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CONDITIONAL MOMENTS FOR A d-DIMENSIONAL CONVEX BODY

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Abstract. For a d-dimensional convex body we define new integral geometric concepts: conditional moments of the random chord length and conditional moments of the distance of two independent uniformly distributed points in the body. Also in this article the relations between the concepts are found.

MSC2010 numbers: 53C65; 53C60; 31A10.
Keywords: integral geometry; convex body; geometric tomography; conditional moments; random chord length.

1. Introduction

Geometric tomography (the term introduced by R. Gardner in [7]) is a field of mathematics engaged in extracting information about a geometric object from data on its sections or projections to reconstruct the geometric object. The reconstruction of convex domain using random sections makes it possible to simplify the calculation, since mathematical statistics methods can be used to estimate the geometric characteristics of random sections. The integral geometric concepts such as the distribution of the chord length, the distribution of the distance between two random points in a convex body and many others carry some information about the body. In this article for a d-dimensional convex body D we define two new integral geometric concepts: conditional moments of the chord length distribution of a convex body and conditional moments of the distribution of the distance of two random points in D. Also in this article we find the relation between the two concepts.
By $\mathbf{R}^d$ ($d \geqslant 2$) we denote the d-dimensional Euclidean space, by S^{d-1} the unit sphere in $\mathbf{R}^d$ centered at the origin. Let L_d be the Lebesgue measure on $\mathbf{R}^d$. For $\omega \in S^{d-1}$ by e_ω we denote the hyperplane containing the origin and orthogonal to ω. Let $\mathcal{N}$ be the set of nonnegative integers. Let $\mathbf{G}^d$ be the space of all lines in $\mathbf{R}^d$. We use the usual parametrization of a line $g = (\omega, P)$, where $\omega \in S^{d-1}$ is the direction of g and P is the intersection point of g and e_ω. By $[D]$ we denote the set of lines intersecting D. In $\mathbf{G}^d$ we consider the invariant measure (with respect to the group of Euclidean motions) $\mu(\cdot)$. It is known that the element dg of the

measure, up to a constant, has the following form ([8], [1], [3])

$$dg = d\omega dP, \tag{1.1}$$

here $d\omega$ and dP are elements of the Lebesgue measure on S^{d-1} and the hyperplane, respectively.

Definition 1.1. *Let D be a compact convex set in $\mathbf{R}^d$ below we call D a convex body. We consider the random line g with normed invariant measure ($\frac{dg}{\mu([D])}$, here $\mu([D])$ is the invariant measure of lines intersecting $[D]$). For a random line g intersecting D by $X(g)$ we denote the length of the chord $D \cap g$. The conditional n-th moment of the distribution of the chords length (with respect to condition $X > u \geqslant 0$) we define as:*

$$I_{n,u} = \frac{1}{\mu([D])} \int_{X(g)>u} X(g)^n dg, \quad n = 1, 2, ... \tag{1.2}$$

Lemma 2.2 (below) gives the explicit formula for $\mu([D])$. In the sequel by $F_X(t)$ we denote the distribution function of $X(g)$.

Definition 1.2. *For two independent uniformly distributed points Q_1, Q_2 in a convex domain D we denote the distance between the points by $r = |Q_1 - Q_2|$. The conditional n-th moment of the distribution of the distance (with respect to condition $r > u \geqslant 0$) we define as:*

$$J_{n,u} = \frac{1}{L_d(D)^2} \int_{|Q_1-Q_2|>u} r^n dQ_1 dQ_2 \tag{1.3}$$

here $L_d(D)$ is the volume of D, dQ_i $(i = 1, 2)$ is the usual Lebesgue's measure in $\mathbf{R}^d$. Also, in the sequel by $F_r(u)$ we denote the distribution function of the distance of two uniformly distributed points Q_1, Q_2 in a convex body D.

In the following theorem we obtain relation between the conditional moments of the distribution of the distance of two random points in D and the conditional moments of the distribution of the chords length.

Theorem 1.1. *Let D be a convex domain and $u \geqslant 0$. For any $n \in \mathcal{N}$*

$$J_{n,u} = \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{I_{0,u} u^{n+d+1}}{(n+d+1)} - \frac{I_{1,u} u^{n+d}}{n+d} + \frac{I_{n+d+1,u}}{(n+d)\,(n+d+1)}\right). \tag{1.4}$$

The moments of the distribution of the chords length and the distribution of the distance between two independent uniformly distributed point in a convex domain was considered in [8] and [6].

For the planar case $d = 2$ (1.4) was proved in [4].

For the distribution function of the distance between two independent uniformly distributed points in a convex body in $\mathbf{R}^d$ we have

Theorem 1.2.

(1.5)

$$F_r(u) = 1 - J_{0,u} = 1 - \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{I_{0,u} u^{d+1}}{d+1} - \frac{I_{1,u} u^d}{d} + \frac{I_{d+1,u}}{d(d+1)} \right).$$

2. Preliminary results

To prove Theorem 1.1 we need to prove the following lemmas. Let $D \subset \mathbf{R}^d$ be a convex body.

Lemma 2.1. *For the invariant measure of the lines intersecting D we have*

$$\mu([D]) = \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{2\,(d-1)}. \tag{2.1}$$

Proof. By definition we have

(2.2)

$$\mu([D]) = \int_{[D]} dg = \int_{[D]} d\omega dP = \frac{1}{2} \int_{S^{d-1}} d\omega \int_{D_\omega} dP = \frac{1}{2} \int_{S^{d-1}} L_{d-1}(D_\omega)\, d\omega,$$

where D_ω is the orthogonal projection of D onto hyperplane e_ω. For $\xi \in S^{d-1}$ we denote by $s(\xi)$ the point on ∂D the outer normal of which is ξ. In [6] (see also [2]) was proved that

$$L_{d-1}(D_\omega) = \frac{1}{2} \int_{\partial D} |\cos \widehat{(\omega, \xi)}| ds_\xi, \tag{2.3}$$

where ds_ξ is the element of $(d-1)$-dimensional Lebesgue's measure on ∂D and $\widehat{(\omega, \xi)}$ is the angle between two directions ω and ξ. Substituting (2.3) into (2.2) and using the Fubini's theorem we obtain

$$\mu([D]) = \frac{1}{4} \int_{S^{d-1}} \int_{\partial D} |\cos \widehat{(\omega, \xi)}| ds_\xi d\omega = \frac{1}{4} \int_{\partial D} \int_{S^{d-1}} |\cos \widehat{(\omega, \xi)}| d\omega ds_\xi. \tag{2.4}$$

For any $\xi \in S^{d-1}$ we have (see [2])

$$\int_{S^{d-1}} |\cos \widehat{(\omega, \xi)}| d\omega = \frac{2L_{d-2}\left(S^{d-2}\right)}{d-1}. \tag{2.5}$$

Finally substituting (2.5) into (2.4) we obtain

$$\mu([D]) = \frac{L_{d-2}\left(S^{d-2}\right)}{2\,(d-1)} \int_{\partial D} ds_\xi = \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{2\,(d-1)} \tag{2.6}$$

Lemma 2.1 is proved.

Now we consider a pair of points (Q_1, Q_2) in $\mathbf{R}^d$. There are two equivalent representations of (Q_1, Q_2).

1. A pair of points Q_1, Q_2 can be determined by the usual cartesian coordinates.

2. A pair of points Q_1, Q_2 can be determined by the line $g = (\omega, P)$ passing through the points and pair of two one dimensional coordinates (t_1, t_2) which determine Q_1 and Q_2 on the line g (for 3-dimensional case see [8]). Thus

$$(Q_1, Q_2) = (g, t_1, t_2) = (\omega, P, t_1, t_2). \tag{2.7}$$

Note that as a reference point on g one can take the point P on g.

Lemma 2.2. *The Jacobian of that transform* (2.7) *is*

$$dQ_1 dQ_2 = |t_1 - t_2|^{d-1} dt_1 dt_2 d\omega dP. \tag{2.8}$$

Proof. For a fixed Q_1 we represent Q_2 by polar coordinates with respect to Q_1. It is known that

$$dQ_2 = r^{d-1} dr d\omega \tag{2.9}$$

where $r = |Q_1 - Q_2|$ and ω is the direction of the vector $\overrightarrow{Q_1 Q_2}$. For a fixed ω the point Q_1 can be represented by P and t_1. Thus

$$dQ_1 = dt_1 dP \tag{2.10}$$

and by multiplying (2.9) and (2.10) and taking into account that $r = |t_1 - t_2|$, we get

$$dQ_1 dQ_2 = |t_1 - t_2|^{d-1} dt_1 dt_2 d\omega dP. \tag{2.11}$$

Lemma 2.2 is proved.

In the sequel also, we use the following lemma. For a random line g intersecting a convex body $D \subset \mathbf{R}^d$ we have the following lemma.

Lemma 2.3. *Let $X(g)$ be the length of the chord $D \cap g$. We have*

$$\int_{[D]} X(g) dg = \frac{L_d(D) L_{d-1}\left(S^{d-1}\right)}{2} \tag{2.12}$$

Proof. By definition we have $(g = (\omega, P))$

$$\int_{[D]} X(g) dg = \frac{1}{2} \int_{S^{d-1}} d\omega \int_{D_\omega} X(\omega, P) dP. \tag{2.13}$$

For any $\omega \in S^{d-1}$ it is obvious that $X(\omega, P) dP$ is the element of d-dimensional volume of D, hence the integrating by dP over D_ω we get $L_d(D)$.

$$\begin{aligned} \int_{[D]} X(g) dg &= \frac{1}{2} \int_{S^{d-1}} d\omega \int_{D_\omega} X(\omega, P) dP = \\ &= \frac{L_d(D)}{2} \int_{S^{d-1}} d\omega = \frac{L_d(D) L_{d-1}\left(S^{d-1}\right)}{2} \end{aligned} \tag{2.14}$$

3. Proof of Theorem 1.1

Let Q_1, Q_2 are two independent uniformly distributed points in a convex body D. For a random line g intersecting D we denote by $X(g) = |g \cap D|$ the length of the intersection. For $u \geqslant 0$ using (2.8), (2.1) and taking into account $r = |Q_1 - Q_2| = |t_1 - t_2|$ we have

$$J_{n,u} = \frac{1}{L_d(D)^2} \int_{|P_1 - P_2| > u} |P_1 - P_2|^n \, dQ_1 dQ_2 = \frac{1}{L_d(D)^2} \int_{X(g)>u} \int_{|t_1 - t_2|>u} |t_1 - t_2|^{n+d-1} dt_1 dt_2 dg. \tag{3.1}$$

Consider the internal integral of (3.1). For two points t_1 and t_2 chosen at random, independently with uniform distribution in a segment of length $X > u$ we have.

$$\int_{|t_1 - t_2|>u} |t_1 - t_2|^{n+d-1} dt_1 dt_2 = 2 \int_0^{X-u} dt_1 \int_{t_1+u}^{X} (t_2 - t_1)^{n+d-1} \, dt_2 = \tag{3.2}$$
$$= 2 \int_0^{X-u} \frac{(X - t_1)^{n+d}}{n+d} - \frac{u^{n+d}}{n+d} dt_1 = 2 \left(\frac{u^{n+d+1}}{(n+d+1)} - \frac{X u^{n+d}}{n+d} + \frac{X^{n+d+1}}{(n+d)\,(n+d+1)} \right)$$

Substituting (3.2) into (3.1) we get

(3.3)
$$J_{n,u} = \frac{2}{L_d(D)^2} \int_{X(g)>u} \left(\frac{u^{n+d+1}}{(n+d+1)} - \frac{X(g) u^{n+d}}{n+d} + \frac{X(g)^{n+d+1}}{(n+d)\,(n+d+1)} \right) dg$$
$$= \frac{L_{d-1}\,(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{I_{0,u} u^{n+d+1}}{(n+d+1)} - \frac{I_{1,u} u^{n+d}}{n+d} + \frac{I_{n+d+1,u}}{(n+d)\,(n+d+1)} \right).$$

Theorem 1.1. is proved.

Not that for $u \geqslant Diam(D)$, both sides of (3.3) are 0.

Corollary 3.1. *For $u = 0$ and $d = 2$ from* (3.1) *for a convex domain D we get the following well known formula (see* [8]*)*

$$J_{n,0} = \frac{2L_1\,(\partial D)}{L_2(D)^2} \left(\frac{I_{n+3,0}}{(n+2)\,(n+3)} \right). \tag{3.4}$$

Corollary 3.2. *For $n = 0$ we get*

$$J_{0,u} = \frac{L_{d-1}\,(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{I_{0,u} u^{d+1}}{d+1} - \frac{I_{1,u} u^d}{d} + \frac{I_{d+1,u}}{d(d+1)} \right). \tag{3.5}$$

Taking into account

$$J_{0,u} = \frac{1}{L_d(D)^2} \int_{|Q_1 - Q_2|>u} dQ_1 dQ_2 = 1 - F_r\,(u) \tag{3.6}$$

we get the following theorem.

4. A representation for $I_{n,u}$

Now we are going to find a representations for $I_{0,u}, I_{1,u}, I_{d+1,u}$.

1. For $I_{0,u}$ we have

$$I_{0,u} = \frac{2(d-1)}{L_{d-1}(\partial D)\, L_{d-2}(S^{d-2})} \int_{X(g)>u} dg = P\left(X(g) > u\right) = \left(1 - P\left(X(g) \leqslant u\right)\right) = 1 - F_X\left(u\right). \tag{4.1}$$

Here $F_X\left(t\right)$ is the chord length distribution function.

2. For the derivative of $I_{1,u}$ we have

$$\left(I_{1,u}\right)' = \left(\int_{X(g)>u} X(g)\, \frac{dg}{\mu([D])}\right)' = - \lim_{\Delta u \to 0} \frac{\int_{u<X(g)<u+\Delta u} X(g)\, \frac{dg}{\mu([D])}}{\Delta u} = - \lim_{\Delta u \to 0} \frac{\mathrm{uP}\left(u < X(g) < u + \Delta u\right)}{\Delta u} = -\mathrm{uf}_X\left(u\right), \tag{4.2}$$

here $f_X\left(t\right)$ is the density function of the chord length distribution of $X(g)$.
It follows from Lemma 2.3 that

$$I_{1,0} = \frac{(d-1)L_d(D)L_{d-1}(S^{d-1})}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})}. \tag{4.3}$$

Integrating (4.2) and taking into account (4.3) we get

$$I_{1,u} = \frac{(d-1)L_d(D)L_{d-1}(S^{d-1})}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})} - \int_0^u v f_x\left(v\right) \mathrm{dv}. \tag{4.4}$$

3. By the same way (see (4.2)) for the derivative of $I_{d+1,u}$ we have

$$\left(I_{d+1,u}\right)' = \left(\int_{X(g)>u} X(g)^{d+1}\, \frac{dg}{\mu([D])}\right)' = -u^{d+1} f_X\left(u\right). \tag{4.5}$$

It follows from (3.3) that

$$I_{d+1,0} = \frac{(d-1)d(d+1)L_d(D)^2}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})}. \tag{4.6}$$

Integrating (4.5) and taking into account (4.6) we get

$$I_{d+1,u} = \frac{(d-1)d(d+1)L_d(D)^2}{L_{d-1}(\partial D)L_{d-2}(S^{d-2})} - \int_0^u v^{d+1} f_X((v)\, \mathrm{dv}. \tag{4.7}$$

Finally substituting (4.7), (4.4), (4.1) into (3.5) we obtain the following relation between the distribution function of the distance of two uniformly distributed points of D and the chord length distribution function of D.

Theorem 4.1. *Let D be a convex body in $\mathbf{R}^d$. For $u \geqslant 0$*

(4.8)

$$F_r(u) = 1 - \frac{L_{d-1}(\partial D)\, L_{d-2}\left(S^{d-2}\right)}{L_d(D)^2\,(d-1)} \left(\frac{u^{d+1}}{d+1} - \frac{(d-1)L_d(D)L_{d-1}(S^{d-1})}{d\, L_{d-1}(\partial D)\, L_{d-2}(S^{d-2})} u^d + \frac{(d-1)L_d(D)^2}{L_{d-1}(\partial D)\, L_{d-2}(S^{d-2})} - \frac{u^d}{2} \int_0^u F_X(v)dv + \frac{1}{d} \int_0^u v^d F_X(v) dv \right).$$

For the planar case $d = 2$ (4.8) was proved in [5] (see also [4]).

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A PRINCIPLE RELATED TO ZEROS OF REAL FUNCTIONS

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Abstract. In this paper we present a result of a general nature related to arbitrary enough smooth real functions. It is a simple principle (in the spirit of 17th century) which permits to give (for the first time) bounds for the number of zeros of similar functions. It is interesting that there is a version of the principle (for real functions) similar to the classical Nevanlinna deficiency relation (in complex analysis).

MSC2010 numbers: 26A99; 26D99; 30D99.
Keywords: zeros of real functions.

1. The principle

In this paper we present a general method permitting to give bounds for the number of zeros of "enough smooth" real function. The method was found in 1970s, presented first in the book [1], see item 5.3.2.

Below we give some modifications of this method and apply them to establish a result for real function which is an analogue of the classical Nevanlinna deficiency relation (in complex analysis).

Consider a real function $f(t)$, $t \in [a,b]$ with continuos f'' (i.e. $f(t) \in C^2[a,b]$) and denote by $N_{[a,b]}\,(f=0)$ the number of zeros of $f(t)$ in $[a,b]$; here we count in $N_{[a,b]}\,(f=0)$ each possible interval, where $f \equiv 0$ as one zero.

Further, we take an arbitrary function $\varphi(t) \in C^2[a,b]$ satisfying $\varphi(t) > 0$ and $\varphi'(t) > 0$ for $t \in [a,b]$ and compose the following function $F(f(t)) = f(t) + i\varphi(t)f(t)$, where i is the complex unit. (Obviously $F(f(t))$ is a complex function of one real variable t).

Theorem 1.1 (*principle of zeros of real functions*). *For an arbitrary function $f(t) \in C^2[a,b]$ we have*

$$N_{[a,b]}\,(f=0) \leq \frac{1}{\pi}\int_a^b \left|\frac{d}{dt}\arg\frac{d}{dt}F(f(t))\right| dt + 1. \tag{1.1}$$

Notice that the inequality is true also when $N_{[a,b]}\,(f=0) = \infty$.

The inequality is sharp; see below.

Applying (1.1) with $\varphi(t) = t - a$ (i.e. for function $F(f(t)) = f(t) + i(t-a)f(t))$ we get

$$N_{[a,b]}\,(f=0) \le \frac{1}{\pi}\int\limits_a^b \frac{|f''(t)||f(t)| + 2\,[f'(t)]^2}{[f'(t)]^2 + [f(t) + (t-a)f'(t)]^2}\,dt + 1. \tag{1.2}$$

2. An analogue of Nevanlinna second fundamental theorem for real functions

We aren't going to present here Nevanlinna theory and mention here just what we need for our purpose. The famous *deficiency relation* (following from the second fundamental theorem) asserts that

$$\sum_a \delta(a,w) \le 2,$$

where $\delta(a,w)$ is the deficiency (at the complex point a) of meromorphic function w in the complex plane and the sum is taken for all complex values a. This implies the following amazing consequence: *deficiencies of all values a are equal to zero ($\delta(a,w)=0$) except not more that a countable set of values a.*

Below we present a result of similar nature for real functions $f(t) \in C^2[a,b]$ of one variable, which deals with the number $N_{[a,b]}\,(f=A)$ of zeros of $f-A$ (i.e. solutions of $f - A = 0$) for different real values A.

Denote

$$E_{[a,b]} := \sup_{-1\le A\le 1} \frac{1}{\pi}\int\limits_a^b \left|\frac{d}{dt}\arg\frac{d}{dt}F(f(t)-A)\right| dt$$

and denote

$$H_{[a,b]} := \int\limits_a^b ||f(t)|'_t|\,dt.$$

Notice that the last magnitude represent the total length of projections of the curve $f(t)$, $t \in [a.b]$ (in the (t,f)-plane) on the axis f.

The following result represents an analogue of Nevanlinna second fundamental theorem for real functions.

Theorem 2.1. *For an arbitrary $f(t) \in C^2[a.b]$ and an arbitrary finite set of values $A_1, A_2, ..., A_q \in [-1,1]$, $q \ge 2$, we have*

$$\sum_{\nu=1}^{q} N_{[a,b]}\,(f=A_\nu) \le E_{[a,b]} + \frac{2}{\Delta}H_{[a,b]} + 2, \tag{2.1}$$

where $\Delta = \min_{k\ne j}|A_k - A_j|$.

3. An analogue of Nevanlinna deficiency relation for real functions

Here we follow ideas of Nevanlinna theory. Assume $f(t) \in C^2[0,\infty)$, $(t \in [0,\infty))$, $E_{[a,T]} \to \infty$ and $H_{[a,T]}E^{-1}_{[a,T]} \to 0$ when $T \to \infty$. The class of similar functions denote by D. Define deficiency $\delta(A,f)$ of value A for function $f(t)$ as

$$\delta(A,f) = \liminf_{T\to\infty} \frac{N_{[0,T]}\,(f=A)}{E_{[0,T]}}.$$

Theorem 2 implies the following assertion resembling Nevanlinna deficiency relation.
Theorem 3.1 (deficiency relation for real functions). *For an arbitrary $f(t) \in D$ we have*

$$\sum_A \delta(A,f) \le 1; \tag{3.1}$$

here in the sum we count all values $A \in [-1,1]$.

From (3.1) immediately follow the following Nevanlinna type consequence:

for all values $A \in [-1,1]$ except not more than a countable set of values A we have $\delta(A,f) = 0$.

4. Proofs

Proof of Theorem 1.1. Consider the curve γ determined by $x = f(t)$, $y := \varphi(t)f(t)$, $t \in [a,b]$ and denote by t_j the set of points $a \le t_1 < t_2 ... < t_N \le b$, where $u = v = 0$. Since $f(t)$, $\varphi(t)f(t) \in C^2[a,b]$ we have at each point $t \in [a,b]$

$$\Omega(t) := [f'(t)]^2 + \left[(\varphi(t)f(t))'\right]^2 := [f'(t)]^2 + \left[(\varphi'(t)f(t)) + \varphi(t)f'(t)\right]^2.$$

Notice that $\Omega(t) > 0$ in each point $t \in (t_j, t_{j+1})$. Indeed $\Omega(t)$ can be equal to zero only when we have simultaneously $f(t) = 0$ and $f'(t) = 0$ (since we assumed that $\varphi(t) > 0$ and $\varphi'(t) > 0$). This cannot happen for $t \in (t_j, t_{j+1})$ (since due to definition $f(t) \ne 0$ in this interval). Due to well known result (see [2], theorem on page 13), $\Omega(t) > 0$ implies that the curve is locally topological for $t \in (t_j, t_{j+1})$; consequently has no singular points for any $t \in (t_j, t_{j+1})$.

Now we consider occurring in Theorem 1 complex function $F(f(t)) := x + iy := f(t) + i\varphi(t)f(t)$ which obviously represents the same curve γ. For a given point t^* denote by $\beta(t^*)$ the tangential angle at the point $t^* \in (t_j, t_{j+1})$ (that is the angle formed by the tangent to γ at the point $F(f(t^*))$ and real axis y.

Since $\beta(t) = \arg \frac{d}{dt}F(f(t))$ we have for its derivative

$$\frac{d}{dt}\arg\frac{d}{dt}F(f(t)) = \frac{d}{dt}\arctan\frac{[\varphi(t)f(t)]'}{f'(t)} =$$
$$\frac{[\varphi(t)f(t)]''\,f'(t) - f''(t))\,[\varphi(t)f(t)]'}{\Omega^*(t)}.$$

It follows that the derivative of $\beta(t)$ exists and finite for $t \in (t_j, t_{j+1})$.

Further, since for each j we have $F(f(t_j)) = 0$ we conclude that the part of the curve $F(f(t))$ corresponding to the interval (t_j, t_{j+1}) is a closed curve with continuos tangent which starts and ends at the origin of the complex plane (x, y), where $x = \mathrm{Re}F(f(t))$ and $y = \mathrm{Im}F(f(t))$. It follows that

$$\pi \leq \int\limits_{t_j}^{t_{j+1}} \left| \frac{d}{dt} \arg \frac{d}{dt} F(f(t)) \right| dt.$$

Now we assume that the number N of zeros is finite; denote N by $N_{[a,b]}\,(f = 0)$. Summing up by j we obtain that the number of all possible intervals (t_j, t_{j+1}) cannot exceed

$$\frac{1}{\pi} \sum_{j=1}^{t_N - 1} \int\limits_{t_j}^{t_{j+1}} \left| \frac{d}{dt} \arg \frac{d}{dt} F(f(t)) \right| dt \leq \frac{1}{\pi} \int\limits_{a}^{b} \left| \frac{d}{dt} \arg \frac{d}{dt} F(f(t)) \right| dt.$$

Taking into account that the number of these intervals is equal to the number of their endpoints (i.e. $N_{[a,b]}\,(f = 0)$) minus 1 we get (1) when $N_{[a,b]}\,(f = 0) < \infty$. The last two inequalities imply that (1) is true also when $N_{[a,b]}\,(f = 0) = \infty$.

To derive (1.2) (from (1.1)) it is enough to notice that for $F(f(t)) = f(t) + i(t - a)f(t)$ we have

$$\left| \frac{d}{dt} \arg \frac{d}{dt} F(f(t)) \right| dt = \frac{|f''(t)||f(t)| + 2\,[f'(t)]^2}{[f'(t)|^2 + [f(t) + (t-a)f'(t)]^2}.$$

Sharpness of the principle. Denote the positive integers by n. Take a (small) number $\varepsilon > 0$. Assume $f(t) \equiv -1$ on $(n + \varepsilon, n + 1 - \varepsilon)$ for odd n and $f(t) \equiv 1$ on $(n + \varepsilon, n + 1 - \varepsilon)$ for even n. Then we connect the point $(n + 1 - \varepsilon, -1)$ with the point $(n+1+\varepsilon, 1)$ by the segment of a straight line. Make similar connections for all integer (odd and even) points in a given segment $[a, b]$. By rounding off smuttily all similar connection points we can get a function $f_\varepsilon(t)$ defined on $[a, b]$ which belongs to $C^2[a, b]$. Take a function a monotone functions $\varphi(t)$ (very) close to 1 with $\varphi'(t)$, $\varphi''(t)$ (very) close to 0 and compose the function $F_{\varepsilon,\varphi}(f(t)) = f_\varepsilon(t) + i\varphi(t)f_\varepsilon(t)$. Notice that for any n we have exactly one point $t_n \in (n, n+1)$ where $F_{\varepsilon,\varphi}(f(t)) = 0$. Consider behavior of the curve $F_{\varepsilon,\varphi}(f(t))$, $t \in (t_n, t_{n+1})$ for odd n. We can divide the curve onto three parts: part 1 of the curve starts at the point 0, moves very closely to the segment connecting $(0, 0)$ and (1.1); part 2 of the curve make a turn around the point $(1, 1)$; part 3 of the curve returns very closely to the segment connecting $(1, 1)$ with the point $(0, 0)$.

It is easy to see that the variation of the tangential angle of the curve $F_{\varepsilon,\varphi}(f(t))$ on each interval $(n, n+1)$ tends to π when $\varepsilon \to 0$ and $\varphi(t) \to 1$. Since the angle is $\arg \frac{d}{dt} F_{\varepsilon,\varphi}(f(t))$ we obtain for each n that

$$\frac{1}{\pi} \int_{n}^{n+1} \left| \frac{d}{dt} \arg \frac{d}{dt} F_{\varepsilon,\varphi}(f(t)) \right| dt$$

tends to 1 when $\varepsilon \to 0$ and $\varphi(t) \to 1$. Now we consider function $F_{\varepsilon,\varphi}(f(t))$ on the segment $[n + n^*]$, where n^* is a positive integer. Due to above arguments we have

$$\frac{1}{\pi} \int_{n}^{n+n^*} \left| \frac{d}{dt} \arg \frac{d}{dt} F_{\varepsilon,\varphi}(f(t)) \right| dt \to n^*$$

and noticing that in each interval $(n, n+1)$ we have exactly one zero of $F_{\varepsilon,\varphi}(f(t))$ we obtain that $N_{[n,n^*]}(f = 0) = n^*$. Taking $n^* \to \infty$ we see that the ratio of the left and the right sides of (1) tends to 1; what means that The inequality (1.1) is asymptotically sharp.

Proof of theorem 2.1. By analogy with a-points in complex analysis we will refer the solutions of $f(t) = A$ in $[a, b]$ as A-points. For a given A_ν we can meet some sets (or chains) $\Psi_\tau(A_\nu)$, $(\tau = 1, 2, ..., \Phi_\tau(A_\nu))$ which we define as follows: the chain $\Psi_\tau(A_\nu)$ consists of A_ν-points $t_{1,\tau}(A_\nu)$, $t_{2,\tau}(A_\nu)$,...,$t_{K(\tau,A_\nu),\tau}(A_\nu)$ such that $K(\tau, A_\nu) \geq 2$ and $f(t)$ cannot be equal to another A_μ, $\mu \neq \nu$, on the set $[t_{1,\tau}(A_\nu), t_{K(\tau,A_\nu),\tau}(A_\nu)]$. Applying inequality (1) for function $f(t) - A_\nu$ on the set $[t_{1,\tau}(A_\nu), t_{K(\tau,A_\nu),\tau}(A_\nu)]$ we get

$$N_{[t_{1,\tau}(A_\nu), t_{K(\tau,A_\nu),\tau}(A_\nu)]}(f = A_\nu) \leq$$

$$\frac{1}{\pi} \int_{t_{1,\tau,}(A_\nu)}^{t_{K(\tau,A_\nu),\tau}(A_\nu)} \left| \frac{d}{dt} \arg \frac{d}{dt} F(f(t) - A_\nu) \right| dt + 1.$$

Denoting by $N^*_{[a,b].}(f = A_\nu)$ the total number of all A_ν-points taken for all possible chains $\Psi_\tau(A_\nu)$ and summing up the last inequality for all $\tau = 1, 2, ..., \Phi_\tau(A_\nu)$ and for all $\nu = 1, 2, ..., q$ we get

$$\sum_{\nu=1}^{q} N^*_{[a,b].}(f = A_\nu) \leq$$

$$\sum_{\nu=1}^{q} \sum_{\tau=1}^{\Phi_\tau(A_\nu)} \int_{t_{1,\tau,}(A_\nu)}^{t_{K(\tau,A_\nu),\tau}(A_\nu)} \left| \frac{d}{dt} \arg \frac{d}{dt} F(f(t) - A_\nu) \right| dt + \sum_{\nu=1}^{q} \sum_{\tau=1}^{\Phi_\tau(A_\nu)} 1.$$

Taking into account that $\cup_\nu \cup_\tau [t_{1,\tau}(A_\nu), t_{K(\tau,A_\nu),\tau}(A_\nu)] \subset [a,b]$ we get

$$\sum_{\nu=1}^{q} \sum_{\tau=1}^{\Phi_\tau(A_\nu)} \int\limits_{t_{1,\tau,}(A_\nu)}^{t_{K(\tau,A_\nu),\tau}(A_\nu)} \left| \frac{d}{dt} \arg \frac{d}{dt} F(f(t) - A_\nu) \right| dt \leq E_{[a,b]}$$

so that obtain

$$\sum_{\nu=1}^{q} N^*_{[a,b].}(f = A_\nu) \leq E_{[a,b]} + \sum_{\nu=1}^{q} \sum_{\tau=1}^{\Phi_\tau(A_\nu)} 1. \tag{4.1}$$

Then we notice that for each chain $\Psi_\tau(A_\nu)$ we have either an interval $(t^-, t_{1,\tau}(A_\nu))$, which don't involve any A_ν-point meantime $f(t^-) = A_\mu \neq A_\nu$ or we have an interval $(t_{K(\tau,A_\nu),\tau}(A_\nu), t^+)$, which don't involve any A_ν-point meantime $f(t^+) = A_\mu \neq A_\nu$. Consequently we have either $H_{[t^-, t_{1,\tau}(A_\nu)]} \geq \Delta$ or $H_{[t_{K(\tau,A_\nu),\tau}(A_\nu), t^+]} \geq \Delta$. This implies that for the number of all possible chains taken for all ν we have

$$\sum_{\nu=1}^{q} \sum_{\tau=1}^{\Phi_\tau(A_\nu)} 1 \leq \frac{1}{\Delta} H_{[a.b]} + 1. \tag{4.2}$$

Further we consider the *lonely* A_ν-points, i.e. those points t^*_j, where $f(t^*_j) = A_\nu$ and which don't involved in any chain; in other words at the closest to t^*_j point t_{j+1} (from the right side) and t_{j-1} (from the left side), where we meet another solution of $f(t) = A_\mu$ we should have $A_\mu \neq A_\nu$.

Consider the following two possibilities. (1) The next point t_{j+1} is also a lonely point (what means $t_{j+1} := t^*_{j+1}$), consequently we deal with the interval (t^*_j, t^*_{j+1}), where both these points are lonely; denote by $N^1_{[a,b].}(f = A_\nu)$ the number of all similar lonely points. (2) The next point t_{j+1} belongs to a chain, (say chain $\Psi_\tau(A_\mu)$, $\tau = 1, 2, ..., \Phi_\tau(A_\mu)$), consequently we deal with the interval $(t^*_j, t_{1,\tau}(A_\mu))$; denote by $N^2_{[a,b].}(f = A_\nu)$ the number of all similar lonely points. This means that for each of the mentioned possible lonely points we have corresponding intervals either of the type (t^*_j, t^*_{j+1}) or of the type $(t^*_j, t_{1,\tau}(A_\nu))$. For each (t^*_j, t^*_{j+1}) or of the type $(t^*_j, t_{1,\tau}(A_\mu))$ we have $H_{(t^*_j, t^*_{j+1})} \geq \Delta$ or $H_{(t^*_j, t_{1,\tau}(A_\mu))} \geq \Delta$. It follows

$$\sum_{\nu=1}^{q} N^1_{[a,b].}(f = A_\nu) + \sum_{\nu=1}^{q} N^2_{[a,b].}(f = A_\nu) \leq$$
$$\frac{1}{\Delta} \left\{ \sum H_{(t^*_j, t^*_{j+1})} + \sum H_{(t^*_j, t_{1,\tau}(A_\mu))} \right\} \leq \frac{1}{\Delta} H_{[a.b]};$$

here in the first sum we count all point t^*_j which involved in the segments $[t^*_j, t^*_{j+1}]$ and in the second sum we count all point t^*_j which involved in the segments $[t^*_j, t_{1,\tau}(A_\mu)]$.

Notice that above we considered all possible lonely points but possibly the last lonely point (if there is such a point). It follows that for the number $N^{**}_{[a,b].}(f = A_\nu)$

of all lonely A_ν-points we have

$$\sum_{\nu=1}^{q} N^{**}_{[a,b].}\,(f=A_\nu) \leq \frac{1}{\Delta} H_{[a.b]} + 1. \tag{4.3}$$

Since

$$\sum_{\nu=1}^{q} N_{[a,b]}\,(f=A_\nu) = \sum_{\nu=1}^{q} N^{*}_{[a,b].}(f=A_\nu) + \sum_{\nu=1}^{q} N^{**}_{[a,b].}\,(f=A_\nu)$$

we obtain from (4.1)-(4.3) inequality (2.1) of Theorem 2.1.

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A UNIQUENESS THEOREM FOR MULTIPLE ORTHONORMAL SPLINE SERIES

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Abstract. In this paper we obtain recovery formulas for coefficients of multiple Ciesielski series by means of its sum, if the square partial sums of a Ciesielski series converge in measure to a function f and the majorant of partial sums satisfies some necessary condition.

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1. Introduction

The uniqueness problem and reconstruction of coefficients of series by various orthogonal systems has been considered in a number of papers. Uniqueness theorems for almost everywhere convergent or summable trigonometric series were obtained in the papers [1] and [4], under some additional conditions imposed on the series. Results on uniqueness and restoration of coefficients for series by Haar and Franklin systems have been obtained, for instance, in the papers [3], [6], [7] and [11]-[14]. Here we quote a result by G. Gevorkyan [3] on restoration of coefficients of series by Franklin system.

Specifically, in [3] it was proved that if the Franklin series $\sum_{n=0}^{\infty} a_n f_n(x)$ converges a.e. to a function $f(x)$ and

$$\lim_{\lambda\to\infty}\left(\lambda\cdot\left|\{x\in[0,1]:\sup_{k\in\mathbb{N}}|S_k(x)|>\lambda\}\right|\right)=0,$$

where $|A|$ denotes the Lebesgue measure of a set A and

$$S_k(x)=\sum_{j=0}^{k}a_j f_j(x),$$

then the coefficients a_n of the Franklin series can be reconstructed by the following formula

$$a_n=\lim_{\lambda\to\infty}\int_0^1\left[f(x)\right]_\lambda f_n(x)dx,$$

where

$$\left[f(x)\right]_\lambda=\begin{cases}f(x), & \text{if } |f(x)|\le\lambda,\\ 0, & \text{if } |f(x)|>\lambda.\end{cases}$$

Similar result on uniqueness is also obtained for the Haar system (see [5]).
Afterwards Gevorkyan's result was extended by V. Kostin [12] to the series by generalized Haar system.
Consider the d-dimensional Franklin series

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}),$$

where $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}_0^d$ is a vector with non-negative integer coordinates, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbf{x} = (x_1, \cdots, x_d) \in [0,1]^d$ and

$$f_{\mathbf{n}}(\mathbf{x}) = f_{n_1}(x_1) \cdot \ldots \cdot f_{n_d}(x_d).$$

The following theorem for multiple Franklin series was proved in [7].
Theorem A.([7]) If the partial sums

$$\sigma_{2^k}(\mathbf{x}) = \sum_{\mathbf{n}: n_i \le 2^k, i=1,\cdots,d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x})$$

converge in measure to a function f and

$$\lim_{m\to\infty} \left(\lambda_m \cdot |\{\mathbf{x} \in [0,1]^d : \sup_k |\sigma_{2^k}(\mathbf{x})| > \lambda_m\}| \right) = 0$$

for some sequence $\lambda_m \to +\infty$, then for any $\mathbf{n} \in \mathbb{N}_0^d$

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_m} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}.$$

In this theorem instead of the partial sums $\sigma_{2^k}(\mathbf{x})$ one can take cubic partial sums $\sigma_{q_k}(\mathbf{x})$, where $\{q_k\}$ is any increasing sequence of natural numbers, for which the ratio q_{k+1}/q_k is bounded. The following theorem is proved in [13].
Theorem B.([13]) Let $\{q_k\}$ be an increasing sequence of natural numbers such that the ratio q_{k+1}/q_k is bounded. If the partial sums $\sigma_{q_k}(\mathbf{x})$ converge in measure to a function f and there exists a sequence $\lambda_m \to +\infty$ so that

$$\lim_{m\to\infty} \left(\lambda_m \cdot |\{\mathbf{x} \in [0,1]^d : \sup_k |\sigma_{q_k}(\mathbf{x})| > \lambda_m\}| \right) = 0,$$

then for any $\mathbf{n} \in \mathbb{N}_0^d$

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_m} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}.$$

Note that similar questions for series by Franklin system was considered by K. Keryan in [11].
In this paper we generalize the Theorem A for multiple Ciesielski series.
We are concerned with orthonormal spline systems of order k with dyadic partitions. Let $k \ge 2$ be an integer. For n in the range $-k+2 \le n \le 1$, let $\mathcal{S}_n^{(k)}$ be the space of polynomials of order not exceeding $n+k-1$ (or degree not exceeding $n+k-2$) on the interval $[0,1]$ and $\{f_n^{(k)}\}_{n=-k+2}^{1}$ be the collection of orthonormal polynomials

in $L^2 \equiv L^2[0,1]$ such that the degree of $f_n^{(k)}$ is $n+k-2$. For $n \geq 2$, let $n = 2^\nu + j$, where $\nu \geq 0$, $1 \leq j \leq 2^\nu$. Denote

$$s_{n,i} = \begin{cases} 0, & -k+1 \leq i \leq 0 \\ \frac{i}{2^{\nu+1}}, & 1 \leq i \leq 2j \\ \frac{i-j}{2^\nu}, & 2j+1 \leq i \leq n-1 \\ 1, & n \leq i \leq n+k-1, \end{cases}$$

and let $\mathcal{T}_n$ be the ordered sequence of points $s_{n,i}$. Note that $\mathcal{T}_n$ is obtained from $\mathcal{T}_{n-1}$ by adding the point $s_{n,2j-1}$. In that case, we also define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with grid points $\mathcal{T}_n$. For each $n \geq 2$, the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$ and, therefore, there exists a function $f_n^{(k)} \in \mathcal{S}_n^{(k)}$, that is orthogonal to the space $\mathcal{S}_{n-1}^{(k)}$ and $\|f_n^{(k)}\|_2 = 1$. Observe that this function $f_n^{(k)}$ is unique up to the sign.

The system of functions $\{f_n^{(k)}\}_{n=-k+2}^{\infty}$ is called the Ciesielski system of order k.

Let us note that the case $k=2$ corresponds to orthonormal systems of piecewise linear functions, i.e., the Franklin system.

Let d be a natural number. Consider the d-dimensional Ciesielski series

$$\sum_{\mathbf{n} \in \Lambda^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{n} = (n_1, \cdots, n_d) \in \Lambda^d$ is a vector with integer coordinates,
$\Lambda := \{n \in \mathbb{Z} \mid n \geq -k+1\}$, $\mathbf{x} = (x_1, \cdots, x_d) \in [0,1]^d$ and

$$f_{\mathbf{n}}(\mathbf{x}) = f_{n_1}(x_1) \cdot \ldots \cdot f_{n_d}(x_d).$$

Denote by $\sigma_{2^\mu}(\mathbf{x})$ the cubic partial sums of the series (1.1) with indices 2^μ, that is

$$\sigma_{2^\mu}(\mathbf{x}) = \sum_{\mathbf{n}: n_i \leq 2^\mu, i=1,\cdots,d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}). \tag{1.2}$$

The main result of this paper is the following theorem:

Theorem 1.1. *If the partial sums $\sigma_{2^\mu}(\boldsymbol{x})$ converge in measure to a function f and*

$$\lim_{q \to \infty} \left(\lambda_q \cdot |\{\boldsymbol{x} \in [0,1]^d : \sup_\mu |\sigma_{2^\mu}(\boldsymbol{x})| > \lambda_q\}| \right) = 0 \tag{1.3}$$

for some sequence $\lambda_q \to +\infty$, then for any $\boldsymbol{n} \in \Lambda^d$

$$a_{\boldsymbol{n}} = \lim_{q \to \infty} \int_{[0,1]^d} \left[f(\boldsymbol{x}) \right]_{\lambda_q} f_{\boldsymbol{n}}(\boldsymbol{x}) d\boldsymbol{x}. \tag{1.4}$$

2. Properties of B-spline functions and auxiliary lemmas

We define the functions $(N_{n,i})_{i=-k+1}^{n-1}$ to be the collection of L^∞-normalized B-spline functions of order k corresponding to the partition $\mathcal{T}_n$. The functions $(N_{n,i})_{i=-k+1}^{n-1}$ form a basis for $\mathcal{S}_n^{(k)}$. Those functions are non-negative and are normalized in such a way that they form a partition of unity, i.e.,

$$N_{n,i}(x) \geq 0 \text{ and } \sum_{i=-k+1}^{n-1} N_{n,i}(x) = 1 \text{ for all } x \in [0,1].$$

Moreover

$$\delta_{n,i} := \operatorname{supp} N_{n,i} = [s_{n,i}, s_{n,i+k}] \text{ and } \int_0^1 N_{n,i}(x)dx = \frac{|\delta_{n,i}|}{k}.$$

The L^1-normalized B-spline functions $M_{n,i}$ in $\mathcal{S}_n^{(k)}$ are given by the formula

$$M_{n,i}(x) = \frac{k}{|\delta_{n,i}|} N_{n,i}(x),$$

and satisfy the inequalities

$$0 \leq M_{n,i}(x) \leq \frac{k}{|\delta_{n,i}|}, \quad x \in [0,1].$$

Let $n = 2^\mu + j$, with $\mu \geq 0$, $1 \leq j \leq 2^\mu$. Clearly we have that $N_{n-1,i}(x) = N_{n,i}(x)$, if $-k+1 \leq i \leq 2j-k-2$ and $N_{n-1,i}(x) = N_{n,i+1}(x)$, if $2j-1 \leq i \leq n-2$. Böhm formula (see [15]) gives us the following relationship between the B-splines $N_{n,i}$ and $N_{n-1,i}$, if $2j-k-1 \leq i \leq 2j-2$

$$N_{n-1,i}(x) = a_{n,i} N_{n,i}(x) + (1 - a_{n,i+1}) N_{n,i+1}(x). \tag{2.1}$$

Later we shall mostly deal with the $n = 2^\mu$, so let us introduce the following notation

$$N_i^{(\mu)}(x) := N_{2^\mu,i}(x), \quad M_i^{(\mu)}(x) := M_{2^\mu,i}(x), \quad \delta_i^{(\mu)} := \delta_{2^\mu,i}.$$

For any natural μ we set

$$\Lambda_\mu := \{-k+1, \cdots, 2^\mu\}.$$

It is clear that

$$\sigma_{2^\mu}(\mathbf{x}) = \sum_{\mathbf{n} \in \Lambda_\mu^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}).$$

For any vector $\mathbf{i} = (i_1, \cdots, i_d) \in \Lambda_\mu^d$ denote

$$\Delta_{\mathbf{i}}^{(\mu)} := \delta_{i_1}^{(\mu)} \times \cdots \times \delta_{i_d}^{(\mu)},$$

$$N_{\mathbf{i}}^{(\mu)}(\mathbf{x}) = N_{\mathbf{i}}^{(\mu)}(x_1, \cdots, x_d) = N_{i_1}^{(\mu)}(x_1) \cdot \ldots \cdot N_{i_d}^{(\mu)}(x_d).$$

Obviously

$$\operatorname{supp}(N_{\mathbf{i}}^{(\mu)}) = \Delta_{\mathbf{i}}^{(\mu)}.$$

Let us notice that

$$\int_{[0,1]^d} N_{\mathbf{i}}^{(\mu)}(\mathbf{x})d\mathbf{x} = \int_{\Delta_{\mathbf{i}}^{(\mu)}} N_{\mathbf{i}}^{(\mu)}(\mathbf{x})d\mathbf{x} = \prod_{j=1}^{d} \int_{\delta_{i_j}^{(\mu)}} N_{i_j}^{(\mu)}(x_j)dx_j = \prod_{j=1}^{d} \frac{|\delta_{i_j}^{(\mu)}|}{k} = \frac{|\Delta_{\mathbf{i}}^{(\mu)}|}{k^d}.$$

Hence for $M_{\mathbf{i}}^{(\mu)}(\mathbf{x})$ we have

$$0 \le M_{\mathbf{i}}^{(\mu)}(\mathbf{x}) \le \frac{k^d}{|\Delta_{\mathbf{i}}^{(\mu)}|}, \quad \mathbf{x} \in [0,1]^d.$$

To prove Theorem 1 we will need the following two lemmas.

Lemma 2.1. *Let $P_k(\mathbf{x})$ be a polynomial of degree k defined on $\Delta := [a_1, b_1] \times \cdots \times [a_d, b_d]$, $d \in \mathbb{N}$, then*

$$\left| \left\{ \mathbf{x} \in \Delta : \ \ |P_k(\mathbf{x})| \ge \frac{\max_{\mathbf{x}\in\Delta} |P_k(\mathbf{x})|}{2^d} \right\} \right| \ge \frac{|\Delta|}{(4k^2)^d}.$$

This lemma is a generalization of Corollary 3.1 from [9] and the proof of one dimensional case can be found in [8].

Proof. The proof will be carried out by induction. The case $d = 1$ coincides with Corollary 3.1 ([9]). Suppose that lemma is valid for dimension d, and let us prove it for dimension $d + 1$.

Let the function $P_k(\mathbf{x})$ be defined on $\Delta := [a_1, b_1] \times \cdots \times [a_d, b_d]$ and let $|P_k(\mathbf{x})|$ attains its greatest value at the point $(\alpha_1, \cdots, \alpha_{d+1})$. The function $P_k(\alpha_1, \cdots, \alpha_d, x)$, $x \in [a_{d+1}, b_{d+1}]$, satisfies the assumptions of Corollary 3.1 from [9]. Therefore

$$\left| \left\{ x \in [a_{d+1}, b_{d+1}] : |P_k(\alpha_1, \cdots, \alpha_d, x)| \ge \frac{1}{2} \cdot \max_{\mathbf{x}\in\Delta} |P_k(\mathbf{x})| \right\} \right| \ge \frac{b_{d+1} - a_{d+1}}{4k^2}. \tag{2.2}$$

For a fixed $x \in [a_{d+1}, b_{d+1}]$, the function

$$P_k(x_1, \cdots, x_d, x), \ \ (x_1, \cdots, x_d) \in [a_1, b_1] \times \cdots \times [a_d, b_d]$$

satisfies the induction assumption. Therefore

$$\left| \left\{ (x_1, \cdots, x_d) : x_i \in [a_i, b_i], |P_k(x_1, \cdots, x_d, x)| \ge \frac{|P_k(\alpha_1, \cdots, \alpha_d, x)|}{2^d} \right\} \right| \ge \frac{(b_1 - a_1) \cdots (b_d - a_d)}{(4k^2)^d}. \tag{2.3}$$

It follows from relations (2.2) and (2.3) that

$$\left| \left\{ (x_1, \cdots, x_d, x_{d+1}) : x_i \in [a_i, b_i], |P_k(x_1, \cdots, x_d, x_{d+1})| \ge \frac{\max_{\mathbf{x}\in\Delta} |P_k(\mathbf{x})|}{2^{d+1}} \right\} \right| \ge \frac{(b_1 - a_1) \cdots (b_d - a_d)(b_{d+1} - a_{d+1})}{(4k^2)^{d+1}}.$$

The proof is complete. □

Repeatedly using Böhm formula (2.1) one can proof the following lemma (see [9]), which is the generalization of Lemma 2 from [8].

Lemma 2.2. ([9]) *There exist* $\alpha_{ij}^{(\mu)} \geq 0$ *so that*

$$M_i^{(\mu)}(x) = \sum_{j=-k+1}^{2^{\mu+1}-1} \alpha_{ij} M_j^{(\mu+1)}(x), \quad \textit{with} \quad \alpha_{ij} > 0 \quad \textit{iff} \quad \delta_j^{(\mu+1)} \subset \delta_i^{(\mu)}.$$

Lemma 2.3. *There exist* $\alpha_{\mathbf{j}} \geq 0$ *so that*

$$M_{\mathbf{i}}^{(\mu)}(\mathbf{x}) = \sum_{\mathbf{j} \in \Lambda_\mu^d} \alpha_{\mathbf{j}} M_{\mathbf{j}}^{(\mu+1)}(\mathbf{x}), \quad \textit{with} \quad \alpha_{\mathbf{j}} > 0 \quad \textit{iff} \quad \Delta_{\mathbf{j}}^{(\mu+1)} \subset \Delta_{\mathbf{i}}^{(\mu)}.$$

This lemma is the generalization of the previous lemma (for d-dimensional case).

3. The proof of the main theorem

Let the partial sums (1.2) converge in measure to a function f and the series (1.1) satisfy the condition (1.3). First let's prove that for an arbitrary μ_0 and $\mathbf{i}_0 \in \Lambda_{\mu_0}^d$, the following statement is true:

$$\int_{[0,1]^d} \sigma_{2^{\mu_0}}(\mathbf{x}) M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x} = \lim_{q\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_q} M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x}. \tag{3.1}$$

Denote

$$E_q := \{\mathbf{x} \in \mathrm{supp}(M_{\mathbf{i}_0}^{(\mu_0)}) = \Delta_{\mathbf{i}_0}^{(\mu_0)} : \sup_\mu |\sigma_{2^\mu}(\mathbf{x})| > \lambda_q\}.$$

Let ε be an arbitrary positive number. Under the conditions of the theorem, one can take the natural number q_0 such that the following inequalities hold:

$$2^{5d} \cdot 2^{\mu_0 d} \cdot k^{2d} \cdot \lambda_q \cdot |E_q| < \varepsilon, \quad \text{when} \quad q \geq q_0, \tag{3.2}$$

and

$$|E_q| < \frac{1}{2^{3d} \cdot k^{3d}} \cdot |\Delta_{\mathbf{i}_0}^{(\mu_0)}|, \quad \text{when} \quad q \geq q_0. \tag{3.3}$$

Suppose $\mu \geq \mu_0$. We set

$$\Omega_\mu := \left\{ A : A = \left[\frac{i_1}{2^\mu}, \frac{i_1+1}{2^\mu}\right] \times \cdots \times \left[\frac{i_d}{2^\mu}, \frac{i_d+1}{2^\mu}\right], \quad A \subset \mathrm{supp}(M_{\mathbf{i}_0}^{(\mu_0)}) \right\}.$$

Notice, that if for some $A \in \Omega_\mu$, $\mu \geq \mu_0$, the inequality

$$|E_q \cap A| < \frac{1}{2 \cdot 4^d \cdot k^{2d}} \cdot |A| \tag{3.4}$$

holds, then

$$|\sigma_{2^\mu}(\mathbf{x})| \leq 2^d \lambda_q, \quad \text{for} \quad \mathbf{x} \in A. \tag{3.5}$$

Let suppose that $A \in \Omega_\mu$ and for some point $\mathbf{x}' \in A$ the inequality (3.5) does not hold, i.e.

$$|\sigma_{2^\mu}(\mathbf{x}')| > 2^d \lambda_q.$$

According to the Lemma (2.1), we obtain that

$$\left|\left\{\mathbf{x}\in A:|\sigma_{2^\mu}(\mathbf{x})|>\lambda_q\right\}\right|\geq\frac{|A|}{4^d\cdot k^{2d}},$$

which contradicts (3.4). From (3.3) we have

$$|E_q\cap A|<\frac{1}{2^{3d}\cdot k^{3d}}\cdot|\Delta_{\mathbf{i}_0}^{(\mu_0)}|=\frac{1}{2^{3d}\cdot k^{2d}}\cdot|A|,\quad\text{when}\quad A\in\Omega_{\mu_0}. \tag{3.6}$$

Now let's define by induction the families Ω_μ^1 and Ω_μ^2, $\mu\geq\mu_0$. If $\mu=\mu_0$, then we set

$$\Omega_{\mu_0}^1:=\left\{A\in\Omega_{\mu_0}:|A\cap E_q|>\frac{1}{2^{3d}\cdot k^{2d}}\cdot|A|\right\},\quad Q_{\mu_0}:=\bigcup_{A\in\Omega_{\mu_0}^1}A,$$

and

$$\Omega_{\mu_0}^2:=\{A\in\Omega_{\mu_0}:A\not\subset Q_{\mu_0}\},\quad P_{\mu_0}:=\bigcup_{A\in\Omega_{\mu_0}^2}A.$$

From (3.6) we have, that $Q_{\mu_0}=\emptyset$ and the closure of P_{μ_0} is the $\operatorname{supp}(M_{\mathbf{i}_0}^{(\mu_0)})$. Now suppose we have defined the sets $\Omega_{\mu'}^1$, $\Omega_{\mu'}^2$, $Q_{\mu'}$ and $P_{\mu'}$ for all $\mu'<\mu$.
Let's denote

$$\Omega_\mu^1:=\left\{A\in\Omega_\mu:|A\cap E_q|>\frac{1}{2^{3d}\cdot k^{2d}}\cdot|A|\ \text{ and }\ A\not\subset\bigcup_{\mu'<\mu}Q_{\mu'}\right\}, \tag{3.7}$$

$$Q_\mu:=\bigcup_{A\in\Omega_\mu^1}A,\quad\Omega_\mu^2:=\left\{A\in\Omega_\mu:A\not\subset\bigcup_{\mu'\leq\mu}Q_{\mu'}\right\},\quad P_\mu:=\bigcup_{A\in\Omega_\mu^2}A.$$

Thus we have defined the families $\Omega_\mu^1,\Omega_\mu^2$ and the sets Q_μ,P_μ, satisfying to the following conditions

$$\Omega_\mu^1\subset\Omega_\mu,\quad\Omega_\mu^2\subset\Omega_\mu,$$

$$\operatorname{supp}(M_{\mathbf{i}_0}^{(\mu_0)})=P_\mu\cup\left(\bigcup_{\mu'\leq\mu}Q_{\mu'}\right),\quad P_\mu\cap\left(\bigcup_{\mu'\leq\mu}Q_{\mu'}\right)=\emptyset, \tag{3.8}$$

$$Q_{\mu'}\cap Q_{\mu''}=\emptyset,\quad\text{if}\quad\mu'\neq\mu''. \tag{3.9}$$

Next, it follows from (3.7) and (3.9) that

$$\left|\bigcup_{\mu'\leq\mu}Q_{\mu'}\right|<2^{3d}\cdot k^{2d}\cdot|E_q|,\quad\text{for any}\quad\mu\geq\mu_0. \tag{3.10}$$

For any $\mu>\mu_0$ denote

$$I_\mu=\{\mathbf{i}\in\Lambda_\mu^d:\ \ \Delta_{\mathbf{i}}^{(\mu)}\cap Q_\mu\neq\emptyset\ \text{ and }\ \Delta_{\mathbf{i}}^{(\mu)}\subset P_{\mu-1}\},$$

and observe that if $\mathbf{i}=(i_1,\cdots,i_d)\in I_\mu$ then, for any set B, $B\subset\Delta_{\mathbf{i}}^{(\mu)}$, $B\in\Omega_\mu$ the following inequality holds:

$$|E_q\cap B|<\frac{1}{4^d\cdot k^{2d}}\cdot|B|. \tag{3.11}$$

Indeed, if for some B the inequality (3.11) is not satisfied, then for a cube $D \in \Omega_{\mu-1}$, with $B \subset D$, we would have

$$|E_q \cap D| \geq \frac{1}{2^d \cdot 4^d \cdot k^{2d}} \cdot |D|, \tag{3.12}$$

because

$$|D| = 2^d \cdot |B|.$$

Then, it follows from (3.12) that

$$D \subset \bigcup_{\mu'<\mu} Q_{\mu'}, \quad \text{therefore} \quad B \subset \bigcup_{\mu'<\mu} Q_{\mu'} \quad \text{and} \quad \Delta_{\mathbf{i}}^{(\mu)} \cap \left(\bigcup_{\mu'<\mu} Q_{\mu'} \right) \neq \emptyset,$$

which contradicts the condition $\Delta_{\mathbf{i}}^{(\mu)} \subset P_{\mu-1}$ and the relation (3.12).
Therefore

$$|\sigma_{2^\mu}(\mathbf{x})| \leq 2^d \cdot \lambda_q, \quad \text{if} \quad \mathbf{x} \in \Delta_{\mathbf{i}}^{(\mu)}, \mathbf{i} \in I_\mu. \tag{3.13}$$

Similarly, we can obtain that if $\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu$, then

$$\left| E_q \cap \Delta_{\mathbf{i}}^{(\mu)} \right| \leq \frac{1}{2^{3d} \cdot k^{2d}} \cdot |\Delta_{\mathbf{i}}^{(\mu)}|, \tag{3.14}$$

therefore

$$|\sigma_{2^\mu}(\mathbf{x})| \leq 2^d \cdot \lambda_q, \quad \text{if} \quad \mathbf{x} \in \Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu. \tag{3.15}$$

Now by induction we define expansions ψ_μ for $M_{\mathbf{i}_0}^{(\mu_0)}$, satisfying the conditions:

$$M_{\mathbf{i}_0}^{(\mu_0)} = \psi_\mu = \sum_{n=\mu_0}^{\mu} \sum_{\mathbf{i} \in I_\mu} \alpha_{\mathbf{i}}^{(n)} M_{\mathbf{i}}^{(n)} + \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)}, \tag{3.16}$$

where

$$\sum_{n=\mu_0}^{\mu} \sum_{\mathbf{i} \in I_\mu} \alpha_{\mathbf{i}}^{(n)} + \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} = 1, \quad \alpha_{\mathbf{i}}^{(n)} \geq 0, \ \beta_{\mathbf{i}}^{(\mu)} \geq 0. \tag{3.17}$$

Since $P_{\mu_0} = \operatorname{supp}(M_{\mathbf{i}_0}^{(\mu_0)})$, then $\psi_{\mu_0} = M_{\mathbf{i}_0}^{(\mu_0)}$. Assuming that ψ_μ, satisfying the conditions (3.16), (3.17), is already defined, we define $\psi_{\mu+1}$. By Lemma (2.3), we have

$$M_{\mathbf{i}}^{(\mu)} = \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu+1)} \subset \operatorname{supp} M_{\mathbf{i}}^{(\mu)}} \gamma_{\mathbf{i}}^{(\mu+1)} M_{\mathbf{i}}^{(\mu+1)}, \quad \gamma_{\mathbf{i}}^{(\mu+1)} \geq 0. \tag{3.18}$$

Inserting the expressions (3.18) in (3.16) and grouping similar terms, we obtain

$$M_{\mathbf{i}_0}^{(\mu_0)} = \psi_{\mu+1} = \sum_{n=\mu_0}^{\mu+1} \sum_{\mathbf{i} \in I_\mu} \alpha_{\mathbf{i}}^{(n)} M_{\mathbf{i}}^{(n)} + \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu+1)} \subset P_{\mu+1}} \alpha_{\mathbf{i}}^{(\mu+1)} M_{\mathbf{i}}^{(\mu+1)}. \tag{3.19}$$

Since the integrals of all functions $M_{\mathbf{i}}^{(\mu)}$ are equal to one, from (3.19) we obtain

$$\sum_{n=\mu_0}^{\mu+1}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}+\sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu+1)}\subset P_{\mu+1}}\alpha_{\mathbf{i}}^{(\mu+1)}=1,\quad \alpha_{\mathbf{i}}^{(n)}\geq 0,\alpha_{\mathbf{i}}^{(\mu+1)}\geq 0.$$

Thus, the possibility of representation (3.16) with coefficients satisfying (3.17), is proved.

Suppose we are given a number $\mu\geq\mu_0$ and $\mathbf{p}=(p_1,\cdots,p_d)$ such that $\max_i\{p_i\}>2^\mu$. Then, according to the definition of functions $f_{\mathbf{p}}$ and $M_{\mathbf{i}}^{\mu}$, we get

$$(f_{\mathbf{p}},M_{\mathbf{i}}^{(\mu)})=\int_{[0,1]^d}f_{\mathbf{p}}(\mathbf{x})M_{\mathbf{i}}^{(\mu)}(\mathbf{x})d\mathbf{x}=0,\quad \text{for any }\ \mathbf{i}\in\Lambda_\mu^d.$$

Therefore, for any $n\geq\mu$ and for all $\mathbf{i}\in\Lambda_\mu^d$ one can write

$$(\sigma_{2^n},M_{\mathbf{i}}^{(\mu)})=\sum_{\mathbf{p}\in\Lambda_n^d}a_{\mathbf{p}}(f_{\mathbf{p}},M_{\mathbf{i}}^{(\mu)})=\sum_{\mathbf{p}\in\Lambda_\mu^d}a_{\mathbf{p}}(f_{\mathbf{p}},M_{\mathbf{i}}^{(\mu)})=(\sigma_{2^\mu},M_{\mathbf{i}}^{(\mu)}).$$

Taking into account (3.16), for $n>\mu_0$ we can write

$$(\sigma_{2^{\mu_0}},M_{\mathbf{i}_0}^{(\mu_0)})=\int_{[0,1]^d}\sigma_{2^{\mu_0}}(\mathbf{x})M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x})d\mathbf{x}=\int_{\Delta_{\mathbf{i}_0}^{(\mu_0)}}\sigma_{2^\mu}(\mathbf{x})M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x})d\mathbf{x}$$

$$=\sum_{n=\mu_0}^{\mu}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}(\sigma_{2^n},M_{\mathbf{i}}^{(n)})+\sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)}\subset P_\mu}\beta_{\mathbf{i}}^{(\mu)}(\sigma_{2^\mu},M_{\mathbf{i}}^{(\mu)})=:I_{\mu,1}+I_{\mu,2}.$$

For $I_{\mu,1}$ we will have the inequality

$$|I_{\mu,1}|\leq\sum_{n=\mu_0}^{\mu}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}|(\sigma_{2^n},M_{\mathbf{i}}^{(n)})|\leq 2^d\lambda_q\sum_{n=\mu_0}^{\mu}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}\int_{\Delta_{\mathbf{i}}^{(n)}}M_{\mathbf{i}}^{(n)}(\mathbf{x})d\mathbf{x}.$$

Denote

$$D_\mu=\bigcup_{n=\mu_0}^{\mu}\bigcup_{\mathbf{i}\in I_\mu}\Delta_{\mathbf{i}}^{(n)}. \tag{3.20}$$

From the definition of the set I_μ, it follows that

$$|\Delta_{\mathbf{i}}^{(n)}\cap Q_\mu|\geq k^{-d}\Delta_{\mathbf{i}}^{(n)}.$$

The last relation and (3.9), (3.10) imply

$$|D_\mu|\leq k^d\cdot 2^{3d}\cdot k^{2d}\cdot|E_q|. \tag{3.21}$$

We obtain

$$\sum_{n=\mu_0}^{\mu}\sum_{\mathbf{i}\in I_\mu}\alpha_{\mathbf{i}}^{(n)}\int_{\Delta_{\mathbf{i}}^{(n)}}M_{\mathbf{i}}^{(n)}(\mathbf{x})d\mathbf{x}\leq\int_{D_\mu}M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x})d\mathbf{x}\leq||M_{\mathbf{i}_0}^{(\mu_0)}||_\infty|D_\mu|. \tag{3.22}$$

It is clear that

$$|I_{\mu,1}|\leq 2^d\cdot\lambda_q\cdot||M_{\mathbf{i}_0}^{(\mu_0)}||_\infty|D_\mu|. \tag{3.23}$$

Hence, from (3.23), (3.21) and (3.2), we obtain

$$|I_{\mu,1}| \le 2^d \cdot \lambda_q \cdot \frac{k^d}{|\Delta_{\mathbf{i}_0}^{(\mu_0)}|} \cdot k^d \cdot 2^{3d} \cdot k^{2d} \cdot |E_q| \le \varepsilon \cdot \left(\frac{k}{2}\right)^d . \tag{3.24}$$

For $I_{\mu,2}$ we will have the representation

$$I_{\mu,2} = (\sigma_{2^\mu}, \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)}) = (\sigma_{2^\mu} - \left[f\right]_{\lambda_q}, \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)})$$

$$+(\left[f\right]_{\lambda_q}, \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)}) =: I_{\mu,3} + I_{\mu,4}.$$

Denote

$$H_\mu = \bigcup_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \Delta_{\mathbf{i}}^{(\mu)} \quad \text{and} \quad T_q = \{\mathbf{x} \in \Delta_{\mathbf{i}_0}^{(\mu_0)} : \quad |f(\mathbf{x})| > \lambda_q\}.$$

It is clear that $T_q \subset E_q$, therefore $|T_q| < |E_q|$. From (3.2) we get

$$|T_q| < \frac{\varepsilon}{2^{5d} \cdot 2^{\mu_0 d} \cdot k^{2d} \cdot \lambda_q}. \tag{3.25}$$

Next from (3.15) we have $|\sigma_{2^\mu}(\mathbf{x})| \le 2^d \cdot \lambda_q$, for $\mathbf{x} \in H_\mu$, and hence

$$|\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}| \le (2^d + 1) \cdot \lambda_q, \quad \text{for} \quad \mathbf{x} \in H_\mu. \tag{3.26}$$

It is clear that

$$|I_{\mu,3}| \le (|\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}|, \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{(\mu)} \subset P_\mu} \beta_{\mathbf{i}}^{(\mu)} M_{\mathbf{i}}^{(\mu)}) \tag{3.27}$$

$$\le \int\limits_{H_\mu} |\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}| M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x} \le 2^{\mu_0 d} \int\limits_{H_\mu} |\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}| d\mathbf{x}$$

$$= 2^{\mu_0 d} \left(\int\limits_{H_\mu \setminus T_q} |\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}| d\mathbf{x} + \int\limits_{H_\mu \cap T_q} |\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}| d\mathbf{x} \right).$$

From (3.25) and (3.26) for the second integral on the right-hand side of (3.27), we have

$$2^{\mu_0 d} \int\limits_{H_\mu \cap T_q} |\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}| d\mathbf{x} \le \frac{\varepsilon}{2^{5d}}.$$

From (3.26) we have that the $|\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}|$ is bounded on H_μ and it tends to zero in measure outside the set T_q, since

$$|\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}| = |\sigma_{2^\mu}(\mathbf{x}) - f(\mathbf{x})| \ \text{ on } \ T_q^c.$$

Hence

$$\int\limits_{H_\mu \setminus T_q} |\sigma_{2^\mu}(\mathbf{x}) - \left[f(\mathbf{x})\right]_{\lambda_q}| d\mathbf{x} \xrightarrow[\mu \to \infty]{} 0.$$

Therefore, for sufficiently large μ we have

$$|I_{\mu,3}| < \frac{\varepsilon}{2}. \tag{3.28}$$

For $I_{\mu,4}$, from (3.16) we have

$$I_{\mu,4} = ([f]_{\lambda_q}, M_{\mathbf{i}_0}^{(\mu_0)}) - ([f]_{\lambda_q}, \sum_{n=\mu_0}^{\mu} \sum_{\mathbf{i}\in I_\mu} \alpha_{\mathbf{i}}^{(n)} M_{\mathbf{i}}^{(n)}) \tag{3.29}$$

$$= ([f]_{\lambda_q}, M_{\mathbf{i}_0}^{(\mu_0)}) + I_{\mu,5}.$$

The relations (3.2),(3.21),(3.22) imply that

$$|I_{\mu,5}| \le \varepsilon \cdot \left(\frac{k}{2}\right)^d. \tag{3.30}$$

Therefore by (3.24), (3.28), (3.29), (3.30) we get

$$\left| \int_{[0,1]^d} \sigma_{2^{\mu_0}}(\mathbf{x}) M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x} - \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_q} M_{\mathbf{i}_0}^{(\mu_0)}(\mathbf{x}) d\mathbf{x} \right| < C_{k,d} \cdot \varepsilon \quad \text{for} \quad q \ge q_0.$$

Now let's prove that for any $\mathbf{n} \in \Lambda^d$ the coefficient $a_{\mathbf{n}}$ can be reconstructed by (1.4). First let's fix a number μ satisfying $\max_{1\le i\le d} n_i \le 2^\mu$. Since $f_{\mathbf{n}} \in \mathcal{S}_{2^\mu}$ and the system of functions $\{M_{\mathbf{i}}^{(\mu)}\}_{\mathbf{i}\in\Lambda_\mu^d}$ is a basis in the space $\mathcal{S}_{2^\mu}$, then one can find numbers $\beta_{\mathbf{i}}$, $\mathbf{i} \in \Lambda_\mu^d$, such that

$$f_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{i}\in\Lambda_\mu^d} \beta_{\mathbf{i}} M_{\mathbf{i}}^{(\mu)}(\mathbf{x}).$$

Therefore

$$a_{\mathbf{n}} = (\sigma_{2^\mu}, f_{\mathbf{n}}) = \sum_{\mathbf{i}\in\Lambda_\mu^d} \beta_{\mathbf{i}} (\sigma_{2^\mu}, M_{\mathbf{i}}^{(\mu)}) = \sum_{\mathbf{i}\in\Lambda_\mu^d} \beta_{\mathbf{i}} \lim_{q\to\infty} \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_q} M_{\mathbf{i}}^{(\mu)}(\mathbf{x}) d\mathbf{x}$$

$$= \lim_{q\to\infty} \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_q} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x},$$

which finishes the proof of Theorem 1.1.

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О РАЗРЕШИМОСТИ ОДНОГО КЛАССА МНОГОМЕРНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ В МАТЕМАТИЧЕСКОЙ ТЕОРИИ ГЕОГРАФИЧЕСКОГО РАСПРОСТРАНЕНИЯ ЭПИДЕМИИ

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Аннотация. Исследуется многомерное интегральное уравнение типа свертки с вогнутой нелинейностью. Указанное уравнение возникает в математической теории географического распространения эпидемии. Сочетание известных методов многомерных операторов и методов построения инвариантных конусных отрезков для таких операторов с методами теории интегральных операторов типа свертки и предельных теорем теории функций позволяют доказать существование положительных ограниченных решений для таких уравнений. Также изучается асимптотическое поведение построенных решений. В конкретно выбранном конусном отрезке доказывается также единственность решения. Приводятся конкретные прикладные примеры указанных уравнений.

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Ключевые слова: эпидемия; вогнутость; сходимость; единственность; последовательные приближения.

1. Введение

Рассмотрим следующий класс многомерных нелинейных интегральных уравнений на множестве $(-\infty, T] \times \mathbb{R}^n$:

$$u(t, \mathrm{x}) = \int\limits_{-\infty}^{t} \int\limits_{\mathbb{R}^n} A(t-\tau, \mathrm{x}, \mathrm{y}) g(u(\tau, \mathrm{y})) dy d\tau, \quad t \in (-\infty, T], \quad \mathrm{x} \in \mathbb{R}^n \tag{1.1}$$

относительно искомой функции $u(t, \mathrm{x})$.

Уравнение (1.1) имеет непосредственное применение в математической теории географического распространения эпидемии, где

$$S(t, \mathrm{x}) = S_0 e^{-u(t, \mathrm{x})}, \quad S_0 = const \tag{1.2}$$

[1]Исследование выполнено за счет гранта Российского научного фонда (проект No. 19-11-00223)

представляет собой плотность восприимчивых лиц в момент времени t в точке $\mathrm{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ (см. [1]-[4]) .

Функция $A(\tau, \mathrm{x}, \mathrm{y})$ имеет вероятностный смысл: $A(\tau, \mathrm{x}, \mathrm{y}) d\tau d\mathrm{y}$ представляет собой вероятность того, что восприимчивый человек в точке $\mathrm{x} \in \mathbb{R}^n$ приобретает инфекцию от инфицированных лиц, находящихся в параллелепипеде $(\mathrm{y}, \mathrm{y} + d\mathrm{y})$, $\mathrm{y} \in \mathbb{R}^n$ и зараженных в момент времени из интервала $(\tau - d\tau, \tau)$.

Соответствующее одномерное уравнение исследовалось в работах [1]-[3] в случае, когда функция A допускает следующее представление:

$$A(\tau, x, y) = H(\tau) v(x - y), \quad x, y \in \mathbb{R}, \tag{1.3}$$

где

$$H(\tau) \geq 0,\ \tau \in [0, +\infty),\ \int_0^\infty H(\tau) d\tau = 1,\ \int_0^\infty \tau H(\tau) d\tau < +\infty, \tag{1.4}$$

$$v(-x) = v(x) \geq 0, \quad x \in [0, +\infty),\ \int_{-\infty}^\infty v(x) dx = 1 \tag{1.5}$$

и функция v имеет конечный момент определенного порядка.

В частности, в работе [1] при дополнительных ограничениях на H, v и g построены волновые фронты для следующего одномерного (по координате) нелинейного интегрального уравнения:

$$u(t, x) = \int_{-\infty}^t H(t - \tau) \int_{\mathbb{R}} v(x - y) g(u(\tau, y)) dy d\tau, \quad (t, x) \in (-\infty, T] \times \mathbb{R}. \tag{1.6}$$

В работе [3] построены знакопеременные монотонные (и по времени, и по координате) и ограниченные решения для уравнения (1.6). В работах [1]-[2] особое внимание уделено случаю, когда функция $g(u)$ допускает следующее представление:

$$g(u) = \gamma(1 - e^{-u}), \quad u \geq 0, \quad \gamma > 1, \tag{1.7}$$

где γ — числовой параметр. Условие $\gamma > 1$ называется пороговым условием. Последнее означает, что при $\gamma \leq 1$ невозможно остановить инфекцию (см. [1]-[2]).

В многомерном (n-мерном) случае когда, функция $A(\tau, \mathrm{x}, \mathrm{y})$ представляется в следующем виде:

$$A(\tau, \mathrm{x}, \mathrm{y}) = H(\tau) V(\mathrm{x} - \mathrm{y}) \lambda_0(\mathrm{x}, \mathrm{y}), \tag{1.8}$$

где

I)

$$V(\mathrm{z}) \geq 0,\ \mathrm{z} := (z_1, z_2, \dots, z_n) \in \mathbb{R}^n,\ \int\limits_{\mathbb{R}^n} V(\mathrm{z})d\mathrm{z} = 1, \tag{1.9}$$

II) $V(\mathrm{z})$ — четная функция по каждому аргументу, т.е. $V(\mathrm{z}) = V(|\mathrm{z}_1|, |\mathrm{z}_2|, \dots, |\mathrm{z}_n|)$,

III) $V \in C_M(\mathbb{R}^n)$ и сходятся интегралы: $\int\limits_0^\infty uT_j(u)du < +\infty,\ j = 1, 2, 3, \dots, n$, где

$$\begin{gathered} T_1(u) := \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} V(u, z_2, \dots, z_n)dz_2 \dots dz_n, \\ T_2(u) := \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} V(z_1, u, z_3, \dots, z_n)dz_1 dz_3 \dots dz_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ T_n(u) := \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} V(z_1, \dots, z_{n-1}, u)dz_1 \dots dz_{n-1}, \end{gathered} \tag{1.10}$$

а $C_M(\mathbb{R}^n)$ — пространство непрерывных и ограниченных функций на $\mathbb{R}^n$,

IV) $V \downarrow [0, +\infty)$ по z_j на $j = 1, 2, \dots, n$,

A) $1 \geq \lambda_0(\mathrm{x}, \mathrm{y}) \geq \max\{\mu_1(|y_1|), \mu_2(|y_2|), \dots, \mu_n(|y_n|)\},\ \mathrm{x}, \mathrm{y} \in \mathbb{R}^n$,
$0 < \delta_j \leq \mu_j(u) \leq 1,\ u \in [0, +\infty), \quad \mu_j(u) \uparrow [0, +\infty)$ по u ,
$\lim\limits_{u \to +\infty} \mu_j(u) = 1, \quad 1 - \mu_j \in L_1(0, +\infty), \quad j = 1, 2, \dots, n$,

B) $\lambda_0 \in C(\mathbb{R}^{2n}), \quad \lambda_0(\mathrm{x}, \mathrm{y}) \not\equiv 1, \quad \mathrm{x}, \mathrm{y} \in \mathbb{R}^n$,

уравнение (1.1) изучалось в недавней работе А.Г. Сергеева и автора (см. [4]).

Заметим, что в том случае, когда $\lambda_0(\mathrm{x}, \mathrm{y}) \equiv 1$ (консервативный случай) и функция $g(u)$ допускает представление (1.7), уравнение (1.1) с ядром (1.8) обладает двумя тривиальными (вакуумными) решениями:

$$u^1 \equiv 0, \quad u^2 \equiv \eta,$$

где число η является положительным корнем уравнения

$$\gamma(1 - e^{-u}) = u.$$

Как известно, вопрос о построении нетривиальных и ограниченных решений между вакуумами $u^1 \equiv 0$ и $u^2 \equiv \eta$ в n-мерном случае ($n > 1$), когда в представлении (1.8) функция $\lambda_0 \equiv 1$ до сих пор оставался открытым (см. [1]-[3]).

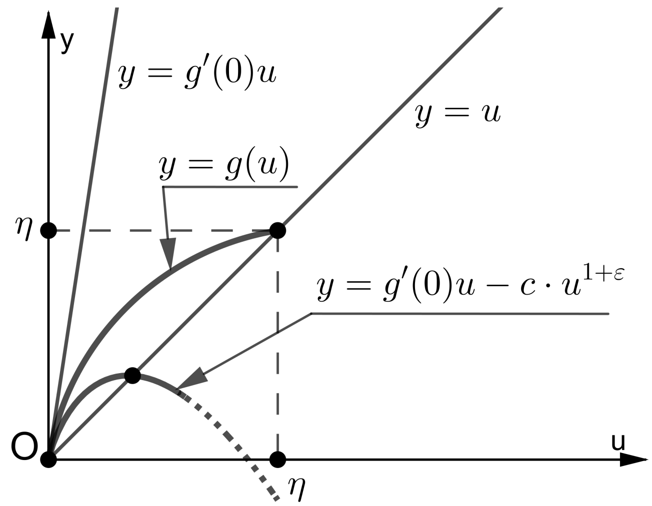

Рис. 1

Настоящая работа посвящена исследованию нелинейного n-мерного интегрального уравнения (1.1) в случае, когда

$$A(\tau, \mathrm{x}, \mathrm{y}) = H(\tau)V(\mathrm{x}-\mathrm{y}) \quad \mathrm{x} \in \mathbb{R}^n,\ \mathrm{y} \in \mathbb{R}^n, \tag{1.11}$$

где функции H и V удовлетворяют соответственно условиям (1.4) и $I) - III)$. Предполагается, что $g(u)$ — вогнутая на некотором отрезке $[0, \eta]$ функция, удовлетворяющая следующим условиям (см. рис. 1):

a) существует производная $1 < g'(0) < +\infty$ такая, что

$$g(u) \le g'(0)u, \quad u \in [0, \eta],$$

b) $g(u) \uparrow$ по u на $[0, \eta]$, $g(0) = 0$, $g(\eta) = \eta$,

c) существуют числа $\varepsilon > 0$ и $c > 0$ такие, что

$$g(u) \ge g'(0)u - cu^{1+\varepsilon}, \quad u \in [0, \eta].$$

В настоящей работе мы займемся построением ограниченного и нетривиального решения между вакуумами 0 и η для уравнения (1.1) с ядром (1.11), а также исследованием некоторых качественных свойств построенного решения. В конкретно выбранном конусном отрезке докажем также единственность решения. В конце приведем конкретные прикладные примеры уравнения (1.1), для которых выполняются все условия сформулированных теорем. Следует отметить, что из доказанных результатов, как частный случай, получается теорема О. Дикмана (см. [1], теорема 6.1) о волновых фронтах для одномерного уравнения (1.6).

2. Обозначения и вспомогательные факты

2.1. **Функция Дикмана.** Рассмотрим следующие функции Дикмана (см. [1]):

$$L_j(\lambda) := g'(0) \int\limits_{-\infty}^{\infty} \widetilde{T}_j(x) e^{-\lambda x} dx, \quad \lambda \in [0, +\infty), \quad j = 1, 2, \ldots, n, \tag{2.1}$$

где

$$\widetilde{T}_j(x) := \int\limits_0^{\infty} H(p) T_j(x - \theta_j p) dp, \quad x \in \mathbb{R}, \quad j = 1, 2, \ldots, n. \tag{2.2}$$

Здесь $\theta_j > 0$, $j = 1, 2, \ldots, n$ — числовые параметры (играют роль волновых скоростей (см. [1])), а функции $\{T_j\}_{j=1}^n$ задаются согласно формулам (1.10).
Ниже перечислим некоторые основные свойства ядерных функций $\{T_j(x)\}_{j=1}^n$, $\{\widetilde{T}_j\}_{j=1}^n$:

$$T_j(-x) = T_j(x),\ x \in [0, +\infty),\ T_j(u) \geq 0,\ u \in \mathbb{R},\ \int\limits_{-\infty}^{\infty} T_j(u) du = 1, \tag{2.3}$$

$$\widetilde{T}_j(u) \geq 0,\ u \in \mathbb{R},\ \int\limits_{-\infty}^{\infty} \widetilde{T}_j(u) du = 1,\ j = 1, 2, \ldots, n. \tag{2.4}$$

Эти свойства сразу следуют из представлений (1.10) и (2.2) с учетом условий I) и II). Из (2.3) и условия III) следует также, что

$$\int\limits_{-\infty}^{\infty} u T_j(u) du = 0, \quad j = 1, 2, \ldots, n. \tag{2.5}$$

Заметим, что в силу (2.4), (2.5) и условий $a), I), III)$, (1.4) имеют место следующие соотношения:

$$L_j(0) = g'(0) \int\limits_{-\infty}^{\infty} \widetilde{T}_j(x) dx > \int\limits_{-\infty}^{\infty} \widetilde{T}_j(x) dx = 1,$$

$$\frac{dL_j(0)}{d\lambda} = -g'(0) \int\limits_{-\infty}^{\infty} \widetilde{T}_j(x) x dx < 0,$$

ибо

$$\int\limits_{-\infty}^{\infty} \widetilde{T}_j(x) x dx = \int\limits_{-\infty}^{\infty} x \int\limits_0^{\infty} H(p) T_j(x - \theta_j p) dp dx =$$

$$= \int_0^\infty H(p) \int_{-\infty}^\infty (\theta_j p + \tau) T_j(\tau) d\tau dp = \theta_j \int_0^\infty p H(p) dp > 0, \quad j = 1, 2, \ldots, n.$$

Очевидно также, что

$$\frac{d^2 L_j(\lambda)}{d\lambda^2} = g'(0) \int_{-\infty}^\infty x^2 \widetilde{T}_j(x) e^{-\lambda x} dx > 0, \text{ (может быть и } +\infty), \quad j = 1, 2, \ldots, n.$$

Из этих соображений немедленно следует, что

- $L_j(\lambda) \downarrow$ по λ в некоторых окрестностях нуля $[0, r_j]$, $j = 1, 2, \ldots, n$ соответственно,
- функции $\{L_j(\lambda)\}_{j=1}^n$ выпуклы (вниз) на $[0, +\infty)$ (см. рис. 2).

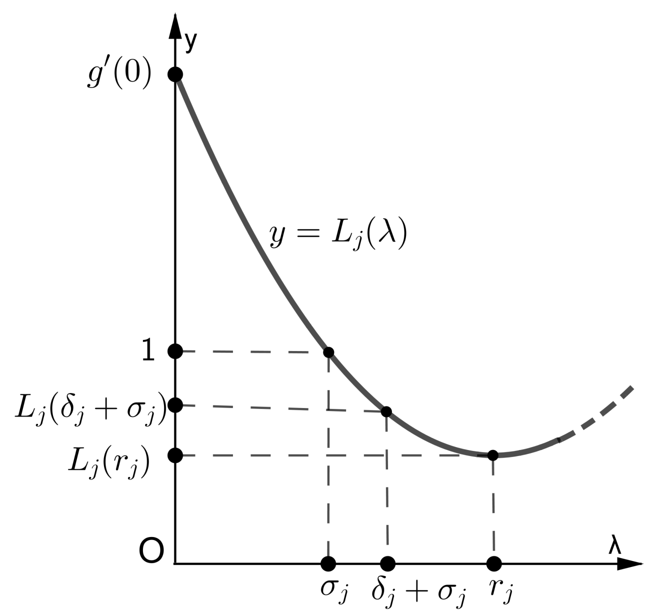

Рис. 2

В дальнейшем, если не будет оговорено противное, будем считать, что

$$L_j(r_j) < 1, \quad j = 1, 2, \ldots, n. \tag{2.6}$$

Тогда согласно теореме Больцано-Коши, существуют числа $\sigma_j \in (0, r_j)$ такие, что

$$L_j(\sigma_j) = 1, \quad j = 1, 2, \ldots, n. \tag{2.7}$$

Из монотонности функций $\{L_j(\lambda)\}_{j=1}^n$ следует, что числа $\{\sigma_j(\lambda)\}_{j=1}^n$ определяются единственным образом.

Учитывая монотонность функций $\{L_j(\lambda)\}_{j=1}^n$, а также формулы (2.6) и (2.7) можем утверждать, что для всех $\delta_j \in (0, r_j - \sigma_j)$ имеют место неравенства

$$L_j(\delta_j + \sigma_j) < 1, \quad j = 1, 2, \ldots, n. \tag{2.8}$$

Рассмотрим следующие вспомогательные функции (см. [1]):

$$\mathcal{L}_j(x) := \max\{\eta e^{\sigma_j x} - M e^{(\delta_j + \sigma_j)x}, 0\}, \quad x \in \mathbb{R}, \quad j = 1, 2, \ldots, n, \tag{2.9}$$

где

$$M > \eta, \quad \delta_j \in (0, \min\{r_j - \sigma_j, \varepsilon\sigma_j\}), \quad j = 1, 2, \ldots, n, \tag{2.10}$$

— числовые параметры.

Из определения функций $\{\mathcal{L}_j(x)\}_{j=1}^n$ сразу следует, что

$$\mathcal{L}_j(x) = 0 \text{ при } x \geq \frac{1}{\delta_j} \ln \frac{\eta}{M}, \quad j = 1, 2, \ldots, n, \tag{2.11}$$

$$\mathcal{L}_j^{1+\varepsilon}(x) \leq \eta^{1+\varepsilon} e^{(\delta_j + \sigma_j)x}, \quad x \in \mathbb{R}, \quad j = 1, 2, \ldots, n. \tag{2.12}$$

2.2. **Последовательные приближения для одномерных вспомогательных уравнений.** Наряду с уравнением (1.1) рассмотрим следующие одномерные нелинейные интегральные уравнения типа свертки на всей прямой:

$$\Phi_j(x) = \int_{-\infty}^{\infty} \widetilde{T}_j(x - t) g(\Phi_j(t)) dt, \quad x \in \mathbb{R}, \quad j = 1, 2, \ldots, n, \tag{2.13}$$

относительно искомых функций $\{\Phi_j(x)\}_{j=1}^n$, где ядра $\{\widetilde{T}_j(x)\}_{j=1}^n$ задаются согласно формуле (2.2). Рассмотрим следующие последовательные приближения Дикмана (см. [1]) для уравнений (2.13):

$$\Phi_j^{(m+1)}(x) = \int_{-\infty}^{\infty} \widetilde{T}_j(x - t) g(\Phi_j^{(m)}(t)) dt, \quad m = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, n,$$

где в качестве нулевого приближения берутся следующие функции:

$$\Phi_j^{(0)}(x) = \begin{cases} \eta, & x \geq 0, \\ \eta e^{\sigma_j x}, & x < 0, \ x \in \mathbb{R}, \ j = 1, 2, \ldots, n. \end{cases} \tag{2.14}$$

Повторяя аналогичные рассуждения как в работе [1], можно убедиться в достоверности следующих утверждений:

1) $\Phi_j^{(m)}(x) \downarrow$ по m, $\quad j = 1, 2, \ldots, n$,
2) $\Phi_j^{(m)}(x) \uparrow$ по x на $\mathbb{R}$, $\quad m = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, n$,

3) $\Phi_j^{(m)} \in C(\mathbb{R}), \quad m = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, n,$

4) при $\delta_j \in (0, \min\{r_j - \sigma_j, \varepsilon\sigma_j\})$ и

$$M > \max\left\{\eta, \max_{1\le j\le n}\left\{\frac{c\eta^{1+\varepsilon}L_j(\delta_j+\sigma_j)}{g\prime(0)(1-L_j(\delta_j+\sigma_j))}\right\}\right\}$$

имеют место следующие неравенства:

$$\Phi_j^{(m)}(x) \ge \mathcal{L}_j(x), \quad x \in \mathbb{R},\ j = 1, 2, \ldots, n,\ m = 0, 1, 2, \ldots, n.$$

Таким образом, из 1)−4) следует, что последовательность непрерывных функций $\{\Phi_j^{(m)}(x)\}_{m=0}^{\infty}$, $j = 1, 2, \ldots, n$ имеет поточечный предел при $m \to \infty$: $\lim\limits_{m\to\infty}\Phi_j^{(m)}(x) = \Phi_j(x)$, $j = 1, 2, \ldots, n$. Согласно предельной теореме Б. Леви (см. [5]) функции $\{\Phi_j(x)\}_{j=1}^{n}$ являются решениями уравнений (2.13), причем из 1), 2) и 4) следует, что

$$\mathcal{L}_j(x) \le \Phi_j(x) \le \Phi_j^{(0)}(x), \quad x \in \mathbb{R}, \quad j = 1, 2, \ldots, n, \tag{2.15}$$

$$\Phi_j(x) \uparrow \ \text{ по } x \ \text{ на } \mathbb{R}, \quad j = 1, 2, \ldots, n. \tag{2.16}$$

Так как свертка суммируемой и ограниченной функций является непрерывной функцией на $\mathbb{R}$ (см. [6]), то в силу (2.15), (2.4) из (2.13) заключаем, что

$$\Phi_j \in C(\mathbb{R}), \quad j = 1, 2, \ldots, n. \tag{2.17}$$

Из (2.15) и (2.9) следует, что

$$\lim_{x\to-\infty}\Phi_j(x) = 0, \quad \Phi_j \in L_1(-\infty, 0), \quad j = 1, 2, \ldots, n. \tag{2.18}$$

Теперь найдем пределы функций $\Phi_j(x)$, $j = 1, 2, \ldots, n$, когда $x \to +\infty$.

В силу (2.15)–(2.17) можем утверждать, что существуют

$$\lim_{x\to+\infty}\Phi_j(x) = l_j \le \eta,$$

причем $l_j > 0$, $j = 1, 2, \ldots, n$.

После перехода к пределу в обеих частях (2.13), когда $x \to +\infty$, с учетом известного предельного соотношения в операции свертки (см. [7]) получим

$$l_j = g(l_j), \quad j = 1, 2, \ldots, n,\ l_j \in (0, \eta_j].$$

В силу условий $a) - c)$ последнее возможно только тогда, когда $l_j = \eta$, $j = 1, 2, \ldots, n$. Итак, мы получили, что

$$\lim_{x\to+\infty}\Phi_j(x) = \eta, \quad j = 1, 2, \ldots, n. \tag{2.19}$$

Совершая аналогичные рассуждения как в доказательстве леммы 2 из работы [8], можно убедиться, что

$$\eta - \Phi_j \in L_1(0, +\infty), \quad j = 1, 2, \dots, n. \tag{2.20}$$

3. Разрешимость уравнения (1.1) с ядром (1.11)

3.1. **Последовательные приближения для уравнения** (1.1) **с ядром** (1.11)**.** Рассмотрим следующие итерации для (1.1) с ядром (1.11):

(3.1)

$$u_{m+1}(t, x_1, \dots, x_n) =$$
$$= \int\limits_{-\infty}^{t} H(t-\tau) \int\limits_{\mathbb{R}^n} V(x_1 - y_1, \dots, x_n - y_n) g(u_m(\tau, y_1, \dots, y_n)) dy_1 \dots dy_n d\tau,$$
$$u_0(t, x_1, \dots, x_n) = \begin{cases} \frac{\eta}{n} \sum\limits_{j=1}^{n} e^{\sigma_j (x_j + \theta_j t)}, & \text{при } x_i + \theta_i t < 0, \quad i = 1, 2, \dots, n \\ \eta, & \text{при остальных} \quad (t, x_1, ..., x_n). \end{cases}$$

Сперва докажем, что

$$0 \le u_m(t, x_1, \dots, x_n) \downarrow \text{ по } m. \tag{3.2}$$

Действительно, учитывая тот факт, что

$$g'(0) \int\limits_{\mathbb{R}} \widetilde{T}_j(x - z) \Phi_j^{(0)}(z) dz \le \Phi_j^{(0)}(x), \quad x \in \mathbb{R} \setminus \mathbb{R}^+, \quad j = 1, 2, \dots, n \tag{3.3}$$

(доказательство (3.3) осуществляется прямой проверкой, рассматривая случаи $z \ge 0$ и $z < 0$), очевидное неравенство $0 \le u_0(t, x_1, \dots, x_n) \le \eta$, с учетом условия a) из (3.1) для $x_j + \theta_j t < 0$, $j = 1, 2, \dots, n$ будем иметь

$$u_1(t, x_1, \dots, x_n) \le \frac{g'(0)}{n} \int\limits_{-\infty}^{t} H(t-\tau) \times$$
$$\int\limits_{\mathbb{R}^n} V(x_1 - y_1, \dots, x_n - y_n)(\Phi_1^{(0)}(y_1 + \theta_1 \tau) + \dots + \Phi_n^{(0)}(y_n + \theta_n \tau)) dy_1 \dots dy_n d\tau =$$
$$= \frac{g'(0)}{n} \int\limits_{-\infty}^{t} H(t-\tau) \int\limits_{\mathbb{R}} T_1(x_1 - y_1) \Phi_1^{(0)}(y_1 + \theta_1 \tau) dy_1 d\tau +$$
$$\dots + \frac{g'(0)}{n} \int\limits_{-\infty}^{t} H(t-\tau) \int\limits_{\mathbb{R}} T_n(x_n - y_n) \Phi_n^{(0)}(y_n + \theta_n \tau) dy_n d\tau =$$
$$= \frac{g'(0)}{n} \int\limits_{\mathbb{R}} \Phi_1^{(0)}(z_1) \int\limits_{-\infty}^{t} H(t-\tau) T_1(x_1 + \theta_1 \tau - z_1) d\tau dz_1 +$$

$$\ldots + \frac{g'(0)}{n}\int\limits_{\mathbb{R}}\Phi_n^{(0)}(z_n)\int\limits_{-\infty}^{t}H(t-\tau)T_n(x_n+\theta_n\tau-z_n)d\tau dz_n =$$

$$= \frac{g'(0)}{n}\int\limits_{\mathbb{R}}\Phi_1^{(0)}(z_1)\widetilde{T}_1(x_1+\theta_1 t-z_1)dz_1+\ldots+\frac{g'(0)}{n}\int\limits_{\mathbb{R}}\Phi_n^{(0)}(z_n)\widetilde{T}_n(x_n+\theta_n t-z_n)dz_n \leq$$

$$\leq \frac{1}{n}(\Phi_1^{(0)}(x_1+\theta_1 t)+\ldots+\Phi_n^{(0)}(x_n+\theta_n t)) = u_0(t,x_1,\ldots,x_n).$$

Неотрицательность функции $u_1(t,x_1,\ldots,x_n)$ сразу следует из неотрицательности функций H, V и $g(u_0)$.

Предполагая, что $u_m(t,x_1,\ldots,x_n)\geq 0$ и $u_m(t,x_1,\ldots,x_n)\leq u_{m-1}(t,x_1,\ldots,x_n)$ при некотором $m\in\mathbb{N}$, из монотонности функции g и неотрицательности ядерных функций H и V будем иметь

$$u_{m+1}(t,x_1,\ldots,x_n)\leq\int\limits_{-\infty}^{t}H(t-\tau)\times$$

$$\int\limits_{\mathbb{R}^n}V(x_1-y_1,\ldots,x_n-y_n)g(u_{m-1}(\tau,y_1,\ldots,y_n))dy_1\ldots dy_n d\tau = u_m(t,x_1,\ldots,x_n).$$

Теперь докажем, что

$$u_m(t,x_1,x_2,\ldots,x_n)\geq\frac{1}{n}\sum_{j=1}^{n}\Phi_j(x_j+\theta_j t),\quad m=0,1,2,\ldots, \tag{3.4}$$

где функции $\{\tilde{\Phi_j}\}_{j=1}^n$ являются решениями уравнений (2.13) и обладают свойствами (2.15), (2.16), (2.17), (2.18) и (2.19), (2.20).

Неравенство (3.4) очевидным образом выполняется при $m=0$, ибо имеет место (2.15). Предполагая, что (3.4) выполняется при некотором натуральном $m\in\mathbb{N}$ и учитывая неравенство Иенсена для вогнутой функции g, в силу (2.2), (2.13) из (3.1) получим

$$u_{m+1}(t,x_1,\ldots,x_n)\geq\int\limits_{-\infty}^{t}H(t-\tau)\times$$

$$\int\limits_{\mathbb{R}^n}V(x_1-y_1,\ldots x_n-y_n)g\left(\frac{1}{n}\sum_{j=1}^{n}\Phi_j(y_j+\theta_j\tau)\right)dy_1\ldots dy_n d\tau\geq$$

$$\geq\frac{1}{n}\int\limits_{-\infty}^{t}H(t-\tau)\int\limits_{\mathbb{R}^n}V(x_1-y_1,\ldots x_n-y_n)g\left(\Phi_1(y_1+\theta_1\tau)\right)dy_1\ldots dy_n d\tau+$$

$$+\frac{1}{n}\int\limits_{-\infty}^{t} H(t-\tau)\int\limits_{\mathbb{R}^n} V(x_1-y_1,\dots x_n-y_n) g\left(\Phi_2(y_2+\theta_2\tau)\right) dy_1\dots dy_n d\tau+$$

$$+\dots+\frac{1}{n}\int\limits_{-\infty}^{t} H(t-\tau)\int\limits_{\mathbb{R}^n} V(x_1-y_1,\dots x_n-y_n) g\left(\Phi_n(y_n+\theta_n\tau)\right) dy_1\dots dy_n d\tau=$$

$$=\frac{1}{n}\int\limits_{-\infty}^{t} H(t-\tau)\int\limits_{\mathbb{R}} T_1(x_1-y_1)) g\left(\Phi_1(y_1+\theta_1\tau)\right) dy_1 d\tau+$$

$$+\frac{1}{n}\int\limits_{-\infty}^{t} H(t-\tau)\int\limits_{\mathbb{R}} T_2(x_2-y_2)) g\left(\Phi_2(y_2+\theta_2\tau)\right) dy_2 d\tau+\dots+$$

$$+\frac{1}{n}\int\limits_{-\infty}^{t} H(t-\tau)\int\limits_{\mathbb{R}} T_n(x_n-y_1)) g\left(\Phi_n(y_n+\theta_n\tau)\right) dy_n d\tau=$$

$$=\frac{1}{n}\int\limits_{\mathbb{R}} \tilde{T}_1(x_1+\theta_1 t-z_1) g(\Phi_1(z_1)) dz_1 d\tau+\dots+\frac{1}{n}\int\limits_{\mathbb{R}} \tilde{T}_n(x_n+\theta_n t-z_n) g(\Phi_n(z_n)) dz_n d\tau=$$

$$=\frac{1}{n}\sum_{j=1}^{n}\Phi_j(x_j+\theta_j t).$$

Итак, из (3.2) и (3.4) получаем поточечную сходимость последовательности $\{u_m(t,x_1,\dots,x_n)\}_{m=0}^{\infty}:\ \lim\limits_{m\to\infty} u_m(t,x_1,\dots,x_n)=u(t,x_1,\dots,x_n)$, причем предельная функция $u(t,x_1,\dots,x_n)$ согласно теореме Б. Леви удовлетворяет уравнению (1.1) с ядром (1.11). Из (3.2) и (3.4) следует также двусторонняя оценка для построенного решения $u:$

$$\frac{1}{n}\sum_{j=1}^{n}\Phi_j(x_j+\theta_j t)\le u(t,x_1,\dots,x_n)\le u_0(T,x_1,\dots,x_n),\quad t\in(-\infty,T],\quad (x_1,x_2\dots,x_n)\in\mathbb{R}^n. \tag{3.5}$$

Так как $u\in M((-\infty,T]\times\mathbb{R}^n),\ H\in L_1(\mathbb{R}^+)$ и $V\in L_1(\mathbb{R}^n)$ (здесь $M((-\infty,T]\times\mathbb{R}^n)$ — пространство ограниченных функций на множестве $(-\infty,T]\times\mathbb{R}^n$), то из (1.1) и (1.11), в силу непрерывности сверки суммируемых и ограниченных функций следует, что

$$u\in C((-\infty,T]\times\mathbb{R}^n). \tag{3.6}$$

В (3.5) вместо переменной t взяв $t=T$, получим

$$\frac{1}{n}\sum_{j=1}^{n}\Phi_j(x_j+\theta_j T)\le u(T,x_1,\dots,x_n)\le u_0(T,x_1,\dots,x_n),$$

из чего в силу (2.19) и (2.14) следует, что

$$\lim_{x_1\to+\infty}\lim_{x_2\to+\infty}\dots\lim_{x_n\to+\infty}u(T,x_1,\dots,x_n)=\eta. \tag{3.7}$$

Так как $t\in(-\infty,T]$, $T<+\infty$, то из (2.18) и (2.14) следует, что

$$\lim_{x_1\to-\infty}\lim_{x_2\to-\infty}\dots\lim_{x_n\to-\infty}u(t,x_1,\dots,x_n)=0. \tag{3.8}$$

В силу (3.7) и (3.8) из (2.18) и (2.20) можем утверждать также, что

(3.9)
$$\eta-u(T,+\infty,\dots,+\infty,x_j,+\infty,\dots,+\infty)\in L_1(0,+\infty),$$
$$u(t,-\infty,\dots,-\infty,x_j,-\infty,\dots,-\infty)\in L_1(-\infty,0),\ \ t\in(-\infty,T],\ \ j=1,\dots,n.$$

Итак, на основе вышеизложенного справедлива следующая

Теорема 3.1. *Пусть выполняются условия* (1.4), *$I)-III)$,* (2.6) *и $a)-c)$. Тогда уравнение* (1.1) *с ядром* (1.11) *обладает положительным нетривиальным ограниченным и непрерывным на $(-\infty,T]\times\mathbb{R}^n$ решением $u(t,x_1,\dots,x_n)$. Более того, данное решение обладает дополнительными свойствами* (3.5), (3.7)–(3.9).

Замечание 3.1. *Следует отметить, что теорема 1 дополняет теорему существования, доказанной в работе* [4]. *Дело в том, что в работе* [4] *функция λ_0 удовлетворяет сильному ограничению: $\lambda_0\not\equiv 1$.*

Замечание 3.2. *Нетрудно убедиться, что всевозможные сдвиги решения $u(t, x_1,\dots,x_n)$ (по переменным $(x_1,\dots,x_n)$) также являются решениями уравнения* (1.1) *с ядром* (1.11). *Действительно, из* (1.1) *и* (1.11) *с помощью замены переменной будем иметь*

$$\int\limits_{-\infty}^{t}H(t-\tau)\int\limits_{\mathbb{R}^n}V(x_1-z_1,x_2-z_2,\dots,x_n-z_n)g(u(\tau,z_1+\beta_1,\dots,z_n+\beta_n))dz_1\dots dz_nd\tau=$$
$$=\int\limits_{-\infty}^{t}H(t-\tau)\int\limits_{\mathbb{R}^n}V(x_1+\beta_1-y_1,x_2+\beta_2-y_2,\dots,x_n+\beta_n-y_n)\times$$
$$\times g(u(\tau,y_1,\dots,y_n))dy_1\dots dy_nd\tau=u(t,x_1+\beta_1,\dots,x_n+\beta_n).$$

Таким образом, мы получаем n-параметрическое семейство решений вида

$$u_{\beta_1,\dots,\beta_n}(t,x_1,\dots,x_n):=u(t,x_1+\beta_1,\dots,x_n+\beta_n),$$

где $u(t,x_1,\dots,x_n)$ — основное решение уравнения (1.1) *с ядром* (1.11), *построенное при помощи последовательных приближений* (3.1), *а $\beta_1,\dots,\beta_n$ — произвольные вещественные параметры.*

3.2. **Единственность решения в определенном конусном отрезке.** Возникает естественный вопрос: является ли единственным решение $u(t, \mathrm{x})$ уравнения (1.1) с ядром (1.11) в конусном отрезке

$$\left[\frac{1}{n}\sum_{j=1}^{n}\Phi_j(x_j+\theta_j t),\, u_0(t, x_1, ..., x_n)\right]$$

Ответ на этот вопрос дает следующая теорема:

Теорема 3.2. *При условиях теоремы 1 уравнение* (1.1) *с ядром* (1.11) *не может иметь более одного решения в следующем классе непрерывных функций:*

$$\mathfrak{M} := \{u \in C((-\infty, T]\times\mathbb{R}^n) :$$

$$\left.\frac{1}{n}\sum_{j=1}^{n}\Phi_j(x_j+\theta_j t) \le u(t, x_1, \dots, x_n) \le u_0(t, x_1, \dots, x_n)\right\}$$

Доказательство. Предположим обратное: уравнение (1.1) с ядром (1.11) обладает двумя решениями u, $\tilde{u} \in \mathfrak{M}$. Сперва, учитывая следующие простые неравенства:

$$\mathcal{L}_j(x) \ge \eta e^{\sigma_j x} - M e^{(\delta_j+\sigma_j)x},\ \ \mathcal{L}_j(x) \ge 0,\ x \in \mathbb{R},\ j = 1, 2, \dots, n,$$

оценим разность

$$\left(\Phi_j^{(0)}(x_j+\theta_j t) - \Phi_j(x_j+\theta_j t)\right) e^{-(x_j+\theta_j t)(\delta_j+\sigma_j)} \le$$

$$\le \begin{cases} \eta e^{-(x_j+\theta_j t)\delta_j} - \eta e^{-(x_j+\theta_j t)\delta_j} + M, & \text{если } x_j+\theta_j t \le \frac{1}{\delta_j}\ln\frac{\eta}{M}, \\ \eta e^{-(x_j+\theta_j t)(\delta_j+\sigma_j)}, & \text{если } x_j+\theta_j t > \frac{1}{\delta_j}\ln\frac{\eta}{M} \end{cases} \le$$

$$\le \begin{cases} M, & \text{если } x_j+\theta_j t \le \frac{1}{\delta_j}\ln\frac{\eta}{M}, \\ \eta e^{\frac{-(\delta_j+\sigma_j)}{\delta_j}\ln\frac{\eta}{M}}, & \text{если } x_j+\theta_j t > \frac{1}{\delta_j}\ln\frac{\eta}{M} \end{cases} \le \eta \max_{1\le j\le n}\left(\frac{M}{\eta}\right)^{1+\frac{\sigma_j}{\delta_j}} := \Lambda,$$

ибо

$$\eta \max_{1\le j\le n}\left(\frac{M}{\eta}\right)^{1+\frac{\sigma_j}{\delta_j}} \ge M.$$

Итак, мы получили следующую априорную оценку сверху:

$$\begin{gathered}\left(\Phi_j^{(0)}(x_j+\theta_j t) - \Phi_j(x_j+\theta_j t)\right) e^{-(x_j+\theta_j t)(\delta_j+\sigma_j)} \le \Lambda < +\infty, \\ x_j \in \mathbb{R},\ \theta_j > 0,\ t \in (-\infty, T],\ \ j = 1, 2, \dots, n. \end{gathered} \tag{3.10}$$

Так как u, $\tilde{u} \in \mathfrak{M}$, то в силу (3.10) имеем

$$|u(t, x_1, \dots, x_n) - \tilde{u}(t, x_1 \dots, x_n)| \le \frac{1}{n}\Lambda\sum_{j=1}^{n} e^{(x_j+\theta_j t)(\delta_j+\sigma_j)},$$

из которого следует, что

$$\alpha := \sup_{(t,x_1,\ldots,x_n)\in(-\infty,T]\times\mathbb{R}^n} \left\{ \left[\sum_{j=1}^{n} e^{(x_j+\theta_j t)(\delta_j+\sigma_j)} \right]^{-1} \times \right. \tag{3.11}$$
$$\left. |u(t,x_1,\ldots,x_n) - \tilde{u}(t,x_1,\ldots,x_n)| \right\} \le \frac{\Lambda}{n} < +\infty.$$

С другой стороны, поскольку конусной отрезок $[0,\eta]$ в себе содержит

$$\left[\frac{1}{n}\sum_{j=1}^{n} \Phi_j(x_j+\theta_j t),\, u_0(x_1,...,x_n) \right]$$

и функция g обладает свойствами $a) - c)$ (см. рис. 3), то для $u,\, \tilde{u} \in \mathfrak{M}$ имеет место следующее неравенство:

$$|g(u) - g(\tilde{u})| \le g^{'}(0)|u-\tilde{u}| \tag{3.12}$$

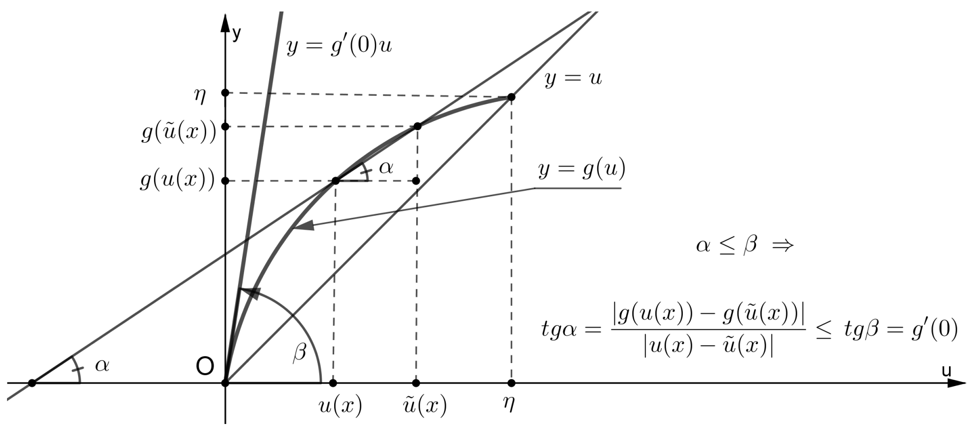

Рис. 3

Таким образом, учитывая (3.11), (3.12), из (1.1) и (1.11), будем иметь

$$|u(t,x_1,\ldots,x_n) - \tilde{u}(t,x_1,\ldots,x_n)| \le g^{'}(0) \int_{-\infty}^{t} H(t-\tau) \int_{\mathbb{R}^n} V(\mathrm{x}-\mathrm{y})|u(\tau,\mathrm{y}) - \tilde{u}(\tau,\mathrm{y})| d\mathrm{y} d\tau \le$$

$$\le \alpha g^{'}(0) \int_{-\infty}^{t} H(t-\tau) \int_{\mathbb{R}^n} V(x_1-y_1,\ldots,x_n-y_n) \sum_{j=1}^{n} e^{(y_j+\theta_j\tau)(\delta_j+\sigma_j)} dy_1\ldots dy_n d\tau =$$

$$= \alpha g^{'}(0) \sum_{j=1}^{n} \int_{-\infty}^{t} H(t-\tau) \int_{\mathbb{R}} T_j(x_j-y_j) e^{(\delta_j+\sigma_j)(y_j+\theta_j\tau)} dy_j d\tau =$$

$$= \alpha g^{'}(0)\sum_{j=1}^{n}\int\limits_{-\infty}^{t} H(t-\tau)\int\limits_{\mathbb{R}} T_j(x_j-z_j+\theta_j\tau)e^{(\delta_j+\sigma_j)z_j}dz_jd\tau =$$

$$= \alpha g^{'}(0)\sum_{j=1}^{n}\int\limits_{\mathbb{R}} e^{(\delta_j+\sigma_j)z_j}\int\limits_{-\infty}^{t} H(t-\tau)T_j(x_j-z_j+\theta_j\tau)d\tau dz_j =$$

$$= \alpha g^{'}(0)\sum_{j=1}^{n}\int\limits_{\mathbb{R}} e^{(\delta_j+\sigma_j)z_j}\int\limits_{0}^{\infty} H(p)T_j(x_j+\theta_jt-z_j-\theta_jp)dpdz_j =$$

$$= \alpha g^{'}(0)\sum_{j=1}^{n}\int\limits_{\mathbb{R}} \tilde{T}(x_j+\theta_jt-z_j)e^{(\delta_j+\sigma_j)z_j}dz_j =$$

$$= \alpha g^{'}(0)\sum_{j=1}^{n}\int\limits_{\mathbb{R}} \tilde{T}(l_j)e^{(\delta_j+\sigma_j)(x_j+\theta_jt)}e^{-(\delta_j+\sigma_j)l_j}dl_j =$$

$$= \alpha\sum_{j=1}^{n} e^{(\delta_j+\sigma_j)(x_j+\theta_jt)}L_j(\delta_j+\sigma_j) \le \alpha\max_{1\le j\le n}L_j(\delta_j+\sigma_j)\sum_{j=1}^{n}e^{(\delta_j+\sigma_j)(x_j+\theta_jt)}.$$

Из полученной оценки следует, что

$$\alpha \le \alpha\max_{1\le j\le n}L_j(\delta_j+\sigma_j). \tag{3.13}$$

Так как $\delta_j\in(0,\min\{\varepsilon\sigma_j, r_j-\sigma_j\})$ для всех $j=1,2,\dots,n$, то

$$\rho := \max_{1\le j\le n}L_j(\delta_j+\sigma_j) < 1. \tag{3.14}$$

Из (3.13) и (3.14) получаем, что $\alpha=0$. Следовательно,

$$u(t,x_1,\dots,x_n)=\tilde{u}(t,x_1,\dots,x_n),\ \ t\in(-\infty,T],\ \ (x_1,\dots,x_n)\in\mathbb{R}^n.$$

Тем самым, теорема полностью доказана.

Замечание 3.3. *Следует отметить, что если первые моменты ядер* $\{T_j(x)\}_{j=1}^n$ *отрицательны и существуют числа* $\tilde{r}_1,\tilde{r}_2,\dots,\tilde{r}_n$ $(\tilde{r}_j>0,\ j=1,2,\dots,n)$ *такие, что* $L_j(-\tilde{r}_j)<1,\ \ j=1,2,\dots,n,$ *то тогда сперва при помощи следующих последовательных приближений:*

$$\Phi_j^{(m+1)}(x)=\int\limits_{-\infty}^{\infty}\tilde{T}_j(x-t)\Phi_j^{(m)}(t)dt,\quad j=1,2,\dots,n,$$

$$\Phi_j^{(0)}(x)=\begin{cases}\eta, & \text{если } x\le 0,\\ \eta e^{-\tilde{\sigma}_jx}, & \text{если } x>0,\end{cases}\quad m=0,1,2,\dots$$

(где $L_j(-\tilde{\sigma}_j)=1,\ \ j=1,2,\dots,n$*) строятся ограниченные решения одномерных уравнений* (2.13) $(\Phi_j^{(m)}(x)\ge\mathcal{L}_j(-x),\ x\in\mathbb{R},\ m=0,1,2,\dots,\ j=1,2,\dots,n)$, *затем опять с применением итерационного процесса* (3.1) *для уравнения* (1.1)

(с ядром (1.11)) *доказывается существование ограниченного положительного непрерывного на* $(-\infty,T]\times\mathbb{R}^n$ *решения* $u(t,x_1,\dots,x_n)$. *Данное решение обладает следующими свойствами:*

1) $\frac{1}{n}\sum\limits_{j=1}^{n}\tilde{\Phi}_j(x_j+\theta_j t)\le u(t,x_1,\dots,x_n)\le\frac{1}{n}\sum\limits_{j=1}^{n}\tilde{\Phi}_j^{(0)}(x_j+\theta_j t),$

$$t\in(-\infty,T],\ (x_1,\dots,x_n)\in\mathbb{R}^n,\ \theta_j>0,\ j=1,2,\dots,n,$$

где $\tilde{\Phi}_j(x):=\lim\limits_{m\to\infty}\Phi_j^{(m)}(x),\quad j=1,2,\dots,n,$

2) $\lim\limits_{x_1\to-\infty}\lim\limits_{x_2\to-\infty}\dots\lim\limits_{x_n\to-\infty}u(T,x_1,x_2,\dots,x_n)=\eta,$

3) $\lim\limits_{x_1\to+\infty}\lim\limits_{x_2\to+\infty}\dots\lim\limits_{x_n\to+\infty}u(t,x_1,x_2,\dots,x_n)=0,\ \forall t\in(-\infty,T],$

4) $\eta-u(T,-\infty,\dots,-\infty,x_j,-\infty,\dots,-\infty)\in L_1(0,+\infty),$
$u(t,+\infty,\dots,+\infty,x_j,+\infty,\dots,+\infty)\in L_1(-\infty,0)$
по x_j *для всех* $t\in(-\infty,T],\ j=1,2,\dots,n,$

5) *Решение единственно в следующем классе функций:* $\mathfrak{M}^*:=$

$$\left\{u\in C((-\infty,T]\times\mathbb{R}^n):\ \frac{1}{n}\sum_{j=1}^{n}\tilde{\Phi}_j(x_j+\theta_j t)\le u(t,x_1,\dots,x_n)\le\frac{1}{n}\sum_{j=1}^{n}\tilde{\Phi}_j^{(0)}(x_j+\theta_j t)\right\}$$

Замечание 3.4. *Совершая аналогичные рассуждения как при доказательстве теоремы 2, для последовательных приближений* (3.1) *можно получить следующую оценку:*

$$\alpha_m\le\alpha_{m-1}\rho,\ \ m=1,2,\dots, \tag{3.15}$$

где

$$\alpha_m:=\sup_{(t,x_1,\dots,x_n)\in(-\infty,T]\times\mathbb{R}^n}\left(\sum_{j=1}^{n}e^{-(\delta_j+\sigma_j t)(x_j+\theta_j t)}\right)^{-1}\times$$
$$|u_{m+1}(t,x_1,\dots,x_n)-u_m(t,x_1,\dots,x_n)|,$$

а число $\rho<1$ *задается согласно* (3.14).

Из оценки (3.15) методом математической индукции легко можно доказать, что

$$\alpha_m\le\alpha_0\rho^m,\quad m=1,2,\dots. \tag{3.16}$$

Таким образом, в силу (3.16) можем утверждать, что последовательности функций

$$Q_m(t,\mathrm{x}):=\left(\sum_{j=1}^{n}e^{-(\delta_j+\sigma_j)(x_j+\theta_j t)}\right)^{-1}u_m(t,\mathrm{x}),\quad \mathrm{x}=(x_1,\dots,x_n),\ \ m=1,2,\dots$$

являются равномерно сходящимися на множестве $(-\infty,T]\times\mathbb{R}^n$.

Замечание 3.5. *Заметим, что при условиях теоремы 1 уравнение* (1.1) *с ядром* (1.11) *в следующем классе непрерывных функций:*

$$\mathfrak{P} := \left\{ f(t,\mathrm{x}) : \ 0 < \inf_{(t,\mathrm{x})\in(-\infty,T]\times\mathbb{R}^n} f(t,\mathrm{x}) \le f(t,\mathrm{x}) \le \eta, \ t \in (-\infty, T], \ \mathrm{x} \in \mathbb{R}^n \right\}$$

имеет только одно тривиальное решение $u(t, x_1, \dots, x_n) \equiv \eta$.

Действительно, если обозначить через

$$\alpha := \inf_{(t,\mathrm{x})\in(-\infty,T]\times\mathbb{R}^n} u(t,\mathrm{x}) > 0,$$

то из уравнения (1.1) *с учетом* (1.11), (1.4), (1.9) *и монотонности функции* g *на отрезке* $[0,\eta]$ *будем иметь*

$$u(t,x_1,\dots,x_n) \ge \int_{-\infty}^{t} H(t-\tau) \int_{\mathbb{R}^n} V(x_1-y_1, x_2-y_2, \dots, x_n-y_n) g(\alpha) dy_1 \dots dy_n d\tau = g(\alpha),$$

откуда, в частности, следует, что $\alpha \ge g(\alpha)$. *Но, с другой стороны, в силу вогнутости функции* g *и свойств* $a) - c)$:

$$g(\alpha) \ge \alpha$$

(*ибо* $\alpha \in (0,\eta]$). *Следовательно,* $g(\alpha) = \alpha$. *Так как* $\alpha \in (0,\eta]$ *и* g *— вогнутая функция на отрезке* $[0,\eta]$, *то* $\alpha = \eta$. *Прямой проверкой можно убедиться, что* $u \equiv \eta$ *удовлетворяет уравнению* (1.1). *Таким образом, если* $u \in \mathfrak{P}$ *и является решением уравнения* (1.1), *то* $u \equiv \eta$.

4. Приложение

Ниже приведем прикладные примеры ядерных функций H, V и нелинейности g, для которых выполняются все условия доказанных теорем.

В математической теории географического распространения эпидемии возникают нелинейные интегральные уравнения вида (1.1) с ядром (1.11), в которых функции H и V допускает следующие представления (см. [1]–[2]):

$$H(\tau) = ae^{-a\tau}, \ \tau \ge 0, \ a > 0, \ n = 2, \ V(z_1, z_2) = \frac{1}{\pi} e^{-(z_1^2+z_2^2)}, \ z_1, \ z_2 \in \mathbb{R}, \tag{4.1}$$

а нелинейность $g(u)$ имеет вид:

$$g(u) = \frac{\mu S_0}{a}(1 - e^{-u}), \ u \ge 0, \ \mu > 0, \ S_0 > 0. \tag{4.2}$$

В теории географического распространения эпидемии неравенство

$$\frac{a}{\mu S_0} < 1 \tag{4.3}$$

называется пороговым условием: это критическое число зараженных лиц, выше которого эпидемию невозможно остановить.

На этом примере убедимся, что все условия теорем 1 и 2 выполняются.

Очевидным образом приведенные функции H и V удовлетворяют условиям (1.4), $I)-III)$. Из порогового условия (4.3) сразу следует, что для функции вида (4.2) существует

$$g^{'}(0) := \gamma = \frac{\mu S_0}{a} > 1.$$

Убедимся, что

$$g(u) \leq \frac{\mu S_0}{a} u, \quad u \geq 0.$$

Действительно, последнее неравенство сразу следует из следующего известного неравенства: $e^{-u} \geq 1-u,\ u \in \mathbb{R}$. Так как $g^{''}(u) = -\dfrac{\mu S_0}{a} e^{-u} < 0$, то функция $g(u)$ будет вогнутой. Для функции вида (4.2) проверим теперь неравенство

$$g(u) \geq g^{'}(0)u - cu^{1+\varepsilon},\ \ u \geq 0, \tag{4.4}$$

Последнее равносильно неравенству

$$\frac{\mu S_0}{a}(1-e^{-u}) \geq \frac{\mu S_0}{a} u - cu^{1+\varepsilon},\ u \geq 0. \tag{4.5}$$

При $\varepsilon = 1,\ c = \frac{\mu S_0}{2a}$ данное неравенство примет следующий вид:

$$1 - e^{-u} \geq u - \frac{u^2}{2}, \quad u \geq 0. \tag{4.6}$$

Докажем неравенство (4.6). Рассмотрим функцию

$$\chi(u) = 1 - e^{-u} - u + \frac{u^2}{2}, \quad u \geq 0.$$

Заметим, что $\chi(0) = 0,\ \chi^{'}(u) = e^{-u} - 1 + u \geq 0,\ u \geq 0$. Следовательно, (4.4) выполняется.

Проверим теперь выполнение условия (2.6) для ядерных функций вида (4.1). Во-первых заметим, что тогда функции $\{T_j(x)\}_{j=1,2}$ имеют следующий вид:

$$T_1(x) = T_2(x) = \frac{1}{\sqrt{\pi}} e^{-x^2},\ \ x \in \mathbb{R}$$

и, следовательно, в силу (2.2) имеем

$$L_j(\lambda) = g^{'}(0) \int\limits_{-\infty}^{\infty} \tilde{T}_j(x) e^{-\lambda x} dx = \frac{\mu S_0}{a} \int\limits_{-\infty}^{\infty} e^{-\lambda x} \int\limits_{0}^{\infty} H(p) T_j(x - \theta_j p) dp dx =$$

$$= \frac{\mu S_0}{\sqrt{\pi}} \int\limits_{0}^{\infty} e^{-ap} \int\limits_{-\infty}^{\infty} e^{-(x-\theta_j p)^2} e^{-\lambda x} dx dp = \frac{\mu S_0}{\sqrt{\pi}} \int\limits_{0}^{\infty} e^{-(a+\lambda\theta_j)p} dp \int\limits_{-\infty}^{\infty} e^{-z^2} e^{-\lambda z} dz$$

$$= \frac{\mu S_0}{(a+\lambda\theta_j)\sqrt{\pi}} e^{\lambda^2/4} \int\limits_{-\infty}^{\infty} e^{-(z+\lambda/2)^2} dz = \frac{\mu S_0}{a+\lambda\theta_j} e^{\lambda^2/4}.$$

Зафиксируем параметры $\{\theta_j\}_{j=1,2},\ \mu,\ a,\ S_0$ и исследуем функции $L_j(\lambda) = \dfrac{\mu S_0 e^{\lambda^2/4}}{a+\lambda\theta_j}$, $j = 1,2$. Заметим, что $L_j(0) = \dfrac{\mu S_0}{a} > 1$ и $\dfrac{dL_j(\lambda)}{d\lambda} = \dfrac{\mu S_0 e^{\lambda^2/4}(a\lambda + \lambda^2\theta_j - 2\theta_j)}{2(a+\lambda\theta_j)^2}$.
Следовательно, функции $L_j(\lambda)\ \downarrow$ по λ на $\left[\dfrac{-a-\sqrt{a^2+8\theta_j^2}}{2\theta_j}, \dfrac{-a+\sqrt{a^2+8\theta_j^2}}{2\theta_j}\right]$, $j = 1,2$ и $r_j := \frac{\sqrt{a^2+8\theta_j^2}-a}{2\theta_j}$ являются точками минимума функций $L_j(\lambda)$, $j = 1,2$. Теперь числа $\{\theta_j\}_{j=1,2}$ выберем так, чтобы

$$L_j(r_j) < 1,\ j = 1,2.$$

Используя следующее простое неравенство:

$$e^{\left(\frac{\sqrt{a^2+8\theta_j^2}-a}{4\theta_j}\right)^2} < e^{1/2}$$

имеем

$$L_j(r_j) < \frac{2\mu S_0 e^{1/2}}{a+\sqrt{a^2+8\theta_j^2}} < 1$$

при

$$\theta_j > \frac{\sqrt{\mu S_0 e^{1/2}(\mu S_0 e^{1/2} - a)}}{\sqrt{2}},\ j = 1,2$$

ибо $\mu S_0 > a$ (см. пороговое условие).

В конце приведем еще один пример нелинейности $g(u)$, имеющей приложение в теории географического распространения эпидемии:

$$g(u) = \gamma u - \gamma u^2, \quad u \geq 0,$$

где $\gamma > 1$ — числовой параметр.

Очевидно, что если в этом случае в качестве η выбрать $\eta = \frac{\gamma-1}{\gamma}$, то $g\ \uparrow$ на $[0,\eta],\ g(0) = 0,\ g(\eta) = \eta,\ g$ — вогнутая функция на отрезке $[0,\eta],\ g'(0) = \gamma > 1,\ c = \gamma,\ \varepsilon = 1$. Здесь также волновые скорости $\{\theta_j\}_{j=1,2}$ можно подобрать так, чтобы выполнялись условия (2.6).

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Abstract. A multidimensional integral equation of the convolution type with concave nonlinearity is investigated. This equation meets in the mathematical theory of the geographical spread of the epidemic. The combination of well-known multidimensional methods and operator methods for constructing invariant cone segments for such operators with methods of the theory of integral operators of convolution type and limit theorems of function theory allow us to prove the existence of positive bounded solutions for such equations. The asymptotic behavior of the constructed solutions is also studied. In a concretely chosen cone segment, the uniqueness of the solution is also proved. Specific applied examples of these equations are given.

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VOLTERRA INTEGRAL OPERATORS FROM CAMPANATO SPACES INTO GENERAL FUNCTION SPACES

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Abstract. In this paper, the boundedness and compactness of embedding from Campanato spaces $\mathcal{L}_{p,\lambda}$ into tent spaces $\mathcal{T}_{p,s}(\mu)$ are investigated. As an application, we give a characterization for the boundedness of the Volterra integral operator J_g from $\mathcal{L}_{p,\lambda}$ to general function spaces $F(p, p-1-\lambda, s)$. Meanwhile, the operator I_g and the multiplication operator M_g from $\mathcal{L}_{p,\lambda}$ to $F(p, p-1-\lambda, s)$ are studied. Furthermore, the essential norm of J_g and I_g from $\mathcal{L}_{p,\lambda}$ to $F(p, p-1-\lambda, s)$ are also considered.

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1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ its boundary. Let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space H^p is the set of all $f \in H(\mathbb{D})$ satisfying (see [1])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p d\zeta < \infty.$$

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space, denoted by A^p_α, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha}^p = (\alpha+1) \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. When $\alpha = 0$, A^p_α is the Bergman space, denoted by A^p. As usual, H^∞ denotes the space of bounded analytic function.

In 1996, Zhao [26] introduced the general family of function spaces $F(p, q, s)$. Namely, for $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$, the space $F(p, q, s)$ consists of

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functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformation of $\mathbb{D}$ interchanging a and 0. It is known that, for $p \ge 1$, $F(p,q,s)$ is a Banach space under the above norm. Also, it is known that $F(p,q,s)$ contains only constant functions if $s+q \le -1$. Thus, it is natural to study $F(p,q,s)$ spaces under the assumption that $s+q > -1$. $F(p,p,0)$ is just the Bergman space. When $p=2$ and $q=0$, it gives the Q_s space (see [22]). Especially, Q_1 is the $BMOA$ space, the space of analytic functions in the Hardy space whose boundary functions have bounded mean oscillation. When $s>1$, Q_s is the Bloch space, denoted by $\mathcal{B}$, which is the space of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The little Bloch space $\mathcal{B}_0$, consists of all $f \in \mathcal{B}$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. See [13, 26] for more results of $F(p,q,s)$ spaces.

Let I be an arc of $\partial\mathbb{D}$ and $|I|$ be the normalized Lebesgue arc length of I. The Carleson square based on I, denoted by $S(I)$, is defined by

$$S(I) = \left\{ z = re^{i\theta} \in \mathbb{D} : 1-|I| \le r < 1, e^{i\theta} \in I \right\}.$$

Let $0 < p < \infty$, $0 < s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. The tent space $\mathcal{T}_{p,s}(\mu)$ consists of all μ-measurable functions f such that

$$\|f\|^p_{\mathcal{T}_{p,s}(\mu)} = \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^p d\mu(z) < \infty.$$

Let $p \ge 1$ and $0 \le \lambda < \infty$. We say that an $f \in H^p$ belongs to the analytic Campanato space $\mathcal{L}_{p,\lambda}$ if (see [25])

$$\|f\|_{\mathcal{L}_{p,\lambda}} = |f(0)| + \left(\sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^p \frac{|d\zeta|}{2\pi} \right)^{\frac{1}{p}} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}, \; I \subseteq \partial\mathbb{D}.$$

When $p=2$, the space $\mathcal{L}_{2,\lambda}$ is called the Morrey space, which was studied by Wu and Xie in [20]. When $\lambda = 0$, $\mathcal{L}_{p,0}$ is just the Hardy space H^p. $\mathcal{L}_{p,1}$ is the $BMOA$ space. Recently, some fundamental function and operator-theoretic properties on $\mathcal{L}_{p,\lambda}$ have been investigated in [5, 10, 14, 18, 19, 20, 21, 24, 25]

Let $f, g \in H(\mathbb{D})$. The Volterra integral operator J_g and the integral operator I_g are defined by

$$J_g f(z) = \int_0^z g'(w) f(w) dw, \quad I_g f(z) = \int_0^z g(w) f'(w) dw, \quad z \in \mathbb{D},$$

respectively. The multiplication operator M_g is defined by $M_g f(z) = g(z)f(z),\ \ f \in H(\mathbb{D}),\ z \in \mathbb{D}$.

The operator J_g was introduced by Pommerenke in [12]. Pommerenke showed that $J_g : H^2 \to H^2$ is bounded if and only if $g \in BMOA$. Furthermore, in [3], Aleman and Siskakis proved that $J_g : H^p \to H^p$ is bounded if and only if $g \in BMOA$. In [4], Aleman and Siskakis showed that $J_g : A^p \to A^p$ is bounded if and only if $g \in \mathcal{B}$. For more information on Volterra integral operators, see [2] - [9], [11, 14, 15, 23] and the references therein.

Recently, Li, Liu and Lou in [5] proved that $J_g : \mathcal{L}_{2,\lambda} \to \mathcal{L}_{2,\lambda}$ is bounded if and only if $g \in BMOA$. In [18], Wang generalized the result in [5] and proved that $J_g : \mathcal{L}_{p,\lambda} \to \mathcal{L}_{2,1-2/p(1-\lambda)}$ is bounded if and only if $g \in BMOA$ under the assumption that $2 \le p < \infty$ and $0 \le \lambda < 1$. An interesting and nature question is to find an analytic function space X for which

$$J_g : \mathcal{L}_{p,\lambda} \to X \ \text{ is bounded if and only if } \ g \in \mathcal{B}.$$

In this paper, we prove that $J_g : \mathcal{L}_{p,\lambda}\ \ to\ \ F(p, p-1-\lambda, s)$ is bounded if and only if $g \in \mathcal{B}$. Moreover, we show that the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is bounded (resp.compact) if and only if μ is a $s-\lambda+1$-Carleson measure(resp. a vanishing $s-\lambda+1$-Carleson measure) under the assumption that $2 \le p < \infty$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. The essential norm of the operator J_g is also investigated. Furthermore, we study the boundedness and compactness of the operators I_g and M_g from $\mathcal{L}_{p,\lambda}$ to $F(p, p-1-\lambda, s)$.

Throughout this paper, we say that $A \lesssim B$, if there exists a constant C such that $A \le CB$. The symbol $A \asymp B$ means that $A \lesssim B \lesssim A$.

2. Embedding from $\mathcal{L}_{p,\lambda}$ to tent spaces

An important tool to study function spaces is Carleson type measure. For $s > 0$, a positive Borel measure μ on $\mathbb{D}$ is said to be an s-Carleson measure if $\sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty$. For $s = 1$, we get the classical Carleson measures (see [1]). If μ is an s-Carleson measure, then we set

$$\|\mu\|_s = \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s}.$$

If $\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0$, then μ is called a vanishing s-Carleson measure. It is well known (see [25]) that μ is an s-Carleson measure if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}} d\mu(z) < \infty.$$

Moreover,

$$\sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^s}\asymp\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}}d\mu(z). \tag{2.1}$$

Now we are in a position to state and prove the main results in this section.

Theorem 2.1. *Let $2\le p<\infty$, $0\le\lambda<1$, $\lambda<s<\infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the identity operator $i:\mathcal{L}_{p,\lambda}\to\mathcal{T}_{p,s}(\mu)$ is bounded if and only if μ is a $(s+1-\lambda)$-Carleson measure.*

Proof. Assume that μ is a $(s+1-\lambda)$-Carleson measure. Let I be any arc on $\partial\mathbb{D}$ and $a=(1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the midpoint of I. Let $f\in\mathcal{L}_{p,\lambda}$. From [18, Lemma 2.5], we get

$$|f(a)|\lesssim\frac{\|f\|_{\mathcal{L}_{p,\lambda}}}{(1-|a|)^{\frac{1-\lambda}{p}}}=\frac{\|f\|_{\mathcal{L}_{p,\lambda}}}{|I|^{\frac{1-\lambda}{p}}}.$$

Then

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^pd\mu(z)&\lesssim\frac{1}{|I|^s}\int_{S(I)}|f(a)|^pd\mu(z)+\frac{1}{|I|^s}\int_{S(I)}|f(z)-f(a)|^pd\mu(z)\\&=M+N.\end{aligned}$$

It is obvious that

$$M\lesssim\frac{\mu(S(I))}{|I|^{s-\lambda+1}}\|f\|^p_{\mathcal{L}_{p,\lambda}}\lesssim\|f\|^p_{\mathcal{L}_{p,\lambda}}.$$

Now we turn to estimate N. The estimate will be divided into two cases.

Case 1: $s-\lambda\ge1$.

By the assumed condition and Theorem 7.4 in [27], we know that the identity operator $i:A^p_{s-\lambda-1}\to L^p(d\mu)$ is bounded. Then

$$\begin{aligned}N&\asymp\int_{S(I)}\frac{|f(z)-f(a)|^p}{|1-\bar{a}z|^s}d\mu(z)\\&\asymp(1-|a|^2)^{1-\lambda}\int_{S(I)}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^{3-\lambda+s}}d\mu(z)\\&\lesssim(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{\left|(1-\bar{a}z)^{\frac{3-\lambda+s}{p}}\right|^p}d\mu(z)\\&\lesssim(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^{3-\lambda+s}}(1-|z|^2)^{s-\lambda-1}dA(z)\\&\lesssim(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p(1-|a|^2)^2}{|1-\bar{a}z|^4}dA(z)\\&=(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}|f\circ\sigma_a(w)-f(a)|^pdA(w)\\&\lesssim(1-|a|^2)^{1-\lambda}\int_{\partial\mathbb{D}}|f\circ\sigma_a(\zeta)-f(a)|^pd\zeta\le\|f\|^p_{\mathcal{L}_{p,\lambda}}<\infty.\end{aligned}$$

The last second inequality is come from [25, Theorem 1].

Case 2: $0 < s - \lambda < 1$.

Since $H^p \subseteq A^p_{s-\lambda-1}$, we have

$$\begin{aligned}
N &\asymp (1-|a|^2)^{-s} \int_{S(I)} |f(z) - f(a)|^p d\mu(z) \\
&\asymp (1-|a|^2)^{2-s} \int_{S(I)} \frac{|f(z) - f(a)|^p (1-|a|^2)^2}{|1-\bar{a}z|^4} d\mu(z) \\
&\lesssim (1-|a|^2)^{2-s} \int_{\mathbb{D}} \frac{|f(z) - f(a)|^p (1-|a|^2)^2}{|1-\bar{a}z|^4} d\mu(z) \\
&\lesssim (1-|a|^2)^{2-s} \int_{\mathbb{D}} \frac{|f(z) - f(a)|^p (1-|a|^2)^2}{|1-\bar{a}z|^4} (1-|z|^2)^{s-\lambda-1} dA(z) \\
&= (1-|a|^2)^{2-s} \int_{\mathbb{D}} |f \circ \sigma_a(w) - f(a)|^p (1-|\sigma_a(w)|^2)^{s-\lambda-1} dA(w) \\
&\lesssim (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f \circ \sigma_a(w) - f(a)|^p (1-|w|^2)^{s-\lambda-1} dA(w) \\
&\lesssim (1-|a|^2)^{1-\lambda} \int_{\partial\mathbb{D}} |f \circ \sigma_a(\zeta) - f(a)|^p d\zeta \lesssim \|f\|^p_{\mathcal{L}_{p,\lambda}} < \infty.
\end{aligned}$$

Combining the estimates M and N, we conclude that the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is bounded.

Conversely, suppose that the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is bounded. For $a \in \mathbb{D}$, let

$$f_a(z) = \frac{(1-|a|^2)^{1+\frac{\lambda-1}{p}}}{(1-\bar{a}z)}, \quad z \in \mathbb{D}. \tag{2.2}$$

By [18, Lemma 2.3], we have that $f_a \in \mathcal{L}_{p,\lambda}$ with $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathcal{L}_{p,\lambda}} \lesssim 1$. Fixed an arc $I \subseteq \partial\mathbb{D}$. Let $e^{i\theta}$ be the center of I and $a = (1-|I|)e^{i\theta}$. Then

$$|1-\bar{a}z| \asymp 1-|a| = |I|, \quad |f_a(z)|^p \asymp |I|^{\lambda-1},$$

whenever $z \in S(I)$. So

$$\frac{\mu(S(I))}{|I|^{s+1-\lambda}} \asymp \frac{1}{|I|^s} \int_{S(I)} |f_a(z)|^p d\mu(z) \leq \|f_a\|^p_{\mathcal{T}_{p,s}(\mu)} < \infty.$$

Consequently, μ is a $(s+1-\lambda)$-Carleson measure. □

Theorem 2.2. *Let $2 \leq p < \infty$, $0 \leq \lambda < 1$, $\lambda < s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$ such that point evaluation is a bounded functional on $\mathcal{T}_{p,s}(\mu)$. Then the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is compact if and only if μ is a vanishing $(s-\lambda+1)$-Carleson measure.*

Proof. Assume that μ is a vanishing $(s-\lambda+1)$-Carleson measure. It is clear that μ is a $(s-\lambda+1)$-Carleson measure. Hence $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu)$ is bounded. For $0 < r < 1$, let $\chi_{\{z:|z|<r\}}$ be the characteristic function of the set $\{z : |z| < r\}$. From

[6] we see that $\lim_{r\to 1}\|\mu-\mu_r\|_{s-\lambda+1}=0$, where $d\mu_r=\chi_{\{z:|z|<r\}}d\mu$. Let $\{f_k\}$ be a bounded sequence in $\mathcal{L}_{p,\lambda}$ such that $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. We have

$$\begin{aligned}\frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d\mu(z) &\lesssim \frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d\mu_r(z)+\frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d(\mu-\mu_r)(z)\\ &\lesssim \frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d\mu_r(z)+\|\mu-\mu_r\|_{s-\lambda+1}\|f_k\|^p_{\mathcal{L}_{p,\lambda}}\\ &\lesssim \frac{1}{|I|^s}\int_{S(I)}|f_k(z)|^p d\mu_r(z)+\|\mu-\mu_r\|_{s-\lambda+1}\to 0,\end{aligned}$$

as $r\to 1$ and $k\to\infty$. Therefore, $\lim_{k\to\infty}\|f_k\|_{\mathcal{T}_{p,s}(\mu)}=0$, which means that the identity operator $i:\mathcal{L}_{p,\lambda}\to\mathcal{T}_{p,s}(\mu)$ is compact.

Conversely, suppose that the identity operator $i:\mathcal{L}_{p,\lambda}\to\mathcal{T}_{p,s}(\mu)$ is compact. Let $\{I_k\}$ be a sequence arcs with $\lim_{k\to\infty}|I_k|=0$. We denote the center of I_k by $e^{i\theta_k}$. Set $a_k=(1-|I_k|)e^{i\theta_k}$ and

$$f_k(z)=\frac{(1-|a_k|^2)^{1+\frac{\lambda-1}{p}}}{(1-\bar{a_k}z)},\quad z\in\mathbb{D}. \tag{2.3}$$

It is easy to check that $\{f_k\}$ is bounded in $\mathcal{L}_{p,\lambda}$ and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Then $\lim_{k\to\infty}\|f_k\|_{\mathcal{T}_{p,s}(\mu)}=0$ by the assumption. Since

$$|f_k(z)|\asymp(1-|a_k|)^{\frac{\lambda-1}{p}}=|I_k|^{\frac{\lambda-1}{p}},$$

when $z\in S(I_k)$, we obtain

$$\frac{\mu(S(I_k))}{|I_k|^{s-\lambda+1}}\asymp\frac{1}{|I_k|^s}\int_{S(I_k)}|f_k(z)|^p d\mu(z)\le\|f_k\|^p_{\mathcal{T}_{p,s}(\mu)}\to 0,\quad k\to\infty,$$

which implies that μ is a vanishing $(s-\lambda+1)$-Carleson measure. □

3. Boundedness of J_g, I_g and M_g

In this section, via the embedding theorem (Theorem 2.1), we provide a characterization for the boundedness of Volterra integral operator J_g from $\mathcal{L}_{p,\lambda}$ to $F(p,p-1-\lambda,s)$. We also study the boundedness of the operators I_g and M_g.

Theorem 3.1. *Let $2\le p<\infty$, $0\le\lambda<1$ and $\lambda<s<\infty$. If $g\in H(\mathbb{D})$, then $J_g:\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)$ is bounded if and only if $g\in\mathcal{B}$. Moreover, $\|J_g\|\asymp\|g\|_{\mathcal{B}}$.*

Proof. Let $g\in\mathcal{B}$. Using the equivalent norm of Bloch function (see [26]), we obtain

$$\begin{aligned}\|g\|^p_{\mathcal{B}} &\asymp \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^{s-\lambda+1}dA(z)\\ &=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g'(z)|^p(1-|z|^2)^{p-1+s-\lambda}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{s-\lambda+1}dA(z)\\ &\asymp\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^{s-\lambda+1}}\int_{S(I)}|g'(z)|^p(1-|z|^2)^{p-1+s-\lambda}dA(z)\asymp\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu_g(S(I))}{|I|^{s-\lambda+1}},\end{aligned}$$

which implies that $d\mu_g(z) = |g'(z)|^p(1-|z|^2)^{p-1+s-\lambda}dA(z)$ is a $(s-\lambda+1)$-Carleson measure. By Theorem 2.1, the identity operator $i : \mathcal{L}_{p,\lambda} \to \mathcal{T}_{p,s}(\mu_g)$ is bounded. Let $f \in \mathcal{L}_{p,\lambda}$. We deduce that

$$\begin{aligned}
\|J_g f\|^p_{F(p,p-1-\lambda,s)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p |g'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p |g'(z)|^p (1-|z|^2)^{p-1-\lambda+s} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^s dA(z) \\
&\asymp \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^p d\mu_g(z) \\
&= \|f\|^p_{\mathcal{T}_{p,s}(\mu_g)} \lesssim \|\mu_g\|_{s-\lambda+1} \|f\|^p_{\mathcal{L}_{p,\lambda}} \asymp \|g\|^p_{\mathcal{B}} \|f\|^p_{\mathcal{L}_{p,\lambda}} < \infty.
\end{aligned}$$

That is, $J_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded and $\|J_g\| \lesssim \|g\|_{\mathcal{B}}$.

Conversely, suppose that $J_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded. For any $a \in \mathbb{D}$, let f_a be defined as in (2.2). Then $f_a \in \mathcal{L}_{p,\lambda}$ and $\|f_a\|_{\mathcal{L}_{p,\lambda}} \lesssim 1$. Thus,

$$\|J_g f_a\|_{F(p,p-1-\lambda,s)} \leq \|J_g\|\ \|f_a\|_{\mathcal{L}_{p,\lambda}} \lesssim \|J_g\|.$$

By Lemma 4.12 of [27], we have

$$\begin{aligned}
\|J_g f_a\|^p_{F(p,p-1-\lambda,s)} &\geq \int_{\mathbb{D}} |g'(z)|^p \frac{(1-|a|^2)^{p-1+\lambda}}{|1-\bar{a}z|^p} (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z) \\
&= \int_{\mathbb{D}} |g'(z)|^p \frac{(1-|a|^2)^{p-1+\lambda+s}(1-|z|^2)^{p-1-\lambda+s}}{|1-\bar{a}z|^{2s+p}} dA(z) \\
&\geq \int_{D(a,r)} |g'(z)|^p \frac{(1-|a|^2)^{p-1+\lambda+s}(1-|z|^2)^{p-1-\lambda+s}}{|1-\bar{a}z|^{2s+p}} dA(z) \\
&\gtrsim |g'(a)|^p(1-|a|^2)^p.
\end{aligned}$$

Hence, for any $a \in \mathbb{D}$,

$$|g'(a)|(1-|a|^2) \lesssim \|J_g f_a\|_{F(p,p-1-\lambda,s)} \lesssim \|J_g\|,$$

which implies that $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \lesssim \|J_g\|$. □

Theorem 3.2. *Suppose that $2 \leq p < \infty$, $0 \leq \lambda < 1 < s < \infty$ or $p = 2$, $0 \leq \lambda < 1$ and $\lambda < s < \infty$. If $g \in H(\mathbb{D})$, then $I_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded if and only if $g \in H^\infty$. Furthermore, $\|I_g\| \asymp \|g\|_{H^\infty}$.*

Proof. Assume that $I_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded. For any $a \in \mathbb{D}$, set $h_a = \frac{(1-|a|^2)^{1+\frac{\lambda-1}{p}}}{\bar{a}(1-\bar{a}z)}$. It is easy to see that $h_a \in \mathcal{L}_{p,\lambda}$ and $\sup_{a\in\mathbb{D}} \|h_a\|_{\mathcal{L}_{p,\lambda}} \lesssim 1$. Hence

$$\|I_g h_a\|_{F(p,p-1-\lambda,s)} \leq \|I_g\|\ \|h_a\|_{\mathcal{L}_{p,\lambda}} \lesssim \|I_g\|.$$

Lemma 4.12 of [27] gives

$$\begin{aligned}\|I_g h_a\|^p_{F(p,p-1-\lambda,s)} &\gtrsim \int_{\mathbb{D}} |g(z)|^p \frac{(1-|a|^2)^{p+\lambda-1}}{|1-\bar{a}z|^{2p}}(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\gtrsim \int_{D(a,r)} |g(z)|^p \frac{(1-|a|^2)^{p+\lambda-1}}{|1-\bar{a}z|^{2p}}(1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\gtrsim |g(a)|^p,\end{aligned}$$

which implies that $g \in H^\infty$ and $\|g\|_{H^\infty} \lesssim \|I_g\|$.

Conversely, suppose that $g \in H^\infty$. First we consider the case $2 \le p < \infty$, $0 \le \lambda < 1 < s < \infty$. Let $f \in \mathcal{L}_{p,\lambda}$. Then by [18, Lemma 2.4],

$$|f'(z)|^p \lesssim \frac{\|f\|^p_{\mathcal{L}_{p,\lambda}}}{(1-|z|^2)^{p+1-\lambda}}.$$

Combined with Lemma 3.10 of [27], we have

$$\begin{aligned}\|I_g f\|^p_{F(p,p-1-\lambda,s)} &\le \|g\|^p_{H^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\le \|g\|^p_{H^\infty}\|f\|^p_{\mathcal{L}_{p,\lambda}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1-|z|^2)^{-2}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\le \|g\|^p_{H^\infty}\|f\|^p_{\mathcal{L}_{p,\lambda}} \sup_{a\in\mathbb{D}}(1-|a|^2)^s \int_{\mathbb{D}} \frac{(1-|z|^2)^{s-2}}{|1-\bar{a}z|^{2s}} dA(z) \quad (s>1)\\ &\le \|g\|^p_{H^\infty}\|f\|^p_{\mathcal{L}_{p,\lambda}}.\end{aligned}$$

Thus, $I_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded and $\|I_g\| \le \|g\|_{H^\infty}$.

When $p = 2$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. From above, we have

$$\begin{aligned}\|I_g f\|^2_{F(2,1-\lambda,s)} &\le \|g\|^2_{H^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\le \|g\|^2_{H^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{1-\lambda}(1-|\sigma_a(z)|^2)^\lambda dA(z)\\ &\lesssim \|g\|^2_{H^\infty} \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2) dA(z) \le \|g\|^2_{H^\infty}\|f\|^2_{\mathcal{L}_{2,\lambda}}.\end{aligned}$$

The proof is complete. □

Using Theorems 3.1 and 3.2, we get the characterization of the boundedness of the multiplication operator $M_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$.

Theorem 3.3. *Suppose that* $2 \le p < \infty$, $0 \le \lambda < 1 < s < \infty$ *or* $p = 2$, $0 \le \lambda < 1$ *and* $\lambda < s < \infty$. *Then* $M_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ *is bounded if and only if* $g \in H^\infty$.

Proof. Assume that $M_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded. Let $h \in F(p,p-1-\lambda,s)$ and $b \in \mathbb{D}$. We have (see [26])

$$|h'(b)| \lesssim \frac{\|h\|_{F(p,p-1-\lambda,s)}}{(1-|b|^2)^{1+\frac{1-\lambda}{p}}},$$

and hence

$$|h(b)| \lesssim \frac{\|h\|_{F(p,p-1-\lambda,s)}}{(1-|b|^2)^{\frac{1-\lambda}{p}}}.$$

For any $a \in \mathbb{D}$, let f_a be defined as in (2.2). Then $\{f_a\}$ is bounded in $\mathcal{L}_{p,\lambda}$. By the assumption we see that $M_g f_a \in F(p, p-1-\lambda, s)$. Hence

$$|M_g f_a(z)| \lesssim \frac{\|M_g f_a\|_{F(p,p-1-\lambda,s)}}{(1-|z|^2)^{\frac{1-\lambda}{p}}} \lesssim \frac{\|M_g\| \, \|f_a\|_{\mathcal{L}_{p,\lambda}}}{(1-|z|^2)^{\frac{1-\lambda}{p}}} \lesssim \frac{\|M_g\|}{(1-|z|^2)^{\frac{1-\lambda}{p}}},$$

which implies that

$$\left| \frac{1-|a|^2}{(1-\bar{a}z)^{1+\frac{1-\lambda}{p}}} g(z) \right| \lesssim \frac{\|M_g\|}{(1-|z|^2)^{\frac{1-\lambda}{p}}}.$$

By the arbitrariness of $z, a \in \mathbb{D}$, let $a = z$, we obtain that $g \in H^\infty$ and $\|g\|_{H^\infty} \lesssim \|M_g\|$.

Conversely, assume that $g \in H^\infty$. It follows from Theorems 3.1 and 3.2 that

$$J_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s) \;\text{ and }\; I_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$$

are bounded. So by the following relation

$$J_g f + I_g f = M_g f - f(0)g(0),$$

we see that $M_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is bounded. □

4. Essential norm of J_g and I_g

In this section, we give an estimation of the essential norm of J_g and I_g. First, let us recall the definition of the essential norm of a operator. Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. The essential norm of $T : X \to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined by

$$\|T\|_{e,X\to Y} = \inf_S \{\|T - S\|_{X\to Y} : \quad S \text{ is compact from } X \text{ to } Y\}.$$

Lemma 4.1. [17] *If $f \in \mathcal{B}$, then*

$$\limsup_{|z|\to 1} (1-|z|^2)|f'(z)| \asymp \limsup_{r\to 1} \|f - f_r\|_{\mathcal{B}}.$$

Here $f_r(z) = f(rz)$, $0 < r < 1, z \in \mathbb{D}$.

Lemma 4.2. *Let $2 \le p < \infty$, $0 \le \lambda < 1$ and $\lambda < s < \infty$. If $0 < r < 1$ and $g \in \mathcal{B}$, then $J_{g_r} : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is compact.*

Proof. Let $\{f_k\}$ be a bounded sequence in $\mathcal{L}_{p,\lambda}$ such that $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ and $\sup_k \|f_k\|_{\mathcal{L}_{p,\lambda}} \leq 1$. Then

$$\begin{aligned}\|J_{g_r} f_k\|^p_{F(p,p-1-\lambda,s)} &\leq \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p |g_r'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\lesssim \frac{\|g\|^p_{\mathcal{B}}}{(1-r^2)^p} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\lesssim \frac{\|g\|^p_{\mathcal{B}}}{(1-r^2)^p} \int_{\mathbb{D}} |f_k(z)|^p (1-|z|^2)^{p-1-\lambda} dA(z)\\ &\lesssim \frac{\|g\|^p_{\mathcal{B}}\|f_k\|^p_{\mathcal{L}_{p,\lambda}}}{(1-r^2)^p} \int_{\mathbb{D}} (1-|z|^2)^{p-2} dA(z).\end{aligned}$$

By the dominated convergence theorem, we get the result. □

Theorem 4.1. *Let $2 \leq p < \infty$, $0 \leq \lambda < 1$ and $\lambda < s < \infty$. If $g \in H(\mathbb{D})$ such that $J_g : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is bounded, then*

$$\|J_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \asymp \limsup_{|z|\to 1}(1-|z|^2)|g'(z)|.$$

Proof. By Lemma 4.2, $J_{g_r} : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$ is compact. Hence

$$\begin{aligned}\|J_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} &\leq \|J_g - J_{g_r}\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)}\\ &= \|J_{g-g_r}\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \asymp \|g-g_r\|_{\mathcal{B}}.\end{aligned}$$

Using Lemma 4.1, we have

$$\|J_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \lesssim \limsup_{r\to 1} \|g-g_r\|_{\mathcal{B}} \asymp \limsup_{|z|\to 1}(1-|z|^2)|g'(z)|.$$

Next we prove that

$$\|J_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \gtrsim \limsup_{|z|\to 1}(1-|z|^2)|g'(z)|.$$

Let $\{a_k\}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty} |a_k| = 1$ and f_k be defined as in (2.3). Then $\{f_k\}$ is bounded in $\mathcal{L}_{p,\lambda}$ and converges to zero uniformly on each compact subset of $\mathbb{D}$. For any given compact operator $S : \mathcal{L}_{p,\lambda} \to F(p,p-1-\lambda,s)$, by [16, Lemma 2.10] we have $\lim_{k\to\infty} \|Sf_k\|_{F(p,p-1-\lambda,s)} = 0$. Then

$$\begin{aligned}&\|J_g - S\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \gtrsim \limsup_{k\to\infty} \|(J_g - S)f_k\|_{F(p,p-1-\lambda,s)}\\ &\gtrsim \limsup_{k\to\infty} \left(\|J_g f_k\|_{F(p,p-1-\lambda,s)} - \|Sf_k\|_{F(p,p-1-\lambda,s)}\right)\\ &\geq \limsup_{k\to\infty} \left(\int_{\mathbb{D}} |f_k(z)|^p |g'(z)|^p (1-|z|^2)^{p-1-\lambda}(1-|\sigma_{a_k}(z)|^2)^s dA(z)\right)^{\frac{1}{p}}\\ &\gtrsim \limsup_{k\to\infty}(1-|a_k|^2)|g'(a_k)|,\end{aligned}$$

which implies the desired result. □

Using Theorem 4.1 and the well-known result that $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$, we easily get the following corollary.

Corollary 4.1. *Let* $2 \le p < \infty$, $0 \le \lambda < 1$ *and* $\lambda < s < \infty$. *If* $g \in H(\mathbb{D})$, *then* $J_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ *is compact if and only if* $g \in \mathcal{B}_0$.

Theorem 4.2. *Suppose that* $2 \le p < \infty$, $0 \le \lambda < 1 < s < \infty$ *or* $p = 2$, $0 < \lambda < 1$ *and* $\lambda < s < \infty$. *If* $g \in H(\mathbb{D})$ *and* $I_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ *is bounded, then*

$$\|I_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \asymp \|g\|_{H^\infty}.$$

Proof. First, Theorem 3.2 gives

$$\|I_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} = \inf_S \|I_g - S\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \le \|I_g\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \lesssim \|g\|_{H^\infty}.$$

Now we prove that

$$\|I_g\|_{e,\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} \gtrsim \|g\|_{H^\infty}.$$

Let $\{a_k\}$, $\{f_k\}$ and S be defined as in the proof of Theorem 4.1. Since $S : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ is compact, by [16, Lemma 2.10] we get $\lim_{k\to\infty} \|Sf_k\|_{F(p,p-1-\lambda,s)} = 0$. Hence,

$$\begin{aligned}\|I_g - S\|_{\mathcal{L}_{p,\lambda}\to F(p,p-1-\lambda,s)} &\gtrsim \limsup_{k\to\infty} \|(I_g - S)f_k\|_{F(p,p-1-\lambda,s)} \\ &\gtrsim \limsup_{k\to\infty} \Big(\|I_g f_k\|_{F(p,p-1-\lambda,s)} - \|Sf_k\|_{F(p,p-1-\lambda,s)}\Big) \\ &= \limsup_{k\to\infty} \|I_g f_k\|_{F(p,p-1-\lambda,s)}.\end{aligned}$$

Similarly to the proof of Theorem 3.2, we get $\|I_g f_k\|_{F(p,p-1-\lambda,s)} \gtrsim |g(a_k)|$, which implies the desired result. □

Using Theorem 4.2, we easily get the following corollary.

Corollary 4.2. *Suppose that* $2 \le p < \infty$, $0 \le \lambda < 1 < s < \infty$ *or* $p = 2$, $0 \le \lambda < 1$ *and* $\lambda < s < \infty$. *If* $g \in H(\mathbb{D})$, *then* $I_g : \mathcal{L}_{p,\lambda} \to F(p, p-1-\lambda, s)$ *is compact if and only if* $g = 0$.

Remark. We conclude the article with a remark. There is a class of Möbius invariant spaces that are closely related to the Bloch space and BMOA, namely, the Q_s space. Let $2 \le p < \infty$, $0 \le \lambda < 1$ and $0 < s < 1$. An interesting and nature question is to find an analytic function space X for which

$$J_g : \mathcal{L}_{p,\lambda} \to X \quad \text{is bounded if and only if} \quad g \in Q_s.$$

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ON THE FREDHOLM PROPERTY OF SEMIELLIPTIC OPERATORS IN ANISOTROPIC WEIGHTED SPACES IN $\mathbb{R}^n$

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Abstract. We study the Fredholm property of semielliptic operators in anisotropic weighted spaces in $\mathbb{R}^n$. In this paper necessary conditions are obtained for fulfillment of a priori estimates for such operators. Necessary and sufficient conditions are obtained for the Fredholm property of semielliptic operators with variable coefficients that have certain rate at infinity.

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1. Introduction, basic notions and definitions

This paper studies the Fredholm property of semielliptic operators with variable coefficients in anisotropic weighted Sobolev spaces in $\mathbb{R}^n$. The class of semielliptic operators is a special subclass of hypoelliptic operators which contains elliptic, parabolic, $2b$–parabolic operators, etc. (see [1]). The analysis of the Fredholm property of semielliptic operators in Sobolev spaces in $\mathbb{R}^n$ has certain difficulties related to the facts that Fredholm theorems for compact manifolds cannot always be used in this case and characteristic polynomials of semielliptic operators are not homogeneous as in elliptic case. The Fredholm property of such operators has been a subject of interest for many authors.

The Fredholm property of elliptic operators in special weighted spaces is studied in the works of L.A. Bagirov [2], R.B. Lockhart, R.C. McOwen [3, 4], E. Schrohe [5] and others.

L.A. Bagirov [6], G.A. Karapetyan, A.A. Darbinyan [7] and A.A. Darbinyan, A.G. Tumanyan [8, 9] studied the Fredholm property of semielliptic operators in anisotropic weighted spaces. In G.V. Demidenko's works [10, 11] the isomorphism properties are obtained on the special scale of weighted spaces for quasi-homogenous semielliptic operator with constant coefficients.

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In this work necessary and sufficient conditions are obtained for the Fredholm property of semielliptic operators with special variable coefficients acting in anisotropic Sobolev spaces with certain weight functions. The classes of considered operators and the weight functions are extended compared to the ones from the works [8, 9].

Definition 1.1. *A bounded linear operator A, acting from a Banach space X to a Banach space Y, is called an n–normal operator, if the following conditions hold:*

(1) *the image of operator A is closed* $\left(\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}\right)$*;*
(2) *the kernel of operator A is finite dimensional* $(\dim \mathrm{Ker}(A) < \infty)$.

An operator A is called a Fredholm operator if conditions 1-2 hold and

(3) *the cokernel of operator A is finite dimensional*
$(\dim \mathrm{coker}(A) = \dim Y/\mathrm{Im}(A) < \infty)$.

The difference between the dimension of the kernel and the cokernel of operator A is called index of the operator:

$$\mathrm{ind}\,(A) = \dim \mathrm{Ker}(A) - \dim \mathrm{coker}(A).$$

Definition 1.2. *For a bounded linear operator A, acting from a Banach space X to a Banach space Y, bounded linear operator $R_1 : Y \to X$ and $R_2 : Y \to X$ are called respectively left and right regularizers if the following holds: $R_1A = I_X + T_1, AR_2 = I_Y + T_2$, where I_X, I_Y– identity operators, $T_1 : X \to X$ and $T_2 : Y \to Y$ are compact operators.*

Definition 1.3. *For a bounded linear operator A, acting from a Banach space X to a Banach space Y, bounded linear operator $R : Y \to X$ is called a regularizer for operator A, if it is left and right regularizer.*

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be Euclidean n-dimensional space, $\mathbb{Z}^n_+$, $\mathbb{N}^n$ be the sets of n-dimensional multiindices and multiindices with natural components respectively.

Consider the differential form

$$P\,(x, \mathbb{D}) = \sum_{(\alpha:\nu)\leq s} a_\alpha(x) D^\alpha, \tag{1.1}$$

where $s \in \mathbb{N}, \alpha \in \mathbb{Z}^n_+, \nu \in \mathbb{N}^n, (\alpha : \nu) = \frac{\alpha_1}{\nu_1} + \cdots + \frac{\alpha_n}{\nu_n}, D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}, D_j = i^{-1}\frac{\partial}{\partial x_j}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, a_\alpha(x) \in C\,(\mathbb{R}^n)\,.$

Denote

$$P_s\,(x, \mathbb{D}) = \sum_{(\alpha:\nu)=s} a_\alpha\,(x)\, D^\alpha \tag{1.2}$$

the principal part of $P\,(x, \mathbb{D})$, and

$$P_s(x,\xi)=\sum_{(\alpha:\nu)=s} a_\alpha(x)\,\xi^\alpha \tag{1.3}$$

the symbol of $P_s(x,\mathbb{D})$.

Definition 1.4. *The differential form $P(x,\mathbb{D})$ is called semielliptic at point $x_0\in\mathbb{R}^n$, if the following is satisfied:*

$$P_s(x_0,\xi)\neq 0,\forall\xi\in\mathbb{R}^n,|\xi|\neq 0.$$

Definition 1.5. *The differential form $P(x,\mathbb{D})$ is called semielliptic in $\mathbb{R}^n$, if $P(x,\mathbb{D})$ is semielliptic at each point $x\in\mathbb{R}^n$.*

For $\xi\in\mathbb{R}^n$ denote by

$$|\xi|_\nu=\left(\sum_{i=1}^n \xi_i^{2\nu_i}\right)^{1/2}.$$

Definition 1.6. *The differential form $P(x,\mathbb{D})$ is called uniformly semielliptic in $\mathbb{R}^n$, if there exists a constant $C>0$ such that:*

$$|P_s(x,\xi)|\geq C\,|\xi|_\nu^s\,,\forall x\in\mathbb{R}^n,\forall\xi\in\mathbb{R}^n.$$

For $k\in\mathbb{R},\nu\in\mathbb{N}^n$ denote by $H^{k,\nu}(\mathbb{R}^n)$ the space

$$H^{k,\nu}(\mathbb{R}^n):=\left\{u\in S^{'}:\widehat{u}-\text{function},\|u\|_{k,\nu}=\left(\int|\widehat{u}(\xi)|^2(1+|\xi|_\nu)^{2k}\,d\xi\right)^{\frac{1}{2}}<\infty\right\},$$

S' is the set of tempered distributions, $\widehat{u}$ is the Fourier transform of function u.

For $r\in\mathbb{Z}_+,\nu\in\mathbb{N}^n$ denote

$$C^{r,\nu}(\mathbb{R}^n):=\left\{a:D^\beta a(x)\in C(\mathbb{R}^n),\ \sup_{x\in\mathbb{R}^n}|D^\beta a(x)|<\infty,\forall\beta\in\mathbb{Z}_+^n\ s.t.\ (\beta:\nu)\leq r\right\},$$

$$Q:=\{g\in C(\mathbb{R}^n):g(x)>0,\forall x\in\mathbb{R}^n\},$$

$$Q^{r,\nu}:=\left\{g\in Q:D^\beta g(x)\in C(\mathbb{R}^n)\text{ and }\frac{1}{g(x)}\rightrightarrows 0,\max_{y,|x-y|\leq 1}\frac{|g(x)-g(y)|}{g(y)}\rightrightarrows 0,\right.$$
$$\left.\frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\nu)}}\rightrightarrows 0\text{ when }|x|\to\infty,\forall\beta\in\mathbb{Z}_+^n,0<(\beta:\nu)\leq r\right\}.$$

Let $\nu_{max}=\max_{1\leq i\leq n}\nu_i$. The examples of weight functions from $Q^{r,\nu}$ include polynomial functions as well as special exponential functions, for example:

$$(1+|x|_\nu)^l\,,l>0,\exp(1+|x|_\nu)^\sigma,0<\sigma<\frac{1}{\nu_{max}}.$$

For $k \in \mathbb{Z}_+, \nu \in \mathbb{N}^n$, $q \in Q$ and domain $\Omega \subset \mathbb{R}^n$ denote by $H_q^{k,\nu}(\mathbb{R}^n)$ and $H_q^{k,\nu}(\Omega)$ respectively the spaces of measurable functions $\{u\}$ with norms

$$\|u\|_{k,\nu,q} := \|u\|_{H_q^{k,\nu}(\mathbb{R}^n)} := \sum_{(\alpha:\nu)\leq k} \|D^\alpha u \cdot q^{k-(\alpha:\nu)}\|_{L_2(\mathbb{R}^n)} < \infty,$$

$$\|u\|_{H_q^{k,\nu}(\Omega)} := \sum_{(\alpha:\nu)\leq k} \|D^\alpha u \cdot q^{k-(\alpha:\nu)}\|_{L_2(\Omega)} < \infty.$$

Let $k \in \mathbb{N}, k \geq s, q \in Q$ and the coefficients of differential expression $P(x, \mathbb{D})$ of the form (1.1) satisfy the following conditions:

(1.4)

$$|D^\beta a_\alpha(x)| \leq C_{\alpha,\beta}\, q(x)^{s-(\alpha:\nu)+(\beta:\nu)} \quad \left(\forall \alpha, \beta \in \mathbb{Z}_+^n \; (\alpha:\nu) \leq s, (\beta:\nu) \leq k-s\right).$$

Then $P(x, \mathbb{D})$ generates a bounded linear operator, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k-s,\nu}(\mathbb{R}^n)$.

In the paper [8] the fulfillment of special a priori estimate and the Fredholm property of semielliptic operators are studied in anisotropic Sobolev spaces. The following theorem is proved:

Theorem 1.1. *Let the differential form $P(x, \mathbb{D})$ with some constant $C > 0$ satisfies the following estimate:*

$$\|u\|_{k,\nu,q} \leq C \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(\mathbb{R}^n)}\right), \forall u \in H_q^{k,\nu}(\mathbb{R}^n). \tag{1.5}$$

Then $P(x, \mathbb{D})$ is uniformly semielliptic in $\mathbb{R}^n$.

It is easy to check that in the case $q \equiv 1$ inverse statement is true with some smoothness conditions on the coefficients of the principal part of the differential form. In this paper it is proved that under the special conditions on the weight function and coefficients of the differential form $P(x, \mathbb{D})$ uniform semiellipticity in $\mathbb{R}^n$ does not imply the fulfillment of a priori estimate of the form (1.5) and stronger conditions are necessary for it. The results related to a priori estimates are further used to establish necessary conditions for the Fredholm property of the considered class of operators.

In this work necessary and sufficient conditions are obtained for the Fredholm property of semielliptic operators with special variable coefficients acting in anisotropic spaces $H_q^{k,\nu}(\mathbb{R}^n)$.

2. Main results

Let $k, s \in \mathbb{N}, k \geq s$. Consider the differential form

$$P(x, \mathbb{D}) = \sum_{(\alpha:\nu)\leq s} a_\alpha q(x)^{s-(\alpha:\nu)} D^\alpha, \tag{2.1}$$

where a_α – some constant numbers, $q \in Q^{k-s,\nu}$ and denotations from (1.1) are used.

For $N > 0$ and $x_0 \in \mathbb{R}^n$ denote

$$K_N(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \leq N\}, \quad K_N := K_N(0).$$

Theorem 2.1. *Let $P(x, \mathbb{D})$ be the differential form* (2.1) *and $k \in \mathbb{N}, k \geq s, q \in Q^{k-s,\nu}$. Let the differential form $P(x, \mathbb{D})$ with some constant $\kappa > 0$ satisfies the following estimate:*

$$\|u\|_{k,\nu,q} \leq \kappa \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(\mathbb{R}^n)}\right), \quad u \in H_q^{k,\nu}(\mathbb{R}^n). \tag{2.2}$$

Then there exists a constant $\delta > 0$ such that

$$\left|\sum_{(\alpha:\nu)\leq s} a_\alpha \lambda^{s-(\alpha:\nu)} \xi^\alpha\right| \geq \delta(\lambda + |\xi|_\nu)^s, \quad \xi \in \mathbb{R}^n, \lambda > 0.$$

Proof. Let $M > 0$, $x_M \in \mathbb{R}^n \backslash K_M$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \varphi \subset K_1(x_M)$, $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$ and $\xi \in \mathbb{R}^n$. Consider the function $\tilde{u}(x) = e^{i(q(x_M)^{\frac{1}{\nu}} \xi, x)} \varphi(x)$.

Since $\lim\limits_{|x|\to\infty} \max\limits_{|x-y|\leq 1} \frac{|q(x)-q(y)|}{q(y)} = 0$, then for any $r > 0$ the following inequality is fulfilled

$$|q(x)^r - q(x_M)^r| \leq \varepsilon_r(M) q(x_M)^r, \quad x \in K_1(x_M), \tag{2.3}$$

where $\varepsilon_r(M) \to 0$ when $M \to \infty$.

Using (2.3) and the fact that $\operatorname{supp} \widetilde{u} \subset K_1(x_M)$ it is easy to see that there exists a function $\varepsilon(M)$ such that $\varepsilon(M) \to 0$ when $M \to \infty$ and the following inequalities hold:

$$\|\tilde{u}\|_{k,\nu,q} \geq (1 - \varepsilon(M)) \|\tilde{u}\|_{k,\nu,q(x_M)}, \tag{2.4}$$

$$\|P\tilde{u}\|_{k-s,\nu,q} \leq (1 + \varepsilon(M)) \|P\tilde{u}\|_{k-s,\nu,q(x_M)}. \tag{2.5}$$

Taking into consideration the definition of function $\tilde{u}$ one can check that for any $\alpha \in \mathbb{Z}_+^n, (\alpha : \nu) \leq k$ with some constant $C_1 = C_1(\varphi) > 0$ the following holds

$$\|D^\alpha \tilde{u}\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\alpha:\nu)} \geq |\xi^\alpha| \|\varphi\|_{L_2(\mathbb{R}^n)} q(x_M)^k - C_1(1 + |\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}}.$$

Using previous inequality and the fact that $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$ we get that with some constant $C_2 = C_2(\varphi) > 0$ the following holds

$$\|\tilde{u}\|_{k,\nu,q(x_M)} \geq \sum_{(\alpha:\nu)\leq k} |\xi^\alpha| \, q(x_M)^k - C_2(1 + |\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}}. \tag{2.6}$$

For $\beta \in \mathbb{Z}_+^n, (\beta : \nu) \leq k - s$ with some constant $C_3 = C_3(P) > 0$ we have the following estimates

(2.7)
$$\left\|D^{\beta}(P(x,\mathbb{D})\tilde{u})\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)} \le \left\|D^{\beta}(P(x_M,\mathbb{D})\tilde{u})\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$+ \left\|D^{\beta}\left((P(x,\mathbb{D}) - P(x_M,\mathbb{D}))\tilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$\le \left\| \sum_{(\alpha:\nu)\le s} a_\alpha q(x_M)^{s-(\alpha:\nu)} D^{\alpha+\beta}\tilde{u} \right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$+ C_3 \sum_{(\alpha:\nu)\le s} \left\|D^{\beta}\left((q(x)^{s-(\alpha:\nu)} - q(x_M)^{s-(\alpha:\nu)})D^{\alpha}\tilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}.$$

Taking into account $q \in Q^{k-s,\nu}$, inequality (2.3), the definition of function $\tilde{u}$ and the fact that $\operatorname{supp}\widetilde{u} \subset K_1(x_M)$, then for all $\alpha, \beta \in \mathbb{Z}_+$ such that $(\alpha:\nu) \le s, (\beta:\nu) \le k-s$ with some constants $C_4 > 0, C_5 = C_5(\varphi) > 0$ we get the following estimate

(2.8)
$$\left\|D^{\beta}\left(\left(q(x)^{s-(\alpha:\nu)} - q(x_M)^{s-(\alpha:\nu)}\right) D^{\alpha}\widetilde{u}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$\le \left\|\left(q(x)^{s-(\alpha:\nu)} - q(x_M)^{s-(\alpha:\nu)}\right) D^{\beta+\alpha}\widetilde{u}\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$+ C_4 \sum_{0\le\gamma<\beta} \left\|D^{\beta-\gamma}\left(q(x)^{s-(\alpha:\nu)}\right) D^{\gamma+\alpha}\widetilde{u}\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$\le \tau(M)(1+|\xi|_\nu)^k q(x_M)^k + C_5(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}},$$

where $\tau(M)$ is such a function that $\tau(M) \to 0$ when $M \to \infty$.

Similarly, using the definition of function $\tilde{u}$, with some constant $C_6 = C_6(P,\varphi) > 0$ we can get

(2.9)
$$\left\| \sum_{(\alpha:\nu)\le s} a_\alpha q(x_M)^{s-(\alpha:\nu)} D^{\alpha+\beta}\widetilde{u} \right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$\le \left| \sum_{(\alpha:\nu)\le s} a_\alpha \xi^\alpha \right| \left|\xi^\beta\right| q(x_M)^k + C_6(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}}.$$

Then from (2.7), (2.8) and (2.9), with some constant $C_7 = C_7(P,\varphi) > 0$ we get

(2.10)
$$\left\|D^{\beta}(P(x,\mathbb{D})\tilde{u})\right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$
$$\le \left| \sum_{(\alpha:\nu)\le s} a_\alpha \xi^\alpha \right| \left|\xi^\beta\right| q(x_M)^k + \omega(M)(1+|\xi|_\nu)^k q(x_M)^k + C_7(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}},$$

where $\omega(M)$ is such a function that $\omega(M) \to 0$ when $M \to \infty$.

Therefore, with some constant $C_8 = C_8(P,\varphi) > 0$ the following holds

$$\|P\tilde{u}\|_{k-s,\nu,q(x_M)} \le \sum_{(\beta:\nu)\le k-s} |\xi^\beta| \left| \sum_{(\alpha:\nu)\le s} a_\alpha \xi^\alpha \right| q(x_M)^k + C_8(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}} + \tilde{\omega}(M)(1+|\xi|_\nu)^k q(x_M)^k, \tag{2.11}$$

where $\tilde{\omega}(M) \to 0$ when $M \to \infty$.

From (2.2), according to inequalities (2.4)–(2.6), (2.11) and the definition of the function $\tilde{u}$ we get

$$(1-\varepsilon(M)) \left(\sum_{(\alpha:\nu)\le k} |\xi^\alpha| \, q(x_M)^k - C_2(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}} \right) \le \kappa \left((1+\varepsilon(M)) \left(\sum_{(\beta:\nu)\le k-s} |\xi^\beta| \left| \sum_{(\alpha:\nu)\le s} a_\alpha \xi^\alpha \right| q(x_M)^k + C_8(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}} + \tilde{\omega}(M)(1+|\xi|_\nu)^k q(x_M)^k \right) + 1 \right).$$

From the last inequality, according to the facts that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$ and $\varepsilon(M) \to 0, \tilde{\omega}(M) \to 0$ when $M \to \infty$, dividing by $(q(x_M))^k$ and tending $M \to \infty$ we get

$$\sum_{(\beta:\nu)\le k} |\xi^\beta| \le \kappa \sum_{(\beta:\nu)\le k-s} |\xi^\beta| \left| \sum_{(\alpha:\nu)\le s} a_\alpha \xi^\alpha \right|.$$

Since $k, s \in \mathbb{N}, k \ge s, \nu \in \mathbb{N}^n$, then there exist the constants $\delta_1, \delta_2 > 0$ such that

$$\sum_{(\alpha:\nu)\le k} |\xi^\alpha| \ge \delta_1(1+|\xi|_\nu)^k, \quad \sum_{(\alpha:\nu)\le k-s} |\xi^\alpha| \le \delta_2(1+|\xi|_\nu)^{k-s}, \quad \xi \in \mathbb{R}^n. \tag{2.12}$$

Then with some constant $\delta = \frac{\delta_1}{\kappa\delta_2} > 0$ we get

$$\left| \sum_{(\alpha:\nu)\le s} a_\alpha \xi^\alpha \right| \ge \delta(1+|\xi|_\nu)^s, \quad \xi \in \mathbb{R}^n.$$

Let $\lambda > 0$. By substituting $\xi \in \mathbb{R}^n$ in the last inequality with $\frac{\xi}{\lambda^{\frac{1}{\nu}}} = \left(\frac{\xi_1}{\lambda^{\frac{1}{\nu_1}}}, \dots, \frac{\xi_n}{\lambda^{\frac{1}{\nu_n}}} \right)$, it is easy to get the following estimate:

$$\left| \sum_{(\alpha:\nu)\le s} a_\alpha \lambda^{s-(\alpha:\nu)} \xi^\alpha \right| \ge \delta(\lambda+|\xi|_\nu)^s, \quad \xi \in \mathbb{R}^n, \ \lambda > 0.$$

□

Let $k, s \in \mathbb{N},\ k \geq s$. Consider the differential form $P(x,\mathbb{D})$ (see (1.1)), which is expressed in the following way:

$$P(x,\mathbb{D}) = \sum_{(\alpha:\nu)\leq s} a_\alpha(x) D^\alpha = \sum_{(\alpha:\nu)\leq s} \Big(a^0_\alpha(x) q(x)^{s-(\alpha:\nu)} + b_\alpha(x) \Big) D^\alpha, \tag{2.13}$$

where $a_\alpha(x) = a^0_\alpha(x) q(x)^{s-(\alpha:\nu)} + b_\alpha(x)$, $a^0_\alpha(x) \in C^{k-s,\nu}(\mathbb{R}^n), q \in Q^{k-s,\nu}$ and

$$D^\beta(b_\alpha(x)) = o(q(x)^{s-(\alpha:\nu)+(\beta:\nu)}),\ \text{when } |x| \to \infty, \quad (\alpha:\nu) \leq s,\ (\beta:\nu) \leq k-s.$$

Denote

$$L(x,\mathbb{D}) = \sum_{(\alpha:\nu)\leq s} b_\alpha(x) D^\alpha. \tag{2.14}$$

Theorem 2.2. *Let $k, s \in \mathbb{N}, k \geq s$, $q \in Q^{k-s,\nu}$ and $P(x,\mathbb{D})$ be the differential form (2.13) with the coefficients that satisfy $\lim\limits_{|x|\to\infty} \max\limits_{|x-y|\leq 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for $\alpha \in \mathbb{Z}^n_+$, $(\alpha:\nu) \leq s$. Let there exists a constant $\kappa > 0$ such that:*

$$\|u\|_{k,\nu,q} \leq \kappa \left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(\mathbb{R}^n)} \right), \forall u \in H^{k,\nu}_q(\mathbb{R}^n). \tag{2.15}$$

Then there exist constants $\delta > 0$ and $M > 0$ such that

$$\left| \sum_{(\alpha:\nu)\leq s} a^0_\alpha(x) \lambda^{s-(\alpha:\nu)} \xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu)^s, \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| \geq M.$$

Proof. Let $M > 0, x_M \in \mathbb{R}^n \backslash K_M, \varphi \in C^\infty_0(\mathbb{R}^n), \operatorname{supp}\varphi \subset K_1(x_M)$, $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$ and $\xi \in \mathbb{R}^n$. Consider the function $\tilde{u}(x) = e^{i(q(x_M)^{\frac{1}{\nu}}\xi, x)}\varphi(x)$.

Similar to the proof of Theorem 2.1 it is easy to check, that there exists a function $\varepsilon(M)$ such that $\varepsilon(M) \to 0$ when $M \to \infty$ and the following inequalities hold:

$$\|\tilde{u}\|_{k,\nu,q} \geq (1 - \varepsilon(M)) \|\tilde{u}\|_{k,\nu,q(x_M)}, \tag{2.16}$$

$$\|P\tilde{u}\|_{k-s,\nu,q} \leq (1 + \varepsilon(M)) \|P\tilde{u}\|_{k-s,\nu,q(x_M)}. \tag{2.17}$$

Taking into account the definition of the function $\tilde{u}$ one can check that with some constant $C_1 = C_1(\varphi) > 0$ the following holds:

$$\|\tilde{u}\|_{k,\nu,q(x_M)} \geq \sum_{(\alpha:\nu)\leq k} |\xi^\alpha|\, q(x_M)^k - C_1(1 + |\xi|_\nu)^k q(x_M)^{k - \frac{1}{\nu_{max}}}. \tag{2.18}$$

For any $\beta \in \mathbb{Z}_+^n, (\beta : \nu) \leq k - s$

$$\left\| D^\beta \left(P(x, \mathbb{D}) \widetilde{u} \right) \right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)} \leq \tag{2.19}$$

$$\leq \left\| \sum_{(\alpha:\nu) \leq s} a_\alpha^0(x_M) q(x_M)^{s-(\alpha:\nu)} D^{\alpha+\beta} \widetilde{u} \right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$

$$+ \sum_{(\alpha:\nu) \leq s} \left\| D^\beta \left(\left[a_\alpha^0(x) q(x)^{s-(\alpha:\nu)} - a_\alpha^0(x_M) q(x_M)^{s-(\alpha:\nu)} \right] D^\alpha \widetilde{u} \right) \right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}$$

$$+ \sum_{(\alpha:\nu) \leq s} \left\| D^\beta (b_\alpha(x) D^\alpha \widetilde{u}) \right\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-s-(\beta:\nu)}.$$

Since $a_\alpha^0(x) \in C^{k-s,\nu}(\mathbb{R}^n)$, $\lim\limits_{|x| \to \infty} \max\limits_{|x-y| \leq 1} \left| a_\alpha^0(x) - a_\alpha^0(y) \right| = 0$ and $q \in Q^{k-s,\nu}$, it is easy to check that for $\beta \in \mathbb{Z}_+^n$, $0 < (\beta : \nu) \leq k-s$ there exist functions $\tilde{\varepsilon}_1(M), \tilde{\varepsilon}_2(M)$ such that $\tilde{\varepsilon}_1(M), \tilde{\varepsilon}_2(M) \to 0$ when $M \to \infty$ and some constant $C_2 = C_2(P) > 0$ that the following inequalities hold

$$\left| a_\alpha^0(x) q(x)^{s-(\alpha:\nu)} - a_\alpha^0(x_M) q(x_M)^{s-(\alpha:\nu)} \right| \tag{2.20}$$

$$\leq \left| a_\alpha^0(x) - a_\alpha^0(x_M) \right| q(x)^{s-(\alpha:\nu)} + \left| a_\alpha^0(x_M) \left(q(x)^{s-(\alpha:\nu)} - q(x_M)^{s-(\alpha:\nu)} \right) \right|$$

$$\leq \tilde{\varepsilon}_1(M) q(x_M)^{s-(\alpha:\nu)}, \forall x \in K_1(x_M),$$

$$\left| D^\beta \left(a_\alpha^0(x) q(x)^{s-(\alpha:\nu)} \right) \right| \leq \tilde{\varepsilon}_2(M) q(x_M)^{s-(\alpha:\nu)+(\beta:\nu)} \tag{2.21}$$

$$+ C_2 q(x_M)^{s-(\alpha:\nu)+(\beta:\nu)-\frac{1}{\nu_{max}}}, \forall x \in K_1(x_M).$$

Taking into account that $D^\beta(b_\alpha(x)) = o(q(x)^{s-(\alpha:\nu)+(\beta:\nu)})$ when $|x| \to \infty$ and the definition of function $\tilde{u}$, for multiindices $\alpha, \beta \in \mathbb{Z}_+^n$ such that $(\alpha : \nu) \leq s, (\beta : \nu) \leq k - s$ with some constants $C_3 > 0, C_4 = C_4(P, \varphi) > 0$ we get the following estimate

$$\left\| D^\beta \left(b_\alpha(x) D^\alpha \widetilde{u} \right) \right\|_{L_2(\mathbb{R}^n)} \leq C_3 \sum_{0 \leq \gamma \leq \beta} \left\| D^\gamma (b_\alpha(x)) D^{\beta-\gamma+\alpha} \widetilde{u} \right\|_{L_2(\mathbb{R}^n)} \tag{2.22}$$

$$\leq \delta(M)(1+|\xi|_\nu)^{(\alpha:\nu)+(\beta:\nu)} q(x_M)^{s+(\beta:\nu)} + C_4 (1+|\xi|_\nu)^{(\alpha:\nu)+(\beta:\nu)} q(x_M)^{s+(\beta:\nu)-\frac{1}{\nu_{max}}},$$

where $\delta(M)$ is such a function that $\delta(M) \to 0$ when $M \to \infty$. Then from the estimates (2.19)–(2.22) with some constant $C_5 = C_5(\varphi, P) > 0$ we get

$$\|P\tilde{u}\|_{k-s,\nu,q(x_M)} \leq \sum_{(\beta:\nu) \leq k-s} \left| \xi^\beta \right| \left| \sum_{(\alpha:\nu) \leq s} a_\alpha^0(x_M) \xi^\alpha \right| q(x_M)^k \tag{2.23}$$

$$+ C_5 (1 + |\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}} + \tilde{\omega}(M)(1 + |\xi|_\nu)^k q(x_M)^k,$$

where $\tilde{\omega}(M)$ is such a function that $\tilde{\omega}(M) \to 0$ when $M \to \infty$.

From the estimate (2.15), according to (2.16)–(2.18), (2.23) and the definition of the function $\tilde{u}$, we get

$$(1-\varepsilon(M))\left(\sum_{(\beta:\nu)\le k}\left|\xi^{\beta}\right| q(x_M)^k - C_1(1+|\xi|_\nu)^k\right) q(x_M)^{k-\frac{1}{\nu_{max}}}$$
$$\le \kappa\left((1+\varepsilon(M))\left(\sum_{(\beta:\nu)\le k-s}\left|\xi^{\beta}\right|\left|\sum_{(\alpha:\nu)\le s} a^0(x_M)\xi^{\alpha}\right| q(x_M)^k\right.\right.$$
$$\left.\left.+C_5(1+|\xi|_\nu)^k q(x_M)^{k-\frac{1}{\nu_{max}}}+\tilde{\omega}(M)(1+|\xi|_\nu)^k q(x_M)^k\right)+1\right).$$

From the last inequality, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x|\to\infty$, $\tilde{\omega}(M)\to 0, \varepsilon(M)\to 0$ when $M\to\infty$, dividing by $(q(x_M))^k$, we get

$$\sum_{(\beta:\nu)\le k}\left|\xi^{\beta}\right| - \tau(M)(1+|\xi|_\nu)^k \le \kappa \sum_{(\beta:\nu)\le k-s}\left|\xi^{\beta}\right|\left|\sum_{(\alpha:\nu)\le s} a^0_\alpha(x_M)\xi^{\alpha}\right|,$$

where $\tau(M)$ is such a function that $\tau(M)\to 0$ when $M\to\infty$.

From the last estimate, using inequalities (2.12), we get

$$\left|\sum_{(\alpha:\nu)\le s} a^0_\alpha(x_M)\xi^{\alpha}\right| \ge \frac{\delta_1}{\kappa\delta_2}(1+|\xi|_\nu)^s - \frac{\tau(M)}{\kappa\delta_2}(1+|\xi|_\nu)^s. \tag{2.24}$$

Since $\tau(M)\to 0$ when $M\to\infty$, then there exists $M_0 = M_0(P,\varphi,\delta_1,\delta_2,\kappa)>0$ such that for any $M\ge M_0$ with some constant $\delta=\delta(\kappa,\delta_1,\delta_2)>0$ the following is true

$$\left|\sum_{(\alpha:\nu)\le s} a^0_\alpha(x)\xi^{\alpha}\right| \ge \delta(1+|\xi|_\nu)^s, \forall\xi\in\mathbb{R}^n, |x|\ge M.$$

Similarly to the proof of Theorem 2.1 from the last inequality it is easy to get the following

$$\left|\sum_{(\alpha:\nu)\le s} a^0_\alpha(x)\lambda^{s-(\alpha:\nu)}\xi^{\alpha}\right| \ge \delta(\lambda+|\xi|_\nu)^s, \forall\xi\in\mathbb{R}^n, \lambda>0, |x|\ge M.$$

□

Theorem 2.3. (see [14], theorem 7.1). *Let E, F and E_0 be Banach spaces such that E is compactly embedded in E_0. Let A be a bounded linear operator acting from E to F. Operator $A: E\to F$ is an $n-$normal if and only if there exists a constant $C>0$ such that*

$$\|x\|_E \le C\left(\|Ax\|_F + \|x\|_{E_0}\right), \quad x\in E.$$

Applying the previous theorem for operator $P(x,\mathbb{D})$, acting from $H^{k,\nu}_q(\mathbb{R}^n)$ to $H^{k-s,\nu}_q(\mathbb{R}^n)$, we get

Theorem 2.4. *Let $P(x,\mathbb{D})$ be differential form* (1.1). *Then operator $P(x,\mathbb{D})$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k-s,\nu}(\mathbb{R}^n)$, is an $n-$normal if and only if there exist constants $\kappa>0$ and $R>0$ such that the following holds*

$$\|u\|_{k,\nu,q} \le \kappa\left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(K_R)}\right), \quad u \in H_q^{k,\nu}(\mathbb{R}^n).$$

Corollary 2.1. *Let $k,s\in\mathbb{N}, k\ge s$, $q\in Q^{k-s,\nu}$ and $P(x,\mathbb{D})$ be the differential form* (2.13) *with the coefficients that satisfy $\lim\limits_{|x|\to\infty}\max\limits_{|x-y|\le 1}|a_\alpha^0(x)-a_\alpha^0(y)|=0$ for $\alpha\in\mathbb{Z}_+^n$, $(\alpha:\nu)\le s$. Let operator $P(x,\mathbb{D})$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k-s,\nu}(\mathbb{R}^n)$, be a Fredholm operator. Then there exist constants $\delta>0$ and $M>0$ such that*

$$\left|\sum_{(\alpha:\nu)\le s} a_\alpha^0(x)\lambda^{s-(\alpha:\nu)}\xi^\alpha\right| \ge \delta(\lambda+|\xi|_\nu)^s, \quad \xi\in\mathbb{R}^n,\ \lambda>0,\ |x|\ge M.$$

Proof. Since operator $P(x,\mathbb{D})$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k-s,\nu}(\mathbb{R}^n)$, is a Fredholm operator, then it is an $n-$normal operator. From Theorem 2.4 we get that there exist such constants $\kappa>0$ and $R>0$ that the following estimate holds

$$\|u\|_{k,\nu,q} \le \kappa\left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(K_R)}\right) \le \kappa\left(\|Pu\|_{k-s,\nu,q} + \|u\|_{L_2(\mathbb{R}^n)}\right),\ u\in H_q^{k,\nu}(\mathbb{R}^n).$$

From last estimate and the conditions on the coefficients of $P(x,\mathbb{D})$ using Theorem 2.2 we obtain that there exist constants $\delta>0$ and $M>0$ such that

$$\left|\sum_{(\alpha:\nu)\le s} a_\alpha^0(x)\lambda^{s-(\alpha:\nu)}\xi^\alpha\right| \ge \delta(\lambda+|\xi|_\nu)^s, \quad \xi\in\mathbb{R}^n,\ \lambda>0,\ |x|\ge M.$$

□

Theorem 8.5.14 from [12] can be formulated in the following equivalent way:

Theorem 2.5. *Let A be a bounded linear operator acting from a Banach space X to a Banach space Y. Then the following holds:*

(1) *if operator A has left regularizer, then kernel of operator A in X is finite dimensional;*
(2) *if operator A has right regularizer, then the image of operator A is closed in Y and cokernel is finite dimensional;*
(3) *operator A has left and right regularizers if and only if A is a Fredholm operator.*

It is easy to check that the following proposition holds:

Proposition 2.1. *Let $k,s\in\mathbb{N}, k\ge s, q\in Q^{k-s,\nu}$, $P(x,\mathbb{D})$ be the differential expression of the form* (1.1) *with the coefficients that satisfy conditions* (1.4) *and $\varphi\in C_0^\infty(\mathbb{R}^n)$. Then operator*

$$Tu := P(u\varphi) - \varphi Pu, \quad u\in H_q^{k,\nu}(\mathbb{R}^n)$$

is a compact operator acting from $H_q^{k,\nu}(\mathbb{R}^n)$ *to* $H_q^{k-s,\nu}(\mathbb{R}^n)$.

Theorem 2.6. *Let* $k, s \in \mathbb{N}, k \geq s, q \in Q^{k-s,\nu}$ *and the differential form* $P(x, \mathbb{D})$ *(see (2.13)) be semielliptic in* $\mathbb{R}^n$ *with the coefficients that satisfy*

$$\lim_{|x|\to\infty} \max_{|x-y|\leq 1} |a_\alpha^0(x) - a_\alpha^0(y)| = 0, \quad \alpha \in \mathbb{Z}_+^n, (\alpha : \nu) \leq s.$$

Then the operator $P(x, \mathbb{D}) : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ *is a Fredholm operator if and only if there exist constants* $\delta > 0$ *and* $M > 0$ *such that*

$$\left| \sum_{(\alpha:\nu)\leq s} a_\alpha^0(x)\lambda^{s-(\alpha:\nu)}\xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu)^s, \quad \xi \in \mathbb{R}^n,\ \lambda > 0,\ |x| \geq M. \tag{2.25}$$

Proof. Let's first prove sufficient part.

Let $\delta_0 > 0, \varphi(x) \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^n$ and $\varphi(x) = 1$ for $x \in K_{\frac{\delta_0}{2}}$, $\varphi(x) = 0$ for $|x| \geq \delta_0$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset K_{2\delta_0}$ and $\psi(x) = 1$ for $x \in K_{\delta_0}$. Let $\omega > 0$ be such that $\omega\sqrt{n} < \delta_0$. Let's denote $\{z_m\}_{m=0}^\infty$ points on the lattice in $\mathbb{R}^n$ with a side equals to ω.

Denote

$$\varphi_m(x) := \varphi(x - z_m) \left(\sum_{l=0}^\infty \varphi(x - z_l) \right)^{-1}, \quad \psi_m(x) := \psi(x - z_m), \quad m \in \mathbb{Z}_+.$$

Then $\{\varphi_m\}_{m=0}^\infty$ is a partition of unity that satisfies the following condition:

(i) $\max_{x,y\in\operatorname{supp}\varphi_m} |x - y| < \delta_0$,

(ii) there exists $r \in \mathbb{N}$ such that for any number i there are no more than r functions $\varphi_j(x)$ such that $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j \neq \varnothing$;

(iii) for any $\alpha \in \mathbb{Z}_+^n$ there exists some constant $C_\alpha > 0$ such that $|D^\alpha\varphi_m(x)| \leq C_\alpha$, ny $x \in \mathbb{R}^n,\ m \in \mathbb{Z}_+$.

Denote $W_m = \operatorname{supp}\varphi_m,\ m \in \mathbb{Z}_+$. Let $x_m \in W_m$ and $m_0 \in \mathbb{N}$. For $m \leq m_0$ denote

$$P^m(x, \mathbb{D}) := \sum_{(\alpha:\nu)\leq s} \left(\psi_m(x)\left(a_\alpha(x) - a_\alpha(x_m)\right) + a_\alpha(x_m)\right) D^\alpha.$$

For $m > m_0$ denote

$$P^m(x, \mathbb{D}) := \sum_{(\alpha:\nu)\leq s} \left(\psi_m(x)\left(a_\alpha^0(x)q(x)^{s-(\alpha:\nu)} - a_\alpha^0(x_m)q(x_m)^{s-(\alpha:\nu)}\right) + a_\alpha^0(x_m)q(x_m)^{s-(\alpha:\nu)}\right) D^\alpha.$$

Since $q \in Q^{k-s,\nu}$ and $\lim_{m\to\infty} \max_{|x-x_m|\leq 1} |a_\alpha^0(x) - a_\alpha^0(x_m)| = 0$, according to Theorem 2.2 from [7], we can choose m_0 big enough such that for $m > m_0$ operator P^m : $H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ has the inverse operator $R^m : H_q^{k-s,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$.

For $m \le m_0$ consider

$$R_0^m := F^{-1} \frac{|\xi|_\nu^s}{(1+|\xi|_\nu^s) P_s^m(x_m,\xi)} F.$$

Since $P(x,\mathbb{D})$ is semielliptic in $\mathbb{R}^n$, then using Lemma 4.3 from work [13] we get that for a small enough δ_0 from condition (i) the following holds

(2.26)

$$R_0^m P^m(x,\mathbb{D}) = R_0^m P^m(x_m,\mathbb{D}) + R_0^m\left(P^m(x,\mathbb{D}) - P^m(x_m,\mathbb{D})\right) = I + T_1^m + T_2^m,$$

where $T_1^m : H^{k,\nu}(\mathbb{R}^n) \to H^{k+\sigma,\nu}(\mathbb{R}^n)$ with some number $\sigma = \sigma(\nu) > 0$ and operator $T_2^m : H^{k,\nu}(\mathbb{R}^n) \to H^{k,\nu}(\mathbb{R}^n)$ satisfies $\|T_2^m\| < 1$.

For $m \le m_0$ let $R^m := (I + T_2^m)^{-1} R_0^m$. From (2.26) we have

$$R^m P^m(x,\mathbb{D}) = I + T^m, \tag{2.27}$$

where $T^m : H^{k,\nu}(\mathbb{R}^n) \to H^{k+\sigma,\nu}(\mathbb{R}^n)$ with some number $\sigma = \sigma(\nu) > 0$. Denote

$$Rf := \sum_{l=0}^{\infty} \psi_l R^l(\varphi_l f),\, f \in H_q^{k-s,\nu}(\mathbb{R}^n).$$

Since (2.25) holds one can check that the norms of operators R^l, acting from $H_q^{k-s,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, are uniformly bounded. From this fact, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$ and properties (i)–(iii) of the functions $\{\varphi_m\}_{m=0}^{\infty}, \{\psi_m\}_{m=0}^{\infty}$, it is easy to check that R is a bounded linear operator, acting from $H_q^{k-s,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$.

For $P(x,\mathbb{D})$ and $RP(x,\mathbb{D})$, taking into account (2.13), (2.14) and definitions of the functions $\{\varphi_m\}_{m=0}^{\infty}, \{\psi_m\}_{m=0}^{\infty}$, we have the following representations

$$P(x,\mathbb{D})u = \sum_{m=0}^{\infty} \varphi_m P(x,\mathbb{D})(\psi_m u)$$
$$= \sum_{m=0}^{m_0} \varphi_m P^m(x,\mathbb{D})(\psi_m u) + \sum_{m=m_0+1}^{\infty} \varphi_m P^m(x,\mathbb{D})(\psi_m u) + \sum_{m=m_0+1}^{\infty} \varphi_m L(x,\mathbb{D})(\psi_m u),$$

(2.28)

$$RP(x,\mathbb{D})u = \sum_{l=0}^{m_0} \sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l \varphi_m P^m(\psi_m u)\right) + \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l \varphi_m P^m(\psi_m u)\right)$$
$$+ \sum_{l=m_0+1}^{\infty} \sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l \varphi_m P^m(\psi_m u)\right) + \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l \varphi_m P^m(\psi_m u)\right)$$
$$+ \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l \varphi_m L(\psi_m u)\right) + \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l \varphi_m L(\psi_m u)\right),$$

where $u \in H_q^{k,\nu}(\mathbb{R}^n)$.

For $m, l \in \mathbb{Z}_+$ such that $l \leq m_0$ and $m \leq m_0$, based on the definitions of $P^m(x,\mathbb{D})$ and the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$, the following holds:

$$\varphi_l\varphi_m P^m(x,\mathbb{D})\,(\psi_m u) = \varphi_l\varphi_m P(x,\mathbb{D})\,(\psi_m u) = \varphi_l\varphi_m P^l(x,\mathbb{D})\,(\psi_m u)\,.$$

From the last equality, using (2.27) and the fact that $\varphi_m(x)\psi_m(x) = \varphi_m(x)$ for all $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}_+$, we get

$$\begin{aligned}
&\sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) = \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)\\
&= \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l P^l(\varphi_l\varphi_m\psi_m u) + \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\right)\\
&\qquad = \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \varphi_l\varphi_m u + \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l T^l(\varphi_l\varphi_m u)\\
&\qquad\qquad + \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\right),
\end{aligned}$$

where $u \in H_q^{k,\nu}(\mathbb{R}^n)$. Consider

$$T_1 := \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l T^l(\varphi_l\varphi_m \cdot) + \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m \cdot) - P^l(\varphi_l\varphi_m\psi_m \cdot)\right).$$

Using Proposition 2.1 we get that $\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m \cdot) - P^l(\varphi_l\varphi_m\psi_m \cdot)\right)$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, is a compact operator. Similarly, since $T^l : H^{k,\nu}(\mathbb{R}^n) \to H^{k+\sigma,\nu}(\mathbb{R}^n)$ with some $\sigma > 0$, it is easy to check that operator $\psi_l T^l(\varphi_l\varphi_m \cdot)$, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$ is a compact operator. As the finite sum of compact operators T_1 is a compact operator, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$. So we get

$$\sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) = \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \varphi_l\varphi_m u + T_1 u, u \in H_q^{k,\nu}(\mathbb{R}^n),$$

where $T_1 : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ is a compact operator.

For $m, l \in \mathbb{Z}_+$ such that $l \leq m_0$ and $m > m_0$, based on the definitions of $P^m(x,\mathbb{D})$, $L(x,\mathbb{D})$ and the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$, the following holds:

$$\begin{aligned}
\varphi_l\varphi_m P^m(x,\mathbb{D})\,(\psi_m u) &= \varphi_l\varphi_m\left(P(x,\mathbb{D}) - L(x,\mathbb{D})\right)(\psi_m u)\\
&= \varphi_l\varphi_m P^l(x,\mathbb{D})\,(\psi_m u) - \varphi_l\varphi_m L(x,\mathbb{D})\,(\psi_m u)\,.
\end{aligned}$$

From the last equality we get

$$\begin{aligned}
(2.29)\quad \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) &= \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)\\
&- \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty} \psi_l R^l\left(\varphi_l\varphi_m L(x,\mathbb{D})\,(\psi_m u)\right), u \in H_q^{k,\nu}(\mathbb{R}^n).
\end{aligned}$$

Now consider

$$\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot)\right).$$

Using (2.27) and the properties of the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$ we can check that the following holds:

$$\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)=\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\varphi_l\varphi_m u+T_2 u,\, u\in H_q^{k,\nu}(\mathbb{R}^n)$$

where

$$T_2:=\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l T^l(\varphi_l\varphi_m\cdot)+\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot)-P^l(\varphi_l\varphi_m\psi_m\cdot)\right)$$

$$=\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{m_1}\psi_l T^l(\varphi_l\varphi_m\cdot)+\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{m_1}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot)-P^l(\varphi_l\varphi_m\psi_m\cdot)\right),$$

where

$$m_1:=\max_{m>m_0}\{m:\operatorname{supp}\varphi_m\bigcap\left(\bigcup_{l=0}^{m_0}\operatorname{supp}\varphi_l\right)\neq\varnothing\}.$$

Since T_2 contains the finite number of terms for which $\varphi_l\varphi_m\neq 0$, similarly as for operator T_1, we can show that T_2 is a compact operator, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$.

For $m,l\in\mathbb{Z}_+$ such that $l>m_0$ and $m\leq m_0$, based on the definitions of $P^m(x,\mathbb{D})$, $L(x,\mathbb{D})$ and the functions $\{\varphi_m\}_{m=0}^{\infty}$, $\{\psi_m\}_{m=0}^{\infty}$, the following holds:

$$\varphi_l\varphi_m P^m(x,\mathbb{D})\,(\psi_m u)=\varphi_l\varphi_m P^l(x,\mathbb{D})\,(\psi_m u)+\varphi_l\varphi_m L(x,\mathbb{D})\,(\psi_m u)\,.$$

Analogously, from the last equality and the fact that for $l>m_0$ operators $R^l: H_q^{k-s,\nu}(\mathbb{R}^n)\to H_q^{k,\nu}(\mathbb{R}^n)$ are the inverse operators of $P^l: H_q^{k,\nu}(\mathbb{R}^n)\to H_q^{k-s,\nu}(\mathbb{R}^n)$ we get

$$\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right)=\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)+\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m L(x,\mathbb{D})(\psi_m u)\right),\tag{2.30}$$

$$\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)=\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\varphi_l\varphi_m u+T_3 u,\ u\in H_q^{k,\nu}(\mathbb{R}^n)\tag{2.31}$$

where

$$T_3:=\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot)-P^l(\varphi_l\varphi_m\psi_m\cdot)\right)=$$

$$\sum_{l=m_0+1}^{m_1}\sum_{m=0}^{m_0}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m\cdot)-P^l(\varphi_l\varphi_m\psi_m\cdot)\right),$$

$$m_1 := \max_{l>m_0}\{l : \operatorname{supp}\varphi_l \cap \left(\bigcup_{j=0}^{m_0} \operatorname{supp}\varphi_j\right) \neq \varnothing\}.$$

As T_3 contains the finite number of terms for which $\varphi_l\varphi_m \neq 0$, taking into account Proposition 2.1, we get that operator T_3 is a compact operator, acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$.

For $l > m_0$ and $m > m_0$, based on the definitions of $P^m(x,\mathbb{D})$ and the functions $\{\varphi_m\}_{m=0}^\infty$, $\{\psi_m\}_{m=0}^\infty$, we have:

$$\varphi_l\varphi_m P^m(x,\mathbb{D})\,(\psi_m u) = \varphi_l\varphi_m P^l(x,\mathbb{D})\,(\psi_m u)\,.$$

From the last equality and the fact that for $m > m_0$ operators $R^m : H_q^{k-s,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ are the inverse operators of $P^m : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ we get

$$\sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^m(\psi_m u)\right) = \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u)\right)$$
$$= \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\varphi_l\varphi_m u + \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l\left(\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\right),$$

where $u \in H_q^{k,\nu}(\mathbb{R}^n)$.

Taking into account (2.13), the definitions of $P^l(x,\mathbb{D})$ and the properties of functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, for $l > m_0$ and $m > m_0$ with some constant $C_1 > 0$ we get

$$\|\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\|_{k-s,\nu,q}$$
$$\leq C_1\left\|\sum_{(\alpha:\nu)\leq s}\sum_{\beta+\gamma=\alpha,|\gamma|>0} a_\alpha^0(x)D^\beta(\psi_m u)D^\gamma(\varphi_l\varphi_m)q(x)^{s-(\alpha:\nu)}\right\|_{k-s,\nu,q}$$
$$\leq C_1\left\|\sum_{(\alpha:\nu)\leq s}\sum_{\beta+\gamma=\alpha,|\gamma|>0} a_\alpha^0(x)D^\gamma(\varphi_l\varphi_m)\frac{1}{q(x)^{(\gamma:\nu)}}D^\beta(\psi_m u)q(x)^{s-(\beta:\nu)}\right\|_{k-s,\nu,q}.$$

From the last inequality, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$, properties (i)–(iii) of the functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$ and the conditions on the coefficients $\{a_\alpha^0(x)\}$ (see (2.13)) we get

$$\|\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\|_{k-s,\nu,q} \leq \omega(m_0)\|u\|_{H_q^{k,\nu}(W_l\cap W_m)}. \tag{2.32}$$

where $\omega(m_0)$ is such a function that $\omega(m_0) \to 0$ when $m_0 \to \infty$.

Since (2.25) holds the norms of operators R^l, acting from $H_q^{k-s,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, are uniformly bounded. Using this fact, inequality (2.32), the properties (i)–(iii) of the functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, it is easy to check that for a big enough m_0

operator

$$T_4 := \sum_{l=m_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left[\varphi_l \varphi_m P^l(\psi_m \cdot) - P^l(\varphi_l \varphi_m \psi_m \cdot)\right],$$

acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, satisfies $\|T_4\| < \frac{1}{2}$.

Similarly for remained terms from (2.28), (2.29) and (2.30), taking into account that $D^\beta(b_\alpha(x)) = o(q(x)^{s-(\alpha:\nu)+(\beta:\nu)})$ when $|x| \to \infty$, $(\alpha:\nu) \le s, (\beta:\nu) \le k-s$ (see (2.13), (2.14)), for a big enough m_0 we get that the operator

$$T_5 := \sum_{l=0}^{\infty} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left(\varphi_l \varphi_m L(\psi_m \cdot)\right) - \sum_{l=0}^{m_0} \sum_{m=m_0+1}^{\infty} \psi_l R^l \left(\varphi_l \varphi_m L(\psi_m \cdot)\right) + \sum_{l=m_0+1}^{\infty} \sum_{m=0}^{m_0} \psi_l R^l \left(\varphi_l \varphi_m L(\psi_m \cdot)\right),$$

acting from $H_q^{k,\nu}(\mathbb{R}^n)$ to $H_q^{k,\nu}(\mathbb{R}^n)$, has a norm that satisfies $\|T_5\| < \frac{1}{2}$.

Denote

$$T^{'} := T_1 + T_2 + T_3, T^{''} := T_4 + T_5.$$

From the representation (2.28) we get

$$RPu = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \varphi_l \varphi_m u + T_1 u + T_2 u + T_3 u + T_4 u + T_5 u = u + T^{'} u + T^{''} u,$$

where $u \in H_q^{k,\nu}(\mathbb{R}^n)$, $T^{'} : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ is a compact operator and for operator $T^{''} : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ we have $\|T^{''}\| < 1$.

Therefore

$$\left(I + T^{''}\right)^{-1} RP = I + \left(I + T^{''}\right)^{-1} T^{'},$$

where $T := \left(I + T^{''}\right)^{-1} T^{'} : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ is a compact operator. So we get that operator $\left(I + T^{''}\right)^{-1} R : H_q^{k-s,\nu}(\mathbb{R}^n) \to H_q^{k,\nu}(\mathbb{R}^n)$ is a left regularizer.

Analogously we can construct a right regularizer.

Since right and left regularizers exist, applying Theorem 2.5, we obtain the Fredholm property of operator $P(x, \mathbb{D}) : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$.

Necessity of condition (2.25) for the Fredholm property of $P(x, \mathbb{D}) : H_q^{k,\nu}(\mathbb{R}^n) \to H_q^{k-s,\nu}(\mathbb{R}^n)$ follows from Corollary 2.1. □

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CAUSALITY BETWEEN STOPPED FILTRATIONS AND SOME APPLICATIONS

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Abstract. In this paper we consider the concept of statistical causality in continuous time between filtrations associated with stopping times, which is based on Granger's definition of causality. Especially, we consider a generalization of a causality relationship "**H** is a cause of **E** within **F**" from fixed to stopping time. Then we apply the given concept of causality to strongly orthogonal stopped martingales. We show the equivalence between the given concept of causality and orthogonality of stopped local martingales, too.

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Keywords: filtration; causality; stopped local martingales; stopped orthogonal martingales.

1. Introduction

We consider causality in continuous time which unifies the nonlinear Granger's causality with some related concepts. Here, the concept of causality is analyzed using the tool of conditional independence among the σ-fields.

The Granger causality is focused on discrete time stochastic processes (time series). But, in many cases, for example in economy and finance, it may be difficult to capture relations of causality in discrete-time model and it may depend on the length of interval between each sampling. So, continuous time models become more and more frequent in econometrics (see, for example, [1] - [6]). In this paper we will consider the continuous time processes. The continuous time framework is fruitful, not only for the internal consistency of economic theories but also for the statistical approach to causality analysis between stochastic processes that rapidly evolve (see [7]).

The paper is organized as follows. After Introduction, in the Section 2 we present a generalization of a causality concept "**H** is a cause of **E** within **F**", which involves prediction in any horizon in continuous time. This concept is based on Granger's definition of causality (see [3]). The concept of causality in continuous time associated with stopping times with some basic properties is introduced in [8]. In this paper

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we give some new properties of causality concept between stopped filtrations and between stopped processes.

The given concept of causality can be connected with the stable subspaces of H^p (see [9]) and with the orthogonality of martingales (see [10]). Also, weak solutions and local weak solutions of the stochastic differential equations driven with semimartingales, as well as solutions of martingale problem can be expressed using the given concept of causality (see [6, 11]). The preservation of the martingale property is directly connected with the concept of causality (see [12]).

The Section 3 and Section 4 contain our main results. The Section 4 relates the given concept of statistical causality in continuous time to the orthogonality of stopped martingales and stopped local martingales. Also, we investigate the case when the processes are stopped by the different stopping times.

Some applications in finance are given in the Section 5. More specifically, we showed that the given concept of causality is strongly connected with the question of locally risk minimization strategy for defaultable claims.

2. Preliminaries and notation

Causality is, in any case, a prediction property and the central question is: is it possible to reduce available information in order to predict a given filtration?
A probabilistic model for a time-dependent system is described by $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathbf{F} = \{\mathcal{F}_t,\ t \in I, I \subseteq R^+\}$ is a "framework" filtration that satisfies the usual conditions of right continuity and completeness. $\mathcal{F}_\infty = \bigvee_{t\in I} \mathcal{F}_t$ is the smallest σ-algebra containing all the $\{\mathcal{F}_t\}$. An analogous notation will be used for filtrations $\mathbf{H} = \{\mathcal{H}_\mathbf{t}\}$, $\mathbf{G} = \{\mathcal{G}_\mathbf{t}\}$ and $\mathbf{E} = \{\mathcal{E}_\mathbf{t}\}$. It is said that the filtration $\mathbf{G}$ is a subfiltration of $\mathbf{H}$ and written as $\mathbf{G} \subseteq \mathbf{H}$, if $\mathcal{G}_t \subseteq \mathcal{H}_t$ for each t. Given a stochastic process X we denote by $\{\mathcal{F}_t^X\}$ the smallest σ-algebra for which all X_s with $s \leqslant t$, are measurable and $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in I\}$ is the natural filtration of X. The natural filtration $\mathbf{F}^X$ is the smallest filtration that makes X to be adapted.

The intuitive notion of causality in continuous time formulated in terms of Hilbert spaces is given in [5]. We consider the analogous notion of causality for filtrations using the conditional independence between sub-σ-algebras of $\mathcal{F}$ (see [13] and [14]).

Definition 2.1. (see [2] and [5]) It is said that $\mathbf{H}$ is a cause of $\mathbf{E}$ within $\mathbf{F}$ relative to P (and written as $\mathbf{E} \mid\!< \mathbf{H}; \mathbf{F}; P$) if $\mathcal{E}_\infty \subseteq \mathcal{F}_\infty$, $\mathbf{H} \subseteq \mathbf{F}$ and if $\mathcal{E}_\infty$ is conditionally independent of $\{\mathcal{F}_t\}$ given $\{\mathcal{H}_t\}$ for each t, i.e. $\mathcal{E}_\infty \perp \mathcal{F}_t | \mathcal{H}_t$ (i.e. $\mathcal{E}_u \perp \mathcal{F}_t | \mathcal{H}_t$ holds for each t and each u), or

$$(\forall A \in \mathcal{E}_\infty) \quad P(A|\mathcal{F}_t) = P(A|\mathcal{H}_t). \tag{2.1}$$

Intuitively, $\mathbf{E} \mid\!< \mathbf{H}; \mathbf{F}; P$ means that all information about $\mathcal{E}_\infty$ that gives $\{\mathcal{F}_t\}$ comes via $\{\mathcal{H}_t\}$ for arbitrary t; equivalently, $\{\mathcal{H}_t\}$ contains all the information from the $\{\mathcal{F}_t\}$ needed for predicting $\mathcal{E}_\infty$. We can consider subfiltration $\mathbf{H} \subseteq \mathbf{F}$ as a reduced information.

The definition similar to Definition 2.1 was first given in [4]: "It is said that $\mathbf{H}$ entirely causes $\mathbf{E}$ within $\mathbf{F}$ relative to P (and written as $\mathbf{E} \mid\!< \mathbf{H}; \mathbf{F}$;P) if $\mathbf{E} \subseteq \mathbf{F}$, $\mathbf{H} \subseteq \mathbf{F}$ and if $\mathcal{E}_\infty \perp \mathcal{F}_t|\mathcal{H}_t$ for each t". Instead of $\mathcal{E}_\infty \subseteq \mathcal{F}_\infty$ this definition contains the condition $\mathbf{E} \subseteq \mathbf{F}$, or equivalently $\mathcal{E}_t \subseteq \mathcal{F}_t$ for each t, which does not have intuitive justification. Since the Definition 2.1 is a more general than the definition given in [4], all results related to causality in the sense of the Definition 2.1 will also be true in the sense of the Definition from [4] (pg.3), when we add the condition $\mathbf{E} \subseteq \mathbf{F}$.

It should be mentioned that the definition of causality from [4] is equivalent to definition of strong global noncausality as given in [1]. So, the Definition 2.1 is a generalization of the notion of strong global noncausality. The equivalence between the statistical causality concept and the concept of adapted distribution given by Hoover and Keisler in [15] is proven in [16].

If $\mathbf{H}$ and $\mathbf{F}$ are such that $\mathbf{H} \mid\!< \mathbf{H}; \mathbf{F}; P$ we shall say that $\mathbf{H}$ is its own cause (or, self caused) within $\mathbf{F}$ (compare with [4]). It should be noted that the statement "**H** is its own cause" sometimes occurs as a useful assumption in the theory of martingales and stochastic integration (see [12]). The concept of being "its own cause" is equivalent to the hypothesis $(\mathcal{H})$ introduced in [12]. It also, should be mentioned that the notion of subordination (as introduced in [17]) is equivalent to the notion of being "its own cause"as defined here.

If $\mathbf{H}$ and $\mathbf{F}$ are such that $\mathbf{H} \mid\!< \mathbf{H}; \mathbf{H} \bigvee \mathbf{F}$ (where $\mathbf{H} \bigvee \mathbf{F}$ is a family determined by $(H \bigvee F)_t = H_t \bigvee F_t$), we shall say that $\mathbf{F}$ does not cause $\mathbf{H}$. Now, it is clear that the interpretation of Granger–causality is that $\mathbf{F}$ does not cause $\mathbf{H}$ if $\mathbf{H} \mid\!< \mathbf{H}; \mathbf{H} \bigvee \mathbf{F}$ holds (see [4]). Without difficulty, it can be shown that this term and the term "**F** does not anticipate **H**"(as introduced in [17]) are identical.

These definitions can be applied to stochastic processes if we consider corresponding induced filtrations. For example, $\{\mathcal{F}_t\}$-adapted stochastic process X_t is its own cause if $\{\mathcal{F}_t^X\}$ is its own cause within $\{\mathcal{F}_t\}$ i.e. if

$$\mathbf{F}^X \mid\!< \mathbf{F}^X; \mathbf{F}; P.$$

Process X which is its own cause is completely described by its behavior with respect to its natural filtration $\mathbf{F}^X$ (see [10]). For example, process $X = \{X_t, t \in I\}$ is a Markov process with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ if and only if X is a Markov process with respect to $\mathbf{F}^X$ and if it is its own cause within $\mathbf{F}$ relative to P. As a consequence, Brownian motion $W = \{W_t, t \in I\}$ with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ is its own cause within $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ relative to probability P.

In many situations we observe certain systems up to some random time, for example up to time when something happens for the first time. So, it is natural to consider causality in continuous time which involves stopping times, a class of random variables that plays essential role in the theory of martingales (for details see [18] and [19]).

If τ is a stopping time with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t\}$, the associated σ-algebra $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{R}_+\}$ is a set of events that occur up to time τ. For a process X, we set $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, whenever $\tau(\omega) < +\infty$. We define the stopped process $X^\tau = \{X_{t\wedge\tau}, t \in I\}$ with

$$X_t^\tau(\omega) = X_{t\wedge\tau(\omega)}(\omega) = X_t\chi_{\{t<\tau\}} + X_\tau\chi_{\{t\geqslant\tau\}}.$$

Note that if X is adapted and cadlag and if τ is a stopping time, then the stopped process X^τ is adapted, too. The family of σ-fields $\mathbf{F}^\tau = \{\mathcal{F}_{t\wedge\tau}\}$ is a stopped filtration (for details, see [13]).

The generalization of the Definition 2.1 from fixed to stopping time is introduced in [8].

Definition 2.2. ([8]) Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ and $\mathbf{E} = \{\mathcal{E}_t\}, t \in I$, be given filtrations on the probability space $(\Omega, \mathcal{F}, P)$ and let τ be a stopping time with respect to filtration $\mathbf{E}$. The filtration $\mathbf{H}^\tau$ entirely causes $\mathbf{E}^\tau$ within $\mathbf{F}^\tau$ relative to P (and written as $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$) if $\mathbf{E}^\tau \subseteq \mathbf{F}^\tau$, $\mathbf{H}^\tau \subseteq \mathbf{F}^\tau$ and if $\mathcal{E}_\tau$ is conditionally independent of $\{\mathcal{F}_{t\wedge\tau}\}$ given $\{\mathcal{H}_{t\wedge\tau}\}$ for each t, i.e. $\mathcal{E}_\tau \perp \mathcal{F}_{t\wedge\tau} \mid \mathcal{H}_{t\wedge\tau}$ for all t, or

$$(\forall t \in I)(\forall A \in \mathcal{E}_\tau) \qquad P(A \mid \mathcal{F}_{t\wedge\tau}) = P(A \mid \mathcal{H}_{t\wedge\tau}). \tag{2.2}$$

The concept of causality given in the Definition 2.2 is defined up to some specified stopping time τ. It includes the stopped filtrations. The relation (2.2) does not consider the causality up to infinite horizon, so it does not imply (2.1).

Compared to the Definition 2.1, in the Definition 2.2 we have reduced the amount of information needed for predicting some other filtration.

3. Some properties of the stopped causality

Some basic properties of the concept of causality characterized with the stopping time are given in ([8]). We now prove that some new properties holds.

Theorem 3.1. *Let* $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ *and* $\mathbf{E} = \{\mathcal{E}_t\}$ *be given filtrations on the measurable space* $(\Omega, \mathcal{F})$ *and let* τ *be a stopping time with respect to* $\mathbf{E}$. *Let* P *and* Q *be a probability measures on* $\mathcal{F}$ *satisfying* $Q \ll P$ *with* $\dfrac{dQ}{dP}$ *as* $(\mathcal{E}_\tau)$*-measurable. Then*

$$\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P \quad \textit{implies} \quad \mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; Q.$$

Proof. Let E_P and E_Q be a conditional expectation for the measures P and Q respectively, and I parametric set. Since $Q \ll P$, the right regular version of the density process $L_{t\wedge\tau}$, where

$$L_{t\wedge\tau} = L_t \chi_{\{t<\tau\}} + L_\tau \chi_{\{t\geqslant\tau\}}$$

is the cadlag modification of $E_P(L_\infty \mid \mathcal{F}_{t\wedge\tau})$ where $L_\infty = \dfrac{dQ}{dP}$ is the Radon-Nykodim derivative. Obviously, since L_∞ is $\mathcal{E}_\tau$-measurable, we have

$$L_{t\wedge\tau} = E_P(L_\infty \mid \mathcal{F}_{t\wedge\tau}) = E_P(L_\infty \mid \mathcal{H}_{t\wedge\tau}). \tag{3.1}$$

Let $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ holds. Because of causality, process $M_{t\wedge\tau} = P(A \mid \mathcal{F}_{t\wedge\tau})$, for all $A \in \mathcal{E}_\tau$ is $(\mathcal{H}_{t\wedge\tau}, P)$-martingale. Now, we proove that $(ML)^\tau$ is a $(\mathcal{F}_{t\wedge\tau}, P)$-martingale. For $s < t < \tau$ and $F \in \mathcal{F}_{s\wedge\tau}$ we have

$$\begin{aligned} \int_F M_{t\wedge\tau} L_{t\wedge\tau} dP &= \int_F M_{t\wedge\tau} \frac{dQ_t}{dP} dP = \int_F M_{t\wedge\tau} dQ_t = \int_F M_{s\wedge\tau} dQ_s \\ &= \int_F M_{s\wedge\tau} \frac{dQ_s}{dP} dP = \int_F M_{s\wedge\tau} L_{s\wedge\tau} dP, \end{aligned} \tag{3.2}$$

and for $\tau \leqslant t \leqslant r$, by the Optional Sampling Theorem for $F \in \mathcal{F}_{t\wedge\tau}$ and due to (3.2), we have

$$\begin{aligned} \int_F M_{t\wedge\tau} L_{t\wedge\tau} dP &= \int_F M_{t\wedge\tau} L_\tau dP = \int_F M_{t\wedge\tau} L_t dP = \int_F M_{t\wedge\tau} \frac{dQ_t}{dP} dP \\ &= \int_F M_{t\wedge\tau} dQ = \int_F M_{r\wedge\tau} dQ = \int_F M_{r\wedge\tau} L_{r\wedge\tau} dP. \end{aligned}$$

Therefore,

$$M_{t\wedge\tau} L_{t\wedge\tau} = E(M^\tau_\infty L^\tau_\infty \mid \mathcal{F}_{t\wedge\tau}). \tag{3.3}$$

According to Lemma 6.6 in [18], equalities (3.1) and (3.3), $(\forall A \in \mathcal{E}_\tau)$ $(\forall t \in I)$ it follows that

$$\begin{aligned} M_{t\wedge\tau}L_{t\wedge\tau} &= P(A \mid \mathcal{F}_{t\wedge\tau})L_{t\wedge\tau} = E_P(\chi_A \mid \mathcal{F}_{t\wedge\tau})E_P(L_\infty \mid \mathcal{F}_{t\wedge\tau}) \\ &= E_P(\chi_A L_\infty \mid \mathcal{F}_{t\wedge\tau}) = E_P(\chi_A \frac{dQ}{dP} \mid \mathcal{F}_{t\wedge\tau}) = E_Q(\chi_A \mid \mathcal{F}_{t\wedge\tau}) \\ &= Q(A \mid \mathcal{F}_{t\wedge\tau}). \end{aligned}$$

Due to (3.1), for all $A \in \mathcal{E}_\tau$ and $L_\infty = \frac{dQ}{dP}$ we have

$$\begin{aligned} Q(A \mid \mathcal{F}_{t\wedge\tau}) &= E_Q(\chi_A \mid \mathcal{F}_{t\wedge\tau}) = E_P(\chi_A \frac{dQ}{dP} \mid \mathcal{F}_{t\wedge\tau}) \\ &= E_P(\chi_A \mid \mathcal{H}_{t\wedge\tau})E_P(L_\infty \mid \mathcal{H}_{t\wedge\tau}) = E_P(\chi_A L_\infty \mid \mathcal{H}_{t\wedge\tau}) \\ &= E_Q(\chi_A \mid \mathcal{H}_{t\wedge\tau}) = Q(A \mid \mathcal{H}_{t\wedge\tau}). \end{aligned}$$

The result is proved. □

Theorem 3.2. *Let* $\mathbf{F} = \{\mathcal{F}_t\}$*,* $\mathbf{H} = \{\mathcal{H}_t\}$*,* $\mathbf{E} = \{\mathcal{E}_t\}$ *and* $\mathbf{G} = \{\mathcal{G}_t\}$ *be given filtrations on the probability space* $(\Omega, \mathcal{F}, P)$ *and let* τ *be a stopping time with respect to* $\mathbf{E}$*. Assume that* $\mathbf{E}^\tau \subseteq \mathbf{F}^\tau$*,* $\mathbf{H}^\tau \subseteq \mathbf{F}^\tau$ *and* $\mathbf{G}^\tau \subseteq \mathbf{F}^\tau$*. Then, the following assertions hold:*

(i) $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P \implies \mathbf{E}^\tau \subseteq \mathbf{H}^\tau,$

(ii) $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P \ \wedge\ \mathbf{E}^\tau \mid\!< \mathbf{G}^\tau; \mathbf{F}^\tau; P \implies \mathbf{E}^\tau \mid\!< (\mathbf{H}^\tau \wedge \mathbf{G}^\tau); \mathbf{F}^\tau; P.$

Proof. *(i)* Let the $Y_{t\wedge\tau}$ be (cadlag process) $\{\mathcal{E}_{t\wedge\tau}\}$-measurable. Then $Y_{t\wedge\tau}$ is, also, $\mathcal{E}_\tau$-measurable since $\mathcal{E}_{t\wedge\tau} \subset \mathcal{E}_{\infty\wedge\tau} = \mathcal{E}_\tau$. According to $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ follows that for all $Y_{t\wedge\tau}$

$$E(Y_{t\wedge\tau} \mid \mathcal{F}_{t\wedge\tau}) = E(Y_{t\wedge\tau} \mid \mathcal{H}_{t\wedge\tau}). \tag{3.4}$$

Since $\mathcal{E}_{t\wedge\tau} \subseteq \mathcal{F}_{t\wedge\tau}$, it follows that $Y_{t\wedge\tau}$ is $\{\mathcal{F}_{t\wedge\tau}\}$-measurable, too. According to (3.4) we have $Y_{t\wedge\tau} = E(Y_{t\wedge\tau} \mid \mathcal{H}_{t\wedge\tau})$, so $Y_{t\wedge\tau}$ is $\{\mathcal{H}_{t\wedge\tau}\}$-measuarble for all t, and the assertion holds.

(ii) Let $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ and $\mathbf{E}^\tau \mid\!< \mathbf{G}^\tau; \mathbf{F}^\tau; P$ hold. From $\mathbf{E}^\tau \mid\!< \mathbf{H}^\tau; \mathbf{F}^\tau; P$ we have $\mathcal{E}_\tau \subseteq \mathcal{F}_\tau$, $\mathbf{H}^\tau \subseteq \mathbf{F}^\tau$ and $\mathcal{E}_\tau \perp \mathcal{F}_{t\wedge\tau} \mid \mathcal{H}_{t\wedge\tau}, t \in I$, i.e. $\forall A \in \mathcal{E}_\tau$

$$P(A \mid \mathcal{H}_{t\wedge\tau}) = P(A \mid \mathcal{F}_{t\wedge\tau}). \tag{3.5}$$

Also, from $\mathbf{E}^\tau \mid\!< \mathbf{G}^\tau; \mathbf{F}^\tau; P$ we have $\mathbf{G}^\tau \subseteq \mathbf{F}^\tau$ and $\mathcal{E}_\tau \perp \mathcal{F}_{t\wedge\tau} \mid \mathcal{G}_{t\wedge\tau}, t \in I$, i.e. for all $A \in \mathcal{E}_\tau$

$$P(A \mid \mathcal{G}_{t\wedge\tau}) = P(A \mid \mathcal{F}_{t\wedge\tau}). \tag{3.6}$$

The intersection of two σ-algebras is also σ-algebra, so $\mathcal{H}_{t\wedge\tau}\cap\mathcal{G}_{t\wedge\tau}=\mathcal{H}_{t\wedge\tau}\wedge\mathcal{G}_{t\wedge\tau}$. Therefore, because of (3.5) and (3.6), for all $A\in\mathcal{E}_\tau$ we have

$$\begin{aligned}P(A\mid\mathcal{H}_{t\wedge\tau}\cap\mathcal{G}_{t\wedge\tau}) &= E(\chi_A\mid\mathcal{H}_{t\wedge\tau}\cap\mathcal{G}_{t\wedge\tau})=E(E(\chi_A\mid\mathcal{H}_{t\wedge\tau})\mid\mathcal{G}_{t\wedge\tau})\\ &= E(E(\chi_A\mid\mathcal{F}_{t\wedge\tau})\mid\mathcal{G}_{t\wedge\tau})=E(\chi_A\mid\mathcal{F}_{t\wedge\tau}\cap\mathcal{G}_{t\wedge\tau})\\ &= E(\chi_A\mid\mathcal{G}_{t\wedge\tau})=E(\chi_A\mid\mathcal{F}_{t\wedge\tau})=P(A\mid\mathcal{F}_{t\wedge\tau})\end{aligned}$$

so, it follows that $\mathbf{E}^\tau\mid\!<(\mathbf{H}^\tau\wedge\mathbf{G}^\tau);\mathbf{F}^\tau;P$ holds. □

Lemma 3.1. *If $\mathcal{E}_\tau\subseteq\mathcal{H}_\tau$ holds, then from $\mathbf{H}^\tau\mid\!<\mathbf{H}^\tau;\mathbf{F}^\tau;P$ it follows that $\mathbf{E}^\tau\mid\!<\mathbf{H}^\tau;\mathbf{F}^\tau;P$ holds.*

Proof. The result follows directly from $\mathcal{E}_\tau\subseteq\mathcal{H}_\tau$ and $\mathbf{H}^\tau\mid\!<\mathbf{H}^\tau;\mathbf{F}^\tau;P$, since (for all $A\in\mathcal{E}_\tau$) $P(A\mid\mathcal{F}_{t\wedge\tau})=P(A\mid H_{t\wedge\tau})$. □

4. Causality and orthogonality of stopped martingales

The orthogonality of local martingales is considered in [10]. We now consider the orthogonality of stopped martingales and stopped local martingales in the sense of the Definition 2.2. Also, we consider the case when the processes are stopped by the different stopping times.

Let us briefly recall some basics about orthogonal martingales and properties which will be used later (see [19, 20, 21]).

Definition 4.1. ([19]) Two martingales X and Y are said to be weakly orthogonal if $E(X_\infty Y_\infty)=0$.

Definition 4.2. ([19]) Two martingales X and Y are said to be strongly orthogonal if XY is a martingale.

If X and Y are strongly orthogonal martingales they are weakly orthogonal, too. However, the converse is not true.

The definition of orthogonal local martingales is slightly different.

Definition 4.3. ([21]) Two local martingales X and Y are called orthogonal if their product XY is a local martingale.

The equivalence between the concept of causality from the Definition 2.2 and strongly orthogonal stopped martingales is given in the following theorem.

Theorem 4.1. *Let τ be $\{\mathcal{F}_t^X\}$ and $\{\mathcal{F}_t^Y\}$-stopping time and let $X^\tau=X_{t\wedge\tau}$ and $Y^\tau=Y_{t\wedge\tau}$ be two independent $\mathbf{F}^\tau=\{\mathcal{F}_{t\wedge\tau}\}$-stopped martingales. Processes X^τ and Y^τ are strongly orthogonal if and only if each of them is its own cause within $\{\mathcal{F}_{t\wedge\tau}\}$, i.e. if $\mathbf{F}^{X^\tau}\mid\!<\mathbf{F}^{X^\tau};\mathbf{F}^\tau;P$ and $\mathbf{F}^{Y^\tau}\mid\!<\mathbf{F}^{Y^\tau};\mathbf{F}^\tau;P$ hold.*

Proof. Let $X_{t\wedge\tau}$ and $Y_{t\wedge\tau}$ be two strongly orthogonal and independent $\{\mathcal{F}_{t\wedge\tau}\}$ stopped martingales. Then $(XY)^\tau = (XY)_{t\wedge\tau}$ is a stopped martingale, too.

According to Theorem 6 in [8], each of the processes $X^\tau = X_{t\wedge\tau}$ and $Y^\tau = Y_{t\wedge\tau}$ is its own cause within $\mathbf{F}^\tau = \{\mathcal{F}_{t\wedge\tau}\}$, i.e.

$$\mathbf{F}^{X^\tau} \ltimes \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P \quad \text{and} \quad \mathbf{F}^{Y^\tau} \ltimes \mathbf{F}^{Y^\tau}; \mathbf{F}^\tau; P$$

hold.

Conversely, let the processes X^τ and Y^τ be its own cause within $\{\mathcal{F}_{t\wedge\tau}\}$, i.e.

$$\forall A \in \mathcal{F}_\tau^X \qquad P(A \mid \mathcal{F}_{t\wedge\tau}^X) = P(A \mid \mathcal{F}_{t\wedge\tau}), \tag{4.1}$$

$$\forall B \in \mathcal{F}_\tau^Y \qquad P(B \mid \mathcal{F}_{t\wedge\tau}^Y) = P(B \mid \mathcal{F}_{t\wedge\tau}). \tag{4.2}$$

Now, from independence of X^τ and Y^τ, we have

$$\begin{aligned} X_{t\wedge\tau} Y_{t\wedge\tau} &= P(A \mid \mathcal{F}_{t\wedge\tau}^X) P(B \mid \mathcal{F}_{t\wedge\tau}^Y) \\ &= P(A \mid \mathcal{F}_{t\wedge\tau}) P(B \mid \mathcal{F}_{t\wedge\tau}) = P(AB \mid \mathcal{F}_{t\wedge\tau}). \end{aligned}$$

Due to Theorem 3 in [12], from causality it follows that the filtration $\mathbf{F}^\tau = \{\mathcal{F}_{t\wedge\tau}\}$ is generated by processes $X^\tau = X_{t\wedge\tau} = P(A \mid \mathcal{F}_{t\wedge\tau}^X)$. Therefore, since χ_A is $\mathcal{F}_\tau^X$-measurable indicator function of the set $A \in \mathcal{F}_\tau^X$ ($B \in \mathcal{F}_\tau^Y$, so χ_B is $\mathcal{F}_\tau^Y$-measurable) we have

$$\begin{aligned} E(X_\infty^\tau Y_\infty^\tau \mid \mathcal{F}_{t\wedge\tau}) &= E(P(A \mid \mathcal{F}_\tau^X) P(B \mid \mathcal{F}_\tau^Y) \mid \mathcal{F}_{t\wedge\tau}) \\ &= E(E(\chi_A \mid \mathcal{F}_\tau^X) \mid \mathcal{F}_{t\wedge\tau}) E(E(\chi_B \mid \mathcal{F}_\tau^Y) \mid \mathcal{F}_{t\wedge\tau}) \\ &= E(\chi_A \mid \mathcal{F}_{t\wedge\tau}) E(\chi_B \mid \mathcal{F}_{t\wedge\tau}) \end{aligned}$$

So, from causality it follows

$$\begin{aligned} E(X_\infty^\tau Y_\infty^\tau \mid \mathcal{F}_{t\wedge\tau}) &= E(\chi_A \mid \mathcal{F}_{t\wedge\tau}^X) E(\chi_B \mid \mathcal{F}_{t\wedge\tau}^Y) = P(A \mid \mathcal{F}_{t\wedge\tau}^X) P(B \mid \mathcal{F}_{t\wedge\tau}^Y) \\ &= X_{t\wedge\tau} Y_{t\wedge\tau}. \end{aligned}$$

Thus, $(XY)^\tau = (XY)_{t\wedge\tau}$ is a (stopped) martingale with respect to $\{\mathcal{F}_{t\wedge\tau}\}$ and X^τ and Y^τ are two strongly orthogonal stopped martingales. □

The concept of causality from Definition 2.2 can be applied to martingales which are stopped at a different stopping times τ and σ.

Proposition 4.1. *Let τ be a $\{\mathcal{F}_t^X\}$-stopping time, σ be a $\{\mathcal{F}_t^Y\}$-stopping time, $\tau \vee \sigma = max(\tau, \sigma)$ and processes $X^\tau = (X_{t\wedge\tau})$ and $Y^\sigma = (Y_{t\wedge\sigma})$ be two independent $\mathbf{F}^{\tau\vee\sigma} = \{\mathcal{F}^{\tau\vee\sigma}\}$-stopped martingales. Processes X^τ and Y^σ are strongly orthogonal if and only if each of them is its own cause within $\{\mathcal{F}^{\tau\vee\sigma}\}$, i.e. if $\mathbf{F}^{X^\tau} \ltimes \mathbf{F}^{X^\tau}; \mathbf{F}^{\tau\vee\sigma}; P$ and $\mathbf{F}^{Y^\sigma} \ltimes \mathbf{F}^{Y^\sigma}; \mathbf{F}^{\tau\vee\sigma}; P$ hold.*

Proof. Suppose that the processes X^τ and Y^σ are two independent, strongly orthogonal $\{\mathcal{F}^{\tau\vee\sigma}\}$-stopped martingales. Here we have to consider the following two cases: $\sigma \leqslant \tau$ and $\tau < \sigma$. In the first case, $\sigma \leqslant \tau$, we have $\tau \vee \sigma = max(\sigma, \tau) = \tau$ and $\mathbf{F}^{\tau\vee\sigma} = \mathbf{F}^\tau$. So, the process X^τ is its own cause within $\mathbf{F}^\tau$, and according to Theorem 6 in [8] the causality relation $\mathbf{F}^{X^\tau} |\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ holds. Similary we can prove that the process Y^σ is its own cause within $\mathbf{F}^\sigma$, i.e. $\mathbf{F}^{Y^\sigma} |\!< \mathbf{F}^{Y^\sigma}; \mathbf{F}^\tau; P$ holds.

Conversely, let the processes X^τ and Y^σ be two independent $\{\mathcal{F}^\tau\}$-stopped martingales, for which $\mathbf{F}^{X^\tau} |\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ and $\mathbf{F}^{Y^\sigma} |\!< \mathbf{F}^{Y^\sigma}; \mathbf{F}^\tau; P$ holds, i.e

$$\forall A \in \mathcal{F}^X_\tau \qquad P(A \mid \mathcal{F}_{t\wedge\tau}) = P(A \mid \mathcal{F}^X_{t\wedge\tau})$$
$$\forall B \in \mathcal{F}^Y_\sigma \qquad P(B \mid \mathcal{F}_{t\wedge\tau}) = P(B \mid \mathcal{F}^Y_{t\wedge\sigma})$$

Then, for all $A \in \mathcal{F}^X_\tau$ and for all $B \in \mathcal{F}^Y_\tau$ we have

$$\begin{aligned} E(X_\tau Y_\sigma \mid \mathcal{F}_{t\wedge\tau}) &= E(P(A \mid \mathcal{F}^X_{t\wedge\tau})P(B \mid \mathcal{F}^Y_{t\wedge\sigma}) \mid \mathcal{F}_{t\wedge\tau}) = E(\chi_A\chi_B \mid \mathcal{F}_{t\wedge\tau}) \\ &= P(A \mid \mathcal{F}_{t\wedge\tau})P(B \mid \mathcal{F}_{t\wedge\tau}) = P(A \mid \mathcal{F}^X_{t\wedge\tau})P(B \mid \mathcal{F}^Y_{t\wedge\sigma}) \\ &= X_{t\wedge\tau}Y_{t\wedge\sigma}. \end{aligned}$$

So, the processes X^τ and Y^σ are strongly orthogonal stopped martingales.

The proof is similar in the second case if $\tau < \sigma$. □

The Theorem 4.1 can be extended to a larger class of processes, the stopped local martingales.

Theorem 4.2. *Let τ be a $\{\mathcal{F}^X_t\}$ and $\{\mathcal{F}^Y_t\}$-stopping time and let $X^\tau = X_{t\wedge\tau}$ and $Y^\tau = Y_{t\wedge\tau}$ be two independent $\{\mathcal{F}_{t\wedge\tau}\}$-stopped local martingales. Processes X^τ and Y^τ are orthogonal if and only if each of them is its own cause within $\{\mathcal{F}_{t\wedge\tau}\}$, i.e. if $\mathbf{F}^{X^\tau} |\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ and $\mathbf{F}^{Y^\tau} |\!< \mathbf{F}^{Y^\tau}; \mathbf{F}^\tau; P$ hold.*

Proof. Suppose that X^τ and Y^τ are two orthogonal, independent $\{\mathcal{F}_{t\wedge\tau}\}$-stopped local martingales. Then, there exists a sequence of $\{\mathcal{F}^X_{t\wedge\tau}\}$ stopping times $\{\tau_n\} \longrightarrow \infty$, such that process $X_{t\wedge\tau\wedge\tau_n}$ is $\{\mathcal{F}_{t\wedge\tau}\}$-martingale (every martingale is local martingale, but converse is not true). As a consequence, this process is $\{\mathcal{F}^X_{t\wedge\tau}\}$ martingale. Therefore, $E(X_{\tau\wedge\tau_n} \mid \mathcal{F}^X_{t\wedge\tau}) = E(X_{\tau\wedge\tau_n} \mid \mathcal{F}_{t\wedge\tau})$ for all $X_{\tau\wedge\tau_n}$ (which are $\{\mathcal{F}^X_{\tau\wedge\tau_n}\}$-measurable). According to previous equality, it follows that $\mathbf{F}^{X^{\tau\wedge\tau_n}} |\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ holds. Due to invariance of causality under convergence (for details see Theorem 3.5 in [22]), and by Theorem 4.1, we have that $\mathbf{F}^{X^\tau} |\!< \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$ holds.

Similarly, for a sequence of $\{\mathcal{F}^Y_{t\wedge\tau}\}$ stopping times $\{\sigma_n\} \longrightarrow \infty$, we obtain that $\mathbf{F}^{Y^\tau} |\!< \mathbf{F}^{Y^\tau}; \mathbf{F}^\tau; P$ holds.

Conversely, let the causality relations hold. Then, due to Theorem 6 in [8], processes X^τ and Y^τ are independent $\{\mathcal{F}^\tau\}$-stopped martingales. Then

$$E((XY)^\tau_\infty \mid \mathcal{F}_{t\wedge\tau}) \quad = \quad E(X^\tau_\infty \mid \mathcal{F}_{t\wedge\tau})E(Y^\tau_\infty \mid \mathcal{F}_{t\wedge\tau}) = X_{t\wedge\tau}Y_{t\wedge\tau} = (XY)_{t\wedge\tau}.$$

So, XY is stopped martingale (at the same time it is a stopped local martingale) and according to Definition 4.3, X^τ and Y^τ are two orthogonal stopped local martingales. □

5. Example

The concept of causality can be applied to the problem of local risk-minimization, that has become a popular criterion for pricing and hedging in incomplete markets (see [23] - [26]).

The time horizon $T \in (0,\infty)$ is fixed. The random time of default is represented by a stopping time $\tau : \Omega \to [0,T] \cup \{+\infty\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. For default time τ is introduced the associated default process H, given by $H_t = 1_{\{\tau\leqslant t\}}$ and $(\mathcal{F}^H_t)$ is its natural filtration. Let W and B be two one-dimensional independent Brownian motions and $\mathcal{G}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_t$, where $\{\mathcal{F}^W_t\}$ and $\{\mathcal{F}^B_t\}$ denotes the natural filtrations of the processes W and B.

The risky asset price S is represented by a stochastic process on $(\Omega, \mathcal{F}, P)$ whose dynamics is given by

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad S_0 = s_0 > 0$$

where $\sigma_t > 0$ a.s., μ, σ are $\mathcal{F}$-adapted processes and X_t an unobservable exogenous stochastic factor satisfying

$$dX_t = b_t dt + a_t(\rho dW_t + \sqrt{1-\rho^2}dB_t), \quad X_0 = x_0 \in \mathbb{R}.$$

Let $\mathbf{F} = \{\mathcal{F}_t\},\ t \in [0,T]$ be the filtration given by $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{F}^H_t = \mathcal{F}^W_t \vee \mathcal{F}^B_t \vee \mathcal{F}^H_t$. Investors do not have a complete information on the market, they cannot observe neither the stochastic factor X nor the Brownian motions W and B which drive the dynamics of the pair (S, X) and as a consequence they cannot observe the F-hazard rate. At any time t, they may observe the risky asset price and know if default has occurred or not. The available information is given by

$$\widetilde{\mathcal{F}} := \mathcal{F}^S \vee \mathcal{F}^H \quad \subset \quad \mathcal{F} = \mathcal{G} \vee \mathcal{F}^H := \mathcal{F}^W \vee \mathcal{F}^B \vee \mathcal{F}^H.$$

A defaultable claim is a triplet (ξ, Z, τ), where promissed contigent claim ξ is the promised payoff received by the owner at maturity T, Z is the recovery process, which is paid at the default time if default has happened prior to or at time T, τ

is the default time. Process $N = N_t, t \in [0, T]$ models the payment stream arising from the defaultable claim

$$N_t = Z_\tau I_{\tau \leqslant t} = \int_0^T Z_s dH_s,\ 0 \leqslant t < T, \qquad N_T = \xi I_{\tau > T},\ t = T. \tag{5.1}$$

By assumptions in [23] the hedging stops after default, hence is considered the stopped interval $[0, \tau \wedge T]$. If M is an $(\mathbf{G}, P)$-martingale the stopped process $M^\tau = M_{t\wedge\tau}$ is a $(\mathbf{F}, P)$-martingale ($\{\mathcal{G}_t\}$ is increasing and subfiltration of $\{\mathcal{F}_t\}$, see Lemma 5.1.6 in [27]). The stopped processes W^τ and B^τ are $(\mathbf{F}, P)$-Brownian motions. The risky asset price S^τ is a $(\mathbf{F}, P)$-semimartingale with decomposition

$$S_t^\tau = s_0 + \int_0^{t\wedge\tau} S_u^\tau \mu_u du + \int_0^{t\wedge\tau} S_u^\tau \sigma(u, S_u^\tau) dW_u^\tau, \quad t \in [0, T],$$

$$M_t^{\mathcal{F}} = \int_0^{t\wedge\tau} S_u^\tau \sigma(u, S_u^\tau) dW_u^\tau.$$

Risk-minimization approach is introduced in [25].

The risky asset process S is martingale, $\psi = (\theta, \eta)$ is an admissible strategy, $V(\psi) := \theta S + \eta$ its value process, and the cost process: $C_t(\psi) := V_t(\psi) - \int_0^t \theta_u dS_u$. An admissible strategy such that $V_T(\psi) = \xi$ is risk-minimizing if minimizes the risk process: $E[(C_T(\psi) - C_t(\psi))^2 | \mathcal{F}_t]$. Process θ^* is given by the Föllmer-Schweizer decomposition of ξ (see [24]) $\xi = E[\xi] + \int_0^T \theta_u^* dS_u + A_T,\ P - a.s.$ where A is a martingale strongly orthogonal to S.

Strategy ψ^* is mean-self-financing and $C_t(\psi^*) = E[\xi] + A_t$.

In the semimartingale case such a strategy does not exist, hence Schweizer (in [26]) introduced the weaker concept of locally risk-minimizing strategy (under suitable assumptions it is equivalent to pseudo optimality).

This approach in the case of a defaultable claim and in partial information framework is considered in [23]. Here is assumed that hedging stops after default. This allows to work with hedging strategies only up to time $T \wedge \tau$.

The cost process $C(\varphi)$ of a $(\mathcal{F}, L^2)$-strategy (resp. $(\widetilde{\mathcal{F}}, L^2)$-strategy)$\varphi = (\theta, \eta)$ is given by

$$C_t(\varphi) = N_t + V_t(\varphi) - \int_0^t \theta_u dS_u^\tau, t \in [0, T \wedge \tau],$$

where N is defined in (5.1).

Due to Definition 3.3 in [23], φ is mean-self-financing if its cost process $C(\varphi)$ is a $(\mathcal{F}, P)$-martingale (resp. $(\widetilde{\mathcal{F}}, P)$-martingale).

Theorem 1.6 in [26], defines local risk minimization and formulates its equivalent characterisation. The extension of the local risk-minimization approach to payment streams requires to look for admissible strategies with the 0-achieving property, that is $V_{\tau\wedge T}(\varphi) = 0,\ P - a.s.$

Due to Definition 3.5 in [23] about the stopped Föllmer-Schweizer decomposition of random variable $\zeta \in L^2(\mathcal{F}_T, P)$ and Theorem 4.1 we can say that a random variable ζ admits a stopped Föllmer-Schweizer decomposition if it can be written as

$$\zeta = \zeta_0 + \int_0^T \theta_u^{\mathcal{F}} dS_u^\tau + A_{T\wedge\tau}^{\mathcal{F}}, \quad P - a.s.,$$

and if each of processes $A_t^{\mathcal{F}}$ and $M_t^{\mathcal{F}}, t \in [0, T\wedge\tau]$ is its own cause within $\mathcal{F}$, where $M_t^{\mathcal{F}}$ is the martingale part of S^τ.

Adapting the results proved in [23] (see Proposition 3.6) to the concept of causality between stopped filtrations (Theorem 4.1) we get the following characterization.

Proposition 5.1. *Let N be the payment stream associated to the defaultable claim (ξ, Z, τ). Then, N admits an $(\mathcal{F}^S, \widetilde{\mathcal{F}})$-locally risk-minimizing strategy $\varphi^* = (\theta^*, \eta^*)$ if and only if there exists a process $\theta^{\mathcal{F}} \in \Theta^{\mathcal{F},\tau}$ a square integrable $(\mathcal{F}, P)$-martingale, which is its own cause within $\{\mathcal{F}_t\}$, null at zero such that the martingale part of S^τ is its own cause and*

$$N_{T\wedge\tau} = N_0 + \int_0^T \theta_u^{\mathcal{F}^S} dS_u^\tau + A_{t\wedge\tau}^{\widetilde{\mathcal{F}}} P - a.s.$$

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