} \int_{C_i} |b(y)-b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx \leq C \sum_{k=-\infty}^{k_0} 2^{-kpn}$$

$$\times \int_{C_k} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} \big(|b(y)-b_{B_k}||f(y)|\big)^p \omega(y)^{1-p} dy \right)^{\frac{1}{p}} \omega(B_i)^{\frac{1}{p'}} \right|^p \omega(x)^{1-p} dx$$

$$\le C\sum_{k=-\infty}^{k_0} 2^{-kpn}\int_{C_k}\bigg|\sum_{i=-\infty}^{k}\bigg(\int_{B_i}|b(y)-b_{B_k}|^{p_1}\omega(y)^{1-p_1}dy\bigg)^{\frac{1}{p_1}}$$

$$\times\bigg(\int_{B_i}|f(y)|^{p_2}\omega(y)dy\bigg)^{\frac{1}{p_2}}\omega(B_i)^{\frac{1}{p'}}\bigg|^p\omega(x)^{1-p}dx$$

$$\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\sum_{k=-\infty}^{k_0}2^{-knp}\omega(B_k)^{p\lambda_1+\frac{p}{p_1}}$$

$$\times\int_{B_k}\omega(x)^{1-p}dx\bigg|\sum_{i=-\infty}^{k}\omega(B_i)^{\frac{1}{p_2}+\lambda_2+\frac{1}{p'}}\bigg|^p$$

$$\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+\lambda p}$$

$$\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}\sum_{k=-\infty}^{k_0}2^{(k-k_0)n\delta(1+\lambda p)}$$

$$\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}.$$

Based on I and II, we obtain (3.1). For (3.2),

$$\begin{aligned}\int_{B_{k_0}}|H_b^*|^p\omega(x)^{1-p}dx &= \int_{B_{k_0}}\bigg|\int_{|y|>|x|}\frac{\big(b(x)-b(y)\big)}{|y|^n}f(y)dy\bigg|^p\omega(x)^{1-p}dx\\ &\le \int_{B_{k_0}}\bigg|\int_{|x|<|y|<2^{k_0}a}\frac{\big(b(x)-b(y)\big)}{|y|^n}f(y)dy\bigg|^p\omega(x)^{1-p}dx\\ &\quad+\int_{B_{k_0}}\bigg|\int_{2^{k_0}a<|y|}\frac{\big(b(x)-b(y)\big)}{|y|^n}f(y)dy\bigg|^p\omega(x)^{1-p}dx\\ &= I'+II'.\end{aligned}$$

For I', using the same discussion as (3.1), we omit the details. The analysis of II' is different.

$$\begin{aligned}I' &\le \int_{B_{k_0}}\bigg|\frac{1}{|x|^n}\int_{|y|<2^{k_0}a}|b(x)-b(y)||f(y)|dy\bigg|^p\omega(x)^{1-p}dx\\ &\le C\sum_{k=-\infty}^{k_0}2^{-knp}\int_{B_k}\bigg|\sum_{i=-\infty}^{k}\int_{B_i}|b(x)-b(y)||f(y)|dy\bigg|^p\omega(x)^{1-p}dx\\ &\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)}\|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda p}.\end{aligned}$$

For the term II', we proceed to show that

$$\begin{aligned}II' &\le \int_{B_{k_0}}\bigg|\sum_{k=k_0}^{\infty}\int_{C_k}\frac{|b(x)-b_{B_{k_0}}|}{|y|^n}|f(y)|dy\bigg|^p\omega(x)^{1-p}dx\\ &\quad+\int_{B_{k_0}}\bigg|\sum_{k=k_0}^{\infty}\int_{C_k}\frac{|b(y)-b_{B_{k_0}}|}{|y|^n}|f(y)|dy\bigg|^p\omega(x)^{1-p}dx = II_1'+II_2'.\end{aligned}$$

Employing Hölder's inequality, (2.1) and (2.2),

$$\begin{aligned}
II_1' &\le \int_{B_{k_0}} |b(x)-b_{B_{k_0}}|^p \omega(x)^{1-p}dx \bigg| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|f(y)|}{|y|^n} dy \bigg|^p \\
&\le \left(\int_{B_{k_0}} |b(x)-b_{B_{k_0}}|^{p_1} \omega(x)^{1-p_1} dx \right)^{\frac{p}{p_1}} \omega(B_{k_0})^{1-\frac{p}{p_1}} \\
&\times \bigg| \sum_{k=k_0}^{\infty} 2^{-kpn} \left(\int_{C_k} |f(y)|^{p_2} \omega(y) dy \right)^{\frac{1}{p_2}} \left(\int_{C_k} \omega(y)^{1-p_2'} dy \right)^{\frac{1}{p_2'}} \bigg|^p \\
&\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda_1 p} \bigg| \sum_{k=k_0}^{\infty} \omega(B_k)^{\lambda_2} \bigg|^p \\
&\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p} \sum_{k=k_0}^{\infty} 2^{(k-k_0)n\delta\lambda_2} \\
&\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p}.
\end{aligned}$$

To complete the proof, we divide II_2' into two parts:

$$\begin{aligned}
II_2' &\le \int_{B_{k_0}} \bigg| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b(y)-b_{B_k}|}{|y|^n} |f(y)| dy \bigg|^p \omega(x)^{1-p} dx \\
&+ \int_{B_{k_0}} \bigg| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b_{B_k}-b_{B_{k_0}}|}{|y|^n} |f(y)| dy \bigg|^p \omega(x)^{1-p} dx = II_{21}' + II_{22}'.
\end{aligned}$$

For II_{21}', we have

$$\begin{aligned}
II_{21}' &\le \int_{B_{k_0}} \bigg| \sum_{k=k_0}^{\infty} \left(\int_{C_k} \left(\frac{|b(y)-b_{B_{k_0}}|}{|y|^n} |f(y)| \right)^p \omega(y)^{1-p} dy \right)^{\frac{1}{p}} \omega(B_k)^{\frac{1}{p'}} \bigg|^p \omega(x)^{1-p} dx \\
&\le \int_{B_{k_0}} \bigg| \sum_{k=k_0}^{\infty} \left(\int_{C_k} |b(y)-b_{B_{k_0}}|^{p_1} \omega(y)^{1-p_1} dy \right)^{\frac{1}{p_1}} \left(\int_{C_k} \left| \frac{|f(y)|}{|y|^n} \right|^{p_2} \omega(y) dy \right)^{\frac{1}{p_2}} \bigg|^p \omega(x)^{1-p} dx \\
&\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1-p} \bigg| \sum_{k=k_0}^{\infty} 2^{(k_0-k)n} \omega(B_k)^{1+\lambda} \bigg|^p \\
&\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p} \bigg| \sum_{k=k_0}^{\infty} 2^{(k-k_0)n\lambda} \bigg|^p \\
&\le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p}.
\end{aligned}$$

Applying Hölder's inequality, (2.1), (2.2) and $|b_{B_k}-b_{B_{k_0}}| \le C(k-k_0)\|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)} \frac{\omega(B_{k_0})^{1+\lambda_1}}{|B_{k_0}|}$, we can get estimate of II_{22}'. Indeed,

$$|b_{B_k} - b_{B_{k_0}}| \le |b_{B_{k_0}} - b_{B_{k_0+1}}| + \cdots + |b_{B_{k-1}} - b_{B_k}|,$$

$$\begin{aligned}|b_{B_{k_0}} - b_{B_{k_0+1}}| &\le \left| \frac{1}{|B_{k_0}|} \int_{B_{k_0+1}} b(x) - b_{B_{k_0+1}} dx \right| \\ &\le \frac{1}{|B_{k_0}|} \left(\int_{B_{k_0+1}} |b(x) - b_{B_{k_0+1}}|^{p_1} \omega(x)^{1-p_1} dx \right)^{\frac{1}{p_1}} \omega(B_{k_0+1})^{1-\frac{1}{p_1}} \\ &\le \|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)} \frac{\omega(B_{k_0+1})^{1+\lambda_1}}{|B_{k_0}|}.\end{aligned}$$

So, we get the following inequalities,

$$\begin{aligned}|b_{B_k} - b_{B_{k_0}}| &\le \|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)} \sum_{j=k_0}^{k-1} \frac{\omega(B_{j+1})^{1+\lambda_1}}{|B_j|} \\ &\le \|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)} \omega(B_{k_0+1})^{\lambda_1} \sum_{j=k_0}^{k-1} \frac{\omega(B_{j+1})}{|B_j|} \\ &\le (k-k_0) \|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)} \frac{\omega(B_{k_0+1})^{1+\lambda_1}}{|B_{k_0}|} \\ &\le C(k-k_0) \|b\|_{\dot{C}^{p_1,\lambda_1}(\omega)} \frac{\omega(B_{k_0})^{1+\lambda_1}}{|B_{k_0}|}.\end{aligned}$$

In the next step,

$$\begin{aligned}II'_{22} \le\; & C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \left| \frac{\omega(B_{k_0})^{1+\lambda_1}}{|B_{k_0}|} \right|^p \int_{B_{k_0}} \omega(x)^{1-p} dx \\ & \times \left| \sum_{k=k_0}^{\infty} 2^{-kn} (k-k_0) \left(\int_{C_k} |f(y)|^{p_2} \omega(y) dy \right)^{\frac{1}{p_2}} \left(\int_{C_k} \omega(y)^{1-p_2'} dy \right)^{\frac{1}{p_2'}} \right|^p \\ \le\; & C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda_1 p} \left| \sum_{k=k_0}^{\infty} \omega(B_k)^{\lambda_2} (k-k_0) \right|^p \\ \le\; & C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p} \left| \sum_{k=k_0}^{\infty} 2^{(k-k_0)n\delta\lambda_2} (k-k_0) \right|^p \\ \le\; & C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p}.\end{aligned}$$

Summarizing, one has

$$II^{'} \le C\|b\|^p_{\dot{C}^{p_1,\lambda_1}(\omega)} \|f\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \omega(B_{k_0})^{1+\lambda p}.$$

Combining I' with II', we proved (3.2).

Next we prove $(ii) \Rightarrow (i)$. For a fixed ball $B = B(0,r)$, we assume b satisfies reverse Hölder condion (2.3), we just need to prove

$$\frac{1}{\omega(B)^{1+\lambda_1 p_1}} \int_B |b(x) - b_B|^{p_1} \omega(x)^{1-p_1} dx \le C. \tag{3.3}$$

Indeed,

$$\begin{aligned}\frac{1}{\omega(B)^{1+\lambda_1 p_1}}&\int_B |b(x)-b_B|^{p_1}\omega(x)^{1-p_1}dx\\ &\le \omega(B)^{-1-\lambda_1 p_1}\int_B \omega(x)^{1-p_1}dx\big(\sup_{x\in B}|b(x)-b_B|\big)^{p_1}\\ &\le C\omega(B)^{-p_1-\lambda_1 p_1}|B|^{p_1}\left(\frac{1}{|B|}\int_B |b(x)-b_B|dx\right)^{p_1}\\ &\le C\omega(B)^{-\lambda_1 p_1-\frac{p_1}{p}}\left(\int_B |b(x)-b_B|^{p}\omega(x)^{1-p}dx\right)^{\frac{p_1}{p}}.\end{aligned}$$

Next we estimate $\int_B |b(x)-b_B|^p\omega(x)^{1-p}dx$,

$$\begin{aligned}\int_B |b(x)-b_B|^p\omega(x)^{1-p}dx &\le \frac{1}{|B|^p}\int_B\left|\int_B (b(x)-b(y))dy\right|^p\omega(x)^{1-p}dx\\ &\le \frac{1}{|B|^p}\int_B |x|^{np}\left|\frac{1}{|x|^n}\int_{|y|<|x|}(b(x)-b(y))\chi_B(y)dy\right|^p\omega(x)^{1-p}dx\\ &+\frac{1}{|B|^p}\int_B\left|\int_{|y|>|x|}|y|^n\frac{(b(x)-b(y))\chi_B(y)}{|y|^n}dy\right|^p\omega(x)^{1-p}dx\\ &= I+II.\end{aligned}$$

Considering I and II, respectively

$$\begin{aligned}I &\le \int_B |H_b\chi_B(x)|^p\omega(x)^{1-p}dx = \omega(B)^{1+\lambda p}\|H_b\chi_B\|^p_{\dot{M}^{p,\lambda}(\omega,\omega^{1-p})}\\ &\le C\omega(B)^{1+\lambda p}\|\chi_B\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \le C\omega(B)^{1+\lambda_1 p}.\end{aligned}$$

Similarly,

$$\begin{aligned}II &\le \int_B |H_b^*\chi_B(x)|^p\omega(x)^{1-p}dx = \omega(B)^{1+\lambda p}\|H_b^*\chi_B\|^p_{\dot{M}^{p,\lambda}(\omega,\omega^{1-p})}\\ &\le C\omega(B)^{1+\lambda p}\|\chi_B\|^p_{\dot{M}^{p_2,\lambda_2}(\omega,\omega)} \le C\omega(B)^{1+\lambda_1 p}.\end{aligned}$$

Hence,

$$\frac{1}{\omega(B)^{1+\lambda_1 p_1}}\int_B |b(x)-b_B|^{p_1}\omega(x)^{1-p_1}dx \le C\omega(B)^{-\lambda_1 p_1-\frac{p_1}{p}}\left(C\omega(B)^{1+\lambda_1 p}\right)^{\frac{p_1}{p}} \le C.$$

Combining (3.1), (3.2) and (3.3), the proof of Theorem 3.1 is completed.

Proof of Theorem 3.2:

$(i)\Rightarrow(ii)$. For a fixed ball $B(0,r)=B_{k_0}$ with $k_0\in\mathbb{Z}$, we just need to prove that

$$\left(\frac{1}{\omega(B_{k_0})^{1+2\lambda p}}\int_{B_{k_0}} |H_b f(x)|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}} \le C\|f\|_{\dot{M}^{p,\lambda}(\omega,\omega)} \tag{3.4}$$

and

$$\left(\frac{1}{\omega(B_{k_0})^{1+2\lambda p}}\int_{B_{k_0}} |H_b^* f(x)|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}} \le C\|f\|_{\dot{M}^{p,\lambda}(\omega,\omega)}. \tag{3.5}$$

The Hölder's inequality and $1/p+1/p'=1$ show that

$$\int_{B_{k_0}} |H_b f(x)|^p \omega(x)^{1-p} dx = \int_{B_{k_0}} \left| \frac{1}{|x|^n} \int_{|y|<|x|} \big(b(x)-b(y)\big) f(y) dy \right|^p \omega(x)^{1-p} dx$$

$$\leq C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \int_{B_k} |b(x)-b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$+ C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \int_{B_k} |b(y)-b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx = I + II.$$

For I, we show that

$$I \leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} |b(x)-b_{B_k}|^p \omega(x)^{1-p} dx \left| \sum_{i=-\infty}^{k} \int_{B_i} |f(y)| dy \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} 2^{-knp} \omega(B_k)^{1+\lambda p} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \left(\int_{B_i} \omega(y)^{1-p'} dy \right)^{\frac{1}{p'}} \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p,\lambda}(\omega)} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+2\lambda p} \left| \sum_{i=-\infty}^{k} 2^{(i-k)n(1+\lambda)} \right|^p$$

$$\leq C \|b\|^p_{\dot{C}^{p,\lambda}(\omega)} \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \omega(B_{k_0})^{1+2\lambda p}.$$

To get the boundedness for the term II, we require the following decomposition

$$II \leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(y)-b_{B_k}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$\leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(y)-b_{B_i}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$+ C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b_{B_k}-b_{B_i}||f(y)| dy \right|^p \omega(x)^{1-p} dx$$

$$= II' + II''.$$

Discussing II' and II'', respectively, if $p > p'$, then

$$II' \leq C \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |b(y)-b_{B_i}|^{p'} \omega(y)^{1-p'} dy \right)^{\frac{1}{p'}} \right.$$

$$\times \left. \left(\int_{B_i} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \right|^p \omega(x)^{1-p} dx$$

$$\leq C \|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)} \sum_{k=-\infty}^{k_0} 2^{-knp} \int_{B_k} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |b(y)-b_{B_i}|^p \omega(y)^{1-p} dy \right)^{\frac{1}{p}} \right.$$

$$\times \left. \omega(B_i)^{\frac{1}{p'}+\lambda} \right|^p \omega(x)^{1-p} dx$$

$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+2\lambda p}\left|\sum_{i=-\infty}^{k}2^{(i-k)n\delta(1+2\lambda)}\right|^p$$
$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\omega(B_{k_0})^{1+2\lambda p}.$$

On the other hand, applying (2.1), (2.2) and $|b_{B_k}-b_{B_i}|\leq C(k-i)\|b\|_{\dot{C}^{p,\lambda}(\omega)}\frac{\omega(B_i)^{1+\lambda}}{|B_i|}$, we show

$$II''=C\sum_{k=-\infty}^{k_0}2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}\int_{B_i}|b_{B_k}-b_{B_i}||f(y)|dy\right|^p\omega(x)^{1-p}dx$$
$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\sum_{k=-\infty}^{k_0}2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}(k-i)\frac{\omega(B_i)^{1+\lambda}}{|B_i|}\left(\int_{B_i}|f(y)|^p\omega(y)dy\right)^{\frac{1}{p}}\right.$$
$$\times\left.\left(\int_{B_i}\omega(y)^{1-p'}dy\right)^{\frac{1}{p'}}\right|^p\omega(x)^{1-p}dx$$
$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1-p}\left|\sum_{i=-\infty}^{k}(k-i)\omega(B_i)^{1+2\lambda}\right|^p$$
$$\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+2\lambda p}\leq C\|b\|^p_{\dot{C}^{p,\lambda}(\omega)}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\omega(B_{k_0})^{1+2\lambda p}.$$

If $p'>p$, we can obtain that $[b,H]$ are bounded from $\dot{M}^{p',\lambda}(\omega,\omega)$ to $\dot{M}^{p',2\lambda}(\omega,\omega^{1-p'})$.

By slightly modifying Theorem 3.1, we can obtain the proof of $[H^*,b]$. Here, we omit its proof for the similarity.

$(ii)\Rightarrow(i)$, case 1: $p>p'$, we want to get

$$\frac{1}{\omega(B)^{1+\lambda p}}\int_B|b(x)-b_B|^p\omega(x)^{1-p}dx\leq C. \tag{3.6}$$

Indeed,

$$\frac{1}{\omega(B)^{1+\lambda p}}\int_B|b(x)-b_B|^p\omega(x)^{1-p}dx$$
$$\leq\frac{1}{\omega(B)^{1+\lambda p}}\frac{1}{|B|^p}\int_B|x|^{np}\left|\frac{1}{|x|^n}\int_{|y|<|x|}(b(x)-b(y))\chi_B(y)dy\right|^p\omega(x)^{1-p}dx$$
$$+\frac{1}{\omega(B)^{1+\lambda p}}\frac{1}{|B|^p}\int_B\left|\int_{|y|>|x|}|y|^n\frac{(b(x)-b(y))}{|y|^n}\chi_B(y)dy\right|^p\omega(x)^{1-p}dx=K+L.$$

Considering K and L, respectively

$$K\leq\frac{\omega(B)^{1+2\lambda p}}{\omega(B)^{1+\lambda p}}\|H_b\chi_B\|^p_{\dot{M}^{p,2\lambda}(\omega,\omega^{1-p})}\leq C\omega(B)^{\lambda p}\|\chi_B\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\leq C.$$

Also,

$$L\leq\frac{\omega(B)^{1+2\lambda p}}{\omega(B)^{1+\lambda p}}\|H^*_b\chi_B\|^p_{\dot{M}^{p,2\lambda}(\omega,\omega^{1-p})}\leq C\omega(B)^{\lambda p}\|\chi_B\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\leq C.$$

Case 2: $p'>p$, with the $\left(\dot{M}^{p',\lambda}(\omega,\omega),\dot{M}^{p',2\lambda}(\omega,\omega^{1-p'})\right)$ boundedness of H_b and H^*_b, the similar arguments of case 1 can be applied to this and show that

$$\frac{1}{\omega(B)^{1+\lambda p'}}\int_B |b(x)-b_{B_{\gamma_0}}|^{p'}\omega(x)^{1-p'}dx \le C. \tag{3.7}$$

So, combining(3.4), (3.5), (3.6) and (3.7), The proof of Theorem 3.2 is completed.

The Proof of Theorem 3.3 is similar to that of Theorem 3.2.

4. Weighted Lipschitz estimates

Definition 4.1 [13]. Let $1 \le p \le \infty$, $0 < \beta < 1$, and $\omega \in A_\infty$, a locally integrable function f is said to belong to the weighted Lipschitz space $Lip^p_{\beta,\omega}$ if

$$\|f\|_{Lip^p_{\beta,\omega}} = \sup_B \frac{1}{\omega(B)^{\frac{\beta}{n}}}\left(\frac{1}{\omega(B)}\int_B |f(x)-f_B|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}} < \infty.$$

Modulo constants, the Banach space of such functions is denoted by $Lip^p_{\beta,\omega}$. Put $Lip_{\beta,\omega} = Lip^1_{\beta,\omega}$, obviously, if $\omega = 1$, then the $Lip_{\beta,\omega}$ is the classical Lipschitz space Lip_β, if $\omega \in A_1$, J. García-Cuerva [13] proved that the spaces $Lip^p_{\beta,\omega}$ coincide, and the norm of $Lip^p_{\beta,\omega}$ are equivalent with respect to different values of provided that $1 \le p \le \infty$. That is $Lip^p_{\beta,\omega} \sim Lip_{\beta,\omega}$ where $1 \le p \le \infty$.

Theorem 4.1. Let $\omega \in A_1(\mathbb{R}^n)$, $1 < p < \infty$, $-\frac{1}{p} < \lambda < \lambda_1 < 0$, $0 < \beta < 1$, and $\lambda_1 = \lambda + \beta/n$, $b \in Lip_{\beta,\omega}$, then commutators $[b,H]$ and $[b,H^*]$ are bounded from $\dot{M}^{p,\lambda}(\omega,\omega)$ to $\dot{M}^{p,\lambda_1}(\omega,\omega^{1-p})$.

Proof of Theorem 4.1. For a fixed ball $B(0,r) = B_{k_0}$ with $k_0 \in \mathbb{Z}$, applying (2.1), (2.2) and Hölder's inequality,

$$\int_{B_{k_0}} |H_b f(x)|^p\omega(x)^{1-p}dx = \int_{B_{k_0}} \left|\frac{1}{|x|^n}\int_{|y|<|x|}\big(b(x)-b(y)\big)f(y)dy\right|^p \omega(x)^{1-p}dx$$

$$\le C\sum_{k=-\infty}^{k_0}\int_{C_k}\left|\frac{1}{|x|^n}\int_{B_k}|b(x)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx$$

$$+ C\sum_{k=-\infty}^{k_0}\int_{C_k}\left|\frac{1}{|x|^n}\int_{B_k}|b(y)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx = I + II.$$

We firstly prove I,

$$I \le C\sum_{k=-\infty}^{k_0} 2^{-knp}\int_{B_k}|b(x)-b_{B_k}|^p\omega(x)^{1-p}dx\left|\sum_{i=-\infty}^{k}\int_{B_i}|f(y)|dy\right|^p$$

$$\le C\|b\|^p_{Lip_{\beta,\omega}}\sum_{k=-\infty}^{k_0}2^{-knp}\omega(B_k)^{1+\frac{\beta p}{n}}\left|\sum_{i=-\infty}^{k}\left(\int_{B_i}|f(y)|^p\omega(y)dy\right)^{\frac{1}{p}}\left(\int_{B_i}\omega(y)^{1-p'}dy\right)^{\frac{1}{p'}}\right|^p$$

$$\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+\lambda_1 p}\left|\sum_{i=-\infty}^{k}2^{(i-k)n(1+\lambda)}\right|^p$$

$$\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda_1 p}.$$

Breaking II into two parts:

$$\begin{aligned} II &\le C\sum_{k=-\infty}^{k_0} 2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}\int_{B_i}|b(y)-b_{B_k}||f(y)|dy\right|^p\omega(x)^{1-p}dx\\ &\le C\sum_{k=-\infty}^{k_0} 2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}\int_{B_i}|b(y)-b_{B_i}||f(y)|dy\right|^p\omega(x)^{1-p}dx\\ &+C\sum_{k=-\infty}^{k_0} 2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}\int_{B_i}|b_{B_k}-b_{B_i}||f(y)|dy\right|^p\omega(x)^{1-p}dx = II'+II''. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} II' &\le C\sum_{k=-\infty}^{k_0} 2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}\left(\int_{B_i}|b(y)-b_{B_i}|^{p'}\omega(y)^{1-p'}dy\right)^{\frac{1}{p'}}\right.\\ &\quad\left.\times\left(\int_{B_i}|f(y)|^p\omega(y)dy\right)^{\frac{1}{p}}\right|^p\omega(x)^{1-p}dx\\ &\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}2^{-knp}\int_{B_k}\omega(x)^{1-p}dx\left|\sum_{i=-\infty}^{k}\omega(B_i)^{1+\lambda_1}\right|^p\\ &\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+\lambda_1 p}\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda_1 p}. \end{aligned}$$

On the other hand, applying $|b_{B_k}-b_{B_i}|\le C(k-i)\|b\|_{Lip_{\beta,\omega}}\omega(B_k)^{\frac{\beta}{n}}\frac{\omega(B_i)}{|B_i|}$,

$$\begin{aligned} II'' &= C\sum_{k=-\infty}^{k_0} 2^{-knp}\int_{B_k}\left|\sum_{i=-\infty}^{k}\int_{B_i}|b_{B_k}-b_{B_i}||f(y)|dy\right|^p\omega(x)^{1-p}dx\\ &\le C\|b\|^p_{Lip_{\beta,\omega}}\sum_{k=-\infty}^{k_0}2^{-knp}\omega(B_k)^{\frac{\beta p}{n}}\int_{B_k}\left|\sum_{i=-\infty}^{k}(k-i)\frac{\omega(B_i)}{|B_i|}\left(\int_{B_i}|f(y)|^p\omega(y)dy\right)^{\frac{1}{p}}\right.\\ &\quad\left.\times\left(\int_{B_i}\omega(y)^{1-p'}dy\right)^{\frac{1}{p'}}\right|^p\omega(x)^{1-p}dx\\ &\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1-p+\frac{\beta p}{n}}\left|\sum_{i=-\infty}^{k}(k-i)\omega(B_i)^{1+\lambda}\right|^p\\ &\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\sum_{k=-\infty}^{k_0}\omega(B_k)^{1+\lambda_1 p}\le C\|b\|^p_{Lip_{\beta,\omega}}\|f\|^p_{\dot{M}^{p,\lambda}(\omega,\omega)}\omega(B_{k_0})^{1+\lambda_1 p}. \end{aligned}$$

By slightly modifying Theorem 3.1, we can get the proof of $[H^*,b]$. Here, we omit its proof.

Remark 4.1. Since $\|b\|_{\dot{C}^{p,\lambda}(\omega)}\le\|b\|_{C^{p,\lambda}(\omega)}$ and $\|b\|_{C^{p,\lambda}(\omega)}\sim\|b\|_{Lip_{\beta,\omega}}$ when $\omega\in A_1$ and $0<\lambda<1/n$, $b\in Lip_{\beta,\omega}$ is a sufficient condition for the boundedness of the $[b,H]$ and $[b,H^*]$ rather than a necessary condition. However, if $b\in\dot{C}^{p,\lambda}(\omega)$, it is still a necessary and sufficient condition.

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Contents