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ИЗВЕСТИЯ

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ДРОБНОЕ ИНТЕГРИРОВАНИЕ В ВЕСОВЫХ ПРОСТРАНСТВАХ ЛЕБЕГА

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Аннотация. В статье изучается действие оператора дробного интегрирования в весовых лебеговых классах и пространствах со смешанной нормой в единичном шаре из $\mathbb{R}^n$. Мы уточняем и обобщаем некоторые результаты Харди, Литтлвуда и Флетта.

MSC2010 number: 26A33; 26D10.

Ключевые слова: дробное интегрирование; дробное дифференцирование; весовые пространства Лебега; пространства со смешанной нормой.

1. Введение и обозначения

Первые результаты о действии операторов дробного интегрирования в лебеговых классах восходят к Харди и Литтлвуду [1], [2], [3]. В частности, они показали насколько дробный интеграл $D^{-\alpha}f$ "лучше" самой функции $f \in L^p$, именно, оператор дробного интегрирования $D^{-\alpha}$ $(0 < \alpha < 1)$ ограниченно действует из пространства L^p $(1 < p < 1/\alpha)$ в пространство L^q $\left(q = \frac{p}{1-\alpha p}\right)$. Харди и Литтлвуд [1], [2], [3] установили также разнообразные весовые аналоги, в том числе для голоморфных функций. Различные обобщения можно найти в работах Флетта [4], [5], Карапетянца и Рубина [6]. В настоящей статье мы намерены выяснить действие операторов дробного интегрирования на весовых пространствах Лебега со смешанной нормой $L(p, q, \alpha)$.

Пусть $B = B_n$ — открытый единичный шар в $\mathbb{R}^n\,(n \geq 2)$ и $S = \partial B$ — его граница, единичная сфера. Интегральные средние порядка p функции $f(x) = f(r\zeta)$ на сфере $|x| = r$ обозначаются через

$$M_p(f; r) := \left\|f(r\cdot)\right\|_{L^p(S;d\sigma)}, \qquad 0 \leq r < 1,\ 0 < p \leq \infty,$$

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где $d\sigma$ — $(n-1)$-мерная поверхностная мера Лебега на S, нормированная так, что $\sigma(S)=1$.

По определению пространство со смешанной нормой $L(p,q,\alpha)\big(0<p,q\le\infty, \alpha\in\mathbb{R}\big)$ есть множество тех функций f, измеримых в единичном шаре B, для которых конечна норма (квазинорма)

$$\|f\|_{L(p,q,\alpha)} := \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q-1} M_p^q(f;r)dr\right)^{1/q}, & 0<q<\infty,\\ \operatorname*{ess\,sup}_{0\le r<1} (1-r)^{\alpha} M_p(f;r), & q=\infty. \end{cases}$$

Иногда будем обращаться к чуть более общей версии смешанной нормы при $q<\infty$ и $\gamma\in\mathbb{R}$

$$\|f\|_{L(p,q,\alpha;r^\gamma\,dr)} := \left[\int_0^1 (1-r)^{\alpha q-1} M_p^q(f;r)\, r^\gamma\, dr\right]^{1/q} = \left\|r^{\gamma/q} f\right\|_{L(p,q,\alpha)}.$$

Точки в $\mathbb{R}^n$ будем представлять в виде $x=r\zeta$, где $\zeta\in S,\ |x|=r$. Символы $C(\alpha,\beta,\dots), C_\alpha$ и т.п. будут обозначать различные положительные постоянные, зависящие только от указанных параметров. Условимся через $\mathbb{N}$ обозначать множество положительных целых чисел, а через $[\alpha]$ — наибольшее целое число, не превосходящее $\alpha\in\mathbb{R}$, т.е. целую часть числа α. Для вещественных выражений A и B, символ $A\approx B$ означает двустороннее неравенство $c_1|A|\le|B|\le c_2|A|$ с некоторыми несущественными положительными постоянными c_1 и c_2, независимыми от участвующих переменных.

Определение 1.1. *(Дробный интеграл и производная Римана-Лиувилля)* *Для функции $f(r)$ одной переменной $r\in[0,1)$, определим*

$$D^{-\alpha}f(r) := \frac{1}{\Gamma(\alpha)}\int_0^r (r-t)^{\alpha-1} f(t)\,dt = \frac{r^\alpha}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} f(tr)\,dt,$$

$$D^m f(r) := \left(\frac{d}{dr}\right)^m f(r), \qquad D^\alpha f(r) := D^{-(m-\alpha)} D^m f(r), \qquad D^0 f := f,$$

где $\alpha>0,\ m\in\mathbb{N},\ m-1<\alpha\le m.$

Определение 1.2. *(Дробный интеграл и производная Римана-Лиувилля на $\mathbb{R}^n$, $n\ge 2$) Для заданной функции $f(x)$ в единичном шаре B определим*

$$\mathcal{D}^{-\alpha}f(x) \equiv \mathcal{D}_n^{-\alpha}f(x) := r^{-(\alpha+n/2-1)} D^{-\alpha}\Big\{r^{n/2-1}f(x)\Big\} = $$

$$= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\, f(tx)\, t^{n/2-1}\,dt,$$

$$\mathcal{D}^{\alpha}f(x) \equiv \mathcal{D}_n^{\alpha}f(x) := r^{-(n/2-1)} D^{\alpha}\Big\{r^{\alpha+n/2-1}f(x)\Big\}, \qquad r=|x|, \quad \alpha>0.$$

Последняя версия радиальной дробной производной на $\mathbb{R}^n$ была введена в [7] и позволяет упростить ее применение в весовых пространствах. Определим также чуть более общий дробный интеграл на $\mathbb{R}^n$.

Определение 1.3. *Для* $\alpha > 0$, $\gamma \in \mathbb{R}$, *определим*

$$\widetilde{\mathcal{D}}_{n,\gamma}^{-\alpha} f(x) := r^{-(\alpha+\gamma+n/2-1)} D^{-\alpha}\Big\{r^{\gamma+n/2-1} f(x)\Big\} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx)\, t^{\gamma+n/2-1}\, dt. \tag{1.1}$$

Как видим, $\widetilde{\mathcal{D}}_{n,\gamma}^{-\alpha}$ отличается от $\mathcal{D}^{-\alpha}$ лишь несущественным множителем t^γ в подынтегральном выражении. Можно положить $\gamma + n/2 > 0$. Очевидно, что $\widetilde{\mathcal{D}}_{n,0}^{-\alpha} = \mathcal{D}^{-\alpha}$ и $\widetilde{\mathcal{D}}_{n,\gamma}^{-\alpha}|f(x)| \le \mathcal{D}^{-\alpha}|f(x)|$, если $\gamma \ge 0$. Из общей формулы

$$\mathcal{D}^m f(x) = r^{-(n/2-1)} \frac{\partial^m}{\partial r^m}\Big[r^{m-1+n/2}\, f(x)\Big], \qquad m \in \mathbb{N},$$

полезно явно выразить дробные производные первого и второго порядков

$$\mathcal{D}^1 f(x) = \frac{n}{2}\, f + r\, \frac{\partial f}{\partial r}, \tag{1.2}$$
$$\mathcal{D}^2 f(x) = \Big(1+\frac{n}{2}\Big)\, \frac{n}{2}\, f(x) + 2\,\Big(1+\frac{n}{2}\Big)\, r\, \frac{\partial f}{\partial r} + r^2\, \frac{\partial^2 f}{\partial r^2}.$$

Хорошо известно, насколько совершенно действует дробное интегродифференцирование в весовых пространствах Бергмана в единичном круге $\mathbb{D} = B_2$, т.е. весовых классах голоморфных (или гармонических) функций, интегрируемых по площади. В частности,

$$\big\|\mathcal{D}_2^{-\beta} f\big\|_{L(p,p,\alpha-\beta)} \le C(p,\alpha)\|f\|_{L(p,p,\alpha)}, \qquad \alpha > \beta > 0, \quad 0 < p < \infty, \tag{1.3}$$

для всех голоморфных функций в круге (см., например, [2, Теор.8], [5, Теор.6]). Обратное неравенство также верно. Поэтому может показаться несколько странным, что неравенство (1.3) при $p = 1$ нарушается для общих измеримых функций. Действительно, выберем функцию $h_0(x) = h_0(r) := r^{-1}\left(\log\frac{2}{r}\right)^{-2}$, $0 < r < 1$, посчитаем и оценим нормы

$$\|h_0\|_{L(1,1,\alpha)} = \int_0^1 (1-r)^{\alpha-1} h_0(r)\, dr = \int_0^1 \frac{(1-r)^{\alpha-1}}{r\left(\log\frac{2}{r}\right)^2}\, dr < +\infty, \text{ т.е. } h_0 \in L(1,1,\alpha),$$

$$\big\|\mathcal{D}_2^{-\beta} h_0\big\|_{L(1,1,\alpha-\beta)} \ge C \int_0^1 \frac{(1-r)^{\alpha-\beta-1}}{r\, \log\frac{4}{r}}\, dr = +\infty, \quad \text{т.е.} \quad \mathcal{D}_2^{-\beta} h_0 \notin L(1,1,\alpha-\beta).$$

Этот же контрпример h_0 применим, когда рассматриваются пространства со смешанной нормой $L(p,1,\alpha)$, $1 \le p \le \infty$, вместо $L(p,p,\alpha)$ в (1.3).

Цель настоящей статьи — получить неравенства типа (1.3) с дробным интегрированием на пространствах Лебега со смешанной нормой $L(p, q, \alpha)$. Различные дробные операторы, весовые функции и индексы возможно приведут к разным результатам.

2. Неравенства Флетта и их следствия

Приведем некоторые неравенства типа Харди, полученные Флеттом [4, p.490-491], [5, p. 758].

Лемма A. *(*[4], [5]*) Для* $1 \le q < \infty,\ \alpha > \beta > 0,\ \lambda < 1 - \frac{1}{q}$, *и измеримой функции* $h(r) \ge 0$, *имеет место неравенство*

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1}\, r^{q(\lambda-\beta)} \left(D^{-\beta} h(r)\right)^q dr \le C \int_0^1 (1-r)^{\alpha q-1}\, r^{q\lambda}\, h^q(r)\, dr, \tag{2.1}$$

где $C = C(q, \alpha, \beta, \lambda)$, *и более простое неравенство*

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1} \left(D^{-\beta} h(r)\right)^q dr \le C(q, \alpha, \beta) \int_0^1 (1-r)^{\alpha q-1}\, h^q(r)\, dr. \tag{2.2}$$

Замечание 2.1. *Неравенства* (2.1), (2.2) *можно считать обобщениями известных неравенств Харди (см., например,* [3]*) на случай дробных интегралов произвольного порядка. При* $\beta = 1$ *неравенство* (2.2) *как раз сводится к одному из классических неравенств Харди. Тем не менее, неравенства* (2.1), (2.2) *вполне могут быть уточнены и улучшены по нескольким направлениям. Например, для более общих весовых функций (так называемых нормальных функций) неравенства типа* (2.1), (2.2) *были получены в* [8], [9].

С другой стороны, неравенство (2.2) *при* $q = 1$ *можно обобщить до тождества. Кроме того, нас интересует возможность замены оператора дробного интегрирования* $D^{-\beta}$ *на другой. Эти и некоторые другие уточнения содержатся в нижеследующей лемме.*

Лемма 2.1. **(i)** *При* $q = 1,\ \alpha > \beta > 0,\ \gamma \in \mathbb{R},\ n \ge 2$, *справедливы тождества*

$$\int_0^1 (1-r)^{\alpha-\beta-1}\, D^{-\beta} h(r)\, dr = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 (1-r)^{\alpha-1}\, h(r)\, dr, \tag{2.3}$$

$$\int_0^1 (1-r)^{\alpha-\beta-1}\, r^{\beta+\gamma+n/2-1}\, \widetilde{\mathfrak{D}}^{-\beta}_{n,\gamma} h(r)\, dr = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 (1-r)^{\alpha-1}\, r^{\gamma+n/2-1}\, h(r)\, dr, \tag{2.4}$$

(ii) *Если* $1 \le q < \infty,\ \alpha > \beta > 0,\ \lambda < 1 - 1/q,\ \gamma \in \mathbb{R},\ n \ge 2,\ h(r) \ge 0,$ *то*

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1}\, r^{q(\lambda+\gamma+n/2-1)} \left(\widetilde{\mathcal{D}}^{-\beta}_{n,\gamma} h(r)\right)^q dr \le \le C(q,\alpha,\beta,\lambda) \int_0^1 (1-r)^{\alpha q-1}\, r^{q(\lambda+\gamma+n/2-1)}\, h^q(r)\, dr, \tag{2.5}$$

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1}\, r^{q(\beta+\gamma+n/2-1)} \left(\widetilde{\mathcal{D}}^{-\beta}_{n,\gamma} h(r)\right)^q dr \le \le C(q,\alpha,\beta) \int_0^1 (1-r)^{\alpha q-1}\, r^{q(\gamma+n/2-1)}\, h^q(r)\, dr. \tag{2.6}$$

В частности, если $1 < q < \infty$ *либо* $1 \le q < \infty$, $n \ge 3$, *то*

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1} \left(\mathcal{D}^{-\beta} h(r)\right)^q dr \le C(q,\alpha,\beta) \int_0^1 (1-r)^{\alpha q-1} h^q(r) dr. \tag{2.7}$$

Доказательство. **(i)** Теорема Фубини и замена переменных $r - t = \xi(1-t),\ 1 - r = (1-t)(1-\xi)$ приводят к

$$\int_0^1 (1-r)^{\alpha-\beta-1}\, D^{-\beta} h(r)\, dr = \int_0^1 (1-r)^{\alpha-\beta-1} \left[\frac{1}{\Gamma(\beta)} \int_0^r (r-t)^{\beta-1}\, h(t)\, dt\right] dr =$$
$$= \frac{1}{\Gamma(\beta)} \int_0^1 \left[\int_t^1 (r-t)^{\beta-1}\, (1-r)^{\alpha-\beta-1}\, dr\right] h(t)\, dt =$$
$$= \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\alpha-1} \left[\int_0^1 \xi^{\beta-1}\, (1-\xi)^{\alpha-\beta-1}\, d\xi\right] h(t)\, dt =$$
$$= \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1}\, h(t)\, dt.$$

Тождество (2.3) доказано. Далее заменим $h(r)$ на $r^{\gamma+n/2-1}\, h(r)$ в (2.3) и получим тождество (2.4),

$$\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 (1-r)^{\alpha-1}\, r^{\gamma+n/2-1}\, h(r)\, dr =$$
$$= \int_0^1 (1-r)^{\alpha-\beta-1}\, r^{\beta+\gamma+n/2-1}\, r^{-(\beta+\gamma+n/2-1)}\, D^{-\beta} \big\{ r^{\gamma+n/2-1}\, h(r) \big\}\, dr =$$
$$= \int_0^1 (1-r)^{\alpha-\beta-1}\, r^{\beta+\gamma+n/2-1}\, \widetilde{\mathcal{D}}^{-\beta}_{n,\gamma} h(r)\, dr.$$

(ii) Для доказательства (2.5) и (2.6) достаточно заменить $h(r)$ на $r^{\gamma+n/2-1}\, h(r)$ в (2.1) и (2.2), соответственно.

В частном случае $1 < q < \infty$ либо $1 \le q < \infty$, $n \ge 3$, можно выбрать $\gamma = 0,\ \lambda = -(n/2-1) \le 0$ и получить (2.7). □

Замечание 2.2. *Можно допустить упрощающее обозначение* $s := q(\lambda+\gamma+n/2-1)$ *и переписать* (2.5) *в эквивалентном виде*

$$\int_0^1 (1-r)^{(\alpha-\beta)q-1}\, r^s \left(\widetilde{\mathcal{D}}^{-\beta}_{n,\gamma} h(r)\right)^q dr \le C \int_0^1 (1-r)^{\alpha q-1}\, r^s\, h^q(r)\, dr, \tag{2.8}$$

если только $s < q(\gamma+n/2)-1$ *или* $\gamma > \frac{s+1}{q} - \frac{n}{2}$. *При этом постоянная* $C = C(q,\alpha,\beta,\gamma,s,n)$ *зависит только от указанных параметров.*

3. Полугрупповые свойства операторов $\mathcal{D}^{\pm\alpha}$ и $\widetilde{\mathcal{D}}^{-\alpha}_{n,\gamma}$

В этом разделе мы выведем несколько вспомогательных формул типа полугрупповых или коммутационных для дробных операторов $\mathcal{D}^{\pm\alpha}$ и $\widetilde{\mathcal{D}}^{-\alpha}_{n,\gamma}$. Некоторые тождества такого типа уже были нами установлены в [10] при $n = 2$.

Лемма 3.1. *Для* $m \in \mathbb{N}$ *и достаточно гладкой функции* $f(x)$ *в единичном шаре* B *имеют место следующие "полугрупповые" формулы:*

$$\mathcal{D}^{-\alpha-\beta} f = r^{-\beta}\mathcal{D}^{-\alpha}\big\{r^\beta \mathcal{D}^{-\beta} f\big\} = \widetilde{\mathcal{D}}^{-\alpha}_{n,\beta}\mathcal{D}^{-\beta} f, \qquad \alpha,\beta > 0, \tag{3.1}$$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{\beta} f = r^{-\beta}\mathcal{D}^{-(\alpha-\beta)}\big\{r^\beta f\big\} = \widetilde{\mathcal{D}}^{-(\alpha-\beta)}_{n,\beta} f, \qquad \alpha > \beta > 0, \tag{3.2}$$

$$r^{-\beta}\mathcal{D}^{-\alpha}\big\{r^\beta f\big\} = \widetilde{\mathcal{D}}^{-\alpha}_{n,\beta} f, \qquad \alpha > 0,\ \beta \in \mathbb{R}, \tag{3.3}$$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{\beta} f = r^{-\alpha}\mathcal{D}^{\beta-\alpha}\big\{r^\alpha f\big\}, \qquad \beta > \alpha > 0, \tag{3.4}$$

$$\mathcal{D}^{\delta} f = r^{m-\delta}\mathcal{D}^{-(m-\delta)}\mathcal{D}^{m}\big\{r^{-(m-\delta)} f\big\}, \qquad 0 < \delta \le m, \tag{3.5}$$

$$\mathcal{D}^{\delta} f = \widetilde{\mathcal{D}}^{-(1-\delta)}_{n,\delta-1}\Big[(\delta-1) f + \mathcal{D}^1 f\Big], \qquad 0 < \delta \le 1, \tag{3.6}$$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{m} f = \mathcal{D}^{m}\mathcal{D}^{-\alpha} f, \qquad \alpha > 0. \tag{3.7}$$

Доказательство. Поочередно докажем все семь тождеств, сохраняя их нумерацию.

(3.1) Согласно Определениям 1.1–1.3 операторов $D^{\pm\alpha}, \mathcal{D}^{\pm\alpha}, \widetilde{\mathcal{D}}^{\pm\alpha}$, получаем

$$\begin{aligned}
\mathcal{D}^{-\alpha-\beta} f &= r^{-(\alpha+\beta+n/2-1)} D^{-\alpha-\beta}\big\{r^{n/2-1} f(x)\big\} = \\
&= r^{-(\alpha+\beta+n/2-1)} D^{-\alpha}\Big[r^{\beta+n/2-1} r^{-(\beta+n/2-1)} D^{-\beta}\big\{r^{n/2-1} f(x)\big\}\Big] = \\
&= r^{-\beta}\,\mathcal{D}^{-\alpha}\Big[r^\beta\,\mathcal{D}^{-\beta} f(x)\Big] = \\
&= r^{-\beta} r^{\beta}\,\frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\mathcal{D}^{-\beta} f(tx)\, t^{\beta+n/2-1}\, dt = \widetilde{\mathcal{D}}^{-\alpha}_{n,\beta}\,\mathcal{D}^{-\beta} f(x).
\end{aligned}$$

(3.2) Используя формулы обращения для $\mathcal{D}^{\pm\alpha}$, а также (1.1) и (3.1), получаем для $\alpha > \beta > 0$

$$\mathcal{D}^{-\alpha}\mathcal{D}^{\beta} f = \mathcal{D}^{-(\alpha-\beta)-\beta}\mathcal{D}^{\beta} f = r^{-\beta}\mathcal{D}^{-(\alpha-\beta)}\big\{r^\beta \mathcal{D}^{-\beta}\mathcal{D}^{\beta} f\big\} =$$

$$= r^{-\beta}\mathcal{D}^{-(\alpha-\beta)}\big\{r^{\beta}f\big\} = \widetilde{\mathcal{D}}_{n,\beta}^{-(\alpha-\beta)}f.$$

(3.3) Аналогичным образом,

$$r^{-\beta}\mathcal{D}^{-\alpha}\big\{r^{\beta}f(x)\big\} = r^{-\beta}\,\frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\,(tr)^{\beta}\,f(tx)\,t^{n/2-1}\,dt = \widetilde{\mathcal{D}}_{n,\beta}^{-\alpha}f(x).$$

(3.4) Применяя формулы обращения и Определения 1.1–1.3 с $\beta > \alpha > 0$, получаем

$$\begin{aligned}\mathcal{D}^{-\alpha}\mathcal{D}^{\beta}f &= r^{-(\alpha+n/2-1)}D^{-\alpha}\Big\{r^{n/2-1}\mathcal{D}^{\beta}f\Big\} =\\ &= r^{-(\alpha+n/2-1)}D^{\beta-\alpha}D^{-\beta}\Big\{r^{n/2-1}\mathcal{D}^{\beta}f\Big\} =\\ &= r^{-(\alpha+n/2-1)}D^{\beta-\alpha}r^{\beta+n/2-1}\,\mathcal{D}^{-\beta}\Big\{r^{-(n/2-1)}r^{n/2-1}\mathcal{D}^{\beta}f\Big\} =\\ &= r^{-(\alpha+n/2-1)}D^{\beta-\alpha}\Big\{r^{\beta+n/2-1}\,f\Big\} = r^{-\alpha}\mathcal{D}^{\beta-\alpha}\big\{r^{\alpha}f\big\},\end{aligned}$$

что доказывает тождество (3.4).

(3.5) Для доказательства тождества (3.5) достаточно в (3.4) заменить β на m, также α на $m-\delta$, а затем функцию $r^{m-\delta}\,f(x)$ на $f(x)$.

(3.6) Преобразуем

$$\begin{aligned}\mathcal{D}^{1}\big\{r^{-(1-\delta)}f\big\} &= \frac{n}{2}\,r^{-(1-\delta)}f + r\,D^{1}\big\{r^{-(1-\delta)}f\big\} =\\ &= \frac{n}{2}\,r^{-(1-\delta)}f + r\Big[-(1-\delta)r^{-(2-\delta)}f + r^{-(1-\delta)}D^{1}f\Big] =\\ &= r^{-(1-\delta)}\left[\left(\delta+\frac{n}{2}-1\right)\,f + r\,D^{1}f\right] = r^{-(1-\delta)}\Big[(\delta-1)\,f + \mathcal{D}^{1}f\Big].\end{aligned}$$

Отсюда посредством уже доказанных тождеств (3.5) и (3.3) приходим к

$$\begin{aligned}\mathcal{D}^{\delta}f &= r^{1-\delta}\mathcal{D}^{-(1-\delta)}\mathcal{D}^{1}\big\{r^{-(1-\delta)}f\big\} =\\ &= r^{1-\delta}\mathcal{D}^{-(1-\delta)}r^{-(1-\delta)}\Big[(\delta-1)\,f+\mathcal{D}^{1}f\Big] = \widetilde{\mathcal{D}}_{n,\delta-1}^{-(1-\delta)}\Big[(\delta-1)f+\mathcal{D}^{1}f\Big],\end{aligned}$$

что и требовалось.

(3.7) Вначале докажем более простую формулу коммутации

$$r^{m}D^{m}\mathcal{D}^{-\alpha}f(x) = \mathcal{D}^{-\alpha}\big\{r^{m}D^{m}f(x)\big\}, \qquad x = r\zeta. \tag{3.8}$$

Для этого раскроем ее, пользуясь очевидной формулой $\frac{\partial^m}{\partial r^m}f(tr\zeta) = t^m\,\frac{\partial^m}{\partial (tr)^m}f(tr\zeta)$,

$$\begin{aligned}r^{m}D^{m}\mathcal{D}^{-\alpha}f(x) &= r^{m}\frac{\partial^m}{\partial r^m}\left[\frac{1}{\Gamma(\alpha)}\int_0^1(1-t)^{\alpha-1}f(tx)\,t^{n/2-1}\,dt\right] =\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1(1-t)^{\alpha-1}\left[(tr)^m\frac{\partial^m}{\partial(tr)^m}f(tx)\right]\,t^{n/2-1}\,dt = \mathcal{D}^{-\alpha}\big\{r^{m}D^{m}f(x)\big\},\end{aligned}$$

что совпадает (3.8).

Аналогично формулам (1.2) для $m = 1, 2$, теперь для произвольного $m \in \mathbb{N}$ найдутся постоянные $c_k = c_k(n) > 0$ $(k = 0, 1, 2, \ldots, m-1)$, зависящие только от n такие, что

$$\mathcal{D}^m f(x) = r^{-(n/2-1)} D^m \big\{ r^{m+n/2-1} f(x) \big\} = \sum_{k=0}^{n} c_k \, r^k \, D^k f(x). \tag{3.9}$$

Тогда посредством (3.8) и (3.9) получаем

$$\mathcal{D}^m \mathcal{D}^{-\alpha} f(x) = \sum_{k=0}^{n} c_k \, r^k \, D^k \mathcal{D}^{-\alpha} f(x) = \mathcal{D}^{-\alpha} \left\{ \sum_{k=0}^{n} c_k \, r^k \, D^k f(x) \right\} = \mathcal{D}^{-\alpha} \mathcal{D}^m f(x).$$

Лемма 3.1 полностью доказана. □

4. Дробное интегрирование в лебеговых пространствах со смешанной нормой в шаре

Поскольку мы заинтересованы в вопросах дробного интегрирования на функциях, интегрируемых с весом в шаре, то нам необходимы точные оценки некоторых эталонных интегралов. Интеграл в нижеследующей лемме хорошо известен, в частности из теории гипергеометрической функции Гаусса, и мы нуждаемся в двусторонней оценке этого интеграла. Подобные оценки можно найти в [11, Лем.3.1], [12, Лем.6.1,6.3], [13]. Здесь мы дадим иное прямое и самодостаточное доказательство неравенств, основанное лишь на интегральных оценках без привлечения теории специальных функций.

Лемма 4.1. *При $\alpha, \lambda > 0$, $\beta \in \mathbb{R}$ имеют место двусторонние оценки*

$$\int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt \approx \begin{cases} \dfrac{1}{(1-r)^{\beta-\alpha}}, & \beta > \alpha, \\ 1, & \beta < \alpha, \\ \log \dfrac{2}{1-r}, & \beta = \alpha, \end{cases} \qquad 0 \le r < 1, \tag{4.1}$$

где участвующие (но явно не указанные) постоянные зависят лишь от α, β, λ.

Доказательство. Все неравенства в (4.1) тривиальны, если $0 \le r \le \frac{1}{2}$. Поэтому достаточно доказать (4.1) только при $\frac{1}{2} \le r < 1$ или $r \to 1^-$. Вначале, полагая $\beta > \alpha > 0$, оценим интеграл (4.1) сверху

$$\int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt = \int_0^r \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt + \int_r^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt \le$$
$$\le \int_0^r \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-t)^{\beta}}\,dt + \frac{1}{(1-r)^{\beta}} \int_r^1 t^{\lambda-1}\,(1-t)^{\alpha-1}\,dt =$$

$$= \int_0^r \frac{t^{\lambda-1}}{(1-t)^{1+\beta-\alpha}}\,dt + \frac{1}{(1-r)^{\beta}} \int_r^1 t^{\lambda-1}\,(1-t)^{\alpha-1}\,dt.$$

Ввиду двух асимптотических соотношений при $r \to 1^-$

$$\int_0^r \frac{t^{\lambda-1}}{(1-t)^{1+\beta-\alpha}}\,dt \sim \frac{1}{(\beta-\alpha)\,(1-r)^{\beta-\alpha}}, \qquad \int_r^1 t^{\lambda-1}\,(1-t)^{\alpha-1}\,dt \sim \frac{1}{\alpha}\,(1-r)^{\alpha},$$

заключаем, что

$$\int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt \le C(\alpha,\beta,\lambda)\frac{1}{(1-r)^{\beta-\alpha}}, \qquad \frac{1}{2} \le r < 1.$$

Обратно,

$$\int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt \ge \int_r^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt \ge \frac{C_\lambda}{(1-r^2)^{\beta}} \int_r^1 (1-t)^{\alpha-1}\,dt =$$
$$= \frac{C_\lambda}{\alpha(1-r^2)^{\beta}}(1-r)^{\alpha} \ge \frac{C_\lambda}{\alpha\,2^{\beta}}\,\frac{1}{(1-r)^{\beta-\alpha}}.$$

Случай $\beta > \alpha > 0$ доказан. Переходя теперь к случаю $\beta < \alpha$, положим $0 < \beta < \alpha$ и оценим

$$\int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt \le \int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-t)^{\beta}}\,dt = B(\lambda,\alpha-\beta),$$
$$\int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\,dt \ge \int_0^1 t^{\lambda-1}\,(1-t)^{\alpha-1}\,dt = B(\lambda,\alpha),$$

где $B(\cdot,\cdot)$ — бета-функция Эйлера. В случае $\beta < 0$ доказательство аналогично.

Наконец в третьем случае $\beta = \alpha$ заметим, что $\frac{1-t}{1-rt} < 1$ и заменим переменную в интеграле

$$\int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\alpha}}\,dt = \int_0^1 \frac{t^{\lambda-1}}{1-rt}\,\left(\frac{1-t}{1-rt}\right)^{\alpha}\,dt \le \int_0^1 \frac{t^{\lambda-1}}{1-rt}\,dt =$$
$$= \frac{1}{r^{\alpha}} \int_0^r \frac{\eta^{\alpha-1}}{1-\eta}\,d\eta \sim \log\frac{1}{1-r} \qquad \text{при} \quad r \to 1^-.$$

Этим доказана верхняя оценка. Для доказательства нижней оценки воспользуемся монотонным убыванием функции $\frac{1-t}{1-rt}$ по t,

$$\int_0^1 \frac{t^{\lambda-1}\,(1-t)^{\alpha-1}}{(1-rt)^{\alpha}}\,dt = \int_0^1 \frac{t^{\lambda-1}}{1-rt}\,\left(\frac{1-t}{1-rt}\right)^{\alpha}\,dt \ge \int_0^r \frac{t^{\lambda-1}}{1-rt}\,\left(\frac{1-t}{1-rt}\right)^{\alpha}\,dt \ge$$
$$\ge \int_0^r \frac{t^{\lambda-1}}{1-rt}\,\left(\frac{1-r}{1-r^2}\right)^{\alpha}\,dt \ge \frac{1}{2^{\alpha}} \int_0^r \frac{t^{\lambda-1}}{1-rt}\,dt \sim \frac{1}{2^{\alpha}}\log\frac{1}{1-r}$$

при $r \to 1^-$. Этим Лемма 4.1 доказана. □

Далее, нам потребуются точные оценки интегралов, схожих с (4.1). В следующей лемме мы уточняем оценки некоторых интегралов, рассмотренных Флеттом [4, (15.3),(15.4)].

Лемма 4.2. *При $a \in \mathbb{R}$, $b, c > 0$, справедливы двусторонние оценки*

(4.2)

$$J_1 = J_1(r) := \int_0^r \frac{(r-t)^{b-1}\, t^{c-1}}{(1-t)^a}\, dt \approx \begin{cases} \dfrac{r^{b+c-1}}{(1-r)^{a-b}}, & a > b, \\ r^{b+c-1}, & a < b, \\ r^{b+c-1} \log \dfrac{2}{1-r}, & a = b, \end{cases} \quad (0 < r < 1),$$

(4.3)

$$J_2 = J_2(\rho) := \int_\rho^1 \frac{(t-\rho)^{b-1}\,(1-t)^{c-1}}{t^a}\, dt \approx \begin{cases} \dfrac{(1-\rho)^{b+c-1}}{\rho^{a-b}}, & a > b, \\ (1-\rho)^{b+c-1}, & a < b, \\ (1-\rho)^{b+c-1} \log \dfrac{2}{\rho}, & a = b, \end{cases} \quad (0 < \rho < 1),$$

где неявно вовлеченные постоянные зависят лишь от a, b, c.

Доказательство. Первый интеграл J_1 немедленно оценивается заменой переменного $t = r\eta$,

$$J_1 = \int_0^r \frac{(r-t)^{b-1}\, t^{c-1}}{(1-t)^a}\, dt = r^{b+c-1} \int_0^1 \frac{(1-\eta)^{b-1}\, \eta^{c-1}}{(1-r\eta)^a}\, d\eta,$$

с дальнейшим применением Леммы 4.1. Второй интеграл J_2 сводится к J_1 заменой переменных $1 - t = \eta(1-\rho)$ и $r = 1 - \rho$. □

На основе доказанных Лемм 4.1 и 4.2 мы дадим точные оценки интегралов (2.1) и (2.2) при $q = 1$, и тем самым уточним Лемму А Флетта.

Лемма 4.3. (**i**) *При $\lambda < 0 < \beta < \alpha$, справедлива двусторонняя оценка*

$$\int_0^1 (1-r)^{\alpha-\beta-1}\, r^{\lambda-\beta}\, D^{-\beta} h(r)\, dr \approx \int_0^1 (1-r)^{\alpha-1}\, r^{\lambda}\, h(r)\, dr. \tag{4.4}$$

(**ii**) *В предельном случае $\lambda = 0 < \beta < \alpha$, имеем*

$$\int_0^1 (1-r)^{\alpha-\beta-1}\, \mathcal{D}_2^{-\beta} h(r)\, dr \approx \int_0^1 (1-r)^{\alpha-1}\, h(r)\, \log \frac{2}{r}\, dr. \tag{4.5}$$

Доказательство. Из двух похожих оценок (4.4)-(4.5) докажем только вторую (4.5). При $\lambda = 0$ по теореме Фубини и (4.3) для оператора $\mathcal{D}_2^{-\beta} = r^{-\beta} D^{-\beta}$ выводим

$$\int_0^1 (1-r)^{\alpha-\beta-1}\, r^{-\beta}\, D^{-\beta} h(r)\, dr$$
$$= \int_0^1 (1-r)^{\alpha-\beta-1}\, r^{-\beta} \left[\frac{1}{\Gamma(\beta)} \int_0^r (r-t)^{\beta-1}\, h(t)\, dt \right] dr =$$

$$= \frac{1}{\Gamma(\beta)} \int_0^1 \left[\int_t^1 \frac{(r-t)^{\beta-1}\,(1-r)^{\alpha-\beta-1}}{r^{\beta}}\,dr \right] h(t)\,dt \approx$$
$$\approx \int_0^1 (1-t)^{\alpha-1} \log\frac{2}{t}\, h(t)\,dt,$$

что и требовалось. □

Перейдем к формулировке и доказательству основного результата настоящей статьи.

Теорема 4.1. *При $1 \le p, q \le \infty$ справедливы следующие пять утверждений, причем в каждом из них предполагается, что норма в правых частях неравенств (4.6)–(4.11) конечна.*

(**i**) *Для всех $\alpha > \beta > 0$ справедливо неравенство*

$$\left\| \mathcal{D}^{-\beta} f \right\|_{L(p,q,\alpha-\beta)} \le C(q,\alpha,\beta) \|f\|_{L(p,q,\alpha)}, \tag{4.6}$$

за исключением случая $q = 1,\ n = 2$, когда неравенство (4.6) нарушается.

(**ii**) *Для всех $1 \le q < \infty,\ \alpha > \beta > 0,\ \gamma < \frac{qn}{2} - 1$ справедливо неравенство*

$$\left\| \mathcal{D}^{-\beta} f \right\|_{L(p,q,\alpha-\beta;r^{\gamma}dr)} \le C(q,\alpha,\beta,\gamma) \|f\|_{L(p,q,\alpha;r^{\gamma}dr)}. \tag{4.7}$$

В частности, при $q = 1,\ n = 2$, имеем

$$\left\| \mathcal{D}_2^{-\beta} f \right\|_{L(p,1,\alpha-\beta;r^{\gamma}dr)} \le C(\alpha,\beta,\gamma) \|f\|_{L(p,1,\alpha;r^{\gamma}dr)}, \qquad \gamma < 0 < \alpha < \beta. \tag{4.8}$$

(**iii**) *При любых $\alpha > \beta > 0,\ \ell > \frac{1}{q} - \frac{n}{2}$ справедливо*

$$\left\| \widetilde{\mathcal{D}}_{n,\ell}^{-\beta} f \right\|_{L(p,q,\alpha-\beta)} \le C(q,\alpha,\beta,\ell) \|f\|_{L(p,q,\alpha)}. \tag{4.9}$$

(**iv**) *Если $0 < \alpha < \delta \le k,\ j \le \delta\ (j,k \in \mathbb{N})$, то*

$$\|\mathcal{D}^{j} f\|_{L(p,q,\delta-\alpha)} \le C(q,\alpha,\delta,j,k) \left\| \mathcal{D}^{k} f \right\|_{L(p,q,k-\alpha)} \tag{4.10}$$

для всех достаточно гладких функций f в B.

(**v**) *Если $0 < \alpha < \delta \le [\alpha] + 1 < \delta + \dfrac{n}{2} - \dfrac{1}{q}$, то*

$$\left\| \mathcal{D}^{\delta} f \right\|_{L(p,q,\delta-\alpha)} \le C(q,\alpha,\delta) \left\| \mathcal{D}^{[\alpha]+1} f \right\|_{L(p,q,[\alpha]+1-\alpha)} \tag{4.11}$$

для всех достаточно гладких функций f в B.

Доказательство. (**i**) **Случай** $q = \infty$**.** Полагая, что $f(x) \in L(p,\infty,\alpha)$, по определению имеем

$$(1-r)^{\alpha} M_p(f;r) \le \|f\|_{L(p,\infty,\alpha)}, \qquad 0 \le r < 1. \tag{4.12}$$

Последовательно применяя неравенство Минковского и (4.12), получаем

$$M_p(\mathcal{D}^{-\beta}f;r) \le \frac{1}{\Gamma(\beta)}\int_0^1 (1-t)^{\beta-1}M_p(f;tr)dt \le$$
$$\le \|f\|_{L(p,\infty,\alpha)}\frac{1}{\Gamma(\beta)}\int_0^1 \frac{(1-t)^{\beta-1}}{(1-rt)^{\alpha}}dt \le C(\alpha,\beta)\frac{\|f\|_{L(p,\infty,\alpha)}}{(1-r)^{\alpha-\beta}}.$$

Следовательно $\|\mathcal{D}^{-\beta}f\|_{L(p,\infty,\alpha-\beta)} \le C\|f\|_{L(p,\infty,\alpha)}$, как и требовалось.

Случай $1 \le q < \infty$. Для $1 < q < \infty$, либо $1 \le q < \infty$, $n \ge 3$, по неравенству Минковского и (2.7) получаем

$$\begin{aligned}
&\big\|\mathcal{D}^{-\beta}f\big\|^q_{L(p,q,\alpha-\beta)} = \\
&= \int_0^1 (1-r)^{(\alpha-\beta)q-1}\left\|\frac{1}{\Gamma(\beta)}\int_0^1 (1-t)^{\beta-1}\, f(tr\zeta)\, t^{n/2-1}\, dt\right\|^q_{L^p(S;d\sigma)} dr \le \\
&\le \int_0^1 (1-r)^{(\alpha-\beta)q-1}\left(\frac{1}{\Gamma(\beta)}\int_0^1 (1-t)^{\beta-1}\, \big\|f(tr\zeta)\big\|_{L^p(S;d\sigma)}\, t^{n/2-1}\, dt\right)^q dr = \\
&= \int_0^1 (1-r)^{(\alpha-\beta)q-1}\left(\mathcal{D}^{-\beta}\|f\|_{L^p(S)}\right)^q dr \le \\
&\le C(\alpha,\beta,q,n)\int_0^1 (1-r)^{\alpha q-1}\|f\|^q_{L^p(S)}dr = C\|f\|^q_{L(p,q,\alpha)}.
\end{aligned}$$

Нарушение (4.6) при $q=1$, $n=2$ следует из (4.5), или же в этом можно убедиться применив контрпример h_0 из Введения.

(ii) Теперь применим более общее неравенство (2.8) с $\gamma = 0$, а также неравенство Минковского и оценим аналогично предыдущему утверждению **(i)**. В результате получим

$$\|\mathcal{D}^{-\beta}f\|^q_{L(p,q,\alpha-\beta;r^\gamma dr)} \le C\int_0^1 (1-r)^{\alpha q-1}\|f\|^q_{L^p(S)}\, r^\gamma\, dr = C\|f\|^q_{L(p,q,\alpha;r^\gamma\, dr)}.$$

В частности, при $q=1$, $n=2$, имеем $\gamma<0$.

(iii) Случай $1 \le q < \infty$. Вновь по неравенству Минковского и (2.8), учитывая $\ell > \frac{1}{q}-\frac{n}{2}$, получаем

$$\big\|\widetilde{\mathcal{D}}^{-\beta}_{n,\ell}f\big\|^q_{L(p,q,\alpha-\beta)} \le \int_0^1 (1-r)^{(\alpha-\beta)q-1}\left(\widetilde{\mathcal{D}}^{-\beta}_{n,\ell}\|f\|_{L^p(S)}\right)^q dr \le$$
$$\le C(\alpha,\beta,q,\ell,n)\int_0^1 (1-r)^{\alpha q-1}\|f\|^q_{L^p(S)}dr = C\|f\|^q_{L(p,q,\alpha)}.$$

В частности, при $q=1$, $n=2$, получаем $\big\|\widetilde{\mathcal{D}}^{-\beta}_{2,\ell}f\big\|_{L(p,1,\alpha-\beta)} \le C\|f\|_{L(p,1,\alpha)},\ \ell>0.$

Случай $q = \infty$**.** По определению имеем $(1-r)^\alpha M_p(f;r) \le \|f\|_{L(p,\infty,\alpha)}$ при $0 \le r < 1$, предполагая что $f(x) \in L(p,\infty,\alpha)$. Для оценки применяем Лемму 4.1,

$$M_p(\widetilde{\mathcal{D}}^{-\beta}_{n,\ell} f; r) \le \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1} M_p(f;tr)\, t^{\ell+n/2-1}\, dt \le$$
$$\le \|f\|_{L(p,\infty,\alpha)} \frac{1}{\Gamma(\beta)} \int_0^1 \frac{(1-t)^{\beta-1}}{(1-rt)^\alpha}\, t^{\ell+n/2-1}\, dt \le$$
$$\le C(\alpha,\beta,\ell,n) \frac{\|f\|_{L(p,\infty,\alpha)}}{(1-r)^{\alpha-\beta}}, \qquad \alpha > \beta > 0.$$

Таким образом, $\big\|\widetilde{\mathcal{D}}^{-\beta}_{n,\ell} f\big\|_{L(p,\infty,\alpha-\beta)} \le C(\alpha,\beta,\ell,n)\|f\|_{L(p,\infty,\alpha)}$, что и требовалось.

(iv) Для достаточно гладких функций и любых $0 < \alpha < \delta$, $1 \le j \le \delta \le k$ $(j,k \in \mathbb{N})$, оцениваем, следуя формуле обращения $f = \mathcal{D}^{-k}\mathcal{D}^k f$ и формулам (3.7), (3.2), (4.9),

$$\|\mathcal{D}^j f\|_{L(p,q,\delta-\alpha)} = \|\mathcal{D}^j \mathcal{D}^{-k} \mathcal{D}^k f\|_{L(p,q,\delta-\alpha)} = \|\mathcal{D}^{-k} \mathcal{D}^j \mathcal{D}^k f\|_{L(p,q,\delta-\alpha)} =$$
$$= \big\|\widetilde{\mathcal{D}}^{-(k-j)}_{n,j} \mathcal{D}^k f\big\|_{L(p,q,\delta-\alpha)} \le C\|\mathcal{D}^k f\|_{L(p,q,\delta-\alpha+k-j)} \le C\|\mathcal{D}^k f\|_{L(p,q,k-\alpha)}.$$

(v) Случай $1 \le q < \infty$**.** Обозначим $m := [\alpha]+1$, так что $0 < \alpha < \delta \le m = [\alpha]+1 < \delta + \dfrac{n}{2} - \dfrac{1}{q}$, Преобразуем m-ую производную в (3.5) с использованием разложения (3.9) и правила дифференцирования Лейбница,

$$\mathcal{D}^m\big\{r^{-(m-\delta)} f(x)\big\} = r^{-(n/2-1)} D^m \Big[r^{\delta+n/2-1}\, f(x)\Big] =$$
$$= r^{-(n/2-1)} \sum_{j=0}^m \binom{m}{j} \Big[D^{m-j} r^{\delta+n/2-1}\Big]\, D^j f =$$
$$= \sum_{j=0}^m \binom{m}{j} C(\delta,n,j)\, r^{\delta-m+j}\, D^j f =$$
$$= r^{-(m-\delta)} \left[\sum_{j=0}^{m-1} C(\delta,m,n,j)\, r^j\, D^j f + r^m\, D^m f\right],$$

или же в более точной форме в терминах $\mathcal{D}^k$,

$$r^{m-\delta}\, \mathcal{D}^m\big\{r^{-(m-\delta)} f(x)\big\} = \sum_{j=0}^{m-1} C_j(\delta,m)\, \mathcal{D}^j f + \mathcal{D}^m f. \tag{4.13}$$

Теперь преобразуем производную $\mathcal{D}^\delta f$ с помощью (3.5), (4.13), (3.3),

$$\mathcal{D}^\delta f(x) = r^{m-\delta} \mathcal{D}^{-(m-\delta)} \Big[\mathcal{D}^m\big\{r^{-(m-\delta)} f(x)\big\}\Big] =$$
$$= r^{m-\delta} \mathcal{D}^{-(m-\delta)} r^{-(m-\delta)} \left[\sum_{j=0}^{m-1} C_j(\delta,m)\, \mathcal{D}^j f + \mathcal{D}^m f\right] =$$

$$= \widetilde{\mathcal{D}}^{-(m-\delta)}_{n,\delta-m}\left[\sum_{j=0}^{m-1} C_j(\delta,m)\,\mathcal{D}^j f + \mathcal{D}^m f\right], \qquad 0<\delta\le m<\delta+1. \tag{4.14}$$

Перейдем к смешанным нормам, которые оценим используя (4.14), (4.9), (4.10) и условие $\delta - m > \frac{1}{q} - \frac{n}{2}$,

$$\begin{aligned}\left\|\mathcal{D}^\delta f\right\|_{L(p,q,\delta-\alpha)} &= \left\|\widetilde{\mathcal{D}}^{-(m-\delta)}_{n,\delta-m}\left[\sum_{j=0}^{m-1} C_j(\delta,m)\,\mathcal{D}^j f + \mathcal{D}^m f\right]\right\|_{L(p,q,\delta-\alpha)} \\ &\le C\left\|\sum_{j=0}^{m-1} C_j(\delta,m)\,\mathcal{D}^j f + \mathcal{D}^m f\right\|_{L(p,q,m-\alpha)} \\ &\le C\sum_{j=0}^{m-1} C_j(\delta,m)\left\|\mathcal{D}^j f\right\|_{L(p,q,m-\alpha)} + \left\|\mathcal{D}^m f\right\|_{L(p,q,m-\alpha)} \\ &\le C\left\|\mathcal{D}^m f\right\|_{L(p,q,m-\alpha)},\end{aligned}$$

где последняя постоянная $C = C(q,\alpha,\delta,m,n)$ зависит от указанных параметров. Доказан случай $1 \le q < \infty$ неравенства (4.11).

Случай $q = \infty$ аналогичен, детали доказательства опускаем. Теорема 4.1 полностью доказана. □

Abstract. The paper studies the action of a fractional integration operator in weighted Lebesgue classes and mixed norm spaces on the unit ball of $\mathbb{R}^n$. We sharpen and generalize some results of Hardy, Littlewood and Flett.

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ON TWO SHARED VALUES PROBLEM RELATED TO UNIQUENESS OF MEROMORPHIC FUNCTION AND ITS k TH DERIVATIVE

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Abstract. In the paper, we have exhaustively studied about the uniqueness of meromorphic function sharing two values with its k-th derivative counterpart. We have obtained a series of results each of which will improve and extend a number of existing results relevant with the content of the paper. We have also pointed out some gaps in one theorem in a paper due to Chen et. al. (Pure Mathematics, 8(4)(2018), 378 - 382 (in Chinese)) and rectifying those gaps presented the corrected form of the same in a compact manner. Thus we have been able to streamline all the results in this particular section of literature.

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Keywords: meromorphic function; uniqueness; weighted sharing; derivative.

1. Introduction and definitions

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$, where $\mathbb{C}$ and $\mathbb{N}$ denote the set of all complex numbers and natural numbers respectively and by $\mathbb{Z}$ we denote the set of all integers. In this paper by any meromorphic function f we always mean that it is defined on $\mathbb{C}$. We use standard notation of Nevanlinna theory as stated in [5]. For any non-constant meromorphic function $h(z)$ we define $S(r,h) = o(T(r,h)), (r \longrightarrow \infty, r \notin E)$ where E denotes any set of positive real numbers having finite linear measure. We recall that $T(r,f)$ denotes the Nevanlinna characteristic function of the non-constant meromorphic function and $N(r, \frac{1}{f-a}) = N(r,a;f)$ ($\overline{N}(r, \frac{1}{f-a}) = \overline{N}(r,a;f)$) denotes the counting function (reduced counting function) of a-points of meromorphic function f. When $a = \infty$, we use $N(r,f) = N(r,\infty;f)$ $\left(\overline{N}(r,f) = \overline{N}(r,\infty;f)\right)$ to denote counting (reduced counting) function of poles of f.

Let us define $\chi_k = \begin{cases} 1, & if\ k = 1 \\ k+1, & if\ k \geq 2. \end{cases}$

Now we give the following definitions which are used throughout the paper.

Definition 1.1. [6] *For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f \mid\leq$*

$m)(N(r,a;f\mid\geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f\mid\leq m)(\overline{N}(r,a;f\mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid< m), N(r,a;f\mid> m), \overline{N}(r,a;f\mid< m)$ and $\overline{N}(r,a;f\mid> m)$ are defined analogously.

For a constant value a, we denote the set of all a-points (counting multiplicities or CM) of f by $E(a,f)$, and all distinct a-points (ignoring multiplicities or IM) of f by $\overline{E}(a,f)$. As per Ozawa [12], the notation $E(a,f)\subseteq E(a,g)$ means if z_n is a zero of $f-a$ of order $\nu(n)$, then z_n is also a zero of $g-a$ of order at least $\nu(n)$.

Definition 1.2. [5] *For two non-constant meromorphic functions f and g, we say f and g share the value a CM, if $E(a,f)=E(a,g)$. On the other hand, if $\overline{E}(a,f)=\overline{E}(a,g)$ we say f and g share the value a IM.*

Definition 1.3. [7] *Let $k\in\mathbb{Z}^+\cup\{\infty\}$. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity p is counted p times if $p\leq k$ and $k+1$ times if $p>k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k.*

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,l) for any integer l such that $0\leq l<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

In the beginning of nineteenth century R. Nevanlinna inaugurated the value distribution theory with his famous Five value and Four value theorems which were the bases of uniqueness theory. In [13], Rubel and Yang first investigated about the uniqueness of non-constant entire function f and $f^{'}$ sharing two values. This investigation was very important as it first exhibited that in the uniqueness theory, the number of sharing values can be reduced from 5 to 2, for the special class of functions. They proved the following result.

Theorem A. [13] *Let f be non-constant entire function. If f and $f^{'}$ share two distinct values (a,∞), (b,∞), then $f\equiv f^{'}$.*

The following example shows that in the above theorem the two CM value sharing can not be relaxed to one CM value sharing even if the function share the value CM with all its k th derivatives.

Example 1.1. *Consider* $f = Ae^{\theta z} + \frac{B}{2}$*, where* A *and* B *be two non-zero constants and* θ *is the root of the equation* $z^k = 2$*. Then* f *and* $f^{(k)}$ *share* (B, ∞)*, but* $f \neq f^{(k)}$*.*

In 1979, Mues and Steinmetz [10] relaxed the sharing condition in *Theorem A* from CM to IM. Their result is the following.

Theorem B. [10] *Let* f *be non-constant entire function. If* f *and* $f^{'}$ *share two distinct values* $(a, 0)$*,* $(b, 0)$*, then* $f \equiv f^{'}$*.*

In 2000, Li-Yang [8] obtained the following result which settled the conjecture of Frank [3]

Theorem C. [8] *Let* f *be non-constant entire function. If* f *and* $f^{(k)}$ *share two distinct values* $(a, 0)$*,* $(b, 0)$*, then* $f \equiv f^{(k)}$*.*

Earlier in 1983, Mues-Steinmetz [11] and Gundersen [4] independently investigated about the uniqueness of non-constant meromorphic function, when f and $f^{'}$ sharing two values CM.

Theorem D. [4, 11] *Let* f *be non-constant meromorphic function. If* f *and* $f^{'}$ *share values* (a, ∞)*,* (b, ∞)*, then* $f \equiv f^{'}$*.*

Thus *Theorem D* improves *Theorem A*.

After that there was a long recess in this perspective. In 2006, Tanaiadchawoot [14] proved the following result.

Theorem E. [14] *Let* f *be a non-constant meromorphic function,* a, b *be nonzero distinct finite complex constant. If* f *and* $f^{'}$ *share* (a, ∞)*,* $(b, 0)$ *and* $\overline{N}(r, f) = S(r, f)$ *then* $f \equiv f^{'}$*.*

Chundang-Tanaiadchawoot [2] found similar type of result for f and $f^{'}$ share one value CM and another value IM. The result is the following.

Theorem F. [2] *Let* f *be a non-constant meromorphic function let* $a, b \neq 0$ *be distinct finite complex constants. If* f*,* $f^{'}$ *share the values* (a, ∞)*,* $(b, 0)$ *and* $N\big(r, b; f^{'} \mid\, \geq 2\big) = S(r, f)$*, then* $f \equiv f^{'}$*, where* $N\big(r, b; f^{'} \mid\, \geq 2\big)$ *is the counting function which only includes multiple zero of* $f^{'}(z) - b$*.*

Considering the example $f(z) = \Big(\frac{1}{2} - \frac{\sqrt{5}}{2} i tan\big(\frac{\sqrt{5}}{4} iz\big)\Big)^2$, Mues-Steinmetz [10] showed that for a non-constant meromorphic function two IM value sharing is not sufficient enough to make a meromorphic function identical with its derivative. So additional condition is required. In this direction Li [9] obtained the following result.

Theorem G. [9] *Let f be a non-constant meromorphic function such that $\overline{N}(r,f) < \lambda T(r,f)$, where $\lambda \in [0, \frac{1}{9})$, and a, b be two distinct finite values. If f and $f^{'}$ share $(a,0)$, $(b,0)$, then $f \equiv f^{'}$.*

However, without any additional suppositions, Frank [3] (see also [8]) investigated the uniqueness of a meromorphic function f and its k th derivative $f^{(k)}$ sharing two values CM. Below we recall the result of Frank.

Theorem H. [3] *Let f be a non-constant meromorphic function. If f and $f^{(k)}$ share distinct finite values (a,∞), (b,∞), then $f \equiv f^{(k)}$.*

Recently in 2018, Chen et. al [1] investigated *Theorem H* under IM shared values. They proved the following two results.

Theorem I. [1] *Let f be a non-constant meromorphic function and k be a positive integer. If $\overline{N}(r,f) < T(r,f)/(3k+1)$, f and $f^{(k)}$ share two different non-zero values $(a,0),(b,0)$, then $f \equiv f^{(k)}$.*

Theorem J. [1] *Let f be a non-constant meromorphic function, and k be a positive integer. If $\overline{N}(r,f) < T(r,f)/(3k^2+4k+2)$, f and $f^{(k)}$ share $(0,0),(1,0)$ and $E(0,f) \subseteq E(0,f^{(k)})$, $E(1,f) \subseteq E(1,f^{(k)})$, then $f \equiv f^{(k)}$.*

Remark 1.1. *Unfortunately there is a drawback in the statement as well as in the proof of* Theorem I. *In the proof of* Theorem I, *(p. 380) Chen et. al.* [1] *uses the following two inequalities*

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right) \leq T(r,f)+k\overline{N}(r,f)+S(r,f),$$

$$m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right) \leq m\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)$$

to deduce the inequality

$$T(r,f) \leq m\left(r,\frac{1}{f^{(k)}}\right)+k\overline{N}(r,f)+S(r,f).$$

But this is true only when f and $f^{(k)}$ has only simple a, b points. That means a restriction has to be imposed on $a(b)$-points of f and $f^{(k)}$. Consequently Theorem I *cease to be hold for general meromorphic functions.*

Remark 1.2. *In* Theorem J *(p. 379 of* [1]*) the authors assume the conditions f and $f^{(k)}$ share $(0,0)$, $(1,0)$ together with $E(0,f) \subseteq E(0,f^{(k)})$, $E(1,f) \subseteq E(1,f^{(k)})$. From Taylor series expansion it can be easily verified that $E(0,f) \subseteq E(0,f^{(k)})$ implies the multiplicity of the zeros of f is always $\leq k-1$. In particular, if $k=1$, then as f and $f^{'}$ share $(0,0)$, we see that 0 is a Picard exceptional value of both f*

and $f^{'}$*, which means* f *and* $f^{'}$ *practically share* $(0,\infty)$*. So for* $k=1$*, the sharing conditions of* Theorem J *actually reduces to* f *and* $f^{'}$ *share* $(0,\infty)$*,* $(1,0)$ *together with* $E(1,f)\subseteq E(1,f^{'})$*. But we see that* Theorem G *is proved under far weaker sharing hypothesis and so for the case* $k=1$ Theorem J *is redundant.*

We have already seen that the problem initiated by Rubel-Yang has a long history and as the time progressed several elegant results were obtained in different directions. As per the knowledge of the authors till date to find the relation between f and $f^{(k)}$, the researchers have only studied the effect of two extreme sharing conditions namely CM and IM or sometimes with some additional restrictions on the sharing values like [1]. Naturally in view of the sharing conditions adopted by Chen et. al. [1], it will be interesting to deal the situation under a systematic manner of gradation of sharing namely weighted sharing. The prime motivation of writing this paper is to improve all the results by well organizing them under relaxed sharing hypothesis.

It is to be noted that the uniqueness problem of f and $f^{(k)}$ yields the following differential equation

$$f^{(k)}(z)-f(z)=0. \tag{1.1}$$

We observe that all solution of the above differential equation is of the form

$$f(z)=\sum_{i=1}^{n}c_ie^{a_iz} \tag{1.2}$$

where c_i are complex constants and a_i are the k-th roots of unity. For the last few decades, in this particular theory, researchers have been exhaustively involved to determine sufficient conditions under which solutions of (1.1) exists.

Now we are going to state our main theorems. First of all, in *Theorem 1.1* we give the corrected forms of *Theorem I*, where as in *Theorem 1.2* we investigate *Theorem J* under a different sharing condition in a convenient way.

Theorem 1.1. *Let* f *be a non-constant meromorphic function satisfies* $\overline{N}(r,f)<\lambda T(r,f)$*. If* f *and* $f^{(k)}$ *share two distinct non-zero values* $(a,k-1)$*,* $(b,k-1)$ *and* $0\leq\lambda<\frac{1}{3k+1}$ *then* $f\equiv f^{(k)}$*.*

Remark 1.3. *We see that, Theorem 1.1 improves* Theorem G *for* $a.b\neq 0$*.*

Theorem 1.2. *Let* f *be a non-constant meromorphic function such that* f*,* $f^{(k)}$ *share the value* $(0,\chi_{_k}-1)$ *and a non-zero value* $(b,k-1)$ *and satisfies* $\overline{N}(r,f)<\lambda T(r,f)$*,* $0\leq\lambda<\frac{1}{3k^2+4k+2}$*, then* $f\equiv f^{(k)}$*.*

Remark 1.4. *Theorem 1.1 together with Theorem 1.2 improve Theorem G.*

In the next two theorems we shall show that *Theorems 1.1* and *1.2* can further be improved at the expense of different weighted shared values.

Theorem 1.3. *Let f be a non-constant meromorphic function such that f and $f^{(k)}$ share two distinct non-zero values (a,k), (b,k) and satisfies $(k+2)\overline{N}(r,f) < T(r,f)$, then $f \equiv f^{(k)}$.*

Theorem 1.4. *Let f be a non-constant meromorphic function such that f, $f^{(k)}$ share the value $(0,k)$ and a non-zero value (b,k) and satisfies $\frac{(k+1)^2}{k}\overline{N}(r,f) < T(r,f)$, then $f \equiv f^{(k)}$.*

Corollary 1.1. *If f and $f^{(k)}$ share two distinct values (a,k), (b,k) and satisfies $\overline{N}(r,f) < \lambda T(r,f)$, where $0 \leq \lambda < \frac{k}{(k+1)^2}$ then $f \equiv f^{(k)}$. In particular, when $k=1$ if f, $f^{'}$ share two distinct values $(a,1)$, $(b,1)$ and satisfies $\overline{N}(r,f) < \lambda T(r,f)$, where $0 \leq \lambda < \frac{1}{4}$, then $f \equiv f^{'}$.*

In the next two theorems we have fixed one shared value to be 0 and the other to be non-zero and investigated the uniqueness of f and $f^{(k)}$ under different conditions.

Theorem 1.5. *Let f be a non-constant meromorphic function such that f, $f^{(k)}$ share the values $(0,\infty)$, $(b,k-1)$ and satisfies $\overline{N}(r,f) + \overline{N}\big(r,b;f^{(k)}| \geq k+1\big) < T(r,f)$, then $f \equiv f^{(k)}$. In particular, when $k \geq 2$ and $\frac{k}{k-1}\overline{N}(r,f) < T(r,f)$, then $f \equiv f^{(k)}$.*

Theorem 1.6. *Let f be a non-constant meromorphic function such that f, $f^{(k)}$ share the values $(0,\infty)$, $(b,k-1)$ and satisfies $\overline{N}(r,f) < \lambda T(r,f)$, where $0 \leq \lambda < \frac{1}{k+1}$, then $f \equiv f^{(k)}$.*

Example 1.2. *In* [15] *Zhang considered the following example*

$$f(z) = \frac{2A}{1 - Be^{-2z}},\ A \neq 0,\ B \neq 0.$$

It is easy to see that f and $f^{'}$ share the values $(0,\infty)$ and $(A,0)$ but $f \not\equiv f^{'}$. Here $\overline{N}(r,\infty;f) \sim 2T(r,e^z)$. So for the case $k=1$, when the condition in Theorem 1.6 *is not satisfied the conclusion cease to be hold.*

In our last two theorems, in this section, we have paid our attentions to the uniqueness of f and $f^{(k)}$ sharing two distinct values with weight $\leq k-2$ and thus included the case of IM shared values for the case $k \geq 2$ which were never investigated earlier.

Theorem 1.7. *Let f be a non-constant meromorphic function such that f, $f^{(k)}$ share two non-zero values (a,l), (b,l) where $0 \leq l \leq k-2$ and satisfies the condition $\left(1 + \frac{2k^2}{l+1}\right)\overline{N}(r,f) + \frac{2k}{l+1}N_k(r,0;f) < T(r,f)$, then $f \equiv f^{(k)}$.*

Theorem 1.8. *Let f be a non-constant meromorphic function such that f, $f^{(k)}$ share $(0,l)$, (b,l) where $b \neq 0$, $0 \leq l \leq k-2$ and satisfies the condition* $\Big(1+\frac{k^2}{l+1}\Big)\overline{N}(r,f)+(k+1)\overline{N}(r,0;f)+\frac{k}{l+1}N_k(r,0;f)<T(r,f)$ *then* $f \equiv f^{(k)}$.

2. Lemmas

Lemma 2.1. *Let f be a non-constant meromorphic function such that f and $f^{(k)}$ share $(a,0)$, $(b,0)$ where $a \neq 0$ and satisfies $\overline{N}(r,f) < \lambda T(r,f)+S(r,f)$, where $\lambda \in [0,1), k \in \mathbb{Z}^+$. Then*

$$(i)\ T(r,f) \quad \leq \quad \overline{N}(r,f)+N\Big(r,\frac{1}{f-b}\Big)+\overline{N}\Big(r,\frac{1}{f^{(k)}-a}\Big)-N_0\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f),$$

where $N_0\Big(r,\frac{1}{f^{(k+1)}}\Big)$ *is the counting function of those zeros of $f^{(k+1)}$ which are not the zeros of $f^{(k)}-a$.*

$$(ii) \quad \overline{N}\Big(r,\frac{1}{f-a}\Big) > \frac{1-\lambda}{k+1}T(r,f)+S(r,f).$$

Proof. (i) can be obtain from Milloux inequality (see *Theorem 3.2* of [5]) just taking $f-b$ in the place of f and $f^{(k)}-a$ in the place of $\Psi-1$. And (ii) follows from *Lemma 2.1* of [9] by taking $f-a$ in the place of $f-1$.

Remark 2.1. *The sharing conditions for a, b are no longer required for proving (i) of Lemma 2.1. That is this part is independent of sharing.*

Lemma 2.2. *Let f be a non-constant meromorphic function. For a non zero constant a and $k \in \mathbb{Z}^+$, if f and $f^{(k)}$ share $(a,0),(0,0)$ then,*

$$T(r,f) \leq \overline{N}(r,f)+(k+1)\overline{N}(r,0;f)+\overline{N}\Big(r,\frac{1}{f^{(k)}-a}\Big)-N_{\otimes}\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f),$$

where $N_{\otimes}\Big(r,\frac{1}{f^{(k+1)}}\Big)$ *denotes the counting function of those zeros of $f^{(k+1)}$ which are not zeros of f as well as $f^{(k)}-1$.*

Proof. Note that as f and $f^{(k)}$ share $(a,0),(0,0)$ then multiplicity of zeros of f is at least $k+1$. Not only that the zeros with multiplicities $p \geq k+2$ will be counted $p-k-1$ times in $N\Big(r,\frac{1}{f^{(k+1)}}\Big)$. Also the multiplicities of a points of f is at most k and multiple zeros of $f^{(k)}-a$ of order $t(\geq 2)$ are all zeros of $N\Big(r,\frac{1}{f^{(k+1)}}\Big)$ of order $t-1$. Then invoking (i) of Lemma 2.1 for $b=0$ we get

$$T(r,f) \leq \overline{N}(r,f)+(k+1)\overline{N}(r,0;f)+\overline{N}\Big(r,\frac{1}{f^{(k)}-a}\Big)-N_{\otimes}\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f).$$

3. Proofs of the theorems

Proof of Theorem 1.1. Suppose $f \not\equiv f^{(k)}$. As f and $f^{(k)}$ share the values $(a, k-1)$ and $(b, k-1)$, it follows that the a (or b)-points of f are of multiplicity $\leq k$. If z_1 is a zero of $f-a$ (or $f-b$) with multiplicity $p \leq k-1$, then z_1 is also zero of $f^{(k)}-f$ with multiplicity exactly p. If z_2 is a zero of $f-a$ (or $f-b$) with multiplicity k then z_2 is also zero of $f^{(k)}-a$ (or $f^{(k)}-b$) with multiplicity $k+j$, where $j \geq 0$. So z_2 be zero of $f^{(k)}-f$ with multiplicity equals to $min\{k, k+j\} = k$. Thus in both the cases multiplicity of zeros of $f-a$ (or $f-b$) is same as the multiplicity of zeros of $f^{(k)}-f$. This implies $N(r,a;f)+N(r,b;f) \leq N\Big(r,0;f-f^{(k)}\Big)$, using this fact and the First Fundamental Theorem we have

$$
\begin{aligned}
(3.1) \qquad & \overline{N}\Big(r,a;f^{(k)}\Big)+\overline{N}\Big(r,b;f^{(k)}\Big) \leq N(r,a;f)+N(r,b;f) \\
\leq\ & N\Big(r,0;f-f^{(k)}\Big) \leq T\big(r,f-f^{(k)}\big)+S(r,f) \\
\leq\ & m\big(r,f-f^{(k)}\big)+N\big(r,f-f^{(k)}\big)+S(r,f) \\
=\ & m\left(r,f\Big(1-\frac{f^{(k)}}{f}\Big)\right)+N(r,f)+k\overline{N}(r,f)+S(r,f) \\
\leq\ & m(r,f)+N(r,f)+k\overline{N}(r,f)+S(r,f) \\
\leq\ & T(r,f)+k\overline{N}(r,f)+S(r,f).
\end{aligned}
$$

Noting that

$$m(r,a;f)+m(r,b;f) \leq m\Big(r,\frac{1}{f^{(k)}}\Big)+S(r,f),$$

from (3.1) in view of the First Fundamental Theorem we get

$$T(r,f) \leq m\Big(r,\frac{1}{f^{(k)}}\Big)+k\overline{N}(r,f)+S(r,f). \tag{3.2}$$

Again as

$$
\begin{aligned}
& N\left(r,\frac{1}{f^{(k)}-a}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}-a}\right)+N\left(r,\frac{1}{f^{(k)}-b}\right) \\
& \qquad -\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right) \leq N\Big(r,\frac{1}{f^{(k+1)}}\Big),
\end{aligned}
$$

from (3.1) we have

$$
\begin{aligned}
(3.3) \qquad & N\left(r,\frac{1}{f^{(k)}-a}\right)+N\left(r,\frac{1}{f^{(k)}-b}\right) \\
& \leq T(r,f)+k\overline{N}(r,f)+N\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f).
\end{aligned}
$$

Note that

$$m\Big(r,\frac{1}{f^{(k)}}\Big)+m\left(r,\frac{1}{f^{(k)}-a}\right)+m\left(r,\frac{1}{f^{(k)}-b}\right)\leq m\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f).$$

So from (3.3) we have

$$\begin{aligned}
&m\Big(r,\frac{1}{f^{(k)}}\Big)+2T(r,f^{(k)})+O(1)=m\Big(r,\frac{1}{f^{(k)}}\Big)+m\left(r,\frac{1}{f^{(k)}-a}\right)\\
&+m\left(r,\frac{1}{f^{(k)}-b}\right)+N\left(r,\frac{1}{f^{(k)}-a}\right)+N\left(r,\frac{1}{f^{(k)}-b}\right)\\
\leq\;& T(r,f)+T\Big(r,\frac{1}{f^{(k+1)}}\Big)+k\overline{N}(r,f)+S(r,f)\\
\leq\;& T(r,f)+m\big(r,f^{(k+1)}\big)+N\big(r,f^{(k+1)}\big)+k\overline{N}(r,f)+S(r,f)\\
\leq\;& T(r,f)+m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+m\big(r,f^{(k)}\big)+N\big(r,f^{(k)}\big)\\
+\;& (k+1)\overline{N}(r,f)+S(r,f).
\end{aligned}$$

Using (3.2) in the above inequality we get

$$T\big(r,f^{(k)}\big)\leq(2k+1)\overline{N}(r,f)+S(r,f). \tag{3.4}$$

Combining (3.2) and (3.4) we have

$$\begin{aligned}
T(r,f)\;&\leq\; m\Big(r,\frac{1}{f^{(k)}}\Big)+k\overline{N}(r,f)+S(r,f)\\
&\leq\; T(r,f^{(k)})+k\overline{N}(r,f)+S(r,f)\\
&\leq\; (3k+1)\overline{N}(r,f)+S(r,f).
\end{aligned} \tag{3.5}$$

Now (3.5) contradicts the given condition. Therefore $f\equiv f^{(k)}$. □

Proof of Theorem 1.2. For the case $k=1$, we refer to [9]. So we consider the case $k\geq 2$. Clearly f, $f^{(k)}$ share the value $(0,k)$ and a non-zero value $(b,k-1)$. Let us assume $f\not\equiv f^{(k)}$. Without loss of generality we assume that $b=1$. Consider the function

$$U=\frac{f^{(k)}\big(f^{(k)}-f\big)}{f(f-1)}. \tag{3.6}$$

Note that if z_0 is a pole of f with multiplicity p then z_0 is also a pole of U with multiplicity $2k$. Now

$$\begin{aligned}
m(r,U)\;&=\; m\left(r,\Big(\frac{f^{(k)}}{f}-1\Big)\frac{f^{(k)}}{f-1}\right)\\
&\leq\; m\Big(r,\frac{f^{(k)}}{f-1}\Big)+m\Big(r,\frac{f^{(k)}}{f}\Big)=S(r,f).
\end{aligned} \tag{3.7}$$

By (3.6) we get that $U(f^2-f)=f^{(k)}\Big(f^{(k)}-f\Big)$. Differentiating both side we get

$$U^{'}(f^2-f)+U(2ff^{'}-f^{'})=f^{(k+1)}\Big(f^{(k)}-f\Big)+f^{(k)}\Big(f^{(k+1)}-f^{'}\Big). \tag{3.8}$$

Let z_1 be a 1-point of f. As f and $f^{(k)}$ share $(1,k-1)$ then $f(z_1)=f^{(k)}(z_1)=1$. Using it in (3.8) we get

$$f^{(k+1)}(z_1)=(1+U(z_1))f^{'}(z_1).$$

Now we consider a function

$$\sigma=\frac{\Big(f^{(k+1)}-(1+U)f^{'}\Big)\big(f^{(k)}-f\big)}{f(f-1)}.$$

Note that $m(r,\sigma)=S(r,f)$ and pole of f is also pole of σ with multiplicity $3k+1$. As f and $f^{(k)}$ share $(0,k)$ so multiplicity of zeros always $\geq 2k+1$, then it is easy to see that zeros of f will not contribute any poles of σ. Thus

$$T(r,\sigma)=N(r,\sigma)+S(r,f)\leq(3k+1)\overline{N}(r,f)+S(r,f).$$

As f and $f^{(k)}$ share $(1,k-1)$, the multiplicity of zeros of $f-1$ is always $\leq k$. If z_1 is a zero of $f-1$ with multiplicity $p\leq k-1$, then z_1 is also zero of $f^{(k)}-f$ with multiplicity exactly p. If z_2 is a zero of $f-1$ with multiplicity k then z_2 is also zero of $f^{(k)}-1$ with multiplicity $k+j$, where $j\geq 0$. So z_2 be zero of $f^{(k)}-f$ with multiplicity equal to $min\{k,k+j\}=k$. Thus in both the cases zeros of $f-1$ will not contribute to the zeros or poles of $\frac{f^{(k)}-f}{f-1}$. Note that the zeros of $f-1$ must be a zero of $\Big(f^{(k+1)}-(1+U)f^{'}\Big)$ of multiplicity at least one. Thus 1-points of f must be the zeros of σ. So

$$\begin{aligned}\overline{N}\Big(r,\frac{1}{f-1}\Big)\leq N\big(r,\frac{1}{\sigma}\big) &\leq T(r,\sigma)\leq(3k+1)\overline{N}(r,f)+S(r,f)\\ &< \lambda(3k+1)T(r,f)+S(r,f).\end{aligned} \tag{3.9}$$

By *Lemma 2.1* we have

$$\overline{N}\Big(r,\frac{1}{f-1}\Big)>\frac{1-\lambda}{k+1}T(r,f)+S(r,f). \tag{3.10}$$

Combining (3.9) and (3.10) we get

$$\frac{1-\lambda}{k+1}<\lambda(3k+1)\implies\lambda>\frac{1}{3k^2+4k+2}. \tag{3.11}$$

Clearly (3.11) contradicts the given condition. Therefore $f\equiv f^{(k)}$. □

Proof of Theorem 1.3. Suppose $f\not\equiv f^{(k)}$. As f and $f^{(k)}$ share the values (a,k) and (b,k), it follows that the a (b)-points of f and $f^{(k)}$ coincides in location as well as in multiplicity $\leq k$. So $f^{(k)}$ will not have any a (b)-points of multiplicity $\geq k+1$.

In other words f and $f^{(k)}$ share the values (a,∞) and (b,∞). Similar as (3.2) we get

$$\begin{aligned}N\Big(r,a;f^{(k)}\Big)+N\Big(r,b;f^{(k)}\Big) &\leq N(r,a;f)+N(r,b;f)\\ &\leq T(r,f)+k\overline{N}(r,f)+S(r,f).\end{aligned} \tag{3.12}$$

Consider

$$V=\frac{f^{(k+1)}}{f^{(k)}-a}-\frac{f'}{f-a},\ \ V_1=\frac{f^{(k+1)}}{f^{(k)}-b}-\frac{f'}{f-b}. \tag{3.13}$$

First suppose both $V\equiv 0$ and $V_1\equiv 0$. Then integrating we get

$$(f^{(k)}-a)=C_1(f-a),\ (f^{(k)}-b)=C_2(f-b),$$

where C_1,C_2 are two non zero constants. If either of $C_1=1$ or $C_2=1$, then we are done. If $C_1\neq 1$ and $C_2\neq 1$, then from the above two equations, after simple calculations we get

$$(C_1-C_2)f(z)=C_1a-C_2b+b-a.$$

If $C_1\neq C_2$, then f is a constant, a contradiction. Therefore $C_1=C_2$ and hence $C_1(a-b)=(a-b)$. As a and b are distinct, we have $C_1=C_2=1$ and so $f\equiv f^{(k)}$.

Next, we will take one of V, V_1 is equivalent to 0 and other is not equivalent to 0. Without loss of generality, we assume $V\equiv 0$ and $V_1\not\equiv 0$. Then from (3.13) integrating the first equation we get

$$f^{(k)}-a=C_1(f-a).$$

If f has a pole at z_1 of multiplicity p then z_1 is also pole of $f^{(k)}$ of multiplicity $p+k$, but this contradicts the last equation. So f does not have any pole. Thus f is entire. Now if $C_1=1$ then $f\equiv f^{(k)}$. So we assume $C_1\neq 1$. Noting the fact that f and $f^{(k)}$ share (b,∞), from $f^{(k)}-a=C_1(f-a)$ we can conclude $C_1=1$, which is a contradiction. Thus b is a Picard exceptional value of f and $f^{(k)}$. It follows that V_1 does not have any pole. As f and $f^{(k)}$ share (a,∞), an elementary calculation yields a-points of f are zeros of V_1. So by the First Fundamental Theorem we get

$$N\Big(r,\frac{1}{f-a}\Big)\leq N\Big(r,\frac{1}{V_1}\Big)\leq T(r,V_1)+O(1)=m(r,V_1)+N(r,V_1)+O(1)=S(r,f).$$

Next using the Second Fundamental Theorem we obtain

$$T(r,f)\leq N(r,f)+N\Big(r,\frac{1}{f-a}\Big)+N\Big(r,\frac{1}{f-b}\Big)+S(r,f)=S(r,f),$$

a contradiction.

So we assume $V\not\equiv 0$ and $V_1\not\equiv 0$.

From (3.13) it is clear that

$$m(r,V)\leq S(r,f^{(k)})+S(r,f)=S(r,f).$$

Since f and $f^{(k)}$ share the values (a,∞) and (b,∞), we note that

$$N(r,V)\le \overline{N}(r,f)+S(r,f).$$

Now

$$\begin{aligned}(3.14)\qquad m\Big(r,\frac{1}{f-b}\Big) &\le m\big(r,\frac{1}{V}\big)+m\big(r,\frac{V}{f-b}\big)\\
&\le m\big(r,\frac{1}{V}\big)+m\left(r,\frac{f^{(k+1)}f^{(k)}}{f^{(k)}(f^{(k)}-a)(f-b)}\right)+m\Big(r,\frac{f'}{(f-a)(f-b)}\Big)\\
&\le m\big(r,\frac{1}{V}\big)+m\left(r,\frac{f^{(k+1)}}{a}\Big\{\frac{1}{f^{(k)}-a}-\frac{1}{f^{(k)}}\Big\}\right)+m\Big(r,\frac{f^{(k)}}{f-b}\Big)\\
&+ m\left(r,\frac{f'}{b-a}\Big\{\frac{1}{f-a}-\frac{1}{f-b}\Big\}\right)+O(1)\\
&\le T(r,V)+S(r,f)\le \overline{N}(r,f)+S(r,f).\end{aligned}$$

Similarly we can write

$$m\Big(r,\frac{1}{f-a}\Big)\le \overline{N}(r,f)+S(r,f). \tag{3.15}$$

Using (3.12), (3.14), (3.15) and the First Fundamental Theorem we get

$$\begin{aligned}2T(r,f)+O(1) &= N\Big(r,\frac{1}{f-a}\Big)+N\Big(r,\frac{1}{f-b}\Big)+m\Big(r,\frac{1}{f-a}\Big)\\
&+ m\Big(r,\frac{1}{f-b}\Big)+O(1)\le T(r,f)+k\overline{N}(r,f)+2\overline{N}(r,f)+S(r,f)\\
&= T(r,f)+(k+2)\overline{N}(r,f)+S(r,f),\end{aligned}$$

which yields

$$T(r,f)\le (k+2)\overline{N}(r,f)+S(r,f). \tag{3.16}$$

It is clear to see that (3.16) contradict the given condition. Therefore $f\equiv f^{(k)}$. □

Proof of Theorem 1.4. Let us assume $f\not\equiv f^{(k)}$. Without loss of generality we assume that $b=1$. As f and $f^{(k)}$ share the values $(0,k)$ and $(1,k)$, it follows that the zeros of f are of multiplicity $\ge 2k+1$ and as usual f and $f^{(k)}$ share $(1,\infty)$. We consider the function

$$W=\frac{f'(f-f^{(k)})}{f(f-1)}. \tag{3.17}$$

Note that

$$m(r,W)=S(r,f)$$

and

$$m\Big(r,\frac{1}{f}\Big)\leq m\left(r,\frac{1}{W}\right)+m\left(r,\left(\frac{f^{'}(f^{(k)}-f)}{f^{2}(f-1)}\right)\right)+S(r,f)$$
$$\leq m\left(r,\frac{1}{W}\right)+m\left(r,f^{'}\left\{\frac{1}{f-1}-\frac{1}{f}\right\}\right)+m\left(r,\left(\frac{f^{(k)}}{f}-1\right)\right)+S(r,f)$$
$$\leq m\left(r,\frac{1}{W}\right)+S(r,f).$$

Let z_0 be a zero of f with multiplicity $p\geq 2k+1$ then z_0 is also a zero of W with multiplicity $p-(k+1)$. The 1-point of f will not contribute to any zero or pole of W. If z_1 is a pole of f with multiplicity q then z_1 is also a pole of W with multiplicity $(k+1)$.

Considering above we get

$$\begin{aligned}(3.18)\qquad & N(r,\frac{1}{f})-(k+1)\overline{N}(r,\frac{1}{f})\leq N\Big(r,\frac{1}{W}\Big)+S(r,f)\\ \leq\;& T\Big(r,\frac{1}{W}\Big)-m\Big(r,\frac{1}{W}\Big)+S(r,f)\\ \leq\;& N(r,W)+m(r,W)-m\Big(r,\frac{1}{W}\Big)+S(r,f)\\ \leq\;& (k+1)\overline{N}(r,f)+m(r,W)-m\Big(r,\frac{1}{W}\Big)+S(r,f).\\ \leq\;& (k+1)\overline{N}(r,f)-m\Big(r,\frac{1}{f}\Big)+S(r,f).\end{aligned}$$

Therefore by (3.18)

$$\begin{aligned}(3.19)\qquad T(r,f) &= T\Big(r,\frac{1}{f}\Big)+S(r,f)\\ &\leq (k+1)\overline{N}\Big(r,\frac{1}{f}\Big)+(k+1)\overline{N}(r,f)+S(r,f).\end{aligned}$$

Next consider $W_1=\frac{f^{(k+1)}}{f^{(k)}-1}-\frac{f^{'}}{f-1}$.
It is clear that $W_1\not\equiv 0$ otherwise $f\equiv f^{(k)}$. As f and $f^{(k)}$ share $(0,k)$, it follows that f has no zeros of order $\leq 2k$ and $f^{(k)}$ has no zeros of order $\leq k$. Thus by simple calculation we get

$$(3.20)\qquad k\overline{N}\Big(r,0;f|\geq 2k+1\Big)\leq N\Big(r,\frac{1}{W_1}\Big)\leq\overline{N}(r,f)+S(r,f).$$

Combining (3.19) and (3.20) we get

$$\begin{aligned}(3.21)\qquad T(r,f) &\leq \frac{k+1}{k}\overline{N}(r,f)+(k+1)\overline{N}(r,f)+S(r,f)\\ &= \frac{(k+1)^2}{k}\overline{N}(r,f)+S(r,f).\end{aligned}$$

Clearly (3.21) contradicts the given condition. Therefore $f\equiv f^{(k)}$. □

Proof of Theorem 1.5. Let us assume $f \not\equiv f^{(k)}$. Without loss of generality we assume that $b = 1$. According to the condition of the theorem 0 is an exceptional value of Picard for both f and $f^{(k)}$. We consider a function

$$\begin{aligned} \phi &= \frac{f^{(k+1)}}{f\big(f^{(k)}-1\big)} - \frac{f^{'}}{f(f-1)} \\ &= \frac{f^{(k)}}{f}\Big(\frac{f^{(k+1)}}{f^{(k)}-1} - \frac{f^{(k+1)}}{f^{(k)}}\Big) - \Big(\frac{f^{'}}{f-1} - \frac{f^{'}}{f}\Big). \end{aligned} \tag{3.22}$$

From (3.22) it is clear that

$$m(r,\phi) = S(r,f). \tag{3.23}$$

If $\phi \equiv 0$ then integrating (3.22) we get $f^{(k)} - 1 = C(f-1)$, where C is non zero constant. Now counting the order of poles of $f^{(k)} - 1$ and $C(f-1)$ we get a contradiction. Therefore $\phi \not\equiv 0$. We can write (3.22) as

$$f = \frac{1}{\phi}\Big(\frac{f^{(k+1)}}{f^{(k)}-1} - \frac{f^{'}}{f-1}\Big).$$

It is clear that

$$m(r,f) \le m\Big(r,\frac{1}{\phi}\Big) + S(r,f). \tag{3.24}$$

Note that the 1-points of $f^{(k)}$ with multiplicity $\ge k+1$ are the only poles of ϕ and if z_0 is a pole of f with multiplicity p then z_0 is a zero of ϕ with multiplicity $p-1$. Then

$$\begin{aligned} N(r,f) - \overline{N}(r,f) &\le N\Big(r,\frac{1}{\phi}\Big) + S(r,f) \\ &\le T\Big(r,\frac{1}{\phi}\Big) - m\Big(r,\frac{1}{\phi}\Big) + S(r,f) \\ &\le T(r,\phi) - m\Big(r,\frac{1}{\phi}\Big) + S(r,f) \\ &\le m(r,\phi) + N(r,\phi) - m\Big(r,\frac{1}{\phi}\Big) + S(r,f) \\ &\le m(r,\phi) + \overline{N}\big(r,1;f^{(k)}\mid \ge k+1\big) - m\Big(r,\frac{1}{\phi}\Big) + S(r,f) \end{aligned} \tag{3.25}$$

Now using (3.23), (3.24) in (3.25) we get

$$T(r,f) \le \overline{N}(r,f) + \overline{N}\big(r,1;f^{(k)}\mid \ge k+1\big) + S(r,f), \tag{3.26}$$

a contradiction and hence $f \equiv f^{(k)}$.

Next consider $k \ge 2$. Suppose

$$\psi = \frac{f^{(k+1)}}{f^{(k)}} - \frac{f^{'}}{f}.$$

As f is non entire meromorphic function then $\psi \not\equiv 0$. As f and $f^{(k)}$ share 1 with weight $k-1$ then 1-point of $f^{(k)}$ are zeros of ψ. Thus by simple calculation for $k \geq 2$ we get

$$(k-1)\overline{N}\Big(r,1;f^{(k)}\mid\geq k+1\Big) \leq N\Big(r,\frac{1}{\psi}\Big) \leq \overline{N}(r,f)+S(r,f). \tag{3.27}$$

Combining (3.26) and (3.27)

$$T(r,f) \leq \frac{k}{k-1}\overline{N}(r,f)+S(r,f). \tag{3.28}$$

Clearly (3.28) contradicts the given condition. Therefore $f \equiv f^{(k)}$. □

Proof of Theorem 1.6. Let us assume $f \not\equiv f^{(k)}$. Without loss of generality assume that $b=1$. Consider the function

$$\mu = \frac{f^{(k)}}{f}.$$

If f has a pole at z_∞ of any multiplicity, then z_∞ is also a pole of μ with multiplicity k. Note that $m(r,\mu)=S(r,f)$. Thus

$$T(r,\mu)=N(r,\mu)+S(r,f) \leq k\overline{N}(r,f)+S(r,f).$$

It is clear that 1-point of f is also 1-point of μ. So

$$\begin{aligned} N\Big(r,\frac{1}{f-1}\Big) &\leq N\Big(r,\frac{1}{\mu-1}\Big) \leq T(r,\mu-1) \\ &\leq k\overline{N}(r,f)+S(r,f). \end{aligned} \tag{3.29}$$

We note that as $f, f^{(k)}$ share $(0,\infty)$, 0 is an exceptional value of Picard for f and $f^{(k)}$. So from *Lemma 2.2* and (3.29) we have

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f)+\overline{N}\big(r,1;f^{(k)}\big)-N_{\otimes}\Big(r,\frac{1}{f^{(k+1)}}\Big)+S(r,f) \\ &\leq \overline{N}(r,f)+\overline{N}(r,1;f)+S(r,f) \\ &\leq \overline{N}(r,f)+k\overline{N}(r,f)+S(r,f) \\ &\leq (k+1)\overline{N}(r,f)+S(r,f). \end{aligned} \tag{3.30}$$

It is easy to see that (3.30) contradicts the given condition. Therefore $f \equiv f^{(k)}$. □

Proof of Theorem 1.7. Let us assume $f \not\equiv f^{(k)}$. As f, $f^{(k)}$ share (a,l) and (b,l) for $0 \leq l \leq k-2$, so from (i) of *Lemma 2.1* we can write

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f)+N\Big(r,\frac{1}{f-a}\Big)+\overline{N}\Big(r,\frac{1}{f^{(k)}-a}\Big)-N_0\Big(r,\frac{1}{f^{(k+1)}}\Big) \\ &+ S(r,f) \leq \overline{N}(r,f)+N(r,a;f)+\overline{N}\big(r,a;f^{(k)}\big)+S(r,f) \\ &\leq \overline{N}(r,f)+N(r,a;f)+\overline{N}(r,a;f)+S(r,f). \end{aligned} \tag{3.31}$$

Let us consider

$$\tau = \frac{f}{f^{(k)}}.$$

Let z_0 be a-point of f with multiplicity p_0 such that $l+2 \leq p_0 \leq k$, then z_0 will be an a-point of $f^{(k)}$ with multiplicity at least $l+1$ and so it will be counted in the counting function of $\tau - a$ at most $\frac{kp_0}{l+1}$ times. Thus by the first fundamental theorem we get

$$\begin{aligned} N(r,a;f) &\leq \frac{k}{l+1} N\Big(r, \frac{f^{(k)}}{f-f^{(k)}}\Big) \leq \frac{k}{l+1} T(r,\tau) + S(r,f) \\ &\leq \frac{k}{l+1} N\Big(r, \frac{f^{(k)}}{f}\Big) + S(r,f) \\ &\leq \frac{k}{l+1}\big[N_k(r,0;f) + k\overline{N}(r,f)\big] + S(r,f). \end{aligned} \tag{3.32}$$

Now together from (3.31) and (3.32) we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + \frac{2k}{l+1}\big[N_k(r,0;f) + k\overline{N}(r,f)\big] + S(r,f) \\ &\leq \Big(1 + \frac{2k^2}{l+1}\Big)\overline{N}(r,f) + \frac{2k}{l+1} N_k(r,0;f) + S(r,f). \end{aligned} \tag{3.33}$$

Note that we can find same result if we apply the above method using b-points instead of a-points. Now it is clear that (3.33) contradicts the given condition. Therefore $f \equiv f^{(k)}$. □

Proof of Theorem 1.8. Let us assume $f \not\equiv f^{(k)}$. For non zero constant b and $0 \leq l \leq k-2$ we have $f, f^{(k)}$ share $(0,l)$ and (b,l). From *Lemma 2.2* we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + (k+1)\overline{N}(r,0;f) + \overline{N}\Big(r, \frac{1}{f^{(k)}-b}\Big) \\ &- N_{\otimes}\Big(r, \frac{1}{f^{(k+1)}}\Big) + S(r,f). \end{aligned} \tag{3.34}$$

Now similar as in the proof of *Theorem 1.7* we can write

$$\overline{N}\Big(r, \frac{1}{f^{(k)}-b}\Big) \leq \frac{k}{l+1}\big[N_k(r,0;f) + k\overline{N}(r,f)\big] + S(r,f). \tag{3.35}$$

Combining (3.34) and (3.35) we get

$$\begin{aligned} T(r,f) &\leq \Big(1 + \frac{k^2}{l+1}\Big)\overline{N}(r,f) + (k+1)\overline{N}(r,0;f) \\ &+ \frac{k}{l+1} N_k(r,0;f) + S(r,f). \end{aligned} \tag{3.36}$$

Thus one can see that (3.36) contradicts the given condition. Therefore $f \equiv f^{(k)}$.

□

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ON VALUE DISTRIBUTION OF A CLASS OF ENTIRE FUNCTIONS

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Abstract. We study uniqueness problems in terms of shared values or shared sets for a large class of entire functions representable as Dirichlet series in some right half-plane. In this article, we obtain a result that extends a recent result due to Oswald and Steuding [Annales Univ. Sci. Budapest., Sect. Comp., 48 (2018), 117-128]. Our result is also a variant of a result of Yuan-Li-Yi [Lithuanian Math. J., 58 (2018), 249-262], and a result of the present authors [Lithuanian Math. J., 60 (2020), 80-91] for the said class of functions.

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1. Introduction, definitions and results

Suppose that f and g are either meromorphic or entire functions in the complex plane $\mathbb{C}$. Let $c \in \mathbb{C} \cup \{\infty\}$. The functions f and g are said to share the value c IM (ignoring multiplicities) if $f - c$ and $g - c$ have the same set of zeros, or equivalently, if $f^{-1}(c) = g^{-1}(c)$, where $f^{-1}(c)$ denotes the set of preimages of c under f, defined as $f^{-1}(c) := \{s \in \mathbb{C} : f(s) - c = 0\}$. Moreover, f and g are said to share the value c CM (counting multiplicities) if f and g have the same set of zeros and the multiplicities of the corresponding zeros are also equal. In connection to the shared values one must recall a much celebrated result due to R. Nevanlinna (known as Nevanlinna's five value theorem or uniqueness theorem) which tells that two nonconstant meromorphic functions are identical whenever they share five distinct values IM; the number "five" is the best possible, as shown by Nevanlinna (see [5, 11, 19]). Besides Nevanlinna's uniqueness theorem Pólya's theorem [13] can be mentioned as another fundamental result and a forerunner of the above theorem. In [13], the author showed that four distinct shared CM values are required for the uniqueness of entire functions of finite order. For any set $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) := \bigcup_{c \in S} \{s \in \mathbb{C} : f(s) - c = 0\},$$

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where each zero of $f-c$ is counted with multiplicities, that is, $E_f(S)$ is a multi-set. Also, by $\overline{E}_f(S)$ we mean the collection of distinct elements in $E_f(S)$. If $E_f(S) = E_g(S)$, we say f and g share the set S CM; if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that they share the set S IM. Clearly, sharing a singleton set and sharing a value have the same meaning by all means. There are meromorphic functions which have importance in number theory, and so their value distribution is also valuable. During the last decade, shared value problems related to these functions, such as zeta functions and more generally the Selberg class L-functions have been studied extensively (see [3, 8, 10, 16, 18]).

In [16], Steuding investigated on the possible number of shared values for the Selberg class functions. A function of the said class generally means a Dirichlet series $\mathcal{L}(s) = \sum_{n\geq 1} \frac{a(n)}{n^s}$ with coefficients $a(n) \ll n^{\epsilon}$ (for each $\epsilon > 0$) which has a meromorphic continuation of finite order to the entire complex plane $\mathbb{C}$ with only possible pole at $s = 1$, satisfies a Riemann type functional equation, and also might have an Euler product over primes (see [15, 16] for precise definition).

In view of Gross's question for two sets (see [4]), Yuan, Li and Yi [20] asked: *What can be said about the relationship between a meromorphic function f and an L-function $\mathcal{L}$ of Selberg class when they share two finite sets?* The authors [20] also resolved this question by proving the following theorem.

Theorem A. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$, and let $\mathcal{L}$ be a nonconstant L-function of Selberg class. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, where α_1, α_2, $\ldots$, α_l are all distinct roots of the algebraic equation $\omega^p + a\omega^q + b = 0$. Here l is a positive integer satisfying $1 \leq l \leq p$, p and q are relatively prime positive integers with $p \geq 5$ and $p > q$, and a, b, c are three finite nonzero constants, where $c \neq \alpha_j$ for $1 \leq j \leq l$. If f and $\mathcal{L}$ share S CM and c IM, then $f = \mathcal{L}$.*

Recently, in [14], the present authors proved an IM analogue of Theorem A, as shown in the following result.

Theorem B. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$, and let $\mathcal{L}$ be a nonconstant L-function of Selberg class. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are all distinct roots of the algebraic equation $P(\omega) = \omega^p + a\omega^q + b = 0$. Here l is a positive integer satisfying $1 \leq l \leq p$, p and q are relatively prime positive integers with $p > 4k + 9$ and $k = p - q \geq 1$, and a, b, c are three finite nonzero constants, where $c \neq \alpha_j$ for $1 \leq j \leq l$. If f and $\mathcal{L}$ share S IM and c IM, then $f = \mathcal{L}$.*

In [12], Oswald and Steuding considered a more general class of functions, namely the class of entire functions of the form

$$L(s;f) = \sum_{n\geq 1} \frac{f(n)}{n^s}, \tag{1.1}$$

which are representable as Dirichlet series in some right half-plane. Here the coefficients are given by an arithmetical function $f : \mathbb{N} \to \mathbb{C}$. For such functions the authors [12] proved the following result.

Theorem C. *Let $L(s;f_1)$ and $L(s;f_2)$ be two entire functions of finite order so that each of them has a convergent Dirichlet series representation of the form* (1.1) *in some right half-plane. If $L(s;f_1)$ and $L(s;f_2)$ share two distinct complex values a and b CM, then $L(s;f_1) = L(s;f_2)$.*

As Theorem C deals with only the shared values, it would be desirable to explore the problem on the shared sets for the same pair of functions. Moreover, it becomes interesting to investigate how far the conclusions of Theorem A and Theorem B hold for these functions. We prove the following theorem in this regard.

Theorem 1.1. *Let $L(s;f_1)$ and $L(s;f_2)$ be two nonconstant entire functions having convergent Dirichlet series representations of the form* (1.1) *in certain right half-plane, and one of them is of finite order. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are all distinct roots of the algebraic equation $P(\omega) = \omega^p + a\omega^q + b = 0$. Here l is a positive integer satisfying $1 \leq l \leq p$, p and q are relatively prime positive integers with $p > 2$ and $p > q$, and a, b are two finite nonzero constants. If $L(s;f_1)$ and $L(s;f_2)$ share S IM and they assume a common complex value c $(\neq \alpha_j)$ $(1 \leq j \leq l)$ for some $s_0 \in \mathbb{C}$, then $L(s;f_1) = L(s;f_2)$ in some right half-plane.*

It is assumed that the readers are accustomed with Nevanlinna theory, and so with its standard notations for a meromorphic (entire) function f, such as $T(r,f)$ (the Nevanlinna characteristic function), $m(r,f)$ (the proximity function), $N(r,f)$ (the counting function) and $\overline{N}(r,f)$ (the reduced counting function) (for details, we refer the reader to [5], [7], [19]). The notion $S(r,f)$, often used in this theory, will mean any quantity that equals $O(\log(rT(r,f)))$, $(r \to \infty)$ except possibly a set of r of finite Lebesgue measure. In particular, if $\rho(f) < +\infty$ ($\rho(f)$ denotes the order of f), then $S(r,f) = O(\log r)$, $(r \to \infty)$ holds without any exceptional set.

2. Lemmas

The following results are important for the proof of our main theorem.

Lemma 2.1. [6, Satz 12] *Let $F(s)$ be a function represented by a Dirichlet series $F(s)=\sum_{n\geq 1}\frac{f(n)}{n^s}$, convergent and non-vanishing in some right half-plane Re $s>\sigma_0$. Then its reciprocal also obeys a Dirichlet series representation $\frac{1}{F(s)}=\sum_{n\geq 1}\frac{g(n)}{n^s}$ in the same half-plane Re $s>\sigma_0$.*

Lemma 2.2. [7, p. 5] *Let $g,h:(0,+\infty)\to\mathbb{R}$ be monotonically increasing real functions such that $g(r)\leq h(r)$ outside an exceptional set M of finite linear measure. Then, for any $\kappa>1$, there exists $r_0>0$ such that $g(r)\leq h(\kappa r)$ for all $r>r_0$.*

Lemma 2.3. [21, Lemma 8] *Let $p(>0)$ and q be two relatively prime integers, and let a be a finite complex number satisfying $a^p=1$. Then the expressions ω^p-1 and ω^q-a have a unique common zero.*

Lemma 2.4. [9, Lemma 2.7] *Let $P(\omega)=\omega^p+a\omega^q+b$, where p and q are positive integers satisfying $p>q$, $a(\neq 0)$ and $b(\neq 0)$ are finite complex numbers. Then the following cases occur:*

(i) The algebraic equation $P(\omega)=0$ has no root of multiplicity ≥ 3.

(ii) If

$$\frac{b^{p-q}}{a^p}\neq\frac{(-1)^p q^q (p-q)^{p-q}}{p^p}, \tag{2.1}$$

then the algebraic equation $P(\omega)=0$ has exactly p distinct roots which are all simple, and no multiple root exists.

(iii) If

$$\frac{b^{p-q}}{a^p}=\frac{(-1)^p q^q (p-q)^{p-q}}{p^p}, \tag{2.2}$$

and p and q are relatively prime, then the algebraic equation $P(\omega)=0$ has exactly $p-1$ distinct roots which include $p-2$ simple roots and only one double root.

Lemma 2.5. *Let $L(s;f_1)$ and $L(s;f_2)$ be two entire functions of finite order so that each of them has a convergent Dirichlet series representation of the form (1.1) in some right half-plane. Let $R(\omega)=0$ be an algebraic equation with $l(\geq 1)$ distinct roots, where $R(\omega)$ is a monic polynomial. If $L(s;f_1)$ and $L(s;f_2)$ share $S=\{\omega: R(\omega)=0\}$ IM and they assume a common complex value c for some $s_0\in\mathbb{C}$ such that $R(c)\neq 0$, then $R(L(s;f_1))\equiv R(L(s;f_2))$ for all sufficiently large Re s.*

Proof. Suppose that $F(s;f_1)=R(L(s;f_1))$ and $F(s;f_2)=R(L(s;f_2))$. Since $L(s;f_1)$ and $L(s;f_2)$ share S IM, then $F(s;f_1)$ and $F(s;f_2)$ share 0 IM.

We now need an explicit form of $R(\omega)$ to proceed further. Suppose that $R(\omega)$ has the form: $R(\omega) = (\omega-\gamma_1)^{l_1}(\omega-\gamma_2)^{l_2}\dots(\omega-\gamma_k)^{l_k}$, where $\gamma_j \in \mathbb{C}$ are all distinct, $\sum_{j=1}^{k} l_j = l$, and $l_j \in \mathbb{N}$. Therefore

$$F(s;f_i) = (L(s;f_i)-\gamma_1)^{l_1}(L(s;f_i)-\gamma_2)^{l_2}\dots(L(s;f_i)-\gamma_k)^{l_k},\quad i=1,2.$$

Let us define a function $\varepsilon : \mathbb{N} \to \mathbb{C}$ by

$$\varepsilon(n) = \begin{cases} 0, & \text{if } n > 1 \\ 1, & \text{if } n = 1 \end{cases}$$

so that $\varepsilon = \mu * u$ as a Dirichlet convolution of the Möbius μ-function μ with the function u (see [1, p. 31]). Here u is the arithmetical function defined as $u(n) = 1$ for all $n \geq 1$. Then $L(s;f_i) - \gamma_j = L(s; f_i - \gamma_j\varepsilon)$ for $i = 1, 2;\ j = 1, 2, \dots, k$.

Now from the uniqueness theorem for Dirichlet series (see [1, p. 227], [17, p. 309]), it follows that any convergent Dirichlet series is non-vanishing in another right half-plane, and hence $L(s;f_i) - \gamma_j$ $(i = 1, 2; j = 1, 2, \dots, k)$ is also a zero-free Dirichlet series for all s with sufficiently large Re s. Therefore, for the shared value zero, we see that there exists a suitable right half-plane in which each of $F(s;f_1)$ and $F(s;f_2)$ is zero-free. Moreover, $F(s;f_i)$ is an entire function as $L(s;f_i)$ is so.

Let

$$W(s) = \frac{F(s;f_1)}{F(s;f_2)}.$$

Then for all s with sufficiently large Re s, W is an entire function without any zeros. Note that the orders of both $F(s;f_1)$ and $F(s;f_2)$ are finite. If $\hat{\rho}$ =max $\{\rho(F(s;f_1)), \rho(F(s;f_2))\}$, then by Hadamard Factorization Theorem (see [2, p. 384], [17, p. 250]), $W(s)$ must take the form

$$W(s) = \frac{F(s;f_1)}{F(s;f_2)} = e^{P_1(s)}, \tag{2.3}$$

for some polynomial $P_1(s)$ with $\deg(P_1(s)) \leq \hat{\rho}$.

Since $L(s;f_2)-\gamma_j$ is a zero-free Dirichlet series for all s with sufficiently large real part, using Lemma 2.1, we have for all these s with large Re s, $[L(s;f_2)-\gamma_j]^{-1} = [L(s;f_2-\gamma_j\varepsilon)]^{-1} = L(s;g)$, where $(f_2 - \gamma_j\varepsilon) * g = \varepsilon$. As the set of Dirichlet series is closed under multiplication, in view of Dirichlet convolution $*$, we obtain

$$\frac{L(s;f_1)-\gamma_j}{L(s;f_2)-\gamma_j} = L(s;f_1-\gamma_j\varepsilon)L(s;g) = L(s;h_j),$$

where $h_j = (f_1 - \gamma_j\varepsilon) * g$ for $j = 1, 2, \dots, k$. This implies

$$\left(\frac{L(s;f_1)-\gamma_j}{L(s;f_2)-\gamma_j}\right)^{l_j} = [L(s;h_j)]^{l_j} = L(s;\hat{h}_j),$$

where $\hat{h}_j = h_j * h_j * \ldots * h_j$ (l_j times) for $j = 1, 2, \ldots, k$. Therefore, if $x = \hat{h}_1 * \hat{h}_2 * \ldots * \hat{h}_k$, then

$$\frac{F(s; f_1)}{F(s; f_2)} = \prod_{1 \le j \le k} L(s; \hat{h}_j) = L(s; x) = \sum_{n \ge 1} \frac{x(n)}{n^s} = \sum_{n \ge m_1} \frac{x(n)}{n^s},$$

where m_1 is the minimum of all $n \in \mathbb{N}$ such that $x(n) \ne 0$. From (2.3) we have

$$\begin{aligned} P_1(s) &= \log \left[\sum_{n \ge m_1} \frac{x(n)}{n^s} \right] \\ &= \log \frac{x(m_1)}{{m_1}^s} + \log \left[1 + \sum_{n > m_1} \frac{x(n)}{x(m_1)} \left(\frac{m_1}{n} \right)^s \right]. \end{aligned}$$

Clearly, the series on the right-hand side is convergent for all sufficiently large Re s. Since $P_1(s)$ is a polynomial, the series must be identically zero and so

$$P_1(s) = \log \left[\frac{x(m_1)}{{m_1}^s} \right] = -s \log m_1 + \log\{x(m_1)\}, \tag{2.4}$$

which means $P_1(s)$ is a linear polynomial or constant. Now for $s = \sigma + it$, we can write

$$Re\ P_1(\sigma + it) = A(t)\sigma + B(t), \tag{2.5}$$

a polynomial in σ with $A(t), B(t)$ being polynomials in t. We now show that $A(t) \equiv 0$. For this, we first note that $\lim_{\sigma \to +\infty} F(s; f_1) = d_1$ and $\lim_{\sigma \to +\infty} F(s; f_2) = d_2$ for some nonzero constants $d_1, d_2 \in \mathbb{C}$ as $F(s; f_1)$ and $F(s; f_2)$ are non-vanishing and convergent for all sufficiently large Re s. Therefore we get

$$\lim_{\sigma \to +\infty} \frac{F(s; f_1)}{F(s; f_2)} = d_3, \tag{2.6}$$

where $d_3 (\ne 0) \in \mathbb{C}$. Again, from (2.3) and (2.5) we obtain that

$$\left| \frac{F(s; f_1)}{F(s; f_2)} \right| = e^{A(t)\sigma + B(t)}. \tag{2.7}$$

If we assume that $A(t_0) > 0$ for some $t_0 \in \mathbb{C}$, then from (2.6) and (2.7), it follows that for the limit $\sigma \to +\infty$ and $t = t_0$, $|d_3| = \infty$, which is a contradiction. Similarly, if we suppose that $A(t_1) < 0$ for some $t_1 \in \mathbb{C}$, then we get $|d_3| = 0$ as $\sigma \to +\infty$, that is, a contradiction. Therefore $A(t) \equiv 0$ and so from (2.7) we obtain

$$\left| \frac{F(s; f_1)}{F(s; f_2)} \right| = e^{B(t)}. \tag{2.8}$$

Since $e^{B(t)}$ is independent of σ, it has the same value for any arbitrary σ. Taking $\sigma \to +\infty$, we see from (2.6) that the left-hand side of (2.8) is $|d_3|$ for any value of t and hence $e^{B(t)} = |d_3|$. Therefore, we have $|\frac{F(s;f_1)}{F(s;f_2)}| = |d_3|$, which implies that the function $\frac{F(s;f_1)}{F(s;f_2)}$ is a constant. Therefore, from (2.6) we get

$$\frac{F(s;f_1)}{F(s;f_2)} = d_3. \tag{2.9}$$

Since $L(s;f_1)$ and $L(s;f_2)$ assume the common value c at some $s = s_0$ with $R(c) \neq 0$, by (2.9) we deduce that $d_3 = 1$. Therefore $F(s;f_1) = F(s;f_2)$. This completes the proof of the lemma. □

3. Proof of the theorem

Proof of Theorem 1.1. Let $F(s;f_1) = P(L(s;f_1))$ and $F(s;f_2) = P(L(s;f_2))$. Then $F(s;f_1)$ and $F(s;f_2)$ share 0 IM. We first show that $\rho(L(s;f_1)) = \rho(L(s;f_2))$. In view of Lemma 2.4, we know that $P(\omega) = 0$ has at least $p-1$ distinct roots, say $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}$. Since the entire functions $L(s;f_1)$ and $L(s;f_2)$ share S IM and $p > 2$, we get by Nevanlinna's second fundamental theorem that

$$\begin{aligned}(p-2)T(r, L(s;f_1)) &\leq \sum_{j=1}^{p-1} \overline{N}(r, \alpha_j; L(s;f_1)) + O(\log r + \log T(r, L(s;f_1))) \\ &= \sum_{j=1}^{p-1} \overline{N}(r, \alpha_j; L(s;f_2)) + O(\log r + \log T(r, L(s;f_1))),\end{aligned}$$

Therefore

$$T(r, L(s;f_1)) \leq \frac{p-1}{p-2} T(r, L(s;f_2)) + O(\log r + \log T(r, L(s;f_1))) \tag{3.1}$$

as $r \to \infty$ and $r \notin M$, where M is a set of positive real numbers of finite linear measure.

Similarly,

$$T(r, L(s;f_2)) \leq \frac{p-1}{p-2} T(r, L(s;f_1)) + O(\log r + \log T(r, L(s;f_2))) \tag{3.2}$$

as $r \to \infty$ and $r \notin M$.

Using Lemma 2.2, we can remove the exceptional set in (3.1) and (3.2) and thus the inequalities hold for all $r > r_0$ for some $r_0 > 0$. Therefore, we get $\rho(L(s;f_1)) \leq \rho(L(s;f_2))$ and $\rho(L(s;f_2)) \leq \rho(L(s;f_1))$. Consequently, we obtain that both the orders of $L(s;f_1)$ and $L(s;f_2)$ are equal and finite as well. Therefore, by Lemma 2.5 it follows that

$$L^p(s;f_1) - L^p(s;f_2) = -a(L^q(s;f_1) - L^q(s;f_2)), \tag{3.3}$$

and so

$$L^{p-q}(s;f_2) = -a\frac{G^q - 1}{G^p - 1}, \tag{3.4}$$

for all s having sufficiently large real part, where $G = \frac{L(s;f_1)}{L(s;f_2)}$ is a non-vanishing entire function for such s. Now in the common right half-plane of $F(s;f_1)$ and $F(s;f_2)$, we consider the following two cases:

Case 1: Suppose that $G^p = 1$. Then $L^p(s;f_2) = L^p(s;f_1)$. Substituting this in (3.3), we obtain $L^q(s;f_2) = L^q(s;f_1)$. Applying Lemma 2.3, we have $L(s;f_1) = L(s;f_2)$.

Case 2: Suppose that $G^p \neq 1$. Since p and q are relatively prime positive integers, we get by Lemma 2.3 that the numerator and the denominator of right-hand side of (3.4) has exactly one common zero. Therefore, the zeros of the denominator (if exist) produces $p-1$ distinct poles of $L^{p-q}(s;f_2)$ on the left hand-side of (3.4). Since $L^{p-q}(s;f_2)$ has no pole, $p > 2$, and that a nonconstant entire function can possess at most one Picard exceptional value, then it follows that G should have $p-1$ Picard exceptional values. Thus G is a constant and so from (3.4) we get that $L(s;f_2)$ is constant, which is clearly a contradiction.

This completes the proof of Theorem 1.1. □

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THE EXACT ENTIRE SOLUTIONS OF CERTAIN TYPE OF NONLINEAR DIFFERENCE EQUATIONS

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Abstract. In this paper, we consider the entire solutions of nonlinear difference equation $f^3 + q(z)\Delta f = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$, where q is a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ are nonzero constants with $\alpha_1 \neq \alpha_2$. It is showed that if f is a non-constant entire solution of $\rho_2(f) < 1$ to the above equation, then $f(z) = e_1 e^{\frac{\alpha_1 z}{3}} + e_2 e^{\frac{\alpha_2 z}{3}}$, where e_1 and e_2 are two constants. Meanwhile, we give an affirmative answer to the conjecture posed by Zhang et al in [18].

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1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. In order to prove the main results, we will employ Nevanlinna theory. Before to proceed, we spare the reader for a moment and assume his/her familiarity with the basics of Nevanlinna's theory of meromorphic functions in $\mathbb{C}$ such as the *first* and *second* fundamental theorems, and the usual notations such as the *characteristic function* $T(r, f)$, the *proximity function* $m(r, f)$ and the *counting function* $N(r, f)$. $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\,(T(r, f))$ as $r \to \infty$, except possibly on a set of finite logarithmic measure(see e.g., [16, 17]). We also need the following definition.

Definition 1. The order $\rho(f)$, hyper-order $\rho_2(f)$ of the meromorphic function $f(z)$ are defined as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}, \qquad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

Characterizing complex analytic solutions of differential equations has a topic of a long history (see e.g., the monograph [7]). It seems to us that Yang firstly

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started to study the the existence and uniqueness of finite order entire solutions of nonlinear differential equation of the form

$$L(f)(z) - p(z)f^n(z) = h(z),\ n \geq 3,$$

where $L(f)$ is a linear differential polynomial in f with polynomial coefficients, p is a non-vanishing polynomial and h is an entire function. Recently, the difference analogues to Nevanlinna theory was established by Halburd and Korhonen [3, 4], Chiang and Feng [2], independently. With the help of this tool, many scholars have studied the solvability and existence of meromorphic solutions of some non-linear difference equations (see e.g., [1, 5, 6], [8] – [15]).

In 2010, Yang and Laine [15] considered the following difference equation.

Theorem A. *A non-linear difference equation*

$$f^3(z) + q(z)f(z+1) = c \sin bz = c\frac{e^{biz} - e^{-biz}}{2i},$$

where $q(z)$ is a non-constant polynomial and $b, c \in \mathbb{C}$ are nonzero constants, does not admit entire solutions of finite order. If $q(z) = q$ is a nonzero constant, then the above equation possesses three distinct entire solutions of finite order, provided that $b = 3n\pi$ and $q^3 = (-1)^{n+1}c^2 27/4$ for a nonzero integer n.

The follow-up research on this aspect was done by Liu and Lü et al. In [12], they considered the following more general difference equation

$$f^n(z) + q(z)\Delta f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.1}$$

where n is a positive integer, $\Delta f(z) = f(z+1) - f(z)$, $q(z)$ is a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ are nonzero constants with $\alpha_1 \neq \alpha_2$. More specifically, Liu and Lü et al. proved the following.

Theorem B. *Let $n \geq 4$ be an integer, q be a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If there exists some entire solution f of finite order to (1.1), then $q(z)$ is a constant, and one of the following relations holds:*
(1). $f(z) = c_1 e^{\frac{\alpha_1 z}{n}}$, and $c_1(\exp \frac{\alpha_1}{n} - 1)q = p_2,\ \alpha_1 = n\alpha_2$,
(2). $f(z) = c_2 e^{\frac{\alpha_2 z}{n}}$, and $c_2(\exp \frac{\alpha_2}{n} - 1)q = p_1,\ \alpha_2 = n\alpha_1$, where $c_1,\ c_2$ are constants satisfying $c_1^3 = p_1,\ c_2^3 = p_2$.

The study for the case $n = 3$ was due to Zhang et al. [18], who obtained the following result.

Theorem C. *Let q be a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If f is an entire solution of finite order to the following equation:*

$$f^3 + q(z)\Delta f = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.2}$$

then $q(z)$ is a constant, and one of the following relations holds:

(1) $T(r,f) = N_{1)}(r, \frac{1}{f}) + S(r,f)$,

(2) $f(z) = c_1 e^{\frac{\alpha_1 z}{3}}$, *and* $c_1(\exp \frac{\alpha_1}{3} - 1)q = p_2,\ \alpha_1 = 3\alpha_2$,

(3) $f(z) = c_2 e^{\frac{\alpha_2 z}{3}}$, *and* $c_2(\exp \frac{\alpha_2}{3} - 1)q = p_1,\ \alpha_2 = 3\alpha_1$,

where $N_{1)}(r, \frac{1}{f})$ *denotes the counting function corresponding to simple zeros of* f *, and* $c_1,\ c_2$ *are constants satisfying* $c_1^3 = p_1,\ c_2^3 = p_2$.

Remark 1. For the cases (2) and (3) in Theorem C, it is easy to see that 0 is a Picard value of f and $N(r, 1/f) = 0$. So $T(r,f) \neq N_{1)}(r, \frac{1}{f}) + S(r,f) = S(r,f)$. It is natural to ask whether the case (1) occurs or not. The answer is positive. It is showed by the following example, which can be found in [18].

Example 1. Consider $f(z) = e^{\pi i z} + e^{-\pi i z} = 2i\sin(\pi i z)$. Then f is a solution of the following equation:

$$f^3 + \frac{3}{2}\Delta f = e^{3\pi i z} + e^{-3\pi i z}.$$

Obviously, $T(r,f) = N_{1)}(r, \frac{1}{f}) + S(r,f)$. So, the case (1) occurs.

In Theorem C, it seems that the case (1) is unnatural. Meanwhile, Zhang et al. observed that $\alpha_1 + \alpha_2 = 3\pi i + (-3\pi i) = 0$ in Example 1. This observation leaded Zhang et al. to pose the following conjecture.

Conjecture. *If* $\alpha_1 \neq \alpha_2,\ \alpha_1 + \alpha_2 \neq 0$, *then the conclusion (1) of Theorem C is impossible. In fact, any entire solution* f *of (1.2) must have 0 as its Picard exceptional value.*

Remark 2. The conjecture has been studied by many researchers (see [1, 9]). In 2017, Latreuch in [9] has gave an affirmative answer to the conjecture. However, when $\alpha_1 + \alpha_2 \neq 0$ does not hold, Latreuch did not give the specific form of the meromorphic solution of (1.2). In Example 1, we further observe that $f(z) = e^{\pi i z} + e^{-\pi i z} = 2i\sin(\pi i z)$. In [1], one can not get $m(r, \lambda^2 f - n^2 f'') = O(\log r)$ in the proof of Theorem 1.1 directly. This leads us to ask whether any entire solution of the equation (1.2) always is this form when Case (1) occurs. In the present paper, we focus on the problem and give an affirmative answer by the following theorem.

Theorem 1.1. *Let* q *be a polynomial, and* $p_1, p_2, \alpha_1, \alpha_2$ *be nonzero constants such that* $\alpha_1 \neq \alpha_2$. *If* f *is an entire solution of* $\rho_2(f) < 1$ *to the equation (1.2), then* $q(z)$ *is a constant, and one of the following relations holds:*

(1) $f(z) = e_1 e^{\frac{\alpha_1 z}{3}} + e_2 e^{\frac{\alpha_2 z}{3}}$, *where* e_1 *and* e_2 *are two nonzero constants satisfying* $e_1^3 = p_1$, $e_2^3 = p_2$ *(or* $e_1^3 = p_2$, $e_2^3 = p_1$*)*, $3e_1e_2 - 2q = 0$, $\alpha_1 + \alpha_2 = 0$ *and* $e^{\frac{\alpha_1}{3}} = -1$;

(2) $f(z) = c_1 e^{\frac{\alpha_1 z}{3}}$, *and* $c_1(\exp \frac{\alpha_1}{3} - 1)q = p_2,\ \alpha_1 = 3\alpha_2$;

(3) $f(z) = c_2 e^{\frac{\alpha_2 z}{3}}$, *and* $c_2(\exp \frac{\alpha_2}{3} - 1)q = p_1,\ \alpha_2 = 3\alpha_1$.

Remark 3. Clearly, Example 1 satisfies Case (1) of Theorem 1.1, where $\alpha_1 = 3\pi i, \alpha_2 = -3\pi i$; $e_1 = e_2 = 1, p_1 = p_2 = 1$; $q = 3/2$. Next we give two examples to show Cases (2) and (3) indeed occur in Theorem 1.1.

Example 2. Consider the function $f(z) = e^{\pi i z}$, which is a nonconstant entire solution of the following equation

$$f^3(z) - \frac{1}{2}\Delta f(z) = e^{3\pi i z} + e^{\pi i z},$$

where $\alpha_1 = 3\pi i = 3\alpha_2$, $c_1 = 1, q = -1/2, p_2 = 1$. Thus, the case (2) occurs.

Example 3. Consider the function $f(z) = e^{3\pi i z}$, which satisfies the following equation

$$f^3(z) - \frac{1}{2}\Delta f(z) = e^{3\pi i z} + e^{9\pi i z},$$

where $\alpha_2 = 9\pi i = 3\alpha_1$, $c_2 = 1, q = -1/2, p_1 = 1$. Therefore, the case (3) occurs.

By Theorem 1.1, we get an immediate conclusion as follows.

Corollary 1. *Let q be a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If f is a nonconstant entire solution of $\rho_2(f) < 1$ to the equation (1.2), then $q(z)$ is a constant, and*

$$f(z) = e_1 e^{\frac{\alpha_1 z}{3}} + e_2 e^{\frac{\alpha_2 z}{3}},$$

where e_1 and e_2 are two constants.

At the end, we turn attention to the question: What will happen if we replace the function f^3 by f^2 in the equation (1.2). After studying this question, we derive some similar results to Theorem C as follows.

Theorem 1.2. *Let q be a polynomial, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If f is an entire solution of $\rho_2(f) < 1$ to the following equation*

$$f^2 + q(z)\Delta f = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.3}$$

and satisfying $N(r, \frac{1}{f}) = S(r, f)$, then $q(z)$ is a constant, and one of the following relations holds:

(1) $f(z) = c_1 e^{\frac{\alpha_1 z}{2}}$, *and* $c_1(\exp\frac{\alpha_1}{2} - 1)q = p_2,\ \alpha_1 = 2\alpha_2$,

(2) $f(z) = c_2 e^{\frac{\alpha_2 z}{2}}$, *and* $c_2(\exp\frac{\alpha_2}{2} - 1)q = p_1,\ \alpha_2 = 2\alpha_1$, *where* $c_1,\ c_2$ *are constants satisfying* $c_1^2 = p_1,\ c_2^2 = p_2$.

We below offer an example to show that the condition $N(r, \frac{1}{f}) = S(r, f)$ is necessary in Theorem 1.2.

Example 4. Consider the function $f(z) = -2 - \sqrt{2}e^{\pi i z} + \sqrt{2}e^{-\pi i z}$, which satisfies the equation

$$f^2(z) - 2\Delta f(z) = 2e^{2\pi i z} + 2e^{-2\pi i z}.$$

A calculation yields that $T(r,f) = 2r(1+o(1))$ and $N(r,1/f) = 2r(1+o(1))$. Clearly, $N(r,\frac{1}{f}) \neq S(r,f)$ and f does not satisfy any conclusion of Theorem 1.2.

2. Some lemmas

Before to the proofs of main theorems, we firstly give the following result, whcih is a version of the difference analogue of the logarithmic derivative lemma.

Lemma 2.1 ([4]). *Let $f(z)$ be a meormorphic function of $\rho_2(f) < 1$, and let $c \in \mathbb{C}\backslash\{0\}$. Then*

$$m(r, \frac{f(z+c)}{f(z)}) = o(\frac{T(r,f)}{r^{1-\rho_2(f)-\varepsilon}}),$$

outside of an exceptional set of finite logarithmic measure.

In addition, by applying Lemma 2.1 and the same argument as in [8, Theorem 2.3], we get the following lemma, which is a version of the difference analogue of the Clunie lemma. The details are omitted here.

Lemma 2.2. *Let f be a transcendental meromorphic solution of $\rho_2(f) < 1$ to the difference equation*

$$H(z,f)P(z,f) = Q(z,f),$$

where $H(z,f)$, $P(z,f)$, $Q(z,f)$ are difference polynomials in f such that the total degree of $H(z,f)$ in f and its shifts is n, and that the corresponding total degree of $Q(z,f)$ is $\leq n$. If $H(z,f)$ contains just one term of maximal total degree, then for any $\varepsilon > 0$,

$$m(r, P(z,f)) = S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

3. Proof of Theorem 1.1

Suppose that f is an entire solution of $\rho_2(f) < 1$ to Eq (1.2). Obviously, f is a transcendental function. By differentiating both sides of (1.2), one has

$$3f^2f' + (q(z)\Delta f)' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{3.1}$$

Combining (1.2) and (3.1) yields

$$\alpha_2 f^3 + \alpha_2 q\Delta f - 3f^2 f' - (q(z)\Delta f)' = (\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z}. \tag{3.2}$$

By differentiating (3.2), we derive that

$$3\alpha_2 f^2 f' + \alpha_2 (q\Delta f)' - 6f(f')^2 - 3f^2 f'' - (q(z)\Delta f)'' = \alpha_1(\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$f\varphi = T(z,f), \tag{3.4}$$

where

$$\varphi = \alpha_1\alpha_2 f^2 - 3(\alpha_1+\alpha_2)ff' + 6(f')^2 + 3ff'', \tag{3.5}$$

$$T(z,f) = -\alpha_1\alpha_2 q\Delta f + (\alpha_1+\alpha_2)(q\Delta f)' - (q\Delta f)''.$$

Note that $T(z,f)$ is a differential-difference polynomial in f of degree 1. Then by applying Lemma 2.2 to the equation (3.4), one has $m(r,\varphi) = S(r,f)$. Further, $T(r,\varphi) = m(r,\varphi) = S(r,f)$, since φ is an entire function. It means that φ is a small function of f.

Suppose that $\varphi \equiv 0$. Then $\alpha_1\alpha_2 f^2 - 3(\alpha_1+\alpha_2)ff' + 6(f')^2 + 3ff'' \equiv 0$. Rewrite it as $\frac{f''}{f} = (\frac{f'}{f})' + (\frac{f'}{f})^2$, which yields a Riccati equation

$$t' + 3t^2 - (\alpha_1+\alpha_2)t + \alpha_1\alpha_2/3 = 0,$$

where $t = \frac{f'}{f}$. Clearly, the equation has two constant solutions $t_1 = \alpha_1/3$, $t_2 = \alpha_2/3$. We assume $t \not\equiv t_1, t_2$. Then we have

$$\frac{1}{t_1 - t_2}\left(\frac{t'}{t-t_1} - \frac{t'}{t-t_2}\right) = -3.$$

Integrating the above equation yields

$$\ln\frac{t-t_1}{t-t_2} = 3(t_2-t_1)z + C,$$

where C is a constant. Therefore,

$$\frac{t-t_1}{t-t_2} = e^{3(t_2-t_1)z+C}.$$

This immediately yields

$$t = t_2 + \frac{t_2-t_1}{e^{3(t_2-t_1)z+C}-1} = \frac{f'}{f},$$

Note that the zeros of $e^{3(t_2-t_1)z+C}-1$ are the zeros of f. If z_0 is a zero of f with the multiplicity k, then

$$k = \mathrm{Res}[\frac{f'}{f}, z_0] = \mathrm{Res}[t_2 + \frac{t_2-t_1}{e^{3(t_2-t_1)z+C}-1}, z_0] = \frac{1}{3},$$

which is a contradiction. Thus, either $t \equiv t_1 = \alpha_1/3$ or $t \equiv t_2 = \alpha_2/3$.

If $t \equiv t_1 = \alpha_1/3$, then $f(z) = c_1 e^{\frac{\alpha_1}{3}z}$. Substituting the form $f(z) = c_1 e^{\frac{\alpha_1}{3}z}$ into the equation (1.2), we obtain

$$c_1^3 e^{\alpha_1 z} + c_1 q(z) e^{\frac{\alpha_1}{3}z}(e^{\frac{\alpha_1}{3}} - 1) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

which implies that $c_1^3 = p_1$, $c_1 q(e^{\frac{\alpha_1}{3}} - 1) = p_2$ and $\alpha_1 = 3\alpha_2$.

Similarly as above, if $t \equiv t_2 = \alpha_2/3$, then we can derive that $f(z) = c_2 e^{\frac{\alpha_2}{3}z}$ satisfying $c_2^3 = p_2$, $c_2 q(e^{\frac{\alpha_2}{3}} - 1) = p_1$ and $\alpha_2 = 3\alpha_1$.

In the following, based on the idea in [10, Theorem 1.1], we will consider the case $\varphi \not\equiv 0$. By Theorem C, one has

$$T(r,f) = N_{1)}(r,\frac{1}{f}) + S(r,f). \tag{3.6}$$

Differentiating (3.5) yields

$$\varphi' = \alpha_1\alpha_2 2ff' - 3(\alpha_1+\alpha_2)(ff''+(f')^2) + 12f'f'' + 3ff''' + 3f'f''. \tag{3.7}$$

From (3.5) and (3.7), we can obtain that

$$f[A_0 f + A_1 f' + A_2 f'' + A_3 f'''] = f'[B_1 f' + B_2 f''], \tag{3.8}$$

where

$$A_0 = \alpha_1\alpha_2\varphi', \quad A_1 = -3\varphi'(\alpha_1+\alpha_2) - 2\varphi\alpha_1\alpha_2,$$
$$A_2 = 3\varphi' + 3\varphi(\alpha_1+\alpha_2), \quad A_3 = -3\varphi,$$
$$B_1 = -3\varphi(\alpha_1+\alpha_2) - 6\varphi', \quad B_2 = 15\varphi.$$

Obviously, all A_i $(i=0,1,2,3)$, B_j $(j=1,2)$ are small functions of f.

Suppose that z_0 is a zero of f, not a zero of φ. It follows from (3.5) that $6(f')^2(z_0) = \varphi(z_0) \neq 0$, which implies that z_0 is a simple zero of f. Then by (3.8), we have

$$B_1(z_0)f'(z_0) + B_2(z_0)f''(z_0) = 0.$$

Set

$$A = \frac{B_1 f' + B_2 f''}{f}. \tag{3.9}$$

We claim that A is an entire function. Clearly, all the simple zeros of f are not poles of f. Suppose that b_0 is a multiple zero of f. By (3.5), we get b_0 is also a multiple zero of φ. So, b_0 is a zero of B_1 and a multiple zero of B_2. Note that b_0 is a pole of $\frac{f'}{f}$ and $\frac{f''}{f}$ with multiplicity one and two, respectively. Thus, b_0 is not a pole of $B_1\frac{f'}{f}$ and $B_2\frac{f''}{f}$, which implies that b_0 is not a pole of A. Thus, A is an entire function. The claim is proved. Furthermore,

$$T(r,A) = m(r,\frac{B_1 f' + B_2 f''}{f}) = S(r,f).$$

Hence A is a small function of f. We consider two cases below.

Case 1. $A = 0$.

Then, $B_1 f' + B_2 f'' = 0$. Rewrite it as

$$\frac{f''}{f'} = -\frac{B_1}{B_2} = \frac{1}{5}(\alpha_1+\alpha_2) + \frac{2}{5}\frac{\varphi'}{\varphi}.$$

By integrating the above equation, we have

$$f'(z) = \beta e^{\frac{1}{5}(\alpha_1+\alpha_2)z},$$

where β is a small function of f. Obviously, $\alpha_1+\alpha_2\neq 0$. Otherwise, $T(r,f')=T(r,\beta)=S(r,f)$, a contradiction. We below consider two subcases.

Subcase 1.1. $\varphi'=0$.

The equation $B_1f'+B_2f''=0$ yields

$$\frac{f''}{f'}=-\frac{B_1}{B_2}=\frac{1}{5}(\alpha_1+\alpha_2).$$

By integrating the above equation, we derive that $f'(z)=H_1e^{\frac{1}{5}(\alpha_1+\alpha_2)z}$, where H_1 is a nonzero constant.

Integrating the function f' yields

$$f(z)=k_1e^{\frac{1}{5}(\alpha_1+\alpha_2)z}+k_2,$$

where $k_1(\neq 0)$, k_2 are two constants. Obviously, $k_2\neq 0$. Otherwise, f has no zeros, which contradicts with (3.6). Substitute the form of f into the equation (1.2) yields

$$a_3e^{\frac{3}{5}(\alpha_1+\alpha_2)z}+a_2e^{\frac{2}{5}(\alpha_1+\alpha_2)z}$$
$$+a_1e^{\frac{1}{5}(\alpha_1+\alpha_2)z}+k_2^3=p_1e^{\alpha_1z}+p_2e^{\alpha_2z},$$

where a_1,a_2,a_3 are small functions of f. Then, the above equation yields that $k_2=0$, a contradiction. Hence Subcase 1.1 can not occur.

Subcase 1.2. $\varphi'\neq 0$.

By differentiating f' one and two times respectively, we have

$$f''=H_2e^{\frac{1}{5}(\alpha_1+\alpha_2)z},\quad f'''=H_3e^{\frac{1}{5}(\alpha_1+\alpha_2)z},$$

where H_2 and H_3 are two small functions of f. The equation (3.8) implies that

$$A_0f+A_1f'+A_2f''+A_3f'''=0.$$

Furthermore,

$$f=-\frac{A_1f'+A_2f''+A_3f'''}{A_0}=H_0e^{\frac{1}{5}(\alpha_1+\alpha_2)z},$$

where H_0 is a small function of f. So,

$$N(r,\frac{1}{f})=N(r,\frac{1}{H_0})\leq T(r,H_0)=S(r,f),$$

which contradicts with (3.6). Thus, Subcase 1.2 can not occur.

Case 2. $A\neq 0$.

By (3.8) and (3.9), one has

$$\frac{A_0f+A_1f'+A_2f''+A_3f'''}{f'}=A,$$

which yields that

$$A_0f+(A_1-A)f'+A_2f''+A_3f'''=0. \tag{3.10}$$

Rewrite (3.9) as

$$Af-B_1f'-B_2f''=0.$$

Differentiating the above equation as

$$A'f + (A - B_1')f' - (B_1 + B_2')f'' - B_2 f''' = 0. \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$C_0 f + C_1 f' + C_2 f'' = 0, \tag{3.12}$$

where

$$C_0 = A_0 B_2 + A' A_3, \quad C_1 = (A_1 - A)B_2 + A_3(A - B_1'), \quad C_2 = A_2 B_2 - A_3(B_1 + B_2').$$

Obviously, C_i $(i = 0, 1, 2)$ are small functions of f.

We consider two subcases again.

Subcase 2.1. $C_2 = 0$.

It follows that $C_0 = C_1 = 0$. Otherwise, without loss of generality, suppose that $C_0 \neq 0$. By (3.12), we have that $C_1 \neq 0$. Assume that ω_0 is a simple zero of f. Then ω_0 is a zero of C_1. Furthermore,

$$T(r, f) = N_{1)}(r, \frac{1}{f}) + S(r, f) \leq N(r, \frac{1}{C_1}) + S(r, f) \leq T(r, C_1) + S(r, f) = S(r, f),$$

a contradiction. Thus, $C_0 = C_1 = 0$.

The fact $C_2 = 0$ leads to

$$2\varphi' + \varphi(\alpha_1 + \alpha_2) = 0. \tag{3.13}$$

If $\alpha_1 + \alpha_2 \neq 0$, then $\varphi = H_4 e^{-\frac{\alpha_1 + \alpha_2}{2} z}$, where H_4 is a nonzero constant. Therefore, we have

$$\begin{aligned} m(r, \varphi) &= \frac{|\frac{\alpha_1 + \alpha_2}{2}|}{\pi} r(1 + o(1)), \\ m(r, e^{\alpha_1 z}) &= \frac{|\alpha_1|}{\pi} r(1 + o(1)), \\ m(r, e^{\alpha_2 z}) &= \frac{|\alpha_2|}{\pi} r(1 + o(1)). \end{aligned}$$

Note that φ is a small function of f. So $e^{\alpha_1 z}$, $e^{\alpha_2 z}$ are also two small functions of f. Rewrite (1.2) as

$$f^3 = -q(z)\Delta f + p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}.$$

Therefore,

$$\begin{aligned} 3T(r, f) = T(r, f^3) &= T(r, -q(z)\Delta f + p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) \\ &\leq T(r, \Delta f) + S(r, f) \leq T(r, f) + S(r, f), \end{aligned}$$

a contradiction.

Hence $\alpha_1 + \alpha_2 = 0$. Then, (3.13) reduces to $\varphi' = 0$. It implies that φ is a constant and $A_0 = \varphi' \alpha_1 \alpha_2 = 0$. Together with $C_0 = 0$, it is easy to deduce that $A' = 0$ and A

is also a constant. Therefore, B_1 and B_2 become two constants. Then the following equation reduces to a constant coefficient homogeneous linear differential equation

$$Af - B_1 f' - B_2 f'' = 0.$$

Suppose that the characteristic equation $B_2\lambda^2 + B_1\lambda - A = 0$ has two distinct roots $\lambda_1,\ \lambda_2$. Clearly, $\lambda_1,\ \lambda_2$ are nonzero constants. Then, by solving the above equation, one derives

$$f(z) = e_1 e^{\lambda_1 z} + e_2 e^{\lambda_2 z}. \tag{3.14}$$

Clearly, $e_1 e_2 \neq 0$. Otherwise f has no zeros, a contradiction. Substitute the form f into (1.2), we have

$$\begin{aligned} e_1^3 e^{3\lambda_1 z} + & e_2^3 e^{3\lambda_2 z} + 3e_1^2 e_2 e^{(2\lambda_1+\lambda_2)z} + 3e_1 e_2^2 e^{(\lambda_1+2\lambda_2)z} \\ & + qe_1(e^{\lambda_1} - 1)e^{\lambda_1 z} + qe_2(e^{\lambda_2} - 1)e^{\lambda_2 z} = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}. \end{aligned} \tag{3.15}$$

Suppose that $\lambda_1 + \lambda_2 \neq 0$. Observe that $\lambda_1 \neq \lambda_2$. So $3\lambda_1,\ 3\lambda_2,\ 2\lambda_1 + \lambda_2,\ \lambda_1 + 2\lambda_2$ are distinct from each other. Furthermore, by (3.15) and Borel's Theorem, we easily get the following two sets are identity

$$\{3\lambda_1,\ 3\lambda_2,\ 2\lambda_1 + \lambda_2,\ \lambda_1 + 2\lambda_2\} = \{\lambda_1,\ \lambda_2,\ \alpha_1,\ \alpha_2\},$$

which implies that $3\lambda_2 = \lambda_1$ and $3\lambda_1 = \lambda_2$. It is impossible. Thus, $\lambda_1 + \lambda_2 = 0$. Rewrite (3.15) as

$$e_1^3 e^{3\lambda_1 z} + e_2^3 e^{3\lambda_2 z} + q_1 e^{\lambda_1 z} + q_2 e^{\lambda_2 z} = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where $q_1 = 3e_1^2 e_2 + qe_1(e^{\lambda_1} - 1)$, $q_2 = 3e_2^2 e_1 + qe_2(e^{\lambda_2} - 1)$ are two polynomials. Then, it follows from the above equation that $q_1 = q_2 = 0$. Meanwhile, one has

$$3\lambda_1 = \alpha_1, \quad 3\lambda_2 = \alpha_2$$

or

$$3\lambda_1 = \alpha_2, \quad 3\lambda_2 = \alpha_1.$$

Furthermore, we obtain that $e_1^3 = p_1$ and $e_2^3 = p_2$ (or $e_1^3 = p_2$ and $e_2^3 = p_1$). Note that

$$q_1 = 3e_1^2 e_2 + qe_1(e^{\lambda_1} - 1) = 0, \quad q_2 = 3e_2^2 e_1 + qe_2(e^{\lambda_2} - 1) = 0.$$

By the above two equation, $\lambda_1 + \lambda_2 = 0$ and a calculation, we deduce that $e^{\lambda_1} = -1$ and q reduces to a constant satisfying $3e_1 e_2 - 2q = 0$.

Now, we suppose that $B_2\lambda^2 + B_1\lambda - A = 0$ has a multiple root, say λ_3. Then, $f(z) = (e_3 + e_4 z)e^{\lambda_3 z}$. Therefore, f just has one zero, a contradiction.

Subcase 2.2. $C_2 \neq 0$.

Combining (3.9) and (3.12) yields

$$(B_2 C_0 + AC_2)f + (C_1 B_2 - B_1 C_2)f' = 0.$$

Suppose that $C_1B_2 - B_1C_2 \neq 0$. It folllows $B_2C_0 + AC_2 \neq 0$. Assume that σ_0 is a simple zero of f. By the above equation, one has σ_0 is also a zero of $C_1B_2 - B_1C_2$. Then,

$$T(r,f) = N_{1)}(r,\frac{1}{f}) + S(r,f) \leq N(r,\frac{1}{C_1B_2 - B_1C_2}) + S(r,f)$$
$$\leq T(r, C_1B_2 - B_1C_2) + S(r,f) = S(r,f),$$

a contradiction. The above discussion forces that $C_1B_2 - B_1C_2 = 0$ and $B_2C_0 + AC_2 = 0$. By the definitions of $C_1,\ C_2,\ B_1,\ B_2$, a calculation leads to

$$8A\varphi' - 5\varphi A' = -[4\varphi A(\alpha_1 + \alpha_2) + 25\alpha_1\alpha_2\varphi\varphi'] \tag{3.16}$$

and

$$15\varphi A = [6(\alpha_1 + \alpha_2)^2 - 25\alpha_1\alpha_2]\varphi^2 - 21(\alpha_1 + \alpha_2)\varphi\varphi' + 24(\varphi')^2 - 15\varphi\varphi''. \tag{3.17}$$

Suppose that δ_0 is a zero of φ with multiplicity s. The equation (3.17) implies $s \geq 2$. Furthermore, δ_0 is a zero of φ^2 and $\varphi\varphi'$ with multiplicity $2s$ and $2s-1$, respectively. Suppose that the Laurent expansions of φ at δ_0 is as follows

$$\varphi(z) = \mu_s(z - \delta_0)^s + \mu_{s+1}(z - \delta_0)^{s+1} + \cdots,$$

where $\mu_s(\neq 0)$, μ_{s+1} are constants. Then, a calculation yields

$$24(\varphi')^2 - 15\varphi\varphi'' = [24(\mu_s)^2s^2 - 15(\mu_s)^2s(s-1)](z-\delta_0)^{2s-2} + \theta_{2s-1}(z-\delta_0)^{2s-1} + \cdots,$$

where θ_{2s-1} is a constant. Obviously,

$$24\mu_s^2s^2 - 15\mu_s^2s(s-1) = \mu_s^2s[9s + 15] \neq 0,$$

which implies that δ_0 is a zero of $24(\varphi')^2 - 15\varphi\varphi''$ with multiplicity $2s-2$. Suppose that δ_0 is a zero of A with multiplicity l. Then, comparing the multiplicity of both side of equation (3.17) at point δ_0, we have $s + l = 2s - 2$. So, $s = l + 2$.

Assume that $l = 0$. Then, $s = 2$ and $A(\delta_0) \neq 0$. Rewrite (3.16) as

$$8A\varphi' = 5\varphi A' - [4\varphi A(\alpha_1 + \alpha_2) + 25\alpha_1\alpha_2\varphi\varphi']. \tag{3.18}$$

Clearly, δ_0 is a simple zero of $A\varphi'$. However, δ_0 is a multiple zero of $5\varphi A' - [4\varphi A(\alpha_1 + \alpha_2) + 25\alpha_1\alpha_2\varphi\varphi']$, a contradiction. Therefore, $l \geq 1$.

Furthermore, δ_0 is a zero of $4\varphi A(\alpha_1 + \alpha_2) + 25\alpha_1\alpha_2\varphi\varphi'$ with multiplicity $2l + 2$. Suppose that the Laurent expansions of A at δ_0 is

$$A(z) = \nu_l(z - \delta_0)^l + \nu_{s+1}(z - \delta_0)^{l+1} + \cdots,$$

Then,

$$8A\varphi' - 5\varphi A' = \nu_l\mu_{l+2}[8(l+2) - 5l](z - \delta_0)^{2l+1} + \xi_{2l+2}(z - \delta_0)^{2l+2} + \cdots,$$

where ξ_{2l+2} is a constant. Then, δ_0 is a zero of $8A\varphi' - 5\varphi A'$ with multiplicity $2l+1$, since $\nu_l\mu_{l+2}[8(l+2) - 5l] \neq 0$. So, the point δ_0 is a zero of the left side function of (3.16) with multiplicity $2l+1$. On the other hand, δ_0 is a zero of the right side function of (3.16) with multiplicity at least $2l+2$, which is impossible. Therefore, φ has no zeros.

If φ is not a constant, then, we can assume that $\varphi = \phi e^{\omega(z)}$, where ϕ is a constant and $\omega(\neq 0)$ is an entire function. Then, the same argument as in Subcase 2.1 yields that $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ are two small functions of f. Furthermore, we can derive a contradiction. Thus, φ is a constant. Plus (3.17), one has that A is also a constant. Furthermore, it follows from (3.16) that $\alpha_1+\alpha_2=0$. Similarly as the above discussion, we can deduce the desired result.

Thus, we finish the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Suppose that f is an entire solution of $\rho_2(f)<1$ to the equation (1.3). Obviously, f is a transcendental function. By differentiating both sides of (1.3), one has

$$2ff' + (q(z)\Delta f)' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{4.1}$$

Combining (1.3) and (4.1) yields

$$\alpha_2 f^2 + \alpha_2 q\Delta f - 2ff' - (q(z)\Delta f)' = (\alpha_2-\alpha_1)p_1 e^{\alpha_1 z}. \tag{4.2}$$

By differentiating (4.2), we derive that

$$2\alpha_2 ff' + \alpha_2(q\Delta f)' - 2(f')^2 - 2ff'' - (q(z)\Delta f)'' = \alpha_1(\alpha_2-\alpha_1)p_1 e^{\alpha_1 z}. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\varphi_1 = T_1(z,f), \tag{4.4}$$

where

$$\varphi_1 = \alpha_1\alpha_2 f^2 - 2(\alpha_1+\alpha_2)ff' + 2ff'' + 2(f')^2, \tag{4.5}$$

$$T_1(z,f) = -\alpha_1\alpha_2 q\Delta f + (\alpha_1+\alpha_2)(q\Delta f)' - (q\Delta f)''.$$

If $\varphi_1 \not\equiv 0$, then

$$\frac{1}{f^2} = \frac{1}{\varphi_1}(\alpha_1\alpha_2 - 2(\alpha_1+\alpha_2)\frac{f'}{f} + 2\frac{f''}{f} + 2(\frac{f'}{f})^2).$$

By (4.4)-(4.5), and Lemma 2.1, we have

$$m(r,\frac{\varphi_1}{f}) = m(r,\frac{T_1}{f}) = S(r,f) \quad \text{and} \quad m(r,\frac{\varphi_1}{f^2}) = S(r,f). \tag{4.6}$$

Combining $N(r,\frac{1}{f})=S(r,f)$ and (4.6), we obtain

$$\begin{aligned} & 2T(r,f)=2m(r,\frac{1}{f})+S(r,f)=m(r,\frac{\varphi_1}{f^2})+S(r,f) \\ \leq\ & m(r,\frac{\varphi_1}{f^2})+m(r,\frac{1}{\varphi_1})+S(r,f) \\ \leq\ & T(r,\varphi_1)+S(r,f)=m(r,\varphi_1)+S(r,f) \\ =\ & m(r,\frac{\varphi_1}{f})+m(r,f)+S(r,f)=T(r,f)+S(r,f), \end{aligned}$$

which implies $T(r,f)=S(r,f)$, a contradiction.

If $\varphi_1\equiv 0$, then by the similar reasoning as in Theorem 1.1 we can obtain the conclusions (1) and (2). Below, we give the details. By $\varphi_1\equiv 0$, one has the differential equation $\alpha_1\alpha_2f^2-2(\alpha_1+\alpha_2)ff'+2ff''+2(f')^2=0$. Plus the fact $\frac{f''}{f}=(\frac{f'}{f})'+(\frac{f'}{f})^2$, we can rewrite the above equation to a Riccati equation

$$t'+2t^2-(\alpha_1+\alpha_2)t+\alpha_1\alpha_2/2=0,$$

where $t=\frac{f'}{f}$. Clearly, the equation has two constant solutions $t_1=\alpha_1/2$, $t_2=\alpha_2/2$.

Suppose the solution $t\not\equiv t_1,t_2$. Then

$$\frac{1}{t_1-t_2}(\frac{t'}{t_1-t_2}-\frac{t'}{t_1-t_2})=-2.$$

Integrating the above equation yields

$$\ln\frac{t-t_1}{t-t_2}=2(t_2-t_1)z+C,$$

where C is a constant. Therefore,

$$\frac{t-t_1}{t-t_2}=e^{2(t_2-t_1)z+C}.$$

This immediately yields

$$t=t_2+\frac{t_2-t_1}{e^{2(t_2-t_1)z+C}-1}=\frac{f'}{f}.$$

Note that the zeros of $e^{2(t_2-t_1)z+C}-1$ are the zeros of f. If z_0 is the zero of f with the multiplicity k, then

$$k=\mathrm{Res}[\frac{f'}{f},z_0]=\mathrm{Res}[t_2+\frac{t_2-t_1}{e^{2(t_2-t_1)z+C}-1},z_0]=\frac{1}{2}.$$

It is a contradiction. Thus, either $t\equiv t_1=\alpha_1/2$ or $t\equiv t_2=\alpha_2/2$.

If $t\equiv t_1=\alpha_1/2$, then $f(z)=c_1e^{\frac{\alpha_1}{2}z}$. Substituting $f(z)=c_1e^{\frac{\alpha_1}{2}z}$ into (1.3), we obtain

$$c_1^2e^{\alpha_1z}+c_1q(z)e^{\frac{\alpha_1}{2}z}(e^{\frac{\alpha_1}{2}}-1)=p_1e^{\alpha_1z}+p_2e^{\alpha_2z}.$$

Moreover, we have $c_1^2=p_1$, $c_1q(e^{\frac{\alpha_1}{2}}-1)=p_2$ and $\alpha_1=2\alpha_2$.

Similarly, if $t\equiv t_2=\alpha_2/2$, then we have $f(z)=c_2e^{\frac{\alpha_2}{2}z}$ satisfying $c_2^2=p_2$, $c_2q(e^{\frac{\alpha_2}{2}}-1)=p_1$ and $\alpha_2=2\alpha_1$.

Thus, we finish the proof of Theorem 1.2.

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ОБОБЩЕНИЕ ТЕОРЕМЫ АРТИНА

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Аннотация. Известна следующяя теорема Артина об альтернативных линейных алгебр определённых на коммутативном, ассоциативном кольце с единицей: в алтернативной линейной алгебре, если $(a, b, c) = 0$, то подалгебра порождённая элементами a, b, c – ассоциативна. В данной статье мы предлагаем широкое обобщение этого классического результата, используя концепции сверхтождества и котождества. Соответствующие универсальные алгебры мы называем g-алгебрами.

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Ключевые слова: сверхтождество; котождество; альтернативность; сверхассоциативность; сверхальтернативность; g-алгебра.

1. g-алгебры. Введение

Вводится многообразие g-алгебр, являюшийся многообразием мултиоператорных Ω-групп специального типа [1, 2, 3, 19].

Хорошо известна следующяя теорема Артина об альтернативных линейных алгебрах определённых на коммутативном, ассоциативном кольце с единицей: в алтернативной линейной алгебре, если $(a, b, c) = 0$, то подалгебра порождённая элементами a, b, c – ассоциативна [2, 4]. В данной статье мы предлагаем широкое обобщение этого классического результата, используя понятия сверхтождества и котождества. Соответствующие универсальные алгебры мы называем g-алгебрами.

Определение 1.1. *Пусть Φ – ассоциативное, коммутативное кольцо с единичным элементом 1. Множество A называется g-алгеброй над кольцом Φ, если существует структура унитарного Φ-модуля определённого на A и существует множество бинарных операций Σ определённых на A, связанных с*

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модульными операциями следующими равенствами:

$$X(a+b,c) = X(a,c) + X(b,c), \tag{1.1}$$

$$X(a,b+c) = X(a,b) + X(a,c), \tag{1.2}$$

$$\alpha(X(a,b)) = X(\alpha a,b) = X(a,\alpha b), \tag{1.3}$$

для всех $a,b,c \in A$ *и для всех* $\alpha \in \Phi, X \in \Sigma$. *Мы обозначим* g*-алгебру* A *над кольцом* Φ *через* $A(+,\Sigma,\Phi)$ *или, кратко,* A.

Пример 1.1. Приведём пример g-алгебры. Пусть Z – кольцо целых чисел, X,Y – произвольные абелевы группы и $M = Hom(X,Y)$ – соответсвующая абелева группа. Для каждого элемента γ из $Hom(Y,X)$ мы определяем на M следующую бинарную операцию:

$$\overline{\gamma}(a,b) \overset{def}{=} a \circ \gamma \circ b - b \circ \gamma \circ a,$$

где $a,b \in M$ и $\circ$ - обычная суперпозиция отображений. Обозначая через $\Sigma = \{\overline{\gamma}|\gamma \in Hom(Y,X)\}$, мы получаем g-алгебру $M(+,\Sigma,Z)$.

Пример 1.2. Пусть P – поле, n – натуральное число, $P^{n\times n}$ – множество всех квадратных матриц с размерностью n и с элементами из P, $+$ и $\cdot$ есть сложение и умнажение матриц. Определим новую бинарную операцию на $P^{n\times n}$ следующим образом:

$$A \circ B \overset{def}{=} A^T \cdot B,$$

где $A,B \in P^{n\times n}$. Тогда $P^{n\times n}(+,\{\cdot,\circ\},P)$ - g-алгебра.

Аналогично, определяя операцию $\circ$ как

$$A \circ B \overset{def}{=} B^T \cdot A,$$

мы опять получим g-алгебру.

Известны приложения таких операций в теоретической астрономии.

Определение 1.2. *Пусть* $A(+,\Sigma,\Phi)$ *–* g*-алгебра,* $B \subseteq A$. *Подмножество* B *называется подалгеброй* g*-алгебры* $A(+,\Sigma,\Phi)$ *если оно замкнуто относительно модульных операций и бинарных операций из* Σ.

Нам необходимы понятия сверхтождества и котождества. Для формул первой и второй ступени (и языков первой и второй ступени) смотрите [5] - [9]. Напомним, что сверхтождество [10] - [17] (или $\forall(\forall)$-тождество) – формула второй ступени следующего вида:

$$(*) \qquad \forall X_1, \ldots, X_m \forall x_1, \ldots, x_n (\omega_1 = \omega_2),$$

где ω_1, ω_2 - слова (термы) в алфавите функциональных переменных $X_1, \ldots, X_m$ и предметных переменных $x_1, \ldots, x_n$. Сверхтождества обычно записываются без универсальных кванторов: $\omega_1 = \omega_2$. Скажем, что сверхтождество $\omega_1 = \omega_2$ выполняется в алгебре $(Q; \Sigma)$ если это равенство имеет место при замене предметных переменных $x_1, \ldots, x_n$ любыми элементами из Q и функциональных переменных $X_1, \ldots, X_m$ любыми операциями из Σ соответствующих арностей. Возможность такой замены предполагается, то есть:

$$\{|X_1|, \ldots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T_{(Q;\Sigma)} = T_{(\Sigma)},$$

где $|S|$ арность S, и $T_{(Q;\Sigma)}$ называется арифметическим типом $(Q; \Sigma)$. T-алгебра - алгебра с арифметическим типом $T \subseteq N$. Класс алгебр называется классом T-алгебр, если каждая алгебра из этого класса - T-алгебра.

Котождество (или $(\exists)\forall$-тождество, см. [11] - [18]) – формула второй ступени следующего вида:

$$\exists x_1, \ldots, x_n \forall X_1, \ldots, X_m (\omega_1 = \omega_2).$$

Котождества обычно записываются без кванторов: $\omega_1 = \omega_2$. Скажем, что котождество $\omega_1 = \omega_2$ выполняется в алгебре $(Q; \Sigma)$, если существуют значения предметных переменых $x_1, \ldots, x_n$ из множества Q такие, что равенство $\omega_1 = \omega_2$ имеет место при замене функциональных переменных $X_1, \ldots, X_m$ любыми операциями из Σ соответствующих арностей (возможность такой замены также предполагается). Обычно, в записи котождества $\omega_1 = \omega_2$, предметные переменные заменяются соответствующими фиксированными значениями из Q.

Котождества могут быть определены также, как формулы второй ступени следующего вида:

$$\forall X_1, \ldots, X_m (\omega_1 = \omega_2).$$

Пример 1.3. В каждой мультиоператорной Ω-группе выполняется следующее котождество:

$$X(\underbrace{0, \ldots, 0}_{n}) = 0,$$

для всех $n \in T_{(\Omega)}$, где все предметные переменные заменены нулевым элементом Ω-группы [1, 2, 3, 19].

Пример 1.4. (J. von Neumann) Пусть $L(+,\cdot)$ модулярная решетка, a, b, $c \in L$. Подрешетка решетки L, порожденная элементами a, b, c, дистрибутивна тогда и только тогда, когда следующее котождество левой дистрибутивности выполняется в решетке $L(+,\cdot)$:

$$X\left(a,\, Y(b,\, c)\right) = Y\left(X(a,\, b),\, X(a,\, c)\right).$$

Сверхтождество $(*)$ назовём нетривиальным, если $m > 1$, и тривиальным, если $m = 1$. Число m называется функциональным рангом данного сверхтождества.

Алгебра $(Q;\Sigma)$ с бинарными операциями называется бинарной алгеброй. Алгебра (Q,Σ) называется q-алгеброй (e-алгеброй) если существует операция $A \in \Sigma$ такая что $Q(A)$ – квазигруппа (группоид с единицей). Бинарная алгебра (Q,Σ) называется функционально нетривиальной, если $|\Sigma| > 1$. Известно (см. [11, 12], а также [13, 20]) что если ассоциативное нетривиальное сверхтождество выполяется в функционально нетривиальной q-алгебре (e-алгебре) тогда функнциональный ранг этого сверхтождества может быть равен только двум и это сверхтождество имеет одну из следующих видов:

$(ass)_1$ $$X(x,Y(y,z)) = Y(X(x,y),z),$$

$(ass)_2$ $$X(x,Y(y,z)) = X(Y(x,y),z),$$

$(ass)_3$ $$Y(x,Y(y,z)) = X(X(x,y),z).$$

Более того, в классе q-алгебр (e-алгебр) сверхтождество $(ass)_3$ влечёт сверхтождество $(ass)_2$, которое влечёт сверхтождество $(ass)_1$.

Определение 1.3. *Бинарная алгебра $(Q;\Sigma)$ называется сверхассоциативной если в ней выполняется первое сверхтождество ассоциативности $(ass)_1$.*

Следовательно, сверхассоциативные алгебры – это алгебры с полугрупповыми операциями. Сверхассоциативные алгебры под названием Г-полугрупп (гамма-полугрупп) или доппельполугрупп рассматриваются в работах разных авторов [21]-[32].

Пример 1.5. Пусть A, B непустые множества, Σ множество всех отображений из B в A и Q множество всех отображений из A в B. Тогда каждый элемент $\alpha \in \Sigma$ мы можем рассмотреть как бинарную операцию на Q:

$$\alpha(a,b) = a \cdot \alpha \cdot b,$$

где $a, b \in Q$ и $a \cdot \alpha \cdot b$ обычная суперпозиция отображений. В результате, мы получаем сверхассоциативную алгебру $(Q; \Sigma)$. Более того, если $A = B$, мы получаем алгебру второй степени (второго порядка) $(Q; \Sigma; \cdot)$ в смысле [33].

Определение 1.4. *Бинарная алгебра $(Q; \Sigma)$ называется левой сверхальтернативной, если в ней выполняется следующее сверхтождество левой альтернативности:*

$$(alt)_l \qquad X(x, Y(x, y)) = Y(X(x, x), y).$$

Определение 1.5. *Бинарная алгебра $(Q; \Sigma)$ называется правой сверхальтернативной, если в ней выполняется следующее сверхтождество правой альтернативности:*

$$(alt)_r \qquad X(x, Y(y, y)) = Y(X(x, y), y).$$

Определение 1.6. *Бинарная алгебра $(Q; \Sigma)$ называется сверхальтернативной, если она является правой и левой сверхальтернативной.*

Определение 1.7. *g-алгебра $A(+, \Sigma, \Phi)$ называется сверхальтернативной, если бинарная алгебра(редукт) $(A; \Sigma)$ – сверхальтернативна.*

Пример 1.6. Пусть $A(+, \cdot, P)$ альтернативная алгебра и c элемент из ядра алгебры A ([34, 35, 4]), то есть:

$$(x \cdot c) \cdot y = x \cdot (c \cdot y),$$

для всех $x, y \in A$. Определим новую бинарную операцию над A:

$$x \circ y \stackrel{def}{=} x \cdot c \cdot y.$$

Тогда $A(+, \{\cdot, \circ\}, P)$ будет сверхальтернативной g-алгеброй.

Определение 1.8. *g-алгебра $A(+, \Sigma, \Phi)$ называется сверхассоциативной, если бинарная алгебра $(A; \Sigma)$ – сверхассоциативна.*

Если в примере 1.6 мы возьмём ассоциативную алгебру $A(+, \cdot, P)$, тогда полученная g-алгебра $A(+, \{\cdot, \circ\}, P)$ будет сверхассоциативной для каждого элемента $c \in A$.

Пусть $A(+, \Sigma, \Phi)$ – g-алгебра. Обозначим $(x, y, z)_{X,Y} := X(x, Y(y, z)) - Y(X(x, y), z)$, где $x, y, z \in A$, и $X, Y \in \Sigma$. Тогда условия $(alt)_l, (alt)_r$ для $(A; \Sigma)$ могут быть записаны следующим образом

$$(1.4) \qquad (x, x, y)_{X,Y} = 0; \forall x, y \in A, \forall X, Y \in \Sigma,$$

$$(x, y, y)_{X,Y} = 0; \forall x, y \in A, \forall X, Y \in \Sigma. \tag{1.5}$$

Отметим также, что g-алгебра $A(+, \Sigma, \Phi)$ сверхассоциативна тогда и только тогда, когда выполяется следующее условие:

$$(x, y, z)_{X,Y} = 0, \forall x, y, z \in A, \forall X, Y \in \Sigma.$$

Лемма 1.1. *Пусть $A(+, \Sigma, \Phi)$ – g-алгебра. Тогда:*

$$(\alpha_1 a + \alpha_2 b, \beta_1 c + \beta_2 d, \gamma_1 e + \gamma_2 f)_{X,Y} =$$
$$(\alpha_1\beta_1\gamma_1)(a, c, e)_{X,Y} + (\alpha_1\beta_1\gamma_2)(a, c, f)_{X,Y} + (\alpha_1\beta_2\gamma_1)(a, d, e)_{X,Y}+$$
$$(\alpha_1\beta_2\gamma_2)(a, d, f)_{X,Y} + (\alpha_2\beta_1\gamma_1)(b, c, e)_{X,Y} + (\alpha_2\beta_1\gamma_2)(b, c, f)_{X,Y}+$$
$$(\alpha_2\beta_2\gamma_1)(b, d, e)_{X,Y} + (\alpha_2\beta_2\gamma_2)(b, d, f)_{X,Y},$$

$\forall a, b, c, d, e, f \in A, \forall \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \Phi, \forall X, Y \in \Sigma.$

Доказательство.

$$(\alpha_1 a + \alpha_2 b, \beta_1 c + \beta_2 d, \gamma_1 e + \gamma_2 f)_{X,Y} =$$
$$X(\alpha_1 a+\alpha_2 b, Y(\beta_1 c+\beta_2 d, \gamma_1 e+\gamma_2 f))-Y(X(\alpha_1 a+\alpha_2 b, \beta_1 c+\beta_2 d), \gamma_1 e+\gamma_2 f) \overset{(1.1)-(1.3)}{=}$$
$$(\alpha_1\beta_1\gamma_1)(X(a, Y(c, e)) - Y(X(a, c), e)) + (\alpha_1\beta_1\gamma_2)(X(a, Y(c, f)) - Y(X(a, c), f))+$$
$$(\alpha_1\beta_2\gamma_1)(X(a, Y(d, e)) - Y(X(a, d), e)) + (\alpha_1\beta_2\gamma_2)(X(a, Y(d, f)) - Y(X(a, d), f))+$$
$$(\alpha_2\beta_1\gamma_1)(X(b, Y(c, e)) - Y(X(b, c), e)) + (\alpha_2\beta_1\gamma_2)(X(b, Y(c, f)) - Y(X(b, c), f)+$$
$$(\alpha_2\beta_2\gamma_1)(X(b, Y(d, e)) - Y(X(b, d), e)) + (\alpha_2\beta_2\gamma_2)(X(b, Y(d, f)) - Y(X(b, d), f)) =$$
$$(\alpha_1\beta_1\gamma_1)(a, c, e)_{X,Y} + (\alpha_1\beta_1\gamma_2)(a, c, f)_{X,Y} + (\alpha_1\beta_2\gamma_1)(a, d, e)_{X,Y}+$$
$$(\alpha_1\beta_2\gamma_2)(a, d, f)_{X,Y} + (\alpha_2\beta_1\gamma_1)(b, c, e)_{X,Y} + (\alpha_2\beta_1\gamma_2)(b, c, f)_{X,Y}+$$
$$(\alpha_2\beta_2\gamma_1)(b, d, e)_{X,Y} + (\alpha_2\beta_2\gamma_2)(b, d, f)_{X,Y}.$$

□

Лемма 1.2. *Пусть $A(+, \Sigma, \Phi)$ – сверхальтернативная g-алгебра. Тогда выполяются следующие условия:*

$$(x, y, x)_{X,Y} = 0, \tag{1.6}$$

$$(x, y, z)_{X,Y} = -(y, x, z)_{X,Y}, \tag{1.7}$$

$$(x, y, z)_{X,Y} = -(z, y, x)_{X,Y}, \tag{1.8}$$

$$(x, y, z)_{X,Y} = -(x, z, y)_{X,Y}, \tag{1.9}$$

для всех $x, y, z \in A$ и для всех $X, Y \in \Sigma$.

Лемма 1.3. *Пусть $A(+, \Sigma, \Phi)$ – g-алгебра. Тогда выполяется следующее условие:*

$$(Z(x,y),z,t)_{X,Y} - (x, X(y,z),t)_{Z,Y} + (x,y,Y(z,t))_{Z,X} =$$
$$Z(x,(y,z,t)_{X,Y}) + Y((x,y,z)_{Z,X},t),$$

для всех $x, y, z, t \in A$ и для всех $X, Y, Z \in \Sigma$.

Пусть $R(+, \Sigma, \Phi)$ – g-алгебра и $A, B, C \subseteq R$ и $X, Y \in \Sigma$. Обозначая $(A, B, C)_{X,Y} = 0$ будем иметь ввиду $(a, b, c)_{X,Y} = 0, \forall a \in A, \forall b \in B, \forall c \in C$. Скажем подмножество $A \subseteq R$ $*$-множество, если для любых $a_1, a_2 \in A$ и $r \in R$ выполяется следующее котождество: $X(a_1, Y(a_2, r)) = Y(X(a_1, a_2), r)$, т.е.

$$(A, A, R)_{X,Y} = 0, \forall X, Y \in \Sigma. \tag{1.10}$$

Очевидно, что если некое $*$-подмножество g-алгебры R также подалгебра g-алгебры R, тогда эта подалгебра сверхассоциативна. Если R – сверхальтернативная g-алгебра и $A \subseteq R$ – $*$-множество, тогда из условий (1.7) – (1.9) вытекают следующие котождества для любых $a_1, a_2 \in A$ и $r \in R$:

$$X(a_1, Y(r, a_2)) = Y(X(a_1, r), a_2), \qquad X(r, Y(a_1, a_2)) = Y(X(r, a_1), a_2),$$

т.е.

$$(A, R, A)_{X,Y} = 0; \forall X, Y \in \Sigma, \tag{1.11}$$

$$(R, A, A)_{X,Y} = 0; \forall X, Y \in \Sigma. \tag{1.12}$$

Лемма 1.4. *Пусть $R(+, \Sigma, \Phi)$ – сверхальтернативная g-алгебра и $A \subseteq R$ является $*$-подмножеством. Тогда подалгебра g-алгебры R, порождённая подмножеством A, также будет $*$-множеством.*

2. Основной результат

Сформулируем основной результат статьи.

Теорема 2.1. *Пусть $R(+, \Sigma, \Phi)$ – сверхальтернативная g-алгебра, а $A, B, C \subseteq R$ – подалгебры и $*$-множества данной g-алгебры R. Если для любых $a \in A$, $b \in B$, $c \in C$ выполняется следующее котождество в $(R; \Sigma)$:*

$$X(a, Y(b, c)) = Y(X(a, b), c),$$

т.е. $(A, B, C)_{X,Y} = 0$, для всех $X, Y \in \Sigma$, тогда подалгебра g-алгебры R порождённая A, B и C – сверхассоциативна.

Доказательство. Рассмотрим следующее подмножество g-алгебры R:

$$D = \{d \in R | (A, B, d)_{X,Y} = (B, C, d)_{X,Y} = (C, A, d)_{X,Y} = 0, \forall X, Y \in \Sigma\}.$$

Очевидно, что D – непустое множество, так как:

$$A \cup B \cup C \subseteq D. \tag{2.1}$$

Согласно условиям $(1.1)-(1.3)$, множество D замкнуто относительно модульных операций, т.е. для всех $d_1, d_2 \in D$ и $\beta, \gamma \in \Phi$ имеем $\beta d_1 + \gamma d_2 \in D$.

Покажем следующие включения:

$$X(a', d), X(d, a'), X(b', d), X(d, b'), X(c', d), X(d, c') \in D, \tag{2.2}$$

для всех $d \in D, a' \in A, b' \in B, c' \in C, X \in \Sigma$.

Возьмём $a \in A, b \in B$ и $c \in C$, получим:

$$(Z(a, a'), d, b)_{X,Y} - (a, X(a', d), b)_{Z,Y} + (a, a', Y(d, b))_{Z,X} \overset{Lemma 1.3}{=}$$
$$Z(a, (a', d, b)_{X,Y}) + Y((a, a', d)_{Z,X}, b) \overset{(1.9)-(1.10)}{\Rightarrow}$$
$$(a, b, X(a', d))_{Z,Y} = 0; \forall a \in A, b \in B, Y, Z \in \Sigma,$$
$$(Z(a, a'), d, c)_{X,Y} - (a, X(a', d), c)_{Z,Y} + (a, a', Y(d, c))_{Z,X} \overset{Lemma 1.3}{=}$$
$$Z(a, (a', d, c)_{X,Y}) + Y((a, a', d)_{Z,X}, c) \overset{(1.9)-(1.10)}{\Rightarrow}$$
$$(a, c, X(a', d))_{Z,Y} = 0; \forall a \in A, c \in C, Y, Z \in \Sigma,$$
$$(Z(b, a'), d, c)_{X,Y} - (b, X(a', d), c)_{Z,Y} + (b, a', Y(d, c))_{Z,X} \overset{Lemma 1.3}{=}$$
$$Z(b, (a', d, c)_{X,Y}) + Y((b, a', d)_{Z,X}, c) \overset{(1.9)-(1.10)}{\Rightarrow}$$
$$(b, c, X(a', d))_{Z,Y} = 0; \forall b \in B, c \in C, Y, Z \in \Sigma.$$

Следовательно, $X(a', d) \in D$.

Далее, имеем:

$$(Z(b, d), a', a)_{X,Y} - (b, X(d, a'), a)_{Z,Y} + (b, d, Y(a', a))_{Z,X} \overset{Lemma 1.3}{=}$$
$$Z(b, (d, a', a)_{X,Y}) + Y((b, d, a')_{Z,X}, a) \overset{(1.8),(1.9),(1.12)}{\Rightarrow}$$
$$(a, b, X(d, a'))_{Z,Y} = 0; \forall a \in A, b \in B, Y, Z \in \Sigma,$$
$$(Z(c, d), a', a)_{X,Y} - (c, X(d, a'), a)_{Z,Y} + (c, d, Y(a', a))_{Z,X} \overset{Lemma 1.3}{=}$$
$$Z(c, (d, a', a)_{X,Y}) + Y((c, d, a')_{Z,X}, a) \overset{(1.8),(1.9),(1.12)}{\Rightarrow}$$
$$(a, c, X(d, a'))_{Z,Y} = 0; \forall a \in A, c \in C, Y, Z \in \Sigma,$$
$$(Z(b, d), a', c)_{X,Y} - (b, X(d, a'), c)_{Z,Y} + (b, d, Y(a', c))_{Z,X} \overset{Lemma 1.3}{=}$$
$$Z(b, (d, a', c)_{X,Y}) + Y((b, d, a')_{Z,X}, c) \overset{(1.8),(1.9),(1.12)}{\Rightarrow}$$

$$(b, c, X(d, a'))_{Z,Y} = 0; \forall b \in B, c \in C, Y, Z \in \Sigma.$$

Таким образом, $X(d, a') \in D$. Остальные случаи доказываются аналогично.

Обозначим через (A, B, C) подалгебру R, порождённую подалгебрами A, B и C, т.е (A, B, C) есть наименьшая подалгебра g-алгебры R, содержащая подалгебры A, B, C.

Пусть $(A; \Sigma)$ – бинарная алгебра, $a_1, ..., a_n \in A$ и $X_1, ..., X_{n-1} \in \Sigma$. Терм $a_1 a_2 ... a_n$, где скобки и операции $X_1, ..., X_{n-1}$ распределены неким образом, называется произведением или n-произведением элементов $a_1, ..., a_n$ в отношении операций $X_1, ..., X_{n-1}$(или просто n-произведением).

Рассмотрим теперь следующее подмножество g-алгебры R: $P = \{\beta_1 w_1 + \beta_2 w_2 + ... + \beta_n w_n | n \in N, \beta_i \in \Phi, w_i$ есть k_i-произведение $(k_i \in N)$ элементов из $A \cup B \cup C, i \in \{1, 2, ..., n\}\}$. Очевидно, что P содержит A, B, C и $P \subseteq (A, B, C)$. С другой стороны, P замкнут относительно модульных операций и бинарных операций из Σ, т.е. P – подалгебра g-алгебры R содержащая A, B и C, следовательно, $P = (A, B, C)$.

Пусть w есть n-произведение элементов из $A \cup B \cup C$. Скажем, что w – нормальное произведение, если $n = 1$ или для всех k $(n \geq k \geq 2)$ каждое k-произведение, которое есть часть произведения w, представляет из себя произведение элемента из $A \cup B \cup C$ и некоторого $(k-1)$-произведения.

Согласно условиям (2.1), (2.2), каждое нормальное произведение w принадлежит D, т.е.

$$(A, B, w)_{X,Y} = 0, \tag{2.3}$$

$$(A, C, w)_{X,Y} = 0, \tag{2.4}$$

$$(C, B, w)_{X,Y} = 0, \tag{2.5}$$

для всех $X, Y \in \Sigma$.

Покажем, что произведение нормальных произведений представима в виде суммы нормальных произведений. Достаточно показать, что произведение двух нормальных произведений представима в виде суммы нормальных произведений. Докажем это через индукцию.

Пусть n-произведение v и k-произведение w – нормальные произведения.

(1) Пусть $n = 1$, то есть $v \in A \cup B \cup C$ и следовательно, $X(v, w)$ – нормальное произведение для любого $X \in \Sigma$;

(2) Пусть $X(v_1, w_1)$ есть сумма нормальных произведений для каждого n'-произведения v_1 и k'-произведения w_1, где $n > n'$;

(a) $v = X(v', a)$, где $a \in A$. Имеем:

$$Y(v,w) = Y(X(v',a),w) = X(v',Y(a,w)) - (v',a,w)_{X,Y} \overset{(1.9)}{=}$$

$$X(v',Y(a,w)) + (v',w,a)_{X,Y} =$$

$$X(v',Y(a,w)) + X(v',Y(w,a)) - Y(X(v',w),a)$$

(b) $v = X(a, v')$, где $a \in A$. Имеем:

$$Y(v,w) = Y(X(a,v'),w) = X(a,Y(v',w)) - (a,v',w)_{X,Y} \overset{(1.7)}{=}$$

$$X(a,Y(v',w)) + (v',a,w)_{X,Y} =$$

$$X(a,Y(v',w)) + X(v',Y(a,w)) - Y(X(v',a),w)$$

Случаи $v = X(v', b)$ и $v = X(b, v')$, $v = X(v', c)$ и $v = X(c, v')$, где $b \in B, c \in C$, доказываются аналогично.

Каждое n-произведение элеметнов из $A \cup B \cup C$ – нормальное произведение или есть произведение нормальных произведений элементов из $A \cup B \cup C$, следовательно, элементы подалгебры (A, B, C) представимы как сумма нормальных произведений. Справедливы следующие условия:

$$(a,v,w)_{X,Y} = 0, \tag{2.6}$$

$$(b,v,w)_{X,Y} = 0, \tag{2.7}$$

$$(c,v,w)_{X,Y} = 0, \tag{2.8}$$

где n-произведение v и k-произведение w – нормальные произведения и $a \in A, b \in B, c \in C$ и $X, Y \in \Sigma$.

Покажем, что

$$(u,v,w)_{X,Y} = 0, \tag{2.9}$$

где $X, Y \in \Sigma$ и n-произведение u, m-произведение v, k-произведение w – нормальные произведения.

Докажем условие (2.9) индукцией по n. При $n = 1$ доказательство следует из условий (2.6), (2.7), (2.8);

Пусть $(u', v', w')_{X,Y} = 0$ для любого n'-произведения u', m'-произведения v' и k'-произведения w', где $n > n'$;

(1) $u = Z(u', a)$, где $a \in A$. Имеем:

$$(Z(u', a), v, w)_{X,Y} - (u', X(a, v), w)_{Z,Y} + (u', a, Y(v, w))_{Z,X} =$$

$$Z(u', (a, v, w)_{X,Y}) + Y((u', a, v)_{Z,X}, w) \overset{(1.7),(1.9),(2.6)}{\Rightarrow}$$

$$(u, v, w)_{X,Y} = 0.$$

(2) $u = Z(a, u')$, где $a \in A$. Имеем:

$$(Z(a, u'), v, w)_{X,Y} - (a, X(u', v), w)_{Z,Y} + (a, u', Y(v, w))_{Z,X} =$$

$$Z(a, (u', v, w)_{X,Y}) + Y((a, u', v)_{Z,X}, w) \overset{(1.7),(1.9),(2.6)}{\Rightarrow}$$

$$(u, v, w)_{X,Y} = 0.$$

Случаи $u = Z(u', b)$, $u = Z(b, u')$, $u = Z(u', c)$ и $u = Z(c, u')$, где $b \in B, c \in C$, доказываются аналогично.

Итак, каждый элемент из (A, B, C) представим в виде суммы нормальных произведений, следовательно, согласно условию (2.9), получим, что подалгебра (A, B, C) – сверхассоциативна.

□

3. Следствия

Следствие 3.1. *Пусть $R(+, \Sigma, \Phi)$ – сверхальтернативная g-алгебра и $A, B \subseteq R$ подалгебры и $*$-подмножества. Тогда подалгебра (A, B) данной g-алгебры R порождённой подалгебрами A и B – сверхассоциативна.*

Следствие 3.2. *Пусть $R(+, \Sigma, \Phi)$ – сверхальтернативная g-алгебра и для некоторых элементов $a, b, c \in R$ выполняется следующее котождество в $(R; \Sigma)$:*

$$X(a, Y(b, c)) = Y(X(a, b), c),$$

т.е. $(a, b, c)_{X,Y} = 0$ для всех $X, Y \in \Sigma$. Тогда подалгебра g-алгебры R, порождённая элементами a, b, c – сверхассоциативна.

Следствие 3.3. *Пусть $R(+, \Sigma, \Phi)$ – сверхальтернативная g-алгебра и $a, b \in R$. Тогда подалгебра g-алгебры R, порождённая элементами a, b – сверхассоциативна.*

Открытая проблема

Доказать теорему типа Келли для сверхассоциативных(сверхальтернативных) g-алгебр.

Abstract. The following Artin theorem for linear algebras defined on commutative and associative ring with unit is well known: In the alternative linear algebra any two its elements generate an associative subalgebra; moreover if $(a, b, c) = 0$ then the subalgebra generated by the elements a, b, c is associative. In this paper we suggest a larger generalization of this classical result, using the concepts of hyperidentity and coidentity. The corresponding structures we call g-algebras.

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NON-REAL ZEROS OF POLYNOMIALS IN A POLYNOMIAL SEQUENCE SATISFYING A THREE-TERM RECURRENCE RELATION

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Abstract. This paper discusses the location of zeros of polynomials in a polynomial sequence $\{P_n(z)\}_{n=1}^{\infty}$ generated by a three-term recurrence relation of the form $P_n(z) + B(z)P_{n-1}(z)+$ $+A(z)P_{n-k}(z) = 0$ with $k > 2$ and the standard initial conditions $P_0(z) = 1, P_{-1}(z) = \dots$ $= P_{-k+1}(z) = 0$, where $A(z)$ and $B(z)$ are arbitrary coprime real polynomials. We show that there always exist polynomials in $\{P_n(z)\}_{n=1}^{\infty}$ with non-real zeros.

MSC2010 numbers: 12D10; 26C10; 30C15.
Keywords: recurrence relation; hyperbolic polynomials; discriminant.

1. Introduction

For decades, a popular topic of studies in mathematics is related to three-term recurrence relations subject to natural restrictions on their coefficients. By Favard's theorem [1], such recurrences generate orthogonal polynomials and these are of great interest since they are frequently used in many problems in the approximation theory, mathematical and numerical analysis, and their applications (for example, least square approximation of functions, difference and differential equations, Gaussian quadrature processes, etc.), see [2].

In general, the zeros of polynomials $P_n(z)$ generated by recurrences do not exactly lie on a particular curve but are attracted to a curve (which in this paper we shall call the limiting curve) as $n \to \infty$. Such a limiting curve is explicitly described in [3, 4]. Recently, K. Tran in [5, 6] has proved cases where the polynomials $P_n(z)$ generated by three-term recurrences have all their zeros (for all or sufficiently large n) situated exactly on the said limiting curve. We begin with the following conjecture.

Conjecture A ([5]). *For an arbitrary pair of polynomials $A(z)$ and $B(z)$, all zeros of every polynomial in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ satisfying the three-term recurrence relation of length k*

$$P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k}(z) = 0 \tag{1.1}$$

with the standard initial conditions $P_0(z) = 1$, $P_{-1}(z) = \ldots = P_{-k+1}(z) = 0$ which do satisfy $A(z) \neq 0$ lie on the algebraic curve $\Gamma \subset \mathbb{C}$ given by

$$\Im\left(\frac{B^k(z)}{A(z)}\right) = 0 \quad \text{and} \quad 0 \leq (-1)^k \Re\left(\frac{B^k(z)}{A(z)}\right) \leq \frac{k^k}{(k-1)^{k-1}}. \tag{1.2}$$

Moreover, these roots become dense in Γ when $n \to \infty$.

In the same paper, the above conjecture was proven for $k = 2, 3, 4$. In [6], K. Tran settled Conjecture A for polynomials $P_n(z)$ with sufficiently large n. The problems around this area of study have most recently received substantial interest and a number of studies have been carried out, see for example the papers [6] - [11]. In [8], the authors proved the following theorem.

Theorem 1.1 (see [8])**.** *For an arbitrary pair of polynomials $A(z)$ and $B(z)$, all the zeros of every polynomial in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ satisfying the three-term recurrence relation of length k*

$$P_n(z) + B(z)P_{n-\ell}(z) + A(z)P_{n-k}(z) = 0$$

where k and ℓ are coprime and with the standard initial conditions $P_0(z) = 1$, $P_{-1}(z) = \ldots = P_{-k+1}(z) = 0$ which satisfy the condition $A(z)B(z) \neq 0$ lie on the real algebraic curve $\mathcal{C}$ given by

$$\Im\left(\frac{B^k(z)}{A^\ell(z)}\right) = 0. \tag{1.3}$$

The above theorem completely settles the first part of Conjecture A. There has been an initial attempt to obtain the exact portion of the curve $\mathcal{C}$ where the zeros of the polynomials lie by providing in addition to (1.3), an inequality constraint satisfied by the real part of the rational function $\frac{B^k(z)}{A^\ell(z)}$. This has been proven for specific cases namely, $(k, \ell) = (3, 2)$ and $(4, 3)$ respectively and the details of the proofs can be found in [10]. In the same paper based on numerical experiments, a more general conjecture for the real part of $\frac{B^k(z)}{A^\ell(z)}$ has been proposed for this problem.

In the present paper, it is of interest to determine where in complex plane the zeros of every polynomial in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ generated by (1.1) are located. In a particular case of $k = 2$, the author in [7] characterizes real polynomials $A(z)$ and $B(z)$ to ensure that all the generated polynomials $P_n(z)$ are hyperbolic. This paper is a sequel of [7] but for $k > 2$. We aim at proving whether or not it is possible to generalize the former.

Problem 1. *In the above notation, consider the recurrence relation*

$$P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k}(z) = 0 \tag{1.4}$$

where $k > 2$ with the standard initial conditions,

$$P_0(z) = 1, P_{-1}(z) = \ldots = P_{-k+1}(z) = 0, \tag{1.5}$$

where $A(z)$ and $B(z)$ are arbitrary real polynomials. Characterize $A(z)$ and $B(z)$ (if possible) such that all the $P_n(z)$ are hyperbolic.

To formulate our main result, we need to look at the curve defined by the first condition in (1.2). We shall view $\mathbb{C}P^1$ as $\mathbb{C}\cup\{\infty\}$, the extended complex plane and $\mathbb{R}P^1$ as the extended real line.

Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be the rational function defined by $f(z) = \frac{B^k(z)}{A(z)}$ where $A(z)$ and $B(z)$ are real polynomials. Denote by $\widetilde{\Gamma} \subset \mathbb{C}P^1$ the curve given by $\Im(f(z)) = 0$, that is $\widetilde{\Gamma} = \{z \in \mathbb{C}P^1 : \Im(f(z)) = 0\} = f^{-1}(\mathbb{R}P^1)$.

For real polynomials $A(z)$ and $B(z)$, define the curve Γ by the condition (1.2). It is clear that $\Gamma \subset \widetilde{\Gamma}$.

In the remaining part of this section, let us remind the reader of some basic definitions and facts about rational functions. For further details, see [7].

For a non-constant rational function $R(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials with no common zeros, the degree of $R(z)$ is defined as the maximum of the degrees of $P(z)$ and $Q(z)$. A point $z_0 \in \mathbb{C}P^1$ is called a critical point of $R(z)$, (and $R(z_0)$ a critical value) if $R(z)$ fails to be injective in any neighbourhood of z_0, that is, either $R'(z_0) = 0$ or $R'(z_0) = \infty$ (i.e, at the zeros of $Q(z)$). The order of a critical point z_0 of $R(z)$ is the order of zero of $R'(z)$ at z_0.

Given a pair $(P(z), Q(z))$ of polynomials, we define their Wronskian as the polynomial $\mathcal{W}(P, Q) := P'Q - Q'P$ where P' and Q' are derivatives of P and Q with respect to z respectively. If P and Q have no common zeros, then the zeros of $\mathcal{W}(P, Q)$ are exactly the critical points of the rational map $R(z)$. In fact if α is a multiple zero of R, then α is a zero of the Wronskian.

We call a non-zero univariate polynomial with real coefficients hyperbolic if all its zeros are real. In [12, §3.1], we find that the zeros of two hyperbolic polynomimals $P(z), Q(z) \in \mathbb{R}[z]$ interlace if and only if $|\deg P - \deg Q| \leq 1$ and $\mathcal{W}(P, Q)$ is either nonnegative or nonpositive on the whole real axis. Notice that to say that the zeros of P and Q interlace means that each zero of Q lies between two successive zeros of P and there is at most one zero of Q between any two successive zeros of P, [13]. More information about the Wronskian can be found in [14].

Remark 1.1. *For the rational function $f(z) = \frac{B^k(z)}{A(z)}$, we have*

$$f'(z) = \frac{B^{k-1}(z)(kA(z)B'(z) - B(z)A'(z))}{A^2(z)} = \frac{\mathcal{W}(B^k(z), A(z))}{A^2(z)}.$$

We observe that the critical points of $f(z)$ are the zeros of the Wronskian $\mathcal{W}(B^k(z), A(z))$ or the poles of $f(z)$. In particular, if $A(z), B(z) \in \mathbb{R}[z]$ are coprime polynomials where $B(z)$ is hyperbolic with distinct zeros, then all the zeros of $B(z)$ are real critical points of $f(z)$ each with multiplicity $k-1$.

Let $P(x)$ be a univariate polynomial of degree n with zeros $x_1, \ldots, x_n$ and leading coefficient a_n. The ordinary discriminant of $P(x)$ denoted by $\mathrm{Disc}_x(P(x))$ is defined as

$$\mathrm{Disc}_x(P(x)) = a_n^{2n-2} \prod_{1 \le i < j \le n} (x_i - x_j)^2.$$

Generally, the discriminant of a polynomial connects with the ratio of its zeros in the sense that the discriminant is zero if and only if the polynomial has multiple zeros. In particular, the discriminant of a polynomial vanishes whenever there exist at least a zero with multiplicity greater or equal to 2. For more details on the ordinary discriminants, see [15].

Example 1.1. *For coprime $1 \le \ell < k$, discriminant of a trinomial*

$$P(x) = ax^k + bx^\ell + c$$

is given by $k^k c^{k-1} a^{k-1} + (-1)^{k-1} \ell^\ell (k-\ell)^{k-\ell} c^{\ell-1} b^k a^{k-\ell-1}$. In particular,

$$\mathrm{Disc}_x(x^k + Bx + A) = A^{k-2} \left(k^k A + (-1)^{k-1} (k-1)^{k-1} B^k \right). \tag{1.6}$$

The expression (1.6) will be of interest later in this work.

The main result of this paper is as follows.

Theorem 1.2. *In the above notation of Problem 1 for $k > 2$, there always exist polynomials in the sequence $\{P_n(z)\}_{n=1}^{\infty}$ with non-real zeros.*

2. Proofs

Lemma 2.1. *For $k > 2$, consider the recurrence relation*

$$P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k}(z) = 0 \tag{2.1}$$

with the standard initial conditions,

$$P_0(z) = 1, P_{-1}(z) = \ldots = P_{-k+1}(z) = 0,$$

where $A(z)$ and $B(z)$ are arbitrary polynomials. If $f(z) = \frac{B^k(z)}{A(z)}$, then the zeros of $P_1(z)$ and $P_2(z)$ are critical points of $f(z)$.

Proof. Substitution of the initial conditions in the recurrence relation

$$P_n(z) + B(z)P_{n-1}(z) + A(z)P_{n-k}(z) = 0$$

for $n = 1$ gives $P_1(z) = -B(z)$. Similarly since $k > 2$, repeating the same process gives $P_2(z) = B^2(z)$. The zeros of $P_1(z)$ and $P_2(z)$ are the zeros $B(z)$. The conclusion that the zeros of $P_1(z)$ and $P_2(z)$ are critical points of $f(z)$ follows from Remark 1.1. □

Remark 2.1. *For the recurrence* (2.1) *to generate a sequence of hyperbolic polynomials, $B(z)$ must be hyperbolic by Lemma 2.1.*

Let us briefly discuss some facts about the limiting curve Γ. Let $P(\lambda, z) = \lambda^k + B(z)\lambda + A(z)$ be the characteristic polynomial of the recurrence (1.4) and $\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)$ be its distinct non-zero characteristic roots. For $i \neq j$, let $\Gamma_{i,j} := \{\alpha \in \mathbb{C} : |\lambda_i(\alpha)| = |\lambda_j(\alpha)|\}$ be the equimodular curve of $P(\lambda, z)$ associated to the characteristic functions $\lambda_i(z)$ and $\lambda_j(z)$. For a fixed $\alpha \in \mathbb{C}$ with $i \neq j$, we have $|\lambda_i(\alpha)| = |\lambda_j(\alpha)|$ if and only if there exists an $s \in \mathbb{C}$ such that $|s| = 1$ and $\lambda_i(\alpha) = s\lambda_j(\alpha)$.

For each $i \neq j$, let $w = w(z) := \lambda_i(z)/\lambda_j(z)$ and define

$$P_w(\lambda, z) = P(w\lambda, z) = w^k\lambda^k + B(z)w\lambda + A(z).$$

Then it is clear that $\lambda_j(z)$ is a common solution of both $P(\lambda, z) = 0$ and $P_w(\lambda, z) = 0$. A necessary and sufficient condition for $P_w(\lambda, z)$ and $P(\lambda, z)$ to have a non-constant common factor is that their resultant $\rho(w, z)$ vanishes as a function of z. By Lemma 3 [16, §3], $\rho(w, z) = A(z)(w - 1)^k\Delta_k(w, z)$ where $\Delta_k(w, z)$ is a reciprocal polynomial in w of degree $k(k - 1)$. In addition, $\Delta_k(1, z)$ is a multiple of $\mathrm{Disc}_\lambda(P(\lambda, z))$, the discriminant of $P(\lambda, z)$. In the same paper, it is proved that the reciprocal polynomial $\Delta_k(w, z)$ can be written as

$$\Delta_k(w, z) = w^{k(k-1)/2}v(t, z)$$

where $t = w + w^{-1} + 2$. The equimodularity condition $|w| = 1$ corresponds to t being in the real interval $0 \leq t \leq 4$. In particular,

$$v(4, z) = \Delta_k(1, z) = \pm\,\mathrm{Disc}_\lambda(P(\lambda, z)). \tag{2.2}$$

Lemma 2.2. *Let $A(z)$ and $B(z)$ be as defined in* (1.4) *and $R(t, z) = t^k + B(z)t + A(z)$. Further, let $U := \{z : \mathrm{Disc}_t(R(t, z)) = 0\}$ and Γ be the curve defined in* (1.2)*. A point $z_0 \in \mathbb{C}$ is an endpoint of Γ if and only if $z_0 \in U$ and $A(z_0) \neq 0$.*

Proof. Let $E = U \setminus \{z \in \mathbb{C} : A(z) = 0\}$. We observe that $z_0 \in E$ if and only if $\mathrm{Disc}_t(R(t, z_0)) = 0$ and $A(z_0) \neq 0$. This is equivalent to $v(4, z_0) = 0$ and $A(z_0) \neq 0$ by (2.2). But by the results proved in [16, §5], $v(4, z_0) = 0$ and $A(z_0) \neq 0$ if and only if z_0 is an endpoint of segments of Γ. The proof is complete. □

Remark 2.2. *For the recurrence* (2.1) *to generate a sequence of hyperbolic polynomials, it is a necessary condition that the* $z_0 \in \mathbb{C}$ *mentioned in Lemma 2.2 is real.*

We now state the following theorem about the behaviour of analytic functions near a critical point. This will be used in the proof of the main result. For details of the proof, see [17].

For $\delta > 0$ and $z_0 \in \mathbb{C}$, we define $D_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}$.

Theorem 2.1. *Let $g(z)$ be a non-constant analytic, function in a region, $\Omega \subset \mathbb{C}$. Let $z_0 \in \Omega$, $w_0 = g(z_0)$, and suppose that $g(z) - w_0$ has a zero of order $p \geq 2$ at z_0. The following hold;*

(a) There are $\epsilon, \delta > 0$ such that for every $w \in D_\epsilon(w_0) \setminus \{w_0\}$, there are exactly p distinct solutions of

$$g(z) = w \tag{2.3}$$

with $z \in D_\delta(z_0)$. Moreover, for these solutions, $g(z) - w$ has a simple zero.

(b) There is an analytic function, h, on $D_{\epsilon^{1/p}}(0)$ with $h(0) = 0, h'(0) \neq 0$, so that if $w \in D_\epsilon(w_0)$ and

$$w = w_0 + \tau e^{i\theta}, \qquad 0 < \tau < \epsilon, 0 \leq \theta < 2\pi$$

then the p solutions of (2.3) are given by

$$z = z_0 + h(\tau^{1/p} e^{(i(\theta + 2\pi j)/p)}), \qquad j = 0, 1, \ldots, p - 1.$$

(c) There is a power series, $\sum_{n=1}^{\infty} b_n x^n$, with radius of convergence at least ϵ, so the solutions of (2.3) are given by

$$z = z_0 + \sum_{n=1}^{\infty} b_n (w - w_0)^{n/p}$$

where $(w - w_0)^{1/p}$ is interpreted as the pth roots of $(w - w_0)$ (same root taken in all terms of the power series).

Let us finally settle the main result of this paper.

Proof of Theorem 1.2. Suppose that $B(z)$ is hyperbolic, otherwise the theorem follows from Lemma 2.1. Additionally, let z_0 be a zero of $B(z)$ with multiplicity $p > 0$. Then z_0 is a zero of both $P_1(z)$ and $P_2(z)$ by Lemma 2.1. By Remark 1.1, z_0 is a real critical point of $f(z)$. Moreover, $z_0 \in \Gamma$ by (1.2). Theorem 2.1 implies that in the neighbourhood of z_0,

$$f(z) = \frac{B^k(z)}{A(z)} = (z - z_0)^{pk} q(z)$$

where $q(z)$ is analytic at z_0 and $q(z_0) \neq 0$. Pick a $\delta_1 > 0$ such that $q(z)$ is non-vanishing in $D_{\delta_1}(z_0)$. In this neighbourhood, there exists an analytic function $q_1(z)$ such that $q_1(z) = \sqrt[pk]{q(z)}$. Take $q_1(z)$ as a branch of $\sqrt[pk]{q(z)}$. Define $u(z) := (z - z_0)q_1(z)$. Then we have

$$f(z) = u(z)^{pk}, \quad \text{where} \quad u(z_0) = 0, \ \ q_1(z_0) = u'(z_0) \neq 0.$$

Thus for a small positive $\epsilon \in \mathbb{R}$, the equation $f(z) = \pm\varepsilon$ is equivalent to

$$(z - z_0)^{pk} q_1(z)^{pk} = \pm\varepsilon. \tag{2.4}$$

Let $h(z)$ be the inverse function to $u(z)$. Then by applying Theorem 2.1 to (2.4) where the left side of this equation has a zero $z = z_0$ with multiplicity pk, we obtain solutions of the form

$$z = z_0 + h(\varepsilon^{(1/pk)} e^{(2\pi ij/pk)}) \quad \text{or} \quad z = z_0 + h(\varepsilon^{(1/pk)} e^{(i\pi + 2\pi ij)/pk)}),$$

where $\varepsilon^{(1/pk)}$ is the pk-th roots of ε. Using the fact that $pk > 2$ and h has a simple zero at 0, we deduce that (2.4) has pk solutions z and these cannot be all real.

Denote by $\rho = \frac{k^k}{(k-1)^{k-1}}$. Then, by Theorem 2 [7], the zeros of $P_n(z)$ are contained in $\Gamma = f^{-1}([0, \rho])$ or $\Gamma = f^{-1}([-\rho, 0])$ when k is even or odd respectively, and these zeros are dense on Γ as $n \to \infty$. Now for $[0, \varepsilon] \subset [0, \rho]$ and $[-\varepsilon, 0] \subset [-\rho, 0]$, it follows that

$$f^{-1}([0, \varepsilon]) \subset \Gamma \quad \text{or} \quad f^{-1}([-\varepsilon, 0]) \subset \Gamma.$$

For all the polynomials $P_n(z)$ to be hyperbolic, we require Γ to consist only of intervals on the real line in $\mathbb{C}$. From the solutions of (2.4), it is clear that neither $f^{-1}([0, \varepsilon])$ nor $f^{-1}([-\varepsilon, 0])$ is a subset of only real intervals. Thus there will always be at least one non-real curve through z_0 on which non-real zeros of $P_n(z)$ will be located. The conclusion follows. $\square$

3. Examples

In this section we present concrete examples using numerical experiments. In these examples, we consider the sequence of polynomials $\{P_n(z)\}_{n=0}^{\infty}$ generated by the rational function

$$\sum_{n=0}^{\infty} P_n(z) t^n = \frac{1}{1 + B(z)t + A(z)t^k}, \tag{3.1}$$

where $A(z)$ and $B(z)$ are coprime real polynomials. We plot a graph showing

(i) a portion (black curves) of the curve $\widetilde{\Gamma}$ given by $\Im\left(\frac{B^k(z)}{A(z)}\right) = 0$;

(ii) the zeros (circles) of one of the polynomials $P_n(z)$ in (3.1) (of our choice) described by specifying $k, A(z), B(z)$ and a positive integer n. These are located on Γ;

(iii) the points $z^* \in \mathbb{C}$ which are endpoints of the curve Γ (indicated by black dots). Such points are the elements of the set E defined in Lemma 2.2.

Example 3.1. *For* $n = 71, k = 3, A(z) = z^3 - z^2 - 5z + 7$ *and* $B(z) = z^2 - z - 6$, *we obtain Fig. 1.*

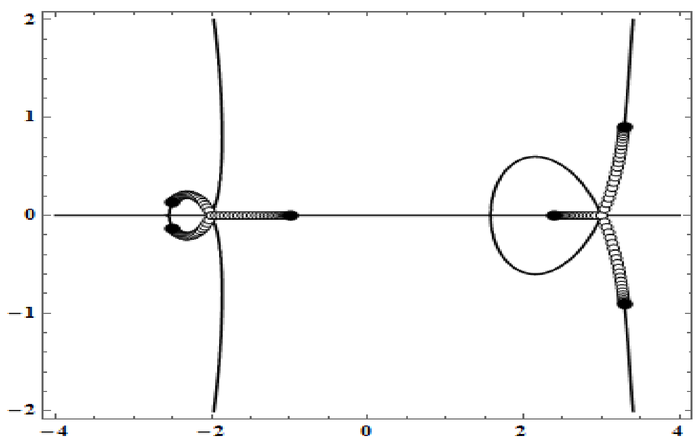

Рис. 1. The rational function is given by $f(z) = 1/(1 + (z^2 - z - 6)t + (z^3 - z^2 - 5z + 7)t^3)$.

Example 3.2. *For* $n = 150, k = 5, A(z) = z^2 + z - 4$ *and* $B(z) = z^2 + z - 2$, *we obtain Fig. 2.*

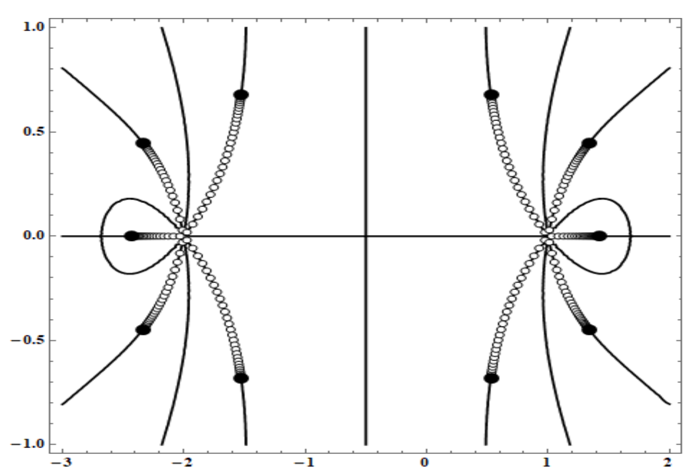

Рис. 2. The rational function is given by $f(z) = 1/(1 + (z^2 + z - 2)t + (z^2 + z - 4)t^5)$.

4. Final remarks

Problem 1 has been settled in the negative in the sense that it is not possible to generate a sequence of hyperbolic polynomials using the recurrence (1.4) with the given initial conditions (1.5).

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