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NON-EXISTENCE OF GLOBAL SOLUTIONS FOR A FRACTIONAL INTEGRO-DIFFERENTIAL PROBLEM WITH A CONVOLUTION KERNEL

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Abstract. In this paper, we investigate the nonexistence of nontrivial global solutions for a fractional integro-differential problem in the space of absolutely continuous functions. We provide criteria under which no nontrivial global solutions exist. It is shown that a dissipation of order between zero and one or even a (frictional) dissipation of order one does not help providing global nontrivial solutions. The test function method is used with several derived estimations. Examples with numerical computations are given to illustrate the results.

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1. Introduction

We consider the following fractional integro-differential inequality

$$u'(t)+\left({}^{C}D_{0+}^{\alpha}u\right)(t)\geq\int_0^t g(t-s)f\left(u\left(s\right)\right)ds,\quad t>0,\ 0\leq\alpha<2, \tag{1.1}$$

subject to

$$u(0)=u_0,\ \text{ when } 0\leq\alpha<1, \tag{1.2}$$

or,

$$u(0)=u_0,\ u'(0)=u_1,\ \text{ when } 1\leq\alpha<2, \tag{1.3}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order α and $u_0,u_1\in\mathbb{R}$ are given initial data.

This initial value problem is a generalization of many interesting initial value problems. When the kernel g represents the Dirac delta function, $f\left(u\right)=u^p\left(t\right)$, $p>1$ and $\alpha=0$, the equality in (1.1) represents the Bernoulli differential equation

$$u'(t)+u(t)=u^p\left(t\right),\ t>0,\ p>1. \tag{1.4}$$

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Equation (1.4) with $u(0) = u_0$ has the solution

$$u(t) = \left(\left(u_0^{1-p} - 1 \right) e^{(p-1)t} + 1 \right)^{\frac{1}{1-p}},$$

that blows up in the finite time

$$T_b = \frac{1}{1-p} \ln \left(1 - u_0^{1-p} \right)$$

if and only if $u_0 > 1$, (see [4]).

The solution of the nonlinear Volterra integro-differential equation

$$u'(t) = -c + \int_0^t u^p(s) ds, \tag{1.5}$$

is given by

$$u(t) = \left(\frac{1-p}{2} \sqrt{\frac{2}{p+1}} t + u_0^{\frac{1-p}{2}} \right)^{\frac{2}{1-p}},$$

and it blows up in the finite time

$$T_b = \frac{2}{p-1} \sqrt{\frac{p+1}{2}} u_0^{\frac{1-p}{2}},$$

when $c = \sqrt{\frac{2}{p+1} u_0^{p+1}}$ and $u_0 > 0$.

When $\alpha = 0$, $u_0 \geq 0$ and the kernel $g(t)$ is positive, locally integrable and $\lim\limits_{t \to \infty} \int_0^t g(s) ds = \infty$, it has been shown in [11] that the solution of

$$u'(t) + u(t) = \int_0^t g(t-s) f(u(s)) \, ds, \quad t > 0, \tag{1.6}$$

blows up in finite time if and only if for some $\beta > 0$,

$$\int_\nu^\infty \left(\frac{s}{f(s)} \right)^{\frac{1}{\beta}} \frac{ds}{s} < \infty, \text{ for any } \nu > 0. \tag{1.7}$$

It has been assumed that $f(t)$ is nonnegative, continuous and nondecreasing for $t > 0$, $f \equiv 0$ for $t \leq 0$, and $\lim\limits_{t \to \infty} \frac{f(t)}{t} = \infty$. Obviously, when $f(u(s)) = |u(s)|^p$ in (1.6), the condition (1.7) is fulfilled if $p > 1$.

By choosing $g(t)$ to be the Dirac delta function and $f(u) = |u(t)|^p, p > 1$ in the equality in (1.1), we obtain

$$u'(t) + \left({}^C D_{0+}^\alpha u \right)(t) = |u(t)|^p, \quad t > 0, \quad p > 1. \tag{1.8}$$

As proven in [12], the solution of the system

(1.9)
$$\begin{cases} u_t + \sum\limits_{i=1}^{n} a_i \left(D_{0^+}^{\alpha_i} u\right)(t) = \int_0^t \frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)} f_1(u(s),v(s))ds, \quad t>0,\ 0<\alpha_i,\gamma_1<1, \\ v_t + \sum\limits_{i=1}^{n} a_i \left(D_{0^+}^{\beta_i} v\right)(t) = \int_0^t \frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_2)} f_2(u(s),v(s))ds, \quad t>0,\ 0<\beta_i,\gamma_2<1, \\ u(0)=u_0,\ v(0)=v_0\ ,\ \ 0<u_0,v_0\in\mathbb{R}, \end{cases}$$

blows up in finite time for the continuously differentiable functions f_1 and f_2 satisfying the growth conditions:

$$f_1(u,v) \geq a\,|v|^p \quad \text{and} \quad f_2(u,v) \geq b\,|u|^q\,, \quad a,b>0 \ \text{ for all } \ u,v\in\mathbb{R}.$$

Although, the authors in [12], treated a system rather than an equation, the kernel there is a special case of ours, that is $k(t-s) = \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}$.

The present authors studied, in [3], the nonexistence of nontrivial global solutions for the fractional integro-differential problem

$$\begin{cases} \left(D_{0^+}^{\alpha} u\right)(t) + \left(D_{0^+}^{\beta} u\right)(t) \geq \int_0^t h(t-s)\,|u(s)|^p\,ds, \quad t>0,\ p>1, \\ \left(I^{1-\alpha}u\right)(0^+) = b,\ b\in\mathbb{R}, \end{cases} \tag{1.10}$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann-Liouville fractional derivatives of orders α and β, respectively, $0\leq\beta<\alpha\leq1$ and h is a nonnegative function different from zero almost everywhere. It has been shown that if $\left(t^{-\alpha p'}+t^{-\beta p'}\right)h^{1-p'}(t)\in L_{loc}^1[0,\infty)$ and

$$\lim_{T\to\infty} T^{1-p'}\left(\int_0^T t^{-\alpha p'}h^{1-p'}(t)dt + \int_0^T t^{-\beta p'}h^{1-p'}(t)dt\right) = 0,$$

where $p' = \frac{p}{p-1}$, then, the problem (1.10) has no nontrivial global solution when $b\geq0$.

In this paper, we prove nonexistence of nontrivial global solutions for Problems $(1.1)-(1.2)$ and $(1.1)-(1.3)$ under some conditions on the functions g and f . The test function method introduced in [13] is adopted to the fractional case and used here, see also [5, 9, 10, 15].

It is well known that lower order derivatives usually represent damping terms and therefore help stabilizing the system in addition to the existence of solutions for all time. On the contrary, polynomial sources destabilize the system and they can even force solutions to blow up in finite time. In fact they are sometimes called blowing up terms. When they are both present in the system we will have a competition between these two terms. When $0<\alpha\leq1$, the fractional derivative acts as a damping term, while when $1<\alpha\leq2$, it is the first derivative which plays this role.

Many results on the existence of solutions for fractional differential equations are available in the literature, (see e.g. [1, 2, 8]). The most important recent results on fractional differential equations with Caputo fractional derivatives are surveyed in [1]. The study of the nonexistence of solutions for differential equations is as important as the study of the existence of solutions. It is particularly capital for the nonlinear differential equations where solutions cannot be found explicitly. We refer the reader to [5, 6, 12, 9, 10, 15] and the references therein.

The rest of this paper is structured as follows. Section 2 is devoted to the required notions and notations from fractional calculus that will be used throughout this paper. Also, we present the test function and some of its properties we use. The statements and proofs of our results are presented in Section 3. In the last section, we provide some examples of special types of kernels with the numerical treatment at various values of the parameters.

2. Preliminaries

In this section, we begin with some fractional-order operators relevant to our study and recall some of their properties. We introduce our selected test function with some of its characteristics.

The Riemann-Liouville left-sided and right-sided fractional derivatives of order $\alpha \geq 0$, are defined by

$$(D_{a^+}^{\alpha} u)(t) = D^n \left(I_{a^+}^{n-\alpha} u\right)(t), \tag{2.1}$$

$$(D_{b^-}^{\alpha} u)(t) = (-1)^n D^n \left(I_{b^-}^{n-\alpha} u\right)(t), \tag{2.2}$$

respectively, where $D^n = \frac{d^n}{dt^n}$, $n = [\alpha] + 1$ and $[\alpha]$ is the integral part of α. I_{a+}^{α} and $I_{b^-}^{\alpha}$ are the Riemann-Liouville left-sided and right-sided fractional integrals of order $\alpha > 0$ defined by

$$\begin{aligned} \left(I_{a+}^{\alpha} u\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \ t > a, \\ (I_{b^-}^{\alpha} u)(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds, \quad t < b, \end{aligned}$$

respectively, provided the right-hand sides exist. The function Γ is the Euler Gamma function. We define $I_{a+}^0 u = I_{b^-}^0 u = u$. In particular, when $\alpha = m \in \mathbb{N}_0$, it follows from the definitions that

$$D_{a^+}^m u = D^m u, \ D_{b^-}^{\alpha} u = (-1)^m D^m u.$$

The Caputo left-sided and right-sided fractional derivatives of order $\alpha \geq 0$, are defined by

$$\left({}^{C}D_{a^+}^{\alpha}u\right)(t) = \left(D_{a^+}^{\alpha}\left(u(s) - \sum_{i=0}^{n-1}\frac{u^{(i)}(a)}{i!}(s-a)^i\right)\right)(t),$$

$$\left({}^{C}D_{b^-}^{\alpha}u\right)(t) = \left(D_{b^-}^{\alpha}\left(u(s) - \sum_{i=0}^{n-1}\frac{u^{(i)}(b)}{i!}(b-s)^i\right)\right)(t),$$

respectively, where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ for $\alpha \in \mathbb{N}_0$.

In particular, when $\alpha = n \in \mathbb{N}_0$, it follows from the definitions that

$${}^{C}D_{a^+}^{0}u = {}^{C}D_{b^-}^{0}u = u, \; {}^{C}D_{a^+}^{n}u = D^n u, \; {}^{C}D_{b^-}^{n}u = (-1)^n D^n u.$$

Notice that if $u^{(i)}(a) = 0$ for all $i = 0, 1, ..., n-1$, then ${}^{C}D_{a^+}^{\alpha}u = D_{a^+}^{\alpha}u$, and if $u^{(i)}(b) = 0$ for all $i = 0, 1, ..., n-1$, then ${}^{C}D_{b^-}^{\alpha}u = D_{b^-}^{\alpha}u$. For more details about fractional operators, we refer to the books [7, 14].

The space of absolutely continuous functions on $[a, b]$ is denoted by $AC[a, b]$. In general, for $n \in \mathbb{N}$,

$$AC^n[a, b] = \left\{u : [a, b] \to \mathbb{R} \text{ such that } D^{n-1}u \in AC[a, b]\right\}.$$

If $u \in AC^n[a, b]$, then ${}^{C}D_{a^+}^{\alpha}u$ and ${}^{C}D_{b^-}^{\alpha}u$ exist almost everywhere on the interval $[a, b]$ and

$$\left({}^{C}D_{a^+}^{\alpha}u\right)(t) = \left(I_{a^+}^{n-\alpha}D^n u\right)(t), \tag{2.3}$$

$$\left({}^{C}D_{b^-}^{\alpha}u\right)(t) = (-1)^n\left(I_{b^-}^{n-\alpha}D^n u\right)(t). \tag{2.4}$$

Lemma 2.1. [7] *If $\alpha \geq 0$, $\beta > 0$, then*

$$\left(I_{b^-}^{\alpha}(b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1},$$

$$\left(D_{b^-}^{\alpha}(b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}.$$

Lemma 2.2. [14] *Let $\alpha \geq 0$, $p \geq 1, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $f \in L^p(a, b)$ and $g \in L^q(a, b)$, then*

$$\int_a^b f(t)\left(I_{a+}^{\alpha}g\right)(t)\,dt = \int_a^b g(t)\left(I_{b-}^{\alpha}f\right)(t)\,dt.$$

Lemma 2.3. *Let $\alpha \geq 0$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ for $\alpha \in \mathbb{N}_0$. For $f \in C[a, b]$ and $g, I_{b-}^{n-\alpha}f \in AC^n[a, b]$, we have*

$$\int_a^b f(t)\left({}^{C}D_{a^+}^{\alpha}g\right)(t)\,dt = \int_a^b g(t)\left(D_{b^-}^{\alpha}f\right)(t)\,dt + \sum_{i=0}^{n-1}\left[\left(D_{b^-}^{\alpha+i-n}f\right)(t)\left(D^{n-1-i}g\right)(t)\right]_a^b.$$

Proof. Since $g \in AC^n[a,b]$, then we have from the definition (2.3),

$$\int_a^b f(t)\left({}^C D_{a+}^{\alpha} g\right)(t)\,dt = \int_a^b f(t)\left(I_{a+}^{n-\alpha} D^n g\right)(t)dt.$$

Because $f \in L^{m_1}(a,b)$ for any $m_1 \geq 1$ and $D^n g \in L^1(a,b)$, we deduce from Lemma 2.2,

$$\int_a^b f(t)\left(I_{a+}^{n-\alpha} D^n g\right)(t)dt = \int_a^b D^n g(t)\left(I_{b-}^{n-\alpha} f\right)(t)dt.$$

As $I_{b-}^{n-\alpha} f \in AC^n[a,b]$ and $D^{n-1} g \in AC[a,b]$, then integrating by parts n times yields

$$\int_a^b f(t)\left({}^C D_{a+}^{\alpha} g\right)(t)\,dt = \sum_{i=0}^{n-1}\left[\left(D_{b^-}^{\alpha+i-n} f\right)(t)\left(D^{n-1-i} g\right)(t)\right]_a^b +$$

$$+(-1)^n \int_a^b g(t) D^n\left(I_{b^-}^{n-\alpha} f\right)(t)\,dt.$$

Owing to (2.2), the proof is complete. □

In this paper, we use the following test function

$$\phi(t) := \begin{cases} \left(1-\frac{t}{T}\right)^{\theta}, & 0 \leq t \leq T, \\ 0, & t > T. \end{cases} \tag{2.5}$$

The function ϕ has the following properties.

Lemma 2.4. *Let ϕ be the function defined in (2.5), then for $\theta > nr-1$, $r > 1$, $n = 0, 1, 2, ...,$ we have*

$$\int_0^T \phi^{1-r}(t)\,|D^n \phi(t)|^r\,dt = C_{n,r} T^{1-nr},\ T > 0,$$

where

$$C_{n,r} = \frac{\Gamma^r(\theta+1)}{(\theta-nr+1)\,\Gamma^r(\theta-n+1)}\ .$$

Proof. Since

$$\begin{aligned} D^n \phi(t) &= (-1)^n \theta(\theta-1)(\theta-2)...(\theta-n+1)\,T^{-\theta}(T-t)^{\theta-n} \\ &= \frac{(-1)^n \Gamma(\theta+1)}{\Gamma(\theta-n+1)} T^{-\theta}(T-t)^{\theta-n}, \end{aligned}$$

it follows that

$$\begin{aligned} \int_0^T \phi^{1-r}(t)\,|D^n \phi(t)|^r\,dt &= \left(\frac{\Gamma(\theta+1)}{\Gamma(\theta-n+1)}\right)^r T^{-\theta} \int_0^T (T-t)^{\theta-nr}\,dt \\ &= C_{n,r} T^{1-nr}. \end{aligned}$$

□

Lemma 2.5. *Let $\alpha \geq 0$ and ϕ be as in (2.5) with $\theta > \alpha - 1$, then we have for all $0 \leq t \leq T$,*

$$\left(D^{\alpha}_{T^-}\phi\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\alpha+1)} T^{-\theta}\left(T-t\right)^{\theta-\alpha}, \tag{2.6}$$

$$\int_0^T t^m \left(D^{\alpha}_{T^-}\phi\right)(t)\,dt = \xi_{m,\theta} T^{m+1-\alpha}, \quad m = 0,1,2,...,n-1,\ \ n = [\alpha]+1, \tag{2.7}$$

where $\xi_{m,\theta} = \frac{(-1)^m m!\Gamma(\theta+1)}{\Gamma(\theta-\alpha+m+2)}$.

Proof. We have from Lemma 2.1,

$$\left(D^{\alpha}_{T^-}\phi\right)(t) = \left(D^{\alpha}_{T^-} T^{-\theta}\left(T-s\right)^{\theta}\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\alpha+1)} T^{-\theta}\left(T-t\right)^{\theta-\alpha}.$$

An integration m times by parts yields

$$\begin{aligned} \int_0^T t^m \left(D^{\alpha}_{T^-}\phi\right)(t)\,dt &= \sum_{i=0}^{m-1}\left[(-1)^i \frac{m!}{(m-i)!} t^{m-i}\left(I^{i+1}_{T^-} D^{\alpha}_{T^-}\phi\right)(t)\right]_0^T \\ &+ (-1)^m m! \int_0^T \left(I^{m}_{T^-} D^{\alpha}_{T^-}\phi\right)(t)\,dt. \end{aligned} \tag{2.8}$$

Using (2.6) and Lemma 2.1, we find

$$\left(I^{i+1}_{T^-} D^{\alpha}_{T^-}\phi\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma\left(\theta-\alpha+i+2\right)} T^{-\theta}\left(T-t\right)^{\theta-\alpha+i+1},$$

$$\left(I^{m}_{T^-} D^{\alpha}_{T^-}\phi\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma\left(\theta-\alpha+m+1\right)} T^{-\theta}\left(T-t\right)^{\theta-\alpha+m}.$$

Therefore

$$\left[t^{m-i}\left(I^{i+1}_{T^-} D^{\alpha}_{T^-}\phi\right)(t)\right]_0^T = 0 \ \text{ for all } i = 0,1,2,...,m-1, \tag{2.9}$$

and

$$\int_0^T \left(I^{m}_{T^-} D^{\alpha}_{T^-}\phi\right)(t)\,dt = \frac{\Gamma(\theta+1)}{\Gamma\left(\theta-\alpha+m+2\right)} T^{m-\alpha+1}. \tag{2.10}$$

Now, by substituting (2.9) and (2.10) in (2.8) we obtain (2.7). □

Lemma 2.6. *Let $\alpha \geq 0$, $n = [\alpha]+1$ and ϕ be as in (2.5) with $\theta > \alpha - 1$, then*

$$\left(I^{n-\alpha}_{T^-}\phi\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta+n-\alpha+1)} T^{-\theta}\left(T-t\right)^{\theta+n-\alpha},$$

for all $0 \leq t \leq T$. Moreover, $I^{n-\alpha}_{T^-}\phi \in AC^n[0,T]$.

Proof. From an application of Lemma 2.1, we deduce that

$$\left(I^{n-\alpha}_{T^-}\phi\right)(t) = \left(I^{n-\alpha}_{T^-}\left(T^{-\theta}\left(T-s\right)^{\theta}\right)\right)(t) = \frac{\Gamma(\theta+1)}{\Gamma(\theta+n-\alpha+1)} T^{-\theta}\left(T-t\right)^{\theta+n-\alpha}.$$

It is clear that $I^{n-\alpha}_{T^-}\phi$ is in the space $AC^n[0,T]$ for $\theta > \alpha - 1$. □

Lemma 2.7. *Let $\alpha \geq 0$ and ϕ be as in (2.5) with $\theta > \max\{0, \alpha - 1\}$. Suppose that $g \in AC^n[0,T]$, $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ for $\alpha \in \mathbb{N}_0$. Then*

$$\int_0^T \phi(t) \left({}^C D_{0+}^{\alpha} g\right)(t)\, dt = \int_0^T g(t) \left(D_{T^-}^{\alpha} \phi\right)(t)\, dt - \sum_{i=0}^{n-1} \bar{\xi}_{i,\alpha} T^{n-\alpha-i} \left(D^{n-1-i} g\right)(0),$$

where $\bar{\xi}_{i,\alpha} = \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\alpha-i+n)}$.

Proof. As a consequence of (2.6) in Lemma 2.5, we get for $i = 0, 1, 2, ..., n-1$,

$$\left(D_{T^-}^{\alpha+i-n} \phi\right)(0) = \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\alpha-i+n)} T^{n-\alpha-i},$$

$$\left(D_{T^-}^{\alpha+i-n} \phi\right)(T) = 0.$$

Since $\phi \in C[0,T]$ for $\theta > 0$ and $I_{T^-}^{n-\alpha} \phi \in AC^n[0,T]$, then the conclusion follows in the light of Lemma 2.3. □

3. The main results

In this section we prove the nonexistence of a nontrivial global solution for the initial value problems (1.1) − (1.2) and (1.1) − (1.3).

Definition 3.1. By a nontrivial global solution of Problem (1.1)−(1.2) or Problem (1.1)−(1.3), we mean a nonzero function u defined on $[0, \infty)$ such that $u \in AC[0,T]$ or $u \in AC^2[0,T]$ for all $T > 0$, for which the inequality (1.1) holds for all $t > 0$, and satisfying (1.2) or (1.3), respectively.

Firstly, we need to prove the following auxiliary lemma.

Lemma 3.1. *Let $\beta \geq 0$, $n = [\beta] + 1$ and $r > 1$. Let ϕ be as in (2.5) with $\theta > nr - 1$. Suppose that g is a nonnegative function that is different from zero almost everywhere and $t^{r(n-\beta-1)} g^{1-r}(t) \in L^1_{loc}[0, +\infty)$. Then, for any $T > 0$*

$$\int_0^T \left(D_{T^-}^{\beta} \phi\right)^r(t) \left(\int_t^T g(s-t)\phi(s)\, ds\right)^{1-r} dt \leq \hat{C}_{\beta,r} T^{1-nr} \int_0^T t^{r(n-\beta-1)} g^{1-r}(t) dt,$$

where $\hat{C}_{\beta,r} = \frac{C_{n,r}}{\Gamma^r(n-\beta)}$, $C_{n,r}$ is given in Lemma 2.4.

Proof. Since $\phi^{(i)}(T) = 0$ for all $i = 0, 1, ..., n-1$, then $D_{T^-}^{\beta} \phi = {}^C D_{T^-}^{\beta} \phi$. Also, since $\phi \in AC^n[0,T]$ for $\theta > n-1$, then ${}^C D_{T^-}^{\beta} \phi = (-1)^n I_{T^-}^{n-\beta} D^n \phi$ and

$$\left(D_{T^-}^{\beta} \phi\right)(t) \leq \left(I_{T^-}^{n-\beta} |D^n \phi|\right)(t) = \frac{1}{\Gamma(n-\beta)} \int_t^T (s-t)^{n-\beta-1} |(D^n \phi)(s)|\, ds$$

$$= \frac{1}{\Gamma(n-\beta)} \int_t^T (s-t)^{n-\beta-1} g^{\frac{1}{r'}}(s-t) \phi^{\frac{1}{r'}}(s)\, g^{-\frac{1}{r'}}(s-t) \phi^{-\frac{1}{r'}}(s) \left|(D^n \phi)(s)\right| ds.$$

Using Hölder inequality with $\frac{1}{r}+\frac{1}{r'}=1$, we find

$$\begin{aligned}\left(D_{T^-}^{\beta}\phi\right)(t) &\leq \frac{1}{\Gamma(n-\beta)}\left(\int_t^T g(s-t)\phi(s)\,ds\right)^{\frac{1}{r'}}\\ &\quad\times\left(\int_t^T (s-t)^{(n-\beta-1)r}g^{-\frac{r}{r'}}(s-t)\phi^{-\frac{r}{r'}}(s)\left|(D^n\phi)(s)\right|^r ds\right)^{\frac{1}{r}}.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^T\left(D_{T^-}^{\beta}\phi\right)^r(t)\left(\int_t^T g(s-t)\phi(s)\,ds\right)^{1-r}dt\\ \leq\;& b_1\int_0^T\int_t^T (s-t)^{r(n-\beta-1)}g^{-\frac{r}{r'}}(s-t)\phi^{-\frac{r}{r'}}(s)\left|(D^n\phi)(s)\right|^r dsdt,\;\; b_1=\frac{1}{\Gamma^r(n-\beta)}\\ =\;& b_1\int_0^T\int_0^s (s-t)^{r(n-\beta-1)}g^{1-r}(s-t)\phi^{1-r}(s)\left|(D^n\phi)(s)\right|^r dtds\\ =\;& b_1\int_0^T\phi^{1-r}(s)\left|(D^n\phi)(s)\right|^r\left(\int_0^s (s-t)^{r(n-\beta-1)}g^{1-r}(s-t)dt\right)ds.\end{aligned}$$

Let $\tau=s-t$, then the following uniform bound is obtained

$$\int_0^s \tau^{r(n-\beta-1)}g^{1-r}(\tau)d\tau\leq\int_0^T\tau^{r(n-\beta-1)}g^{1-r}(\tau)d\tau,$$

and the result follows from Lemma 2.4. □

Now, we are able to prove the nonexistence of solutions for the problem $(1.1)-(1.2)$ when $0\leq\alpha<1$.

Theorem 3.1. *Let $0\leq\alpha<1$and f be $C^1(\mathbb{R},\mathbb{R})$ function satisfies*

$$f(x)\geq a|x|^p \text{ for all } x\in\mathbb{R} \text{ for some positive constant } a \text{ and } p>1.$$

Suppose that g is a nonnegative function different from zero almost everywhere with $g^{1-p'}$, $t^{-\alpha p'}g^{1-p'}(t)\in L^1_{loc}[0,+\infty)$ and

$$\lim_{T\to\infty}T^{1-p'}\left(T^{-p'}\int_0^T g^{1-p'}(t)dt+\int_0^T t^{-\alpha p'}g^{1-p'}(t)dt\right)=0, \tag{3.1}$$

where $p'=\frac{p}{p-1}$. Then the problem $(1.1)-(1.2)$ does not admit any global nontrivial solution when $u_0\geq 0$.

Proof. Assume, on the contrary, that a solution $u\in AC[0,T]$ exists for all $T>0$. Multiplying both sides of (1.1) by the test function ϕ defined in (2.5) with $\theta>2p'-1$

and integrating, we obtain

(3.2)
$$\int_0^T \phi(t)\, u'(t)dt + \int_0^T \phi(t) \left({}^C D_{0+}^{\alpha} u\right)(t)dt \geq \int_0^T \phi(t) \left(\int_0^t g(t-s) f(u(s))\, ds \right) dt.$$

By Lemma 2.7 we have,

$$\int_0^T \phi(t) \left({}^C D_{0+}^{\alpha} u\right)(t)\, dt = \int_0^T u(t) \left(D_{T-}^{\alpha} \phi\right)(t)\, dt - u_0 \bar{\xi}_{0,\alpha} T^{1-\alpha}, \tag{3.3}$$

$$\int_0^T \phi(t)\, u'(t)dt = -\int_0^T u(t)\phi'(t)\, dt - u_0, \tag{3.4}$$

where $\bar{\xi}_{0,\alpha} = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\alpha+2)}$.

Substituting (3.3) and (3.4) in (3.2) yields

$$W + u_0 \left(1 + \bar{\xi}_{0,\alpha} T^{1-\alpha}\right) \leq \int_0^T u(-\phi')\, dt + \int_0^T u D_{T-}^{\alpha} \phi dt \tag{3.5}$$

where

$$W := \int_0^T \phi(t) \left(\int_0^t g(t-s) f(u(s))\, ds \right) dt. \tag{3.6}$$

To have a bound for the integral$W$, we rewrite it as

$$W = \int_0^T f(u(s)) \left(\int_s^T g(t-s)\phi(t)\, dt \right) ds = \int_0^T f(u(s))\, G(s) ds,$$

where

$$G(s) := \int_s^T g(t-s)\phi(t)\, dt, \quad 0 \leq s < t \leq T.$$

Applying Hölder inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ for the two integrals in right hand side of (3.5), we obtain

$$\int_0^T u(-\phi')\, dt \leq \left(\int_0^T |u|^p\, G dt \right)^{\frac{1}{p}} \left(\int_0^T G^{-\frac{p'}{p}} (-\phi')^{p'}\, dt \right)^{\frac{1}{p'}} \leq W^{\frac{1}{p}} U^{\frac{1}{p'}},$$

$$\int_0^T u D_{T-}^{\alpha} \phi dt \leq \left(\int_0^T |u|^p\, G dt \right)^{\frac{1}{p}} \left(\int_0^T G^{-\frac{p'}{p}} \left(D_{T-}^{\alpha} \phi\right)^{p'}\, dt \right)^{\frac{1}{p'}} \leq W^{\frac{1}{p}} V^{\frac{1}{p'}},$$

where

$$U := a^{-\frac{p'}{p}} \int_0^T G^{-\frac{p'}{p}} (-\phi')^{p'}\, dt \quad \text{and} \quad V := a^{-\frac{p'}{p}} \int_0^T G^{-\frac{p'}{p}} \left(D_{T-}^{\alpha} \phi\right)^{p'}\, dt. \tag{3.7}$$

Therefore (3.5) can be rewritten as

$$W + u_0 \left(1 + \bar{\xi}_{0,\alpha} T^{1-\alpha}\right) \leq W^{\frac{1}{p}} \left(U^{\frac{1}{p'}} + V^{\frac{1}{p'}} \right). \tag{3.8}$$

From the positivity of W, u_0 and $\bar{\xi}_{0,\alpha}$, we get from (3.8)

$$W \leq W^{\frac{1}{p}} \left(U^{\frac{1}{p'}} + V^{\frac{1}{p'}} \right),$$

which implies that

$$W \leq 2^{p'-1}(U+V). \tag{3.9}$$

Now, we estimate the integral U defined in (3.7),

$$\begin{aligned} U &= a^{-\frac{p'}{p}} \int_0^T G^{-\frac{p'}{p}}(t)\left(-\phi'(t)\right)^{p'} dt \\ &= a^{-\frac{p'}{p}} \int_0^T \left(\int_t^T g(s-t)\phi(s)\,ds\right)^{1-p'} \left(D^1_{T^-}\phi(t)\right)^{p'} dt \\ &\leq a^{-\frac{p'}{p}} \hat{C}_{1,p'} T^{1-2p'} \int_0^T g^{1-p'}(t)dt, \ \text{(Lemma 3.1 with } \beta = 1). \end{aligned} \tag{3.10}$$

Similarly, we see that

$$\begin{aligned} V &= a^{-\frac{p'}{p}} \int_0^T G^{-\frac{p'}{p}}(t)\left(D^\alpha_{T^-}\phi\right)^{p'}(t)\,dt \\ &= a^{-\frac{p'}{p}} \int_0^T \left(\int_t^T g(s-t)\phi(s)\,ds\right)^{1-p'} \left(D^\alpha_{T^-}\phi\right)^{p'}(t)\,dt \\ &\leq a^{-\frac{p'}{p}} \hat{C}_{\alpha,p'} T^{1-p'} \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt, \end{aligned} \tag{3.11}$$

(Lemma 3.1 with $0 \leq \beta = \alpha < 1$).

Substituting (3.10) and (3.11) in (3.9) we end up with

$$W \leq M\left(T^{1-2p'}\int_0^T g^{1-p'}(t)dt + T^{1-p'}\int_0^T t^{-\alpha p'} g^{1-p'}(t)dt\right), \tag{3.12}$$

where $M = 2^{p'-1}\max\left\{a^{-\frac{p'}{p}}\hat{C}_{1,p'}, a^{-\frac{p'}{p}}\hat{C}_{\alpha,p'}\right\}$. Eventually, we deduce from Condition (3.1) that $u \equiv 0$ and the proof is complete. $\square$

The following result is a corollary of Theorem 3.1.

Corollary 3.1. *Let α and f be as in the assumptions of Theorem 3.1. Suppose that, for any $T>0$, there exist positive constants k_1, k_2,*

$$\omega_1 < \frac{p+1}{p-1} \ \text{and} \ \omega_2 < \frac{1}{p-1} \tag{3.13}$$

such that

$$\int_0^T g^{1-p'}(t)dt \leq k_1 T^{\omega_1}, \ \text{and} \ \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt \leq k_2 T^{\omega_2}, \tag{3.14}$$

where $p' = \frac{p}{p-1}$ and g is a nonnegative function that is different from zero almost everywhere with $g^{1-p'}$, $t^{-\alpha p'}g^{1-p'}(t) \in L^1_{loc}[0,+\infty)$. Then the problem $(1.1)-(1.2)$ has no nontrivial global solution when $u_0 \geq 0$.

Proof. It suffices to show that the assumptions (3.13) and (3.14) imply that (3.1) is fulfilled. We deduce from the hypothesis (3.14), that

$$0 \leq T^{1-p'}\left(T^{-p'}\int_0^T g^{1-p'}(t)dt + \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt\right) \leq k_1 T^{1-2p'+\omega_1} + k_2 T^{1-p'+\omega_2}.$$

We find from (3.13) that $1-2p'+\omega_1<0,\ \ 1-p'+\omega_2<0$ and consequently (3.1) follows. □

The following corollary considers an important type of kernels appear in numerous applications.

Corollary 3.2. *Let α and f be as in the assumptions of Theorem 3.1. Suppose that $g(t) \geq bt^{-\eta}$, $t>0$, for some constant $b>0$, where $1-p(1-\alpha) < \eta < 2+p(\alpha-1)$. Then the problem* (1.1) – (1.2) *has no nontrivial global solution when $u_0 \geq 0$.*

Proof. It suffices to show that Hypothesis (3.1) is satisfied with the function g. Indeed, since $g(t) \geq bt^{-\eta}$; $b,\ \eta>0$, then $g^{1-p'}(t) \leq b^{1-p'} t^{\eta(p'-1)}$ and

$$\int_0^T g^{1-p'}(t)dt \leq b^{1-p'}\int_0^T t^{\eta(p'-1)}dt = \frac{b^{1-p'}}{\eta(p'-1)+1}T^{\eta(p'-1)+1}$$

$$\begin{aligned}\int_0^T t^{-\alpha p'} g^{1-p'}(t)dt &\leq b^{1-p'}\int_0^T t^{\eta(p'-1)-\alpha p'}dt \\ &= \frac{b^{1-p'}}{\eta(p'-1)-\alpha p'+1}T^{\eta(p'-1)-\alpha p'+1}.\end{aligned}$$

Therefore

$$\begin{aligned}&T^{1-p'}\left(T^{-p'}\int_0^T g^{1-p'}(t)dt + \int_0^T t^{-\alpha p'} g^{1-p'}(t)dt\right) \\ &\leq \frac{b^{1-p'}}{\eta(p'-1)+1}T^{\sigma_1} + \frac{b^{1-p'}}{\eta(p'-1)-\alpha p'+1}T^{\sigma_2},\end{aligned}$$

where

$$\sigma_1 = 2-\eta+p'(\eta-2),\ \ \sigma_2 = 2-\eta+p'(\eta-\alpha-1).$$

It follows from $1-p(1-\alpha)<\eta<2+p(\alpha-1)$ that both σ_1 and σ_2 are negative and so (3.1) is satisfied. □

Remark 3.1. Corollary 3.2 can be considered also as a consequence of Corollary 3.1 with

$$\begin{aligned}k_1 &= \frac{b^{1-p'}}{\eta(p'-1)+1},\ k_2 = \frac{b^{1-p'}}{\eta(p'-1)-\alpha p'+1}, \\ \omega_1 &= \eta(p'-1)+1 = \frac{p+\eta-1}{p-1}, \\ \omega_2 &= \eta(p'-1)-\alpha p'+1 = \frac{p(1-\alpha)+\eta-1}{p-1},\quad 0 \leq \alpha < 1.\end{aligned}$$

It is clear from $1-p\,(1-\alpha)<\eta<2-p\,(1-\alpha)$ that $0<\omega_1<\frac{p+1}{p-1}$, $0<\omega_2<\frac{1}{p-1}$.

The next theorem deals with the case $1\leq\alpha<2$.

Theorem 3.2. *Let $1\leq\alpha<2$ and f be as in the assumptions of Theorem 3.1. Assume that g is a nonnegative function that is different from zero almost everywhere with $g^{1-p'}$, $t^{(1-\alpha)p'}g^{1-p'}(t)\in L^1_{loc}\,[0,+\infty)$. Suppose that*

$$\lim_{T\to\infty}T^{1-2p'}\left(\int_0^T g^{1-p'}(t)dt+\int_0^T t^{(1-\alpha)p'}g^{1-p'}(t)dt\right)=0, \tag{3.15}$$

where $p'=\frac{p}{p-1}$. Then (1.1) *subject to* (1.3) *has no nontrivial global solution when $u_0,u_1\geq 0$.*

Proof. Assume, on the contrary, that a solution $u\in AC^2\,[0,T]$ exists for all $T>0$. Then as in the proof of Theorem 3.1, we have

$$W+u_0\left(1+\bar{\xi}_{1,\alpha}T^{1-\alpha}\right)+u_1\bar{\xi}_{0,\alpha}T^{2-\alpha}\leq W^{\frac{1}{p}}\left(U^{\frac{1}{p'}}+V^{\frac{1}{p'}}\right),$$

where W, U and V are as in (3.6) and (3.7). Accordingly, for $1\leq\alpha<2$, from Lemma 3.1 with $1\leq\beta=\alpha<2$, we obtain the following estimates

$$\begin{aligned} U &\leq a^{-\frac{p'}{p}}\hat{C}_{1,p'}T^{1-2p'}\int_0^T g^{1-p'}(t)dt,\\ V &= \int_0^T\left(\int_t^T g(s-t)\phi\,(s)\,ds\right)^{1-p'}\left(D^\alpha_{T-}\phi\right)^{p'}(t)\,dt\\ &\leq a^{-\frac{p'}{p}}\hat{C}_{\alpha,p'}T^{1-2p'}\int_0^T t^{(1-\alpha)p'}g^{1-p'}(t)dt,\quad \text{(Lemma 3.1 with } 1\leq\beta=\alpha<2). \end{aligned}$$

By the assumptions (3.15), we get $u\equiv 0$ and the proof is complete. □

Applying Theorem 3.2 for kernels of the type $g(t)\geq bt^{-\eta}$, we obtain the following result.

Corollary 3.3. *Let α and f be as in the assumptions of Theorem 3.2. Suppose that $g(t)\geq bt^{-\eta}$, $t>0$, for some constant $b>0$, where $1-p\,(2-\alpha)<\eta<2$. Then* (1.1) *subject to* (1.3) *has no nontrivial global solution when $u_0,u_1\geq 0$.*

Proof. The hypotheses of Theorem 3.2 are satisfied with the given kernel g. In fact,

$$\int_0^T g^{1-p'}(t)dt\leq b^{1-p'}\int_0^T t^{\eta\left(p'-1\right)}dt=b_2T^{\eta_1},$$

$$\int_0^T t^{(1-\alpha)p'}g^{1-p'}(t)dt\leq b^{1-p'}\int_0^T t^{\eta\left(p'-1\right)+(1-\alpha)p'}dt=b_3T^{\eta_2},$$

where

$$b_2 = \frac{b^{1-p'}}{\eta(p'-1)+1}, \; b_3 = \frac{b^{1-p'}}{\eta(p'-1)+(1-\alpha)p'+1},$$

$$\eta_1 = \eta(p'-1)+1 = \frac{p+\eta-1}{p-1} > 0,$$

$$\eta_2 = \eta(p'-1)+(1-\alpha)p'+1 = \frac{p(2-\alpha)+\eta-1}{p-1} > 0.$$

It is easy to check that $\eta_1, \eta_2 > 0$ and $1-2p'+\eta_1, 1-2p'+\eta_2 < 0$. □

Remark 3.2. The same results of Theorem 3.2 and Corollary 3.3 can be obtained with more relaxed conditions on the initial data. It is enough to have $a_0 u_0 + a_1 u_1 \geq 0$, for some positive constants a_0 and a_1, instead of $u_0 \geq 0$ and $u_1 \geq 0$. Indeed, a_0 and a_1 can be given in terms of the constants $T, \bar{\xi}_{0,\alpha}$ and $\bar{\xi}_{1,\alpha}$.

4. Applications

In this section, we provide a special case of the kernel $g(t)$ in Corollaries 3.2 and 3.3, when the source term is the Riemann-Liouville fractional integral of a power of the state. We show here by computing the solutions numerically that the solutions can not exist globally.

Example 4.1. *Consider the fractional integro-differential inequality*

$$u'(t) + \left({}^C D^{\alpha}_{0+} u\right)(t) \geq \left(I^{\beta}_{0+} |u(s)|^p\right)(t), \quad t>0,\; \beta>0,\; p>1, \tag{4.1}$$

subject to (1.2) *when* $0 \leq \alpha < 1$,*or,* (1.3) *when* $1 \leq \alpha < 2$. *The problem consists of* (4.1) *subject to* (1.2) *is a special case of* (1.1) – (1.2) *when*

$$g(t) = t^{\beta-1}, \quad 0<\beta<p(1-\alpha), \; 0 \leq \alpha < 1.$$

Therefore, we deduce from Corollary 3.2, when $g(t) = t^{-\eta}$, $\eta = 1-\beta$, *that Problem* (4.1) *has no nontrivial global solutions when* $u_0 \geq 0$. *Similarly,* (4.1) *subject to* (1.3) *has no nontrivial global solution when* $u_0, u_1 \geq 0$. *This result is a special case of Corollary 3.3 when*

$$g(t) = t^{-\eta},\; \eta = 1-\beta, \quad 0<\beta<p(2-\alpha), \; 1 \leq \alpha < 2.$$

For treating the two problems in Example 4.1 numerically, we consider the case of equality and write

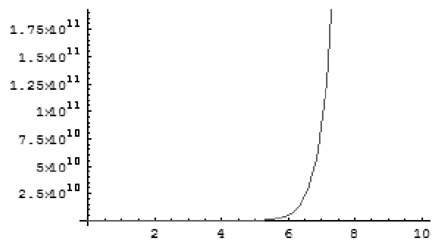
$$u(t) = u_0 - \int_0^t \left({}^C D^{\alpha}_{0+} u\right)(s)ds + \left(I^{\beta+1}_{0+} |u(s)|^p\right)(t).$$

Using the iterative schemes

$$u^{(n)}(t)=u_0-\int_0^t D^{\alpha}_{0+}\left(u^{(n-1)}(\tau)-u_0\right)(s)ds+\left(I^{\beta+1}_{0^+}\left|u^{(n-1)}(s)\right|^p\right)(t),$$

$$u^{(n)}(t)=u_0-\int_0^t \left(D^{\alpha}_{0+}u^{(n-1)}(\tau)-u_0-u_1\tau\right)(s)ds+\left(I^{\beta+1}_{0^+}\left|u^{(n-1)}(s)\right|^p\right)(t),$$

$n=1,2,\ldots$ with $u^{(0)}(t)=u_0$, for (4.1) subject to (1.2) and (1.3), respectively, the curves of the fourth iteration $u^{(4)}$ show, for different values of the parameters, that the solutions can not be extended for all t.



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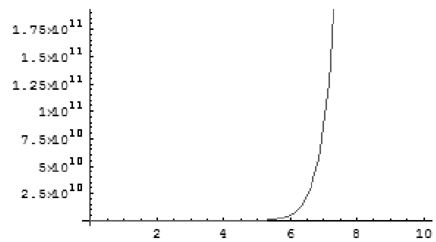

Figure 1: $\alpha=\beta=\frac{1}{2}, p=2, u_0=1$. Figure 2: $\alpha=\frac{3}{2}, \beta=\frac{1}{2}, p=2, u_0=u_1=1$.

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DIFFERENTIAL SANDWICH-TYPE RESULTS FOR SYMMETRIC FUNCTIONS ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

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Abstract. In this paper we obtain some applications of the theory of differential subordination, differential superordination, and sandwich-type results for some subclasses of symmetric functions associated with Pascal distribution series.

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Keywords: symmetric functions; convex functions; univalent functions; Hadamard (convolution) product; differential subordination; differential superordination; sandwich-type results; Pascal distribution

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{H}[a, m]$ denote the subclass of functions $f \in \mathcal{H}(\mathbb{U})$ of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \dots,\ z \in \mathbb{U},$$

with $a \in \mathbb{C}$ and $m \in \mathbb{N} := \{1, 2, \dots\}$.

Also, let $\mathcal{A}(m)$ denote the subclass of functions $f \in \mathcal{H}(\mathbb{U})$ of the form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k,\ z \in \mathbb{U}, \tag{1.1}$$

with $m \in \mathbb{N}$, and let $\mathcal{A} := \mathcal{A}(1)$.

A variable x is said to have the *Pascal distribution* if it takes the values $0, 1, 2, 3, \dots$ with the probabilities

$$(1-q)^r, \quad \frac{qr(1-q)^r}{1!}, \quad \frac{q^2 r(r+1)(1-q)^r}{2!}, \quad \frac{q^3 r(r+1)(r+2)(1-q)^r}{3!}, \dots$$

respectively, where q and r are called the parameters, and thus we have the probability formula

$$P(X = k) = \binom{k+r-1}{r-1} q^k (1-q)^r,\ k \in \mathbb{N}_0.$$

Now, we introduce a power series whose coefficients are probabilities of the *Pascal distribution*, that is

$$\mathrm{Q}_{q,m}^{r}(z) := z + \sum_{n=m+1}^{\infty} \binom{n+r-2}{r-1} q^{n-1}(1-q)^r z^n,\ z \in \mathbb{U},$$
$$(m \in \mathbb{N},\ r \geq 1,\ 0 \leq q \leq 1)\,,$$

and using the ratio test we easily deduce that the radius of convergence of the above power series is at least $\dfrac{1}{q} \geq 1$, hence $\mathrm{Q}_{q,m}^{r} \in \mathcal{A}(m)$.

Defining the functions

$$\begin{aligned}\mathrm{M}_{q,\lambda}^{r,m}(z) &:= (1-\lambda)\mathrm{Q}_{q,m}^{r}(z) + \lambda z \left(\mathrm{Q}_{q,m}^{r}(z)\right)' \\ &= z + \sum_{n=m+1}^{\infty} \binom{n+r-2}{r-1} \left[1+\lambda(n-1)\right] q^{n-1}(1-q)^r z^n,\ z \in \mathbb{U},\end{aligned}$$
$$(m \in \mathbb{N},\ r \geq 1,\ 0 \leq q \leq 1,\ \lambda \geq 0)\,,$$

we introduce the linear operator $\mathcal{N}_{q,\lambda}^{r,m} : \mathcal{A}(m) \to \mathcal{A}(m)$ defined by

$$\begin{aligned}\mathcal{N}_{q,\lambda}^{r,m} f(z) &:= \mathrm{M}_{q,\lambda}^{r,m}(z) * f(z) \\ &= z + \sum_{n=m+1}^{\infty} \binom{n+r-2}{r-1} \left[1+\lambda(n-1)\right] q^{n-1}(1-q)^r a_n z^n,\ z \in \mathbb{U},\end{aligned}$$
$$(m \in \mathbb{N},\ r \geq 1,\ 0 \leq q \leq 1,\ \lambda \geq 0)\,,$$

where f is given by (1.1), and the symbol "$*$" stands for the *Hadamard (or convolution) product*.

Remark that, for $m = 1$ the function $\mathrm{M}_{q,\lambda}^{r,m}$ reduces to $\mathrm{N}_{q,\lambda}^{r} := \mathrm{M}_{q,\lambda}^{r,1}$ introduced and studied by El-Deeb et al. [9].

Definition 1.1. For $f, g \in \mathcal{H}(\mathbb{U})$, we say that f *is subordinate to* g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which is analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [12, 7]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $k, h \in \mathcal{H}(\mathbb{U})$, and let $\varphi(r, s; z) : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$.

(i) If k satisfies *the first order differential subordination*

$$\varphi(k(z), zk'(z); z) \prec h(z), \tag{1.2}$$

then k is said to be *a solution of the differential subordination* (1.2). The function q is called *a dominant of the solutions of the differential subordination* (1.2) if $k \prec q(z)$ for all the functions k satisfying (1.2). A dominant $\widetilde{q}$ is said to be *the best dominant of* (1.2) if $\widetilde{q}(z) \prec q(z)$ for all the dominants q.

(ii) If k satisfies *the first order differential superordination*

$$h(z) \prec \varphi(k(z), zk'(z); z), \tag{1.3}$$

then k is called to be *a solution of the differential superordination* (1.3). The function q is called *a subordinant of the solutions of the differential superordination* (1.3) if $q(z) \prec k(z)$ for all the functions k satisfying (1.3). A subordinant $\widetilde{q}$ is said to be *the best subordinant of* (1.3) if $q(z) \prec \widetilde{q}(z)$ for all the subordinants q.

Miller and Mocanu [13] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(k(z), zk'(z); z) \Rightarrow q(z) \prec k(z).$$

Using the results of Miller and Mocanu [13], Bulboacă [6] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [5]. Ali et al. [1], have used the results of [7] (see also [2, 3, 8]) to obtain sufficient conditions for normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are univalent functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$.

Sakaguchi [15] introduced a class S_s^* of *functions starlike with respect to symmetric points*, which consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$Re \frac{zf'(z)}{f(z) - f(-z)} > 0,\ z \in \mathbb{U},$$

that represents a subclass of *close-to-convex functions*, and hence univalent in $\mathbb{U}$, and moreover, this class includes the class of *convex functions* and *odd starlike functions with respect to the origin* (see [14, 15]).

Also, Aouf et al. [4] introduced and studied the class $S_{s,n}^* T(1,1)$ of *functions n-starlike with respect to symmetric points*, which consists of functions $f \in \mathcal{A}$ with $a_k \leq 0$ for $k \geq 2$, and satisfying the inequality

$$Re \frac{D^{n+1} f(z)}{D^n f(z) - D^n f(-z)} > 0,\ z \in \mathbb{U},$$

where D^n is the *Sălăgean operator* [16].

The classes defined in [14] and [15] could be generalized by introducing the next class of functions, defined with the aid of the $\mathcal{N}_{q,\lambda}^{r,m}$ operator:

Definition 1.2. A function $f \in \mathcal{A}(m)$ with

$$\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z) \neq 0,\ z \in \dot{\mathbb{U}} := \mathbb{U} \setminus \{0\}, \tag{1.4}$$

is said to be in the class $\mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,A,B)$ if it satisfies the subordination condition

$$(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} \tag{1.5}$$

$$-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)\right)'-z\left(\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\prec\frac{1+Az}{1+Bz},$$

$$\left(\gamma\in\mathbb{C},\ 0<\mu<1,\ -1\leq B<A\leq 1,\ m\in\mathbb{N},\ r\geq 1,\ 0\leq q\leq 1,\ \lambda\geq 0\right).$$

In this paper we will obtain some sharp differential subordination and superordination results for the functions belonging to the class $\mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,A,B)$, in order to try to make a connection between a special subclass of analytic functions whose coefficients are probabilities of the *Pascal distribution*, and the differential subordination theory.

2. Preliminaries

In order to prove our results we shall need the following definition and lemmas.

Definition 2.1. [12, Definition 2.2b., p. 21] Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{\mathbb{U}}\setminus E(f)$, where $E(f):=\left\{\zeta\in\partial\mathbb{U}:\lim\limits_{z\to\zeta}f(z)=\infty\right\}$ and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial\mathbb{U}\setminus E(f)$.

Lemma 2.1. [12, Theorem 3.1b., p. 71] *Let the function H be convex in $\mathbb{U}$, with $H(0)=a$, and $\lambda\neq 0$ with* $\mathrm{Re}\lambda\geq 0$. *If $\Phi\in\mathcal{H}[a,m]$ and*

$$\Phi(z)+\frac{z\Phi'(z)}{\lambda}\prec H(z), \tag{2.1}$$

then

$$\Phi(z)\prec\Psi(z):=\frac{\lambda}{mz^{\frac{\lambda}{m}}}\int_0^z t^{\frac{\lambda}{m}-1}H(t)dt\prec H(z),$$

and the function Ψ is convex, $\Psi\in\mathcal{H}[a,m]$, and is the best dominant of (2.1).

Lemma 2.2. [18, Lemma 2.2., p. 3] *Let q be a univalent in $\mathbb{U}$, with $q(0)=1$. Let $\xi,\varphi\in\mathbb{C}$ with $\varphi\neq 0$, and assume that*

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\mathrm{Re}\frac{\xi}{\varphi}\right\},\ z\in\mathbb{U}.$$

If k is analytic in $\mathbb{U}$ and

$$\xi k(z)+\varphi zk'(z)\prec\xi q(z)+\varphi zq'(z), \tag{2.2}$$

then $k(z)\prec q(z)$, and q is the best dominant of (2.2).

From [13, Theorem 6, p. 820] we could easily obtain the following lemma:

Lemma 2.3. *Let q be convex in $\mathbb{U}$, and $k \neq 0$ with $Rek \geq 0$. If $g \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, such that $g(z) + kzg'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + kzq'(z) \prec g(z) + kzg'(z), \tag{2.3}$$

implies that $q(z) \prec g(z)$, and q is the best subordinant of (2.3).

Lemma 2.4. [10] *Let F be analytic and convex in $\mathbb{U}$, and $0 \leq \lambda \leq 1$. If $f, g \in \mathcal{A}$, such that $f(z) \prec F(z)$ and $g(z) \prec F(z)$, then*

$$\lambda f(z) + (1 - \lambda) g(z) \prec F(z).$$

3. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\gamma \in \mathbb{C}$, $0 < \mu < 1$, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}$, $r \geq 1$, $0 \leq q \leq 1$, $\lambda \geq 0$, and the powers are understood as principle values.

Theorem 3.1. *If $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma, \mu, A, B)$ and $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ with $Re\gamma \geq 0$, then*

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} \prec \Psi(z) := \frac{\mu}{\gamma m} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{\mu}{\gamma m} - 1} du \prec \frac{1 + Az}{1 + Bz},$$

and Ψ is convex, $\Psi \in \mathcal{H}[1, m]$, and is the best dominant.

Proof. If we define a function h by

$$h(z) = \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu}, \ z \in \mathbb{U}, \tag{3.1}$$

from (1.4) it follows that h is an analytic function in $\mathbb{U}$, with $h(0) = 1$. Differentiating (3.1) with respect to z, we obtain that

$$(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu}$$

$$-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m} f(z)\right)' - z\left(\mathcal{N}_{q,\lambda}^{r,m} f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu}$$

$$= h(z) + \frac{\gamma}{\mu} z h'(z) \prec \frac{1 + Az}{1 + Bz}. \tag{3.2}$$

Since

$$\mathcal{N}_{q,\lambda}^{r,m} f(z) = z + \sum_{n=m+1}^{\infty} \alpha_n z^n, \quad \text{and} \quad \mathcal{N}_{q,\lambda}^{r,m} f(-z) = -z + \sum_{n=m+1}^{\infty} \alpha_n (-1)^n z^n,$$

where

$$\alpha_n = \binom{n+r-2}{r-1} \left[1 + \lambda(n-1)\right] q^{n-1} (1-q)^r a_n, \ n \geq m+1,$$

we have

$$U(z) := \frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} = \frac{2z}{2z + \sum\limits_{n=m+1}^{\infty} \alpha_n \left[1 + (-1)^{n+1}\right] z^n} = \frac{1}{1 + \sum\limits_{k=m}^{\infty} \beta_k z^k},$$

with

$$\beta_k = \frac{\alpha_{k+1} \left[1 + (-1)^k\right]}{2},\ k \geq m.$$

Moreover,

$$U(z) = \frac{1}{1 + \sum\limits_{k=m}^{\infty} \beta_k z^k} = 1 + \sum_{j=1}^{\infty} \gamma_j z^j,\ z \in \mathbb{U},$$

with unknowns γ_j, $j \geq 1$, we have

$$1 = \left(1 + \beta_m z^m + \beta_{m+1} z^{m+1} + \dots\right) \left(1 + \gamma_1 z + \gamma_2 z^2 + \dots + \gamma_m z^m + \gamma_{m+1} z^{m+1} + \dots\right),$$

and equating the corresponding coefficients it follows that

$$\gamma_1 = \gamma_2 = \dots = \gamma_{m-1} = 0, \quad \gamma_m = -\beta_m, \quad \gamma_{m+1} = -\beta_{m+1}, \dots,$$

hence

$$U(z) = 1 + \sum_{j=m}^{\infty} \gamma_j z^j \in \mathcal{H}[1, m].$$

According to (3.1), we have

$$h = U^{\mu}, \quad \text{with} \quad U \in \mathcal{H}[1, m],$$

and using the binomial power expansion formula we get

$$h = U^{\mu} \in \mathcal{H}[1, m].$$

Now, from the subordination (3.2), using Lemma 2.1 for $\lambda = \dfrac{\mu}{\gamma}$ we obtain our result. □

Remark 3.1. The above theorem shows that

$$\mathcal{N}_{q,\lambda}^{r,m}(\gamma, \mu, A, B) \subset \mathcal{N}_{q,\lambda}^{r,m}(0, \mu, A, B),$$

for all $\gamma \in \mathbb{C}$ with $\mathrm{Re}\gamma \geq 0$.

Moreover, the next inclusion result for the classes $\mathcal{N}_{q,\lambda}^{r,m}(\gamma, \mu, A, B)$ holds:

Theorem 3.2. *If $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $0 \leq \gamma_1 \leq \gamma_2$, and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, then*

$$\mathcal{N}_{q,\lambda}^{r,m}(\gamma_2, \mu, A_2, B_2) \subset \mathcal{N}_{q,\lambda}^{r,m}(\gamma_1, \mu, A_1, B_1). \tag{3.3}$$

Proof. If $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma_2, \mu, A_2, B_2)$, since $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, it is easy to check that

$$(1+\gamma_2)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$-\gamma_2\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$\prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}, \tag{3.4}$$

that is $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma_1, \mu, A_1, B_1)$, hence the assertion (3.3) holds for $\gamma_1 = \gamma_2$.

If $0 \le \gamma_1 < \gamma_2$, from Remark 3.1 and (3.4) it follows $f \in \mathcal{N}_{q,\lambda}^{r,m}(0, \mu, A_1, B_1)$, that is

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} \prec \frac{1+A_1z}{1+B_1z}. \tag{3.5}$$

A simple computation shows that

$$(1+\gamma_1)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$-\gamma_1\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$= \left(1-\frac{\gamma_1}{\gamma_2}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} + \frac{\gamma_1}{\gamma_2}\left[(1+\gamma_2)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\right.$$

$$\left.-\gamma_2\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\right],\ z \in \mathbb{U}. \tag{3.6}$$

Moreover,

$$0 \le \frac{\gamma_1}{\gamma_2} < 1,$$

and the function $\dfrac{1+A_1z}{1+B_1z}$, with $-1 \le B_1 < A_1 \le 1$, is analytic and convex in $\mathbb{U}$. According to (3.6), using the subordinations (3.4) and (3.5), from Lemma 2.4 we deduce that

$$(1+\gamma_1)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$

$$-\gamma_1\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} \prec \frac{1+A_1z}{1+B_1z},$$

that is $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma_1, \mu, A_1, B_1)$. $\square$

Theorem 3.3. *Suppose that q is univalent in $\mathbb{U}$, with $q(0) = 1$, and let $\gamma \in \mathbb{C}^*$ such that*

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -Re\frac{\mu}{\gamma}\right\},\ z \in \mathbb{U}. \tag{3.7}$$

If $f \in \mathcal{A}(m)$ such that (1.4) holds, and satisfies the subordination

$$(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$
$$-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$
$$\prec q(z) + \frac{\gamma}{\mu}zq'(z), \tag{3.8}$$

then

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu} \prec q(z),$$

and q is the best dominant of (3.8).

Proof. Since $f \in \mathcal{A}(m)$ such that (1.4) holds, it follows that the function h defined by (3.1) is analytic in $\mathbb{U}$, and $h(0) = 1$. Like in the proof of Theorem 3.1, differentiating (3.1) with respect to z, we obtain that (3.8) is equivalent to

$$h(z) + \frac{\gamma}{\mu}zh'(z) \prec q(z) + \frac{\gamma}{\mu}zq'(z).$$

Using Lemma 2.2 for $\xi := 1$ and $\varphi := \dfrac{\gamma}{\mu}$, we get that the above subordination implies $h(z) \prec q(z)$, and q is the best dominant of (3.8). $\square$

For the special case $q(z) = \dfrac{1+Az}{1+Bz}$, with $-1 \le B < A \le 1$, Theorem 3.3 reduces to the following corollary:

Corollary 3.1. *Let $\gamma \in \mathbb{C}^*$ and $-1 \le B < A \le 1$, such that*

$$\max\left\{-1; -\frac{1+\mathrm{Re}\frac{\mu}{\gamma}}{1-\mathrm{Re}\frac{\mu}{\gamma}}\right\} \le B \le 0, \quad \textit{or} \quad 0 \le B \le \min\left\{1; \frac{1+\mathrm{Re}\frac{\mu}{\gamma}}{1-\mathrm{Re}\frac{\mu}{\gamma}}\right\}. \tag{3.9}$$

If $f \in \mathcal{A}(m)$ such that (1.4) holds, and satisfies the subordination

$$(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$
$$-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z) - \mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}$$
$$\prec \frac{1+Az}{1+Bz} + \frac{\gamma}{\mu}\frac{(A-B)z}{(1+Bz)^2}, \tag{3.10}$$

then

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\prec\frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant of* (3.10).

Proof. For $q(z)=\dfrac{1+Az}{1+Bz}$, the condition (3.7) reduces to

$$Re\frac{1-Bz}{1+Bz}>\max\left\{0;-Re\frac{\mu}{\gamma}\right\},\ z\in\mathbb{U}. \tag{3.11}$$

Since

$$\inf\left\{Re\frac{1-Bz}{1+Bz}:z\in\mathbb{U}\right\}=\begin{cases}\dfrac{1+B}{1-B}, & \text{if}\quad -1\le B\le 0,\\ \dfrac{1-B}{1+B}, & \text{if}\quad 0\le B<1,\end{cases}$$

we easily check that (3.11) holds if and only if the assumption (3.9) is satisfied, whenever $-1\le B<1$. $\square$

Theorem 3.4. *Let q be convex in $\mathbb{U}$, with $q(0)=1$, and $\gamma\in\mathbb{C}^*$, with $Re\gamma\ge 0$. Also, let $f\in\mathcal{A}(m)$ such that*

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap\mathcal{Q}, \tag{3.12}$$

and assume that the function

$$\begin{aligned}&(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\\ &-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\end{aligned} \tag{3.13}$$

is univalent in $\mathbb{U}$.

If

$$\begin{aligned}q(z)+\frac{\gamma}{\mu}zq'(z)\prec(1+\gamma)&\left(\frac{2z}{D^nf(z)-D^nf(-z)}\right)^{\mu}\\ -\gamma&\left(\frac{D^{n+1}f(z)+D^{n+1}f(-z)}{D^nf(z)-D^nf(-z)}\right)\left(\frac{2z}{D^nf(z)-D^nf(-z)}\right)^{\mu},\end{aligned} \tag{3.14}$$

then

$$q(z)\prec\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu},$$

and q is the best subordinant of (3.14).

Proof. Letting the function h defined by (3.1), then $h \in \mathcal{H}[q(0), m]$, and from (3.12) we have that $h \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$. Like in the proof of Theorem 3.1, differentiating (3.1) with respect to z, we obtain that

$$q(z) + \frac{\gamma}{\mu} z q'(z) \prec h(z) + \frac{\gamma}{\mu} z h'(z).$$

Now, according to Lemma 2.3 for $k := \dfrac{\gamma}{\mu}$ we obtain the desired result. $\square$

Taking $q(z) = \dfrac{1+Az}{1+Bz}$, with $-1 \leq B < A \leq 1$, in Theorem 3.4 we obtain the following corollary:

Corollary 3.2. *Let $\gamma \in \mathbb{C}^*$, with $Re\gamma \geq 0$, and $-1 \leq B < A \leq 1$. If $f \in \mathcal{A}(m)$ such that the assumption* (3.12) *and* (3.13) *hold, and satisfies the subordination*

$$\frac{1+Az}{1+Bz} + \frac{\gamma}{\mu} \frac{(A-B)z}{(1+Bz)^2} \prec (1+\gamma) \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu}$$

$$-\gamma \left(\frac{z \left(\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z) \right)'}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right) \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu}, \tag{3.15}$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best subordinant of* (3.15).

Combining Theorem 3.3 and Theorem 3.4 we obtain the following sandwich-type theorem:

Theorem 3.5. *Let q_1 and q_2 be two convex functions in $\mathbb{U}$, with $q_1(0) = q_2(0) = 1$, and let $\gamma \in \mathbb{C}^*$, with $Re\gamma \geq 0$. If $f \in \mathcal{A}(m)$ such that the assumption* (3.12) *and* (3.13) *hold, then*

$$q_1(z) + \frac{\gamma}{\mu} z q_1'(z) \prec (1+\gamma) \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu}$$

$$-\gamma \left(\frac{z \left(\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z) \right)'}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right) \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu} \prec q_2(z) + \frac{\gamma}{\mu} z q_2'(z), \tag{3.16}$$

implies that

$$q_1(z) \prec \left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)} \right)^{\mu} \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant of (3.16).

Theorem 3.6. *If $f \in \mathcal{N}_{q,\lambda}^{r,m}(0,\mu,1-2\rho,-1)$, with $0 \le \rho < 1$, then $f \in \mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,1-2\rho,-1)$ for $|z| < R$, where*

$$R = \left(\sqrt{\frac{|\gamma|^2 m^2}{\mu^2} + 1} - \frac{|\gamma| m}{\mu}\right)^{\frac{1}{m}}. \tag{3.17}$$

Proof. For $f \in \mathcal{N}_{q,\lambda}^{r,m}(0,\mu,1-2\rho,-1)$, with $0 \le \rho < 1$, let define the function h by

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} = (1-\rho)h(z) + \rho, \ z \in \mathbb{U}. \tag{3.18}$$

Hence, the function h is analytic in $\mathbb{U}$, with $h(0) = 1$, and since $f \in \mathcal{N}_{q,\lambda}^{r,m}(0,\mu,1-2\rho,-1)$ is equivalent to,

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} \prec \frac{1+(1-2\rho)z}{1-z},$$

it follows that Reh(z) > 0, $z \in \mathbb{U}$.

Like in the proof of Theorem 3.1, since $f \in \mathcal{N}_{q,\lambda}^{r,m}(0,\mu,1-2\rho,-1)$, with $0 \le \rho < 1$, we deduce that

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} \in \mathcal{H}[1,m],$$

and from the relation (3.18) we get $h \in \mathcal{H}[1,m]$. Therefore, the following estimate holds

$$|zh'(z)| \le \frac{2mr^m \mathrm{Reh(z)}}{1-r^{2m}}, \ |z| = r < 1,$$

that represents the result of Shah [17] (the inequality (6), p. 240, for $\alpha = 0$), which generalize Lemma 2 of [11].

A simple computation shows that

$$\frac{1}{1-\rho}\left\{(1+\gamma)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu}\right.$$
$$\left.-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m} f(z) - \mathcal{N}_{q,\lambda}^{r,m} f(-z)}\right)^{\mu} - \rho\right\}$$
$$= h(z) + \frac{\gamma}{\mu} zh'(z), \ z \in \mathbb{U},$$

hence, we obtain

$$\mathrm{Re}\left\{\frac{1}{1-\rho}\left[(1+\gamma)\left(\frac{2\mathrm{z}}{\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(z)}-\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(-z)}}\right)^{\mu}\right.\right.$$

$$\left.\left.-\gamma\left(\frac{z\left(\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right)'}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}-\rho\right]\right\}$$

$$\geq \mathrm{Reh(z)}\left[1-\frac{2\left|\gamma\right|\mathrm{mr^m}}{\mu\left(1-\mathrm{r^{2m}}\right)}\right],\ \mathrm{|z|=r<1}, \tag{3.19}$$

and the right-hand side of (3.19) is positive provided that $r<R$, where R is given by (3.17). □

Theorem 3.7. *Let $f\in\mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,A,B)$, let $\gamma\in\mathbb{C}^*$ with $\mathrm{Re}\gamma\geq 0$, and $-1\leq B<A\leq 1$.*

1. Then,

$$\frac{\mu}{\gamma m}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu}{\gamma m}-1}du<\mathrm{Re}\left(\frac{2\mathrm{z}}{\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(z)}-\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(-z)}}\right)^{\mu}$$

$$<\frac{\mu}{\gamma m}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu}{\gamma m}-1}du,\ z\in\mathbb{U}. \tag{3.20}$$

2. For $|z|=r<1$, we have

$$2r\left(\frac{\mu}{\gamma m}\int_0^1\frac{1+Aur}{1+Bur}u^{\frac{\mu}{\gamma m}-1}du\right)^{-\frac{1}{\mu}}<\left|\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right|$$

$$<2r\left(\frac{\mu}{\gamma m}\int_0^1\frac{1-Aur}{1-Bur}u^{\frac{\mu}{\gamma m}-1}du\right)^{-\frac{1}{\mu}}. \tag{3.21}$$

All these inequalities are the best possible.

Proof. From the assumptions, using Theorem 3.1 we obtain that

$$\left(\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right)^{\mu}\prec\Psi(z):=\frac{\mu}{\gamma m}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\gamma m}-1}du, \tag{3.22}$$

and the convex function $\Psi\in\mathcal{H}[1,m]$ is the best dominant. Therefore,

$$\mathrm{Re}\left(\frac{2\mathrm{z}}{\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(z)}-\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(-z)}}\right)^{\mu}<\sup_{\mathrm{z}\in\mathbb{U}}\mathrm{Re}\left(\frac{\mu}{\gamma\mathrm{m}}\int_0^1\frac{1+\mathrm{Azu}}{1+\mathrm{Bzu}}\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du}\right)$$

$$=\frac{\mu}{\gamma\mathrm{m}}\int_0^1\sup_{z\in\mathbb{U}}\mathrm{Re}\left(\frac{1+\mathrm{Azu}}{1+\mathrm{Bzu}}\right)\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du}=\frac{\mu}{\gamma\mathrm{m}}\int_0^1\frac{1+\mathrm{Au}}{1+\mathrm{Bu}}\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du},\ \mathrm{z}\in\mathbb{U},$$

and

$$\mathrm{Re}\left(\frac{2\mathrm{z}}{\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(z)}-\mathcal{N}_{\mathrm{q},\lambda}^{\mathrm{r,m}}\mathrm{f(-z)}}\right)^{\mu}>\inf_{\mathrm{z}\in\mathbb{U}}\mathrm{Re}\left(\frac{\mu}{\gamma\mathrm{m}}\int_0^1\frac{1-\mathrm{Azu}}{1-\mathrm{Bzu}}\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du}\right)$$

$$=\frac{\mu}{\gamma m}\int_0^1\inf_{z\in\mathbb{U}}\mathrm{Re}\left(\frac{1-\mathrm{Azu}}{1-\mathrm{Bzu}}\right)\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du}=\frac{\mu}{\gamma m}\int_0^1\frac{1-\mathrm{Au}}{1-\mathrm{Bu}}\mathrm{u}^{\frac{\mu}{\gamma\mathrm{m}}-1}\mathrm{du},\ \mathrm{z}\in\mathbb{U}.$$

Also, since

$$\left|\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right|^{\mu}<\sup_{z\in\mathbb{U}}\left|\frac{\mu}{\gamma m}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\gamma m}-1}du\right|$$

$$=\frac{\mu}{\gamma m}\int_0^1\sup_{z\in\mathbb{U}}\left|\frac{1+Azu}{1+Bzu}\right|u^{\frac{\mu}{\gamma m}-1}du=\frac{\mu}{\gamma m}\int_0^1\frac{1+Aur}{1+Bur}u^{\frac{\mu}{\gamma m}-1}du,\ |z|=r<1,$$

we get

$$\left|\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right|>2r\left(\frac{\mu}{\gamma m}\int_0^1\frac{1+Aur}{1+Bur}u^{\frac{\mu}{\gamma m}-1}du\right)^{-\frac{1}{\mu}},$$

while

$$\left|\frac{2z}{\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)}\right|^{\mu}>\inf_{z\in\mathbb{U}}\left|\frac{\mu}{\gamma m}\int_0^1\frac{1-Azu}{1-Bzu}u^{\frac{\mu}{\gamma m}-1}du\right|$$

$$=\frac{\mu}{\gamma m}\int_0^1\inf_{z\in\mathbb{U}}\left|\frac{1-Azu}{1-Bzu}\right|u^{\frac{\mu}{\gamma m}-1}du=\frac{\mu}{\gamma m}\int_0^1\frac{1-Aur}{1-Bur}u^{\frac{\mu}{\gamma m}-1}du,\ |z|=r<1,$$

implies

$$\left|\mathcal{N}_{q,\lambda}^{r,m}f(z)-\mathcal{N}_{q,\lambda}^{r,m}f(-z)\right|<2r\left(\frac{\mu}{\gamma m}\int_0^1\frac{1-Aur}{1-Bur}u^{\frac{\mu}{\gamma m}-1}du\right)^{-\frac{1}{\mu}}.$$

The inequalities of (3.20) and (3.21) are the best possible because the subordination (3.22) is sharp. □

Concluding, all the above results give us information about subordination and superordination properties, inclusion results, radius problem, and sharp estimations for the classes $\mathcal{N}_{q,\lambda}^{r,m}(\gamma,\mu,A,B)$, together general sharp subordination and superordination for the operator $\mathcal{N}_{q,\lambda}^{r,m}$. For special choices of the parameters $\gamma\in\mathbb{C}$, $0<\mu<1$, $-1\le B<A\le 1$, $m\in\mathbb{N}$, $r\ge 1$, $0\le q\le 1$, and $\lambda\ge 0$ we may obtain several simple applications connected with the above mentioned classes and operator.

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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN IN BANACH SPACES

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Abstract. Weakly sufficient conditions that guarantee the existence of positive bounded solutions are obtained for the equation: $(\phi_p(u'(t)))' + f(u(t)) = \theta$, $0 < t < 1$, subject to bounded conditions, by a simple application of a recent theoretical result for sum of two operators. The nonlinearity f takes values in a general Banach space.

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Keywords: positive solution; p-Laplacian; sum of two operators; fixed point index; Banach space.

1. Introduction

Let E be a Banach space with a norm $\|\cdot\|$ and zero element θ. Let also,

$$\mathcal{P} = \{u \in E : u \geqslant \theta\}.$$

With $\mathcal{P}^*$ we will denote the dual cone of the cone $\mathcal{P}$. Set $I = [0,1]$. Then $(\mathcal{C}(I,E), \|\cdot\|_c)$ is a Banach space with $\|x\|_c = \max_{t\in I} \|x(t)\|$, and

$$Q = \{x \in \mathcal{C}(I,E) : x(t) \geqslant \theta, \quad t \in I\}$$

is a cone of the Banach space $\mathcal{C}(I,E)$. Let $r > 1$ be arbitrarily chosen and fixed and

$$B_r = \{x \in \mathcal{C}(I,E) : \|x\|_c \leqslant r\}.$$

In this article, we investigate the following boundary value problem (BVP for short):

$$\begin{aligned} (\phi_p(u'(t)))' + f(u(t)) &= \theta, \quad 0 < t < 1, \\ u'(0) = u(1) &= \theta, \end{aligned} \tag{1.1}$$

where

(H1): $\phi_p(s) = |s|^{p-2}s$, $p > 1$ $\quad \phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$,

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(H2): $f \in \mathcal{C}(\mathcal{P}, \mathcal{P})$ and

$$\sup\{\|f(u(t))\| : u \in Q \cap B_r\} \leqslant M < \infty,$$

where $M > 0$ is a given constant such that

$$\frac{M^{q-1}}{q} < 1. \tag{1.2}$$

Our main result is as follows.

Theorem 1.1. *Suppose* (*H*1) *and* (*H*2)*. Then the BVP* (1.1) *has at least one positive bounded solution.*

In this paper, a positive solution u of (1.1) means $u(t) \geqslant \theta,\ t \in (0, 1)$.

Set

$$\begin{aligned} f^{\beta} &= \limsup_{\|u\| \to \beta} \frac{\|f(u)\|}{\phi_p(\|u\|)}, \quad f_{\beta} = \liminf_{\|u\| \to \beta} \frac{\|f(u)\|}{\phi_p(\|u\|)}, \\ (\psi f)_{\beta} &= \liminf_{\|u\| \to \beta} \frac{\psi(f(u))}{\phi_p(\|u\|)}, \end{aligned}$$

where $\beta = 0$ or ∞, $\psi \in \mathcal{P}^*$ and $\|\psi\| = 1$, and for $r_1 > 0$,

$$T_{r_1} = \{x \in E : \|x\| \leqslant r_1\}.$$

Suppose that $\delta \in \left(0, \frac{1}{2}\right)$. If $\mathcal{P}$ is a normal cone, f is uniformly continuous and bounded on $\mathcal{P} \bigcap T_{r_1}$ and there exists a positive constant L_{r_1} with $(q-1)M^{q-2}L_{r_1} < 1$ such that

$$\alpha(f(D)) \leqslant L_r \alpha(D), \quad \forall\, D \in \mathcal{P} \cap T_{r_1},$$

and if

$$\phi_q(f^0) < 1 < \frac{1}{2}\, \delta \phi_q \left(\left(\frac{1}{2} - \delta \right) (\psi f)_{\infty} \right),$$

in [1, Theorem 3.1], it is proved that the BVP (1.1) has at least one non zero positive solution. Here $\alpha(\cdot)$ denotes the Kuratowski measure. Evidently, our main result is better than the result in [1].

The paper is organized as follows. In the next Section, we give some auxiliary results. In Section 3, we prove our main result. In Section 4 we lustrate our main result with some examples.

2. Auxiliary results

Consider the nonlinear equation $Tx + Fx = x$, posed in some closed convex subset of a Banach space, where $(I - T)$ is Lipschitz invertible mapping, in particular T is an expansive operator, and F is a k-set contraction.

A transformation which allows, under certain additional conditions, to derive several existence results for this equation by resorting to the theory of the fixed point index in cones for strict set contractions mappings. Some of these results have been improved in several directions, and they have been applied to obtain existence results of initial and boundary value problems subject to ordinary and partial differential equations (see [1]-[6]).

In what follows, $\mathcal{K}$ will refer to a cone in a Banach space $\mathbb{E}$.
The following Proposition 2.1 will be used to be proved our main result.

Proposition 2.1. [6, 5] *Let Ω be a subset of $\mathcal{K}$ and U be a bounded open subset of $\mathcal{K}$ with $0 \in U$. Assume that the mapping $T : \Omega \subset \mathcal{K} \to \mathbb{E}$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $S : \overline{U} \to \mathbb{E}$ is a k-set contraction with $0 \leqslant k < \gamma^{-1}$, and $S(\overline{U}) \subset (I - T)(\Omega)$. If*

$$Sx \neq (I - T)(\lambda x) \ \text{ for all } x \in \partial U \bigcap \Omega,\ \lambda \geqslant 1 \ \text{ and } \lambda x \in \Omega,$$

then the fixed point index $i_* \left(T + S, U \bigcap \Omega, \mathcal{K}\right) = 1$.

If $\gamma \in \mathcal{C}(I, E)$, in ([1, Lemma 2.1]), is proved that the unique solution of the BVP

$$\begin{aligned} (\phi_p (u'(t)))' + \gamma(t) &= \theta, \quad 0 < t < 1, \\ u'(0) = u(1) &= \theta, \end{aligned} \tag{2.1}$$

is

$$u(t) = \int_t^1 \phi_q \left(\int_0^s \gamma(\tau) d\tau \right) ds, \quad t \in I. \tag{2.2}$$

3. Proof of the Main Result

Take $\epsilon > 0$. Set $R = \frac{M^{q-1}}{q}$ and

$$\Omega = \{u \in Q : \|u\|_c \leqslant R\}, \quad U = \{u \in Q : \|u\|_c \leqslant r\}.$$

For $u \in Q$, define the operators

$$Tu(t) = (1 - \epsilon)u(t),$$
$$Su(t) = \epsilon \int_t^1 \phi_q \left(\int_0^s f(u(\tau)) d\tau \right) ds, \quad t \in I.$$

Note that any fixed point $u \in Q$ of the operator $T + S$ is a solution of the BVP (1.1).

(1) For, $u \in \Omega$, we have that

$$\|(I - T)u(t)\| = \epsilon \|u(t)\|, \quad t \in I.$$

Therefore $I - T : \Omega \to \mathcal{C}(I, E)$ is Lipschitz invertible with a constant $\gamma = \frac{1}{\epsilon}$.

(2) For $u \in \overline{U}$, we have

$$\begin{aligned}\|Su(t)\| &= \epsilon \left\| \int_t^1 \phi_q \left(\int_0^s f(u(\tau)) d\tau \right) ds \right\| \leqslant \epsilon \int_t^1 \phi_q \left(\int_0^s \|f(u(\tau))\| d\tau \right) ds \\ &\leqslant M^{q-1} \epsilon \int_t^1 s^{q-1} ds \leqslant \epsilon \frac{M^{q-1}}{q}, \quad t \in I,\end{aligned}$$

and

$$\|Su\|_c \leqslant \epsilon \frac{M^{q-1}}{q}.$$

Next,

$$\begin{aligned}\left\| \frac{d}{dt} Su(t) \right\| &= \epsilon \left\| -\phi_q \left(\int_0^t f(u(s)) ds \right) \right\| \\ &\leqslant \epsilon \phi_q \left(\int_0^1 \|f(u(s))\| ds \right) \leqslant \epsilon M^{q-1}, \quad t \in [0,1].\end{aligned}$$

Then, $\|(Su)'\|_c \leqslant \epsilon M^{q-1}$. Hence and the Arzela-Ascoli theorem, we conclude that $S : \overline{U} \to \mathcal{C}(I, E)$ is a completely continuous mapping. Therefore $S : \overline{U} \to \mathcal{C}(I, E)$ is a 0-set contraction.

(3) Let $u \in \overline{U}$ be arbitrarily chosen. Take $v = \frac{Su}{\epsilon}$. We have $v \in Q$ and

$$\|v\|_c = \frac{\|Su\|_c}{\epsilon} \leqslant \frac{M^{q-1}}{q},$$

i.e., $v \in \Omega$. Note that $(I - T)v = Su$. Therefore $S(\overline{U}) \subset (I - T)(\Omega)$.

(4) Assume that there are $u \in \partial U$ and $\lambda \geqslant 1$ so that

$$Su = (I - T)(\lambda u) \quad \text{and} \quad \lambda u \in \Omega.$$

We have $Su = \epsilon \lambda u$, $\|u\|_c = r$, and

$$\epsilon \frac{M^{q-1}}{q} \geqslant \|Su\|_c = \epsilon \lambda \|u\|_c \geqslant \epsilon r,$$

whereupon

$$r \leqslant \frac{M^{q-1}}{q} < 1.$$

This is a contradiction, because $r > 1$.

Hence and Proposition 2.1, it follows that the operator $T + S$ has a fixed point $u \in Q$, which is a bounded solution of the BVP (1.1). This completes the proof of the main result.

4. Examples

Example 1. For $m, k \geqslant 0$, consider the BVP:

$$\begin{aligned}\left(|u'|^3 u'\right)'(t) + (u^m(t) + \ln(u^k(t) + 1)) &= 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 0, &\end{aligned} \tag{4.1}$$

Here $E=\mathbb{R}, \mathcal{P}=\mathbb{R}^{+}, \phi_p(s)=|s|^3 s\,(p=5,\ q=\frac{5}{4})$ and $f(y)=y^m+\ln(y^k+1)$. Clearly, the function f is positive continuous and bounded when y is bounded. Moreover, for some $r>1$, the inequality (1.2) in Assumption $(H2)$ is satisfied for all constants m and k satisfying $r^m+\ln(r^k+1)<(\frac{5}{4})^4$.
Therefore, the problem (4.1) has a bounded positive solution.

Example 2. Consider the following BVP of infinite system of scalar differential equations in the infinite-dimensional Banach space $E=l^\infty=\{u=(u_1,\ldots,u_n,\ldots)\mid \sup\limits_n |u_n|<+\infty\}$ with the sup-norm $\|u\|=\sup\limits_n |u_n|$:

$$\begin{aligned}&\left(|u_n'|u_n'\right)'(t)+\tfrac{1}{100}(|\sin u_{n+1}(t)|+5\,u_n^2(t))=0, \quad 0<t<1,\\ &x_n'(0)=0,\ \ x_n(1)=0,\ \ n=1,2,\ldots\end{aligned} \tag{4.2}$$

Let $\mathcal{P}=\{x=(x_n)\in l^\infty\mid x_n\geqslant 0,\ n=1,2,\ldots\}$. It is easy to see that $\mathcal{P}$ is a cone in E. System (4.2) can be regarded as a BVP of the form (1.1) in l^∞ with $\phi_p(s)=|s|s\,(p=3,\ q=\frac{3}{2}),\ u=(u_1,\ldots,u_n,\ldots),\ f=(f_1,\ldots,f_n,\ldots)$,

$$f_n(u(t))=\frac{1}{100}(|\sin u_{n+1}(t)|+5\,u_n^2(t)),\ \text{ for } n=1,2,\ldots.$$

Then $f\in C(\mathcal{P},\mathcal{P})$. Furthermore, for any $r>1$ satisfying $\frac{1}{100}(1+5\,r^2)<\frac{9}{4}$,

$$\sup\{\|f(u(t))\|: u\in Q\cap B_r\}\leqslant M=\frac{1}{100}(1+5\,r^2)<\infty,$$

and $\frac{M^{q-1}}{q}<1$. Therefore, the system (4.2) has a bounded positive solution.

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L^p AND HÖLDER ESTIMATES FOR CAUCHY-RIEMANN EQUATIONS ON CONVEX DOMAIN OF FINITE/INFINITE TYPE WITH PIECEWISE SMOOTH BOUNDARY IN $\mathbb{C}^2$

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Abstract. In this paper, we investigate L^p estimates $(1 \leq p \leq +\infty)$ and f-Hölder estimates for the Cauchy-Riemann equation in a class of convex domains of finite or infinite type with piecewise smooth boundary in $\mathbb{C}^2$.

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Keywords: convex domain; $\bar{\partial}$ equation; finite and infinite type; Berndtsson-Andersson solution.

1. Introduction

Let (z_1, z_2) be the complex Euclidean coordinates of $\mathbb{C}^2$ and let $\Omega \subset \mathbb{C}^2$ be a bounded domain. The Cauchy-Riemann complex on $C^1(\Omega)$-functions is defined as follow:

$$\bar{\partial} u = \sum_{j=1}^{2} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j,$$

where $\dfrac{\partial}{\partial \bar{z}_j} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_j} + \sqrt{-1}\dfrac{\partial}{\partial x_{2+j}}\right)$ with $z_j = x_j + \sqrt{-1}x_{2+j}, j = 1, 2$. One of the most fundamental and important problems in multidimensional complex analysis is to solve the Cauchy-Riemann equation

$$\bar{\partial} u = \varphi$$

for a given $(0,1)$-form $\varphi = \varphi_1 d\bar{z}_1 + \varphi_2 d\bar{z}_2$. In the case when Ω is a smoothly bounded, convex domain, L^p estimates and Hölder estimates of the $\bar{\partial}$-equation were studied by many mathematicians. The book [3] by Chen and Shaw is an excellent reference for this literature. In this paper, we investigate the problem on a class of bounded convex domains with non-smooth boundaries.

For each $j = 1, \dots, N$, let $\Omega_j \subset \mathbb{C}^2$ be a domain with smooth boundary $b\Omega_j$ and $\rho_j : \mathbb{C}^2 \to \mathbb{R}$ is a function of class $C^\infty(\mathbb{C}^2)$. Assume that ρ_j is a defining function of Ω_j in the following sense:

- $\rho_j(z) < 0$ if and only if $z \in \Omega_j$;
- $\{z \in \mathbb{C}^2 : \rho_j(z) = 0\} = b\Omega_j$;

- $|\nabla\rho_j(z)| > 0$ if $z \in b\Omega_j$;
- $\nabla\rho_j \perp b\Omega_j$.

The certain domain in this paper is the transversal intersection of $\Omega_1, \ldots, \Omega_N$, that is defined as follows

$$\Omega = \Omega_1 \cap \Omega_2 \ldots \cap \Omega_N \tag{1.1}$$

so that $d\rho_{j_1} \wedge \ldots \wedge d\rho_{j_l} \neq 0$ on $\bigcap_{k=1}^{l} U_{j_k}$ for $1 \leq j_1 < \ldots < j_l \leq N$, where U_j is a neighborhood of $b\Omega_j$.

For $z \in \Omega$, let us define

$$\frac{1}{\rho(z)} = \sum_{j=1}^{N} \frac{1}{\rho_j(z)}. \tag{1.2}$$

Since $\rho \in C^\infty(\Omega)$ and

$$N^{-1} \inf_{1\leq j\leq N}\{-\rho_j\} \leq -\rho \leq \inf_{1\leq j\leq N}\{-\rho_j\},$$

we have $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ and $-\rho(z) \approx \operatorname{dist}(z, b\Omega)$. Here and in what follows, the notations $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant, and $\approx$ means the combination of $\lesssim$ and $\gtrsim$.

Such domains were firstly considered by M. Range in [13] and then by Berndtsson-Andersson in [1]. The following theorem is the first result of this paper.

Theorem 1.1 (L^p estimates)**.** *Let Ω_j, $j = 1, \ldots, N$, be smooth bounded, convex domains and admit the maximal type F at all boundary points for a same function F (see Definition* (2.1)*). Let Ω be a piecewise smooth domain defined by* (1.1) *and let ρ be defined by* (1.2)*. Let φ be a $(0,1)$-form in $L^p_{(0,1)}(\Omega)$, for $p \in [1, +\infty]$. Then, there is a function $u \in L^p(\Omega)$ satisfying $\bar\partial u = \varphi$ in the weak sense, and*

$$\|u\|_{L^p(\Omega)} \leq C_p \|\varphi\|_{L^p_{(0,1)}}(\Omega).$$

By $\bar\partial u = \varphi$ in the weak sense, we mean that $u = \lim_{\varepsilon\to 0^+} u_\varepsilon$ in $L^p(\Omega)$ (or f-Hölder spaces in Theorem 1.2), where u_ε is the Berndtsson-Andersson solution of $\bar\partial u_\varepsilon = \varphi_\varepsilon$ with smooth φ_ε.

In the lecture of Range [15] given at Cortona (Italy), he proved that on the following smoothly bounded convex domain

$$\Omega^m = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 - 1 < 0\}, \tag{1.3}$$

where $m = 1, 2, \ldots$, the Cauchy-Riemann equation $\bar\partial u = \varphi$ is solvable. This domain is said to be finite type in the sense of Range (see [14, Definition 1.1]). Moreover, the solution u is Hölder continuous of order $\alpha < \frac{1}{2m}$ whenever φ is a $C^1(\overline{\Omega^m})$ $(0,1)$-form. Moreover, he also showed that on the infinite type smooth boundary convex domain

$$\Omega^\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \exp(1+2/s) \cdot \exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\},$$

for $0 < s < 1$, the Cauchy-Riemann equation is although solvable. Nevertheless, there is no solution which is Hölder continuous of any positive order. For this motivation, we need a class of corresponding non-standard Hölder spaces, namely f-Hölder spaces on Ω. That is, for f being an increasing function such that $\lim_{t\to+\infty} f(t) = +\infty$,

$$\Lambda^f(\Omega) = \{u \in L^\infty(\Omega) : \|u\|_f := \|u\|_{L^\infty(\Omega)} + \sup_{z, z+h\in\Omega} f(|h|^{-1})|u(z+h) - u(z)| < +\infty\}.$$

It is clear that if $f(t) = t^\alpha$, for $0 < \alpha < 1$, the space $\Lambda^f(\Omega)$ coincides to $\Lambda^\alpha(\Omega)$-the classical Hölder space of order α. The f-Hölder space was introduced in [10, 9] and extended to study tangential Cauchy-Riemann equations in [6, 7].

Theorem 1.2 (f-Hölder estimates)**.** *Let $\Omega_1, \ldots, \Omega_N$ and Ω be domains defined in Theorem 1.1. Let φ be a continuous $(0,1)$-form. Then, there is a function $u \in \Lambda^f(\Omega)$ satisfying $\bar\partial u = \varphi$ in the weak sense, and*

$$\|u\|_{\Lambda^f(\Omega)} \lesssim \|\varphi\|_{L^\infty_{(0,1)}}(\Omega),$$

where

$$f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} dt\right)^{-1},$$

and F^* is the inverse function of F.

The paper is organized as follows. We recall the construction of Berndtsson-Andersson $\bar\partial$-solution and maximal type F in Section 2. Then, we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

2. Berndtsson-Andersson solution and Maximal type F

In this section, for every $k = 1, \ldots, N$, we assume that Ω_k is a bounded convex domain in $\mathbb{C}^2$ with smooth boundary $b\Omega_k$, and that ρ_k is a defining function for Ω_k. The convexity means

$$\sum_{i,j=1}^4 \frac{\partial^2 \rho_k}{\partial x_i \partial x_j}(\zeta) a_i a_j \geq 0 \quad \text{on } b\Omega_k,$$

for every $a = (a_1, \ldots, a_4) \in \mathbb{R}^4$ with $\sum_{j=1}^4 a_j \frac{\partial \rho_k}{\partial x_j}(\zeta) = 0$ on $b\Omega_k$. Let us define the following support function of Ω_k, for $\zeta, z \in \overline{\Omega}_k$:

$$\Phi_k(\zeta, z) = \Phi_{\Omega_k}(\zeta, z) = \sum_{j=1}^2 \frac{\partial \rho_k}{\partial \zeta_j}(\zeta)(\zeta_j - z_j). \tag{2.1}$$

For each $\zeta \in b\Omega_k$, the condition $\Phi_k(\zeta, z) = 0$ characterizes the translate of the complex tangent space $T_\zeta^{\mathbb{C}}(b\Omega_k)$. The convexity of Ω_k implies

$$\Phi_k(\zeta, z) \neq 0, \quad \text{for all } \zeta \in b\Omega_k, z \in \overline{\Omega}_k.$$

Theorem 2.1 (Berndtsson-Andersson solution [1])**.** *Let $\Omega_1, \dots, \Omega_N$ and Ω be defined as above. Let $\Delta = \{(\zeta, z) \in \Omega \times \Omega | \zeta = z\}$ be the diagonal of Ω. Then for any $r > 1$, there exists a $(2,1)$-form $K^r(\zeta, z)$ defined on $(\Omega \times \Omega) \setminus \Delta$ such that: for any $\bar{\partial}$-closed, continuous $(0,1)$-form φ in $\overline{\Omega}$, the following*

$$S[\varphi](z) = \int_{\zeta \in \Omega} \varphi(\zeta) \wedge K^r(\zeta, z), \quad z \in \Omega,$$

satisfies

$$\bar{\partial}(S[\varphi])(z) = \varphi(z).$$

Moreover, in [4, page 1421], Cho and Park showed that

$$\begin{aligned} |K^r(\zeta, z)| &\lesssim \frac{\prod_{j=1}^{N} |\rho_j(\zeta)|^r}{|\zeta - z|^3 \prod_{j=1}^{N} |\Phi_j(\zeta, z)|^r} + \frac{1}{|\zeta - z|} \sum_{k=1}^{N} \left[\left(\prod_{j \neq k} \frac{|\rho_j(\zeta)|^r . |\rho_k(\zeta)|^{r-1}}{|\Phi_j(\zeta, z)|^r |\Phi_k(\zeta, z)|^{r+1}} \right.\right. \\ &\left.\left. + \frac{\prod_{j=1}^{N} |\rho_j(\zeta)|^r}{\prod_{j \neq k} |\Phi_j(\zeta, z)|^r |\Phi_k(\zeta, z)|^{r+1}} \right) \right] := K_1^r(\zeta, z) + K_2^r(\zeta, z) + K_3^r(\zeta, z). \end{aligned} \tag{2.2}$$

and they also obtained the following L^1-boundedness.

Theorem 2.2 (L^1-estimate ([4]))**.** *Let φ be a $\bar{\partial}$-closed $(0,1)$-form whose coefficients in $L^1(\Omega)$. Then,*

$$\bar{\partial}(S[\varphi]) = \varphi$$

in the weak sense, and $\|S[\varphi]\|_{L^1(\Omega)} \lesssim \|\varphi\|_{L^1_{(0,1)}(\Omega)}$.

In the present work, we are going to study the case L^∞-estimates, for the solution to the $\bar{\partial}$-equation. Hence, we need the following geometric ingredient.

Definition 2.1. Let $F : [0, \infty) \to [0, \infty)$ be a function such that

(1) F is smooth and increasing;

(2) $F(0) = 0$;

(3) $\displaystyle\int_0^\delta |\ln F(t^2)| dt < \infty$ for some small $\delta > 0$;

(4) $\dfrac{F(t)}{t}$ is non-decreasing.

Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded, convex domain. Then, Ω is called a domain admitting the maximal type F at the boundary point $P \in b\Omega$ if there are positive constants c, c', such that, for all $\zeta \in \bar{\Omega} \cap B(P, c')$ we have

$$\rho(z) \gtrsim F(|z-\zeta|^2),$$

for all $z \in B(\zeta, c)$ with $\Phi_\Omega(\zeta, z) = 0$.

In the case $F(t) = t^m$, Ω is called a convex domain of finite type $2m$ in the sense of Range. The maximal type F was introduced in [6, 7] to study tangential Cauchy-Riemann equations, and the global Lipschitz continuity of the Bergman projection weakly pseudoconvex domains in $\mathbb{C}^2$. Some examples are follows.

- Let Ω be a strongly convex domain with its defining function ρ. Then,

$$\Re\Phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + \lambda_0|\zeta - z|^2,$$

for $|\zeta - z|$ and $|\rho(\zeta)|$ small, and $\lambda_0 > 0$ (see [3] for details). Hence, when $\zeta \in b\Omega \cap \{|\zeta - z| < c\}$, and $\Phi(\zeta, z) = 0$, we have

$$\rho(z) \gtrsim F(|z-\zeta|^2),$$

with $F(t) = t$. So, Ω in this case is of maximal type F.

- The complex ellipsoid is

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m_1} + |z_2|^{2m_2} < 1\},$$

where $m_1, m_2 \in \mathbb{N}$. Then Ω is convex of maximal type F with $F(t) = t^m$, for $m = \max\{m_1, m_2\}$, see [15].

- Assume that Ω denote a bounded domain of the type

$$\Omega = \left\{ z = (z_1, z_2) \in \mathbb{C}^n : \rho(z) = \sum_{j=1}^{2} \rho_j(|z_j|^2) - 1 < 0, \right\}$$

where all functions ρ_j are assumed to be real-analytic in $[0, a_j]$ such that

(1) $\rho_j'(t) \geq 0, \rho_j'(t) + 2t\rho_j''(t) \geq 0$ for $0 \leq t \leq a_j$;

(2) $\rho_j'(0) = \rho_j(0) = 0$ and $\rho_j(a_j) > 1$.

In [2], J. Bruna and J. del Castillo obtained that there exists a positive integer m such that

$$\Re\Phi(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + \sum_{k=1}^{2} \frac{\partial^2\rho}{\partial\zeta_k\partial\bar{\zeta}_k}(\zeta)|z_k - \zeta_k|^2 + |\zeta - z|^{2m},$$

for $\zeta, z \in \bar{\Omega}$ (see [2, Formula (7)]). Therefore Ω is a smoothly bounded, admissibly decoupled, convex domain admitting an F-type, with $F(t) = t^m$.

- Let

$$\Omega^\infty = \{(z_1,z_2)\in\mathbb{C}^2 |\, \rho(z) := \exp(1+2/s)\cdot\exp\left(\frac{-1}{|z_1|^s}\right) + |z_2|^2 - 1 < 0\}.$$

Since

$$\Re[\Phi_{\Omega^\infty}(\zeta,z)] \geq \rho(\zeta) - \rho(z) + \exp(1+2/s)\exp\left\{\frac{-1}{32|\zeta-z|^{2s}}\right\},$$

for $0<s<1/2$, Ω^∞ is convex of the maximal type $F(t)=\exp(\frac{-1}{32.t^s})$, see [19]. Note that Ω_∞ is a domain of infinite type.

The most important property of support functions on convex domains admitting a maximal type F is the following.

Lemma 2.1. *[[6, Lemma 3.3, p. 112]] For each $k=1,\dots,N$, let Ω_k be a smoothly bounded, convex domain in $\mathbb{C}^2$ of maximal type F at $P\in b\Omega_k$. Then there is a positive constant c_k such that the support function $\Phi_k(\zeta,z)$ satisfies the following estimate*

$$|\Phi_k(\zeta,z)| \gtrsim |\rho_k(\zeta)| + |\rho_k(z)| + |\Im\Phi_k(\zeta,z)| + F(|z-\zeta|^2), \tag{2.3}$$

for every $\zeta\in\bar\Omega_k\cap B(P,c)$, and $z\in\overline{\Omega}_k$, $|z-\zeta|<c_k$.

3. Proof of L^p-estimate

By Theorem 2.2 and Riesz-Thorin Interpolation Theorem (see Theorem B.6, Appendix B in [3] for more details), we are only going to prove the L^∞-estimate. Let φ be a $(0,1)$-form with L^∞-coefficients on $\overline{\Omega}$. Then by Hölder inequality,

$$|S[\varphi](z)| \lesssim \|\varphi\|_{L^\infty_{(0,1)}(\Omega)}\int_\Omega |K^r(\zeta,z)|dV(\zeta),$$

where $dV(.)$ is the Lebesgue measure in $\mathbb{R}^4$. Next, we will estimate the integral of each term in the right hand side of (2.2).

Firstly, by Lemma 2.1, we obtain

$$\int_\Omega K_1^r(\zeta,z)dV(\zeta) \lesssim \int_\Omega \frac{\prod_{j=1}^N|\rho_j(\zeta)|^r}{|\zeta-z|^3\prod_{j=1}^N|\rho_j(\zeta)|^r}dV(\zeta) \lesssim \int_\Omega\frac{dV(\zeta)}{|z-\zeta|^3}\lesssim 1.$$

Next, for the second term $K_2^r(\zeta,z)$ and $K_3^r(\zeta,z)$, we have

$$\int_\Omega K_2^r(\zeta,z)dV(\zeta) \lesssim \sum_{k=1}^N\int_\Omega\frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^2} \tag{3.1}$$

and

$$\int_{\Omega} K_3^r(\zeta, z) dV(\zeta) \lesssim \sum_{k=1}^{N} \int_{\Omega} \frac{dV(\zeta)}{|\zeta - z||\Phi_k(\zeta, z)|}. \tag{3.2}$$

We are going to estimate the right hand side of (3.1) and the process for (3.2) is similar and more simple. To do this, we recall the Henkin coordinates on each Ω_k.

Lemma 3.1. [5, page 608] *There exist positive constants M, a and $\eta \leq c$, and, for each z with dist$(z, b\Omega_k) \leq a$, there is a smooth local coordinate system $(t_1, t_2, t_3, t_4) = t = t(\zeta, z)$ on the ball $B(z, c)$ such that we have*

$$\begin{cases} t(z,z) = 0, \\ t_1(\zeta) = \rho_k(\zeta) - \rho_k(z), \\ t_2(\zeta) = \Im(\Phi_k(\zeta, z)), \\ |t| < \delta \quad \textit{for } \zeta \in B(z, c), \\ |J_{\mathbb{R}}(t)| \leq M \quad \textit{and} \quad |det J_{\mathbb{R}}(t)| \geq \frac{1}{M}, \end{cases}$$

where $J_{\mathbb{R}}(t)$ is the Jacobian of the transformation at t.

Now, for each fixed k, by Lemma 2.1, we have

$$\begin{aligned} \int_{\Omega} \frac{dV(\zeta)}{|\zeta - z||\Phi_k(\zeta, z)|^2} &\lesssim \int_{\Omega} \frac{dV(\zeta)}{|\zeta_1 - z_1|(|\rho_k(\zeta)| + |\Im\Phi_k(\zeta, z)| + F(|z_1 - \zeta_1|^2))^2} \\ &\lesssim \int_{t^2 < c^2} \frac{dt_1 dt_2 dt_3 dt_4}{|(t_3, t_4)|(t_1 + t_2 + F(t_3^2 + t_4^2))^2} \\ &\lesssim \int_{t_1^2 + t_3^2 + t_4^2 < c^2} \frac{dt_1 dt_3 dt_4}{|(t_3, t_4)|(t_1 + F(t_3^2 + t_4^2))} \\ &\lesssim \int_{t_3^2 + t_4^2 < c^2} \frac{\ln F(t_3^2 + t_4^2)}{|(t_3, t_4)|} dt_3 dt_4 \\ &\lesssim \int_0^c \ln F(s^2) ds \quad (< \infty \text{ since the condition on } F) \\ &\quad \text{(using the polar coodinates } t_3 = s\cos\theta, t_4 = s\sin\theta). \end{aligned}$$

This estimate completes the proof of the L^∞-estimate and so Theorem 1.1.

4. Proof of f-Hölder estimates

Before to prove, we recall the General Hardy-Littlewood Lemma for $\Lambda^f(\Omega)$-continuous which was established by Khanh in [10].

Lemma 4.1. *Let Ω be a smoothly bounded domain in $\mathbb{R}^n$ and let ρ be a defining function of Ω. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\frac{G(t)}{t}$ is* *decreasing and* $\int_0^d \frac{G(t)}{t}dt < \infty$ *for* $d > 0$ *small enough. If* $u \in C^1(\Omega)$ *such that*

$$|\nabla u(x)| \lesssim \frac{G(|\rho(x)|)}{|\rho(x)|} \quad \textit{for every } x \in \Omega,$$

then

$$f(|x-y|^{-1})|u(x)-u(y)| < \infty$$

uniformly in $x, y \in \Omega$, $x \neq y$, *and where* $f(d^{-1}) := \left(\int_0^d \frac{G(t)}{t}dt\right)^{-1}$.

Hence, to prove Theorem 1.2, we need to show that

$$\int_\Omega |\nabla_z K^r(\zeta, z)| dV(\zeta) \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

It is not difficult to show that the term $\int_\Omega |\nabla_z K^r(\zeta, z)|dV(\zeta)$ is bounded from above by

$$C \times \sum_{k=1}^N \left(\int_\Omega \frac{dV(\zeta)}{|\zeta - z|^4} + \int_\Omega \frac{dV(\zeta)}{|\zeta-z|^3|\Phi_k(\zeta,z)|} + \int_\Omega \frac{dV(\zeta)}{|\zeta-z|^2|\Phi_k(\zeta,z)|^2} + \int_\Omega \frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^3}\right).$$

Moreover, in these integrals, it is most difficult to estimate the followings

$$\int_\Omega \frac{dV(\zeta)}{|\zeta-z|^2|\Phi_k(\zeta,z)|^2}, \int_\Omega \frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^3},$$

and the others are similar and bounded from above by $|\ln(-\rho(z))|$. On the other hand, since $|\Phi_k(z,\zeta)| \lesssim |z-\zeta|$,

$$\int_\Omega \frac{dV(\zeta)}{|\zeta-z|^2|\Phi_k(\zeta,z)|^2} \lesssim \int_\Omega \frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^3}.$$

Now, again, by Lemma 2.1 and the Henkin coordinates, we obtain

$$\begin{aligned}
\int_\Omega \frac{dV(\zeta)}{|\zeta-z||\Phi_k(\zeta,z)|^3} &\lesssim \int_{|t|^2<c^2} \frac{dt_1dt_2dt_3dt_4}{|(t_3,t_4)|(|\rho_k(z)| + t_1 + t_2 + F(|(t_3,t_4)|^2))^3} \\
&\lesssim \int_{t_3^2+t_4^22<c^2} \frac{dt_3dt_4}{|(t_3,t_4)|(|\rho_k(z)| + F(|(t_3,t_4)|^2))} \\
&\lesssim \int_0^c \frac{ds}{|\rho(z)| + F(s^2)}.
\end{aligned}$$

Applying the technique introduced in [10], the right-hand-side is split into two parts

$$\int_0^c \frac{dr}{|\rho(z)| + F(r^2)} = \underbrace{\int_0^{\sqrt{F^*(|\rho(z)|)}} \frac{dr}{|\rho(z)| + F(r^2)}}_{\textbf{easy part}} + \underbrace{\int_{\sqrt{F^*(|\rho(z)|)}}^c \frac{dr}{|\rho(z)| + F(r^2)}}_{\textbf{diff. part}}.$$

It is clear that the "easy part" is bounded from above by $\frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}$. For the "diff. part", if $r \geq \sqrt{F^*(|\rho(z)|)}$, by F is increasing,

$$\frac{F(r^2)}{r^2} \geq \frac{F(F^*(|\rho(z)|))}{F^*(|\rho(z)|)} = \frac{|\rho(z)|}{|F^*(|\rho(z)|)|},$$

so we have

$$\frac{F(r^2)}{|\rho(z)|} \geq \frac{r^2}{F^*(|\rho(z)|)}.$$

Therefore,

$$\int_{\sqrt{F^*(|\rho(z)|)}}^{c} \frac{dr}{|\rho(z)| + F(r^2)} \leq \frac{1}{|\rho(z)|} \int_{\sqrt{F^*(|\rho(z)|)}}^{c} \frac{dr}{1 + r^2/F^*(|\rho(z)|)}$$
$$\leq \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|} \int_1^\infty \frac{dy}{1+y^2} = \frac{\pi}{4} \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

Hence we obtain $\displaystyle\int_\Omega \frac{dV(\zeta)}{|\zeta - z\|\Phi_k(\zeta, z)|^3} \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}$. Moreover, since $|\rho(z)| \approx \operatorname{dist}(z, b\Omega)$, we have

$$\int_\Omega \frac{dV(\zeta)}{|\zeta - z| \cdot |\Phi(\zeta, z)|^3} \lesssim \frac{\sqrt{F^*(\operatorname{dist}(z, b\Omega))}}{\operatorname{dist}(z, b\Omega)}.$$

Next, we are going to check that $\dfrac{\sqrt{F^*(\operatorname{dist}(z, b\Omega))}}{\operatorname{dist}(z, b\Omega)}$ satisfies all conditions in General Hardy-Littlewood Lemma. The fact $\frac{\sqrt{F^*(t)}}{t}$ is decreasing is trivial.

For $d > 0$ small enough, by a changing variables, we have

$$\int_0^d \frac{\sqrt{F^*(t)}}{t} dt = \int_0^{\sqrt{F^*(d)}} y(\ln F(y^2))' dy$$
$$= \sqrt{F^*(d)} \ln d - \lim_{t \to 0} t(\ln F(t^2))$$
$$- \underbrace{\int_0^{\sqrt{F^*(d)}} (\ln F(y^2)) dy}_{\text{finite by the hypothesis}}.$$

Since $|\ln(F(t^2))|$ is decreasing when $0 \leq t \leq \delta$, for $\delta > 0$ small enough, so

$$|\ln F(\eta^2)|\eta \leq \int_0^\eta |\ln F(t^2)| dt \leq \int_0^\delta |\ln F(t^2)| dt < \infty$$

uniformly in $0 \leq \eta \leq \delta$. Hence, $\sqrt{F^*(t)}|\ln t| < \infty$ for all $0 \leq t \leq \sqrt{F^*(\delta)}$, and $\lim_{t \to 0} t|\ln F(t^2)| = 0$. These imply

$$\int_0^d \frac{\sqrt{F^*(t)}}{t} dt < \infty.$$

The last inequality completes the proof of Theorem 1.2.

For example, we consider the case

$$\Omega = \Omega^\infty \cap B\left((0,1), \frac{1}{2}\right),$$

for $0 < s < 1/2$, where $B\left((0,1), \frac{1}{2}\right) \subset \mathbb{C}^2$ be the ball with center $(0,1)$ and radius $1/2$. Since $F(t) = \exp\left(\dfrac{-1}{32t^s}\right)$, a direct calculation implies $f(t) = \dfrac{1024^s(1-2s)}{2s}(|\ln t|)^{\frac{1}{2s}-1}$.

Then, if φ be a continuous $(0,1)$-form, the Berndtsson-Andersson solution $S[\varphi]$ of the equation $\bar{\partial}u = \varphi$ belongs to $\Lambda^f(\Omega)$.

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PERIODIZED WAVELET PACKETS ON BOUNDED SUBSETS OF $\mathbb{R}$

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Abstract. In this paper, we study Daubechies periodized wavelet packets (DPWP) and give a necessary condition for it. Also, some properties of DPWP are discussed. Further, we give an estimate for the approximation error related to DPWP. Finally, we use the thresholding technique to study compression errors.

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1. Introduction

The notion of wavelet packets was introduced by Coifman et al. [1] as a family of orthonormal bases for discrete functions in $\mathbb{R}^n$. They split a generalization of the procedure of MRA and constitute the whole set of subband coded decomposition. The “best basis"selection can be easily done, since wavelet packets give quick access to a rich library of orthonormal bases. It proves to be more flexible and useful in application of pyramid algorithm to an image in order to reduce the information into lesser number of coefficients (see [29]).

Zhang and Wu [42] gave a novel image compression technique using wavelet packets and directional decomposition to exploit the image redundancy efficiently and thereby giving high compression ratio. Klappenecker [25] observed that employing periodized wavelet packet transform on quantum computer is much better and economical than the periodized wavelet transform. Kasaei et al. [15] introduced a novel compression algorithm using wavelet packets and lattice vector quantization for fingerprint analysis. Yoon and Vaidyanathan [41] defined a customized thresholding function which significantly improved the performance of powerful wavelet-based denoising scheme known as VisuShrink which uses a single threshold for all the scales. Joseph [14] used wavelet packets for spoken digit compression and employed Malyalam spoken digit for the same.

Later on, Khanna et al. [22] defined the orthogonal Coifman wavelet packet systems and biorthogonal Coifman wavelet packet systems which have good approximation properties with exponential decay and gave wavelet packet approximation theorem. The problem of inadequacy of a wavelet function to study both the symmetries of an asymmetric signal has been addressed by defining wavelets associated with Riesz projectors [23]. Also, wavelet packets and their moments were studied by Khanna et al. [13, 24]. Recently, Khanna and Kaushik [17] gave wavelet packet approximation theorem for H^r type norm which can measure difference of the (weak) derivatives. Uniform approximation of wavelet packet expansions have been studied in [19]. For litrature related to wavelets and wavelet packets one may consult [2], [4 - 13], [16 - 18], [20 - 24], [26 - 30], [32 - 34], [36].

Overview. Inspired from the work of Daubechies [3, 4], Restrepo et al. [35] introduced periodized wavelets by restricting the wavelets on bounded subsets of $\mathbb{R}$. In Section 3, we define wavelet packets associated with the wavelets introduced by Daubechies [3, 4] and called them Daubechies periodized wavelet packets (DPWP) and obtain a necessary condition for it. Also, we give some properties of DPWP. In Section 4, we define and obtain an estimate of the approximation error of a function in $L^2([0,1]) \cap C^g(\mathbb{R})$ $(g > 1)$. Finally, in Section 5, we discuss compression errors using hard thresholding techniques.

2. Preliminaries

In [12], Multiresolution analysis (MRA), is defined as an increasing sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying

$$V_j \subseteq V_{j+1}, \text{ for all } j \in \mathbb{Z}, \tag{2.1}$$

$$f \in V_j \text{ if and only if } f(2(\cdot)) \in V_{j+1}, \text{ for all } j \in \mathbb{Z}, \tag{2.2}$$

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{2.3}$$

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}). \tag{2.4}$$

$$\text{There exists a function } \phi \in V_0 \text{ such that } \phi(\cdot - k) : k \in \mathbb{Z}\} \text{ is an orthonormal basis for } V_0. \tag{2.5}$$

The function ϕ whose existence is asserted in (2.5) is called a scaling function of the given MRA. The scaling function ϕ solves the dilation equation

$$\phi(x) = \sum_{p\in\mathbb{Z}} u_p \ \phi(2x - p) \tag{2.6}$$

with $|\widehat{\phi}(0)| = 1$. But it is convenient to choose the phase of ϕ so that $\int_{\mathbb{R}} \phi(x)\, dx = 1$ and the associated function ψ is defined by

$$\psi(x) = \sum_{p\in\mathbb{Z}} v_p\ \phi(2x - p). \tag{2.7}$$

Note that only finitely many u_p and v_p are non-zero for Daubechies wavelet system. A family of functions $\omega_n,\ n = 0, 1, 2, ...$ defined by

$$\omega_{2n}(x) = \sum_{p=0}^{2g-1} u_p\ \omega_n(2x - p), \tag{2.8}$$

$$\omega_{2n+1}(x) = \sum_{p=0}^{2g-1} v_p\ \omega_n(2x - p), \tag{2.9}$$

where $\omega_1 = \psi$ and $\omega_0 = \phi$ often called mother and father wavelets, are called Daubechies wavelet packets with genus g (see [36]).
Also, the set $\{\omega_n(x - k) : k \in \mathbb{Z},\ n = 0, 1, 2, ...\}$ is an orthonormal basis of $L^2(\mathbb{R})$. The family of wavelet packets $\{\omega_n\}$ define the family of subspaces of $L^2(\mathbb{R})$ corresponding to some orthonormal scaling function $\phi = \omega_0$ given by

$$U_{n,j} = \overline{span}\{\omega_n(2^j x - k) : k \in \mathbb{Z}\},\ j \in \mathbb{Z},\ n = 0, 1, 2, ... \,. \tag{2.10}$$

Note that $U_{0,j} = V_j$ and $U_{1,j} = W_j$ so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be re-written as $U_{0,j+1} = U_{0,j} \oplus U_{1,j},\ j \in \mathbb{Z}$. In general, the above expression is given by $U_{n,j+1} = U_{2n,j} \oplus U_{2n+1,j},\quad$ for $n = 1, 2, 3, ...; j \in \mathbb{Z}$, where $U_{n,j}$ is defined by (2.10).

Proposition 2.1. [36] *Let $\omega_n, n \in \mathbb{N}_0$ be wavelet packets associated with scaling function ω_0. Then, for $j, k, l, m \in \mathbb{Z}$ with $m \geqslant 0$ and $\omega_{j,n,k}(x) = 2^{j/2}\, \omega_n(2^j x - k)$, we have*

(i) $\langle \omega_{j,n,k}, \omega_{j,n,l}\rangle = \delta_{k,l}$,
(ii) $\langle \omega_{j,n,k}, \omega_{j,m,l}\rangle = \delta_{m,n}\ \delta_{k,l}$.

3. Periodized Daubechies wavelet packets

Heretofore, we have seen that the functions which were defined on $\mathbb{R}$ as in some applications such as audio signal processing, where the length of the signal is arbitrarily long and unknown prior to the desistance of its activity. Nevertheless, for many applications, the time domain is a finite interval. One may notice such example in case of data fitting problems, image processing of signal, etc. These problems can be worked out efficiently with the introduction of periodized wavelet packets. Significantly, wavelet packets which are defined in general can be periodized with a technique of Poisson summation and give rise to periodic wavelet packets.

Analogously, to the construction of non-periodic wavelet packets given in [1, 12], the periodized wavelet packets have an exception that they wraps over the edges of the domain, but in computation for large value of j, they reduced to the non-periodic forms. Thus, due to compact support and the construction by the scaling property of the non-periodic functions, many of the properties of wavelet packets are preserved in the periodic case. For various details related to periodized wavelets and wavelet packets, one may refer [4, 12, 31, 35], [37] - [40].
Next, we give the definition of Daubechies periodized wavelet packets (DPWP).

Definition 3.1. The wavelet packets $\omega_n \in L^2(\mathbb{R})$ $(n \in \mathbb{N}_0)$ obtained from scaling function using multiresolution analysis are said to be periodized in the sense of Daubechies (Daubechies periodized wavelet packets) (DPWP) if

$$\omega^{per}_{j,n,k}(x) = \sum_{l=-\infty}^{\infty} \omega_{j,n,k}(x+l), \tag{3.1}$$

where $j, k \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Periodized wavelet packets unlike non-periodic ones, must be first dialated before periodization as periodization does not commute with dialation.

Next, we give a necessary condition for Daubechies wavelet packets associated with scaling function ω_0 such that $\hat{\omega}_1(0) = 0$. More preciously, we prove the following result.

Proposition 3.1. *Let ω_n, $n \in \mathbb{N}_0$ be Daubechies wavelet packets associated with scaling function ω_0 and let $\hat{\omega}_1(0) = 0$. Then*

$$\hat{\omega}_{4q+1}(4\pi r) = 0, \ \textit{for } r \in \mathbb{Z}, \ q \in \mathbb{N}. \tag{3.2}$$

Proof. Taking Fourier transform of $\omega_{2n}(x)$ and using (2.8), we compute

$$\begin{aligned} \hat{\omega}_{2n}(\eta) &= \int_{\mathbb{R}} \omega_{2n}(x)\ e^{-i\eta x}\ dx \\ &= \frac{1}{2}\sum_{p=0}^{2g-1} u_p\ e^{-\frac{i\eta p}{2}} \int_{\mathbb{R}} \omega_n(x)\ e^{-\frac{i\eta x}{2}}\ dx = F(\frac{\eta}{2})\ \hat{\omega}(\frac{\eta}{2}), \end{aligned} \tag{3.3}$$

where

$$F(\eta) = \frac{1}{2}\sum_{p=0}^{2g-1} u_p\ e^{-i\eta p}. \tag{3.4}$$

Applying (3.3) k-times, we have

$$\hat{\omega}_{2n}(\eta) = \prod_{j=1}^{k} F(\frac{\eta}{2^j})\ \hat{\omega}_{\frac{n}{2^k}}(\frac{\eta}{2^k}). \tag{3.5}$$

Since $\hat{\omega}_0(0) = 1$, we have

$$\sum_{p=0}^{2g-1} u_p = 2. \tag{3.6}$$

Using (3.4) and (3.6), we obtain $-1 \leqslant F(\eta) \leqslant 1$ and so the product converges as $k \to \infty$. This yields

$$\hat{\omega}_{2n}(\eta) = \prod_{j=1}^{\infty} F(\frac{\eta}{2^j})\ \hat{\omega}_0(0), \eta \in \mathbb{R}.$$

This further gives

$$\hat{\omega}_{2n}(2\pi r) = \prod_{j=1}^{\infty} F(\frac{2\pi r}{2^j}),\ \ r \in \mathbb{Z}.$$

If $r = 0$, then using (3.4) and (3.6), we have $\hat{\omega}_{2n}(0) = 1$. Let $r \in \mathbb{Z} \setminus \{0\}$ be such that $r = 2^s M$, where $s \in \mathbb{N}_0$ and M is odd integer. Then

$$\begin{aligned}\hat{\omega}_{2n}(2\pi r) &= \prod_{j=1}^{\infty} F(\frac{2^{s+1} M\pi}{2^j})\\ &= F(2^s M\pi)\ F(2^{s-1} M\pi) \ldots F(M\pi) \ldots = 0.\end{aligned}$$

This gives $\hat{\omega}_{2n}(2\pi r) = \delta_{0,r},\ r \in \mathbb{Z}$. Using Proposition 2.1, we get $\sum\limits_{p=0}^{2g-1} u_p\ v_l = 0$. Note that v_p can be expressed in terms of u_p as

$$v_p = (-1)^p\ u_{2g-1-p},\ p = 0, 1, ..., 2g-1. \tag{3.7}$$

Using (2.9), we obtain $\hat{\omega}_{2n+1}(\eta) = G(\frac{\eta}{2})\ \hat{\omega}(\frac{\eta}{2})$, where

$$G(\eta) = \frac{1}{2} \sum_{p=0}^{2g-1} v_p\ e^{-i\eta p},\ \eta \in \mathbb{R}. \tag{3.8}$$

Also, using (3.7) in (3.8), we evaluate

$$\begin{aligned}G(\eta) &= \frac{1}{2} \sum_{p=0}^{2g-1} (-1)^p\ u_{2g-1-p}\ e^{-i\eta p}\\ &= \frac{1}{2} e^{-i(2g-l)(\eta+\pi)} \sum_{q=0}^{2g-1} u_q\ e^{iq(\eta+\pi)} = e^{-i(2g-1)(\eta+\pi)}\ \overline{F(\eta+\pi)}.\end{aligned}$$

This gives

$$\hat{\omega}_{2n+1}(\eta) = e^{-i(2g-1)(\frac{\eta}{2}+\pi)}\ \overline{F(\frac{\eta+\pi}{2})}\ \hat{\omega}_n(\frac{\eta}{2}). \tag{3.9}$$

Taking $n = 2s,\ s \in \mathbb{N}_0$ and $\eta = 4\pi r$, we have

$$\begin{aligned}\hat{\omega}_{4s+1}(4\pi r) &= e^{-i(2g-1)(2\pi r+\pi)}\ \overline{F(2\pi r+\pi)}\ \hat{\omega}_{2s}(2\pi r)\\ &= \begin{cases} 0, & \text{if } r \neq 0;\\ e^{-i(2g-1)\pi}\ \overline{F(\pi)}, & \text{if } r = 0.\end{cases}\end{aligned} \tag{3.10}$$

Since $\hat{\omega}_1(0) = 0$, it follows that

$$0 = \sum_{p=0}^{2g-1} v_p \int_{\mathbb{R}} \omega_0(2x - p)\ dx = \frac{1}{2} \sum_{l=0}^{2g-1} (-1)^l\ u_l. \tag{3.11}$$

Using (3.4) and (3.11) in (3.10), we finally get $\hat{\omega}_{4s+1}(4\pi r) = 0$. □

In the following result, we give some properties of the Daubechies periodized wavelet packets.

Theorem 3.1. *Let $\omega_n \in L^2(\mathbb{R})$ $(n \in \mathbb{N})$ be wavelet packets. Then*

(i) *for any $j,\ k \in \mathbb{Z},\ \omega^{per}_{j,n,k}$ is 1-periodic.*

(ii) *for $j \leqslant -1,\ k \in \mathbb{Z},\ s \in \mathbb{N}_0$ and $x \in \mathbb{R},\ \omega^{per}_{j,4s+1,k}(x) = 0$, but for $j = 0,\ \omega^{per}_{j,4s+1,k}(x)$ is neither zero nor any constant for odd choice of k.*

(iii) *for $j > 0,\ \omega^{per}_{j,n,k}$ is periodic in the shift parameter with period 2^j.*

(iv) *for $j > j' \geqslant \lceil log_2(2g-1) \rceil$ and $x \in [0,1]$ with ω_n having compact support $[0, 2g-1]$,*

$$\omega^{per}_{j,n,k}(x) = \begin{cases} \omega_{j,n,k}(x), & \text{if } x \in I_{j,k} \cap [0,1]; \\ \omega_{j,n,k}(x+1), & \text{if } x \in [0,1] \text{ and } x \notin I_{j,k}. \end{cases} \tag{3.12}$$

Proof. (i) Let $j, k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$\omega^{per}_{j,n,k}(x+1) = \sum_{l=-\infty}^{\infty} \omega_{j,n,k}(x+l+1) = \omega^{per}_{j,n,k}(x).$$

Thus $\omega^{per}_{j,n,k}(x)$ is 1-periodic.
(ii) Let $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$\omega^{per}_{j,n,k}(x+1) = \sum_{l=-\infty}^{\infty} \omega_{j,n,k}(x+l) = \omega^{per}_{j,n,0}(x).$$

Since $\omega^{per}_{j,n,0}(x)$ is a 1-periodic function, it can be expanded using Fourier series expansion, i.e.,

$$\omega^{per}_{j,n,0}(x) = \sum_{r=-\infty}^{\infty} a_r\ e^{2\pi i r x},\ x \in \mathbb{R}, \tag{3.13}$$

where the Fourier coefficients a_r are given by

$$\begin{aligned} a_r &= \int_0^1 \omega^{per}_{j,n,0}(x)\ e^{-2\pi i r x}\ dx \\ &= \int_0^1 \sum_{p=-\infty}^{\infty} \omega_{j,n,0}(x+p)\ e^{-2\pi i r x}\ dx = 2^{-\frac{j}{2}}\ \hat{\omega}_n(2\pi r 2^{-j}),\ r \in \mathbb{Z}. \end{aligned} \tag{3.14}$$

This yields

$$\omega^{per}_{j,n,k}(x) = \sum_{r=-\infty}^{\infty} 2^{\frac{-j}{2}}\ \hat{\omega}_n(2\pi r 2^{-j})\ e^{2\pi i r x},\ x \in \mathbb{R},\ r \in \mathbb{Z}. \tag{3.15}$$

Using (3.10) in (3.15), we have

$$\omega^{per}_{j,4s+1,k}(x) = 0,\ j \leqslant -1,\ k \in \mathbb{Z},\ s \in \mathbb{N}_0 \text{ and } x \in \mathbb{R}.$$

If $j = 0$ in (3.14), then using (3.9), we have $\hat{\omega}_{4s+1}(2\pi k) \neq 0$ for odd k, so $\omega^{per}_{0,4s+1,k}(x)$ is neither 0 nor any constant for such value of k.

(iii) Let $j > 0,\ m \in \mathbb{Z}$ and $0 \leqslant k \leqslant 2^j - 1$. Then

$$\begin{aligned}\omega^{per}_{j,n,k+2^j m}(x) &= \sum_{l=-\infty}^{\infty} \omega_{j,n,k+2^j m}(x+l) \\ &= 2^{\frac{j}{2}} \sum_{l=-\infty}^{\infty} \omega_n(2^j(x+l-m)-k) = \omega^{per}_{j,n,k}(x),\ x \in \mathbb{R}.\end{aligned}$$

(iv) Let $2^j \geqslant 2g-1$. Then, using (3.1), we get

$$\omega^{per}_{j,n,k}(x) = 2^{\frac{j}{2}} \sum_{l=-\infty}^{\infty} \omega_n(2^j x + 2^j l - k)$$

$$= 2^{\frac{j}{2}} \sum_{l=-\infty}^{\infty} \omega_n(2^j x - (k - 2^j l)) = \sum_{l=-\infty}^{\infty} \omega_{j,n,k-2^j l}(x). \tag{3.16}$$

Since ω_n is compactly supported, it follows that the supports of the terms in the above sum do not overlap for sufficiently large value of 2^j. Choose smallest $j' \in \mathbb{Z}$ such that $2^{j'} \geqslant 2g-1$. Now, $supp(\omega_{j,n,k}) = I_{j,k}$, where $I_{j,k} = \left[\frac{k}{2^j}, \frac{k+2g-1}{2^j}\right]$ and for $j > j'$ the width of $I_{j,k} \leqslant 1$ and thus, (3.16) implies that for $x \in [0,1]$, periodized wavelet packets can be expressed as

$$\omega^{per}_{j,n,k}(x) = \begin{cases} \omega_{j,n,k}(x), & \text{if } x \in I_{j,k} \cap [0,1] \\ \omega_{j,n,k}(x+1), & \text{if } x \in [0,1], \text{ and } x \notin I_{j,k}. \end{cases}$$

□

The following result shows that DPWP forms an orthonormal system for $L^2([0,1])$.

Theorem 3.2. *The collection of Daubechies periodized wavelet packets $\{\omega^{per}_{0,n,k}(x)\}_{n\in\mathbb{N}_0,k\in\mathbb{Z}}$ is an orthonormal system for $L^2([0,1])$.*

Proof. The details of the proof can be seen in ([32], Section 9.3).

Corollary 3.1. *For each fixed $j \in \mathbb{Z}$, the collection of Daubechies periodized wavelet packets $\{\omega^{per}_{j,n,k}\}_{n\in\mathbb{N}_0,k\in\mathbb{Z}}$ forms an orthonormal system for $L^2([0,1])$.*

Proof. Proof follows from the Theorem 3.2.

4. Approximation properties of V^{per}_J

The domain of periodized wavelet packets when restricted to $[0,1]$, generate an MRA of $L^2([0,1])$ analogously to that of $L^2(\mathbb{R})$. The significant subspaces involved are defined as

$$V_j^{per} = span\{\omega_{j,0,k}^{per}(x) : x \in [0,1]\}_{k=0}^{2^j-1},$$
$$U_{j,n}^{per} = span\{\omega_{j,n,k}^{per}(x) : x \in [0,1]\}_{n\in\mathbb{N},\ k=0,1,...,2^j-1}.$$

Note that the V_j^{per} are nested similarly as in the case of non-periodic MRA,

$$V_0^{per} \subset V_1^{per} \subset V_2^{per} \subset \cdots \subset L^2([0,1]).$$

So, $\overline{\bigcup_{j=0}^{\infty} V_j^{per}} = L^2([0,1])$. Further, the orthogonality relationship gives

$$L^2([0,1]) = V_{J_1}^{per} \oplus \bigoplus_{j=J_1}^{\infty} \bigoplus_{n=2^j}^{2^{j+1}-1} U_{0,n}^{per}, \text{ for some } J_1 > 0. \tag{4.1}$$

Let $f \in V_J^{per}$ and let $J_1 : 1 \leqslant J_1 \leqslant J$. Then, the periodized wavelet packet expansion is

$$f(x) = \sum_{k=0}^{2^{J_1}-1} c_{J_1,k}\ \omega_{J_1,0,k}^{per}(x) + \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} d_{0,k}^n\ \omega_{0,n,k}^{per}(x),\ x \in [0,1], \tag{4.2}$$

where the coefficients $c_{j,k}$ and $d_{0,k}^n$ are respectively given by

$$c_{j,k} = \int_0^1 f(x)\ \omega_{j,0,k}^{per}(x)\ dx \text{ and } d_{0,k}^n = \int_0^1 f(x)\ \omega_{0,n,k}^{per}\ dx.$$

Let $\omega_n, n \in \mathbb{N}_0$ be wavelet packets. Then the orthogonal projections of $L^2([0,1])$ on V_j^{per} and $U_{0,n}^{per}$ are respectively defined as

$$(P_{V_j^{per}} f)(x) = \sum_{k=-\infty}^{\infty} c_{j,k}\ \omega_{j,0,k}^{per}(x), \tag{4.3}$$

$$(P_{U_{0,n}^{per}} f)(x) = \sum_{k=-\infty}^{\infty} d_{0,k}^n\ \omega_{0,n,k}^{per}(x), \tag{4.4}$$

where

$$c_{j,k} = \int_0^1 f(x)\ \omega_{j,0,k}^{per}(x)\ dx, \qquad d_{0,k}^n = \int_0^1 f(x)\ \omega_{0,n,k}^{per}(x)\ dx$$

and

$$P_{V_J^{per}} f = P_{V_{J_1}^{per}} f + \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} P_{U_{0,n}^{per}} f,\ J \in \mathbb{Z}.$$

For $f \in L^2([0,1]) \cap C^g(\mathbb{R})$ $(g > 1)$ and $x \in [0,1]$, the approximation error is given by $E_J^{per}(x) = f(x) - (P_{V_J^{per}} f)(x)$.

Now, we give the following result related to the approximation error.

Theorem 4.1. *Let $f \in L^2([0,1]) \cap C^g(\mathbb{R})$ $(g > 1)$ be a function and $\omega_n,\ n \in \mathbb{N}_0$ be DPWP such that*

(i) $|\omega_n(t)| = O(2^{-gj})$ *for* $n = 2^j, ..., (2^{j+1} - 1)$, *where* $j \geqslant 0$,

(ii) $\int_{\mathbb{R}} x^p \ \omega_n(x) \ dx = 0,$ *for* $0 \leqslant p \leqslant g-1$.

Let $J \in \mathbb{Z} : J \geqslant J_1 > 0$, *where* $2^{J_1} \geqslant 2g-1$. *Then*

$$\|E_J^{per}(x)\|_\infty = O(2^{-J(g-1)}).$$

Proof. The periodic wavelet packet expansion for $P_{V_J}^{per}$ is

$$(P_{V_J^{per}} f)(x) = \sum_{k=0}^{2^{J_1}-1} c_{J_1,k} \ \omega_{J_1,0,k}^{per}(x) + \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} d_{0,k}^n \ \omega_{0,n,k}^{per}(x). \tag{4.5}$$

Taking $J \to \infty$, the periodic wavelet packet expansion for $f \in L([0,1])$ is given by

$$f(x) = \sum_{k=0}^{2^{J_1}-1} c_{J_1,k} \ \omega_{J_1,0,k}^{per}(x) + \sum_{j=J_1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} d_{0,k}^n \ \omega_{0,n,k}^{per}(x). \tag{4.6}$$

The approximation error is given by

$$E_J^{per}(x) = f(x) - (P_{V_J^{per}} f)(x), \ x \in [0,1].$$

Therefore, we get

$$E_J^{per}(x) = \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} d_{0,k}^n \ \omega_{0,n,k}^{per}(x). \tag{4.7}$$

Let $I = [0, 2g-1]$ be the compact support of ω_n. Then, it follows that $\omega_{0,n,k}$ is supported in the interval $I_k = [k, k+2g-1]$ with length $l(I_k) = 2g-1$ and centre $x_k = k + g - \frac{1}{2}$.

Note that

$$d_{0,k}^n = \int_0^1 f(x) \ \overline{\omega_{0,n,k}^{per}(x)} \ dx = \int_{\mathbb{R}} f(x) \ \overline{\omega_{0,n,k}(x)} \ dx. \tag{4.8}$$

Since $f \in C^g(\mathbb{R})$, using Taylor's expansion of f about the point x_k, it follows that

$$\begin{aligned} |d_{0,k}^n| = \Big| \int_{\mathbb{R}} &[f(x_k) + (x - x_k) \ f^{(1)}(x_k) + \cdots \\ &+ \frac{1}{(g-1)!}(x-x_k)^{g-1} \ f^{(g-1)}(x_k) + R_g(x)] \ \omega_{0,n,k}(x) \Big|, \end{aligned}$$

where $R_g(x) = \frac{1}{g!}(x-x_k)^g \ f^{(g)}(\eta)$ for some number η between x_k and x. If $x \in I_k$, then, we have

$$|R_g(x)| \leqslant \frac{1}{g!}\big(g - \frac{1}{2}\big)^g \ \max_{x \in I_k} |f^{(g)}(x)|.$$

Therefore, we compute

$$
\begin{aligned}
|d^n_{0,k}| &= \Big| \int_{I_k} R_g(x)\ \overline{\omega_{0,n,k}(x)} \Big| \\
&\leqslant \frac{1}{g!}\big(g-\frac{1}{2}\big)^g \max_{x\in I_k} |f^{(g)}(x)| \int_{I_k} |\omega_{0,n,k}(x)|\ dx \\
&\leqslant \frac{2^{\frac{1}{2}}K}{g!}\big(g-\frac{1}{2}\big)^{g+\frac{1}{2}} \max_{x\in I_k} |f^{(g)}(x)| \Big(\int_0^{2g-1} 2^{-2gj}\ dx \Big)^{\frac{1}{2}} \\
&= \frac{2K}{g!}\big(g-\frac{1}{2}\big)^{g+1} \max_{x\in I_k} |f^{(g)}(x)|\ 2^{-gj} = M\ 2^{-gj}, \qquad (4.9)
\end{aligned}
$$

where $M = \frac{2K}{g!}\big(g-\frac{1}{2}\big)^{g+1} \max\limits_{x\in I_k} |f^{(g)}(x)|$. Using (4.9) in (4.7), we obtain

$$
\|E^{per}_J(x)\|_\infty \leqslant \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} M\ 2^{-gj} \max_{x\in I_k} |\omega^{per}_{0,n,k}(x)|.
$$

Define $C_{\omega^{per}_{0,n,k}} = \max\limits_{x\in I_k} |\omega^{per}_{0,n,k}(x)|$. Then, we compute

$$
\begin{aligned}
\|E^{per}_J(x)\|_\infty &\leqslant C_{\omega^{per}_{0,n,k}}\ M \sum_{j=J}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k=0}^{2^j-1} 2^{-gj} \\
&= C_{\omega^{per}_{0,n,k}}\ M \sum_{j=J}^{\infty} 2^{-(g-2)j} = K'\ 2^{-(g-1)j}, \qquad (4.10)
\end{aligned}
$$

where K' is a constant. Thus, we find that with respect to the resolution J, error E^{per}_J shows an exponential decay. Besides, more is the number of vanishing moments, faster will be the decay. □

5. Compression errors

In this section, using hard thresholding technique, we discuss the compression errors.

The information about a signal f is stored in the form of wavelet packet coefficients $\{\langle f, \omega_{j,n,k}\rangle\}_{j,k\in\mathbb{Z}}$ and this knowledge helps us to reconstruct the signal f. Nevertheless, practically it is not possible to store such an infinite sequence of non-zero numbers and thus it is necessary to chose only finite number of such coefficients. This is primarily done by specifying an independent parameter or threshold $\delta > 0$ such that only those coefficients are retained for which $|\langle f, \omega_{j,n,k}\rangle| \geqslant \delta$. Such coefficients are known as significant wavelet packet coefficients, whereas others which do not satisfy the above inequality are quantized to zero and are known as insignificant wavelet packet coefficients. Thus, threshold value δ separates the insignificant wavelet packet coefficients from the significant ones. One may note that the selection of wavelet packets also plays an essential role as we always look for those wavelet packets which correlates well with the signal under consideration or detection. If there is a large amount of signal information present, one can keep large number of wavelet packet coefficients, as compared to lesser number in case of a noisy signal. The above process is known as hard thresholding. The errors which appeared when small wavelet packet coefficients are repudiated are referred to as compression errors.

Let us define a set of significant wavelet packet coeffficients at level j as

$$S_j^\delta = \{k : 0 \leqslant k \leqslant 2^j - 1 \text{ and } |d_{0,k}^n| > \delta \text{ for } n = 2^j, ..., (2^{j+1} - 1)\}.$$

The set of insignificant wavelet packet coefficients are given by $I_j^\delta = S_j^0 \setminus S_j^\delta$. Thus, δ-truncated wavelet packet expansion for f is given by

$$(P_{V_J^{per}} f)^\delta(x) = \sum_{k=0}^{2^{J_1}-1} c_{J_1,k}\ \omega_{J_1,0,k}^{per}(x) + \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k \in S_j^\delta} d_{0,k}^n\ \omega_{0,n,k}^{per}(x).$$

Let $n_S(\delta)$ be the number of all significant wavelet packet coefficients, i.e.,

$$n_S(\delta) = \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \aleph(S_j^\delta) + 2^{J_1},$$

where $\aleph(S_j^\delta)$ denotes the cardinality of S_j^δ. The last term in the above sum is due to the coefficient of scaling function as they contribute the coarse approximation on which the fine structures are built by wavelet packets. Let us suppose that $n = 2^J$ be the dimension of V_J^{per}. Then, define $n_I(\delta) = n - n_S(\delta)$ to be the number of insignificant wavelet packet coefficients in the expansion. Due to this truncation, an error $E_{\delta,J}^{per}$ has been occured and is given by

$$\begin{aligned} E_{\delta,J}^{per}(x) &= (P_{V_J^{per}} f)(x) - (P_{V_J^{per}} f)^\delta(x) \\ &= \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k \in I_j^\delta} d_{0,k}^n\ \omega_{0,n,k}^{per}(x) \end{aligned} \tag{5.1}$$

with $n_I(\delta)$ number of terms. This ensures the inequality

$$\|E_{\delta,J}^{per}(x)\|_2 \leqslant \delta\ (n_I(\delta))^{\frac{1}{2}}. \tag{5.2}$$

Now, if we redefine the set of significant wavelet packet coefficients as

$$S_j^\delta = \{k : 0 \leqslant k \leqslant 2^j - 1 \text{ and } |d_{0,k}^n| > \delta\ 2^{-\frac{j}{2}} \text{ for } n = 2^j, ..., (2^{j+1} - 1)\},$$

then $I_j^\delta = S_j^0 \setminus S_j^\delta$. Therefore the scale j can be employed to transmute the threshold value δ.

Hence using (5.1), we finally obtain

$$\|E^{per}_{\delta,J}(x)\|_\infty = \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k\in I_j^\delta} \max_x(|d^n_{0,k}\ \omega^{per}_{0,n,k}(x)|)$$

$$= C'_{\omega^{per}_{0,n,k}} \sum_{j=J_1}^{J-1} \sum_{n=2^j}^{2^{j+1}-1} \sum_{k\in I_j^\delta} |d^n_{0,k}| = C'_{\omega^{per}_{0,n,k}}\ \delta\ n_I(\delta), \tag{5.3}$$

where $C'_{\omega^{per}_{0,n,k}} = 2^{-\frac{j}{2}}\ C_{\omega^{per}_{0,n,k}}$.

Conclusion

Restrepo et al. [35] studied periodized wavelets by restricting the wavelets on the bounded subsets of $\mathbb{R}$. In the present article, we amalgamated their with that of Daubechies [3, 4] and studied Daubechies periodized wavelet packets and using it obtained approximation of periodic functions. Also, thresholding technique is used to study compression errors. On comparing (5.2) and (5.3), we find that the threshold is scaled in (5.3) which decreases substantially on the increase in the scale resulting in consequence of which the number of wavelet packet coefficients increases at the finer scales. Thus, n_I will be lesser in the latter case. Finally, we have also observed that using wavelet packets instead of just Daubechies wavelet bases, one can expect reduction in the compression errors.

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ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF n-TH ORDER EMDEN-FOWLER TYPE DIFFERENCE EQUATIONS WITH ADVANCED ARGUMENT

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Abstract. We study oscillatory properties of solutions of the Emden-Fowler type difference equation $\Delta^{(n)}u(k) + p(k)\,\big|u(\sigma(k))\big|^{\lambda}\operatorname{sign} u(\sigma(k)) = 0$, where $n \geq 2$, $0 < \lambda < 1$, $p : \mathbb{N} \to \mathbb{R}_+$, $\sigma : \mathbb{N} \to \mathbb{N}$ and $\sigma(k) \geq k+1$ for $k \in \mathbb{N}$. Sufficient conditions of new type for oscillation of solutions of the above equation are established. Analogous results for linear ordinary and nonlinear functional differential equations see in [1–8].

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Keywords: difference equation; proper solution, Property **A**.

1. Introduction

This work is dedicated to the study of oscillatory properties of the difference equation

$$\Delta^{(n)}u(k) + p(k)\,\big|u(\sigma(k))\big|^{\lambda}\operatorname{sign} u(\sigma(k)) = 0, \tag{1.1}$$

where $n \geq 2$, $p : \mathbb{N} \to \mathbb{R}_+$, $\sigma : \mathbb{N} \to \mathbb{N}$ and

$$0 < \lambda < 1, \quad \sigma(k) \geq k+1 \quad \text{for} \quad k \in \mathbb{N}. \tag{1.2}$$

Here $\Delta^{(1)}u(k) = u(k+1) - u(k)$, $\Delta^{(i)} = \Delta^{(1)} \circ \Delta^{(i-1)}$ $(i = 2, \dots, n)$. It will always be assumed that the condition

$$p(k) \geq 0 \quad \text{for} \quad k \in \mathbb{N} \tag{1.3}$$

is fulfilled. The following notation will be used throughout the work:

Let $k_0 \in \mathbb{N}$. By $\mathbb{N}^+_{k_0}$ $(\mathbb{N}^-_{k_0})$ we denote the set of natural number $\mathbb{N}^+_{k_0} = \{k_0, k_0+1, \dots\}$ $(\mathbb{N}^-_{k_0} = \{1, 2, \dots, k_0\})$.

Definition 1.1. Let $k_0 \in \mathbb{N}$. We will call a function $u : \mathbb{N}^+_{k_0} \to \mathbb{R}$ a proper solution of the equation (1.1), if it satisfies (1.1) on $\mathbb{N}^+_{k_0}$ and

$$\sup\left\{\big|u(i)\big| : i \in \mathbb{N}^+_k\right\} > 0 \ \text{ for any } \ k \in \mathbb{N}^+_{k_0}.$$

Definition 1.2. We say that a proper solution $u : \mathbb{N}^+_{k_0} \to \mathbb{R}$ of equation (1.1) is oscillatory, if for any $k \in \mathbb{N}^+_{k_0}$ there exist $k_1; k_2 \in \mathbb{N}^+_k$ such that $u(k_1)\,u(k_2) \leq 0$. Otherwise the solution is called nonoscillatory.

Definition 1.3. We say that equation (1.1) has Property **A** if any its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$|\Delta^{(i)}u(k)| \downarrow 0 \quad \text{as} \quad k \uparrow +\infty, \quad k \in \mathbb{N} \quad (i = 0, \dots, n-1), \quad \text{when } n \text{ is odd.} \tag{1.4}$$

Some results analogous to those of the paper are given without proofs in [9–11]. The problem of establishing sufficient conditions for the oscillation of all solutions to the second order linear and nonlinear difference equations see in [12–16].

2. On some classes of nonoscillatory discrete functions

Lemma 2.1. *Let $n \geq 2$, $k_0 \in \mathbb{N}$, $u : \mathbb{N}^+_{k_0} \to \mathbb{R}$ and $u(k) > 0$, $\Delta^{(n)}u(k) \leq 0$ for $k \in \mathbb{N}^+_{k_0}$, $\Delta^{(n)}u(k) \not\equiv 0$ for any $s \in \mathbb{N}^+_{k_0}$ and $k \in \mathbb{N}^+_s$. Then there exist $k_1 \in \mathbb{N}^+_{k_0}$ and $\ell \in \{0, \dots, n\}$ such that $\ell + n$ is odd and*

$$\begin{gathered} \Delta^{(i)}u(k) > 0 \quad \text{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = 0, \dots, \ell), \\ (-1)^{i+\ell}\Delta^{(i)}u(k) > 0 \quad \text{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = \ell, \dots, n-1), \\ \Delta^{(n)}u(k) \leq 0 \quad \text{for} \quad k \in \mathbb{N}^+_{k_1}. \end{gathered} \tag{2.1}$$

Proof. The Lemma follows immediately from the fact that, if $u(k) > 0$ and $\Delta^{(2)}u(k) \leq 0$ for $k \in \mathbb{N}^+_{k_0}$, then there exist $k_1 \in \mathbb{N}^+_{k_0}$, such that $\Delta^{(1)}u(k) > 0$ for $k \in \mathbb{N}_{k_1}$. $\square$

Remark 2.1. It is obvious that if $u; v : \mathbb{N} \to \mathbb{R}$ and $\Delta^{(i)}u(k_0) = \Delta^{(i)}v(k_0)$ $(i = 0, \dots, m-1)$ and $\Delta^{(m)}u(k) = \Delta^{(m)}v(k)$ for $k \in \mathbb{N}^+_{k_0}$ (for $k \in \mathbb{N}^-_{k_0}$). Then $u(k) = v(k)$ for $k \in \mathbb{N}^+_{k_0}$ (for $k \in \mathbb{N}^-_{k_0}$).

Lemma 2.2. *Let $u : \mathbb{N} \to \mathbb{R}$, $m; s \in \mathbb{N}$. Then*

$$\begin{aligned} \Delta^{(i)}u(k) = \sum_{j=i}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k - s - r + 1) + \frac{1}{(m-i-1)!} \\ \times \sum_{j=s}^{k} \prod_{r=1}^{m-i-1} (k - j - r + 1)\Delta^{(m)}u(j-1), \quad i = 0, \dots, m-1, \quad \text{for} \quad k \in \mathbb{N}^+_s, \end{aligned} \tag{2.2}$$

where

$$\Delta^{(m)}u(s-1) = 0, \quad \prod_{r=1}^{0}(k - s - r + 1) = 1, \tag{2.3}$$

and

$$\begin{aligned} \Delta^{(i)}u(k) = \sum_{j=i}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k - s - r + 1) - \frac{1}{(m-i-1)!} \\ \times \sum_{j=k}^{s} \prod_{r=1}^{m-i-1} (k - j - r + 1)\Delta^{(m)}u(j), \quad i = 0, \dots, m-1 \quad \text{for} \quad k \in \mathbb{N}^-_s, \end{aligned} \tag{2.4}$$

where

$$\Delta^{(m)}u(s)=0,\quad \prod_{r=1}^{0}(k-s-r+1)=1. \tag{2.5}$$

Proof. Denote

$$u_1(k)=\Delta^{(i)}u(k), \tag{2.6}$$

$$u_2(k)=\sum_{j=i}^{m-1}\frac{\Delta^{(j)}u(s)}{(j-i)!}\prod_{r=1}^{j-i}(k-s-r+1)$$
$$+\frac{1}{(m-i-1)!}\sum_{j=s}^{k}\prod_{r=1}^{m-i-1}(k-j-r+1)\Delta^{(m)}u(j-1),\quad k\in\mathbb{N}_s^+. \tag{2.7}$$

Since

$$\Delta^{(1)}\prod_{r=1}^{j-i}(k-s-r+1)=\prod_{r=1}^{j-i-i}(k+2-r-s)-\prod_{r=1}^{j-i}(k+1-r-s)$$
$$=\prod_{r=0}^{j-i-1}(k+1-r-s)-\prod_{r=1}^{j-i}(k+1-r-s)=(j-i)\prod_{r=1}^{j-i-1}(k+1-r-s),$$

according to (2.3), (2.6) and (2.7) we get $\Delta^{(j)}u_1(s)=\Delta^{(j)}u_2(s)$ $(j=0,\dots,m-i-1)$ and $\Delta^{(m-i)}u_1(k)=\Delta^{(m-i)}u_2(k)$ for $k\in\mathbb{N}_s^+$. Therefore, the conditions of Remark 2.1 are fulfilled, which proves that the equality (2.2) is valid.

By (2.5), similarly we can prove that the equality (2.4) is valid, which proves the lemma. □

Lemma 2.3. *Let $u:\mathbb{N}\to\mathbb{R}$, $m;s\in\mathbb{N}$. Then the equality holds*

$$\sum_{i=s}^{k} i^{m-j-1}\Delta^{(m)}u(i)=\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1}$$
$$-\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(s+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1}\quad \textit{for}\quad k\in\mathbb{N}_s^+, \tag{2.8}$$

where

$$\Delta^{(m)}u(s)=0 \tag{2.9}$$

and

$$-\sum_{i=k}^{s}(i+1)^{m-j-1}\Delta^{(m)}u(i+1)$$
$$=\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1}$$
$$-\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(s+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1}\quad \textit{for}\quad k\in\mathbb{N}_s^-, \tag{2.10}$$

where

$$\Delta^{(m)}u(s+1)=0. \tag{2.11}$$

Proof. Let $u,v:\mathbb{N}\to\mathbb{R}$, then $\Delta^{(1)}\big[u(k)\,v(k)\big]=v(k+1)\Delta^{(1)}u(k)+u(k)\Delta^{(1)}v(k)$. Therefore

$$\begin{aligned}&\Delta^{(1)}\Big(\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1}\Big)\\&=\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i+1)}u(k+1)\Delta^{(m-i-1)}(k+i+2-m)^{m-j-1}\\&+\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i)}(k+i+1-m)^{m-j-1}.\end{aligned}$$

Since $\Delta^{(m-j)}(k+i+1-m)^{m-j-1}=0$, then

$$\Delta^{(1)}\Big(\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1}\Big)=(k+1)^{m-j-1}\Delta^{(m)}u(k+1).$$

By (2.9), ((2.11)) the equality (2.8) (the equality (2.10)) holds. $\square$

Lemma 2.4. *Let $u:\mathbb{N}\to\mathbb{R}$, $k_0;n\in\mathbb{N}$ and*

$$(-1)^i\Delta^{(i)}u(k)>0\ \ (i=0,\dots,n-1),\ \ (-1)^n\Delta^{(n)}u(k)\geq 0\ \ \text{for}\ \ k\in\mathbb{N}^+_{k_0}. \tag{2.12}$$

Then

$$\sum_{k=1}^{+\infty}k^{n-1}\big|\Delta^{(n)}u(k)\big|<+\infty, \tag{2.13}$$

$$\big|\Delta^{(i)}u(k)\big|\geq\frac{1}{(n-i-1)!}\sum_{j=k}^{+\infty}\prod_{r=1}^{n-i-1}(j-k+r-1)\big|\Delta^{(n)}u(j)\big| \tag{2.14}$$
$$\text{for}\ \ k\in\mathbb{N}^+_{k_0},\ \ (i=0,\dots.n-1),$$

$$u(k)\geq u(s)+\sum_{j=1}^{n-1}\frac{\big|\Delta^{(j)}u(s)\big|}{j!}\prod_{r=1}^{j}(j-k+r-1)\ \ \text{for}\ \ s\geq k. \tag{2.15}$$

Proof. Let $k_0\leq k<s$. It can be assumed without loss of generality that $\Delta^{(n)}u(s)=0$. Let $m=n$, according to (2.12) from (2.4) with $s\to+\infty$, we can readily obtain (2.13) and (2.14). As to (2.15), it is immediate consequence of (2.4). $\square$

Lemma 2.5. *Let $u:\mathbb{N}\to\mathbb{R}$ and for some $k_1\in\mathbb{N}$ and $\ell\in\{1,\dots,n-1\}$, (2.1) be fulfilled. Then*

$$\sum_{k=1}^{+\infty}k^{n-\ell-1}\big|\Delta^{(n)}u(k)\big|<+\infty, \tag{2.16}$$

there exists $k_2 \in \mathbb{N}^+_{k_1}$ *such that*

$$\big|\Delta^{(i)}u(k)\big| \geq \frac{1}{(n-i-1)!}\sum_{j=k}^{+\infty}\prod_{r=1}^{n-i-1}(j+r-k-1)\big|\Delta^{(n)}u(j)\big| \quad \text{for } k\in\mathbb{N}^+_{k_2} \ (i=\ell,\dots,n-1), \tag{2.17}$$

$$\Delta^{(i)}u(k) \geq \Delta^{(i)}u(k_2) + \frac{1}{(\ell-i-1)!(n-\ell-1)!}\sum_{s=k_2}^{k-1}\prod_{r=1}^{\ell-i-1}(k+r-(1+s))$$
$$\times\sum_{j=s}^{+\infty}\prod_{r=1}^{n-\ell-1}(j+r-s-1)\big|\Delta^{(n)}u(j)\big|, \quad \text{for } k\in\mathbb{N}^+_{k_2+1} \ (i=0,\dots,\ell-1). \tag{2.18}$$

If in addition

$$\sum_{k=1}^{+\infty}k^{n-\ell}\big|\Delta^{(n)}u(k)\big| = +\infty, \tag{2.19}$$

then

$$\frac{u(k)}{\prod\limits_{i=0}^{\ell-1}(k-i)}\downarrow, \qquad \frac{u(k)}{\prod\limits_{i=1}^{\ell-1}(k-i)}\uparrow, \tag{2.20}$$

for large k

$$u(k) \geq \frac{1+o(1)}{\ell!}\,k^{\ell-1}\Delta^{(\ell-1)}u(k) \tag{2.21}$$

and

$$\Delta^{(\ell-1)}u(k) \geq \frac{k}{(n-\ell-1)!}\sum_{i=k}^{+\infty}i^{n-\ell-1}\big|\Delta^{(n)}u(i)\big|$$
$$+\frac{1}{(n-\ell-1)!}\sum_{i=k_2}^{k}i^{n-\ell}\big|\Delta^{(n)}u(i)\big| \quad \text{for } k\in\mathbb{N}^+_{k_2}. \tag{2.22}$$

Proof. Let $s;k \in N^+_{k_2}$ and $s<k$. Assumed that (2.9) be fulfilled. By virtue of (2.1), from the equality (2.8) with $j=\ell$ and $m=n$ we have

$$\sum_{i=s}^{k}(-1)^{n+\ell}i^{n-\ell-1}\Delta^{(n)}u(i) = \sum_{i=\ell}^{n-1}(-1)^{\ell+i}\Delta^{(i)}u(s+1)\Delta^{(n-i-1)}(s+i+1-n)^{n-\ell-1}$$
$$-\sum_{i=\ell}^{n-1}(-1)^{\ell+i}\Delta^{(i)}u(k+1)\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell-1}.$$

Therefore

$$\sum_{i=s}^{k}i^{n-\ell-1}\big|\Delta^{(n)}u(i)\big| \leq \sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(s+1)\big|\Delta^{(n-i-1)}(s+i+1-n)^{n-\ell-1} \quad \text{for } k\in\mathbb{N}^+_s.$$

The last inequality with $k \to +\infty$ we obtain (2.16). The equality (2.10) also implies the inequality

$$\sum_{i=\ell}^{n-1} \big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell-1}$$

$$\geq \sum_{i=k}^{+\infty} i^{n-\ell-1}\big|\Delta^{(n)}u(i+1)\big| \ \text{ for } \ k \in \mathbb{N}_{k_2}^{+}. \tag{2.23}$$

On account of (2.1) and (2.16), from (2.4) we obtain (2.17).

Analogously, equality (2.2) with $s = k_2$ and $m = \ell$, gives

$$\Delta^{(i)}u(k) \geq \Delta^{(i)}u(k_2) + \frac{1}{(\ell-i-1)!}\sum_{j=k_2}^{k}\prod_{r=1}^{\ell-i-1}(k-j+r-1)\Delta^{(\ell)}u(j-1)$$
$$(i = 0, \dots, \ell-1) \ \text{ for } \ k \in \mathbb{N}_{k_2}^{+}.$$

Hence, by (2.17) we obtain (2.18). Using (2.1), from (2.8) with $j = \ell - 1$ and $m = n$, for $s = k_2$ we have

$$\Delta^{(\ell-1)}u(k) = \frac{1}{(n-\ell)!}\sum_{i=k_2}^{k} i^{n-\ell}\big|\Delta^{(n)}u(i)\big|$$
$$+ \frac{1}{(n-\ell)!}\sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell}$$
$$+ \frac{1}{(n-\ell)!}\sum_{i=\ell-1}^{n-1}(-1)^{n+i-1}\Delta^{(i)}u(k_2+1)\Delta^{(n-i-1)}(k_2+i+1-n)^{n-\ell}.$$

Therefore, according to (2.19) there exist $k^* > k_2$ such that

$$\Delta^{(\ell-1)}u(k+1) \geq \frac{1}{(n-\ell)!}\sum_{i=k_*}^{k} i^{n-\ell}\big|\Delta^{(n)}u(i)\big|$$
$$+ \frac{1}{(n-\ell)!}\sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell} \ \text{ for } \ k \in \mathbb{N}_{k^*}^{+}.$$

From the last inequality by (2.19) we have

$$\Delta^{(\ell-1)}u(k+1) - (k+\ell+1-n)\Delta^{(\ell)}u(k+1) \to +\infty \ \text{ for } \ k \to +\infty \tag{2.24}$$

and by (2.23) the inequality (2.22) holds.

Let $k_0 \in \mathbb{N}$ and for any $k \in \mathbb{N}_{k_0}^{+}$ and $i \in \{1, \dots, \ell\}$ put

$$\rho_i(k) = i\Delta^{(\ell-i)}u(k) - (k+1-i)\Delta^{(\ell-i+1)}u(k), \tag{2.25}$$

$$\gamma_i(k) = (k-i)\Delta^{(\ell-i+1)}u(k) - (i-1)\Delta^{(\ell-i)}u(k). \tag{2.26}$$

Applying (2.24) and L'opital rule, we have

$$\lim_{k\to+\infty}\frac{\Delta^{(\ell-i)}u(k)}{\prod_{j=1}^{i-1}(k-j)} = +\infty \ \ (i = 1, \dots, \ell). \tag{2.27}$$

(Here it is meant that $\prod_{j=1}^{0}(k-j)=1$). Since

$$\Delta^{(1)}\left(\frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{j=1}^{i-1}(k-j-1)}\right)=\frac{\gamma_i(k)}{\prod\limits_{j=0}^{i-1}(k-j-1)}\,,$$

by (2.27) there exist $k_\ell > \cdots > k_1 > k_0$ such that $\gamma_i(k_i) > 0$ $(i=1,\dots,\ell)$. Therefore, by (2.24) $\rho_1(k)\to+\infty$ as $k\to+\infty$, $\Delta^{(1)}\rho_{i+1}(k)=\rho_i(k)$, $\Delta^{(1)}\gamma_{i+1}(k)=\gamma_i(k)$ and $\gamma_1(k)=(k-1)\Delta^{(\ell)}u(k)>0$ for $k\in\mathbb{N}^+_{k_0}$ $(i=1,\dots,\ell-1)$, we find that $\rho_i(k)\to+\infty$ as $k\to+\infty$, and $\gamma_i(k)>0$ for $k\in\mathbb{N}^+_{k_i}$ $(i=1,\dots,\ell)$. These fact along with (2.24)–(2.27) prove (2.20).

On the other hand, since $\rho_i(k)\to+\infty$, by (2.25) for large k, $i\Delta^{(\ell-i)}u(k) > (k+1-i)\Delta^{(\ell-i+1)}u(k)$ $(i=1,\dots,\ell)$, which implies (2.21). □

3. Necessary condition for existence of solutions of type 2.1

The results of this section play an important role in establishing sufficient conditions for equation (1.1) to have Property **A**.

Let $k_0\in N$ and $\ell=\{1,\dots,n-1\}$. By U_{ℓ,k_0} we denote the set of all solutions of equation (1.1) satisfying the condition (2.1).

Theorem 3.1. *Let condition* (1.2), (1.3) *be fulfilled,* $\ell\in\{1,\dots,n-1\}$ *with* $\ell+n$ *odd and*

$$\sum_{k=1}^{+\infty}k^{n-\ell}\big(\sigma(k)\big)^{\lambda(\ell-1)}p(k)=+\infty. \tag{3.1}$$

If, moreover, for some $k_0\in\mathbb{N}$, $U_{\ell,k_0}\neq\varnothing$, *then for any* $\delta\in[0,\lambda]$ *and* $i\in\mathbb{N}$ *we have*

$$\sum_{k=1}^{+\infty}k^{n-\ell-1+\lambda-\delta}\big(\sigma(k)\big)^{\lambda(\ell-1)}\big(\rho_{i,\ell}(\sigma(k))\big)^{\delta}p(k)<+\infty, \tag{3.2}$$

where

$$\rho_{1,\ell}(k)=\left(\frac{1-\lambda}{\ell!(n-\ell)!}\sum_{i=1}^{k-1}\sum_{j=i}^{+\infty}j^{n-\ell-1}\big(\sigma(j)\big)^{\lambda(\ell-1)}p(j)\right)^{\frac{1}{1-\lambda}}, \tag{3.3}$$

(3.4)

$$\rho_{s,\ell}(k)=\frac{1}{\ell!(n-\ell)!}\sum_{i=1}^{k-1}\sum_{j=i}^{+\infty}j^{n-\ell-1}\big(\sigma(j)\big)^{\lambda(\ell-1)}p(j)\big(\rho_{s-1,\ell}\big(\sigma(j)\big)\big)^{\lambda}\ (s=2,3,\dots).$$

Proof. Let $k_0\in\mathbb{N}$ and $U_{\ell,k_0}\neq\varnothing$. By definition of the set U_{ℓ,k_0}, equation (1.1) has a proper solution $u\in U_{\ell,k_0}$ satisfying the condition (2.1). By (2.1) and (3.1) it is clear that the condition (2.19) holds. Thus by Lemma 2.5, (2.20)–(2.22) are fulfilled and by (1.1) and (2.21), (2.22) we have

$$\Delta^{(\ell-1)}u(k) \geq \frac{k}{\ell!(n-\ell)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}\sigma^{\lambda(\ell-1)}(i)\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda}p(i)$$

$$+\frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k} i^{n-\ell}\sigma^{\lambda(\ell-1)}(i)\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda}p(i) \ \text{ for } \ k \in \mathbb{N}_{k_*}^{+}, \tag{3.5}$$

where k_* it is sufficiently large natural number. By the identity

$$\sum_{i=k_*}^{k} u(i)\Delta^{(1)}v(i) = u(k)v(k+1) - u(k_*-1)v(k_*) - \sum_{i=k_*}^{k} v(i)\Delta^{(1)}u(i-1)$$

we have

$$\begin{aligned}
&\sum_{i=k_*}^{k} i^{n-\ell}\sigma^{\lambda(\ell-1)}(i)\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda}p(i)\\
&\quad= -\sum_{i=k_*}^{k} i\Delta^{(1)}\sum_{s=i}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\\
&\quad= -k\sum_{s=k}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\\
&\quad\quad+(k_*-1)\sum_{s=k_*}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\\
&\quad\quad+\sum_{i=k_*}^{k}\sum_{s=i}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)
\end{aligned}$$

Therefore, from (3.5) we get

$$\Delta^{(\ell-1)}u(k) \geq \frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k}\sum_{s=i}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s) \tag{3.6}$$

for $k \in \mathbb{N}_{k*}^{+}$. Denote

$$x(k) = \frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k-1}\sum_{s=i}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s).$$

Since $\Delta^{(\ell-1)}u(k)$ is nondecreasing and $\sigma(k) \geq k+1$, by (3.6) we have

$$\begin{aligned}
\Delta^{(1)}x(k) &\geq \frac{\big(\Delta^{(\ell-1)}u(k+1)\big)^{\lambda}}{\ell!(n-\ell)!}\sum_{s=k}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)p(s)\\
&\geq \frac{x^{\lambda}(k+1)}{\ell!(n-\ell)!}\sum_{s=k}^{+\infty} s^{n-\ell-1}\sigma^{\lambda(\ell-1)}(s)p(s) \ \text{ for } \ k \in \mathbb{N}_{k*}^{+}.
\end{aligned}$$

Therefore

$$\sum_{j=k_*}^{k-1}\frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)} \geq \frac{1}{\ell!(n-\ell)!}\sum_{j=k_*}^{k-1}\sum_{i=j}^{+\infty} i^{n-\ell-1}\sigma^{\lambda(\ell-1)}(i)p(i). \tag{3.7}$$

Since

$$\sum_{j=k_*}^{k-1} \frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)} = \sum_{i=k_*}^{k-1} x^{-\lambda}(j+1) \int_{x(j)}^{x(j+1)} dt$$

and $x^{-\lambda}(j+1) \le t^{-\lambda}$ when $x(j) \le t \le x(j+1)$, we have

$$\sum_{j=k_*}^{k-1} \frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)} \le \sum_{j=k_*}^{k-1} \int_{x(j)}^{x(j+1)} t^{-\lambda}dt = \int_{x(k_*)}^{x(k)} t^{-\lambda}dt.$$

That's why, from (3.7) we get

$$x(k) \ge \left(\frac{1-\lambda}{\ell!(n-\ell)!} \sum_{j=k_*}^{k-1} \sum_{i=j}^{+\infty} i^{n-\ell-1} \sigma^{\lambda(\ell-1)}(i)p(i) \right)^{\frac{1}{1-\lambda}}. \tag{3.8}$$

I.e.

$$\Delta^{(\ell-1)}u(k) \ge \rho_{1,\ell,k_*}(k) \ \text{ for } \ k \in \mathbb{N}_{k_*}^+, \tag{3.9}$$

where

$$\rho_{1,\ell,k_*}(k) = \left(\frac{1-\lambda}{\ell!(n-\ell)!} \sum_{j=k_*}^{k-1} \sum_{i=j}^{+\infty} i^{n-\ell} \sigma^{\lambda(\ell-1)}(i)p(i) \right)^{\frac{1}{1-\lambda}}.$$

Thus, by (3.6), (3.9) we get

$$\Delta^{(\ell-1)}u(k) \ge \rho_{s,\ell,k_*}(k) \ \text{ for } \ k \in \mathbb{N}_{k_*}^+ \ \ (s=2,3,\dots), \tag{3.10}$$

where

$$\rho_{s,\ell,k_*}(k) = \frac{1}{\ell!(n-\ell)!} \sum_{j=k_*}^{k-1} \sum_{i=j}^{+\infty} i^{n-\ell-1} \sigma^{\lambda(\ell-1)}(i)p(i) \big(\rho_{s-1,\ell,k_*}(\sigma(i))\big)^{\lambda}.$$

On the other hand, by (1.2), (2.1), (3.9) and (3.10) from (3.6) for any $\delta \in [0,\lambda]$ we have

$$\begin{aligned} \Delta^{(\ell-1)}u(k+1) \ge \frac{1}{\ell!(n-\ell)!} & \sum_{i=k_*}^{k} \sum_{j=i}^{+\infty} j^{n-\ell-1} \sigma^{\lambda(\ell-1)}(j)p(j) \\ & \times \big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta} |\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda-\delta}, \ \ s=1,2,\dots \end{aligned}$$

and

$$\begin{aligned} \Delta^{(\ell-1)}u(k+1) \ge \frac{k-k_*}{\ell!(n-\ell)!} & \sum_{j=k}^{+\infty} j^{n-\ell-1} \sigma^{\lambda(\ell-1)}(j)p(j) \\ & \times \big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta} \big(\Delta^{(\ell-1)}u(\sigma(j))\big)^{\lambda-\delta}, \ \ s=1,2,\dots \end{aligned} \tag{3.11}$$

If $\delta = \lambda$, then from the last inequality we get

$$\sum_{j=k}^{+\infty} j^{n-\ell-1} \sigma^{\lambda(\ell-1)}(j)p(j) \big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\lambda} \le \frac{\ell!(n-\ell)!(k+1)}{k-k_*} \cdot \frac{\Delta^{(\ell-1)}u(k+1)}{k+1}.$$

By first condition of (2.20) we have

$$\sum_{j=k}^{+\infty} j^{n-\ell-1} \sigma^{\lambda(\ell-1)}(j)p(j) \big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\lambda} < +\infty \ \ (s=1,2,\dots). \tag{3.12}$$

Let $\delta \in [0,\lambda)$. Then from (3.11) implies

$$\frac{\Delta^{(\ell-1)}u(k+1)}{\sum\limits_{j=k}^{+\infty} j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(\sigma(j))\big)^{\lambda-\delta}} \ge \frac{k-k_*}{\ell!(n-\ell)!}$$

for $k \in \mathbb{N}_{k_*}^{+}$. Therefore

$$\frac{\big(\Delta^{(\ell-1)}u(k+1)\big)^{\lambda-\delta}k^{n-\ell-1}p(k)\sigma^{\lambda(\ell-1)}(k)\big(\rho_{s,\ell,k_*}(\sigma(k))\big)^{\delta}}{\Big(\sum\limits_{j=k}^{+\infty} j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(\sigma(j))\big)^{\lambda-\delta}\Big)^{\lambda-\delta}}$$

$$\ge \Big(\frac{k-k_*}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}k^{n-\ell-1}p(k)\sigma^{\lambda(\ell-1)}(k)\big(\rho_{s,\ell,k_*}(\sigma(k))\big)^{\delta}. \tag{3.13}$$

Denote

$$a_k=\sum_{j=k}^{+\infty} j^{n-\ell-1}\sigma^{\lambda(\ell-1)}(j)p(i)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(j+1)\big)^{\lambda-\delta}. \tag{3.14}$$

Since $\Delta^{\ell-1)}u(k)$ is nondecreasing function, according to (3.14), from (3.13) we get

$$\frac{a_k-a_{k+1}}{a_k^{\lambda-\delta}} \ge \Big(\frac{k-k_*}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}k^{n-\ell-1}p(k)\sigma^{\lambda(\ell-1)}(k)\Big(\rho_{s,\ell,k_*}(\sigma(k))\Big)^{\delta}.$$

Thus, from the last inequality we get

$$\sum_{i=k_*}^{k}\frac{a_i-a_{i+1}}{a_i^{\lambda-\delta}} \ge \Big(\frac{1}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}\sum_{i=k_*}^{k}(i-k_*)^{\lambda-\delta}i^{n-\ell-1}p(i)\sigma^{\lambda(\ell-1)}(i)\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta}. \tag{3.15}$$

Since

$$\sum_{i=k_*}^{k}\frac{a_i-a_{i+1}}{a_i^{\lambda-\delta}} = \sum_{i=k_*}^{k}a_i^{\delta-\lambda}\int_{a_{i+1}}^{a_i}dt \le \sum_{i=k_*}^{k}\int_{a_{i+1}}^{a_i}t^{\delta-\lambda}dt \le \int_0^{a_{k_*}}t^{\delta-\lambda}dt = \frac{a_{k_*}^{1+\delta-\lambda}}{1+\delta-\lambda},$$

from (3.15) we get

$$\sum_{i=k_*}^{k}(i-k_*)^{\lambda-\delta}i^{n-\ell-1}p(i)\sigma^{\lambda(\ell-1)}(i)\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta} \le \frac{a_{k_*}^{1+\delta-\lambda}\big(\ell!(n-\ell)!\big)^{\lambda-\delta}}{1+\delta-\lambda}. \tag{3.16}$$

Without loss of generality, by (3.14) we can assume that $a_{k_*} \le 1$. Thus from (3.16) we have

$$\sum_{i=k_*}^{k}(i-k_*)^{\lambda-\delta}i^{n-\ell-1}p(i)\sigma^{\lambda(\ell-1)}(i)\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta} \le \frac{\big(\ell!(n-\ell)!\big)^{\lambda-\delta}}{1+\delta-\lambda}. \tag{3.17}$$

According to (3.12) and (3.17), for any $\delta \in [0,\lambda]$ and $s \in \mathbb{N}$ we have

$$\sum_{i=k_*}^{k} i^{n-\ell-1+\lambda-\delta}p(i)\sigma^{\lambda(\ell-1)}(i)\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta} < +\infty. \tag{3.18}$$

Since $\frac{\rho_{s,\ell}(k)}{\rho_{s,\ell,k_*}(k)} \longrightarrow 1$ for $k \to +\infty$, by (3.18) it is obvious that (3.2) holds, which proves the validity of the theorem. ☐

4. Sufficient conditions of nonexistence of solutions of type (2.1)

Theorem 4.1. *Let conditions* (1.2), (1.3), (3.1) *be fulfilled,* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd and for some* $\delta \in [0, \lambda]$ *and* $s \in \mathbb{N}$

$$\sum_{k=1}^{+\infty} k^{n-\ell-1+\lambda-\delta} \sigma^{\lambda(\ell-1)}(k) \big(\rho_{s,\ell}(\sigma(k))\big)^{\delta} p(k) = +\infty, \tag{4.1}$$

where $\rho_{s,\ell}$ *is defined by* (3.3) *and* (3.4). *Then* $U_{\ell,k_0} = \varnothing$ *for any* $k_0 \in \mathbb{N}$.

Proof. Assume the contrary. Let there exists $k_0 \in \mathbb{N}$ such that $U_{\ell,k_0} \neq \varnothing$. Thus equation (1.1) has a proper solution $u : \mathbb{N}^+_{k_0} \to (0, \infty)$ satisfying the condition (2.1). Since conditions of Theorem 3.1 are fulfilled, (3.2) holds for any $\delta \in [0, \lambda]$ and $s \in \mathbb{N}$, which contradicts (4.1). The obtained contradiction proves the validity of the theorem. $\square$

Theorem 4.2. *Let conditions* (1.2), (1.3) *be fulfilled,* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd and for some* $\alpha \in (1, +\infty)$ *and* $\gamma \in (\lambda, 1)$

$$\liminf_{k \to +\infty} k^{\gamma} \sum_{j=k}^{+\infty} j^{n-\ell-1} \sigma^{\lambda(\ell-1)}(j) p(j) > 0, \quad \liminf_{k \to +\infty} \frac{\sigma(k)}{k^{\alpha}} > 0. \tag{4.2}$$

If moreover, at last one of the conditions

$$\alpha\lambda \geq 1, \tag{4.3}$$

or if $\alpha\lambda < 1$, *for some* $\varepsilon > 0$

$$\sum_{k=1}^{+\infty} k^{n-\ell-1+\frac{\alpha\lambda(1-\gamma}{1-\alpha\lambda}-\varepsilon} \big(\sigma(k)\big)^{\lambda(\ell-1)} p(k) = +\infty \tag{4.4}$$

holds, then $U_{\ell,k_0} = \varnothing$ *for any* $k_0 \in \mathbb{N}$.

Proof. It suffices to show that the condition (4.1) is satisfies for some $s \in \mathbb{N}$ and $\delta = \lambda$. Indeed, according to (4.2) there exist $\alpha > 1$, $\gamma \in (\lambda, 1)$, $c > 0$ and $k_0 \in \mathbb{N}$ such that

$$k^{\gamma} \sum_{j=k}^{+\infty} j^{n-\ell-1} \big(\sigma(j)\big)^{\lambda(\ell-1)} p(j) \geq c \ \text{ for } \ k \in \mathbb{N}^+_{k_0} \tag{4.5}$$

and

$$\sigma(k) \geq c k^{\alpha} \ \text{ for } \ k \in \mathbb{N}^+_{k_0}. \tag{4.6}$$

By (3.3) and (4.2) it is obvious that $\lim\limits_{k \to +\infty} \rho_{1,\ell}(k) = +\infty$. Therefore, without loss of generality we can assume that $\rho_{1,\ell}(k) \geq 1$ for $k \in \mathbb{N}^+_{k_0}$. Thus, by (4.6) from (3.4) we get

$$\rho_{2,\ell}(k) \geq \frac{c}{\ell!(n-\ell)!} \sum_{i=k_0}^{k-1} i^{-\gamma} = \frac{c}{\ell!(n-\ell)!} \sum_{i=k_0}^{k-1} i^{-\gamma} \int_i^{i+1} dt$$

$$\geq \frac{c}{\ell!(n-\ell)!}\sum_{i=k_0}^{k-1}\int_i^{i+1} t^{-\gamma}dt = \frac{c}{\ell!(n-\ell)!}\int_{k_0}^{k} t^{-\gamma}dt = \frac{c}{\ell!(n-\ell)!(1-\gamma)}(k^{1-\gamma}-k_0^{1-\gamma}).$$

We can choose $k_1 \in \mathbb{N}_{k_0}^+$ such that $\rho_{2,\ell}(k) \geq \frac{c}{2\ell!(n-\ell)!(1-\gamma)}k^{1-\gamma}$ for $k \in \mathbb{N}_{k_1}^+$. Thus, by (4.6) from (3.4), for $s = 3$ we have

$$\rho_{3,\ell}(k) \geq \left(\frac{c}{2\ell!(n-\ell)!(1-\gamma)}\right)^{1+\lambda} \cdot k^{(1-\gamma)(1+\alpha\lambda)} \text{ for } k \in \mathbb{N}_{k_2}^+.$$

where $k_2 \in \mathbb{N}_{k_1}^+$ is a sufficiently large natural number. Therefore, for any $s \in \mathbb{N}$ there exists $k_s \in \mathbb{N}$ such that for $k \in \mathbb{N}_s^+$

(4.7)
$$\rho_{s,\ell}(k) \geq \left(\frac{c}{2\ell!(n-\ell)!(1-\gamma)}\right)^{1+\lambda+\cdots+\lambda^{s-2}} k^{(1-\gamma)(1+\alpha\lambda+\cdots+(\alpha\lambda)^{s-2})}, \quad k \geq k_s.$$

Assume that (4.3) be fulfilled. Choose $s_0 \in \mathbb{N}$ such that $(1-\gamma)(s_0-1) \geq \frac{1}{\lambda}$. Then, according to (4.7), $\rho_{s_0,\ell}(k) \geq c_0 k$ for $k \in \mathbb{N}_{k_{s_0}}$, where $c_0 > 0$. Therefore, by (4.7) it is obvious that (4.1) hold, for $\delta = \lambda$ and $s = s_0$. In the case, when (4.3) holds, the validity of the theorem has been already proved.

Assume now that $0 < \alpha\lambda < 1$ and (4.4) holds. Let $\varepsilon > 0$ and by (4.7), choose $s_0 \in \mathbb{N}$ such that $\rho_{s_0,\ell}(k) \geq c_1 k^{\frac{\alpha\lambda(1-\gamma)}{1-\alpha\lambda}-\varepsilon}$ for $k \in \mathbb{N}_{k_{s_0}}^+$, where $c_1 > 0$. Therefore, by (4.4), (4.1) holds for $s = s_0$. The proof of the theorem is proved. □

5. Difference equations with property **A**

Theorem 5.1. *Let the conditions* (1.2), (1.3) *be fulfilled and for any* $\ell \in \{1,\ldots,n-1\}$ *with* $\ell+n$ *odd, let* (3.1) *as well as* (4.1) *hold for some* $\delta \in [0,\lambda]$ *and* $s \in \mathbb{N}$. *Let moreover*

$$\sum_{k=1}^{+\infty} k^{n-1}p(k) = +\infty, \tag{5.1}$$

when n *is odd, then equation* (1.1) *has Property* **A**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbb{N}_{k_0} \to (0,+\infty)$ (the case $u(k) < 0$ is similar). Then by (1.1), (1.3) and Lemma 1.1, there exist $\ell \in \{0,\ldots,n-1\}$ such that $\ell+n$ is odd and the condition (2.1) holds. Since conditions of the Theorem 4.1 are fulfilled, for any $\ell \in \{1,\ldots,n-1\}$ with $\ell+n$ odd, we have $\ell \notin \{1,\ldots,n-1\}$. Therefore, n is odd and $\ell = 0$. Then we will show that the conditions (1.5) hold. If that is not the case, there exists $c > 0$ such that $u(k) \geq c$ for sufficiently large k. According to 2.1, with $\ell = 0$, from (1.1) we have

$$\sum_{j=k_0}^{k} j^{n-1}\Delta^{(n)}u(j) + c\sum_{j=k_0}^{k} j^{n-1}p(j) \leq 0, \tag{5.2}$$

where $k \in \mathbb{N}$ is sufficiently large natural number. On the other hand in view of identity

$$\sum_{j=k_0}^{k} j^{n-1}\Delta^{(n)}u(j) = k^{n-1}\Delta^{(n-1)}u(k+1) - (k_0-1)^{n-1}\Delta^{(n-1)}u(k_0) - \sum_{j=k_0}^{k} \Delta^{(n-1)}u(j)\Delta(j-1)^{n-1}$$

it is easy to show that

$$\sum_{j=k_0}^{k} j^{n-1}\Delta^{(n)}u(j) = \sum_{j=0}^{n-1}(-1)^j\Delta^{(j)}(k-j)^{n-1}\Delta^{(n-j-1)}u(k+1) - \sum_{j=0}^{n-1}(-1)^j(k_0-j-1)^{(n-j-1)}\Delta^{(n-j-1)}u(k_0).$$

From (5.2), by (2.1) with $\ell = 0$

$$c\sum_{j=k_0}^{k} j^{n-1}p(j) \le \sum_{j=0}^{n-1}(k_0-j-1)^{n-j-1}\big|\Delta^{(n-j-1)}u(k_0)\big|.$$

Therefore $\sum_{j=1}^{+\infty} j^{n-1}p(j) < +\infty$, which contradict the condition (5.1). Therefore, equation (1.1) has Property **A**. $\square$

From this theorem, with $\delta = 0$, immediately follow

Theorem 5.1′. *Let the conditions* (1.2), (1.3) *be fulfilled and for any* $\ell \in \{1,\dots,n-1\}$ *with* $\ell + n$ *odd,* (3.1) *as well as*

$$\sum_{k=1}^{+\infty} k^{n-\ell-1+\lambda}\sigma^{\lambda(\ell-1)}(k)p(k) = +\infty \tag{5.3}$$

holds. Then in the case of odd n *condition* (5.1) *is sufficient for equation* (1.1) *to have Property* **A**.

Theorem 5.2. *Let the condition* (1.2), (1.3) *as well as* (5.1) *be fulfilled for odd* n *and*

$$\liminf_{k\to+\infty} \frac{\sigma^{\lambda}(k)}{k} > 0. \tag{5.4}$$

Then the condition

$$\sum_{k=1}^{+\infty} k^{n-2+\lambda}p(k) = +\infty, \tag{5.5}$$

for even n *and the condition*

$$\sum_{k=1}^{+\infty} k^{n-3+\lambda}\big(\sigma(k)\big)^{\lambda}p(k) = +\infty, \tag{5.6}$$

for odd n *is sufficient for equation* (1.1) *to have property* **A**.

Proof. It is obvious that, according to (5.4)–(5.6), for any $\ell = \{1, \dots, n-1\}$, where $\ell + n$ odd, the conditions (5.3) hold. Therefore, all conditions of the Theorem 5.1′ hold, which proves the validity of the theorem. □

Theorem 5.3. *Let the conditions* (1.2), (1.3) *be fulfilled and let*

$$\limsup_{k\to+\infty} \frac{\sigma^{\lambda}(k)}{k} < +\infty. \tag{5.7}$$

Then for equation (1.1) *to have Property* **A** *it is sufficient that*

$$\sum_{k=1}^{+\infty} k^{\lambda}\big(\sigma(k)\big)^{\lambda(n-2)} p(k) = +\infty. \tag{5.8}$$

Proof. It is obvious that, according to (5.7), (5.8) and first condition of (1.2), the condition (5.1) and for any $\ell = \{1, \dots, n-1\}$, where $\ell + n$ is odd, the conditions (5.3) hold. Therefore, all conditions of the Theorem 5.1′ hold, which proves the validity of the theorem. □

Theorem 5.4. *Let the conditions* (1.2), (1.3), (4.3), (4.6) *and* (5.4) *or if* $0 < \alpha\lambda < 1$, *for some* $\varepsilon > 0$

$$\sum_{k=1}^{+\infty} k^{n-2+\frac{\alpha\lambda(1-\gamma)}{1-\alpha\lambda}-\varepsilon} p(k) = +\infty \tag{5.9}$$

be fulfilled. If moreover, there exist $\gamma \in (\lambda, 1)$ *such that*

$$\liminf_{k\to+\infty} k^{\gamma} \sum_{j=k}^{+\infty} j^{n-2} p(j) > 0, \tag{5.10}$$

then equation (1.1) *has Property* **A**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbb{N}_{k_0} \to (0, +\infty)$ (the case $u(k) < 0$ is similar). Then by(1.1), (1.3) and Lemma 1.1, there exist $\ell \in \{0, \dots, n-1\}$ such that $\ell + n$ is odd and the condition (2.1) holds. Since by (4.3), (4.6), (5.4), (5.9) and (5.10) all conditions of the Theorem 4.2 are fulfilled. So for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd, we have $\ell \notin \{1, \dots, n-1\}$. Therefore, n is odd and $\ell = 0$. It is obvious that, since $\gamma \in (0, 1)$, by (5.10) satisfying the condition (5.1). Therefore, analogously Theorem 5.1, we can proved the condition (1.5) hold. That is equation (1.1) has Property **A**. The proof of the theorem is complete. □

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ON SHARED-VALUE PROPERTIES OF $f'(z) = f(z+c)$

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Abstract. This research is a continuation of the recent papers [20, 21]. In this paper, we deal with the uniqueness problems on the derivative of $f(z)$ with its shift $f(z+c)$, and give a new perspective on discussing the complex differential-difference equation $f'(z) = f(z+c)$.

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1. Introduction

It is well known that Nevanlinna theory has a wide range of applications in considering the value distribution of meromorphic solutions of complex differential equations. In addition, with the difference correspondence of the logarithmic derivative lemma obtained by Chiang-Feng [3], and Halburd-Korhonen [7] respectively, the complex domain differences and the complex difference equations also developed rapidly. The related results, readers can refer to [2].

Although the research of complex differential-difference equations can be traced back to Naftalevich's work in [5, 16, 17], the investigations on complex differential-difference field using Nevanlinna theory are still very few. Therefore, the relevant results are very limited, the reader is invited to see [11, 12, 14, 15, 19, 22].

In comparison, in real analysis, the researches on differential-difference equations are too numerous to enumerate. For example, there are extensive studies on the delay equations $f'(x) = f(x-k)$, $(k > 0)$ in real analysis. The related results can be found in [1]. Inspired by such results, Liu and Dong [13] discussed the properties of the solutions of complex differential-difference equations $f'(z) = f(z+c)$. Recently, we looked at this equation from another point of view, that is, "under what sharing value conditions, does $f'(z) = f(z+c)$ hold?" And in [20], we obtained

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Theorem A. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0, a CM, then $f'(z) = f(z+c)$.*

Here, we pose a list of questions related to Theorem A. These questions will be considered in the following.

1. If the condition "$f'(z)$ and $f(z+c)$ share 0, a CM" is changed to "$f'(z)$ and $f(z+c)$ share two distinct values a, b CM", is Theorem A still true?

2. Can value sharing condition or the restriction on the order of $f(z)$ be improved in Theorem A?

Remark. In fact, the solutions of $f'(z) = f(z+c)$ must be transcendental entire functions. Otherwise, suppose that z_0 is a pole of $f(z)$, then from $f'(z) = f(z+c)$, we know $z_0 + nc$ are poles of $f(z)$ also. Hence, $f(z)$ must have infinitely many poles. If m is the minimum order of all poles of $f(z)$, then m is the minimum order of all poles of $f(z+c)$ as well. However, the minimum order of all poles of $f'(z)$ is $1+m$, which contradicts $f'(z) = f(z+c)$. Hence, we just need to consider the condition that $f(z)$ is a transcendental entire function in the following.

In this paper, we will continue to consider the uniqueness problem for the derivative of $f(z)$ with its shift $f(z+c)$. The reminder of this paper is organized as follows: In Section 2, for Question 1, we will give a positive answer by giving Theorem 2.1. In Section 3, we will give two uniqueness results for $f'(z)$ sharing one value with $f(z+c)$, under some appropriate deficiency assumptions.

2. Functions share two values CM

Theorem 2.1. *Let $f(z)$ be a transcendental entire function of hyper-order strictly less than 1. If $f'(z)$ and $f(z+c)$ share two distinct values a, b CM, then $f'(z) = f(z+c)$.*

The following lemma plays a key role in proving Theorem 2.1.

Lemma 2.1. [10, Theorem 1] *Suppose that $f(z)$ and $g(z)$ are two distinct non-constant entire functions. If $f(z)$ and $g(z)$ share the values 0 and 1 CM, then they assume one of the following cases:*

(1) $f(z) = d(1 - e^{A(z)})$, $g(z) = (1-d)(1 - e^{-A(z)})$;
(2) $f(z) = e^{-nA(z)} \sum_{j=0}^{n} e^{jA(z)}$, $g(z) = \sum_{j=0}^{n} e^{jA(z)}$, $n = 1, 2, \ldots$;
(3) $f(z) = -e^{-(n+1)A(z)} \sum_{j=0}^{n} e^{jA(z)}$, $g(z) = -e^{A(z)} \sum_{j=0}^{n} e^{jA(z)}$, $n = 0, 1, 2, \ldots$,
where $d(\neq 0, 1)$ is a constant, and $A(z)$ is a non-constant entire function.

Lemma 2.2. [23, Theorem 1.51] *Suppose that $f_j(z)$ $(j = 1, \dots n)$ $(n \geq 2)$ are meromorphic functions and $g_j(z)$ $(j = 1, \dots, n)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0$.

(2) $1 \leq j < k \leq n$, *$g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.*

(3) *For $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad r \to \infty, r \notin E,$$

where $E \subset (1, \infty)$ is of finite linear measure.

Then $f_j(z) = 0$.

Lemma 2.3. [2, Theorem 1.3] *Let $h_2(z) \not\equiv 0$, $h_1(z), F(z)$ be polynomials, $c_2, c_1(\neq c_2)$ be constants. Suppose that $f(z)$ is a transcendental meromorphic solution of difference equation*

$$h_2(z)f(z + c_2) + h_1(z)f(z + c_1) = F(z).$$

Then, $\rho(f) \geq 1$, where $\rho(f)$ is the order of $f(z)$.

Lemma 2.4. [23, Lemma 5.1] *Let $f(z)$ be a non-constant periodic meromorphic function. Then, $\rho(f) \geq 1$.*

Proof of Theorem 2.1. Suppose that $f'(z) \not\equiv f(z + c)$. Set

$$F(z) = \frac{f'(z) - a}{b - a}, \quad G(z) = \frac{f(z + c) - a}{b - a}. \tag{2.1}$$

Then, from the value sharing assumption and Lemma 2.1, one of the following cases holds:

Case 1. If

$$f'(z) = (b - a)d(1 - e^{A(z)}) + a \tag{2.2}$$

and

$$f(z + c) = (b - a)(1 - d)(1 - e^{-A(z)}) + a. \tag{2.3}$$

Here and below, $A(z)$ is a non-constant entire function of order less than 1. Then, (2.2) and (2.3) give

$$de^{A(z+c)} + (1 - d)A'e^{-A(z)} - (\frac{a}{b - a} + d) = 0. \tag{2.4}$$

Subcase 1.1. If $A(z)$ is a non-constant polynomial, then we have $A(z)$, $A(z+c)$ and $A(z + c) + A(z)$ are non-constant polynomials. Applying Lemma 2.2 to (2.4), we have a contradiction.

Subcase 1.2. If $A(z)$ is a transcendental entire function of order less than 1. Then, we confirm that $A(z+c)+A(z)$ must be transcendental. Otherwise, we suppose that

$A(z+c)+A(z)$ is a polynomial, then from Lemma 2.3, we deduce that $\rho(A) \geq 1$, which is a contradiction. Further, applying Lemma 2.2 to (2.4) again, we obtain a contradiction as well.

Case 2. If

$$f'(z) = (b-a)(1+e^{-A}+e^{-2A}+\cdots+e^{-nA})+a \tag{2.5}$$

and

$$f(z+c) = (b-a)(1+e^{A}+e^{2A}+\cdots+e^{nA})+a. \tag{2.6}$$

Then, combining (2.5) and (2.6), we have

$$nA'e^{nA}+\cdots 2A'e^{2A}+A'e^{A}-\frac{b}{b-a}-e^{-A(z+c)}-e^{-2A(z+c)}-\cdots-e^{-nA(z+c)}=0. \tag{2.7}$$

Subcase 2.1. If $A(z)$ is a non-constant polynomial, then we obtain that $sA(z)+tA(z+c)$ is a non-constant polynomial, where s, $t(\neq -s)$ are two integers such that $s^2+t^2 \neq 0$. Hence, by Lemma 2.2 and (2.7), we have a contradiction.

Subcase 2.2. If $A(z)$ is a transcendental entire function of order less than 1. Then, using the same way of Subcase 1.2, we have $\lambda A(z)+\mu A(z+c)$ must be transcendental, where λ, μ are two integers such that $\lambda^2+\mu^2 \neq 0$. Hence, applying Lemma 2.2 to (2.7), we obtian a contradiction.

Case 3. If

$$f'(z) = (a-b)(e^{-A}+e^{-2A}+\cdots+e^{-(n+1)A})+a, \tag{2.8}$$

and

$$f(z+c) = (a-b)(e^{A}+e^{2A}+\cdots+e^{(n+1)A})+a. \tag{2.9}$$

Then, by (2.8) and (2.9), it follows that

$$\begin{aligned}(n+1)A'e^{(n+1)A}+\cdots+2A'e^{2A}+A'e^{A}+\frac{a}{b-a}\\ -e^{-A(z+c)}-e^{-2A(z+c)}+\cdots-e^{-(n+1)A(z+c)}=0,\end{aligned} \tag{2.10}$$

and as in Case 2, we get a contradiction. Therefore, $f'(z)=f(z+c)$.

Remark. From the proof of the Theorem 2.1, we can find that Lemma 2.1 can make our proof of Theorem 2.1 very simple. However, without the application of Lemma 2.1, our proof will be very cumbersome. In fact, we have already given a complicated proof before. In addition, using Lemma 2.1, we can not only give a very simple proof of Theorem B [25, Theorem 1.1], but also improve Theorem B.

Theorem B. *Let $f(z)$ be a transcendental entire function of finite order and a, b be two distinct constants. If $\triangle f(z)=f(z+c)-f(z)(\not\equiv 0)$ and $f(z)$ share a, b CM, then $\triangle f(z)=f(z)$.*

In fact, we have

Theorem 2.2. *Let $f(z)$ be a transcendental entire function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $\triangle f(z)(\not\equiv 0)$ and $f(z)$ share a, b CM, then $\triangle f(z) = f(z)$.*

The proof of Theorem 2.2 is similar to the proof of Theorem 2.1. For the convenience of the reader, we will give a brief proof here.

Proof of Theorem 2.2. Similarly as in Theorem 2.1, if $f(z) \not\equiv \triangle f(z)$, then we have three possibilities:

Case 1.

$$(1-d)e^{A(z+c)} - de^{A} - (1-d)e^{-A} + d + \frac{a}{b-a} = 0. \tag{2.11}$$

Case 2.

$$\begin{aligned} &e^{nA} + e^{(n-1)A} + \cdots e^{A} + \frac{b}{b-a} + e^{-A} + e^{-2A} + \cdots + e^{-nA} \\ &- e^{-A(z+c)} + e^{-2A(z+c)} + \cdots + e^{-nA(z+c)} = 0. \end{aligned} \tag{2.12}$$

Case 3.

$$\begin{aligned} &e^{(n+1)A} + e^{nA} + \cdots e^{A} - \frac{a}{b-a} + e^{-A} + e^{-2A} + \cdots + e^{-(n+1)A} \\ &- e^{-A(z+c)} + e^{-2A(z+c)} + \cdots + e^{-(n+1)A(z+c)} = 0. \end{aligned} \tag{2.13}$$

The only difference the proof of Theorem 2.1 is that, we need to prove one more case: $A(z+c) - A(z)$ **is not a constant, when** $A(z)$ **is a non-constant polynomial**. Here, we only prove the Case 1, as for the Cases 2 and 3, we can prove similarly.

Otherwise, we suppose $A(z+1) - A(z) = \alpha$, where α is a constant. Then, from Lemma 2.4, we have $\alpha \neq 0$. Further, we have

$$A(z) = \alpha z + \beta, \tag{2.14}$$

where β is a constant. Substituting (2.14) into (2.11), it follows that

$$((1-d)e^{\alpha c+\beta} - de^{\beta})e^{\alpha z} + d + \frac{a}{b-a} - (1-d)e^{-\beta}e^{-\alpha z} = 0. \tag{2.15}$$

Applying Lemma 2.2 to (2.15), we get a contradiction. Thus, $A(z+c) - A(z)$ is not a constant.

3. Functions share one value CM or IM

First of all, let's give the definitions that we need in the following proof.

Definitions. Suppose that z is a zero of $F - 1$ with multiplicity m, meanwhile, a zero of $G - 1$ with multiplicity n. Then, we denote by $N_L(r, \frac{1}{F-1})$ the reduced counting function of those 0-points of $F - 1$ when $m > n$; by $N_E^{(2}(r, \frac{1}{F-1})$ the reduced counting function of those 0-points of $F - 1$ when $m = n \geq 2$. In addition,

$\overline{N}_{(2}(r, \frac{1}{F})$ is the counting function of zeros of F whose multiplicities are greater than 2, $N_0(r, \frac{1}{F'})$ is the counting function of zeros of F' but not the zeros of F and $F-1$. Notations $N_L(r, \frac{1}{G-1})$, $N_E^{(2}(r, \frac{1}{G-1})$, $\overline{N}_{(2}(r, \frac{1}{G})$ and $N_0(r, \frac{1}{G'})$ can be similarly defined. Moreover, we define $\delta(0, f)$ as following

$$\delta(0, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f})}{T(r, f)}.$$

Since in [21], we have given partial results for cases "1 CM + 1IM"and "2 IM". Hence, in the following, we just give the result of $f'(z)$ share one value with $f(z+c)$, under the deficiency assumption.

Theorem 3.1. *Let $f(z)$ be a transcendental entire function of hyper-order strictly less than 1, and let $a(\not\equiv 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a CM and $\delta(0, f) > \frac{1}{2}$. Then, $f'(z) = f(z+c)$.*

For the sharing assumption "1 IM", we obtain

Theorem 3.2. *Let $f(z)$ be a transcendental entire function of hyper-order strictly less than 1, and let $a(\not\equiv 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a IM and $\delta(0, f) > \frac{4}{5}$. Then, $f'(z) = f(z+c)$.*

In order to prove Theorems 3.1-3.2, we need the following lemmas. From Theorem 5.1 in [8], we can immediately obtain the following result:

Lemma 3.1. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1. Then,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Remark. Here and below, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. Meanwhile, by $S_1(r, f)$ we denote any quantity satisfying $S_1(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set of finite linear measure.

From Lemma 8.3 in [8] and Lemma 3.1, we have the following lemma:

Lemma 3.2. [3, Lemma 5.1] *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, then we have*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

The following result is just a simple modification of the result of meromorphic functions with finite order in Lemma 2.5 [18]:

Lemma 3.3. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, then*

$$N\left(r, \frac{1}{f(z+c)}\right) \leq N(r, \frac{1}{f}) + S(r, f).$$

Lemma 3.4. [23, Theorem 1.24] *Suppose $f(z)$ is a non-zero meromorphic function in the complex plane and k is a positive integer. Then,*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S_1(r, f).$$

Lemma 3.5. [24, Lemma 3] *Let $F(z)$ and $G(z)$ be two non-constant meromorphic functions, and let*

$$\Phi(z) = \left(\frac{F''(z)}{F'(z)} - \frac{2F'(z)}{F(z)-1}\right) - \left(\frac{G''(z)}{G'(z)} - \frac{2G'(z)}{G(z)-1}\right). \tag{3.1}$$

If $F(z)$ and $G(z)$ share 1 IM and $\Phi(z) \not\equiv 0$. Then,

$$N_E^{1)}(r, \frac{1}{F-1}) \leq N(r, \Phi) + S_1(r, F) + S_1(r, G), \tag{3.2}$$

where $N_E^{1)}(r, \frac{1}{F-1})$ is the reduced counting function of the common simple zeros of $F-1$ and $G-1$.

Proof of Theorem 3.2. Set

$$F(z) = \frac{f'(z)}{a}, \quad G(z) = \frac{f(z+c)}{a}. \tag{3.3}$$

Then, by the sharing values assumption, we get $F(z)$ and $G(z)$ share 1 IM. Moreover,

$$T(r, F) = T(r, f') + S(r, f) \leq T(r, f) + S(r, f).$$

And Lemma 3.2 gives

$$T(r, G) = T(r, f(z+c)) + S(r, f) = T(r, f) + S(r, f).$$

Hence,

$$S(r, F) = S(r, f), \quad S(r, G) = S(r, f).$$

Further, from Lemma 3.4, it follows that

$$N(r, \frac{1}{F}) \leq N(r, \frac{1}{f'}) + S(r, f) \leq N(r, \frac{1}{f}) + S(r, f). \tag{3.4}$$

And Lemma 3.3 leads to

$$N(r, \frac{1}{G}) \leq N\left(r, \frac{1}{f(z+c)}\right) + S(r, f) \leq N(r, \frac{1}{f}) + S(r, f). \tag{3.5}$$

Let $\Phi(z)$ be given by (3.1). Then, we will discuss two cases as follows.

Case 1. Suppose $\Phi(z) \not\equiv 0$. Then, from (3.1) and the sharing values assumption, we have

$$\begin{aligned} N(r,\Phi) \le\ & \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + N_L(r,\frac{1}{F-1}) \\ & + N_L(r,\frac{1}{G-1}) + N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + S(r,f). \end{aligned} \tag{3.6}$$

Moreover, we have

$$\overline{N}(r,\frac{1}{F-1}) = N_E^{1)}(r,\frac{1}{F-1}) + N_E^{(2}(r,\frac{1}{F-1}) + N_L(r,\frac{1}{F-1}) + N_L(r,\frac{1}{G-1}).$$

Noting $F(z)$ and $G(z)$ share 1 IM, and so

$$\begin{aligned} & \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \\ & = 2N_E^{1)}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{F-1}) + 2N_L(r,\frac{1}{F-1}) + 2N_L(r,\frac{1}{G-1}). \end{aligned} \tag{3.7}$$

Thus, combining (3.2), (3.6) and (3.7) yields

$$\begin{aligned} & \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \\ & \le N(r,\Phi) + N_E^{1)}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{F-1}) \\ & + 2N_L(r,\frac{1}{F-1}) + 2N_L(r,\frac{1}{G-1}) + S(r,f) \\ & \le \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + 3N_L(r,\frac{1}{F-1}) + 3N_L(r,\frac{1}{G-1}) \\ & + N_E^{1)}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{F-1}) + N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + S(r,f). \end{aligned} \tag{3.8}$$

Obviously,

$$\begin{aligned} & N_L(r,\frac{1}{G-1}) + 2N_L(r,\frac{1}{F-1}) + N_E^{1)}(r,\frac{1}{F-1}) + 2N_E^{(2}(r,\frac{1}{F-1}) \\ & \le N(r,\frac{1}{F-1}) \le T(r,F) + S(r,f). \end{aligned}$$

Substituting the above inequality into (3.8) yields

$$\begin{aligned} & \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \\ & \le \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + 2N_L(r,\frac{1}{G-1}) + N_L(r,\frac{1}{F-1}) \\ & + N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + T(r,F) + S(r,f), \end{aligned} \tag{3.9}$$

On the other hand, applying the second main theorem, we derive that

$$\begin{aligned} & T(r,F) + T(r,G) \\ & < \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1}) \\ & - N_0(r,\frac{1}{F'}) - N_0(r,\frac{1}{G'}) + S(r,f). \end{aligned} \tag{3.10}$$

It is easy to see

$$\overline{N}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{F}) \le N(r, \frac{1}{F}), \quad \overline{N}(r, \frac{1}{G}) + \overline{N}_{(2}(r, \frac{1}{G}) \le N(r, \frac{1}{G}). \tag{3.11}$$

Hence, it follows from (3.4), (3.5) and (3.9)–(3.11), that

$$\begin{aligned} T(r, f) &= T(r, G) + S(r, f) \\ &\le N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + 2N_L(r, \frac{1}{G-1}) + N_L(r, \frac{1}{F-1}) + S(r, f) \\ &\le 2N(r, \frac{1}{f}) + 2N_L(r, \frac{1}{G-1}) + N_L(r, \frac{1}{F-1}) + S(r, f). \end{aligned} \tag{3.12}$$

Furthermore, by Lemma 3.4 and (3.4), we obtain

$$N_L(r, \frac{1}{F-1}) \le N(r, \frac{1}{F'}) \le N(r, \frac{1}{F}) + S(r, f) \le N(r, \frac{1}{f}) + S(r, f). \tag{3.13}$$

Similarly, we have

$$N_L(r, \frac{1}{G-1}) \le N(r, \frac{1}{f}) + S(r, f). \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) yields that

$$T(r, f) \le 5N(r, \frac{1}{f}) + S(r, f),$$

which contradicts the assumption $\delta(0, f) > \frac{4}{5}$.

Case 2. Suppose $\Phi(z) = 0$. Then, integrating twice, it follows from (3.1) that

$$\frac{1}{G-1} = \frac{\alpha}{F-1} + \beta, \tag{3.15}$$

where $\alpha(\neq 0)$ and β are constants. Rewrite (3.15) as

$$F = \frac{(\beta - \alpha)G + (\alpha - \beta - 1)}{\beta G - (\beta + 1)}. \tag{3.16}$$

Subcase 2.1. If $\beta \neq 0, -1$. Then, by (3.16), we have

$$\overline{N}\left(r, \frac{1}{G - \frac{\beta+1}{\beta}}\right) = \overline{N}(r, F).$$

From the second main theorem and (3.5), we obtain

$$\begin{aligned} T(r, f) &= T(r, G) + S(r, f) \\ &\le \overline{N}(r, \frac{1}{G}) + \overline{N}\left(r, \frac{1}{G - \frac{B+1}{B}}\right) + S(r, f) \\ &\le N(r, \frac{1}{G}) + \overline{N}(r, F) + S(r, f) \le N(r, \frac{1}{f}) + S(r, f), \end{aligned} \tag{3.17}$$

which contradicts the assumption $\delta(0, f) > \frac{4}{5}$.

Subcase 2.2. If $\beta = 0$. Then, we rewrite (3.16) as

$$F = \alpha G - (\alpha - 1). \tag{3.18}$$

If $\alpha \neq 1$, then by (3.18), we have

$$\overline{N}\left(r, \frac{1}{G - \frac{\alpha-1}{\alpha}}\right) = \overline{N}(r, \frac{1}{F}).$$

Similarly as Subcase 2.1, we get a contradiction as well.

If $\alpha = 1$, then by (3.18), we have $F = G$. That is, $f'(z) = f(z+c)$.

Subcase 2.3. If $\beta = -1$. Then, (3.16) can be rewritten as

$$F = \frac{(\alpha+1)G - \alpha}{G}. \tag{3.19}$$

If $\alpha \neq -1$, then by (3.19), it follows that

$$\overline{N}\left(r, \frac{1}{G - \frac{\alpha}{\alpha+1}}\right) = \overline{N}(r, \frac{1}{F}),$$

Using the same reasoning as in Subcase 2.1, we also get a contradiction.

If $\alpha = -1$. then (3.19) leads to $FG = 1$, which means that

$$f' f(z+c) = a^2. \tag{3.20}$$

By $f'(z)$ and $f(z+c)$ share ∞ CM and (3.20), we deduce that

$$N\left(r, \frac{1}{f(z+c)}\right) = S(r, f).$$

Moreover, from Lemma 3.1, Lemma on the logarithmic derivative and (3.20), it follows that

$$\begin{aligned} m\left(r, \frac{1}{f(z+c)}\right) &= \frac{1}{2} m\left(r, \frac{1}{f(z+c)^2}\right) + S(r, f) \\ &\leq m\left(r, \frac{f' f(z+c)}{f(z+c)^2}\right) + m\left(r, \frac{1}{f' f(z+c)}\right) + S(r, f) \\ &\leq m\left(r, \frac{f'}{f(z+c)} \frac{f}{f}\right) + m(r, \frac{1}{a^2}) + S(r, f) \\ &\leq m(r, \frac{1}{a^2}) + S(r, f) = S(r, f). \end{aligned}$$

Therefore, by Lemma 3.2, we have

$$T(r, f) = T(r, f(z+c)) + S(r, f) = S(r, f),$$

which is a contradiction.

Proof of Theorem 3.1. Using the same way of Theorem 3.2, we also obtain (3.12), i.e.,

$$T(r, f) \leq 2N(r, \frac{1}{f}) + 2N_L(r, \frac{1}{G-1}) + N_L(r, \frac{1}{F-1}) + S(r, f).$$

From the assumption that $f(z)$ and $f(z+c)$ share a CM, we know that $F(z)$ and $G(z)$ share 1 CM. Thus,

$$2N_L(r, \frac{1}{F-1}) + N_L(r, \frac{1}{G-1}) = 0.$$

And so,

$$T(r, f) \leq 2N(r, \frac{1}{f}) + S(r, f),$$

which contradicts the assumption that $\delta(0, f) > \frac{1}{2}$.

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