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ON THE FERMAT-TYPE DIFFERENCE EQUATION $f^3(z) + [c_1 f(z+c) + c_0 f(z)]^3 = e^{\alpha z+\beta}$

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Abstract. In this article, we deal with the solutions of the difference analogue of Fermat-type equation of the form $f^3(z) + [c_1 f(z+c) + c_0 f(z)]^3 = e^{\alpha z+\beta}$ and prove a result generalizing a result of Han and Lü [J. Contm. Math. Anal. 2019] and Ma *et al.* [J. Func. Spaces, Vol. 2020, Article ID 3205357]. Furthermore, we explore the class of functions satisfying the Fermat-type difference equation. A considerable number of examples have been exhibited throughout the paper pertinent with the different issues. We characterized all possible non-constant solutions of the Fermat-type difference equation $f^2(z) + f^2(z+c) = e^{\alpha z+\beta}$.

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1. Introduction

The so called Fermat's Last Theorem, which was proved by Wiles [30], Taylor and Wiles [29] in 1995, states that there do not exist non-zero rational numbers x and y and an integer $n \geq 3$, for which $x^n + y^n = 1$. There is a close relationship between Fermat's Last Theorem and family of solutions (f, g) of the following functional equation

$$f^n + g^n = 1. \tag{1.1}$$

For $n = 1$, finding the solution is effortless, and for $n = 2$, it is easy to see that the pairs $(\sin(\alpha), \cos(\alpha))$ and

$$\left(\frac{1}{\sqrt{2}}[\sin(\alpha) \pm \cos(\alpha)], \frac{1}{\sqrt{2}}[\sin(\alpha) \mp \cos(\alpha)]\right)$$

always solves the equation for an entire function α. For $n \geq 2$, Gross [8] proved that all the meromorphic solutions are of the form

$$f(z) = \frac{2\beta(z)}{1+\beta^2(z)} \text{ and } g(z) = \frac{1-\beta^2(z)}{1+\beta^2(z)}.$$

For $n \geq 3$, it has no transcendental entire solutions proved in [Gauthier-Villars, Paris, (1927), 135—136] but meromorphic solutions exists which is confirmed by

Gross in [8] and one such solution is

$$\begin{aligned} f(z) &= 4^{-1/6}(\wp')^{-1}\left(1+3^{-1/2}\cdot 4^{1/3}\wp\right)\\ g(z) &= 4^{-1/6}(\wp)^{-1}\left(1-3^{-1/2}\cdot 4^{1/3}\wp\right), \end{aligned}$$

where $\wp$ is a Weierstrass $\wp$-function. For $n \geq 4$, it has no transcendental meromorphic solutions confirmed in [8]. No other solutions of the equation (1.1) exist which is confirmed by Gross in [9].

It has been determined for which positive integers n, the equation (1.1) has non-constant solutions f and g in each of the following four function classes (i) meromorphic functions, (ii) rational functions, (iii) entire functions, and (iv) polynomials; (see [11, 12]. The study of the functions analogous to the Fermat-type diophantine equations $x^n + y^n = 1$ was initiated by Gross [8] and Baker [2]. They actually proved that the equation

$$f^n + g^n = 1 \tag{1.2}$$

does not admit any non-constant meromorphic solutions in the complex plane $\mathbb{C}$ if $n > 3$, and does not admit any entire solutions if $n > 2$. For the possible non-constant meromorphic solutions of (1.2), they also characterized it in the case of when $n = 2, 3$. In fact, for the case $n = 3$, Gross [8] and Baker [2] proved that the following pair (f, g), where

$$f(z) = \left(\frac{1}{2} + \frac{\wp'(z)}{2\sqrt{3}}\right)/\wp(z) \tag{1.3}$$

and

$$g(z) = \left(\frac{1}{2} - \frac{\wp'(z)}{2\sqrt{3}}\right)/\wp(z), \tag{1.4}$$

are meromorphic solution of equation (1.2), where $\wp$ is Weierstrass $\wp$-function.

It is worth to observe that the equation $x^3 + y^3 = 1$ defines an algebraic function whose Reimann surface has genus 1, and there is accordingly a uniformization by Weierstrass elliptic function. Weierstrass elliptic function $\wp(z) := \mathcal{P}(z, \omega_1, \omega_2)$ is a doubly periodic meromorphic function with periods ω_1 and ω_2, and this function is defined by

$$\wp(z, \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{\substack{\mu,\nu\in\mathbb{Z}\\ \mu^2+\nu^2\neq 0}} \left(\frac{1}{(z+\mu\omega_1+\nu\omega_2)^2} - \frac{1}{(\mu\omega_1+\nu\omega_2)^2}\right),$$

which is even and satisfies, after appropriate choosing ω_1 and ω_2,

$$(\wp')^2 = 4\wp^3 - 1. \tag{1.5}$$

In the same paper, Gross conjectured that every meromorphic solutions of $f^3 + g^3 = 1$ are necessarily elliptic function of entire functions. Later, Baker [2] confirmed the conjecture and established the following result.

Theorem A. [2] *Each pair of meromorphic solutions f and g to the following equation*

$$f^3(z) + g^3(z) = 1 \tag{1.6}$$

over $\mathbb{C}$ must be of the form $f = f_1(h(z))$ and $g(z) = \omega g_1(h(z)) = \omega f_1(-h(z))$, where h is an entire function in $\mathbb{C}$ and ω is a cube root of unity.

In this paper, a meromorphic function will always be non-constant and meromorphic in the complex plane $\mathbb{C}$, unless specifically stated otherwise. In what follows, we assume that the reader is familiar with the elementary Nevanlinna theory (see [7, 33, 35]). In particular, for a meromorphic function f, we denote $\mathcal{S}(f)$ the family of all meromorphic function ω for which $T(r, \omega) = S(r, f) = o(T(r, f))$, where $r \to \infty$ outside of a possible set of finite logarithmic measure. For convenience, we agree that $\mathcal{S}(f)$ includes all constant functions and $\overline{\mathcal{S}}(f) := \mathcal{S}(f) \cup \{\infty\}$. Here, the order $\rho(f)$ of a meromorphic function is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

In 2016, Lü and Han [20] proved that the equation $f(z) + f'(z) = 1$ has the general solution $f(z) = 1 - ae^{-z}$ for $a \in \mathbb{C}$ and $f^2(z) + (f'(z))^2 = 1$ has the general solution $f(z) = \pm\sin(z + b)$ for some $b \in \mathbb{C}$. Nevertheless, $f^n(z) + (f'(z))^n = 1$ can not have any non-constant meromorphic solution when $n > 2$.
Below, we recall a well-known facts about the order of composite meromorphic functions which have been established by Edrei and Fuchs [6], and by Bergweiler [4].

Theorem B. *Let f be a meromorphic functions and h be an entire function in $\mathbb{C}$. When $0 < \rho(f), \rho(h) < \infty$, then $\rho(f \circ h) < \infty$, and h is transcendental, then $\rho(f) = 0$.*

In the recent years, Nevanlinna characteristic of $f(z+c)$ $(c \in \mathbb{C} \setminus \{0\})$, the value distribution theory of difference polynomials, Nevanlinna theory of the difference operator and the difference analogue of the lemma of the logarithmic derivative has been established (see [5, 14, 15]). Due to this development of theories, there has been a recent study on whether the derivative f' of f can be replaced by the shift $f(z+c)$ or difference operator $\Delta_c f$. The difference analogues of the Fermat type functional equations have been investigated in a number of papers (see [19, 24, 26, 27, 31, 34]).

For a meromorphic function f, we define its difference operators by

$$\Delta_c f = f(z+c) - f(z)$$
$$\Delta_c^n f = \Delta_c^{n-1}\left(\Delta_c f\right), \; n \in \mathbb{N}, \; n \geq 2.$$

In 2016, Lü and Han [20] described a property of meromorphic solutions to the equation (1.6) with $g(z) := f(z+c)$, for $c \in \mathbb{C} \setminus \{0\}$ as the following.

Theorem C. [20] *The difference equation $f^3(z) + f^3(z+c) = 1$ does not have meromorphic solutions of finite order.*

For $n \geq 4$ and $\gamma \not\equiv 0$, if we consider the meromorphic solution of the equations $f^n(z)+(f')^n = \gamma^n$, then by the Proposition 1.1 in [16] we see that both the functions f/γ and f'/γ must be constants. Therefore, if we assume $f = c_1\gamma$ and $f' = c_2\gamma$, then a simple computation shows that $c_1^n + c_2^n = 1$. Observe that $c_1 \neq 0$, otherwise $f \equiv 0$, hence $\gamma = 0$. Similarly, $c_2 \neq 0$, otherwise, f and γ will be constants. Therefore, when $c_1c_2 \neq 0$, then γ cannot have any zeros and poles. Hence $\gamma^n(z) = e^{\alpha z+\beta}$ where $\alpha = nc_2/c_1$.

Motivated by the above observations, Han and Lü [16] have investigated the above equation with $f(z+c)$ in the place of $f'(z)$ for the case $n = 3$ and proved the following interesting result.

Theorem D. [16] *The difference equation $f^3(z)+f^3(z+c) = e^{\alpha z+\beta}$, where $\alpha, \beta \in \mathbb{C}$, does not have meromorphic solutions of finite order.*

Regarding existence of solutions of the difference equation $f^n(z) + [\Delta_c f]^n = 1$ for a positive integer n, we have the following note.

Remark 1.1. A simple computation shows that the difference equation $f(z) + \Delta_c f = 1$ has no non-constant meromorphic solutions. Following the proof of Theorem 1.5 of Liu *et al.* in [18, Theorem 1.5], one can observe that there does not exist any non-constant meromorphic solutions of the difference equation $f^2(z) + [\Delta_c f]^2 = 1$.

Therefore, a natural question arises as the following.

Question 1.1. *Does there exist any non-constant meromorphic solutions of the difference equation $f^3(z) + [\Delta_c f]^3 = 1$?*

Recently, Ma *et al.* [21] have investigated Theorem B by considering the difference operator $\Delta_c f$ and proved the following result which answers Question 1.1.

Theorem E. [21] *The difference equation $f^3(z) + [\Delta_c f(z)]^3 = 1$ does not have meromorphic solutions of finite order.*

In the same paper, Han and Lü [16] proved the next result by producing a complete characterization of the solutions.

Theorem F. [16] *The meromorphic solutions f of the following differential equation*

$$f^n(z) + [f'(z)]^n = e^{\alpha z+\beta} \tag{1.7}$$

must be entire functions and the following assertions hold.

(i) *For $n = 1$, the general solution of (1.7) are $f(z) = e^{\alpha z+\beta}/(\alpha+1)$, when $\alpha \neq -1$, and $f(z) = ze^{-z+\beta} + ae^{-z}$.*

(ii) *For $n = 2$, either $\alpha = 0$, and the general solution of (1.7) are $f(z) = e^{\beta/2}\sin(z+b)$, or $f(z) = de^{(\alpha z+\beta)/2}$.*

(iii) *For $n \geq 3$, the general solution of (1.7) is $f(z) = de^{(\alpha z+\beta)/n}$,*

where $a, b, d, \alpha, \beta \in \mathbb{C}$ with $d^n\left(1 + (\alpha/n)^n\right) = 1$, for $n \geq 2$.

The paper is organized as follows. In Section 2, we prove a result generalizing the Theorem D and Theorem E. In Subsection 2.1, the characterization of the solutions of $f^2(z) + f^2(z+c) = e^{\alpha z+\beta}$ is discussed and a result is proved. In Section 3, the claim of Han and Lü in [16, page 102] is disproved exhibiting several counter examples. Section 4 is devoted mainly to prove the main results of this paper. Future course of work on the results of this paper has been discussed in Section 5.

2. Main result

Motivating from Remark 1.1, we are interested to investigate for the non-constant meromorphic solutions of general difference equations. Henceforth, we recall here $L_c(f)$ defined by the present author in [1] as $L_c(f) := c_1 f(z+c) + c_0 f(z)$, $c_1 (\neq 0)$, $c_0 \in \mathbb{C}$. It is easy to see that the shift $f(z+c)$ and difference operator $\Delta_c f$ are the particular cases of $L_c(f)$. With this setting, in this paper, our aim is to investigate Theorems D and E further to establish a combined result. Before state the main result of this paper, we have the following remark.

Remark 2.1. The equation $f^n(z)+[L_c(f)]^n = e^{\alpha z+\beta}$, may consists of non-constant entire as well as meromorphic solutions for $n = 1$ and $n = 2$, from the following examples we ensure this fact.

Example 2.1. *Let*

$$f(z) = \left(-\frac{c_0+1}{c_1}\right)^{z/c} h(z) + \delta e^{\alpha z+\beta},$$

where h is c-periodic finite order entire functions like $h(z)=\sin(2\pi z/c)$ or $\cos(2\pi z/c)$ or $e^{2\pi iz/c}$ etc. and their linear combinations and c be such that $e^{\alpha c}=(1-\delta(c_0+1))/c_1\delta$. It is easy to verify that $f(z)$ solves the equation $f(z)+L_c(f(z))=e^{\alpha z+\beta}$.

Example 2.2. *Let*

$$f(z)=\left(-\frac{c_0+1}{c_1}\right)^{z/c}\frac{g(z)+1}{g(z)-1}+\delta e^{\alpha z+\beta}.$$

where g is c-periodic finite order entire or meromorphic functions like in Example 2.1 and c be such that $e^{\alpha c}=(1-\delta(c_0+1))/c_1\delta$. It is easy to see that $f(z)$ solves the equation $f(z)+L_c(f(z))=e^{\alpha z+\beta}$.

Example 2.3. *Let $f(z)=(1/2)e^{(\alpha z+\beta)/3}\left(e^{(\alpha z+\beta)/3}+1\right)$. We choose $c\in\mathbb{C}$ such that $e^{\alpha c/3}\neq 1$. Let*

$$L_c(f)=\frac{2i}{e^{\frac{\alpha c}{3}}\left(e^{\frac{\alpha c}{3}}-1\right)}f(z+c)+\frac{i\left(e^{\frac{\alpha c}{3}}+1\right)}{1-e^{\frac{\alpha c}{3}}}f(z).$$

Clearly, $f(z)$ solves the equation $f^2(z)+[L_c(f(z))]^2=e^{\alpha z+\beta}$.

Example 2.4. *Let*

$$f(z)=\frac{1}{2}\left(e^{\gamma(\alpha z+\beta)}\sin\left(\frac{2\pi z}{c}\right)+\frac{e^{(1-\gamma)(\alpha z+\beta)}}{\sin\left(\frac{2\pi z}{c}\right)}\right)\ \textit{where}\ \gamma\in\mathbb{C}\setminus\left\{\frac{1}{2}\right\}.$$

Let

$$L_c(f)=\frac{2}{i\left(e^{(1-\gamma)\alpha c}-e^{\gamma\alpha c}\right)}f(z+c)+\frac{i\left(e^{(1-\gamma)\alpha c}+e^{\gamma\alpha c}\right)}{\left(e^{(1-\gamma)\alpha c}-e^{\gamma\alpha c}\right)}f(z).$$

It is easy to verify that $f(z)$ solves the equation $f^2(z)+[L_c(f(z))]^2=e^{\alpha z+\beta}$.

The observations from the above examples motivate us to establish a single result combining the results of Lü and Han [16], and Ma *et al.* [21] (*i.e.,* for the case $n=3$). Therefore, the following question is inevitable.

Question 2.1. *Does there exist any non-constant meromorphic solution of the equation of $f^3(z)+[L_c(f(z))]^3=e^{\alpha z+\beta}$?*

In this paper, with the help of some ideas of [16], we establish Theorem 2.1 which answers Question 2.1 completely.

Theorem 2.1. *The difference equation*

$$f^3(z)+[L_c(f(z))]^3=e^{\alpha z+\beta} \tag{2.1}$$

does not have infinite order meromorphic solutions.

Remark 2.2. In case of meromorphic function of infinite order, the next example evidents that (2.1) may admit solution.

Example 2.5. *Let $f(z)$ be given by (4.2) with $h(z) = e^z$. Therefore, we have $\rho(f) = \infty$ and for $c = \pi i$, each α with $e^{c\alpha/3} = \{1,\ \omega,\ \omega^2\}$ where ω is a non-real cube root of unity. It is easy to see that $f^3(z) + [L_c(f(z))]^3 = e^{\alpha z+\beta}$.*

Our aim is to generalize Theorem F for general setting of the equation. In order to generalize Theorem F, we would like to explore the meromorphic solutions of the following Fermat-type differential equation

$$f^n(z) + \left(f^{(k)}(z)\right)^n = e^{\alpha z+\beta} \text{ for } k \in \mathbb{N}. \tag{2.2}$$

Henceforth, to this end, we denote θ by $\theta = \cos(3\pi/k) + i\sin(3\pi/k)$ where k is a positive integer such that $\theta^k = -1$.

Theorem 2.2. *Let k be any positive integer. Then the meromorphic solutions f of the differential equation*

$$f^n(z) + [f^{(k)}(z)]^n = e^{\alpha z+\beta} \tag{2.3}$$

must be entire functions. Furthermore,

(i) *When $n = 1$, the general solution of (2.3) is*

$$f(z) = \begin{cases} \displaystyle\sum_{j=1}^{k} a_j e^{\theta^j z} + \frac{ze^{\alpha z+\beta}}{\alpha^k + 1}, & \text{for } \alpha \neq \theta, \theta^2, \dots, \theta^{k-1} \\ \displaystyle\sum_{j=1}^{k} a_j e^{\theta^j z} + \frac{ze^{\alpha z+\beta}}{k\alpha^{(k-1)}}, & \text{for } \alpha \in \{\theta, \theta^2, \dots, \theta^{k-1}\}, \\ \displaystyle\sum_{j=1}^{k} a_j e^{\theta^j z} + \frac{ze^{-z+\beta}}{k}, & \text{for } \alpha = -1 \text{ and } k \text{ is odd,} \\ \displaystyle\sum_{j=1}^{k} a_j e^{\theta^j z} - \frac{ze^{-z+\beta}}{k}, & \text{for } \alpha = -1 \text{ and } k \text{ is even,} \end{cases}$$

(ii) *When $n = 2$, one of the following holds: Either*

 (a) *$\alpha = 0$, and the general solution of (2.3) are $f(z) = e^{\beta/2}\sin(z + b)$, only when k is odd but when k is even, then f must be constant, $e^{\beta/2}$, or*

 (b) *$f(z) = de^{(\alpha z+\beta)/2}$.*

(iii) *When $n \geq 3$, the general solution of (2.3) is $f(z) = de^{(\alpha z+\beta)/n}$,*

where $a, b, d, \alpha, \beta \in \mathbb{C}$ are such that $d^n\left(1 + (\alpha/n)^{nk}\right) = 1$, for $n \geq 2$.

2.1. **Characterization of the solutions of $f^2(z)+f^2(z+c) = e^{\alpha z+\beta}$.** In contrast to Theorem 2.1 in [16], Han and Lü have shown that even though the existence of finite or infinite order meromorphic solutions of the difference equation

$$f^2(z) + f^2(z + c) = e^{\alpha z+\beta} \tag{2.4}$$

can be described but they could not prove a result finding the general solution of (2.4). Therefore, it is interesting to seek the possible general meromorphic solutions of the difference equation (2.4). In this paper, we take this opportunity to find out the possible general meromorphic solutions of the above Fermat-type difference equation. Consequently, we prove the following result which may give a complete characterization of the solutions of the difference equation (2.4).

Theorem 2.3. *The general meromorphic solutions of the Fermat-type difference equation* $f^2(z) + f^2(z+c) = e^{\alpha z+\beta}$ *are the following:*

(i) *If f is a non-constant entire function, then*

$$f(z) = \begin{cases} de^{\frac{\alpha z+\beta}{2}}, \text{ where } d \neq \pm 1,\ d^2 = \frac{1}{e^{\alpha c}+1} \text{ with } e^{\alpha c} \neq -1, & \text{when order of } f \text{ is finite.} \\ e^{\frac{\alpha z+\beta}{2}} \sin\left(\frac{(4k+1)\pi z}{2c} + \eta\right), & \text{when order of } f \text{ is finite,} \\ e^{\frac{\alpha z+\beta}{2}} \sin\left(\frac{(4k+1)\pi z}{2c} + \mathcal{H}(z)\right), & \text{when order of } f \text{ is infinite.} \end{cases}$$

(ii) *If f is a non–constant meromorphic function, then*

$$f(z) = \begin{cases} \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(g(z) + \dfrac{e^{\frac{1}{2}(\alpha z+\beta)}}{g(z)}\right), \\ \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(e^{\frac{1}{2}(\alpha z+\beta)}g(z) + \dfrac{1}{g(z)}\right), \end{cases}$$

where g is a meromorphic function, $\mathcal{H}$ is a c-periodic entire function, η is a complex number and $e^{\alpha c} = 1$.

Remark 2.3. If g is a constant or an exponential function, then the solution becomes transcendental entire.

Example 2.6. *Let*

(i) $f_1(z) = \dfrac{1}{9}e^{\frac{\alpha z+\beta}{2}}$, *with* $e^{\alpha c} = 8$, $\rho(f) \leq 1$,

(ii) $f_2(z) = e^{\frac{\alpha z+\beta}{2}} \cos\left(\dfrac{\pi z}{2c} + 1\right)$, $\rho(f) \leq 1$,

(iii) $f_3(z) = e^{\frac{\alpha z+\beta}{2}} \sin\left(\dfrac{\pi z}{2c} - 1\right)$, $\rho(f) \leq 1$,

(iv) $f_4(z) = \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(\dfrac{1}{3} + 3e^{\frac{1}{2}(\alpha z+\beta)}\right)$, *with* $e^{\alpha c} = 1$, $\rho(f) \leq 1$,

(v) $f_5(z) = \dfrac{1}{2}\left(e^{\frac{3}{4}(\alpha z+\beta)+\frac{2\pi i z}{c}} + e^{\frac{1}{4}(\alpha z+\beta)-\frac{2\pi i z}{c}}\right)$, *with* $e^{\alpha c} = 1$, $\rho(f) \leq 1$,

(vi) $f_6(z) = e^{\frac{\alpha z+\beta}{2}} \sin\left(e^{\frac{2\pi i z}{c}} + \dfrac{\pi z}{2c} + 1\right)$, *with* $e^{\alpha c} = 1$, $\rho(f) = \infty$,

It is easy to verify that $f_j^2(z) + f_j^2(z+c) = e^{\alpha z+\beta}$ *for all* $j = 1,\ 2,\ \ldots, 6$.

3. Remarks on the general solution of Fermat-type difference equations

In their paper, Han and Lü [16] have discussed briefly about the meromorphic solutions of the difference equation

$$f(z) + f(z+c) = e^{\alpha z+\beta}. \tag{3.1}$$

In [16, page 102], Han and Lü claimed that the general solution of the difference equation (3.1) is either of the form $f(z) = \delta(z) + de^{\alpha z+\beta}$ or $f(z) = \delta(z) - (z/c)e^{\alpha z+\beta}$, where $\delta(z)$ is a meromorphic function satisfying $\delta(z+c) = -\delta(z)$.

In this paper, after a careful investigation on the functional equation (3.1), we found the following list of counter examples confirming that $f(z) = \delta(z) + de^{\alpha z+\beta}$ or $f(z) = \delta(z) - (z/c)e^{\alpha z+\beta}$ are not the general solution rather some particular solutions of the difference equation $f(z) + f(z+c) = e^{\alpha z+\beta}$.

Example 3.1. *Let*

$$f(z) = \frac{e^{\frac{\pi i z}{c}}}{\sin\left(\frac{2\pi z}{c}\right) - 1} + e^{\alpha z+\beta}\cos^2\left(\frac{\pi z}{2c}\right),$$

where c be so chosen that $e^{\alpha c} = 1$. We verify that $f(z)$ solves the equation $f(z) + f(z+c) = e^{\alpha z+\beta}$ and f is neither in the specific forms suggested by Lü and Han.

Example 3.2. *Let*

$$f(z) = e^{\frac{\pi i z}{c}}\frac{g(z)+1}{g(z)-1} + e^{\alpha z+\beta}\sin^2\left(\frac{\pi z}{2c}\right),$$

where c be such that $e^{\alpha c} = 1$, and g is any c-periodic finite order entire or meromorphic functions like $g(z) = \sin(2\pi z/c)$ or $\cos(2\pi z/c)$ or $\tan(\pi z/c)$ or $\cot(\pi z/c)$ etc. Evidently, $f(z) + f(z+c) = e^{\alpha z+\beta}$ and f is neither in the specific forms claimed by Lü and Han.

Remark 3.1. In connection with the existence of solutions, we see that, in page 148, Liu *et al.* [18] have investigated to find non-constant solutions of the difference equation

$$f^n(z) + f^m(z+c) = 1$$

for different range of values of m and n, where $m,\ n \in \mathbb{N}$. But in particular, when $m = 1 = n$, Liu *et al.* have claimed that the general entire solutions are of the form $f(z) = 1/2 + e^{\pi i z/c}h(z)$, where h is a c-periodic entire function. In the following, we construct examples to show that the general solution is not always of that form. Therefore, we consider the function $g(z) = \sin z$ or $\cos z$.

Example 3.3. *Let $f(z) = g^2(\pi z/2c) + e^{\pi iz/c}h(z)$, where h is a c-periodic entire function. We see that although $f(z)$ solves the equation $f(z)+f(z+c)=1$ but not in the said form.*

Example 3.4. *Let $f(z) = (3/5)g^2(\pi z/2c) + 1/5$. Clearly, $f(z)$ solves the equation $f(z)+f(z+c)=1$ without being of the said form.*

4. Proof of the main result

Proof of Theorem 2.1. The difference equation $f^3(z) + [L_c(f)]^3 = e^{\alpha z+\beta}$ of the theorem, can be expressed as

$$\left(\frac{f(z)}{e^{\frac{\alpha z+\beta}{3}}}\right)^3 + \left(\frac{L_c(f)}{e^{\frac{\alpha z+\beta}{3}}}\right)^3 = 1.$$

By the Proposition 1.1 in [16], it is known that the only non-constant meromorphic solutions of $F^3(z) + G^3(z) = 1$ are

$$F(z) = \frac{1}{2\wp(h)}\left(1+\frac{1}{\sqrt{3}}\wp'(h)\right) \text{ and } G(z) = \frac{\omega}{2\wp(h)}\left(1-\frac{1}{\sqrt{3}}\wp'(h)\right),$$

where h is an entire function, ω is a cube root of unity and $\wp$ denotes the Weierstrass $\wp$-function. Therefore, in view of the Proposition 1.1, we obtain

$$f(z) = \frac{1}{2\wp(h)}\left(1+\frac{1}{\sqrt{3}}\wp'(h)\right)e^{\frac{\alpha z+\beta}{3}} \tag{4.1}$$

and

$$L_c(f) = \frac{\omega}{2\wp(h)}\left(1-\frac{1}{\sqrt{3}}\wp'(h)\right)e^{\frac{\alpha z+\beta}{3}}. \tag{4.2}$$

From (4.2), we obtain

$$f(z+c) = \frac{\dfrac{\omega - c_0}{2} - \dfrac{\omega + c_0}{2\sqrt{3}}\wp'(z)}{c_1\wp(h(z))}e^{\frac{\alpha z+\beta}{3}}. \tag{4.3}$$

A routine computation using (4.1) and (4.3) shows that

$$\frac{(\omega - c_0) - \dfrac{\omega + c_0}{\sqrt{3}}\wp'(h(z))}{\wp(h(z))} = \frac{c_1\left(1+\dfrac{\wp'(h(z+c))}{\sqrt{3}}\right)}{\wp(h(z+c))}e^{\frac{\alpha c}{3}}. \tag{4.4}$$

Equation (4.1) can be written as

$$\frac{\wp'(h(z))}{\sqrt{3}} = 2\wp(h(z))f(z) - 1. \tag{4.5}$$

Assuming $\rho(f) < \infty$, then in view of (1.5) and (4.5), we obtain

$$\frac{3f^2(z)\wp^2(h(z))}{e^{\frac{2}{3}(\alpha z+\beta)}} - \frac{3f(z)\wp(h(z))}{e^{\frac{1}{3}(\alpha z+\beta)}} + 1 = \wp^3(h(z)). \tag{4.6}$$

We recall here the estimate (2.7) of Bank and Langley [3] which states that

$$T(r,\wp) = \frac{\pi r^2}{A}(1+o(1)) \text{ and } \rho(\wp) = 2, \tag{4.7}$$

where A is the area of the parallelogram $\mathfrak{P}$ with the vertices $0,\ \omega_1,\ \omega_2$ and $\omega_1+\omega_2$. Therefore, taking into account that $T\left(r,e^{\alpha z}\right)=(\alpha r/\pi)(1+O(1))$, combining (4.5) and (4.7), we obtain

$$T(r,\wp(h))\leq 2T(r,f)+\frac{2}{3}T\left(r,e^{\alpha z}\right)+O(1), \tag{4.8}$$

and hence $\rho(\wp(h))<\infty$ as well.

By Corollary 1.2 of Edrei and Fuchs [6] (see also Theorem of Bergweiler [4]), h must be a polynomial.

Actually, we have $T(r,\wp(h))=O\left(r^{2q}\right)$, for $q\geq 1$. It is easy to see that if $\wp(z_0)=0$, then from (1.5), we obtain $\left(\wp'(z_0)\right)^2=-1$ which shows that $\wp'(z_0)=\pm i$. We now denote $\{z_n\}_{n\in\mathbb{N}}$ by all the zeros of $\wp(z)$ that satisfy $z_n\to\infty$ when $n\to\infty$ and assume that $h(a_{n,k})=z_n$, for $k=1,\ 2,\ldots,\deg(h)$. Thus we have $\left(\wp'\right)^2(h(a_{n,k}))=\left(\wp'\right)^2(z_n)=-1$. Suppose there is a sub-sequence $\{a_{n,k}\}_{n\in\mathbb{N}}$ with respect to n such that $\wp(h(a_{n,k}+c))=0$. We denote this sub-sequence still by $\{a_{n,k}\}_{n\in\mathbb{N}}$ and fixed the index k below. Therefore, we have $\left(\wp'\right)^2(h(a_{n,k}+c))=-1$. Differentiating both sides of (4.4), we obtain

$$\begin{aligned}&\left(-\frac{\omega+c_0}{\sqrt{3}}\wp''(h(z))h'(z)\right)\wp(h(z+c))\\&\quad+\left((\omega-c_0)-\frac{\omega+c_0}{\sqrt{3}}\wp'(h(z))\right)\wp'(h(z+c))h'(z+c)\\&=\left(\frac{c_1}{\sqrt{3}}\wp''(h(z+c))h'(z+c)\right)\wp(h(z))e^{\frac{\alpha c}{3}}+c_1\left(1+\frac{\wp'(h(z+c))}{\sqrt{3}}\right)\wp'(h(z))h'(z)e^{\frac{\alpha c}{3}}.\end{aligned} \tag{4.9}$$

Substituting $a_{n,k}$ (for sufficiently large n) into the equation (4.9) and by using $\wp(h(a_{n,k}+c))=0$ and $\wp(h(a_{n,k}))=0$, we obtain

$$\begin{aligned}&\left((\omega-c_0)-\frac{\omega+c_0}{\sqrt{3}}\wp'(h(a_{n,k}))\right)\wp'(h(a_{n,k}+c))h'(a_{n,k}+c)\\&\quad=c_1\left(1+\frac{\wp'(h(a_{n,k}+c))}{\sqrt{3}}\right)\wp'(h(a_{n,k}))h'(a_{n,k})e^{\frac{\alpha c}{3}}.\end{aligned} \tag{4.10}$$

Noting that $\wp'(h(a_{n,k}))=\pm i$ and $\wp'(h(a_{n,k}+c))=\pm i$, without any loss of generality, together with (4.4), we assume that there exists a sub-sequence $\{a_{n,k}\}_{n\in\mathbb{N}}$ (here we still denote it by $\{a_{n,k}\}_{n\in\mathbb{N}}$) such that the following four possible cases may occur.

Case 1. If $\wp'(h(a_{n,k}))=i$ and $\wp'(h(a_{n,k}+c))=i$, then in view of (4.10), we obtain

$$\left(\omega-c_0-\frac{\omega+c_0}{\sqrt{3}}i\right)h'(a_{n,k}+c)=c_1\left(1+\frac{i}{\sqrt{3}}\right)h'(a_{n,k})e^{\frac{\alpha c}{3}}. \tag{4.11}$$

Case 2. If $\wp'(h(a_{n,k})) = -i$ and $\wp'(h(a_{n,k}+c)) = i$, then we get from (4.10),

$$\left(\omega - c_0 + \frac{\omega + c_0}{\sqrt{3}} i\right) h'(a_{n,k}+c) = -c_1 \left(1 + \frac{i}{\sqrt{3}}\right) h'(a_{n,k}) e^{\frac{\alpha c}{3}}. \tag{4.12}$$

Case 3. If $\wp'(h(a_{n,k})) = i$ and $\wp'(h(a_{n,k}+c)) = -i$, then we obtain from (4.10),

$$\left(\omega - c_0 - \frac{\omega + c_0}{\sqrt{3}} i\right) h'(a_{n,k}+c) = -c_1 \left(1 - \frac{i}{\sqrt{3}}\right) h'(a_{n,k}) e^{\frac{\alpha c}{3}}. \tag{4.13}$$

Case 4. If $\wp'(h(a_{n,k})) = i$ and $\wp'(h(a_{n,k}+c)) = i$, then (4.10) yields

$$\left(\omega - c_0 + \frac{\omega + c_0}{\sqrt{3}} i\right) h'(a_{n,k}+c) = c_1 \left(1 - \frac{i}{\sqrt{3}}\right) h'(a_{n,k}) e^{\frac{\alpha c}{3}}. \tag{4.14}$$

Since $h(z)$ and $h(z+c)$ are polynomials of same degree with same leading coefficient and there are infinitely many $a_{n,k}$ (with $|a_{n,k}| \to \infty$), we would have to conclude

$$\begin{cases} \left(\omega - c_0 - \frac{\omega + c_0}{\sqrt{3}} i\right) h'(z+c) = c_1 \left(1 + \frac{i}{\sqrt{3}}\right) h'(z) e^{\frac{\alpha c}{3}} \\ \left(\omega - c_0 + \frac{\omega + c_0}{\sqrt{3}} i\right) h'(z+c) = -c_1 \left(1 + \frac{i}{\sqrt{3}}\right) h'(z) e^{\frac{\alpha c}{3}} \\ \left(\omega - c_0 - \frac{\omega + c_0}{\sqrt{3}} i\right) h'(z+c) = -c_1 \left(1 - \frac{i}{\sqrt{3}}\right) h'(z) e^{\frac{\alpha c}{3}} \\ \left(\omega - c_0 + \frac{\omega + c_0}{\sqrt{3}} i\right) h'(z+c) = c_1 \left(1 - \frac{i}{\sqrt{3}}\right) h'(z) e^{\frac{\alpha c}{3}} \end{cases}$$

This is possible only when

$$e^{\frac{\alpha c}{3}} = \begin{cases} -\frac{2c_0+1}{2c_1} + \frac{\sqrt{3}}{2c_1} i,\ \frac{2c_0+1}{2c_1} - \frac{\sqrt{3}}{2c_1} i,\ -\frac{2+c_0}{2c_1} - \frac{\sqrt{3}c_0}{2c_1} i, \\ \frac{2c_0+1}{2c_1} + \frac{\sqrt{3}}{2c_1} i,\ \frac{1-2c_0}{2c_1} + \frac{\sqrt{3}}{2c_1} i,\ \frac{c_0+1}{2c_1} - \frac{\sqrt{3}(c_0+1)}{2c_1} i, \\ \frac{c_0+1}{2c_1} + \frac{\sqrt{3}(1-c_0)}{2c_1} i,\ -\frac{c_0+1}{c_1},\ -\frac{c_0+1}{2c_1} - \frac{\sqrt{3}}{2c_1} i, \\ \frac{c_0+1}{2c_1} - \frac{\sqrt{3}(c_0-1)}{2c_1} i,\ \frac{c_0+1}{2c_1} + \frac{\sqrt{3}(c_0+1)}{2c_1} i,\ \frac{1-2c_0}{2c_1} - \frac{\sqrt{3}}{2c_1} i, \end{cases}$$

since $\omega = 1$, $\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

Therefore, there exists a positive integer m_0 satisfying $P(h(a_n+c)) \neq 0$ for $n > m_0$.

When this is true, one has uniformly following the above set of equations (which are in terms of $h'(z+c)$ and $h'(z)$) that $h(z) = az + b$ for $ac \neq 0$. Again we know that the function $\wp(z)$ has two distinct zeros in $\mathfrak{P}$, and hence in each associated lattice, we see that all the zeros $\{z_n\}_{n\in\mathbb{N}}$ of $\wp(z)$ are transferred to each other through (an integral multiple) of ac. Therefore, for the simplicity, we can consider two cases: either $ac = \omega_1,\ \omega_2,\ \omega_1+\omega_2$ or $ac \neq \omega_1,\ \omega_2,\ \omega_1+\omega_2$ and $ac \in \mathfrak{P}$. It is worth noticing that the former cannot occur in view of (4.4) and the periodicity of $\wp(z)$ and $\wp'(z)$, and the later cannot occur either $\wp(z)$ has a unique double pole in each lattice. We now substitute $z_\infty = -(b/a)$ into (4.4), and obtain the following

$$\infty = \frac{(\omega - c_0) - \dfrac{\omega + c_0}{\sqrt{3}}\wp'(0)}{\wp(0)} = \frac{c_1\left(1 + \dfrac{\wp'(ac)}{\sqrt{3}}\right)}{\wp(ac)} e^{\frac{\alpha c}{3}} < \infty$$

which leads to a contradiction.

It is easy to see that $\wp(h(a_{n,k}+c)) = 0$ may occur only for finitely $a_{n,k}$'s. Without loss of generality, we assume that $\wp(h(a_{n,k}+c)) \neq 0$ for $k = 1,\ 2,\ \ldots, \deg(h)$ and all $n > N$, with N being a sufficiently large positive integer. Again since $\wp(h(a_{n,k})) = 0$ and $(\wp')^2(h(a_{n,k})) = -1$, hence by (4.4) we must have $\wp(h(a_{n,k})) = \infty$ for $n > N$. This implies that the zeros of $\wp(h(z))$ are the poles of $\wp(h(z+c))$ except for finitely many points. We observe that $O(\log r) = S(r, \wp(h))$, and hence we can write

$$\begin{aligned} N\left(r, \frac{1}{\wp(h(z))}\right) &\leq \bar{N}\left(r, \frac{1}{\wp(h(z))}\right) + 2N\left(r, \frac{1}{h'(z)}\right) \\ &\leq \bar{N}\left(r, \wp(h(z+c))\right) + 2T(r, h'(z)) + O(\log r) \\ &\leq \bar{N}\left(r, \wp(h(z+c))\right) + S\left(r, \wp(h(z))\right). \end{aligned} \tag{4.15}$$

In view of equation (4.1) and the estimate in (4.27), we obtain

$$\begin{aligned} T(r, f) &\leq T(r, \wp(h)) + T(r, \wp'(h)) + \frac{1}{3}T\left(r, e^{\alpha z}\right) + O(1) \\ &\leq O(T(r, \wp(h))). \end{aligned} \tag{4.16}$$

Hence in view of (4.8) and the estimate $T(r, \wp(h)) = O\left(r^{2q}\right)$, we have $\rho(f) = \rho(\wp(h))$ and also $S(r, f) = S(r, \wp(h))$. So we have $T\left(r, e^{\alpha z}\right) = S(r, f)$. From the equation

$$-[\mathcal{L}_c(f)]^3 = f^3(z) - \left(e^{\frac{\alpha z+\beta}{3}}\right)^3 = \left(f(z) - e^{\frac{\alpha z+\beta}{3}}\right)\left(f(z) - \omega e^{\frac{\alpha z+\beta}{3}}\right)\left(f(z) - \omega^2 e^{\frac{\alpha z+\beta}{3}}\right),$$

we deduce that all the zeros of each of the following functions

$$\left(f(z) - e^{\frac{\alpha z+\beta}{3}}\right), \left(f(z) - \omega e^{\frac{\alpha z+\beta}{3}}\right) \text{ and } \left(f(z) - \omega^2 e^{\frac{\alpha z+\beta}{3}}\right)$$

are of multiplicities at least 3.

By Yamanoi's Second Fundamental Theorem (see [32]), we obtain

$$\begin{aligned}
2T(r,f) &\leq \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{\left(f(z)-e^{\frac{\alpha z+\beta}{3}}\right)}\right) + \bar{N}\left(r, \frac{1}{\left(f(z)-\omega e^{\frac{\alpha z+\beta}{3}}\right)}\right)\\
&\quad + \bar{N}\left(r, \frac{1}{\left(f(z)-\omega^2 e^{\frac{\alpha z+\beta}{3}}\right)}\right) + S(r,f)\\
&\leq \frac{1}{3}N\left(r, \frac{1}{\left(f(z)-e^{\frac{\alpha z+\beta}{3}}\right)}\right) + \frac{1}{3}N\left(r, \frac{1}{\left(f(z)-\omega e^{\frac{\alpha z+\beta}{3}}\right)}\right)\\
&\quad + \frac{1}{3}N\left(r, \frac{1}{\left(f(z)-\omega^2 e^{\frac{\alpha z+\beta}{3}}\right)}\right) + N(r,f) + S(r,f)\\
&\leq T(r,f) + T\left(r, e^{\alpha z}\right) + N(r,f) + S(r,f)\\
&\leq T(r,f) + +N(r,f) + S(r,f)
\end{aligned}$$

which implies that $T(r,f) = N(r,f) + S(r,f)$. It leads to $m(r,f) = S(r,f) = S(r,\wp(h))$. On the other hand, the form of the function f in (4.1) shows that

$$\frac{1}{2\wp(h(z))} = f(z)e^{-\frac{\alpha z+\beta}{3}} - \frac{\wp'(h(z))}{2\sqrt{3}\wp(h(z))}.$$

Therefore, by the lemma of the logarithmic derivative, it is easy to see that

$$\begin{aligned}
m\left(r, \frac{1}{\wp(h(z))}\right) &= m\left(r, \frac{1}{2\wp(h(z))}\right) + O(1) \qquad (4.17)\\
\leq m(r,f) + m\left(r, e^{-\frac{\alpha z+\beta}{3}}\right) &+ m\left(r, \frac{h'(z)\wp'(h(z))}{\wp(h(z))}\right) + m\left(r, \frac{1}{h'(z)}\right) + O(1)\\
&\leq T\left(r, e^{-\frac{\alpha z+\beta}{3}}\right) + T\left(r, \frac{1}{h'(z)}\right) + S(r,\wp(h(z)))\\
&\leq T\left(r, e^{\alpha z}\right) + T(r,h'(z)) + S(r,\wp(h(z))) \leq S(r,\wp(h(z))).
\end{aligned}$$

Combining equations (4.15) and (4.17) and observing that each pole of $\wp(z)$ is of multiplicity is exactly 2 (so that each pole $P(h)$ has multiplicity $2k$ for some integer $k \geq 1$), by applying Theorem 2.1 of Chiang and Feng [5], we obtain

$$\begin{aligned}
&T(r,\wp(h(z))) = T\left(r, \frac{1}{\wp(h(z))}\right) + O(1)\\
&= m\left(r, \frac{1}{\wp(h(z))}\right) + N\left(r, \frac{1}{\wp(h(z))}\right) + O(1)\\
&\leq \bar{N}\left(r, \frac{1}{\wp(h(z))}\right) + S(r,\wp(h(z))) \leq \bar{N}(r,\wp(h(z+c))) + S(r,\wp(h(z)))\\
&\leq \frac{1}{2}N(r,\wp(h(z+c))) + S(r,\wp(h(z))) \leq \frac{1}{2}T(r,\wp(h(z+c))) + S(r,\wp(h(z)))\\
&\leq \frac{1}{2}T(r,\wp(h(z))) + S(r,\wp(h(z))) + O\left(r^{\rho(\wp(h))-1+\epsilon}\right)
\end{aligned}$$

which yields that $T(r,\wp(h)) \leq S(r,\wp(h(z)))+O\left(r^{\rho(\wp(h))-1+\epsilon}\right)$. Therefore, we arrive at a contradiction. The proof of the theorem is complete. $\square$

Proof of Theorem 2.2. For the details of proof of Theorem 2.2, we discuss here the case $n=1$ only because the cases $n \geq 2$ will follow from Theorem F of Han and Lü [16]. For $n=1$, equation (2.3) becomes

$$f(z)+f^{(k)}(z)=e^{\alpha z+\beta}. \tag{4.18}$$

The general solution of the differential equation (4.18) consist of two parts: one is complementary function $f_c(z)$ and the other is particular solution $f_p(z)$. The auxiliary equation here is $m^k+1=0$ which implies $m=\theta,\theta^2,\ldots,\theta^{k-1}$. It is easy to see that m can take value -1 also for the case when k is odd. Therefore, we have $f_c(z)=\sum_{j=1}^{k} a_j e^{\theta^j z}$, where a_j's are complex constants. Let us denote the differential operator D as $D \equiv d/dz$. Then equation (4.18) can be expressed as $\left(D^k+1\right)f(z)=e^{\alpha z+\beta}$. Therefore, we have

$$f_p(z)=\frac{1}{D^k+1}e^{\alpha z+\beta}.$$

If $\alpha \notin \{\theta,\theta^2,\ldots,\theta^{k-1}\}$, then a simple computations shows that the particular solution in this case is $f_p(z)=e^{\alpha z+\beta}/(\alpha^k+1)$. Hence the general solution is

$$f(z)=f_c(z)+f_p(z)=\sum_{j=1}^{k} a_j e^{\theta^j z}+\frac{ze^{\alpha z+\beta}}{\alpha^k+1}.$$

If $\alpha \in \{\theta,\theta^2,\ldots,\theta^{k-1}\}$, then we see that $\alpha^k=-1$. Therefore, we have

$$\begin{aligned}
f_p(z)&=\frac{1}{D^k+1}e^{\alpha z+\beta}=e^{\alpha z+\beta}\frac{1}{(D+\alpha)^k+1}(1)\\
&=e^{\alpha z+\beta}\frac{1}{D^k+\binom{k}{1}D^{k-1}\alpha+\binom{k}{2}D^{k-1}\alpha^2+\ldots+\binom{k}{k-1}D\alpha^{k-1}}(1)\\
&=e^{\alpha z+\beta}\frac{1}{\binom{k}{k-1}D\alpha^{k-1}}\left(1+\frac{1}{\binom{k}{k-1}\alpha^{k-1}}\left(D^{k-1}+\binom{k}{1}D^{k-2}+\ldots+1\right)\right)^{-1}(1)\\
&=e^{\alpha z+\beta}\frac{1}{\binom{k}{k-1}\alpha^{k-1}}\frac{1}{D}(1)=\frac{ze^{\alpha z+\beta}}{k\alpha^{k-1}}.
\end{aligned}$$

Hence, the general solution is

$$f(z)=f_c(z)+f_p(z)=\sum_{j=1}^{k} a_j e^{\theta^j z}+\frac{ze^{\alpha z+\beta}}{k\alpha^{k-1}}.$$

When in particular $\alpha=-1$, this case can be handled easily considering k as odd or even separately. $\square$

Proof of Theorem 2.3. We split the whole proof into the follows two cases.

Case 1. Let the solution f be a transcendental entire function. Let us first consider the exponential case *i.e.,* $f(z) = de^{P(z)}$, where $P(z)$ is a polynomial in z. Then we have

$$d^2\left(e^{2P(z)-(\alpha z+\beta)} + e^{2P(z+c)-(\alpha z+\beta)}\right) = 1. \tag{4.19}$$

A simple computations shows that both the functions $2P(z) - (\alpha z + \beta)$ and $2P(z + c) - (\alpha z + \beta)$ must be constants, say, c_1 and c_2, respectively. Then an elementary calculation shows that

$$\alpha c = c_2 - c_1 = 2\left(P(z + c) - P(z)\right). \tag{4.20}$$

By the assumption, f is a finite order entire function and in view of (4.20), $\deg(P)$ must be equal to 1. Hence we can show that $P(z)$ takes the form $P(z) = (\alpha z + \beta)/2$. Thus it follows from (4.19) that $d^2 = 1/e^{\alpha c}$ with $d \neq \pm 1$ and α, c be such that $e^{\alpha c} \neq -1$.

Let $f(z)$ is not of the form $f(z) = de^{P(z)}$. We know from the result of Gross that any entire solution of $f^2(z) + g^2(z) = 1$ is of the form $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where h is a an entire function.

The difference equation $f(z)^2 + f^2(z + c) = e^{\alpha z+\beta}$ can be written as

$$\left(\frac{f(z)}{e^{\frac{\alpha z+\beta}{2}}}\right)^2 + \left(\frac{f(z+c)}{e^{\frac{\alpha z+\beta}{2}}}\right)^2 = 1.$$

Therefore, by the result of Gross [8], it is easy to see that the general solution of $f(z)^2 + f^2(z + c) = e^{\alpha z+\beta}$ must be

$$f(z) = e^{\frac{\alpha z+\beta}{2}} \sin\left(h(z)\right) \quad \text{and} \quad f(z + c) = e^{\frac{\alpha z+\beta}{2}} \cos\left(h(z)\right)$$

for an entire function h. Therefore, we obtain $h(z + c) = h(z) + 2k\pi + \pi/2$ and $e^{\alpha c/2} = 1$, where k is an integer. Writing $h(z) = (4k + 1)\pi z/2c + \mathcal{H}(z)$, it is easy to verify that $\mathcal{H}(z)$ is a c-periodic entire function. Therefore, the general non-constant entire solution can be written as

$$f(z) = e^{\frac{\alpha z+\beta}{2}} \sin\left(\frac{(4k + 1)\pi z}{2c} + \mathcal{H}(z)\right).$$

In particular, if f is a finite order transcendental entire function, then by Pólya's theorem [25], the function $\mathcal{H}(z)$ must be constant, say, η. Hence, the general non-constant transcendental entire solution becomes

$$f(z) = e^{\frac{\alpha z+\beta}{2}} \sin\left(\frac{(4k + 1)\pi z}{2c} + \eta\right).$$

Case 2. Let f be a meromorphic function.

The difference equation $f(z)^2 + f^2(z+c) = e^{\alpha z+\beta}$ can be written as

$$[f(z)+if(z+c)][f(z)-if(z+c)] = e^{\alpha z+\beta}. \tag{4.21}$$

From (4.21), it is easy to see that the functions $[f(z)+if(z+c)]$ and $[f(z)-if(z+c)]$ may have zeros and poles. Therefore, there exists a meromorphic function g and a complex number δ such that $[f(z)+if(z+c)]$ and $[f(z)-if(z+c)]$ can be expressed as

$$f(z)+if(z+c) = e^{\delta(\alpha z+\beta)}g(z) \tag{4.22}$$

and

$$f(z)-if(z+c) = e^{(1-\delta)(\alpha z+\beta)}\frac{1}{g(z)}. \tag{4.23}$$

Solving equations (4.22) and (4.23) for $f(z)$ and $f(z+c)$, we obtain

$$f(z) = \frac{1}{2}\left(e^{\delta(\alpha z+\beta)}g(z) + \frac{e^{(1-\delta)(\alpha z+\beta)}}{g(z)}\right) \tag{4.24}$$

and

$$f(z+c) = \frac{1}{2i}\left(e^{\delta(\alpha z+\beta)}g(z) - \frac{e^{(1-\delta)(\alpha z+\beta)}}{g(z)}\right). \tag{4.25}$$

Combining (4.24) and (4.25), it is easy to see that

$$\begin{aligned} &e^{\delta(\alpha z+\beta)}e^{\alpha\delta c}g(z+c) + \frac{e^{(1-\delta)(\alpha z+\beta)}e^{\alpha(1-\delta)c}}{g(z+c)} \\ &= -i\left(e^{\delta(\alpha z+\beta)}g(z) - \frac{e^{(1-\delta)(\alpha z+\beta)}}{g(z)}\right). \end{aligned} \tag{4.26}$$

Clearly, (4.26) shows that the functions $g(z)$ and $g(z+c)$ have the same set of zeros and poles with the same multiplicities, otherwise, comparing the zeros and poles of $g(z)$ and $g(z+c)$ from both sides of (4.26), we can arrive at a contradiction. Therefore, there exists a polynomial $\mathcal{Q}(z)$ in z such that

$$\frac{g(z+c)}{g(z)} = e^{\mathcal{Q}(z)}. \tag{4.27}$$

If $e^{\mathcal{Q}(z)} \equiv 1$, then g becomes a c-periodic function. Now equating the coefficients in (4.26), we obtain,

$$ie^{\delta\alpha c} = 1 \text{ and } ie^{(1-\delta)\alpha c} = -1.$$

Therefore, we have $e^{\alpha c} = 1$ and $e^{\delta\alpha c} = -i$, which shows that $\delta = 1/4$ or $3/4$. Hence the possible forms of the function f is one of the following:

$$\begin{cases} f(z) = \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(g(z) + \dfrac{e^{\frac{1}{2}(\alpha z+\beta)}}{g(z)}\right) \\ f(z) = \dfrac{e^{\frac{1}{4}(\alpha z+\beta)}}{2}\left(e^{\frac{1}{2}(\alpha z+\beta)}g(z) + \dfrac{1}{g(z)}\right). \end{cases}$$

If $e^{\mathcal{Q}(z)} \neq 1$, then substituting $g(z+c) = e^{\mathcal{Q}(z)} g(z)$ in (4.26), we obtain that

$$g^2(z) e^{(2\delta-1)(\alpha z+\beta)} = -\frac{ie^{(1-\delta)\alpha c} + e^{\mathcal{Q}(z)}}{e^{\mathcal{Q}(z)} \left(ie^{\delta\alpha c} - 1\right)}. \tag{4.28}$$

Clearly, the function g in (4.28) cannot have any poles, hence g must be a transcendental entire function. But note that, all the zeros of $ie^{(1-\delta)\alpha c} + e^{\mathcal{Q}(z)}$ are the zeros of $g(z)$ are of multiplicities at least 2, which leads to a contradiction. This completes the proof. □

5. Future study

To continue the study, one can turn attention to the solutions of more general Fermat-type equations. For example, Ramanujan observed that $x = 9$, $y = 10$ and $z = -12$ is a solution of $x^n + y^n + z^n = 1$ for the case $n = 3$. Therefore, looking for the solutions of equation $x^n + y^n + z^n = 1$ for $n \geq 4$ will of great interests, and the study will become more effective if x, y and z be non-constant functions. Since the problem of finding solutions of (1.1) have been settled for the classes (i)-(iv) mentioned above, it is therefore natural to turn attention to the functional equation

$$f^n + g^n + h^n = 1, \tag{5.1}$$

where n is a positive integer and f, g and h are functions in any one of the above four function classes.

Finding non-constant entire as well as meromorphic solutions are effortless for $n = 1$. For example, for $n = 2$, one can verify that

$$(f, g, h) = (\sin(\phi)\cos(\psi), \sin(\phi)\sin(\psi), \cos(\phi))$$

is an immediate entire solution and

$$(f, g, h) = (i\sin(\phi)\tan(\phi), i\cos(\phi)\tan(\phi), \sec(\phi))$$

is a meromorphic solution of the equation (5.1), where ϕ and ψ are two entire functions. For $n \geq 3$, looking for non-constant entire as well as meromorphic solutions will be of utmost interest. For future course of work and to study Fermat-type functional equations, we refer the reader to go through the article of Gundersen [13] and references there in.

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О ВЕСОВЫХ РЕШЕНИЯХ $\overline{\partial}$-УРАВНЕНИЯ В ВЕРХНЕЙ ПОЛУПЛОСКОСТИ

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Аннотация. В статье рассматривается уравнение $\partial f(w)/\partial\overline{w} = u(w)$ в верхней полуплоскости Π_+. Для функций u класса C^k $(k = 1, 2, 3, \ldots, \infty)$ из весовых L^p пространств $(1 \le p < \infty)$ с весовой функций типа $(Imw)^\alpha \cdot |w+i|^{-\gamma}$, $w \in \Pi_+$, строится семейство решений f_β, зависящее от комплексного параметра β.

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Ключевые слова: $\overline{\partial}$-уравнение; весовые пространства гладких функций.

1. Введение

В работе [1] приводится обобщение интегральной формулы Коши для гладких функций. А именно, если Ω является ограниченной областью с кусочно-гладкой границей и $f \in C^1(\overline{\Omega})$, то справедлива следующая формула:

$$f(z) = \frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\iint\limits_{\Omega} \frac{\frac{\partial f(\zeta)}{\partial\overline{\zeta}}}{\zeta - z}dm(\zeta), \quad z \in \Omega, \tag{1.1}$$

где m - двумерная мера Лебега в комплексной области, а

$$\frac{\partial}{\partial\overline{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad (\zeta = x + iy) \tag{1.2}$$

представляет собой известный оператор Коши-Римана, обращающийся в 0 на голоморфных функциях. Поскольку первое слагаемое в (1.1) голоморфно в Ω, мы можем заключить, что решение $\overline{\partial}$-уравнения

$$\frac{\partial g(z)}{\partial\overline{z}} = v(z), \quad z \in \Omega, \tag{1.3}$$

где функция $v \in C^1(\Omega)$ задана, а функция $g \in C^1(\Omega)$ искомая, **может быть** представлено как

$$g(z) = -\frac{1}{\pi} \cdot \iint\limits_{\Omega} \frac{v(\zeta)}{\zeta - z} dm(\zeta), \quad z \in \Omega. \tag{1.4}$$

Уравнение (1.3) играет важную роль во многих задачах комплексного анализа (особенно в случае многих комплексных переменных).

В следующей теореме (см. [2, Теорема 1.2.2]) рассматривается важный случай, когда формула (1.4) действительно даёт решение $\overline{\partial}$-уравнения.

Теорема 1.1. *Пусть* Ω*- открытое ограниченное множество в* $\mathbb{C}$*,* $k = 1, 2, 3, ..., \infty$ *и* $v \in C_c^k(\Omega)$*, т.е. функция* $v \in C^k(\Omega)$ *и имеет компактный носитель, целиком находящийся в* Ω*. Тогда функция* g*, определяемая формулой* (1.4)*, принадлежит* $C^k(\Omega)$ *и удовлетворяет уравнению* (1.3)*.*

Замечание. В [3, Предложение 16.3.2], [4, Теорема 1.1.3] рассмотрен случай, когда $v \in C^k(\Omega) \cap L^\infty(\Omega)$ или $v \in C^k(\Omega) \cap L^1(\Omega)$.

В работах [5], [6] отмечается следующее обобщение формулы (1.1) для единичного круга $\mathbb{D} = \{\zeta : |\zeta| < 1\}$ $(Re\beta > -1)$:

$$\begin{aligned} f(z) = {} & \frac{\beta + 1}{\pi} \iint\limits_{\mathbb{D}} \frac{f(\zeta)(1 - |\zeta|^2)^\beta}{(1 - z\overline{\zeta})^{2+\beta}} dm(\zeta) \\ & - \frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{\dfrac{\partial f(\zeta)}{\partial \overline{\zeta}}}{\zeta - z} \cdot \left(\frac{1 - |\zeta|^2}{1 - z\overline{\zeta}} \right)^{\beta+1} dm(\zeta), \quad z \in \mathbb{D}, \end{aligned} \tag{1.5}$$

где первое слагаемое голоморфно по $z \in \mathbb{D}$ и впервые появилось в работах [7], [8], где рассмотрены классы голоморфных в $\mathbb{D}$ функций из весовых пространств $L^p_\alpha(\mathbb{D})$, порождённых нормой

$$\|f\|_{p,\alpha} = \left(\iint\limits_{\mathbb{D}} |f(\zeta)|^p (1 - |\zeta|)^\alpha dm(\zeta) \right)^{\frac{1}{p}}. \tag{1.6}$$

Естественно, по аналогии с (1.4), второе слагаемое в (1.5) может быть использовано в качестве формульного решения уравнения (1.3):

$$g_\beta(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{v(\zeta)}{\zeta - z} \cdot \left(\frac{1 - |\zeta|^2}{1 - z\overline{\zeta}} \right)^{\beta+1} dm(\zeta), \quad z \in \mathbb{D}. \tag{1.7}$$

На самом деле справедливо следующее утверждение:

Теорема 1.2. *Пусть* $1 \le p < +\infty$, $\alpha > -1$ *и* $Re\beta > \alpha$. *Если* $v \in C^1(\mathbb{D}) \cap L^p_{\alpha+1}(\mathbb{D})$, *то функция* g_β, *определяемая по формуле* (1.7), *принадлежит* $C^1(\mathbb{D}) \cap L^p_\alpha(\mathbb{D})$ *и удовлетворяет уравнению* (1.3). *Более того,*

$$\|g_\beta\|_{p,\alpha} \le const(\alpha,\beta) \cdot \|v\|_{p,\alpha+1}. \tag{1.8}$$

Эта теорема является следствием соответствующих многомерных результатов работы [5], где рассматриваются случаи единичного шара $B_n \subset \mathbb{C}^n$ и единичного полидиска $U^n \subset \mathbb{C}^n$.

Отметим, что различные многомерные аналоги формулы (1.5) были получены в [9], [10].

Дальнейшие обобщения формулы (1.5) для единичного круга $\mathbb{D}$ были получены в [11], [12], [13], [14] (при различных условиях, налагаемых на $f(\zeta)$ и $\partial f(\zeta)/\partial\overline{\zeta}$) и могут быть записаны следующим образом:

$$\begin{aligned} f(z) = & \iint\limits_{\mathbb{D}} f(\zeta) S_{\beta,\rho,\varphi}(z;\zeta) \cdot (1-|\zeta|^{2\rho})^\beta \cdot |\zeta|^{2\varphi} dm(\zeta) \\ & - \frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{\frac{\partial f(\zeta)}{\partial\overline{\zeta}}}{\zeta - z} \cdot Q_{\beta,\rho,\varphi}(z;\zeta) dm(\zeta), \qquad z \in \mathbb{D}, \end{aligned} \tag{1.9}$$

где ядра S и Q записываются в явной форме (в виде интегралов или рядов).
В работах [12], [13], [15] было установлено, что формула второго слагаемого в (1.9) (с заменой $\frac{\partial f(\zeta)}{\partial\overline{\zeta}}$ на $v(\zeta)$) порождает семейство решений уравнения (1.3) в $\mathbb{D}$.

Формулы типа (1.1), (1.5) представляют интерес так же в случае неограниченных областей. В случае верхней полуплоскости Π_+ справедлива следующая формула, являющаяся следствием соответствующего многомерного результата [16, Теорема 2.2]:

$$\begin{aligned} f(w) = & \frac{2^\beta(\beta+1)}{\pi} \iint\limits_{\Pi_+} \frac{f(\eta)(Im\eta)^\beta}{[i(\overline{\eta}-w)]^{2+\beta}} dm(\eta) \\ & - \frac{2^{\beta+1}}{\pi} \iint\limits_{\Pi_+} \frac{\partial f(\eta)/\partial\overline{\eta}}{\eta - w} \cdot \frac{(Im\eta)^{\beta+1}}{[i(\overline{\eta}-w)]^{\beta+1}} dm(\eta), \quad w \in \Pi_+, \end{aligned} \tag{1.10}$$

где $f(\eta)$ и $\frac{\partial f(\eta)}{\partial\overline{\eta}}$ принадлежат определённым весовым L^p-пространствам в верхней полуплоскости.

В случае голоморфных функций, когда второе слагаемое отсутствует, формула (1.10) получена в [17], [18].

В настоящей работе будет показано, что формула второго слагаемого в (1.10) даёт семейство решений уравнения (1.3) при определённых условиях, налагаемых на правую часть.

Отметим, что в [19] приводятся решения уравнения (1.3) в Π_+ в предположении, что правая часть суть комплексная мера Карлесона, решения понимаются в смысле обобщённых функций, при этом решения записываются в виде нелинейных интегральных операторов.

В [20], [21] в многомерном случае приводятся решения $\overline{\partial}$-уравнения с равномерными оценками в трубе будущего (многомерном аналоге Π_+), но при этом обязательно предполагается, что правая часть имеет ограниченный носитель.

2. Предварительные результаты

Начнём с простых утверждений, доказательство которых не представляет труда.

Предложение 2.1. *Пусть* $\Omega \subset \mathbb{C}$, $\Omega_1 \subset \mathbb{C}$, $\varphi : \Omega \to \Omega_1$, $f : \Omega_1 \to \mathbb{C}$ *и* $\varphi(\zeta) \in H(\Omega)$, $f(\eta) \in C^1(\Omega_1)$. *Тогда*

$$\frac{\partial f(\varphi(\zeta))}{\partial \overline{\zeta}} = \frac{\partial f(\eta)}{\partial \overline{\eta}}\bigg|_{\eta=\varphi(\zeta)} \cdot \overline{\varphi'(\zeta)}.$$

Предложение 2.2. *Пусть* $\eta, w \in \Pi_+$, *тогда:*

(a) $Re[i(\overline{\eta} - w)] = Im\eta + Imw > 0$.

(б) $|i(\overline{\eta} - w)|^2 \geq (Im\eta + Imw)^2 \geq 4 \cdot Im\eta \cdot Imw$, *причём равенство достигается только при* $\eta = w$.

Напомним, что биголоморфный изоморфизм единичного круга $\mathbb{D}$ и верхней полуплоскости Π_+ осуществляется посредством известных преобразований Кэли:

$$\varPhi(\zeta) = i \cdot \frac{1+\zeta}{1-\zeta}, \quad \zeta \in \mathbb{D}, \qquad \varPhi^{-1}(\eta) = \frac{\eta - i}{\eta + i}, \quad \eta \in \Pi_+. \tag{2.1}$$

В следующем утверждении приводятся основные свойства преобразований Кели, необходимые для дальнейшего.

Предложение 2.3. *Пусть* $\zeta \in \mathbb{D}$ *и* $\eta \in \Pi_+$, *тогда:*

$$1^\circ. \quad \varPhi'(\zeta) = \frac{2i}{(1-\zeta)^2}, \qquad \left(\varPhi^{-1}(\eta)\right)' = \frac{2i}{(\eta+i)^2}. \tag{2.2}$$

$$2^\circ. \quad dm(\varPhi(\zeta)) = \frac{4}{|1-\zeta|^4} dm(\zeta), \qquad dm(\varPhi^{-1}(\eta)) = \frac{4}{|\eta+i|^4} dm(\eta). \tag{2.3}$$

3°. Если $\eta = \varPhi(\zeta)$, то

$$1 - |\zeta|^2 = \frac{4Im\eta}{|\eta+i|^2}, \qquad Im\eta = \frac{1-|\zeta|^2}{|1-\zeta|^2}. \tag{2.4}$$

4°. Если $w, \eta \in \Pi_+$, то

$$1 - \Phi^{-1}(w) \cdot \overline{\Phi^{-1}(\eta)} = \frac{2i(\overline{\eta} - w)}{(w+i)(\overline{\eta} - i)}, \tag{2.5}$$

$$1 - |\Phi^{-1}(\eta)|^2 = \frac{4Im\eta}{|\eta + i|^2}, \tag{2.6}$$

$$\Phi^{-1}(w) - \Phi^{-1}(\eta) = \frac{2i(w - \eta)}{(w+i)(\eta+i)}. \tag{2.7}$$

Доказательство. Соотношения (2.2), (2.3) и (2.5), (2.6) следуют из [22, Лемма 1.1], а (2.4) и (2.7) проверяются непосредственно.

Предложение 2.4. *Пусть $w_0 \in \Pi_+$ фиксировано и $Im(w_0) > r_1 > r_2 > 0$. Положим $G(w_0; r_1) \equiv \{w : |w - w_0| < r_1\}$ и $G(w_0; r_2) \equiv \{w : |w - w_0| < r_2\}$. Тогда при $\eta \in \overline{\Pi_+} \backslash G(w_0; r_1)$ и $w \in \overline{G(w_0; r_2)}$ имеем*

$$|\eta - w| \asymp |\eta + i|, \quad \text{т.е.} \quad 0 < A \leq \frac{|\eta - w|}{|\eta + i|} \leq B < +\infty. \tag{2.8}$$

Доказательство. Возьмём $R > 0$ так, что $\overline{G(w_0; r_1)} \subset \{w : |w| \leq R\}$ и при этом $\left|\frac{i}{\eta}\right| < \frac{1}{2}$ и $\left|\frac{w}{\eta}\right| < \frac{1}{2}$ при $|\eta| > R$ и $w \in \overline{G(w_0; r_2)}$. Тогда имеем:

$$\frac{|\eta - w|}{|\eta + i|} = \frac{\left|\frac{\eta}{|\eta|} - \frac{w}{|\eta|}\right|}{\left|\frac{\eta}{|\eta|} + \frac{i}{|\eta|}\right|} \leq \frac{3/2}{1/2} = 3$$

и

$$\frac{|\eta - w|}{|\eta + i|} = \frac{\left|\frac{\eta}{|\eta|} - \frac{w}{|\eta|}\right|}{\left|\frac{\eta}{|\eta|} + \frac{i}{|\eta|}\right|} \geq \frac{1/2}{3/2} = \frac{1}{3}.$$

Если же $|\eta| \leq R$, то рассмотрим отношение $\dfrac{|\eta - w|}{|\eta + i|}$ как функцию двух переменных η и w, которая непрерывна и положительна на компактном множестве $\{(\eta, w) \in \mathbb{C}^2 : \eta \in \overline{\Pi_+} \backslash G(w_0; r_1), |\eta| \leq R \text{ и } w \in \overline{G(w_0; r_2)}\}$. Следовательно, на этом множестве рассматриваемое отношение находится между двумя фиксированными положительными числами. Утверждение доказано.

Определение 2.1. *Для произвольных функций $g(\zeta)$, $\zeta \in \mathbb{D}$, и $f(\eta)$, $\eta \in \Pi_+$, и для $\beta \in \mathbb{C}$ будем писать*

$$g \overset{\beta}{\sim} f \quad \text{или} \quad f \overset{\beta}{\sim} g$$

если

$$g(\zeta) \equiv \frac{f(\Phi(\zeta))}{(1-\zeta)^{2+\beta}}, \quad \zeta \in \mathbb{D}, \tag{2.9}$$

или, что то же самое,

$$f(\eta) \equiv g(\Phi^{-1}(\eta)) \cdot \left(\frac{2i}{\eta + i}\right)^{2+\beta}, \quad \eta \in \Pi_+. \tag{2.10}$$

Очевидно, что если $g \overset{\beta}{\sim} f$, то условие $g \in C^k(\mathbb{D})$ эквивалентно условию $f \in C^k(\Pi_+)$ ($k = 1, 2, 3, \ldots, \infty$). Кроме того, если $g \overset{\beta}{\sim} f_1$ и $g \overset{\beta}{\sim} f_2$, то $f_1 \equiv f_2$ в Π_+ и наоборот: если $g_1 \overset{\beta}{\sim} f$ и $g_2 \overset{\beta}{\sim} f$, то $g_1 \equiv g_2$ в $\mathbb{D}$.

Определение 2.2. *Для произвольных функций $v(\zeta)$, $\zeta \in \mathbb{D}$, и $u(\eta)$, $\eta \in \Pi_+$, и для $\beta \in \mathbb{C}$ будем писать*

$$v \overset{\beta}{\approx} u \quad \text{или} \quad u \overset{\beta}{\approx} v$$

если

$$v(\zeta) = \frac{u(\Phi(\zeta))}{(1-\zeta)^{2+\beta}} \cdot \frac{-2i}{(1-\bar{\zeta})^2}, \quad \zeta \in \mathbb{D}, \tag{2.11}$$

или, что то же самое,

$$u(\eta) = v(\Phi^{-1}(\eta)) \cdot \left(\frac{2i}{\eta + i}\right)^{2+\beta} \cdot \frac{-2i}{(\overline{\eta} - i)^2}, \quad \eta \in \Pi_+. \tag{2.12}$$

Очевидно, что если $v \overset{\beta}{\approx} u$, то условие $v \in C^k(\mathbb{D})$ эквивалентно условию $u \in C^k(\Pi_+)$ ($k = 1, 2, 3, \ldots, \infty$). Кроме того, если $v \overset{\beta}{\approx} u_1$ и $v \overset{\beta}{\approx} u_2$, то $u_1 \equiv u_2$ в Π_+ и наоборот: если $v_1 \overset{\beta}{\approx} u$ и $v_2 \overset{\beta}{\approx} u$, то $v_1 \equiv v_2$ в $\mathbb{D}$.

Предложение 2.5. *Пусть $g \in C^k(\mathbb{D}), f \in C^k(\Pi_+)$, $k = 1, 2, 3, \ldots, \infty$ и $g \overset{\beta}{\sim} f$ ($\beta \in \mathbb{C}$). Тогда для функций $v(\zeta), \zeta \in \mathbb{D}$, и $u(\eta), \eta \in \Pi_+$, справедливы следующие утверждения:*

(а) Если $v(\zeta) \equiv \dfrac{\partial g(\zeta)}{\partial \bar{\zeta}}$ и $u(\eta) \equiv \dfrac{\partial f(\eta)}{\partial \overline{\eta}}$, то $v \overset{\beta}{\approx} u$.

(б) Если $v \overset{\beta}{\approx} u$, тогда равенство $v(\zeta) \equiv \dfrac{\partial g(\zeta)}{\partial \bar{\zeta}}$ эквивалентно равенству $u(\eta) \equiv \dfrac{\partial f(\eta)}{\partial \overline{\eta}}$.

Доказательство. Очивидно, что (б) является следствием (а).
Для доказательства (а) воспользуемся соотношениями (2.9) и (2.10). Тогда ввиду Предложения 2.1

$$v(\zeta) \equiv \frac{\partial g(\zeta)}{\partial \bar{\zeta}} = \frac{\left.\frac{\partial f(\eta)}{\partial \overline{\eta}}\right|_{\eta = \Phi(\zeta)}}{(1-\zeta)^{2+\beta}} \cdot \frac{-2i}{(1-\bar{\zeta})^2},$$
$$u(\eta) \equiv \frac{\partial f(\eta)}{\partial \overline{\eta}} = \left(\frac{2i}{\eta + i}\right)^{2+\beta} \cdot \left.\frac{\partial g(\zeta)}{\partial \bar{\zeta}}\right|_{\zeta = \Phi^{-1}(\eta)} \cdot \frac{-2i}{(\overline{\eta} - i)^2}.$$

Чтобы установить соотношение $v \overset{\beta}{\approx} u$, достаточно показать, в силу (2.11), что

$$v(\zeta) \equiv \frac{u(\Phi(\zeta))}{(1-\zeta)^{2+\beta}} \cdot \frac{-2i}{(1-\overline{\zeta})^2}.$$

Действительно,

$$\frac{u(\Phi(\zeta))}{(1-\zeta)^{2+\beta}} \cdot \frac{-2i}{(1-\overline{\zeta})^2}$$

$$= \frac{1}{(1-\zeta)^{2+\beta}} \cdot \frac{-2i}{(1-\overline{\zeta})^2} \cdot \left(\frac{2i}{i\frac{1+\zeta}{1-\zeta}+i}\right)^{2+\beta} \cdot \frac{\partial g(\zeta)}{\partial \overline{\zeta}} \cdot \frac{-2i}{\left(-i\frac{1+\overline{\zeta}}{1-\overline{\zeta}}-i\right)^2} \equiv \frac{\partial g(\zeta)}{\partial \overline{\zeta}} \equiv v(\zeta).$$

При $Re\beta > -1$ введём два интегральных оператора.

Для комплекснозначной измеримой функции $v(\zeta), \zeta \in \mathbb{D}$, формально положим

$$T_\beta(v)(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{v(\zeta)}{\zeta - z} \left(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\right)^{\beta+1} dm(\zeta), \quad z \in \mathbb{D}. \tag{2.13}$$

Для комплекснозначной измеримой функции $u(\eta), \eta \in \Pi_+$, формально положим

$$T^*_\beta(u)(w) = -\frac{2^{\beta+1}}{\pi} \cdot \iint\limits_{\Pi_+} \frac{u(\eta)}{\eta - w} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta}-w))^{\beta+1}} dm(\eta), \quad w \in \Pi_+. \tag{2.14}$$

Заметим, что (2.13) и (2.14) вполне соответствуют структуре вторых слагаемых в итегральных представлениях (1.5) и (1.10).

Теорема 2.1. *Если $v \overset{\beta}{\approx} u$, то $T_\beta(v) \overset{\beta}{\sim} T^*_\beta(u)$ при условии, что для заданных функций v и u соответствующие интегралы в* (2.13) *и* (2.14) *абсолютно сходятся.*

Доказательство. По ходу доказательства будет показано, что при условиях теоремы абсолютная сходимость одного из интегралов в (2.13) и (2.14) влечёт абсолютную сходимость другого.

В силу (2.10) нужно доказать, что

$$T^*_\beta(u)(w) \equiv T_\beta(v)(\Phi^{-1}(w)) \cdot \left(\frac{2i}{w+i}\right)^{2+\beta}, \quad w \in \Pi_+.$$

Действительно, в силу (2.3) - (2.7)

$$T_\beta(v)(\Phi^{-1}(w)) \cdot \left(\frac{2i}{w+i}\right)^{2+\beta}$$

$$= -\frac{1}{\pi} \cdot \left(\frac{2i}{w+i}\right)^{2+\beta} \iint\limits_{\mathbb{D}} \frac{v(\zeta)}{\zeta - \Phi^{-1}(w)} \cdot \frac{(1-|\zeta|^2)^{\beta+1}}{(1-\Phi^{-1}(w)\cdot\overline{\zeta})^{\beta+1}} dm(\zeta)$$

$$= -\frac{1}{\pi} \cdot \left(\frac{2i}{w+i}\right)^{2+\beta} \iint\limits_{\Pi_+} \frac{v(\Phi^{-1}(\eta))}{\Phi^{-1}(\eta) - \Phi^{-1}(w)} \cdot \frac{(1-|\Phi^{-1}(\eta)|^2)^{\beta+1} dm(\Phi^{-1}(\eta))}{(1-\Phi^{-1}(w)\cdot\overline{\Phi^{-1}(\eta)})^{\beta+1}}$$

$$= -\frac{1}{\pi} \cdot \left(\frac{2i}{w+i}\right)^{2+\beta} \iint\limits_{\Pi_+} \frac{v(\Phi^{-1}(\eta))}{\frac{2i(\eta-w)}{(w+i)(\eta+i)}} \cdot \frac{\left(\frac{4Im\eta}{|\eta+i|^2}\right)^{\beta+1}}{\left(\frac{2i(\overline{\eta}-w)}{(w+i)(\overline{\eta}-i)}\right)^{\beta+1}} \cdot \frac{4dm(\eta)}{|\eta+i|^4}$$

$$= -\frac{1}{\pi} \cdot \left(\frac{2i}{w+i}\right)^{2+\beta} \cdot \frac{1}{2i} \cdot \frac{4^{\beta+2}}{2^{\beta+1}} \cdot \iint\limits_{\Pi_+} \frac{v(\Phi^{-1}(\eta))}{\eta-w} \cdot (w+i)(\eta+i)$$

$$\cdot \frac{(Im\eta)^{\beta+1}}{(\eta+i)^{\beta+3}(\overline{\eta}-i)^{\beta+3}} \cdot \frac{(w+i)^{\beta+1}(\overline{\eta}-i)^{\beta+1}}{(i(\overline{\eta}-w))^{\beta+1}} dm(\eta)$$

$$= -\frac{1}{\pi} \cdot (2i)^{2+\beta} \cdot 2^{\beta+2} \cdot \frac{1}{i} \cdot \iint\limits_{\Pi_+} \frac{v(\Phi^{-1}(\eta))}{\eta-w} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta}-w))^{\beta+1}}$$

$$\cdot \frac{1}{(\eta+i)^{\beta+2}} \cdot \frac{-2i}{(\overline{\eta}-i)^2} \cdot \frac{1}{-2i} dm(\eta)$$

$$= -\frac{2^{\beta+1}}{\pi} \cdot \iint\limits_{\Pi_+} \frac{u(\eta)}{\eta-w} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta}-w))^{\beta+1}} dm(\eta) \equiv T^*_\beta(u)(w), \quad w \in \Pi_+.$$

3. Весовые решения $\overline{\partial}$-уравнения

Начнём с общих рассуждений. Пусть $u \in C^k(\Pi_+), (k = 1, 2, 3 \ldots, \infty)$, тогда, как уже отмечалось, существует единственная функция v, так что $v \overset{\beta}{\approx} u$, причём $v \in C^k(\mathbb{D})$. Положим

$$g(z) = T_\beta(v)(z), \quad z \in \mathbb{D},$$

и пусть известно, что $g \in C^k(\mathbb{D})$ и

$$\frac{\partial g(z)}{\partial \overline{z}} \equiv v(z), \quad z \in \mathbb{D}.$$

Затем положим

$$f(w) = T^*_\beta(u)(w), \quad w \in \Pi_+.$$

Ввиду Теоремы 2.1 получаем, что $f \overset{\beta}{\sim} g$, и при этом $f \in C^k(\Pi_+)$. Более того, в силу Предложения 2.5 (б) имеем

$$u(w) \equiv \frac{\partial f(w)}{\partial \overline{w}}, \quad w \in \Pi_+.$$

Следовательно оператор T^*_β решает $\overline{\partial}$- уравнения в Π_+.

Важным следствием приведённых рассуждений является следующее утверждение.

Теорема 3.1. *Пусть* $u \in C_c^k(\Pi_+)$, $k = 1, 2, 3, \ldots, \infty$, $Re\beta > -1$. *Положим* $f_\beta(w) \equiv T_\beta^*(u)(w)$, $w \in \Pi_+$, *тогда*

$$f_\beta \in C^k(\Pi_+) \quad \text{и} \quad \frac{\partial f_\beta(w)}{\partial \overline{w}} \equiv u(w), \quad w \in \Pi_+. \tag{3.1}$$

Доказательство. Выберем $v(\zeta)$, $\zeta \in \mathbb{D}$ так, что $v \overset{\beta}{\approx} u$. Очевидно, имеем $v \in C_c^k(\mathbb{D})$. Положим

$$g(z) = T_\beta(v)(z), \quad z \in \mathbb{D}.$$

Тогда $g \in C^k(\mathbb{D})$ и $\frac{\partial g(z)}{\partial \overline{z}} \equiv v(z), z \in \mathbb{D}$ (это следует из [5], когда $k = 1$, и из [15], когда $k \geq 1$). В силу приведённых в начале параграфа рассуждений мы непосредственно получаем (3.1).

Теорема 3.2. *Пусть* $u \in C^k(\Pi_+)$, $k = 1, 2, 3, \ldots, \infty$, $Re\beta > -1$ *и*

$$\frac{|u(\eta)| \cdot (Im\eta)^{Re\beta+1}}{|\eta + i|^{Re\beta+2}} \in L^1(\Pi_+). \tag{3.2}$$

Положим $f_\beta(w) \equiv T_\beta^*(u)(w)$, $w \in \Pi_+$, *тогда имеет место* (3.1).

Доказательство. Очевидно, что достаточно установить (3.1) локально, то есть в окрестности произвольной точки из верхней полуплоскости. Зафиксируем произвольное $w_0 \in \Pi_+$ и пусть $Imw_0 > r_1 > r_2 > 0$. Положим $G_1 \equiv \{w : |w - w_0| < r_1\}$, $G_2 \equiv \{w : |w - w_0| < r_2\}$. Мы покажем, что (3.1) имеет место в G_2. Очевидно, что существует функция $\psi \in C_c^\infty(\mathbb{C})$ такая, что

$$\psi|_{G_2} \equiv 1, \tag{3.3}$$

$$\psi|_{\mathbb{C} \setminus G_1} \equiv 0, \tag{3.4}$$

$$\psi|_{G_1 \setminus G_2} \in [0, 1]. \tag{3.5}$$

Следовательно,

$$f_\beta(w) = -\frac{2^{\beta+1}}{\pi} \iint\limits_{\Pi_+} \frac{u(\eta)\psi(\eta)}{\eta - w} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta} - w))^{\beta+1}} dm(\eta)$$
$$- \frac{2^{\beta+1}}{\pi} \iint\limits_{\Pi_+} \frac{u(\eta)(1 - \psi(\eta))}{\eta - w} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta} - w))^{\beta+1}} dm(\eta) \equiv f_1(w) + f_2(w).$$

В силу Теоремы 3.1 имеем, что $f_1 \in C^k(\Pi_+)$ и $\dfrac{\partial f_1(w)}{\partial \overline{w}} \equiv u(w) \cdot \psi(w) \equiv u(w)$, $w \in G_2$. Если мы покажем, что f_2 голоморфна в G_2, то очевидным образом (3.1) будет установлено. Заметим, что

$$f_2(w) = -\frac{2^{\beta+1}}{\pi} \iint\limits_{\Pi_+\setminus G_2} \frac{u(\eta)(1-\psi(\eta))}{\eta - w} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta}-w))^{\beta+1}} dm(\eta), \quad w \in G_2.$$

Поскольку подинтегральное выражение в формуле f_2 голоморфно по $w \in G_2$ для любого фиксированного $\eta \in \Pi_+\setminus G_2$, достаточно найти $F(\eta) \in L^1(\Pi_+\setminus G_2)$ так, что

$$\left| \frac{u(\eta)(1-\psi(\eta))}{(\eta - w)} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta}-w))^{\beta+1}} \right| \le F(\eta),$$

для любого $\eta \in \Pi_+\setminus G_2$ равномерно по $w \in G_3 \equiv \{w : |w-w_0| < r_3\}$, где $r_3 \in (0, r_2)$ произвольно. Прежде всего заметим, что согласно Предложению 2.4

$$|\eta - w| \asymp |\eta + i|, \quad \eta \in \Pi_+\setminus G_2, \quad w \in G_3.$$

Кроме того, очевидно, что $|\overline{\eta} - w| > |\eta - w|$. Следовательно

$$\left| \frac{u(\eta)(1-\psi(\eta))}{(\eta - w)} \cdot \frac{(Im\eta)^{\beta+1}}{(i(\overline{\eta}-w))^{\beta+1}} \right| \le const(w_0, r_2, r_3, \beta) \cdot |u(\eta)| \cdot \frac{(Im\eta)^{Re\beta+1}}{|\eta+i|^{Re\beta+2}} \equiv F(\eta).$$

Из условия теоремы получаем $F \in L^1(\Pi_+\setminus G_2)$.

Пусть $1 \le p < \infty, \alpha > -1$ и $\gamma \in \mathbb{R}$. Для комплекснозначной измеримой функции u, заданной в Π_+, положим

$$\|u\|_{p,\alpha,\gamma} = \left(\iint\limits_{\Pi_+} \frac{|u(\eta)|^p (Im\eta)^\alpha}{|\eta+i|^\gamma} dm(\eta) \right)^{\frac{1}{p}}. \tag{3.6}$$

Соответственно,

$$L^p_{\alpha,\gamma}(\Pi_+) = \{u : \|u\|_{p,\alpha,\gamma} < +\infty\}. \tag{3.7}$$

Отметим, что пространства подобного типа в верхней полуплоскости уже рассматривались в [23].

Теорема 3.3. *Пусть функция* $u \in C^k(\Pi_+)$ $(k = 1, 2, 3, \ldots, \infty)$ *и удовлетворяет одному из следующих условий:*
(а) $u(\eta) \cdot Im\eta \in L^1_{\alpha,\gamma}(\Pi_+)$, $\alpha > -1, \gamma \le 2+\alpha, Re\beta \ge \alpha$,
(б) $u(\eta) \cdot Im\eta \in L^p_{\alpha,\gamma}(\Pi_+)$, $1 < p < \infty, \alpha > -1, \gamma < 2+\alpha, Re\beta > \frac{\alpha+1}{p} - 1$. *Положим* $f_\beta(w) \equiv T^*_\beta(u)(w)$, $w \in \Pi_+$, *тогда имеет место* (3.1).

Доказательство. Ввиду Теоремы 3.2 достаточно показать, что условия (а) или (б) обеспечивают выполнение условия

$$\frac{u(\eta)\cdot Im\eta\cdot(Im\eta)^{Re\beta}}{|\eta+i|^{Re\beta+2}}\in L^1(\Pi_+).$$

Положим $u(\eta)\cdot Im\eta=g(\eta)$, $\eta\in\Pi_+$. Если имеет место (а), то есть $g\in L^1_{\alpha,\gamma}(\Pi_+)$, то

$$\left|\frac{g(\eta)\cdot(Im\eta)^{Re\beta}}{|\eta+i|^{Re\beta+2}}\right|=\frac{|g(\eta)|\cdot(Im\eta)^{\alpha}}{|\eta+i|^{\gamma}}\cdot\frac{(Im\eta)^{Re\beta-\alpha}}{|\eta+i|^{Re\beta-\alpha}}\cdot\frac{1}{|\eta+i|^{\alpha-\gamma+2}}$$
$$\le\frac{|g(\eta)|\cdot(Im\eta)^{\alpha}}{|\eta+i|^{\gamma}}\in L^1(\Pi_+),$$

что и требовалось доказать. Если же имеет место (б), то есть $g\in L^p_{\alpha,\gamma}(\Pi_+)$, то в силу интегрального неравенство Гёльдера $(1/p+1/q=1)$

$$\iint\limits_{\Pi_+}\frac{g(\eta)\cdot(Im\eta)^{Re\beta}}{|\eta+i|^{Re\beta+2}}dm(\eta)=\iint\limits_{\Pi_+}\frac{g(\eta)\cdot(Im\eta)^{\frac{\alpha}{p}}}{|\eta+i|^{\frac{\gamma}{p}}}\cdot\frac{(Im\eta)^{Re\beta-\frac{\alpha}{q}}}{|\eta+i|^{Re\beta-\frac{\gamma}{p}+2}}dm(\eta)\tag{3.8}$$
$$\le\|g\|_{p,\alpha,\gamma}\cdot\left(\iint\limits_{\Pi_+}\frac{(Im\eta)^{(Re\beta-\frac{\alpha}{p})q}}{|\eta+i|^{(Re\beta-\frac{\gamma}{p}+2)q}}dm(\eta)\right)^{\frac{1}{q}}.$$

Для сходимости последнего интеграла необходимо выполнение следующих условий (см., например, [22, Лемма 3.1]):

$$\left(Re\beta-\frac{\alpha}{p}\right)q>-1\quad\text{и}\quad\left(Re\beta-\frac{\gamma}{p}+2\right)q>\left(Re\beta-\frac{\alpha}{p}\right)q+2,$$

которые соответственно эквивалентны условиям теоремы : $Re\beta>\frac{\alpha+1}{p}-1$ и $\gamma<\alpha+2$. Теорема доказана.

4. Весовые L^p-оценки решений $\overline{\partial}$-урвнения

Предложение 4.1. *Пусть для комплекснозначных измеримых функций $v(\zeta),\zeta\in\mathbb{D}$, и $u(\eta),\eta\in\Pi_+$, имеем $v\overset{\beta}{\approx}u$. Если $1\le p<+\infty$, $\alpha>-1$, $\gamma\in\mathbb{R}$, тогда при условии*

$$4p+pRe\beta+\gamma-4-2\alpha\ge0\tag{4.1}$$

справедлива оценка

$$\|v\|_{p,4p+pRe\beta+\gamma-4-\alpha}\le const(p,\beta,\alpha,\gamma)\cdot\|u\|_{p,\alpha,\gamma}.\tag{4.2}$$

Доказательство. Отметим, что нормы в (4.2) понимаются соответственно в смысле (1.6) и (3.6). В силу (2.11), поскольку β фиксировано, имеем

$$|v(\zeta)|^p \asymp \frac{|u(\varPhi(\zeta))|^p}{|1-\zeta|^{p(4+Re\beta)}}, \qquad \zeta \in \mathbb{D}.$$

Следовательно, с учётом неравенства $1-|\zeta|^2 \leq 2\cdot|1-\zeta|$, $\zeta\in\mathbb{D}$, и условия (4.1), приходим к следующей цепочке неравенств:

$$\begin{aligned}
\|u\|^p_{p,\alpha,\gamma} &\overset{\eta=\varPhi(\zeta)}{=\!=\!=} \iint\limits_{\mathbb{D}} \frac{|u(\varPhi(\zeta))|^p\cdot\left(\frac{1-|\zeta|^2}{|1-\zeta|^2}\right)^\alpha}{\left|\frac{2i}{1-\zeta}\right|^\gamma}\cdot\frac{4}{|1-\zeta|^4}dm(\zeta)\\
&= const(\gamma)\iint\limits_{\mathbb{D}}\frac{|u(\varPhi(\zeta))|^p\cdot(1-|\zeta|^2)^\alpha dm(\zeta)}{|1-\zeta|^{4+2\alpha-\gamma}}\\
&\geq const(p,\beta,\gamma)\iint\limits_{\mathbb{D}}\frac{|v(\zeta)|^p\cdot(1-|\zeta|^2)^\alpha dm(\zeta)}{|1-\zeta|^{4+2\alpha-\gamma-4p-pRe\beta}}\\
&\geq const(p,\beta,\alpha,\gamma)\iint\limits_{\mathbb{D}}|v(\zeta)|^p\cdot(1-|\zeta|^2)^\alpha(1-|\zeta|^2)^{4p+pRe\beta+\gamma-4-2\alpha}dm(\zeta)\\
&= const(p,\beta,\alpha,\gamma)\cdot\|v\|^p_{p,\ 4p+pRe\beta+\gamma-4-\alpha}.
\end{aligned}$$

Таким образом, утверждение доказано. По ходу отметим, что (4.1) обеспечивает выполнение условия $4p+pRe\beta+\gamma-4-\alpha>-1$.

Теорема 4.1. *Пусть $1\leq p<\infty$, $\alpha>-1$, $\gamma\in\mathbb{R}$. Предположим также выполнение условия* (4.1) *и, кроме того,*

$$Re\beta > 4p+pRe\beta+\gamma-5-\alpha>-1. \tag{4.3}$$

Тогда для произвольной функции $u\in C^k(\Pi_+)\cap L^p_{\alpha,\gamma}(\Pi_+)$, $k=1,2,3,\ldots,\infty$, интегральный оператор T^*_β решает соответствующее $\overline{\partial}$-уравнение в Π_+, т.е. для функции $f_\beta(w)\equiv T^*_\beta(u)(w)$, $w\in\Pi_+$, имеет место (3.1). *Более того, справедлива оценка*

$$\|f_\beta\|_{p,\ 4p+pRe\beta+\gamma-5-\alpha,\ 6p+pRe\beta+2\gamma-6-2\alpha}\leq const(p,\beta,\alpha,\gamma)\cdot\|u\|_{p,\alpha,\gamma}. \tag{4.4}$$

Доказательство. Исходя из функции u, заданной в Π_+, построим функцию v, заданную в $\mathbb{D}$, так что $v\overset{\beta}{\approx}u$ и при этом справедлива оценка (4.2). Положим $g_\beta(z)=T_\beta(v)(z), z\in\mathbb{D}$, тогда $g_\beta\in C^k(\mathbb{D})$ (см. [5], [15]). Более того, в силу Теоремы 1.2, условий (4.3) и оценки (4.2) имеем:

$$\begin{aligned}
\|g_\beta\|_{p,\ 4p+pRe\beta+\gamma-5-\alpha} &\leq const(p,\beta,\alpha,\gamma)\cdot\|v\|_{p,\ 4p+pRe\beta+\gamma-4-\alpha}\\
&\leq const(p,\beta,\alpha,\gamma)\cdot\|u\|_{p,\alpha,\gamma}.
\end{aligned} \tag{4.5}$$

Остаётся связать друг с другом нормы f_β и g_β. Поскольку $f_\beta \overset{\beta}{\sim} g_\beta$, и β фиксировано, в силу (2.10)

$$|f_\beta(\eta)|^p \asymp \frac{|g_\beta(\Phi^{-1}(\eta))|^p}{|\eta+i|^{p(2+Re\beta)}}, \quad \eta \in \Pi_+.$$

Следовательно, имеем:

$$\begin{aligned}
\|g_\beta\|^p_{p,\ 4p+pRe\beta+\gamma-5-\alpha} &= \iint\limits_{\mathbb{D}} |g_\beta(\zeta)|^p(1-|\zeta|^2)^{4p+pRe\beta+\gamma-5-\alpha}dm(\zeta) \\
&\overset{\zeta=\Phi^{-1}(\eta)}{=\!=\!=} \iint\limits_{\Pi_+} \frac{|g_\beta(\Phi^{-1}(\eta))|^p \cdot (4Im\eta)^{4p+pRe\beta+\gamma-5-\alpha}}{|\eta+i|^{2(4p+pRe\beta+\gamma-5-\alpha)}} \cdot \frac{4}{|\eta+i|^4}dm(\eta) \\
&= const(p,\beta,\alpha,\gamma) \cdot \iint\limits_{\Pi_+} \frac{|g_\beta(\Phi^{-1}(\eta))|^p \cdot (Im\eta)^{4p+pRe\beta+\gamma-5-\alpha}}{|\eta+i|^{2(4p+pRe\beta+\gamma-3-\alpha)}}dm(\eta) \\
&\geq const(p,\beta,\alpha,\gamma) \cdot \iint\limits_{\Pi_+} \frac{|f_\beta(\eta))|^p \cdot (Im\eta)^{4p+pRe\beta+\gamma-5-\alpha}}{|\eta+i|^{6p+pRe\beta+2\gamma-6-2\alpha}}dm(\eta) \\
&= const(p,\beta,\alpha,\gamma) \cdot \|f_\beta\|_{p,\ 4p+pRe\beta+\gamma-5-\alpha,\ 6p+pRe\beta+2\gamma-6-2\alpha}. \qquad (4.6)
\end{aligned}$$

Комбинируя (4.6) и (4.5), получаем (4.4).

Замечание 4.1. Обилие условий на параметры p, β, α, γ (см (4.1) и (4.3)) на самом деле даёт возможность (варьируя их) получать разнообразные оценки. Например, полагая $p = 1, \gamma = 0$, согласно (4.4) получаем:

$$\|f_\beta\|_{1,\ Re\beta-\alpha-1,\ Re\beta-2\alpha} \leq const(\beta,\alpha) \cdot \|u\|_{1,\alpha}$$

при условиях $Re\beta > \alpha > -1$, $Re\beta \geq 2\alpha$.

Abstract. The paper considers the equation $\partial f(w)/\partial\overline{w} = u(w)$ in the upper semiplane Π_+. For a function u belonging to the class C^k $(k = 1, 2, 3, \ldots, \infty)$ and the weighted space L^p, $1 \leq p < \infty$ with a weight of type $(Imw)^\alpha \cdot |w+i|^{-\gamma}$, $w \in \Pi_+$, a family of solutions f_β depending on the complex parameter β is constructed.

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ON PLANE ALGEBRAIC CURVES PASSING THROUGH n-INDEPENDENT NODES

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Abstract. Let a set of nodes $\mathcal{X}$ in the plane be n-independent, i.e., each node has a fundamental polynomial of degree n. Assume that $\#\mathcal{X} = d(n, k-3) + 3 = (n+1) + n + \cdots + (n-k+5) + 3$ and $4 \le k \le n-1$. In this paper we prove that there are at most seven linearly independent curves of degree less than or equal to k that pass through all the nodes of $\mathcal{X}$. We provide a characterization of the case when there are exactly seven such curves. Namely, we prove that then the set $\mathcal{X}$ has a very special construction: all its nodes but three belong to a (maximal) curve of degree $k-3$. Let us mention that in a series of such results this is the third one. At the end an important application to the bivariate polynomial interpolation is provided, which is essential also for the study of the Gasca-Maeztu conjecture.

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1. Introduction

Denote the space of all bivariate polynomials of total degree not exceeding n by

$$\Pi_n = \left\{ \sum_{i+j\le n} a_{ij} x^i y^j \right\}.$$

We have that

$$N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).$$

Denote by Π the space of all bivariate polynomials.

Consider a set of s distinct nodes $\mathcal{X} = \mathcal{X}_s = \{(x_1, y_1), (x_2, y_2), \dots, (x_s, y_s)\}$. The problem of finding a polynomial $p \in \Pi_n$, which satisfies the conditions

$$p(x_i, y_i) = c_i, \quad i = 1, \dots, s, \tag{1.1}$$

is called interpolation problem.

A polynomial $p \in \Pi_n$ is called a fundamental polynomial for a node $A \in \mathcal{X}$ if $p(A) = 1$ and $p\big|_{\mathcal{X}\setminus\{A\}} = 0$, where $p\big|_{\mathcal{X}}$ means the restriction of p on $\mathcal{X}$. We denote this n-fundamental polynomial by $p^\star_A := p^\star_{A,\mathcal{X}}$.

Definition 1.1. *The interpolation problem with a set of nodes $\mathcal{X}_s$ is called n-poised if for any data $(c_1, \ldots, c_s)$ there is a unique polynomial $p \in \Pi_n$ satisfying the interpolation conditions* (1.1).

A necessary condition of poisedness is $\#\mathcal{X}_s = s = N$.

Next, let us consider the concept of n-independence (see [2, 4]).

Definition 1.2. *A set of nodes $\mathcal{X}_s$ is called n-independent, if all its nodes have n-fundamental polynomials. Otherwise, it is called n-dependent.*

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence for $\mathcal{X}_s$ is $s \le N$.

1.1. **Some properties of n-independent nodes.** Let us start with the following

Lemma 1.1 (Lemma 2.2, [6])**.** *Suppose that a set of nodes $\mathcal{X}$ is n-independent and the nodes of another set $\mathcal{Y}$ have n-fundamental polynomials with respect to the set $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$. Then the set $\mathcal{Z}$ is n-independent too.*

Denote the distance between the points A and B by $\rho(A, B)$. Let us recall the following (see [3])

Lemma 1.2. *Suppose that $\mathcal{X}_s = \{A_i\}_{i=1}^s$ is an n-independent set. Then there is a number $\epsilon > 0$ such that any set $\mathcal{X}'_s = \{A'_i\}_{i=1}^s$, with the property that $\rho(A_i, A'_i) < \epsilon,\ i = 1, \ldots, s$, is n-independent too.*

Next result concerns the extensions of n-independent sets.

Lemma 1.3 (Lemma 2.1, [4])**.** *Any n-independent set $\mathcal{X}$ with $\#\mathcal{X} < N$ can be enlarged to an n-poised set.*

Denote the linear space of polynomials of total degree at most n vanishing on $\mathcal{X}$ by

$$\mathcal{P}_{n,\mathcal{X}} := \left\{ p \in \Pi_n : p\big|_{\mathcal{X}} = 0 \right\}.$$

The following two propositions are well-known (see, e.g.,[4]).

Proposition 1.1. *For any node set $\mathcal{X}$ we have that*

$$\dim \mathcal{P}_{n,\mathcal{X}} = N - \#\mathcal{Y},$$

where $\mathcal{Y}$ is a maximal n-independent subset of $\mathcal{X}$.

Proposition 1.2. *If a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ, then we have that $p\big|_\ell = 0$ and $p = \ell r$, where $r \in \Pi_{n-1}$.*

A plane algebraic curve is the zero set of some bivariate polynomial of degree $\geq$ 1. To simplify notation, we shall use the same letter, say p, to denote the polynomial p and the curve given by the equation $p(x, y) = 0$.

In the sequel we will need the following

Proposition 1.3 (Prop. 1.10, [6])**.** *Let $\mathcal{X}$ be a set of nodes. Then $\mathcal{P}_{n,\mathcal{X}} = \{0\}$ if and only if $\mathcal{X}$ has an n-poised subset.*

Set $d(n, k) := N_n - N_{n-k} = (1/2)k(2n+3-k)$. The following is a generalization of Proposition 1.2.

Proposition 1.4 (Prop. 3.1, [9])**.** *Let q be an algebraic curve of degree $k \leq n$ with no multiple components. Then the following hold:*

i) any subset of q containing more than $d(n, k)$ nodes is n-dependent;

ii) any subset $\mathcal{X}$ of q containing exactly $d(n, k)$ nodes is n-independent if and only if the following condition holds:

$$p \in \Pi_n \quad and \quad p|_{\mathcal{X}} = 0 \implies p = qr, \ \ where \ r \in \Pi_{n-k}. \tag{1.2}$$

Thus, according to Proposition 1.4, i), at most $d(n, k)$ n-independent nodes can lie in a curve q of degree $k \leq n$. This motivates the following

Definition 1.3 (Def. 3.1, [9])**.** *Given an n-independent set of nodes $\mathcal{X}$ with $\#\mathcal{X} \geq d(n, k)$. A curve of degree $k \leq n$ passing through $d(n, k)$ points of $\mathcal{X}$ is called maximal.*

Let us bring a characterization of maximal curves:

Proposition 1.5 (Prop. 3.3, [9])**.** *Given an n-independent set of nodes $\mathcal{X}$ with $\#\mathcal{X} \geq d(n, k)$. Then a curve μ of degree k, $k \leq n$, is a maximal curve if and only if $p \in \Pi_n$, $p|_{\mathcal{X}\cap\mu} = 0 \implies p = \mu s$, $s \in \Pi_{n-k}$.*

Next result concerns maximal independent sets in curves.

Proposition 1.6 (Prop. 3.5, [8])**.** *Assume that σ is an algebraic curve of degree k with no multiple components and $\mathcal{X}_s \subset \sigma$ is any n-independent node set of cardinality s, $s < d(n, k)$. Then the set $\mathcal{X}_s$ can be enlarged to a maximal n-independent set $\mathcal{X}_d \subset \sigma$ of cardinality $d = d(n, k)$.*

Below a replacement of a node in an n-independent set is described such that the set remains n-independent.

Lemma 1.4 (Lemma 6, [5]). *Assume that $\mathcal{X}$ is an n-independent node set and a node $A \in \mathcal{X}$ has an n-fundamental polynomial $p_A^\star$ such that $p_A^\star(A') \neq 0$. Then we can replace the node A with A' such that the resulted set $\mathcal{X}' := \mathcal{X} \cup \{A'\} \setminus \{A\}$ is n-independent too. In particular, such replacement can be done in the following two cases:*

i) if a node $A \in \mathcal{X}$ belongs to several components of σ, then we can replace it with a node A', which belongs to only one (desired) component,

ii) if a curve q is not a component of an n-fundamental polynomial $p_A^\star$ then we can replace the node A with a node A' lying in q.

Next result from Algebraic Geometry will be used in the sequel:

Theorem 1.1 (Th. 2.2, [10]). *If $\mathcal{C}$ is a curve of degree n with no multiple components, then through any point O not in $\mathcal{C}$ there pass lines which intersect $\mathcal{C}$ in n distinct points.*

Let us mention also that, as it follows from the proof, if a line ℓ through a point O intersects $\mathcal{C}$ in n distinct points then any line through O, sufficiently close to ℓ, has the same property. Finally, let us present a well-known

Lemma 1.5. *Suppose that m linearly independent polynomials vanish at the set $\mathcal{X}$. Then for any node $A \notin \mathcal{X}$ there are $m-1$ linearly independent polynomials, in their linear span, vanishing at A and the set $\mathcal{X}$.*

2. Main results and a series of results

Let us start with the first result of a series of results:

Theorem 2.1 (Th. 1, [7]). *Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-1)+2$ nodes lying in a curve of degree k with $k \le n$. Then the curve is determined uniquely by these nodes.*

The second result in this series is the following

Theorem 2.2 (Th. 4.2, [8]). *Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-1)+1$ nodes with $2 \le k \le n-1$. Then at most two different curves of degree $\le k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such two curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but one lie in a maximal curve of degree $k-1$.*

Next result is the following

Theorem 2.3 (Th. 3, [5]). *Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-2)+2$ nodes with $3 \le k \le n-1$. Then at most four linearly independent curves of degree*

$\leq k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such four curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but two lie in a maximal curve of degree $k-2$.

Now let us present the main result of this paper:

Theorem 2.4. *Assume that $\mathcal{X}$ is an n-independent set of $d(n,k-3)+3$ nodes with $4 \leq k \leq n-1$. Then at most seven linearly independent curves of degree $\leq k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such seven curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but three lie in a maximal curve of degree $k-3$.*

Let us mention that the inverse implication in the "Moreover" part is straightforward. Indeed, assume that $d(n,k-3)$ nodes of $\mathcal{X}$ are located in a curve μ of degree $k-3$. Therefore, the curve μ is maximal and the remaining three nodes of $\mathcal{X}$, denoted by A, B and C, are outside of it: $A, B, C \notin \mu$. Hence, in view of Proposition 1.5, we have that

$$\mathcal{P}_{k,\mathcal{X}} = \{p : p \in \Pi_k, p_{\mathcal{X}} = 0\} = \{q\mu : q \in \Pi_3, q(A) = q(B) = q(C) = 0\}.$$

Thus we get readily that $\dim \mathcal{P}_{k,\mathcal{X}} = \dim \{q \in \Pi_3 : q(A) = q(B) = q(C) = 0\} = \dim \mathcal{P}_{3,\{A,B,C\}} = 10 - 3 = 7$. Note that in the last equality we use Proposition 1.1 and the fact that any three nodes are 3-independent.

We get also that it is enough to prove only the "Moreover" part. Indeed, assume that the "Moreover" part is proved. Assume also that there are ≥ 7 linearly independent curves satisfying the hypothesis of Theorem 2.4. Then, as we showed above, we have that $\dim \mathcal{P}_{k,\mathcal{X}} = 7$, i.e., there are exactly 7 such curves, Q.E.D.

It is worth mentioning that to prove of Theorem 2.4 we establish an interesting version of Theorem 2.3, where we increase the number of nodes by one and decrease the number of linearly independent curves by one:

Theorem 2.5. *Assume that $\mathcal{X}$ is an n-independent set of $d(n,k-2)+3$ nodes with $3 \leq k \leq n-2$. Then at most three linearly independent curves of degree $\leq k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such three curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ lie in a curve of degree $k-1$, or all the nodes of $\mathcal{X}$ but three lie in a (maximal) curve of degree $k-2$.*

3. Some preliminaries

We will start the proof of Theorem 2.4 in Section 5. Since then we need to do considerable amount of preliminary work.

Lemma 3.1. *Assume that the hypotheses of Theorem 2.4 hold and assume additionally that there is a curve $\sigma_0 \in \Pi_{k-2}$ passing through all the nodes of $\mathcal{X}$. Then all the nodes of $\mathcal{X}$ but three (collinear) lie in a maximal curve μ of degree $k-3$.*

Proof. First note that the curve σ_0 is of exact degree $k-2$, since it passes through more than $d(n,k-3)$ n-independent nodes. This implies also that σ_0 has no multiple components. Therefore, in view of Proposition 1.6, we can enlarge the set $\mathcal{X}$ to a maximal n-independent set $\mathcal{Z} \subset \sigma_0$, by adding $d(n,k-2)-d(n,k-3)-3 = n-k+1$ nodes, i.e., $\mathcal{Z} = \mathcal{X} \cup \mathcal{A},$ where $\mathcal{A} = \{A_0, \ldots, A_{n-k}\}$.

In view of Lemma 1.4, *i*), we may suppose that the nodes from $\mathcal{A}$ are not intersection points of the components of σ_0.

Next, we are going to prove that these $n-k+1$ nodes are collinear together with $m \geq 3$ nodes from $\mathcal{X}$. To this end denote the line through the nodes A_0 and A_1 by ℓ_{01}. Then for each $i = 2\ldots, n-k$, choose a line ℓ_i passing through the node A_i, which is not a component of σ_0. We require also that ℓ_i does not pass through other nodes of $\mathcal{A}$ and therefore the lines are distinct.

Now suppose that $\sigma^* \in \Pi_k$ vanishes on $\mathcal{X}$. Consider the polynomial $p = \sigma^*\ell_{01}\ell_2\cdots\ell_{n-k}$. We have that $p \in \Pi_n$ and p vanishes on the node set $\mathcal{Z}$, which is a maximal n-independent set in the curve σ_0. Therefore, we obtain that

$$p = \sigma^*\ell_{01}\ell_2\cdots\ell_{n-k} = \sigma_0 r, \text{ where } r \in \Pi_{n-k+2}.$$

The lines $\ell_i,\ i = 2,\ldots,n-k$, are not components of σ_0. Therefore, they are components of the polynomial r. Hence we obtain that

$$\sigma^*\ell_{01} = \sigma_0\gamma, \text{ where } \gamma \in \Pi_3.$$

Now let us verify that ℓ_{01} is a component of σ_0. Indeed, otherwise it is a component of the cubic γ and we get that

$$\sigma^* \in \Pi_k,\ \sigma^*\big|_{\mathcal{X}} = 0 \implies \sigma^* = \sigma\beta, \text{ where } \beta \in \Pi_2.$$

Therefore, we obtain that $\dim \mathcal{P}_{k,\mathcal{X}} \leq 6$, which contradicts the hypothesis.

Thus we have that

$$\sigma_0 = \ell_{01}\sigma_{k-3}, \text{ where } \sigma_{k-3} \in \Pi_{k-3}. \tag{3.1}$$

Now let us show that all the nodes of $\mathcal{A}$ belong to ℓ_{01}. Suppose conversely that a node from $\mathcal{A}$, say A_2, does not belong to the line ℓ_{01}. Then in the same way as in the case of the line ℓ_{01} we get that ℓ_{02} is a component of σ_0. Therefore the node A_0 is an intersection point of two components of σ_0, i.e., ℓ_{01} and ℓ_{02}, which contradicts our assumption.

Thus we get that $\mathcal{A} \subset \ell_{01}$. Note that ℓ_{01} is not a component of σ_{k-3} since then it will be a multiple component of σ_0.

Next, let us verify that when enlarging the set $\mathcal{X} \subset \sigma_0$ to an n-maximal set one has to locate the added nodes outside the component σ_{k-3}. Indeed, what was proved already implies that the only possible location of such a node in σ_{k-3} is an intersection point with ℓ_{01}. But in the latter case, by using Lemma 1.4, we can replace the node, say A_1, with one belonging only to the component σ_{k-3}, say A_1', which is a contradiction. Indeed, again A_0 is the intersection point of two components of σ_0, the line through A_0, A_1 and the line through A_0, A_1'.

Hence, in view of Proposition 1.6 we get that $\mu = \sigma_{k-3}$ is a maximal curve for $\mathcal{X}$. Therefore, it vanishes at exactly $d(n, k-3)$ nodes of $\mathcal{X}$. The remaining three nodes, according to (3.1), belong to the line ℓ_{01}. □

The next result we prove with tools of mathematical analysis.

Proposition 3.1. *Assume that $p_1, p_2 \in \Pi$, $\deg p_2 \le \deg p_1 + 1$, and p_1 has no multiple factors. Then, for sufficiently small ϵ, the polynomial $p_1 + \epsilon p_2$ has no multiple factors either.*

Proof. Assume by way of contradiction that there is a sequence ϵ_n such that

$$p_1 + \epsilon_n p_2 = q_n r_n^2, \text{ where } q_n, r_n \in \Pi, \ \deg r_n \ge 1, \text{and } \epsilon_n \to 0. \tag{3.2}$$

We have that $\deg(p_1 + \epsilon_n p_2) \le \max(\deg p_1, \deg p_2)$, and hence

$$\deg q_n + 2 \deg r_n \le \max(\deg p_1, \deg p_2) \le \deg p_1 + 1. \tag{3.3}$$

We deduce from here that there is a subsequence n_k such that

$$\deg q_{n_k} = m_1 = const. \text{ and } \deg r_{n_k} = m_2 = const.$$

Without loss of generality assume that

$$\{\epsilon_n\} \equiv \{\epsilon_{n_k}\}. \tag{3.4}$$

Thus we have that

$$q_n = \sum_{i+j \le m_1} a_{ij}^{(n)} x^i y^j, \quad r_n = \sum_{i+j \le m_2} b_{ij}^{(n)} x^i y^j.$$

In view of (3.2), by a normalization of r_n, i.e., by multiplying it by a constant c and dividing q_n by c^2, we may assume that

$$\max |b_{ij}^{(n)}| = 1 \ \forall n. \tag{3.5}$$

Now, let us denote $M_n := \max |a_{ij}^{(n)}|$.

Case 1. Assume that (a subsequence of) M_n is bounded: $M_n \le M$. Note that in the case of the subsequence we may use again a replacement (3.4) and have that the whole sequence M_n is bounded. In this case, by using the Bolzano–Weierstrass theorem, we have for a subsequence $\{n_k\}$ that

$$a_{ij}^{(n_k)} \to a_{ij}^0 \text{ and } b_{ij}^{(n_k)} \to b_{ij}^0, \ \forall i, j.$$

Here, we use the fact that the number of the coefficients is finite.

By setting $n = n_k$ in (3.2) and tending $k \to \infty$ we obtain that $p_1 = q_0 r_0^2$, where

$$q_0 = \sum_{i+j \le m_1} a_{ij}^0 x^i y^j, \quad r_0 = \sum_{i+j \le m_2} b_{ij}^0 x^i y^j.$$

This contradicts the hypothesis for p_1 if $\deg r_0 \ge 1$.

Let us verify the latter inequality. Since $\deg r_n \ge 1$, we get from (3.3) that $\deg q_n \le \deg p_1 - 1$. Therefore $m_1 \le \deg p_1 - 1$ and hence $\deg r_0 \ge 1$.

Case 2. By taking into account a replacement (3.4) it remains to consider the case $M_n \to +\infty$.

There are numbers i_0, j_0, i_1, j_1 and a subsequence $n = \{n_k\}$, such that

$$|a_{i_0 j_0}^{(n_k)}| = \max_{i,j} |a_{ij}^{(n_k)}| \text{ and } |b_{i_1 j_1}^{(n_k)}| = \max_{i,j} |b_{ij}^{(n_k)}| = 1 \ \forall k. \tag{3.6}$$

Here, again we use the fact that the number of the coefficients is finite. In the last equality we use (3.5).

Now, let us set $n = n_k$ in (3.2) and divide both sides by M_{n_k} to get

$$\frac{1}{M_{n_k}} p_1 + \frac{\epsilon_{n_k}}{M_{n_k}} p_2 = \left(\frac{1}{M_{n_k}} q_{n_k} \right) r_{n_k}^2. \tag{3.7}$$

Evidently, the left hand side here tends to zero. For the right hand side we have that the coefficients of the polynomials $\frac{1}{M_{n_k}} q_{n_k}$ and r_{n_k} are bounded by 1. As above by using the Bolzano–Weierstrass theorem and passing to a new subsequence $\{n_k'\} \subset \{n_k\}$ we obtain that

$$\frac{1}{M_{n_k'}} a_{ij}^{(n_k')} \to a_{ij}^* \text{ and } b_{ij}^{(n_k')} \to b_{ij}^*, \ \forall i, j.$$

In view of (3.6) we have that

$$|a_{i_0 j_0}^*| = 1 \text{ and } |b_{i_1 j_1}^*| = 1. \tag{3.8}$$

Now, by setting $n = n_k'$ in (3.2) and tending $k \to \infty$ we get that $0 = q_* r_*^2$, where

$$q_* = \sum_{i+j \le m_1} a_{ij}^* x^i y^j, \quad r_* = \sum_{i+j \le m_2} b_{ij}^* x^i y^j.$$

In view of (3.8) this is a contradiction. □

Remark 3.1. *In the same way one can prove the following statement: Assume that* $p_1, p_2 \in \Pi$, $\deg p_2 \le \deg p_1$, *and* p_1 *is not reducible. Then, for sufficiently small* ϵ, *the polynomial* $p_1 + \epsilon p_2$ *is not reducible either.*

Note that, as the example of $p_2 = xp_1$ shows, the condition $\deg p_2 \le \deg p_1$ is essential here.

Next result will help to make the hypotheses of Theorem 2.4 more precise.

Proposition 3.2. *Suppose that there are seven linearly independent polynomials from Π_k vanishing on a set $\mathcal{X}$. Then, there are seven linearly independent polynomials vanishing on a set $\mathcal{X}$, each of which is of exact degree k and has no multiple factors, or, alternatively there are three linearly independent polynomials from Π_{k-1} vanishing on $\mathcal{X}$.*

Proof. Let $\sigma_i \in \Pi_k, 0 \le i \le 6$, be the given polynomials. We may assume that a polynomial, say σ_0, is of exact degree k. Indeed, if the degree of each of seven polynomials is less than k then the conclusion of Proposition holds.

Therefore we may assume that all the polynomials $\sigma_i, 0 \le i \le 6$ are of exact degree k. Indeed, it suffices to replace these polynomials with the seven polynomials σ_0 and $\sigma_0 + \epsilon\sigma_i, 1 \le i \le 6$, for some $\epsilon \neq 0$.

Next, let us prove that a polynomial, say σ_0, has no multiple factors. Indeed assume conversely that each of the seven polynomials has a multiple factor. In view of Lemma 3.1 the multiple factors are lines with multiplicity two. Thus, we have that

$$\sigma_i = \ell_i^2 q_i,\ 0 \le i \le 6,\ \text{where } \ell_i \in \Pi_1,\ q_i \in \Pi_{k-2}. \tag{3.9}$$

Then we replace these polynomials with the seven polynomials $\breve{\sigma}_i = \ell_i q_i \in \Pi_{k-1},\ 0 \le i \le 6$, which clearly vanish at the node set $\mathcal{X}$. Let us verify that among these latter seven polynomials there are at least three linearly independent ones. Conversely assume that the seven polynomials are linear combinations of two of them, say $\breve{\sigma}_i,\ i = 0, 1$. Then we get readily that the seven linearly independent polynomials in (3.9) are linear combinations of the following six polynomials:

$$\breve{\sigma}_i, x\breve{\sigma}_i, y\breve{\sigma}_i,\ i = 0, 1,$$

which is a contradiction. Indeed, assume that $\ell_i = A_i x + B_i y + C_i,\ i = 0, \ldots, 6$. Then for $i = 0, 1$, we have that

$$\sigma_i = \ell_i^2 q_i = (A_i x + B_i y + C_i)\breve{\sigma}_i = Ax\breve{\sigma}_i + B_i y\breve{\sigma}_i + C_i\breve{\sigma}_i.$$

Now, assume that $\breve{\sigma}_i = a_i\breve{\sigma}_0 + b_i\breve{\sigma}_1$, for $i = 2, \ldots, 6$. Then we have that

$$\sigma_i = \ell_i^2 q_i = (A_i x + B_i y + C_i)\breve{\sigma}_i = (A_i x + B_i y + C_i)(a_i\breve{\sigma}_0 + b_i\breve{\sigma}_1)$$

$$= a_i A_i x\breve{\sigma}_0 + a_i B_i y\breve{\sigma}_0 + a_i C_i\breve{\sigma}_0 + b_i A_i x\breve{\sigma}_1 + b_i B_i y\breve{\sigma}_1 + b_i C_i\breve{\sigma}_1.$$

Finally, by assuming that σ_0, has no multiple factors, let us again replace the seven polynomials $\sigma_i, 0 \le i \le 6$, with the seven polynomials σ_0 and $\sigma_0 + \epsilon\sigma_i, 1 \le i \le 6$, for a sufficiently small $\epsilon > 0$. This, in view of Proposition 3.1, completes the proof. $\square$

Proposition 3.3. *Suppose that $\sigma_i,\ i = 0, \ldots, 6$, are linearly independent polynomials of exact degree k and have no multiple factors. Then there is a polynomial in the linear span of $\sigma_i,\ i = 1, \ldots, 6$, which has no multiple factors and differs from σ_0 with a factor of degree at least three.*

Lemma 3.2. *Let σ_0, s_1, s_2, be linearly independent polynomials of exact degree k, with no multiple factors. Suppose also that any linear combination of $s_i,\ i = 1, 2$, differs from σ_0 with a factor from Π_2. Then we have that*

$$\sigma_0 = \tilde\sigma_0\beta_0, \quad s_1 = \tilde\sigma_0\beta_1, \quad s_2 = \tilde\sigma_0\beta_2, \ \textit{where } \tilde\sigma_0 \in \Pi_{k-1},\ \beta_i \in \Pi_2. \tag{3.10}$$

Moreover, $\tilde\sigma_0$ is uniquely determined from the first two relations here, if β_0 and β_1 are relatively prime.
Furthermore, if β_0 has a common factor with β_1 and a common factor with β_2 then the following alternative takes place: Either,
(i) $\beta_i = \ell\ell_i,\ i = 0, 1, 2$, i.e., they have a common linear factor, or
(ii) β_0 and $\beta_1 + \epsilon\beta_2$ are relatively prime $\forall\epsilon > 0$.

Proof. Consider the polynomials σ_0, s_1 and s_2. In view of the hypotheses and Proposition 3.1 for sufficiently small $c > 0$ we have that

$$(s_1 + cs_2)\beta(c) = \sigma_0\overline{\beta}(c), \tag{3.11}$$

where $\beta(c), \overline{\beta}(c) \in \Pi_2$ are relatively prime.

Then we have that $\beta(c)$ is a linear or conic component of σ_0. Suppose that σ_0 has k such components. By considering $k + 1$ sufficiently small values of c we get that there are constants c_1 and c_2 such that $\beta(c_1) = \beta(c_2) =: \beta_0$.

Then we readily obtain from (3.11) that

$$s_1\beta_0 = \sigma_0\beta_1 \text{ and } s_2\beta_0 = \sigma_0\beta_2, \text{ where } \beta_1, \beta_2 \in \Pi_2. \tag{3.12}$$

In the case when β_0 is relatively prime with β_1 or β_2 then it clearly divides σ_0. By denoting $\tilde\sigma_0 = \sigma_0/\beta_0 \in \Pi_{k-1}$, we get (3.10) from (3.12).
It remains to consider the case when β_0 is a reducible conic and has a common linear component with β_1 as well as with β_2. Below everywhere the letter ℓ denotes a linear polynomial. Thus suppose that $\beta_0 = \ell_0\ell_0'$. After a cancellation with a linear polynomial in (3.12) two cases are possible:
Case 1. $s_1\ell_0 = \sigma_0\ell_1$ and $s_2\ell_0' = \sigma_0\ell_2$; Case 2. $s_1\ell_0 = \sigma_0\ell_1$ and $s_2\ell_0 = \sigma_0\ell_2$.

In Case 1 $\beta_0 = \ell_0\ell_0'$ again divides σ_0 and we get (3.10). In Case 2 $\beta_0 = \ell_0$ divides σ_0 and we get (3.10), where β_0 therefore β_1 and β_2 are linear. Thus (3.10) is proved.

Note that if β_0 and β_1 are relatively prime then $\tilde{\sigma}_0$ is uniquely determined from the first two relations in (3.10) as the greatest common divisor of σ_0 and s_1.

Now, consider the "Furthermore" statement. Assume that the pairs β_0, β_1, and β_0, β_2, have a common factor. Set $\beta_0 = \ell\ell_0$ and $\beta_1 = \ell\ell_1$. Then we have that either $\beta_2 = \ell\ell_2$, or $\beta_2 = \ell_0\ell_3$. The first case reduces to the item (i). Let us consider the second case. It is easily seen that the polynomials $\beta_0 = \ell\ell_0$ and $\beta_1 + \epsilon\beta_2 = \ell\ell_1 + \epsilon\ell_0\ell_3$ have no common factor.

Indeed, conversely suppose that ℓ is a common factor. Then the last equality implies that $\ell = \ell_0$, or $\ell = \ell_3$. In the first case we get that β_0 and hence, in view of (3.10), σ_0 has a double component ℓ, while in the second case we get that $\beta_0 = \beta_2$ and hence $\sigma_0 = \sigma_2$.

Now conversely suppose that ℓ_0 is a common factor. In this case the same equality implies that $\ell_0 = \ell$, or $\ell_0 = \ell_1$. The first case was considered already, while the second case implies that $\beta_0 = \beta_1$ and hence $\sigma_0 = \sigma_1$. □

Proof of Proposition 3.3. Assume by way of contradiction that any polynomial from $S :=$ Linear span$\{\sigma_1, \ldots, \sigma_6\}$, differs from σ_0 with a factor of degree at most two. By Lemma 3.2, for the polynomial σ_0 and any two polynomials from S, the relation (3.10) holds.

Case 1. Assume that there is a polynomial $s_1 \in S$, say it is $s_1 = \sigma_1$, for which the relation (3.10) holds with β_1 being relatively prime with β_0. Note that this evidently takes place if β_0 is linear.

Then, according to Lemma 3.2, $\tilde{\sigma}_0$ is determined uniquely.

Now, let us apply Lemma 3.2 successively with the triples of polynomials $\sigma_0, \sigma_1, \sigma_i,\ i = 2, \ldots, 6$. Then we get that

$$\sigma_i = \tilde{\sigma}_0\beta_i,\ i = 0, \ldots, 6,\ \text{where } \beta_i \in \Pi_2.$$

Clearly the seven polynomials β_i here, and consequently the seven polynomials σ_i are linearly dependent, which contradicts our assumption.

Case 2. Assume that for any triple of polynomials $\sigma_0, s_1 := \sigma_i, s_2 := \sigma_j$ the relation (3.10) holds with β_0 having a common factor with β_i as well as with β_j. Hence all three are of degree two.

Now, if for some triple the alternative (ii) holds then we have Case 1 with $s_1 := \sigma_i + \epsilon\sigma_j$. Note that, in view of Proposition 3.1, s_1 has no multiple factors if ϵ is sufficiently small.

Next, suppose that the alternative (i) holds: $\beta_0 = \ell\ell_0, \beta_i = \ell\ell_i, \beta_j = \ell\ell_j$.

This reduces to Case 1 since here (3.10) holds also with linear β's:

$$\sigma_0 = \overline{\sigma}_0 \ell_0,\ \sigma_i = \overline{\sigma}_0 \ell_i,\ \sigma_j = \overline{\sigma}_0 \ell_j,\ \text{where } \overline{\sigma} = \tilde{\sigma}\ell. \qquad \square$$

4. The existence of three curves of degree $k-1$

Proposition 4.1. *Assume that the hypotheses of Theorem 2.4 hold. Then, there are three linearly independent curves of degree $k-1$ passing through all the nodes of the set $\mathcal{X}$.*

Proof. Let $\sigma_0, \ldots, \sigma_6$, be the seven curves of degree $\leq k$ that pass through all the nodes of the n-independent set $\mathcal{X}$ with $\#\mathcal{X} = d(n, k-3) + 3$.

In view of Proposition 3.2 assume, without loss of generality, that each of these polynomials is of exact degree k and has no multiple factors.

Step 1. Here we will prove that there is at least one curve of degree $\leq k-1$ passing through all the nodes of the set $\mathcal{X}$.

We start by choosing two nodes $B_1, B_2 \notin \mathcal{X}$ such that the following two conditions are satisfied:

i) the set $\mathcal{X} \cup \{B_1, B_2\}$ is n-independent;

ii) the line ℓ_0 through B_1 and B_2 does not pass through any node from $\mathcal{X}$.

Let us verify that one can find such nodes. Indeed, in view of Lemma 1.3, we can start by choosing some nodes $B_i = B_i',\ i = 1, 2$, satisfying the condition *i*). Then, according to Lemma 1.2, for some positive ϵ all the nodes $B_i,\ i = 1, 2$, in ϵ neighborhoods of B_i', $i = 1, 2$, respectively, satisfy the condition *i*). Finally, from these neighborhoods we can choose the nodes B_i, $i = 1, 2$, satisfying the condition *ii*) too.

Next we find one more node $B_3 \in \ell_0$ such that the set $\mathcal{X} \cup \{B_1, B_2, B_3\}$ is n-independent. Indeed, if there is no such node then we obtain that

$$p \in \Pi_k,\ p|_{\mathcal{X} \cup \{B_1, B_2\}} = 0 \Rightarrow p|_{\ell_0} = 0.$$

Therefore $p = \ell_0 q$, where $q \in \Pi_{k-1}$ and, in view of the condition ii), $q|_{\mathcal{X}} = 0$. Hence, if there is no B_3 then, according to Lemma 1.5, there are five linearly independent polynomials $p \in \Pi_k$ satisfying the condition $p|_{\mathcal{X} \cup \{B_1, B_2\}} = 0$. Therefore, there are five linearly independent $q \in \Pi_{k-1}$ satisfying the condition $q|_{\mathcal{X}} = 0$.

Next, we find successively two more nodes $B_4, B_5 \in \ell_0$ such that the set $\mathcal{X} \cup \mathcal{B}_5$ is n-independent, where $\mathcal{B}_5 := \{B_1, B_2, B_3, B_4, B_5\}$. Indeed, if one cannot find the node B_4 or B_5 then, in the same way as above, we obtain that there are four or three linearly independent polynomials $q \in \Pi_{k-1}$ satisfying the condition $q|_{\mathcal{X}} = 0$, respectively.

Then, in view of Lemma 1.5, there are two curves of degree $\leq k$, which pass through all the nodes of $\mathcal{X} \cup \mathcal{B}_5$. Denote one of them by σ_0. We may assume that it is of exact degree k and has no multiple factors. We may assume also that ℓ_0 is not a component of σ_0. Otherwise as above, we find a desired polynomial q.

Now, in view of Proposition 1.6, we enlarge the set $\mathcal{X} \cup \mathcal{B}_5$ to a maximal n-independent set $\mathcal{Z} \subset \sigma_0$, by adding $d(n,k) - (d(n,k-3)+3) - 5 = 3(n-k)+1$ nodes, i.e.,

$$\mathcal{Z} = \mathcal{X} \cup \mathcal{B}_5 \cup \mathcal{A}, \text{ where } \#\mathcal{A} = 3(n-k)+1 = [3(n-k-1)-1]+5.$$

Let us start with the description of the choice of $3(n-k-1)-1$ nodes of $\mathcal{A}$. By using Proposition 3.3 we find a curve σ in the linear span of $\sigma_i,\ i=1,\ldots,6$, which has no multiple factors and differs from σ_0 with a factor of degree at least three: $\sigma = \gamma r,\ \sigma_0 = \gamma_0 r$, with $d := \deg\gamma = \deg\gamma_0 \geq 3$ and $r \in \Pi_{k-d}$. We have that γ_0 and σ are relatively prime.

Below we are using Theorem 1.1 with respect to the curve $\mathcal{C} := \gamma_0$. Choose a point $O \notin \gamma_0 \cup \sigma$. Since $O \notin \sigma_0$ no line through the point O will be a component of σ_0. Consider a line ℓ_1 through O which intersects $\mathcal{C}$ at distinct points not belonging to $\ell_0 \cup \sigma$. Let A_1, A_2 and A_3, be three of those intersection points. By using a continuity argument we may assume that the lines $\ell_i,\ i=2,\ldots,n-k-1$, pass through O and are enough close to ℓ_1 so that each of them intersects $\mathcal{C}$ at distinct points, which do not belong to $\ell_0 \cup \sigma$. We assume also that $\ell_i \cap (\mathcal{X} \cup \mathcal{B}_5) = \emptyset,\ i=1,\ldots,n-k-1$. As in the case of the line ℓ_1 let A_{3i-2}, A_{3i-1} and A_{3i}, be three of those intersection points belonging to $\gamma_0 \cap \ell_i,\ i=2,\ldots,n-k+1$. Finally, let us dismiss an intersection point, say A_1, and denote the desired set of the remaining $3(n-k-1)-1$ intersection nodes $\{A_i\}$ by $\mathcal{A}(-1)$.

Let us prove that the set $\mathcal{Y} := \mathcal{X} \cup \mathcal{B}_5 \cup \mathcal{A}(-1)$ is n-independent.

We have that the set $\mathcal{A}(-1)$ is a subset of Berzolari-Radon construction of degree $n-k-1$. Hence it is $(n-k-1)$-independent. Now suppose that $p^\star_{A,\mathcal{A}(-1)}$ is a fundamental polynomial of a node $A \in \mathcal{A}(-1)$ of degree $n-k-1$. Then the polynomial $\sigma\ell_0 p^\star_{A,\mathcal{A}(-1)}$ is an n-fundamental polynomial of the node A for the set $\mathcal{Y}$. Here we use the fact that no node from $\mathcal{A}(-1)$ belongs to ℓ_0 or σ. Thus, according to Lemma 1.1, the set $\mathcal{Y}$ is n-independent.

Finally, in view of Proposition 1.6, we enlarge the set $\mathcal{Y} \subset \sigma_0$ with a set $\mathcal{A}_5$ of the last 5 nodes to a maximal n-independent set $\mathcal{Z} \subset \sigma_0$. Thus we have that $\mathcal{Z} := \mathcal{Y} \cup \mathcal{A}_5$ and $\mathcal{A} = \mathcal{A}(-1) \cup \mathcal{A}_5$.

Now suppose that $\sigma^* \in \Pi_k$ vanishes on $\mathcal{X}$ and $\mathcal{A}_5$. According to Lemma 1.5 there are $2 = 7-5$ such polynomials. Hence we may assume that $\sigma^* \neq \sigma_0$. Then consider the polynomial $p = \sigma^* \ell_0 \ell_1 \cdots \ell_{n-k-1}$. We have that $p \in \Pi_n$ vanishes on the maximal n-independent set $\mathcal{Z} \subset \sigma_0$. Therefore, we have that $p = \sigma^* \ell_0 \ell_1 \cdots \ell_{n-k-1} = \sigma_0 s$, where $s \in \Pi_{n-k}$.

The lines ℓ_i, $i = 1, \ldots, n-k-1$, are not components of σ_0 since they pass through $O \notin \sigma_0$. Therefore, they are components of the polynomial s. Thus we obtain

$$\sigma^* \ell_0 = \sigma_0 \ell, \text{ where } \ell \in \Pi_1.$$

Since $\sigma^* \neq \sigma_0$ therefore $\ell_0 \neq \ell$. Whence ℓ_0 is a component of $\sigma_0 : \sigma_0 = \ell_0 q_0$, where $q_0 \in \Pi_{k-1}$. As above we get that q_0 vanishes on $\mathcal{X}$.

Step 2. Here we will prove that there are three linearly independent curves of degree $\leq k-1$ passing through all the nodes of the set $\mathcal{X}$.

We find a line ℓ_0 and collinear nodes $B_1, \ldots, B_4 \in \ell_0$, in the same way as in the Step 1, such that $\ell_0 \cap \mathcal{X} = \emptyset$ and the set $\mathcal{X} \cup \mathcal{B}_4$ is n-independent, where $\mathcal{B}_4 := \{B_1, B_2, B_3, B_4\}$.

Next, in view of Proposition 1.5, there are three linearly independent curves of degree at most k, which pass through all the nodes of the set $\mathcal{X} \cup \mathcal{B}_4$. Denote these curves by $\sigma_0, \sigma_0', \sigma_0''$. If a curve here, say σ_0, is of degree $\leq k-1$ and has no multiple components then instead of given triple of curves we consider the curves $\ell_1\sigma_0, \ell_2\sigma_0, \ell_3\sigma_0$, where the lines ℓ_i are chosen such that these three curves are linearly independent and have no multiple factors.

Next, if a curve $\sigma_0, \sigma_0', \sigma_0''$, has a multiple factor then by throwing away the excessed factor we are in the situation considered in the previous paragraph. Hence, we may consider only the case when each of theses three polynomials is of exact degree k and has no multiple components.

Now consider the curve σ_0. In view of Proposition 1.6 we enlarge the set $\mathcal{X} \cup \mathcal{B}_4$ to a maximal n-independent set $\mathcal{Z} \subset \sigma_0$, by adding $d(n,k) - (d(n,k-3)+3) - 4 = 3(n-k)+2$ nodes, i.e.,

$$\mathcal{Z} = \mathcal{X} \cup \mathcal{B}_4 \cup \mathcal{A}, \text{ where } \#\mathcal{A} = 3(n-k)+2 = [3(n-k-1)-1]+1+5.$$

We find the set of $3(n-k-1)-1$ points from $\mathcal{A}$ in the same way as in Step 1 and denote it again by $\mathcal{A}(-1)$. Then, in the same way as in Step 1, we prove the independence of the set $\mathcal{Y} := \mathcal{X} \cup \mathcal{B}_4 \cup \mathcal{A}(-1)$.

Next, in view of Theorem 1.1, we choose a node $\tilde{A}_1 \in \ell_1$ such that $\tilde{A}_1 \in \sigma_0 \setminus q_0$, where q_0 is the polynomial of degree $\leq k-1$ vanishing on $\mathcal{X}$, found in Step 1. Note that the line ℓ_1 is not a component of q_0 since $\ell_1 \cap \mathcal{X} = \emptyset$.

Then consider the case when $\tilde{A}_1 \in \mathcal{A}(-1)$, i.e., $\tilde{A}_1$ coincides with one of the nodes $A_2, A_3 \in \mathcal{A}(-1) \cap \ell_1$, say $\tilde{A}_1 = A_2$. In this case instead of $\mathcal{A}(-1)$ we would start with the set $\mathcal{A}(-1)' = \mathcal{A}(-1) \cup \{A_1\} \setminus \{A_2\}$ and we will have already that $\tilde{A}_1 \notin \mathcal{A}(-1)'$.

Since ℓ_0 is not a component of σ_0 therefore the set $F := \ell_0 \cap \sigma_0$ is a finite set and we could suppose beforehand that $\ell_1 \cap F = \emptyset$. This will ensure that $\tilde{A}_1 \notin \ell_0$. Also we have that $\tilde{A}_1 \neq O$ since $O \notin \sigma_0$.

Now let us prove the independence of the set $\tilde{\mathcal{Y}} := \mathcal{Y} \cup \{\tilde{A}_1\}$. For this end, in view of Lemma 1.1, it suffices to find a fundamental polynomial of the node $\tilde{A}_1$ with respect to the set $\tilde{\mathcal{Y}}$. We readily verify that $p^\star_{\tilde{A}_1,\tilde{\mathcal{Y}}} = q_0\ell_0\ell_2\cdots\ell_{n-k-1}\ell'\ell''$, where ℓ' and ℓ'' are lines different from ℓ_1 and pass through the nodes A_2 and A_3, respectively.

Finally, according to Proposition 1.6, let us enlarge the set $\tilde{\mathcal{Y}} \subset \sigma_0$ with the set of last 5 nodes, denoted by $\mathcal{A}_5$, to a maximal n-independent set. Thus the set $\mathcal{Z} := \tilde{\mathcal{Y}} \cup \mathcal{A}_5$ is a maximal n-independent set in σ_0.

Now suppose that $\sigma^* \in \Pi_k$ vanishes on $\mathcal{X}$ and the 5 nodes of $\mathcal{A}_5$. According to Lemma 1.5 there are at least two such polynomials. Hence we may assume that $\sigma^* \neq \sigma_0$. Then consider the polynomial $p = \sigma^*\ell_0\ell_1\cdots\ell_{n-k-1}$. We have that $p \in \Pi_n$ and p vanishes on the node set $\mathcal{Z}$, which is a maximal n-independent set in the curve σ_0. Therefore, we have that

$$p = \sigma^*\ell_0\ell_1\cdots\ell_{n-k-1} = \sigma_0 s, \text{ where } s \in \Pi_{n-k}.$$

The lines $\ell_i,\ i = 1, \ldots, n-k-1$, are not components of σ_0. Therefore, they are components of the polynomial s. Thus we get that $\sigma^*\ell_0 = \sigma_0\ell,$ where $\ell \in \Pi_1$. Since $\sigma^* \neq \sigma_0$ therefore $\ell_0 \neq \ell$. Hence ℓ_0 is a component of σ_0 :

$$\sigma_0 = \ell_0 q_{k-1}, \text{ where } q_{k-1} \in \Pi_{k-1}.$$

In the same way for the curves σ_0' and σ_0'' we get $\sigma_0' = \ell_0 q_{k-1}',$ where $q_{k-1}' \in \Pi_{k-1}$, and $\sigma_0'' = \ell_0 q_{k-1}'',$ where $q_{k-1}'' \in \Pi_{k-1}$.

Obviously the curves $q_{k-1}, q_{k-1}', q_{k-1}''$, are linearly independent. □

5. Proofs of Theorems 2.4 and 2.5

Proof of Theorem 2.5. Assume by way of contradiction that there are four curves passing through all the nodes of the set $\mathcal{X}$. Then, according to Theorem 2.3, all the nodes of $\mathcal{X}$ but three belong to a maximal curve μ of degree $k-2$. The curve μ is maximal and the remaining three nodes of $\mathcal{X}$, denoted by A, B and C, are outside of it: $A, B, C \notin \mu$. Hence we have that

$$\mathcal{P}_{k,\mathcal{X}} = \{p : p \in \Pi_k, p|_{\mathcal{X}} = 0\} = \{q\mu : q \in \Pi_2, q(A) = q(B) = q(C) = 0\}.$$

Thus we get readily that $\dim \mathcal{P}_{k,\mathcal{X}} = \dim \{q \in \Pi_2 : q(A) = q(B) = q(C) = 0\} = \dim \mathcal{P}_{2,\{A,B,C\}} = 6 - 3 = 3$, which contradicts our assumption. Note that in the last equality we use Proposition 1.1 and the fact that any three nodes are 2-independent.

Now, let us verify the part "if". By assuming that there is a curve σ of degree $k-1$ passing through the nodes of $\mathcal{X}$ we find readily three linearly independent curves of degree $\le k$: $\sigma, x\sigma, y\sigma$, passing through $\mathcal{X}$. While if we assume that all the nodes of $\mathcal{X}$ but three lie in a curve μ of degree $k-2$ then above evaluation shows that $\dim \mathcal{P}_{k,\mathcal{X}} = 3$.

Finally, let us verify the part "only if". Denote the three curves passing through all the nodes of $\mathcal{X}$ by $\sigma_0, \sigma_0', \sigma_0''$. If one of them is of degree $k-1$ then the conclusion of Theorem is satisfied and we are done. Thus, we may assume that each curve is of degree k and has no multiple components. Now consider the curve σ_0.

By using Proposition 1.6 let us enlarge the set $\mathcal{X}$ to a maximal n-independent set $\mathcal{Z} \subset \sigma_0$. Since $\#\mathcal{Z} = d(n,k)$, we need to add a set of $d(n,k) - (d(n,k-2)+3) = 2(n-k)+2$ nodes, denoted by

$$\mathcal{A} := \{A_1, \dots, A_{2(n-k)+2}\}.$$

Thus we have that $\mathcal{Z} := \mathcal{X} \cup \mathcal{A}$. In view of Lemma 1.4, i), we require that each node of $\mathcal{A}$ may belong only to one component of the curve σ_0.

Case 1, $n = k+2,\ \mathcal{A} := \{A_1, \dots, A_6\}$.

Consider 5 nodes from $\mathcal{A}$ and a conic β^* passing through them. Denote the sixth node by A^*. We have three polynomials from Π_k vanishing on $\mathcal{X}$. By using Lemma 1.5 we get two linearly independent curves of degree at most k, that pass through all the nodes of $\mathcal{X}$ and the node $A^* \in \mathcal{A}$. Thus we may consider a such curve $\sigma^* \in \Pi_k$ by assuming that $\sigma^* \neq \sigma_0$. Now, notice that the polynomial $\sigma^* \beta^*$ of degree n vanishes at all the nodes of $\mathcal{Z} \subset \sigma_0$. Consequently, according to Proposition 1.4, σ_0 divides this polynomial:

$$\sigma^* \beta^* = \sigma_0 \beta, \quad \beta \in \Pi_2. \tag{5.1}$$

We have that $\beta^* \neq \beta$ since $\sigma^* \neq \sigma_0$. Hence if β^* is irreducible then it divides σ_0. Now suppose that β^* is reducible: $\beta^* = \ell_1 \ell_2$, where $\ell_i \in \Pi_1$. Then we have that both lines ℓ_1, ℓ_2, cannot divide β, hence either $\ell_1 \ell_2$ or only one of them is a component of σ_0.

Let us consider the latter case. Suppose that the line ℓ_1 is a component of σ_0 and ℓ_2 is a component of β. Then we get from (5.1) that

$$\sigma^* \ell_1 = \sigma_0 \ell, \text{ where } \ell \in \Pi_1. \tag{5.2}$$

Now, we have that $\sigma_0 = \ell_1 q$, where $\deg q = k-1$. Then we get from (5.2) that $\sigma^* = \ell q$. From the last two equalities we conclude that $\mathcal{X} \subset q \cup \{E\}$, where $E = \ell_1 \cap \ell$.

Therefore all the nodes of $\mathcal{X}$, except possibly E, belong to the curve q. Here q is a component of σ_0 of degree $k-1$ and E belongs to its line component ℓ_1.

We briefly express the above conditions by saying that the line component ℓ_1 of σ_0 satisfies (-1)-node condition for $\mathcal{X}$.

At the end we will see that if this property holds for all three given curves $\sigma_0, \sigma_0', \sigma_0''$, then we can readily complete the proof of Theorem.

Therefore, from now on we may assume that the equality (5.1) implies that $\deg \beta^* = 2$ and β^* is a component of σ_0. Thus we obtain also that β^* is determined uniquely by the 5 nodes from $\mathcal{A}$.

Next, we are going to prove that there is a conic passing through all the six nodes of $\mathcal{A}$. Assume conversely that there is no such conic. Denote by β_i the conic passing through the five nodes of $\mathcal{A} \setminus \{A_i\},\ i = 1, 2.$

We have that these two conics are different components of σ_0. First assume that one of these two conics, say, β_1, is irreducible. Then consider a common node of β_1 and β_2, say, A_3. It is easily seen that A_3 belongs to two different components of σ_0, which contradicts our assumption. Indeed, one is β_1 and another is β_2 if it is irreducible or a line component of β_2 if it is reducible.

Now, assume that both β_1 and β_2 are reducible: $\beta_1 = \ell_1 \ell_1', \quad \beta_2 = \ell_2 \ell_2'$. Without loss of generality assume that

$$\ell_1 \neq \ell_2, \quad \ell_1 \neq \ell_2'. \tag{5.3}$$

We have that ℓ_1 passes through at least one of the common nodes $A_3, \ldots, A_6$, say A_3. Then A_3 belongs either to ℓ_2 or to ℓ_2'. In both cases, in view of (5.3), we have that A_3 belongs to two different line components of σ_0, which is a contradiction. Thus we proved that $\mathcal{A} \subset \beta_0$, where $\beta_0 \in \Pi_2$.

Next let us show that β_0 divides σ_0. Consider a polynomial $\sigma \in \Pi_k$ that vanishes on $\mathcal{X}$ and $\sigma \neq \sigma_0$. Notice that the following polynomial $\sigma\, \beta_0$ of degree $k+2 = n$ vanishes at all the $d(n,k)$ nodes of $\mathcal{Z} \subset \sigma_0$. Consequently, according to Proposition 1.4, σ_0 divides this polynomial:

$$\sigma\, \beta_0 = \sigma_0 \beta, \quad \beta \in \Pi_2. \tag{5.4}$$

This is a type (5.1) equality which, as we mentioned above, implies that $\deg \beta_0 = 2$ and β_0 is a component of σ_0, i.e., $\sigma = \beta_0 q, \quad q \in \Pi_{k-2}$. We conclude also that β_0 is uniquely determined by any 5 nodes from $\mathcal{A}$.

Thus to enlarge the set $\mathcal{X} \subset \sigma_0$ to a maximal n-independent set $\mathcal{Z} = \mathcal{X} \cup \mathcal{A}$ we have to add all the six nodes of $\mathcal{A}$ to the conic β_0. Let us verify that the added nodes cannot belong to the component q. Indeed, suppose conversely that a node belongs to $\beta_0 \cap q$. Then, in view of Lemma 1.4, we can move the node to $q \setminus \beta_0$ such that the resulted set is also n-independent. This is a contradiction, since now the six nodes do not belong to a conic. Indeed, the five nodes determine a unique conic and the sixth node is outside of it. Thus the factor $q \in \Pi_{k-2}$ to which one can not add a new independent node is merely maximal with respect to $\mathcal{X}$. This means that q passes through exactly $d(n, k)$ nodes of $\mathcal{X}$.

Case 2, $n \geq k + 3$.

Consider a subset of $\mathcal{A}$ of cardinality 4 and denote it by $\mathcal{A}_4$. Denote also by $\bar{\mathcal{A}} := \mathcal{A} \setminus \mathcal{A}_4$. We have that $\#\bar{\mathcal{A}} = 2(n-k) - 2$.

There are three linearly independent polynomials $\sigma_0, \sigma_0', \sigma_0'' \in \Pi_k$, vanishing on $\mathcal{X}$. Now suppose that $\sigma^* \in \Pi_k$ vanishes on $\mathcal{X}$ and at an arbitrary node $A^* \in \bar{\mathcal{A}}$, which will be specified below. According to Lemma 1.5 there are two such polynomials. Hence we may assume that $\sigma^* \neq \sigma_0$. We call the node A^* associated with σ^*.

We associate another node $A' \in \bar{\mathcal{A}}$ with the set $\mathcal{A}_4$ and denote by β' a conic that passes through A' and the four nodes of $\mathcal{A}_4$.

For any line component ℓ of σ_0 denote by $r_\ell \in \Pi_{k-1}$ for which

$$\sigma_0 = \ell r_\ell. \tag{5.5}$$

Assume that a line component ℓ of the curve σ_0, passes through exactly m nodes from $\mathcal{X}$, at which r_ℓ does not vanish. Then we obtain from (5.5) that $r_\ell \in \Pi_{k-1}$ vanishes at the all nodes of the set $\mathcal{X}$ except m nodes, which belong to ℓ.

Note that if for a line ℓ we have that $m \leq 1$, then the line component ℓ of σ_0 satisfies the (-1)-node condition for $\mathcal{X}$.

Therefore we may suppose that $m \geq 2$ for all lines ℓ, meaning that the following condition takes place:

(**C**) Any line component of the curve σ_0, passes through at least two nodes from $\mathcal{X}$, at which r_ℓ does not vanish.

Later, in Section 5.1, by using the condition (C), we divide the set of nodes $\bar{\mathcal{A}}$ into $n-k-2$ pairs such that the lines $\ell_1, \ldots, \ell_{n-k-2}$, through them, respectively, are not components of σ_0. The remaining two nodes denoted by A^* and A', are associated with the curve σ^* and $\mathcal{A}_4$, respectively.

Now, let us continue the proof by assuming that the above-described division of $\bar{\mathcal{A}}$ is established.

Notice that the following polynomial $\sigma^* \beta' \ell_1 \ldots \ell_{n-k-2}$ of degree n vanishes at all the $d(n,k)$ nodes of $\mathcal{Z} \subset \sigma_0$. Consequently, according to Proposition 1.4, σ_0 divides this polynomial:

$$\sigma^* \beta' \ell_1 \ldots \ell_{n-k-2} = \sigma_0 r, \quad r \in \Pi_{n-k}. \tag{5.6}$$

The distinct lines $\ell_1, \ldots, \ell_{n-k-2}$ do not divide the polynomial $\sigma_0 \in \Pi_k$, therefore, all they have to divide r. Hence, we get from (5.6) that $\sigma^* \beta' = \sigma_0 \beta,$ where $\beta \in \Pi_2$. Then, we have that $\beta' \neq \beta$ since $\sigma^* \neq \sigma_0$. Now, in the same way as in Case 1 we obtain that $\sigma_0 = \beta' q$ where $q \in \Pi_{k-2}$.

Next, we are going to prove that there is a conic passing through all the nodes of $\mathcal{A}$. Assume by way of contradiction that there is no such conic. Then, in view of Proposition 1.3, we have that there is a set of six nodes, say $\mathcal{A}_6 := \{A_1, \ldots, A_6\} \subset \mathcal{A}$, that does not lie in a conic.

Now, let us choose three noncollinear nodes in $\mathcal{A}_6$, say A_1, A_2, A_3, and consider the following sets of four nodes:

$$A_1, A_2, A_3, A_4; \quad A_1, A_2, A_3, A_5; \quad A_1, A_2, A_3, A_6.$$

Then, consider these three sets with the respective associated nodes:

$$A_1, A_2, A_3, A_4, A'; \quad A_1, A_2, A_3, A_5, A''; \quad A_1, A_2, A_3, A_6, A'''. \tag{5.7}$$

We have that the three conics through these sets are components of σ_0. Since $\mathcal{A}_6$ does not lie in a conic we obtain that these three conics cannot coincide. Hence there are two different conics, say the conics β' and β'', passing through the first two sets in (5.7), respectively.

First assume that one of these two conics, say, β', is irreducible. Then consider a common node, say, A_1. It is easily seen that A_1 belongs to two different components of σ_0, which contradicts our assumption. Indeed, one is β' and another is β'', if it is irreducible too, or a line component of β'', if it is reducible.

Next, assume that both β' and β'' are reducible: $\beta' = \ell_1 \ell_1', \quad \beta'' = \ell_2 \ell_2'$. Without loss of generality assume that

$$\ell_1 \neq \ell_2, \quad \ell_1 \neq \ell_2'. \tag{5.8}$$

Note that ℓ_1 passes through at least one of the common nodes A_1, A_2, A_3, say A_1. Indeed, if ℓ_1 passes through only A' and A_4 then we obtain that ℓ_1' passes through the three noncolinear nodes A_1, A_2, A_3. Now, we have that A_1 belongs either to ℓ_2 or ℓ_2'. In both cases, in view of (5.8), we have that A_1 belongs to two different line components of σ_0, which is a contradiction.

Thus we proved that $\mathcal{A} \subset \beta_0$, where $\beta_0 \in \Pi_2$. Next, in the same way as in Case 1, we show that β_0 divides $\sigma_0 : \ \sigma_0 = \beta_0 q, \quad q \in \Pi_{k-2}$. Also we have that β_0 is uniquely determined by the nodes of $\mathcal{A} \setminus \{A\},\ \forall A \in \mathcal{A}$.

Indeed, assume conversely that β_0 is not uniquely determined by the nodes from $\mathcal{A} \setminus \{A_0\}$, where $A_0 \in \mathcal{A}$. Therefore there are infinitely many conics β_0 passing through the nodes of $\mathcal{A} \setminus \{A_0\}$. Recall that for (any) A_0 one can find a curve, denoted by σ^*, of degree at most k, that passes through all the nodes of $\mathcal{X}$ and is different from σ_0. Then, as in Case 1, we readily get $\sigma^* \beta_0 = \sigma_0 \beta$, where $\beta \in \Pi_2$. This implies that β_0 is a component of σ_0. Therefore σ_0 has infinitely many components, which is a contradiction.

Thus to enlarge the set $\mathcal{X} \subset \sigma_0$ to a maximal n-independent set $\mathcal{Z} = \mathcal{X} \cup \mathcal{A}$ we have to add all the nodes of $\mathcal{A}$ to the conic β_0. Let us verify that the added nodes do not belong to the component q. Suppose conversely that a node $A_0 \in \mathcal{A}$ belongs to $\beta_0 \cap q$. Then, in view of Lemma 1.4, let us move A_0 to $q \setminus \beta_0$ such that the resulted set $\mathcal{A}$ remains n-independent. This is a contradiction, since now the nodes of $\mathcal{A}$ do not belong to a conic. Indeed, the nodes $\mathcal{A} \setminus \{A_0\}$ determine a unique conic and the moved node is outside of it. Therfore, the factor $r \in \Pi_{k-2}$ to which one cannot add a new independent node is merely maximal with respect to $\mathcal{X}$. Hence, r passes through exactly $d(n,k)$ nodes of $\mathcal{X}$.

At the end, before establishing the division of the set $\bar{\mathcal{A}}$, it remains to consider the case when the division may be not possible for all three curves $\sigma_0, \sigma_0', \sigma_0''$, i.e., the case when the condition (C) does not hold. Then, we obtain three curves q, q', q'', which are components of degree $k-1$ of the curves $\sigma_0, \sigma_0', \sigma_0''$, respectively, passing through all the nodes of $\mathcal{X}$ except possibly one.

Assume that q, q', q'', pass through all the nodes of $\mathcal{X}$ except E, E', E'', respectively. First assume that two of these three nodes are different, say $E \neq E'$. We have that q and q' pass through all the nodes of the set $\mathcal{Y} := \mathcal{X} \setminus \{E, E'\},\ \#\mathcal{Y} = d(n, k-3) + 1$. If $q = q'$ then we have that $E = E'$, contradicting our assumption. If $q \neq q'$ then, according to Theorem 2.2, all the nodes of $\mathcal{Y}$ except one belong to a (maximal) curve μ of degree $k-2$. Thus all the nodes of $\mathcal{X}$ except three belong to μ.

It remains to consider the case $E = E' = E''$. Then we have that q, q', q'', pass through all the nodes of the set $\mathcal{Y} := \mathcal{X} \setminus \{E\},\ \#\mathcal{Y} = d(n, k-2) + 2$. We get from Theorem 2.1 that $q = q' = q'' =: q$.

Next, in view of the condition (C), we get that $\sigma = \ell q,\ \sigma' = \ell' q,\ \sigma'' = \ell''$, where $\ell, \ell' \ell'' \in \Pi_1$. This contradicts the linear independence of $\sigma, \sigma' \sigma''$, since we have that $E \in \ell \cap \ell' \cap \ell''$. $\square$

Proof of Theorem 2.4. It is easily seen that Theorem 2.4 follows from Proposition 4.1, Theorem 2.5 and Lemma 3.1. □

5.1. **The division of the set $\bar{\mathcal{A}}$.** Next let us establish the above mentioned division of the node set $\bar{\mathcal{A}} := \mathcal{A} \setminus \mathcal{A}_4$ in the case $n \geq k+3$. Note that this is the case when we need the division.

Recall that each node of $\mathcal{A}$ belongs only to one component of the curve σ_0. By using induction on m one can prove easily the following

Lemma 5.1 (Proof of Th. 3, [5]). *Suppose that a finite set of lines $\mathcal{L}$ and $2m$ nodes lying in these lines are given. Suppose also that no node is an intersetion point of two lines. Then one can divide the node set into m pairs such that no pair belongs to the same line from $\mathcal{L}$ if and only if each line from $\mathcal{L}$ contains no more than m nodes.*

Thus the above mentioned division of the node set $\bar{\mathcal{A}}$ into $n-k-2$ pairs is possible if and only if no $n-k-1$ nodes of $\bar{\mathcal{A}}_0 := \bar{\mathcal{A}} \setminus \{A^*, A'\}$ are located in a line component of σ_0, where the nodes A^* and A' are the nodes associated with the curve σ^* and $\mathcal{A}_4$, respectively. Observe also that we may associate any two nodes A^* and A' of $\mathcal{A}$ with σ^* and $\mathcal{A}_4$,

Now notice that, in view of $\#\bar{\mathcal{A}} = 2(n-k-1)$, there can be at most two undesirable line components for the set $\bar{\mathcal{A}}$, i.e., lines containing at least $n-k-1$ nodes from it. In this case a node from each line we assign as associated and leave in the two lines $\leq n-k-2$ nodes.

Then assume that we have one undesirable line component for the set $\bar{\mathcal{A}}$, containing $\leq n-k$ nodes from it. In this case two nodes from this line we nominate as associated and leave in the line $\leq n-k-2$ nodes.

Finally consider the case of one undesirable line component ℓ of σ_0 with $m \geq n-k+1$ nodes. We have that

$$\sigma_0 = \ell r_\ell, \text{ where } r_\ell \in \Pi_{k-1}.$$

Now we are going to move $m-n+k$ nodes, one by one, from ℓ to the other component r_ℓ such that the set $\mathcal{Z} := \mathcal{X} \cup \mathcal{A}$ remains n-independent. Again, in view of Lemma 1.4, i), we require that each moved node belongs only to one component of the curve σ_0.

To establish each described movement, in view of Lemma 1.4, ii), it suffices to prove that during this process each node $A \in \ell \cap \mathcal{A}$, has no n-fundamental polynomial for which the curve r_ℓ is a component. Suppose conversely that

$$p_A^\star = r_\ell s, \; s \in \Pi_{n-k+1}. \tag{5.9}$$

Now, we have that s vanishes at $\geq n-k$ nodes in $\ell \cap \mathcal{A} \setminus \{A\}$. Indeed, the nodes of the set $\mathcal{A}$ in the line ℓ do not belong to another component. Therefore, r_ℓ does not vanish at these nodes and hence, in view of (5.9), s vanishes. According to the condition (C) r_ℓ does not vanish also at least at two nodes from $\ell \cap \mathcal{X}$, and hence s vanishes there too. Thus the number of zeroes of s in the line ℓ is greater or equal to $n-k+2$ and s together with $p^\star_A$ vanishes at the whole line ℓ, including the node A, which is a contradiction.

It remains to note that there will be no more undesirable lines, except ℓ, in the resulted set $\mathcal{A}$, after the described movement of the nodes, since we finish by keeping exactly $n-k$ nodes in $\ell \cap \mathcal{A}$ and outside of it there are only $n-k-2$ nodes.

6. An application to bivariate interpolation

A GC_n set $\mathcal{X}$ in the plane is an n-poised set of nodes, where the fundamental polynomial of each node is a product of n linear factors. The Gasca–Maeztu conjecture states that any GC_n-set possesses a subset of $n+1$ collinear nodes.

Recall that a node $A \in \mathcal{X}$ uses a line ℓ means that ℓ is a factor of the fundamental polynomial, i.e., $p^\star_A = \ell r$ for some $r \in \Pi_{n-1}$.

It was proved by Carnicer and Gasca in [1], that any line passing through exactly 2 nodes of a GC_n set $\mathcal{X}$ can be used at most by one node from $\mathcal{X}$. Next, it was proved in [8] that any used line passing through exactly 3 nodes of an n-poised set $\mathcal{X}$ can be used either by exactly one or three nodes from $\mathcal{X}$. In [5] was proved that a line ℓ passing through exactly 4 nodes can be used at most by six nodes from $\mathcal{X}$. Moreover, if it is used by at least four nodes then it is used by exactly six nodes from $\mathcal{X}$.

Below we consider the case of lines passing through exactly 5 nodes.

Corollary 6.1. *Let $\mathcal{X}$ be an n-poised set of nodes and ℓ be a line which passes through exactly 5 nodes. Then ℓ can be used at most by ten nodes from $\mathcal{X}$. Moreover, if ℓ is used by at least seven nodes from $\mathcal{X}$ then it is used by exactly ten nodes from $\mathcal{X}$. Furthermore, if it is used by ten nodes, then they form a 3-poised set. In the latter case, if $\mathcal{X}$ is a GC_n set then the ten nodes form a GC_3 set too.*

Proof. Assume that $\ell \cap \mathcal{X} = \{A_1, \ldots, A_5\} =: \mathcal{A}$. Assume also that the seven nodes in $\mathcal{B} := \{B_1, \ldots, B_7\} \in \mathcal{X}$ use the line $\ell : p^\star_{B_i} = \ell\, q_i,\ i = 1, \ldots, 7$, where $q_i \in \Pi_{n-1}$.

The polynomials $q_1, \ldots, q_7$, vanish at $N-12$ nodes of the set $\mathcal{X}' := \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})$. Hence through these $N-12 = d(n, n-4)+3$ nodes pass seven linearly independent curves of degree $n-1$. By Theorem 2.4 there exists a maximal curve μ of degree $n-4$ passing through $N-15$ nodes of $\mathcal{X}'$ and the remaining three nodes denoted by

C_1, C_2, C_3, are outside of it. Now, according to Proposition 1.5, the nodes C_1, C_2, C_3, use $\mu : p^\star_{C_i} = \mu r_i,\ r_i \in \Pi_4,\ i = 1, 2, 3$.

These polynomials r_i have to vanish at the five nodes of $\mathcal{A} \subset \ell$. Hence $r_i = \ell\gamma_i,\ i = 1, 2, 3$, with $\gamma_i \in \Pi_3$. Therefore, the nodes C_1, C_2, C_3, use the line $\ell : p^\star_{C_i} = \mu\ell\gamma_i,\ i = 1, 2, 3$. Hence if seven nodes in $\mathcal{B} \subset \mathcal{X}$ use the line ℓ then there exist three more nodes $C_1, C_2, C_3 \in \mathcal{X}$ using it and all the nodes of $\mathcal{Y} := \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B} \cup \{C_1, C_2, C_3\})$ lie in a maximal curve μ of degree $n - 4$:

$$\mathcal{Y} \subset \mu. \tag{6.1}$$

Next, let us show that there is no eleventh node using ℓ. Assume conversely that except of the ten nodes in $\mathcal{S} := \{B_1, \dots, B_7, C_1, C_2, C_3\}$, there is an eleventh node D using ℓ. Of course we have that $D \in \mathcal{Y}$.

Then we have that the seven nodes $B_1, \dots, B_6$ and D are using ℓ therefore, as was proved above, there exist three more nodes $E_1, E_2, E_3 \in \mathcal{X}$ (which may coincide or not with B_7 or C_1, C_2, C_3) using it and all the nodes of $\mathcal{Y}' := \mathcal{X} \setminus (\mathcal{A} \cup \{B_1, \dots, B_6, D, E_1, E_2, E_3\})$ lie in a maximal curve μ' of degree $n - 4$.

We have also that

$$p^\star_D = \mu' q',\ q' \in \Pi_4. \tag{6.2}$$

Now, notice that both the curves μ and μ' pass through all the nodes of the set $\mathcal{Z} := \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B} \cup \{C_1, C_2, C_3, D, E_1, E_2, E_3\})$ with $\#\mathcal{Z} \geq N - 19$.

Then, we get from Theorem 2.1, with $k = n - 5$, that $N - 19 = d(n, n - 5) + 2$ nodes determine the curve of degree $n - 4$ passing through them uniquely. Thus μ and μ' coincide.

Therefore, in view of (6.1) and (6.2), $p^\star_D$ vanishes at all the nodes of $\mathcal{Y}$, which is a contradiction since $D \in \mathcal{Y}$.

Now, let us verify the "Moreover" statement. Suppose ten nodes in $\mathcal{S} \subset \mathcal{X}$ use the line ℓ. Then, as we obtained earlier, the nodes $\mathcal{Y} := \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{S})$ are located in a maximal curve μ of degree $n - 4$. Therefore the fundamental polynomial of each $A \in \mathcal{S}$ uses μ and hence ℓ :

$$p^\star_A = \mu\ell q_A,\ \text{where } q_A \in \Pi_3.$$

It is easily seen that q_A is a 3-fundamental polynomial of $A \in \mathcal{S}$. □

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EMBEDDING OF BESOV SPACES INTO TENT SPACES AND APPLICATIONS

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Abstract. In this paper, we study the boundedness and compactness of the inclusion mapping from Besov spaces to tent spaces. Meanwhile, the boundedness, compactness and essential norm of Volterra integral operators from Besov spaces to general function spaces are also investigated.

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Keywords: Besov space; Volterra integral operator; Carleson measure; tent space.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. The Hardy space H^p $(0 < p < \infty)$ is the set of all $f \in H(\mathbb{D})$ with (see [4])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let H^∞ be the space of all bounded analytic functions with the supremum norm $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$.

For $1 < p < \infty$, the Besov space, denoted by B_p, is the space of all functions $f \in H(\mathbb{D})$ satisfy

$$\|f\|_{B_p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty.$$

Let $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$. The space $F(p,q,s)$ is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. This space was first introduced by Zhao in [22]. $F(2,0,s)$ is the Q_s space (see [18]). $F(2,0,1)$ is the $BMOA$ space. $F(p,\alpha,0)$ is called the Dirichlet type space, denoted by $\mathcal{D}_\alpha^p$. In particular, $F(p,p-2,0)$ is the Besov space

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B_p. $F(p,p,0)$ is just the classical Bergman space A^p. When $s>1$, from [22] we see that $F(p,p-2,s)$ is equivalent to the Bloch space, denoted by $\mathcal{B}$, which consisting of all $f\in H(\mathbb{D})$ such that $\|f\|_{\mathcal{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty$.
The Volterra integral operator T_g was introduced by Pommerenke in [13]. Pommerenke showed that T_g is bounded on H^2 if and only if $g\in BMOA$, where

$$T_gf(z)=\int_0^z f(w)g'(w)dw, \qquad f\in H(\mathbb{D}).$$

The companion operator I_g induced by $g\in H(\mathbb{D})$ is defined by

$$I_gf(z)=\int_0^z f'(w)g(w)dw, \qquad f\in H(\mathbb{D}).$$

The multiplication operator M_g is defined by $M_gf(z)=f(z)g(z)$. It is easy to see that $M_gf(z)=f(0)g(0)+I_gf(z)+T_gf(z)$. Recently, much attention has been paid to the operators T_g and I_g.

See [1, 2], [5]-[9], [11]-[16], [20, 21] and the references therein for more study of the operators T_g and I_g.

For any arc $I\subseteq\partial\mathbb{D}$, the boundary of $\mathbb{D}$, let $|I|=\frac{1}{2\pi}\int_I|d\zeta|$ denote the normalized length of I and $S(I)$ be the Carleson box defined by

$$S(I)=\{z\in\mathbb{D}:1-|I|\le|z|<1, \quad z/|z|\in I\}.$$

Let $0\le s<\infty, 0<q<\infty$ and μ be a positive Borel measure on $\mathbb{D}$. Let $T_s^q(\mu)$ be the space of all μ-measurable functions f such that (see, e.g., [12])

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^q d\mu(z)<\infty.$$

Let $0\le\alpha<\infty$, $0<s<\infty$ and μ be a positive Borel measureon $\mathbb{D}$. We say that μ is a α-logarithmic s-Carleson measure if (see [21])

$$\|\mu\|_{LCM_{\alpha,s}}:=\sup_{I\subseteq\partial\mathbb{D}}\frac{(\log\frac{2}{|I|})^\alpha\mu(S(I))}{|I|^s}<\infty.$$

When $\alpha=0$, it gives the s-Carleson measure. When $\alpha=0, s=1$, it gives the classical Carleson measure. μ is said to be a vanishing α-logarithmic s-Carleson measure if (see [11])

$$\lim_{|I|\to0}\frac{(\log\frac{2}{|I|})^\alpha\mu(S(I))}{|I|^s}=0.$$

The Carleson measure is very useful in the theory of function spaces and operator theory. The famous embedding theorem say that the inclusion mapping $i:H^p\to L^p(d\mu)$ is bounded if and only if μ is a Carleson measure (see [4]). See [3] for the study of Carleson measure for the Besov space B_p. In [5], Girela and Peláez studied the Carleson measure for Dirichlet type spaces. Among others, under the assumption that $0<p<q<\infty$, they showed that the inclusion mapping $i:B_p\to$

$L^q(d\mu)$ is bounded if and only if μ is $q(1-\frac{1}{p})$-logarithmic 0-Carleson measure. In [20], Xiao proved that the inclusion mapping $i : Q_s \to T^2_s(\mu)$ is bounded if and only if μ is 2-logarithmic s-Carleson measure. In [10], Liu and Lou showed that the inclusion mapping $i : \mathcal{L}^{2,s} \to T^2_s(\mu)$ is bounded if and only if μ is a Carleson measure, where $\mathcal{L}^{2,s}$ is the Morrey space. The main ideas and methods used in [10] more or less are motivated by the three sections 3.2, 4.3, 6.4 of [19]. In [12], Pau and Zhao showed that the inclusion mapping $i : F(p, p-2, s) \to T^p_s(\mu)$ is bounded if and only if μ is p-logarithmic s-Carleson measure. In [7], Li, Liu and Yuan proved that the inclusion mapping $i : \mathcal{D}^p_{p-1} \to T^p_s(\mu)$ is bounded if and only if μ is a $(s+1)$-Carleson measure by using the Carleson embedding theorem for Bergman spaces.

Motivated by [5, 7, 10, 12, 20], in this paper, we study the boundedness and compactness of the inclusion mapping from B_p into $T^q_s(\mu)$. More precisely, we show that the inclusion mapping $i : B_p \to T^q_s(\mu)$ is bounded (resp. compact) if and only if μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure (resp. vanishing $q(1-\frac{1}{p})$-logarithmic s-Carleson measure) under the assumption that $1 < p < q < \infty$ and $0 < s < \infty$. Moreover, we study the boundedness, compactness and essential norm of the operators T_g and I_g acting from B_p to $F(q, q-2, s)$.

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$.

2. Embedding from Besov spaces B_p to $T^q_s(\mu)$

We need the following equivalent description of p-logarithmic s-Carleson measure, see Lemma 2.2 in [12].

Lemma 2.1. *Let $0 \leq \alpha < \infty, 0 < s, t < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then μ is a α-logarithmic s-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^{\alpha} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{s+t}} d\mu(z) < \infty.$$

Moreover,

$$\|\mu\|_{LCM_{\alpha,s}} \approx \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^{\alpha} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{s+t}} d\mu(z).$$

Using [23, Lemma 3.10], we can easily obtain the following result.

Lemma 2.2. *Let $1 < p < \infty$ and $w \in \mathbb{D}$. Set*

$$f_w(z) = \left(\frac{1}{\log \frac{2}{1-|w|^2}}\right)^{1/p} \log \frac{2}{1-\overline{w}z}, \qquad F_w(z) = \frac{1-|w|^2}{\overline{w}(1-\overline{w}z)}, \qquad z \in \mathbb{D}.$$

Then $f_w, F_w \in B_p$.

Lemma 2.3. *Let $1 < p \leq q < \infty$, $0 < s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Suppose that $f \in B_p$ and μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure. Then*

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} (\log \frac{2}{1-|z|^2})^{\frac{q}{p}} dA(z).$$

Proof. Suppose that $f \in B_p$. For any fixed q, s, let α be big enough such that $q\alpha - s > 0$ and $q\alpha + 2 - q - 2s > 0$. From the proof of [12, Lemma 3.2] we have

$$|f(z)|^q \lesssim \int_{\mathbb{D}} \frac{|f'(w)|^q (1-|w|^2)^{q\alpha}}{|1-\overline{w}z|^{q\alpha+2-q}} \left(\log \frac{2}{1-|w|^2}\right)^q dA(w).$$

Since μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure, combining with Lemma 2.1 and the fact that $B_p \subseteq \mathcal{B}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^q d\mu(z) \lesssim \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f'(w)|^q (1-|w|^2)^{q\alpha}}{|1-\overline{w}z|^{q\alpha+2-q}} \left(\log \frac{2}{1-|w|^2}\right)^q dA(w) d\mu(z) \\
&\lesssim \int_{\mathbb{D}} |f'(w)|^q (1-|w|^2)^{q-2+s} (\log \frac{2}{1-|w|^2})^{\frac{q}{p}} \left(\left(\log \frac{2}{1-|w|^2}\right)^{q(1-\frac{1}{p})} \times \right. \\
&\left. \times \int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\overline{w}z|^{2s}} d\mu(z) \right) dA(w) \lesssim \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2+s} (\log \frac{2}{1-|w|^2})^{\frac{q}{p}} dA(w).
\end{aligned}$$

The proof is complete. $\square$

Theorem 2.1. *Let $1 < p < q < \infty$, $0 < s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the inclusion mapping $i : B_p \to T_s^q(\mu)$ is bounded if and only if μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure.*

Proof. First we assume that $i : B_p \to T_s^q(\mu)$ is bounded. For any given arc $I \subseteq \partial\mathbb{D}$, set $a = (1-|I|)\eta$ and η is the center point of I. It is easy to see that

$$|1-\overline{a}z| \approx 1-|a|^2 \approx |I|, \quad z \in S(I).$$

Let

$$f_a(z) = \left(\frac{1}{\log \frac{2}{1-|a|^2}}\right)^{1/p} \log \frac{2}{1-\overline{a}z}.$$

By Lemma 2.2, we see that $f_a \in B_p$. From the boundedness of $i : B_p \to T_s^q(\mu)$, we have

$$\|f_a\|^q_{T_s^q(\mu)} = \sup_{I \subseteq \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f_a(z)|^q d\mu(z) < \infty.$$

By the fact that $|f_a(z)| \approx (\log \frac{2}{|I|})^{1-\frac{1}{p}}$ when $z \in S(I)$, we get

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{(\log \frac{2}{|I|})^{q(1-\frac{1}{p})} \mu(S(I))}{|I|^s} < \infty.$$

Hence μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure.

Conversely, assume that μ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure. Let $f \in B_p$. For any given arc $I \subseteq \partial\mathbb{D}$, set $w = (1-|I|)\eta$ and η is the center point of I. Then

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^q d\mu(z) \lesssim & \frac{1}{|I|^s}\int_{S(I)}|f(z)-f(w)|^q d\mu(z) + \frac{1}{|I|^s}\int_{S(I)}|f(w)|^q d\mu(z)\\ =& A+B,\end{aligned}$$

where

$$A = \frac{1}{|I|^s}\int_{S(I)}|f(z)-f(w)|^q d\mu(z), \qquad B = \frac{1}{|I|^s}\int_{S(I)}|f(w)|^q d\mu(z).$$

Since

$$|f(w)| \lesssim \left(\log\frac{2}{1-|w|^2}\right)^{1-\frac{1}{p}}\|f\|_{B_p} \lesssim \left(\log\frac{2}{|I|}\right)^{1-\frac{1}{p}}\|f\|_{B_p},$$

we get

$$B \lesssim \frac{(\log\frac{2}{|I|})^{q(1-\frac{1}{p})}\mu(S(I))}{|I|^s}\|f\|_{B_p}^q \lesssim \|f\|_{B_p}^q.$$

By Lemma 2.3, we have

$$\begin{aligned}A \lesssim & (1-|w|^2)^s\int_{S(I)}\left|\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{2s}{q}}}\right|^q d\mu(z)\\ \lesssim & (1-|w|^2)^s\int_{\mathbb{D}}\left|\left(\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{2s}{q}}}\right)'\right|^p(1-|z|^2)^{p-2+s}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}}dA(z).\end{aligned}$$

Since

$$\left(\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{2s}{q}}}\right)' = \frac{f'(z)(1-\overline{w}z)^{\frac{2s}{q}}+\overline{w}(\frac{2s}{q})(f(z)-f(w))(1-\overline{w}z)^{\frac{2s}{q}-1}}{(1-\overline{w}z)^{\frac{4s}{q}}},$$

we deduce that $A \lesssim W_1 + W_2$, where

$$W_1 = (1-|w|^2)^s\int_{\mathbb{D}}\frac{|f'(z)|^p}{|1-\overline{w}z|^{\frac{2ps}{q}}}(1-|z|^2)^{p-2+s}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}}dA(z)$$

and

$$W_2 = (1-|w|^2)^s\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\overline{w}z|^{\frac{2ps}{q}+p}}(1-|z|^2)^{p-2+s}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}}dA(z).$$

Since $p<q$ and $\sup_{z\in\mathbb{D}}(1-|z|^2)^{2s(1-\frac{p}{q})}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}} < \infty$, we get that

$$W_1 \lesssim \|f\|_{B_p}^p.$$

Let $0<\epsilon<\min\{\frac{p}{2}, s, 2s(1-\frac{p}{q})\}$. Combining with the fact that $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\epsilon}(\log\frac{2}{1-|z|^2})^{\frac{q}{p}} < \infty$, we obtain

$$W_2 = (1-|w|^2)^s\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\overline{w}z|^{\frac{2ps}{q}+p}}(1-|z|^2)^{p-2+s-\epsilon}dA(z).$$

Making the change of variable $\eta=\varphi_w(z)$ and combining with [23, Proposition 4.2], we have

$$\begin{aligned}
W_2=&(1-|w|^2)^s\int_{\mathbb{D}}\frac{|(f\circ\varphi_w)(\eta)-(f\circ\varphi_w)(0)|^p}{|1-\overline{w}\varphi_w(\eta)|^{\frac{2ps}{q}+p}}(1-|\varphi_w(\eta)|^2)^{p-2+s-\epsilon}\\
&\times\frac{(1-|w|^2)^2}{|1-\overline{w}\eta|^4}dA(\eta)\\
=&(1-|w|^2)^{2s-\frac{2ps}{q}-\epsilon}\int_{\mathbb{D}}|(f\circ\varphi_w)(\eta)-(f\circ\varphi_w)(0)|^p\frac{(1-|\eta|^2)^{p-2+s-\epsilon}}{|1-\overline{w}\eta|^{p+2s-\frac{2ps}{q}-2\epsilon}}dA(\eta)\\
\lesssim&(1-|w|^2)^{2s-\frac{2ps}{q}-\epsilon}\int_{\mathbb{D}}|(f\circ\varphi_w)'(\eta)|^p\frac{(1-|\eta|^2)^{2p-2+s-\epsilon}}{|1-\overline{w}\eta|^{p+2s-\frac{2ps}{q}-2\epsilon}}dA(\eta)\\
\lesssim&(1-|w|^2)^{2s-\frac{2ps}{q}-\epsilon}\int_{\mathbb{D}}|f'(\varphi_w(\eta))|^p(1-|\varphi_w(\eta)|^2)^p\frac{(1-|\eta|^2)^{p-2+s-\epsilon}}{|1-\overline{w}\eta|^{p+2s-\frac{2ps}{q}-2\epsilon}}dA(\eta)\\
\lesssim&(1-|w|^2)^{2s-\frac{2ps}{q}-\epsilon}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^p\frac{(1-|\varphi_w(z)|^2)^{p-2+s-\epsilon}}{|1-\overline{w}\varphi_w(z)|^{p+2s-\frac{2ps}{q}-2\epsilon}}\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}dA(z)\\
\lesssim&(1-|w|^2)^s\int_{\mathbb{D}}|f'(z)|^p\frac{(1-|z|^2)^{2p-2+s-\epsilon}}{|1-\overline{w}z|^{p+\frac{2ps}{q}}}dA(z)\lesssim\|f\|_{B_p}^p.
\end{aligned}$$

Therefore,

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^q d\mu(z)\lesssim\|f\|_{B_p}^p,$$

which implies the desired result. The proof is complete. □

We say that the inclusion mapping $i:B_p\to T_s^q(\mu)$ is compact if

$$\lim_{n\to\infty}\frac{1}{|I|^s}\int_{S(I)}|f_n(z)|^q d\mu(z)=0$$

whenever $I\subseteq\partial\mathbb{D}$ and $\{f_n\}$ is a bounded sequence in B_p that converges to 0 uniformly on compact subsets of $\mathbb{D}$.

Theorem 2.2. *Let $1<p<q<\infty$, $0<s<\infty$. Let μ be a nonnegative Borel measure on $\mathbb{D}$ such that point evaluation is a bounded functional on $T_s^q(\mu)$. Then the inclusion mapping $i:B_p\to T_s^q(\mu)$ is compact if and only if μ is a vanishing $q(1-\frac{1}{p})$-logarithmic s-Carleson measure.*

Proof. First we assume that $i:B_p\to T_s^q(\mu)$ is compact. Let $\{I_k\}$ be a sequence arcs with $\lim_{k\to\infty}|I_k|=0$. Set $a_k=(1-|I_k|)\eta_k$, where η_k is the midpoint of arc I_k. Take

$$f_k(z)=\left(\frac{1}{\log\frac{2}{1-|a_k|^2}}\right)^{1/p}\log\frac{2}{1-\overline{a_k}z}.$$

We see that $f_k \in B_p$ and $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ when $k \to \infty$. Then we get

$$\frac{\left(\log \frac{2}{|I_k|}\right)^{q(1-\frac{1}{p})} \mu(S(I_k))}{|I_k|^s} \lesssim \frac{1}{|I_k|^s} \int_{S(I_k)} |f_k(z)|^q d\mu(z) \to 0,$$

as $k \to \infty$, which implies that μ is a vanishing $q(1-\frac{1}{p})$-logarithmic s-Carleson measure.

Conversely, assume that μ is a vanishing $q(1-\frac{1}{p})$-logarithmic s-Carleson measure. From [12] we see that

$$\|\mu - \mu_r\|_{LCM_{q(1-\frac{1}{p}),s}} \to 0, r \to 1.$$

Here $\mu_r(z) = \mu(z)$ for $|z| < r$ and $\mu_r(z) = 0$ for $r \leq |z| < 1$. Let $\|f_k\|_{B_p} \lesssim 1$ and $\{f_k\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. Then

$$\begin{aligned} \frac{1}{|I|^s} \int_{S(I)} |f_k(z)|^q d\mu(z) &\lesssim \frac{1}{|I|^s} \int_{S(I)} |f_k(z)|^q d\mu_r(z) + \frac{1}{|I|^s} \int_{S(I)} |f_k(z)|^q d(\mu - \mu_r)(z) \\ &\lesssim \frac{1}{|I|^s} \int_{S(I)} |f_k(z)|^q d\mu_r(z) + \|\mu - \mu_r\|_{LCM_{q(1-\frac{1}{p}),s}} \|f_k\|_{B_p}^q \\ &\lesssim \frac{1}{|I|^s} \int_{S(I)} |f_k(z)|^q d\mu_r(z) + \|\mu - \mu_r\|_{LCM_{q(1-\frac{1}{p}),s}}. \end{aligned}$$

Letting $k \to \infty$ and then $r \to 1$, we have $\lim_{k\to\infty} \|f_k\|_{T_s^q(\mu)} = 0$. Therefore $i : B_p \to T_s^q(\mu)$ is compact. □

3. The operators T_g and I_g from B_p to $F(q, q-2, s)$

In this section, we consider the boundedness, compactness and essential norm of operators T_g and I_g from B_p to $F(q, q-2, s)$. Before we state our results in this section, let us recall some definitions.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator. The essential norm of $T : X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf_K \{\|T - K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

Let Φ be a closed subspace of X. Given $f \in X$, the distance from f to Φ, denoted by $\mathrm{dist}_X(f, \Phi)$, is defined by $\mathrm{dist}_X(f, \Phi) = \inf_{g\in\Phi} \|f - g\|_X$.

Suppose that $0 \leq \alpha < \infty, 0 < q, s < \infty$. The space $F_L(q, q-2, s, \alpha)$ is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_L^q = \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^\alpha \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{q-2} (1-|\varphi_a(z)|^2)^s dA(z) < \infty.$$

It is easy to check that $F_L(q, q-2, s, \alpha)$ is a Banach space under the norm $\|f\|_{F_L(q,q-2,s,\alpha)}^q = |f(0)|^q + \|f\|_L^q$ when $q \geq 1$. When $\alpha = 0$, $F_L(q, q-2, s, 0)$ is just the space

$F(q,q-2,s)$. Let $F_L^0(q,q-2,s,\alpha)$ denote the space of all $f \in F_L(q,q-2,s,\alpha)$ such that

$$\lim_{|a|\to 1}\left(\log \frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|f'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)=0.$$

By Lemma 2.1 we easily obtain the following equivalent characterization of the space $F_L(q,q-2,s,\alpha)$.

Lemma 3.1. *Let $0 \le \alpha < \infty, 0 < q,s < \infty$. Then $f \in F_L(q,q-2,s,\alpha)$ if and only if*

$$\sup_{I\subseteq \partial\mathbb{D}}\frac{\left(\log \frac{2}{|I|}\right)^{\alpha}}{|I|^s}\int_{S(I)}|f'(z)|^q(1-|z|^2)^{q-2+s}dA(z)<\infty.$$

Moreover,

$$\|f\|^q_{F_L(q,q-2,s,\alpha)} \approx \sup_{I\subseteq \partial\mathbb{D}}\frac{\left(\log \frac{2}{|I|}\right)^{\alpha}}{|I|^s}\int_{S(I)}|f'(z)|^q(1-|z|^2)^{q-2+s}dA(z).$$

Lemma 3.2. *Let $0 \le \alpha < \infty, 0 < q,s < \infty$. If $g \in F_L(q,q-2,s,\alpha)$, then*

$$\begin{aligned}&\limsup_{|a|\to 1}\left(\left(\log \frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{1/q}\\ &\qquad\approx \mathrm{dist}_{F_L(q,q-2,s,\alpha)}(g,F_L^0(q,q-2,s,\alpha)) \approx \limsup_{r\to 1^-}\|g-g_r\|_{F_L(q,q-2,s,\alpha)}.\end{aligned}$$

Here $g_r(z)=g(rz)$, $0<r<1, z\in\mathbb{D}$.

Proof. For any given $g \in F_L(q,q-2,s,\alpha)$, then $g_r \in F_L^0(q,q-2,s,\alpha)$ and

$$\|g_r\|_{F_L(q,q-2,s,\alpha)} \lesssim \|g\|_{F_L(q,q-2,s,\alpha)}.$$

Let $\delta \in (0,1)$. We choose $a \in (0,\delta)$. Then $\varphi_a(z)$ lies in a compact subset of $\mathbb{D}$. So $\lim_{r\to 1}\sup_{z\in\mathbb{D}}|g'(\varphi_a(z))-rg'(r\varphi_a(z))|=0$. Making a change of variables, we have

$$\begin{aligned}&\lim_{r\to 1}\sup_{|a|\le\delta}\left(\log \frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)-g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\\ =\;&\lim_{r\to 1}\sup_{|a|\le\delta}\left(\log \frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(\sigma_a(z))-g_r'(\sigma_a(z))|^q(1-|z|^2)^{q+s-2}|\varphi_a'(z)|^q dA(z)\\ =\;&\lim_{r\to 1}\sup_{|a|\le\delta}\sup_{z\in\mathbb{D}}|g'(\varphi_a(z))-g_r'(\varphi_a(z))|^q\left(\log \frac{2}{1-|a|^2}\right)^{\alpha}\times\\ \times\;&\int_{\mathbb{D}}(1-|z|^2)^{q+s-2}|\varphi_a'(z)|^q dA(z)=0.\end{aligned}$$

By the definition of distance, we obtain

$$\begin{aligned}&\mathrm{dist}_{F_L(q,q-2,s,\alpha)}(g,F_L^0(q,q-2,s,\alpha)) = \inf_{f\in F_L^0(q,q-2,s,\alpha)}\|g-f\|_{F_L(q,q-2,s,\alpha)}\\ \le\;&\lim_{r\to 1}\|g-g_r\|_{F_L(q,q-2,s,\alpha)}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{r\to 1}\left(\sup_{|a|>\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)-g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\
&+ \lim_{r\to 1}\left(\sup_{|a|\le\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)-g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\
&\lesssim \left(\sup_{|a|>\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\
&+ \lim_{r\to 1}\left(\sup_{|a|>\delta}\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{\frac{1}{q}}.
\end{aligned}
$$

Let $\psi_{r,a}(z)=\varphi_{ra}\circ r\varphi_a(z)$. Then $\psi_{r,a}$ is an analytic self-map of $\mathbb{D}$ and $\psi_{r,a}(0)=0$. Making a change variable of $z=\varphi_a(z)$ and applying the Littlewood's subordination theorem (see Theorem 1.7 of [4]), we have

$$
\begin{aligned}
&\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
=&\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g_r'(\varphi_a(z))|^q(1-|\varphi_a(z)|^2)^q(1-|z|^2)^{s-2} dA(z)\\
\le&\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'\circ\varphi_{ra}\circ\psi_{r,a}(z)|^q(1-|\varphi_{ra}\circ\psi_{r,a}(z)|^2)^q(1-|z|^2)^{s-2} dA(z)\\
\le&\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'\circ\varphi_{ra}\circ\psi_{r,a}(z)|^q(1-|\varphi_{ra}\circ\psi_{r,a}(z)|^2)^q(1-|z|^2)^{s-2} dA(z)\\
\le&\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'\circ\varphi_{ra}(z)|^q(1-|\varphi_{ra}(z)|^2)^q(1-|z|^2)^{s-2} dA(z)\\
\le&\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_{ra}(z)|^2)^s dA(z).
\end{aligned}
$$

Since δ is arbitrary, we get

$$
\begin{aligned}
&\operatorname{dist}_{F_L(q,q-2,s,\alpha)}(g,F_L^0(q,q-2,s,\alpha))\\
\lesssim&\limsup_{|a|\to 1}\left(\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{1/q}.
\end{aligned}
$$

On the other hand, for any $g\in F_L(q,q-2,s,q(1-\frac{1}{p}))$,

$$
\begin{aligned}
&\operatorname{dist}_{F_L(q,q-2,s,\alpha)}(g,F_L^0(q,q-2,s,\alpha))=\inf_{f\in F_L^0(q,q-2,s,\alpha)}\|g-f\|_{F_L(q,q-2,s,\alpha)}\\
\gtrsim&\limsup_{|a|\to 1}\left(\left(\log\frac{2}{1-|a|^2}\right)^{\alpha}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^s dA(z)\right)^{1/q},
\end{aligned}
$$

implies the desired result. □

Lemma 3.3. *Let $1 < p < q < \infty$, $0 < s < \infty$. If $0 < r < 1$ and $g \in F_L(q, q-2, s, q(1-\frac{1}{p}))$, then $T_{g_r} : B_p \to F(q, q-2, s)$ is compact.*

Proof. Given $\{f_k\} \subset B_p$ such that $\{f_k\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ and $\sup_k \|f_k\|_{B_p} \leq 1$. For each $a \in \mathbb{D}$,

$$\begin{aligned}
&\|T_{g_r} f_k\|^q_{F(q,q-2,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^q |g_r'(z)|^q (1-|z|^2)^{q-2} (1-|\varphi_a(z)|^2)^s dA(z)\\
\lesssim& \frac{\|g\|^q_{F_L(q,q-2,s,q(1-\frac{1}{p}))}}{\left(\log \frac{2}{1-r^2}\right)^{q(1-\frac{1}{p})} (1-r^2)^q} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^q (1-|z|^2)^{q-2} (1-|\varphi_a(z)|^2)^s dA(z)\\
\lesssim& \frac{\|g\|^q_{F_L(q,q-2,s,q(1-\frac{1}{p}))}}{\left(\log \frac{2}{1-r^2}\right)^{q(1-\frac{1}{p})} (1-r^2)^q} \int_{\mathbb{D}} |f_k(z)|^q (1-|z|^2)^{q-2} dA(z)\\
\lesssim& \frac{\|g\|^q_{F_L(q,q-2,s,q(1-\frac{1}{p}))}}{\left(\log \frac{2}{1-r^2}\right)^{q(1-\frac{1}{p})} (1-r^2)^q} \int_{\mathbb{D}} |f_k'(z)|^q (1-|z|^2)^{q-2} dA(z)\\
\lesssim& \frac{\|g\|^q_{F_L(q,q-2,s,q(1-\frac{1}{p}))}}{\left(\log \frac{2}{1-r^2}\right)^{q(1-\frac{1}{p})} (1-r^2)^q} \|f_k\|^q_{B_p} \int_{\mathbb{D}} 1 dA(z).
\end{aligned}$$

By the dominated convergence theorem, we get

$$\begin{aligned}
\lim_{k\to\infty} \|T_{g_r} f_k\|^q_{F(q,q-2,s)} &\lesssim \lim_{k\to\infty} \int_{\mathbb{D}} |f_k(z)|^q (1-|z|^2)^{q-2} dA(z)\\
&\lesssim \int_{\mathbb{D}} \lim_{k\to\infty} |f_k(z)|^q (1-|z|^2)^{q-2} dA(z) = 0,
\end{aligned}$$

as desired. The proof is complete. □

The following result is very useful to study the essential norm of operators on some analytic function spaces, see [17].

Lemma 3.4. *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

1. *The point evaluation functionals on Y are continuous.*
2. *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
3. *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if for any bounded sequence $\{f_n\}$ in X such that $\{f_n\}$ converges to zero uniformly on every compact set of $\mathbb{D}$, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Theorem 3.1. *Let $1 < p < q < \infty$, $0 < s < \infty$ and $g \in H(\mathbb{D})$. Then $T_g : B_p \to F(q, q-2, s)$ is bounded if and only if $g \in F_L(q, q-2, s, q(1-\frac{1}{p}))$.*

Proof. Suppose that $f \in B_p$ and $g \in F_L(q, q-2, s, q(1-\frac{1}{p}))$. From Lemma 3.2 we see that $d\mu_g(z) = |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)$ is a $q(1-\frac{1}{p})$-logarithmic s-Carleson measure. By Theorem 1, for any $I \subseteq \partial\mathbb{D}$ we deduce that

$$\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)} |(T_g f)'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\
=&\frac{1}{|I|^s}\int_{S(I)} |f(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+s+\frac{q}{p}}dA(z)\\
=&\frac{1}{|I|^s}\int_{S(I)} |f(z)|^q d\mu_g(z) \lesssim \|f\|_{B_p}^q \|g\|_{F_L(q,q-2,s)}^q < \infty,
\end{aligned}$$

which implies that $T_g : B_p \to F(q, q-2, s)$ is bounded by Lemma 3.1 again.

Conversely, suppose that $T_g : B_p \to F(q, q-2, s)$ is bounded. For any $I \subseteq \partial\mathbb{D}$, let $a = (1-|I|)\zeta$, where ζ is the center of I. Then $1-|a| \approx |1-\overline{a}z| \approx |I|, \ \ z \in S(I)$. Let f_a be defined as in Lemma 2.2. We have

$$\begin{aligned}
&\frac{\left(\log\frac{2}{|I|}\right)^{q(1-\frac{1}{p})}}{|I|^s}\int_{S(I)} |g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\
\lesssim&\frac{1}{|I|^s}\int_{S(I)} |f_a(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\
\lesssim&\frac{1}{|I|^s}\int_{S(I)} |(T_g f_a)'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\
\lesssim&\|T_g f_a\|_{F(q,q-2,s)}^q < \infty,
\end{aligned}$$

which implies that $g \in F_L(q, q-2, s, q(1-\frac{1}{p}))$ by Lemma 3.1. $\square$

Theorem 3.2. *Let $1 < p < q < \infty$, $0 < s < \infty$ and $g \in H(\mathbb{D})$. Then $I_g : B_p \to F(q, q-2, s)$ is bounded if and only if $g \in H^\infty$.*

Proof. Let $f \in B_p$ and $g \in H^\infty$. By the fact that $B_p \subset \mathcal{B}$, we get

$$\begin{aligned}
&\int_{\mathbb{D}} |(I_g f)'(z)|^q(1-|z|^2)^{q-2}\left(1-|\varphi_a(z)|^2\right)^s dA(z)\\
=&\int_{\mathbb{D}} |f'(z)|^q|g(z)|^q(1-|z|^2)^{q-2}\left(1-|\varphi_a(z)|^2\right)^s dA(z)\\
=&\|g\|_{H^\infty}^q\|f\|_{\mathcal{B}}^{q-p}\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z) \lesssim \|g\|_{H^\infty}^q\|f\|_{B_p}^q < \infty,
\end{aligned}$$

which implies that $I_g : B_p \to F(q, q-2, s)$ is bounded.

Conversely, assume that $I_g : B_p \to F(q, q-2, s)$ is bounded. For $a \in \mathbb{D}$ and $r > 0$, let $\mathbb{D}(a, r) = \{z \in \mathbb{D} : \beta(a, z) < r\}$ denote the Bergman metric disk centered at a with radius r. Here $\beta(a, z)$ is the Bergman metric between z and a. For any

$w \in \mathbb{D}$, let F_w be defined as in Lemma 2.2. Using the subharmonic property of $|g|^q$ and the fact that (see [23])

$$\frac{(1-|w|^2)^2}{|1-\bar{w}z|^4} \approx \frac{1}{(1-|z|^2)^2} \approx \frac{1}{(1-|w|^2)^2} \approx \frac{1}{|\mathbb{D}(w,r)|}, \qquad z \in \mathbb{D}(w,r),$$

where $|\mathbb{D}(w,r)|$ denotes the area of the Bergman disk $\mathbb{D}(w,r)$, we have

$$\begin{aligned}
\infty >& \|I_g F_w\|^q_{F(q,q-2,s)} \\
\gtrsim& \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_w'(z)|^q |g(z)|^q (1-|z|^2)^{q-2} \left(1-|\varphi_a(z)|^2\right)^s dA(z) \\
\gtrsim& \int_{\mathbb{D}} |F_w'(z)|^q |g(z)|^q (1-|z|^2)^{q-2} \left(1-|\varphi_w(z)|^2\right)^s dA(z) \\
\gtrsim& \int_{\mathbb{D}(w,r)} |g(z)|^q (1-|z|^2)^{-2} \left(1-|\varphi_w(z)|^2\right)^s dA(z) \\
\gtrsim& \frac{1}{(1-|w|^2)^2} \int_{\mathbb{D}(w,r)} |g(z)|^q dA(z) \gtrsim |g(w)|^q,
\end{aligned}$$

which implies

$$\infty > \|I_g F_w\|^q_{F(q,q-2,s)} \gtrsim \|g\|^q_{H^\infty},$$

as desired. The proof is complete. □

Remark. Let $1 < p < q < \infty$, $0 < s < \infty$ and $g \in H(\mathbb{D})$. From the fact that

$$M_g f(z) = f(0)g(0) + I_g f(z) + T_g f(z),$$

we see that $M_g : B_p \to F(q,q-2,s)$ is bounded if and only if

$$g \in F_L(q,q-2,s,q(1-\frac{1}{p})) \cap H^\infty.$$

Theorem 3.3. *Let $1 < p < q < \infty$, $0 < s < \infty$ and $g \in H(\mathbb{D})$. If $T_g : B_p \to F(q,q-2,s)$ is bounded, then*

$$\|T_g\|_{e,B_p\to F(q,q-2,s)} \approx \operatorname{dist}_{F_L(q,q-2,s,q(1-\frac{1}{p}))}(g, F_L^0(q,q-2,s,q(1-\frac{1}{p}))).$$

Proof. Let $\{a_k\}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty}|a_k| = 1$. For each k, set

$$f_{a_k}(z) = \left(\frac{1}{\log\frac{2}{1-|a_k|^2}}\right)^{1/p} \log\frac{2}{1-\overline{a_k}z}.$$

Then $\{f_{a_k}\}$ is bounded in B_p and $\{f_{a_k}\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$. For any given compact operator $K : B_p \to F(q,q-2,s)$, by

Lemma 3.4 we have $\lim_{k\to\infty}\|Kf_{a_k}\|_{F(q,q-2,s)}=0$. So

$$\begin{aligned}
&\|T_g-K\|\gtrsim\limsup_{k\to\infty}\|(T_g-K)f_{a_k}\|_{F(q,q-2,s)}\\
&\gtrsim\limsup_{k\to\infty}\Big(\|T_gf_{a_k}\|_{F(q,q-2,s)}-\|Kf_{a_k}\|_{F(q,q-2,s)}\Big)\\
&=\limsup_{k\to\infty}\|T_gf_{a_k}\|_{F(q,q-2,s)}\\
&\geq\limsup_{k\to\infty}\left(\int_{\mathbb{D}}|f_{a_k}(z)|^q|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_{a_k}(z)|^2)^s dA(z)\right)^{\frac{1}{q}}\\
&\gtrsim\limsup_{|a_k|\to1}\left(\left(\log\frac{2}{1-|a_k|^2}\right)^{q(1-\frac{1}{p})}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_{a_k}(z)|^2)^s dA(z)\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|T_g\|_{e,B_p\to F(q,q-2,s)}\\
&\gtrsim\limsup_{k\to\infty}\left(\left(\log\frac{2}{1-|a_k|^2}\right)^{q(1-\frac{1}{p})}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\varphi_{a_k}(z)|^2)^s dA(z)\right)^{\frac{1}{q}}.
\end{aligned}$$

By Lemma 3.2 and the arbitrariness of $\{a_k\}$, we get that

$$\|T_g\|_{e,B_p\to F(q,q-2,s)}\gtrsim \mathrm{dist}_{F_L(q,q-2,s,q(1-\frac{1}{p}))}(g,F_L^0(q,q-2,s,q(1-\frac{1}{p}))).$$

On the other hand, by Lemma 3.3, $T_{g_r}:B_p\to F(q,q-2,s)$ is compact. Then

$$\|T_g\|_{e,B_p\to F(q,q-2,s)}\leq\|T_g-T_{g_r}\|=\|T_{g-g_r}\|\approx\|g-g_r\|_{F_L(q,q-2,s,q(1-\frac{1}{p}))}.$$

Using Lemma 3.2 again, we get

$$\begin{aligned}
\|T_g\|_{e,B_p\to F(q,q-2,s)}&\lesssim\limsup_{r\to1^-}\|g-g_r\|_{F_L(q,q-2,s,q(1-\frac{1}{p}))}\\
&\approx \mathrm{dist}_{F_L(q,q-2,s,q(1-\frac{1}{p}))}(g,F_L^0(q,q-2,s,q(1-\frac{1}{p}))).
\end{aligned}$$

The proof is complete. □

By the well-known result that $T:X\to Y$ is compact if and only if $\|T\|_{e,X\to Y}=0$, we get the following result by Theorem 3.3 directly.

Corollary 3.1. *Let $1<p<q<\infty$ and $0<s<\infty$. If $g\in H(\mathbb{D})$, then $T_g:B_p\to F(q,q-2,s)$ is compact if and only if*

$$g\in F_L^0(q,q-2,s,q(1-\frac{1}{p})).$$

Theorem 3.4. *Let $1<p<q<\infty$ and $0<s<\infty$. If $g\in H(\mathbb{D})$ such that $I_g:B_p\to F(q,q-2,s)$ is bounded, then*

$$\|I_g\|_{e,B_p\to F(q,q-2,s)}\approx\|g\|_{\mathcal{H}^\infty}.$$

Proof. Let $\{a_k\}$ and K be defined as in the proof of Theorem 3.3. Set

$$F_{a_k}(z) = \frac{1-|a_k|^2}{\overline{a_k}(1-\overline{a_k}z)}, \quad z \in \mathbb{D}.$$

By Lemma 2.2 we see that $F_{a_k} \in B_p$. By Lemma 3.4 we get $\lim_{k\to\infty} \|KF_{a_k}\|_{F(q,q-2,s)} = 0$. Hence,

$$\begin{aligned}
\|I_g - K\| &\gtrsim \limsup_{k\to\infty} \|(I_g - K)F_{a_k}\|_{F(q,q-2,s)} \\
&\gtrsim \limsup_{k\to\infty} \left(\|I_g F_{a_k}\|_{F(q,q-2,s)} - \|KF_{a_k}\|_{F(q,q-2,s)}\right) \\
&= \limsup_{k\to\infty} \|I_g F_{a_k}\|_{F(q,q-2,s)},
\end{aligned}$$

which implies

$$\|I_g\|_{e,B_p\to F(q,q-2,s)} \gtrsim \limsup_{k\to\infty} \|I_g F_{a_k}\|_{F(q,q-2,s)}.$$

Similarly to the proof of Theorem 3.2 we get that $\|I_g F_{a_k}\|_{F(q,q-2,s)} \gtrsim |g(a_k)|$, which implies that

$$\|I_g\|_{e,B_p\to F(q,q-2,s)} \gtrsim \|g\|_{H^\infty}.$$

On the other hand, by Theorem 3.2 we obtain

$$\|I_g\|_{e,B_p\to F(q,q-2,s)} = \inf_K \|I_g - K\| \le \|I_g\| \lesssim \|g\|_{H^\infty}.$$

The proof is complete. □

From Theorem 3.4 we get the following result.

Corollary 3.2. *Let $1 < p < q < \infty$ and $0 < s < \infty$. If $g \in H(\mathbb{D})$, then $I_g : B_p \to F(q, q-2, s)$ is compact if and only if $g = 0$.*

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NONHOMOGENEOUS DUAL WAVELET FRAMES WITH THE p-REFINABLE STRUCTURE IN $L^2(\mathbb{R}^+)$

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Abstract. In recent years, nonhomogeneous wavelet frames have been widely studied by many researchers, while the ones in $L^2(\mathbb{R}^+)$ have not. Some practical applications indicate that it is desirable to have a nonhomogeneous dual wavelet frame in $L^2(\mathbb{R}^+)$ because of the time variable can not take negative values in signal sampling. In addition, similar to the homogeneous dual wavelet frames, the nonhomogeneous ones derived from refinable functions have fast wavelet algorithms. In view of this, under the setting of $L^2(\mathbb{R}^+)$, we study the properties of nonhomogeneous dual wavelet frames, and obtain a construction of nonhomogeneous dual wavelet frames from a pair of p-refinable functions.

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1. Introduction

The concept of frames was introduced already in 1952 by Duffin and Schaeffer [10] in the study of nonharmonic Fourier series, but the importance of this concept was not recognized by mathematicians until the ground-breaking work of Daubechies et al. [7]. In the past three decades, the theory of frames has attracted many mathematicians and engineers, and has achieved fruitful results (see [5, 6, 27, 28] and many references therein).

An important example about frames is wavelet frames, which are generated by translation and dilation of a finite number of functions. Wavelet frames have many good properties that make them useful in the study of signal processing, image restorations, sampling theory, function spaces [2, 17, 24, 32] and so forth. In order to make the wavelet frames have more applications, several generalized notions of wavelet frames are proposed and studied, namely tight wavelet frames [18], dual wavelet frames [19], (quasi) affine frames and (quasi) affine dual frames [3, 27]. One of the fundamental methods to construct tight wavelet frames from refinable functions is the unitary extension principle (UEP) which was proposed

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by Ron and Shen [27, 28], and then was extended by Daubechies et al. [5] in the form of the oblique extension principle (OEP). They gave sufficient conditions for constructing tight affine frames and affine dual frames from any given refinable functions. From then on, many works along this direction can be found in [1, 4, 25, 34]. Observe that all above works main focus on homogeneous (dual) wavelet frames. In applications, fast wavelet transforms are our main concern, and nonhomogeneous (dual) wavelet frames derived from refinable functions have fast wavelet algorithms. Han in [20–22] comprehensive studied nonhomogeneous (dual) wavelet frames and they connect with homogeneous ones. Similar to the homogeneous dual wavelet frames, the nonhomogeneous ones derived from refinable functions have fast wavelet algorithms, which play an important role in wavelet analysis.

Wavelets and frames have been generalized in many different settings. For example, Lang [23] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group by following the procedure of Daubechies [8] (or see [9]) via scaling filters, and these wavelets turn out to be certain lacunary Walsh series on the real line. Recent works about wavelets and frames on the Cantor group and Vilenkin groups can be found in [12–16]. It is worth noting that the first constructions of wavelet frames on the positive half line with binary addition were proposed by Farkov [11], in which wavelets and frames on the half line $\mathbb{R}^+$ related to the Walsh-Dirichlet kernel and its modification are considered. Shah and Debnath [30] studied Dyadic wavelet frames on a half-line using the Walsh-Fourier transform. Shah in [31] give an explicit construction of tight wavelet frames generated by the Walsh polynomials on positive half-line $\mathbb{R}^+$ using the extension principles, and derive the wavelet frames decomposition and reconstruction formulas.

Intuitively, we can obtain $L^2(\mathbb{R}^+)$ wavelet frames by projection from $L^2(\mathbb{R})$ ones, while it is not the case for $L^2(\mathbb{R}^+)$ since the projections do not have complete affine structure. Furthermore, in many practical problems of nature and physics, the time variable can not take negative values in signal sampling; and in mathematics, $\mathbb{R}^+$ is not closed according to the usual addition "+". As a result, the classical Fourier transform method can not be directly applied to the wavelet frames in $L^2(\mathbb{R}^+)$. However, $\mathbb{R}^+$ is closed in terms of the operation "$\oplus$", and the Walsh-Fourier transform is defined by $\oplus$.

Inspired by the above observation, in this paper we investigate nonhomogeneous dual wavelet frames under the setting of $L^2(\mathbb{R}^+)$. In Section 2 we give some preliminaries and notations. In Section 3 we present some properties of nonhomogeneous dual wavelet frames in $L^2(\mathbb{R}^+)$. Section 4 is devoted to constructing nonhomogeneous dual wavelet frames from a pair of general p-refinable functions.

2. Preliminaries and notations

We first recall some basics of addition "$\oplus$"and subtraction "$\ominus$". We denote by $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{N}$ the set of integers, the set of nonnegative integers and the set of positive integers, respectively; and by $\mathbb{N}_t$ the set of $\{0, 1, \cdots, t-1\}$ for $t \in \mathbb{N}$. Let $p > 1$ be a fixed integer. For $x, y \in \mathbb{N}_p$, we define the $\oplus$ and $\ominus$ on $\mathbb{N}_p$ respectively by

$$x \oplus y = (x + y)(\mathrm{mod}\ p) = \begin{cases} x + y, & x + y < p, \\ x + y - p, & x + y > p, \end{cases}$$

and

$$x \ominus y = (x - y)(\mathrm{mod}\ p) = \begin{cases} x - y, & x > y, \\ x - y + p, & x < y. \end{cases}$$

Given $x \in \mathbb{R}^+$, we denote by $[x]$ its integer part, and by $\{x\}$ its fraction part. Then we have

$$x = \sum_{j=1}^{k_x} x_{-j} p^{j-1} + \sum_{j=1}^{\infty} x_j p^{-j} = [x] + \{x\}, \tag{2.1}$$

where $k_x \in \mathbb{Z}^+, x_j, x_{-j} \in \mathbb{N}_p$ for $j \in \mathbb{N}$, and the sequence $\{x_j\}_{j=1}^{\infty}$ is required to have only finitely many nonzero terms when x is rational. For $y, \omega \in \mathbb{R}^+$, we define y_j, y_{-j} and ω_j, ω_{-j} similarly. Using the above operations on $\mathbb{N}_p$, we define the $\oplus$ and $\ominus$ on $\mathbb{R}^+$ respectively by

$$x \oplus y = \sum_{j=1}^{\infty} (x_j \oplus y_j) p^{j-1} + \sum_{j=1}^{\infty} (x_{-j} \oplus y_{-j}) p^{-j} \tag{2.2}$$

and

$$x \ominus y = \sum_{j=1}^{\infty} (x_j \ominus y_j) p^{j-1} + \sum_{j=1}^{\infty} (x_{-j} \ominus y_{-j}) p^{-j} \tag{2.3}$$

for $x, y \in \mathbb{R}^+$. Note that $z = x \ominus y$ if $z \oplus y = x$, and it is easy to check that $\mathbb{R}^+$ is a group under the operation "$\oplus$". Given $x, \omega \in \mathbb{R}^+$, write

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j \omega_{-j} + x_{-j} \omega_j) \right). \tag{2.4}$$

For a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, its Walsh-Fourier transform is defined by

$$\mathcal{F} f(\cdot) = \int_{\mathbb{R}^+} f(x) \overline{\chi(x, \cdot)} dx,$$

and is extended uniquely to the whole space $L^2(\mathbb{R}^+)$. The details of the Walsh-Fourier transform and Walsh series can be found in [29]. Similarly to the classical Fourier transform, the Walsh-Fourier transform is an unitary operator on $L^2(\mathbb{R}^+)$, and the system $\{\chi(k, \cdot) : k \in \mathbb{Z}^+\}$ is an orthonormal basis for $L^2(\mathbb{T})$ with $\mathbb{T} = [0, 1)$.

We define the dilation operator D and the translation operator $\mathcal{T}_k$ with $k \in \mathbb{Z}^+$ respectively by

$$Df(\cdot) = p^{1/2} f(p\,\cdot) \text{ and } \mathcal{T}_k f(\cdot) = f(\cdot \ominus k) \text{ for } f \in L^2(\mathbb{R}^+).$$

Obviously, they are both unitary operators on $L^2(\mathbb{R}^+)$. And we write

$$f_{j,k} = D^j \mathcal{T}_k f \text{ for } j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.$$

Let $J \in \mathbb{Z}, \psi_0 \in L^2(\mathbb{R}^+)$ and $\Psi = \{\psi_1, \psi_2, \cdots, \psi_L\}$ with $L \in \mathbb{N}$ be a finite subset in $L^2(\mathbb{R}^+)$. We define the homogeneous wavelet system $X(\Psi)$ and the nonhomogeneous wavelet system $X_J(\psi_0,\, \Psi)$ respectively by

$$X(\Psi) = \{\psi_{l,j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^+, 1 \leq l \leq L\} \tag{2.5}$$

and

$$X_J(\psi_0,\, \Psi) = \{\psi_{0,J,k} : k \in \mathbb{Z}^+\} \cup \{\psi_{l,j,k} : j \geq J, k \in \mathbb{Z}^+, 1 \leq l \leq L\}. \tag{2.6}$$

And we write $X_0(\psi_0,\, \Psi) = X(\psi_0,\, \Psi)$ for simplicity. Let $X(\widetilde{\Psi})$ and $X_J(\tilde{\psi}_0,\, \widetilde{\Psi})$ be defined similarly. We say $X(\Psi)$ is a *homogeneous wavelet frame* (HWF) in $L^2(\mathbb{R}^+)$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^+} |\langle f,\, \psi_{l,j,k}\rangle|^2 \leq B\|f\|^2 \text{ for } f \in L^2(\mathbb{R}^+), \tag{2.7}$$

where A, B are called frame bounds. It is called a Bessel sequence in $L^2(\mathbb{R}^+)$ if only the right-hand side of (2.7) holds, where B is called a Bessel bound. We say $(X(\Psi),\, X(\widetilde{\Psi}))$ is a *homogeneous dual wavelet frame* (HDWF) in $L^2(\mathbb{R}^+)$ if $X(\Psi)$ and $X(\widetilde{\Psi})$ are both Bessel sequences in $L^2(\mathbb{R}^+)$, and the identity

$$\langle f,\, g\rangle = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,j,k}\rangle \langle \psi_{l,j,k},\, g\rangle \tag{2.8}$$

holds for $f, g \in L^2(\mathbb{R}^+)$. Similarly, we say $X_J(\psi_0,\, \Psi)$ is a *nonhomogeneous wavelet frame* (NWF) in $L^2(\mathbb{R}^+)$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k \in \mathbb{Z}^+} |\langle f,\, \psi_{0,J,k}\rangle|^2 + \sum_{l=1}^{L} \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^+} |\langle f,\, \psi_{l,j,k}\rangle|^2 \leq B\|f\|^2 \text{ for } f \in L^2(\mathbb{R}^+), \tag{2.9}$$

where A, B are called frame bounds. It is called a Bessel sequence in $L^2(\mathbb{R}^+)$ if only the right-hand side of (2.9) holds, where B is called a Bessel bound. We say $\left(X_J(\psi_0; \Psi),\, X_J(\tilde{\psi}_0; \widetilde{\Psi})\right)$ is a *nonhomogeneous dual wavelet frame* (NDWF) in $L^2(\mathbb{R}^+)$ if $X_J(\psi_0; \Psi)$ and $X_J(\tilde{\psi}_0; \widetilde{\Psi})$ are both Bessel sequences in $L^2(\mathbb{R}^+)$, and the identity

$$\langle f,\, g\rangle = \sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{0,J,k}\rangle\langle \psi_{0,J,k},\, g\rangle + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle \tag{2.10}$$

holds for $f, g \in L^2(\mathbb{R}^+)$. It is easy to check that both $X_J(\psi_0;\Psi)$ and $X_J(\tilde{\psi}_0;\widetilde{\Psi})$ are frames for $L^2(\mathbb{R}^+)$, and reconstruction formula

$$f = \sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{0,J,k}\rangle \psi_{0,J,k} + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,j,k}\rangle \psi_{l,j,k},$$

or

$$f = \sum_{k\in\mathbb{Z}^+} \langle f,\, \psi_{0,J,k}\rangle \tilde{\psi}_{0,J,k} + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle f,\, \psi_{l,j,k}\rangle \tilde{\psi}_{l,j,k}$$

holds for $f \in L^2(\mathbb{R}^+)$ if $\Big(X_J(\psi_0;\Psi),\, X_J(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a NDWF in $L^2(\mathbb{R}^+)$.

Nonhomogeneous (dual) wavelet frames play an important role in frame theory because they are related to filter banks and have a natural relationship with refinable structures as pointed out in [26] where this type of wavelet frames was introduced for the first time. It is worth noting that Han named the term 'nonhomogeneous' for this type of frames and widely studied them in the distribution space and in $L^2(\mathbb{R}^d)$ [21, 22]. In particular, Han proved that if $\Big(X_{J_0}(\psi_0;\Psi),\, X_{J_0}(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a NDWF in $L^2(\mathbb{R}^d)$ for some $J_0 \in \mathbb{Z}$, then $\Big(X_J(\psi_0;\Psi),\, X_J(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a NDWF in $L^2(\mathbb{R}^d)$ for a general $J \in \mathbb{Z}$, and $(X(\Psi),\, X(\widetilde{\Psi}))$ is a HDWF in $L^2(\mathbb{R}^d)$.

3. Some properties of NDWFs in $L^2(\mathbb{R}^+)$

This section is devoted to some properties of NDWFs in $L^2(\mathbb{R}^+)$. Observe that the dilation operator and the Walsh-Fourier transforms are unitary operator on $L^2(\mathbb{R}^+)$. Let $\{\mathcal{T}_k\psi_0 : k \in \mathbb{Z}^+\}$ and $\{\mathcal{T}_k\tilde{\psi}_0 : k \in \mathbb{Z}^+\}$ be Bessel sequences in $L^2(\mathbb{R}^+)$, define a quasi-interpolatory operator P_J on $L^2(\mathbb{R}^+)$ with $J \in \mathbb{Z}$ by

$$P_J f = \sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{0,J,k}\rangle \psi_{0,J,k} \text{ for } f \in L^2(\mathbb{R}^+). \tag{3.1}$$

It is not difficult to prove that $\{\psi_{0,J,k} : k \in \mathbb{Z}^+\}$ and $\{\tilde{\psi}_{0,J,k} : k \in \mathbb{Z}^+\}$ are also Bessel sequences for each $J \in \mathbb{Z}$ under the Bessel assumptions of integer translation of ψ_0 and $\tilde{\psi}_0$. Therefore, P_J is a bounded operator by the Cauchy-Schwarz inequality, and is well defined. Also we have next result.

Lemma 3.1. *Given $J \in \mathbb{Z}$, let $\{\mathcal{T}_k\psi_0 : k \in \mathbb{Z}^+\}$ and $\{\mathcal{T}_k\tilde{\psi}_0 : k \in \mathbb{Z}^+\}$ be Bessel sequences in $L^2(\mathbb{R}^+)$, then we have*

$$\lim_{J\to-\infty} P_J f = 0 \text{ for } f \in L^2(\mathbb{R}^+). \tag{3.2}$$

Proof. Fix $f \in L^2(\mathbb{R}^+)$. For an arbitrary $\epsilon > 0$, let $g \in L^2(\mathbb{R}^+)$ with $\operatorname{supp}(g) \subset [0,\, R]$ for some $R > 0$ such that $\|f - g\| < \epsilon$. Then by the above argument, we have

$$\|P_J f\| \leq \|P_J(f-g)\| + \|P_J g\| \leq C\epsilon + \|P_J g\| \text{ for some constant } C > 0.$$

Next, we prove $\lim\limits_{J\to-\infty} P_J g = 0$ to complete the proof. We estimate

$$\begin{aligned}\|P_J g\|^2 &\leq C \sum_{k\in\mathbb{Z}^+} |\langle g,\, \tilde{\psi}_{0,J,k}\rangle|^2 \leq C\|g\|^2 \sum_{k\in\mathbb{Z}^+} \int_{[0,\, R]} |\tilde{\psi}_{0,J,k}(x)|^2 dx \\ &= C\|g\|^2 \sum_{k\in\mathbb{Z}^+} \int_{[0,\, R]} |p^{J/2}\tilde{\psi}_0(p^J x \ominus k)|^2 dx = C\|g\|^2 \int_{\cup_{k\in\mathbb{Z}^+}[0,\, p^J R+k]} |\tilde{\psi}_0(y)|^2 dy,\end{aligned} \tag{3.3}$$

it tends to 0 as $J \to -\infty$ by Lebesgue's dominate convergence theorem, and thus $\lim\limits_{J\to-\infty} P_J g = 0$. □

The following theorem shows that the equivalence of NDWFs between different scale levels, and an NDWF in $L^2(\mathbb{R}^+)$ can derive an HDWF.

Theorem 3.1. *Given an integer J_0. Let $\psi_0 \in L^2(\mathbb{R}^+)$ and $\Psi = \{\psi_1, \psi_2, \cdots, \psi_L\}$ be a finite subset in $L^2(\mathbb{R}^+)$. Suppose $\left(X_{J_0}(\psi_0;\Psi),\, X_{J_0}(\tilde{\psi}_0;\widetilde{\Psi})\right)$ is a* NDWF *for $L^2(\mathbb{R}^+)$, then $\left(X_J(\psi_0;\Psi),\, X_J(\tilde{\psi}_0;\widetilde{\Psi})\right)$ is a* NDWF *for $L^2(\mathbb{R}^+)$ for all integer J. In particular, $\left(X(\Psi),\, X(\widetilde{\Psi})\right)$ is a* HDWF *for $L^2(\mathbb{R}^+)$.*

Proof. For any integer J and $f, g \in L^2(\mathbb{R}^+)$, we have

$$\langle f,\, \tilde{\psi}_{0,J,k}\rangle = \langle D^{J_0-J}f,\, \tilde{\psi}_{0,J_0,k}\rangle,\ \langle \psi_{0,J,k},\, g\rangle = \langle \psi_{0,J_0,k},\, D^{J_0-J}g\rangle \tag{3.4}$$

and

$$\langle f,\, \tilde{\psi}_{l,j,k}\rangle = \langle D^{J_0-J}f,\, \tilde{\psi}_{l,j+J_0-J,k}\rangle,\ \langle \psi_{l,j,k},\, g\rangle = \langle \psi_{l,j+J_0-J,k},\, D^{J_0-J}g\rangle \tag{3.5}$$

due to D is a unitary operator on $L^2(\mathbb{R}^+)$. And thus, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{0\,J,k}\rangle\langle \psi_{0,J,k},\, g\rangle + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle \\ &= \sum_{k\in\mathbb{Z}^+} \langle D^{J_0-J}f,\, \tilde{\psi}_{0\,J_0,k}\rangle\langle \psi_{0,J_0,k},\, D^{J_0-J}g\rangle \\ &+ \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle D^{J_0-J}f,\, \tilde{\psi}_{l,j+J_0-J,k}\rangle\langle \psi_{l,j+J_0-J,k},\, D^{J_0-J}g\rangle \\ &= \sum_{k\in\mathbb{Z}^+} \langle D^{J_0-J}f,\, \tilde{\psi}_{0,J_0,k}\rangle\langle \psi_{0,J_0,k},\, D^{J_0-J}g\rangle \\ &\quad + \sum_{l=1}^{L}\sum_{j=J_0}^{\infty}\sum_{k\in\mathbb{Z}^+} \langle D^{J_0-J}f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, D^{J_0-J}g\rangle,\end{aligned} \tag{3.6}$$

it equals to $\langle D^{J_0-J}f,\, D^{J_0-J}g\rangle$, and then equals to $\langle f,\, g\rangle$, since $(X_{J_0}(\psi_0;\Psi),\, X_{J_0}(\tilde{\psi}_0;\widetilde{\Psi}))$ is a NWDF for $L^2(\mathbb{R}^+)$. So $\Big(X_J(\psi_0;\Psi),\, X_J(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ is a NDWF for $L^2(\mathbb{R}^+)$ for all integer J, and thus

$$\langle P_J f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle = \langle f,\, g\rangle \text{ for } f,g\in L^2(\mathbb{R}^+). \tag{3.7}$$

Letting $J\to-\infty$ in (3.7) and using Lemma 3.1, we obtain

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle = \langle f,\, g\rangle \text{ for } f,g\in L^2(\mathbb{R}^+). \tag{3.8}$$

Therefore, $\Big(X(\Psi),\, X(\widetilde{\Psi})\Big)$ is a HDWF for $L^2(\mathbb{R}^+)$. The proof is completed. □

Theory 3.1 tells us that the study of NDWFs of the form $\Big(X_{J_0}(\psi_0;\Psi),\, X_{J_0}(\tilde{\psi}_0;\widetilde{\Psi})\Big)$ with general $J_0\in\mathbb{Z}$ can reduces to the study of NDWFs with $J_0=0$. The next theorem characterizes NDWFs in $L^2(\mathbb{R}^+)$ under the general Bessel assumption.

Theorem 3.2. *Let $\psi_0\in L^2(\mathbb{R}^+)$ and $\Psi=\{\psi_1,\psi_2,\cdots,\psi_L\}$ be a finite subset in $L^2(\mathbb{R}^+)$. Suppose $\{\mathcal{T}_k\psi_l : k\in\mathbb{Z}^+, 0\le l\le L\}$ and $\{\mathcal{T}_k\tilde{\psi}_l : k\in\mathbb{Z}^+, 0\le l\le L\}$ are Bessel sequences in $L^2(\mathbb{R}^+)$. Then $\Big(X(\psi_0,\,\Psi),\, X(\tilde{\psi}_0,\,\widetilde{\Psi})\Big)$ is a* NDWF *for $L^2(\mathbb{R}^+)$ if and only if*

$$\lim_{J\to\infty}\langle P_J f,\, g\rangle = \langle f,\, g\rangle \tag{3.9}$$

and

$$\langle P_{J+1} f,\, g\rangle = \langle P_J f,\, g\rangle + \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,J,k}\rangle\langle \psi_{l,J,k},\, g\rangle \tag{3.10}$$

for $f,g\in L^2(\mathbb{R}^+)$ and $J\in\mathbb{Z}$, where P_J is defined as in (3.1).

Proof. "⇐": It follows from (3.10) that

$$\langle P_{J+1} f,\, g\rangle = \langle P_0 f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{J}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle \tag{3.11}$$

for $f,g\in L^2(\mathbb{R}^+)$ and $J\in\mathbb{Z}$. Letting $J\to\infty$ in (3.11) and using (3.9), we have

$$\langle f,\, g\rangle = \langle P_0 f,\, g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\, \tilde{\psi}_{l,j,k}\rangle\langle \psi_{l,j,k},\, g\rangle \tag{3.12}$$

for $f,g\in L^2(\mathbb{R}^+)$. Therefore, $\Big(X(\psi_0,\,\Psi),\, X(\tilde{\psi}_0,\,\widetilde{\Psi})\Big)$ is a NDWF for $L^2(\mathbb{R}^+)$.

"⇒": Suppose $\Big(X(\psi_0,\,\Psi),\,X(\tilde{\psi}_0,\,\widetilde{\Psi})\Big)$ is a NDWF for $L^2(\mathbb{R}^+)$, then $(X_J(\psi_0,\,\Psi),$ $X_J(\tilde{\psi}_0,\,\widetilde{\Psi}))$ is a NDWF for $L^2(\mathbb{R}^+)$ for all integer J by Theory 3.1. It follows that

$$\begin{aligned}\langle f,\,g\rangle &= \langle P_{J+1}f,\,g\rangle + \sum_{l=1}^{L}\sum_{j=J+1}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\,\tilde{\psi}_{l,j,k}\rangle\langle\psi_{l,j,k},\,g\rangle\\ &= \langle P_J f,\,g\rangle + \sum_{l=1}^{L}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^+}\langle f,\,\tilde{\psi}_{l,j,k}\rangle\langle\psi_{l,j,k},\,g\rangle \end{aligned}\tag{3.13}$$

for $f,g\in L^2(\mathbb{R}^+)$ and $J\in\mathbb{Z}$, which leads to (3.10), and thus

$$\langle P_{J+1}f,\,g\rangle = \langle P_0 f,\,g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{J}\sum_{k\in\mathbb{Z}^+}\langle f,\,\tilde{\psi}_{l,j,k}\rangle\langle\psi_{l,j,k},\,g\rangle \tag{3.14}$$

for $f,g\in L^2(\mathbb{R}^+)$ and $J\in\mathbb{Z}$. Also, observe that $\Big(X(\psi_0,\,\Psi),\,X(\tilde{\psi}_0,\,\widetilde{\Psi})\Big)$ is a NDWF for $L^2(\mathbb{R}^+)$. Letting $J\to\infty$ in (3.14), we obtain (3.9). The proof is completed. □

4. Refinable functions based construction of NDWFs in $L^2(\mathbb{R}^+)$

This section is devoted to constructing NDWFs from a pair of general refinable functions.

For $f,g\in L^2(\mathbb{R}^+)$, we define

$$[f,\,g](\cdot) = \sum_{k\in\mathbb{Z}^+} f(\cdot\oplus k)\overline{g(\cdot\oplus k)}\ \text{ a.e. on } \mathbb{R}^+, \tag{4.1}$$

then it belongs to $L^1(\mathbb{T})$, and is well defined. And we write

$$\mathcal{D} := \{f\in L^2(\mathbb{R}^+) : \mathcal{F}f\in L^\infty(\mathbb{R}^+) \text{ and } \mathrm{supp}(\mathcal{F}f) \text{ is bounded}\}, \tag{4.2}$$

where $\mathrm{supp}(\mathcal{F}f) = \{\xi\in\mathbb{R}^+ : \mathcal{F}f(\xi)\neq 0\}$ for $f\in L^2(\mathbb{R}^+)$ and is well defined up to a set 0. It is not difficult to verify that $\mathcal{D}$ is dense in $L^2(\mathbb{R}^+)$.

Now, let us make some assumptions:

Assumption 1. $\psi_0,\tilde{\psi}_0\in L^2(\mathbb{R}^+)$ are p-refinable functions with symbols in $L^\infty(\mathbb{T})$, i.e., there exist $m_0,\widetilde{m}_0\in L^\infty(\mathbb{T})$ such that

$$\mathcal{F}\psi_0(p\,\cdot) = m_0(\cdot)\mathcal{F}\psi_0(\cdot) \text{ and } \mathcal{F}\tilde{\psi}_0(p\,\cdot) = \widetilde{m}_0(\cdot)\mathcal{F}\tilde{\psi}_0(\cdot) \text{ a.e. on } \mathbb{R}^+. \tag{4.3}$$

Assumption 2. $\lim\limits_{j\to\infty}\mathcal{F}\psi_0(p^{-j}\cdot)\mathcal{F}\tilde{\psi}_0(p^{-j}\cdot) = 1$ a.e. on $\mathbb{R}^+$.

Assumption 3. $[\mathcal{F}\psi_0,\,\mathcal{F}\psi_0],[\mathcal{F}\tilde{\psi}_0,\,\mathcal{F}\tilde{\psi}_0]\in L^\infty(\mathbb{T})$.

Given $L\in\mathbb{N}$, let $m_l,\widetilde{m}_l\in L^\infty(\mathbb{T})$ with $1\le l\le L$, and define ψ_l and $\tilde{\psi}_l$ by

$$\mathcal{F}\psi_l(p\,\cdot) = m_l(\cdot)\mathcal{F}\psi_0(\cdot) \text{ and } \mathcal{F}\tilde{\psi}_l(p\,\cdot) = \widetilde{m}_l(\cdot)\mathcal{F}\tilde{\psi}_0(\cdot) \text{ a.e. on } \mathbb{R}^+. \tag{4.4}$$

With m_l and $\widetilde{m}_l, l = 0, 1, \cdots, L$ as the framelet symbols, we write

(4.5)
$$\mathcal{M}(\cdot) = \begin{pmatrix} m_0(\cdot) & m_1(\cdot) & \cdots & m_L(\cdot) \\ m_0(\cdot \oplus 1/p) & m_1(\cdot \oplus 1/p) & \cdots & m_L(\cdot \oplus 1/p) \\ \vdots & \vdots & \ddots & \vdots \\ m_0(\cdot \oplus (p-1)/p) & m_1(\cdot \oplus (p-1)/p) & \cdots & m_L(\cdot \oplus (p-1)/p) \end{pmatrix}$$

and

(4.6)
$$\widetilde{\mathcal{M}}(\cdot) = \begin{pmatrix} \widetilde{m}_0(\cdot) & \widetilde{m}_1(\cdot) & \cdots & \widetilde{m}_L(\cdot) \\ \widetilde{m}_0(\cdot \oplus 1/p) & \widetilde{m}_1(\cdot \oplus 1/p) & \cdots & \widetilde{m}_L(\cdot \oplus 1/p) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{m}_0(\cdot \oplus (p-1)/p) & \widetilde{m}_1(\cdot \oplus (p-1)/p) & \cdots & \widetilde{m}_L(\cdot \oplus (p-1)/p) \end{pmatrix}$$

We will study what m_l, $\widetilde{m}_l \in L^\infty(\mathbb{T})$ with $0 \le l \le L$ are qualified for $(X(\psi_0, \Psi)$, $X(\tilde{\psi}_0, \widetilde{\Psi}))$ to be a NDWF in $L^2(\mathbb{R}^+)$. We begin with some lemmas for latter use.

The following lemma shows that Assumption 3 is equivalent to the fact that $\{\mathcal{T}_k\psi_0 : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$.

Lemma 4.1. ([33, Theorem 2.1]) *Let $\psi_0 \in L^2(\mathbb{R}^+)$. Then $\{\mathcal{T}_k\psi_0 : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$ with Bessel bound B if and only if*

$$[\mathcal{F}\psi_0, \mathcal{F}\psi_0](\cdot) \le B \text{ a.e. on } \mathbb{T}.$$

Observe that $\{\chi(k, \cdot) : k \in \mathbb{Z}^+\}$ is an orthonormal basis for $L^2(\mathbb{T})$ and the Walsh-Fourier transform is a unitary operator on $L^2(\mathbb{R}^+)$.

Lemma 4.2. *Let $k \in \mathbb{Z}^+$ and $f, \psi \in L^2(\mathbb{R}^+)$. Then, $\langle f, \psi_{j,k}\rangle$ is the k-th Walsh Fourier coefficient of $[p^{j/2}\mathcal{F}f(p^j\cdot), \mathcal{F}\psi(\cdot)]$ for each $j \in \mathbb{Z}^+$. In particular, we have*

$$[p^{j/2}\mathcal{F}f(p^j\cdot), \mathcal{F}\psi(\cdot)](\xi) = \sum_{k\in\mathbb{Z}^+} \langle f, \psi_{j,k}\rangle \chi(k, \xi) \text{ a.e. } \xi \in \mathbb{R}^+, \tag{4.7}$$

if $\{\mathcal{T}_k\psi : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$.

Proof. Since $f, \psi \in L^2(\mathbb{R}^+)$, we have $\mathcal{F}f(p^j\cdot)\overline{\mathcal{F}\psi(\cdot)} \in L^1(\mathbb{R}^+)$, and thus

$$\begin{aligned} \int_{\mathbb{T}} [p^{j/2}\mathcal{F}f(p^j\cdot), \mathcal{F}\psi(\cdot)](\xi)\chi(k, \xi)d\xi =& p^{j/2}\int_{\mathbb{R}^+} \mathcal{F}f(p^j\xi)\overline{\mathcal{F}\psi(\xi)}\chi(k, \xi)d\xi \\ =& p^{-j/2}\int_{\mathbb{R}^+} \mathcal{F}f(\xi)\overline{\mathcal{F}\psi(p^{-j}\xi)}\chi(k, p^{-j}\xi)d\xi \\ =& \int_{\mathbb{R}^+} \mathcal{F}f(\xi)\overline{[\mathcal{F}(\psi_{j,k})(\cdot)](\xi)}d\xi = \langle f, \psi_{j,k}\rangle, \end{aligned} \tag{4.8}$$

so $\langle f, \psi_{j,k}\rangle$ is the k-th Walsh-Fourier coefficient of $[p^{j/2}\mathcal{F}f(p^j\cdot), \mathcal{F}\psi(\cdot)]$ for each $j \in \mathbb{Z}^+$.

If $\{\mathcal{T}_k\psi : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$, then $\{D^j\mathcal{T}_k\psi : k \in \mathbb{Z}^+\}$, that is, $\{\psi_{j,k} : k \in \mathbb{Z}^+\}$ is a Bessel sequence in $L^2(\mathbb{R}^+)$ for each $j \in \mathbb{Z}^+$ due to D^j being unitary, it follows that $\{\langle f,\, \psi_{j,k} : k \in \mathbb{Z}^+\rangle\} \in \ell^2(\mathbb{Z}^+)$, and thus (4.7) holds. □

As an application of Lemma 4.2, we have the following lemma immediately

Lemma 4.3. *Let $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ satisfy Assumption 3. Then we have*

$$\langle P_n f,\, g\rangle = p^n \int_{\mathbb{T}} [\mathcal{F}f(p^n\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0,\, \mathcal{F}g(p^n\cdot)](\xi)d\xi$$

for $f, g \in L^2(\mathbb{R}^+)$ and $n \in \mathbb{Z}$, where P_n is defined as in (3.1).

The following two lemmas are necessary for us to prove the main result.

Lemma 4.4. *Let $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ satisfy Assumptions 2 and 3. Then*

$$\lim_{n\to\infty} \langle P_n f,\, g\rangle = \langle f,\, g\rangle \text{ for } f, g \in \mathcal{D},$$

where $\mathcal{D}$ is defined as in (4.2).

Proof. By Lemma 4.3, we have

$$\langle P_n f,\, g\rangle = p^n \int_{[0,\,1]} [\mathcal{F}f(p^n\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0,\, \mathcal{F}g(p^n\cdot)](\xi)d\xi.$$

Since $p > 1$ and $\operatorname{supp}(\mathcal{F}f)$ and $\operatorname{supp}(\mathcal{F}g)$ are bounded, then there exists $N > 0$ such that $\operatorname{supp}(\mathcal{F}f(p^n\cdot)), \operatorname{supp}(\mathcal{F}g(p^n\cdot)) \subset [0,\, 1)$ when $n > N$, and thus

$$[\mathcal{F}f(p^n\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi) = \mathcal{F}f(p^n\xi)\overline{\mathcal{F}\tilde{\psi}_0(\xi)}$$

and

$$[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p^n\cdot)](\xi) = \overline{\mathcal{F}g(p^n\xi)}\mathcal{F}\psi_0(\xi)$$

for a.e. $\xi \in (0,\, 1)$ and $n > N$. So

$$\begin{aligned} \langle P_n f,\, g\rangle &= p^n \int_{[0,\,1]} \mathcal{F}f(p^n\xi)\overline{\mathcal{F}g(p^n\xi)\mathcal{F}\tilde{\psi}_0(\xi)}\mathcal{F}\psi_0(\xi)d\xi \\ &= \int_{\mathbb{R}^+} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)\mathcal{F}\tilde{\psi}_0(p^{-n}\xi)}\mathcal{F}\psi_0(p^{-n}\xi)\chi_{[0,\,1]}(p^{-n}\xi)d\xi \qquad (4.9) \end{aligned}$$

when $n > N$. By Assumption 3 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left|\overline{\mathcal{F}\tilde{\psi}_0(\cdot)}\mathcal{F}\psi_0(\cdot)\right| &\le \sum_{l\in\mathbb{Z}^+} \left|\overline{\mathcal{F}\tilde{\psi}_0(\cdot \oplus l)}\mathcal{F}\psi_0(\cdot \oplus l)\right| \\ &\le \left([\mathcal{F}\tilde{\psi}_0,\, \mathcal{F}\tilde{\psi}_0](\cdot)\right)^{1/2} \left([\mathcal{F}\psi_0,\, \mathcal{F}\psi_0](\cdot)\right)^{1/2} \le C \end{aligned}$$

for some constant $C > 0$. Therefore, the integrand in (4.9) is dominated in module by $C|\mathcal{F}f(\cdot)\mathcal{F}g(\cdot)|$, which belongs to $L^1(\mathbb{R}^+)$. Applying the Lebesgue dominated convergence theorem to (4.9), we obtain

$$\lim_{n\to\infty} \langle P_n f,\, g\rangle = \langle f,\, g\rangle$$

by Assumption 2. □

Lemma 4.5. *Let $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ satisfy Assumptions 1 and 3. Assume that $m_l, \widetilde{m}_l \in L^\infty(\mathbb{T})$ with $1 \le l \le L$, are such that*

$$\mathcal{M}(\cdot)\widetilde{\mathcal{M}}^*(\cdot) = I_p \text{ a.e. on } \mathbb{T}, \tag{4.10}$$

where $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are defined as in (4.5) and (4.6). Define $\psi_l, \tilde{\psi}_l, 1 \le l \le L$ as in (4.4). Then

$$\langle P_{n+1} f,\, g\rangle = \langle P_n f,\, g\rangle + \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,n,k}\rangle\langle \psi_{l,n,k},\, g\rangle \tag{4.11}$$

for $f, g \in L^2(\mathbb{R}^+)$ and $n \in \mathbb{Z}$.

Proof. First, we claim that (4.11) is equivalent to

$$\langle P_1 f,\, g\rangle = \langle P_0 f,\, g\rangle + \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^+} \langle f,\, \tilde{\psi}_{l,0,k}\rangle\langle \psi_{l,0,k},\, g\rangle \tag{4.12}$$

for $f, g \in L^2(\mathbb{R}^+)$. Indeed, if (4.12) holds, we can get (4.11) by replacing f by $D^{-n}f$ and g by $D^{-n}g$ in (4.12), respectively. And, by Lemma 4.3, (4.12) can be written as

$$p\int_{\mathbb{T}} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p^n\cdot)](\xi)d\xi = \int_{\mathbb{T}} \sum_{l=0}^{L} [\mathcal{F}f,\, \mathcal{F}\tilde{\psi}_l](\xi)[\mathcal{F}\psi_l,\, \mathcal{F}g](\xi)d\xi \tag{4.13}$$

for $f, g \in L^2(\mathbb{R}^+)$.

Next, we prove (4.13) to complete the proof. Note that, $m_l, \widetilde{m}_l, 1 \le l \le L$ are 1-periodic functions. By the definitions of $\tilde{\psi}_l, 1 \le l \le L$ and Assumption 1, we have

$$\begin{aligned}
[\mathcal{F}f,\, \mathcal{F}\tilde{\psi}_l](\xi) &= \sum_{k\in\mathbb{Z}^+} \mathcal{F}f(\xi\oplus k)\overline{\widetilde{m}_l(p^{-1}(\xi\oplus k))\mathcal{F}\tilde{\psi}_0(p^{-1}(\xi\oplus k))} \\
&= \sum_{i=0}^{p-1} \overline{\widetilde{m}_l(p^{-1}(\xi\oplus i/p))} \sum_{k\in\mathbb{Z}^+} \mathcal{F}f(\xi\oplus i/p\oplus pk)\mathcal{F}\tilde{\psi}_0(p^{-1}(\xi\oplus i/p)\oplus k) \\
&= \sum_{i=0}^{p-1} \overline{\widetilde{m}_l(p^{-1}(\xi\oplus i/p))}[\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](p^{-1}(\xi\oplus i/p))
\end{aligned} \tag{4.14}$$

for $0 \le l \le L$. Similarly, we have

$$[\mathcal{F}\psi_l,\, \mathcal{F}g](\xi) = \sum_{i'=0}^{p-1} m_l(p^{-1}(\xi\oplus i'/p))[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p\,\cdot)](p^{-1}(\xi\oplus i'/p)) \tag{4.15}$$

for $0 \le l \le L$. By a simple computation, we obtain

$$\begin{aligned}
\sum_{l=0}^{L} [\mathcal{F}f,\, \mathcal{F}\tilde{\psi}_l](\xi)[\mathcal{F}\psi_l,\, \mathcal{F}g](\xi) &= \sum_{i=0}^{p-1} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](p^{-1}(\xi\oplus i/p))\times \\
&\quad\times \sum_{i'=0}^{p-1} \left(\mathcal{M}\widetilde{\mathcal{M}}^*(p^{-1}\xi)\right)_{i,i'} [\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p\,\cdot)](p^{-1}(\xi\oplus i'/p)),
\end{aligned} \tag{4.16}$$

where $\left(\mathcal{M}\widetilde{\mathcal{M}}^*(\cdot)\right)_{i,i'}$ denotes the $(i,\, i')$-entry of $\mathcal{M}\widetilde{\mathcal{M}}^*(\cdot), 0 \le i, i' \le p-1$. By (4.10), (4.13) therefore follows that

$$\begin{aligned}
&\int_{\mathbb{T}} \sum_{l=0}^{L} [\mathcal{F}f,\, \mathcal{F}\tilde{\psi}_l](\xi)[\mathcal{F}\psi_l,\, \mathcal{F}g](\xi)d\xi \\
=&\int_{\mathbb{T}} \sum_{i=0}^{p-1} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](p^{-1}(\xi\oplus i/p))[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p\,\cdot)](p^{-1}(\xi\oplus i/p))d\xi \\
=&p\sum_{i=0}^{p-1} \int_{p^{-1}(\mathbb{T}+i/p)} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p\,\cdot)](\xi)d\xi \\
=&p\int_{\mathbb{T}} [\mathcal{F}f(p\,\cdot),\, \mathcal{F}\tilde{\psi}_0(\cdot)](\xi)[\mathcal{F}\psi_0(\cdot),\, \mathcal{F}g(p^n\cdot)](\xi)d\xi. \qquad (4.17)
\end{aligned}$$

Therefore, (4.13) holds. The proof is completed. □

The following theorem gives a sufficient condition for $\left(X(\psi_0,\, \Psi),\, X(\tilde{\psi}_0,\, \widetilde{\Psi})\right)$ to be a NDWF in $L^2(\mathbb{R}^+)$.

Theorem 4.1. *Let $\psi_0, \tilde{\psi}_0 \in L^2(\mathbb{R}^+)$ satisfy Assumptions* 1-3*. Assume that $m_l, \widetilde{m}_l, \in L^\infty(\mathbb{T})$ with $1 \le l \le L$, are such that*

$$\mathcal{M}(\cdot)\widetilde{\mathcal{M}}^*(\cdot) = I_p \ \textit{a.e. on}\ \mathbb{T}. \qquad (4.18)$$

where $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are defined as in (4.5) *and* (4.6)*. Define ψ_l and $\tilde{\psi}_l, 1 \le l \le L$ as in* (4.4)*. Then $\left(X(\psi_0,\, \Psi),\, X(\tilde{\psi}_0,\, \widetilde{\Psi})\right)$ is a* NDWF *for $L^2(\mathbb{R}^+)$.*

Proof. Since $m_l, \widetilde{m}_l, \in L^\infty(\mathbb{T})$ for $1 \le l \le L$, by Lemma 4.1 and Assumptions 1 and 3, then we have

$$\{\mathcal{T}_k\psi_l : k \in \mathbb{Z}^+, 1 \le l \le L\} \quad \text{and} \quad \{\mathcal{T}_k\tilde{\psi}_l : k \in \mathbb{Z}^+, 1 \le l \le L\}$$

are Bessel sequences in $L^2(\mathbb{R}^+)$. Therefore, the conclusion follows directly by Theory 3.2, Lemmas 4.4 and 4.5. The proof is completed. □

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