ISSN 00002-3043

ՀԱՅԱՍՏԱՆԻ ԳԱԱ
ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ
НАН АРМЕНИИ

ՄԱԹԵՄԱՏԻԿԱ
МАТЕМАТИКА

2021

Известия НАН Армении, Математика, том 56, н. 1, 2021, стр. 3 – 9

ОБ ОДНОЙ АССОЦИАТИВНОЙ ФОРМУЛЕ С ФУНКЦИОНАЛЬНЫМИ ПЕРЕМЕННЫМИ

Д. Н. АРУТЮНЯН

Ереванский государственный университет
E-mail: *david.harutyunyan96@gmail.com*

Аннотация. В работе [1] доказывается что обратимая алгебра (Q,Σ) удовлетворяющая следующей формуле второго порядка

$$\forall X,Y,\exists X',Y'\forall x,y,z(X(Y'(x,y),z)=Y(x,X'(y,z))),$$

является линейной над группой. В настоящей статье доказывается более общий результат, а именно, что регулярная и делимая алгебра (Q,Σ) с указанной формулой второго порядка является эндо-линейной над группой.

MSC2010 number: 03C05, 03C85, 20N05.

Ключевые слова: $\forall\exists(\forall)$-тождество; регулярная и делимая алгебра; эндо-линейная алгебра; квазиэндоморфизм.

1. Введение и предварительные результаты

Пустъ $(Q,\cdot)$ группоид и $a\in Q$. Обозначим через L_a,R_a следующие отображения: $L_a x=ax, R_a x=xa$ и назовем соотвественно левым и правым умножением.

Группоид $(Q,\cdot)$ называется с делением, если для любого $a\in Q$ L_a и R_a сюръективные отображения. Бинарная алгебра (Q,Σ) называется с делением, если (Q,A) является группоидом с делением для любого $A\in\Sigma$.

Назовем группоид $(Q,\cdot)$ лево-регулярным, если:

$$ca=cb\Rightarrow R_a=R_b,$$

где $a,b,c\in Q$. Аналогично определяется право-регулярный группоид. Назавем групоид регулярным, если он одновременно лево-регулярный и право-регулярный. Бинарная алгебра (Q,Σ) называется регулярным, если (Q,A) является регулярным группоидом для любого $A\in\Sigma$.

Определение 1.1. [2][3] *Группоид (Q,A) гомотопен группоиду (Q,B), елси существуют такие отображения $\alpha,\beta,\gamma:Q\to Q$, что имеет место равенство $\gamma A(x,y)=B(\alpha x,\beta y)$, для любых $x,y\in Q$. Тогда тройка (α,β,γ) называется гомотопей из (Q,A) в (Q,B). Если $\gamma=id_Q$ тогда скажем, что эти группоиды* *главно гомотопны. Если α, β, γ сюръективные отображения тогда назовем эти группоиды эпитопными или главно эпитопными соотвественно.*

Определение 1.2. *Отображение $\gamma : Q \to Q$ называется гомотопей групоида (Q, A), если существуют такие отоброжения $\alpha, \beta : Q \to Q$, что тройка (α, β, γ) будет гомотопей из (Q, A) в (Q, A).*

Определение 1.3. [4] *Бинарная алгебра (Q, Σ) гомотопна группоиду $(Q, \cdot)$, если группоид (Q, A) гомотопен группоиду $(Q, \cdot)$ для любого $A \in \Sigma$. Таким же образом определяется главная гомотопность, эпитопность и главная эпитопность бинарной алгебры (Q, Σ) группоиду $(Q, \cdot)$.*

Определение 1.4. *Бинарная алгебра (Q, Σ) называется эндо-линейной над группоидом $(Q, \cdot)$, если каждая операция $A \in \Sigma$ является эндо-линейной над группоидом $(Q, \cdot)$, т. е. для каждого $A \in \Sigma$ существуют сюръективные эндоморфизмы ϕ_A, ψ_A группы $(Q, \cdot)$ и элемент t_A из Q, для которых имеет место равенство:*

$$A(x, y) = \phi_A x \cdot (\psi_A y \cdot t_A)$$

для любого $x, y \in Q$.

Определение 1.5. *Отображение $\phi : Q \to Q$ является квазиэндоморфизмом группы $(Q, \cdot)$, если*

$$\phi(xy) = \phi x \cdot (\phi 1)^{-1} \cdot \phi y$$

для всех $x, y \in Q$, где 1 - единица группы $(Q, \cdot)$.

Лемма 1.1. *Любой квазиэндоморфизм ϕ группы $(Q, \cdot)$ имеет вид:*

$$\phi = \widetilde{R}_a \phi',$$

где $\widetilde{R}_a = x \cdot a, a \in Q$, и ϕ' является эндоморфизмом группы $(Q, \cdot)$.

Доказательство. Пусть $\phi 1 = k$. Покажем, что $\phi' = \widetilde{R}_{k^{-1}} \phi$ является эндоморфизмом. Имеем:

$$\begin{aligned} \phi'(ab) = \widetilde{R}_{k^{-1}} \phi(ab) &= \phi a \cdot (\phi 1)^{-1} \cdot \phi b \cdot k^{-1} \\ &= (\phi a \cdot k^{-1}) \cdot (\phi b \cdot k^{-1}) = \phi' a \cdot \phi' b. \end{aligned}$$

□

Лемма 1.2. *Любая гомотопия α группы $(Q, \cdot)$ является квазиэндоморфизмом $(Q, \cdot)$.*

Доказательство. Согласно определению гомотопии, должны существовать две такие отображения β и γ, чтобы имело место равенство

$$\alpha(ab) = \beta a \cdot \gamma b, \tag{1.1}$$

для любых $a, b \in Q$.

Сделав по очереди в (1.1) замены: 1) $a = 1$, 2) $b = 1$, 3) $a = b = 1$, получаем:

$$\alpha b = \beta 1 \cdot \gamma b, \quad \alpha a = \beta a \cdot \gamma 1, \quad \alpha 1 = \beta 1 \cdot \gamma 1 : \tag{1.2}$$

Преобразуем равенство (1.1), учитывая равенства (1.2):

$$\alpha(ab) = \alpha a \cdot (\gamma 1)^{-1} \cdot (\beta 1)^{-1} \alpha b = \alpha a \cdot (\beta 1 \cdot \gamma 1)^{-1} \cdot \alpha b =$$
$$= \alpha a \cdot (\alpha 1)^{-1} \cdot \alpha b,$$

т. е. α квазиэндоморфизм. □

Теорема 1.1. [5] *Пусть* $(Q, A), (Q, B), (Q, C), (Q, D)$ *группоиды с делением, а* $(Q, A), (Q, C)$ *- регулярны. Тогда, если имеет место тождество*

$$A(x, B(y, z)) = C(D(x, y), z), \tag{1.3}$$

то алгебра $(Q, \{A, B, C, D\})$ *будет эпитопно некоторой группе* $(Q, \cdot)$. *Более того, существуют такие* $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ *сюръективные отображения, что имели место равенства:*

$$\begin{cases} A(x, y) = A_1 x \cdot A_2 x, \\ A_2 B(x, y) = A_2 B_1 x \cdot A_2 B_2 y, \\ C(x, y) = C_1 x \cdot C_2 x, \\ C_2 D(x, y) = C_2 D_1 x \cdot C_2 D_2 y, \end{cases}$$

и

$$\begin{cases} A_1 = C_1 D_1, \\ A_2 B_1 = C_1 D_2, \\ A_2 B_2 = C_2. \end{cases}$$

2. Основные результаты

Теперь мы готовы сформулировать и доказывать основные результаты статьи.

Теорема 2.1. *Пусть* (Q, Σ) *регулярная алгебра с делением, и для любых* $A, C \in \Sigma$ *существуют* $B, D \in \Sigma$ *такие, что имеет место тождество* (1.3). *Тогда, существует такая группа* $(Q, \cdot)$, *что* (Q, Σ) *будет главно эпитопно этой группе.*

Доказательство. Пусть $A \in \Sigma$, тогда из теоремы 1.1 следует, что существуют операции $A', A'' \in \Sigma$ и такая группа $(Q, \cdot)$, что имеют место тождество $A(x, A'(y, z)) = A(A''(x, y), z)$, и

$$\begin{cases} A(x, y) = \alpha x \cdot \beta y, \\ \beta A'(x, y) = \beta\lambda x \cdot \beta\theta y, \\ A(x, y) = \phi x \cdot \psi y, \\ \psi A''(x, y) = \psi\gamma x \cdot \psi\delta y, \end{cases}$$

где $\alpha, \beta, \gamma, \lambda, \theta, \phi, \psi, \delta$ - сюръективные отображения.

Из теоремы 1.1 следует, что для любого $C \in Q$ существуют такие операции $B, D \in \Sigma$ и группа $(Q, \cdot_C)$ такая, что имеют место равенства $A(x, B(y, z)) = C(D(x, y), z)$, и

$$\begin{cases} A(x, y) = \alpha_C x \cdot_C \beta_C y \\ \beta_C B(x, y) = \beta_C \lambda_C x \cdot_C \beta_C \theta_C y \\ C(x, y) = \phi_C x \cdot_C \psi_C y \\ \psi_C D(x, y) = \psi_C \gamma_C x \cdot_C \psi_C \delta_C y \end{cases},$$

где $\alpha_C, \beta_C, \gamma_C, \lambda_C, \theta_C, \phi_C, \psi_C, \delta_C$ - сюръективные отображения.

Так как α_C, β_C сюръекции, то существуют такие $h_{\alpha_C}, h_{\beta_C}$ отображения, что $\alpha_C h_{\alpha_C} = id_Q$, и $\beta_C h_{\beta_C} = id_Q$.

Сделав замены $x \to h_{\alpha_C} x, \ \ y \to h_{\beta_C} y$ в равенстве $A(x, y) = \alpha_C x \cdot_C \beta_C y$, получаем $A(h_{\alpha_C} x, h_{\beta_C} y) = x \cdot_C y$, и так как $A(x, y) = \alpha x \cdot \beta y$, то из этих двух равенств следует

$$x \cdot_C y = \alpha h_{\alpha_C} x \cdot \beta h_{\beta_C} y. \tag{2.1}$$

Докажем, что $\alpha h_{\alpha_C}, \beta h_{\beta_C}$ будут биекциями.

Сделав замену $y = 1_C$ в равенстве (2.1), где 1_C - единица группы $(Q, \cdot_C)$, получаем:

$$x = \alpha h_{\alpha_C} x \cdot c_1 = \widetilde{R}_{c_1} \alpha h_{\alpha_C} x,$$

где $c_1 = \beta h_{\beta_C} 1_C$, и $\widetilde{R}_{c_1}$ является левым умножением группы $(Q, \cdot_C)$ с элементом c_1. Так как $\widetilde{R}_{c_1}$ является биективным отображением, и $\widetilde{R}_{c_1}^{-1} = \widetilde{R}_{c_1^{-1}}$, то αh_{α_C} тоже будет биективным отображением. Таким же образом βh_{β_C} тоже будет биективным отображением.

Следует, что для любых $x, y \in Q$, и для любого $C \in \Sigma$ имеет место равенство $x \cdot_C y = \sigma_C x \cdot \mu_C y$, где $\sigma_C = \alpha h_{\alpha_C}, \ \ \mu_C = \beta h_{\beta_C}$, то есть σ_C и μ_C биекции. Так как

$C(x, y) = \phi_C x \cdot_C \psi_C y$, то сделав замену получим:

$$C(x, y) = \sigma_C \phi_C x \cdot \mu_C \psi_C y,$$

где, $\sigma_C \phi_C, \mu_C \psi_C$ сюръекции. То есть для любого $C \in \Sigma$, (Q, C) будет главно эпитопно группе $(Q, \cdot)$. □

Теорема 2.2. *Пусть* (Q, Σ) *- регулярная алгебра с делением, и для любых* $A, C \in \Sigma$ *существуют* $B, D \in \Sigma$ *такие, что имеет место тождество* (1.3). *Тогда, существует такая группа* $(Q, \cdot)$*, что алгебра* (Q, Σ) *будет эндо-линейной над этой группой.*

Доказательство. Имеем, что каковы бы ни были операции $A, C \in \Sigma$, найдутся две операции $B, D \in \Sigma$ такие, что имеет место тождество (1.3).

Согласно теореме 2.2, все операции из Σ должны быть главно эпитопны одной и той же группе $(Q, \cdot)$, т. е. имеем:

$$\begin{cases} A(x, y) = \alpha_A x \cdot \beta_A y \\ B(x, y) = \alpha_B x \cdot \beta_B y \\ C(x, y) = \alpha_C x \cdot \beta_C y \\ D(x, y) = \alpha_D x \cdot \beta_D y \end{cases},$$

где $\alpha_A, \beta_A, \alpha_B, \beta_B, \alpha_C, \beta_C, \alpha_D, \beta_D$ - сюръективные отображения.

Сделав замены в тождестве (1.3), получим

$$\alpha_C(x \cdot y) = \alpha_A h_{\alpha_D} x \cdot \beta_A(\alpha_B h_{\beta_D} y \cdot \cdot_B h_{\beta_C} 1).$$

Сделаем замены $z \to h_{\beta_C} 1$, $x \to h_{\alpha_D} x$ и $y \to h_{\beta_D} y$, где $\beta_C h_{\beta_C} = \alpha_D h_{\alpha_D} = \beta_D h_{\beta_D} = id_Q$, где 1 является единицей группы $(Q, \cdot)$. Тогда из последнего равенства получим

$$\alpha_C(x \cdot y) = \alpha_A h_{\alpha_D} x \cdot \beta_A(\alpha_B h_{\beta_D} y \cdot \beta_B h_{\beta_C} 1),$$

или

$$\alpha_C(x \cdot y) = \sigma x \cdot \mu y,$$

где $\sigma = \alpha_A h_{\alpha_D}$ и $\mu = \beta_A \widetilde{R}_l \alpha_B h_{\beta_D} y$, где $l = \beta_B h_{\beta_C} 1$.

Согласно лемме 1.2, α_C будет квазиэндоморфизмом $(Q, \cdot)$, а из леммы 1.1 получим, что

$$\alpha_C = \widetilde{R}_t \alpha'_C,$$

где $\widetilde{R}_t$ - правое умножение группы $(Q, \cdot)$, а α'_C является эндоморфизмом группы $(Q, \cdot)$.

Следовательно получаем, что для любого $C \in \Sigma$ имеет место равенство

$$C(x, y) = \alpha_C x \cdot \beta_C y = \widetilde{R}_t \alpha'_C x \cdot \beta_C y = \alpha'_C x \cdot t \cdot \beta_C y = \\ = \alpha'_C x \cdot \widetilde{L}_t \beta_C y = \alpha'_C x \cdot \gamma_C y, \tag{2.2}$$

где α'_C - сюръективный эндоморфизм, а γ_C - сюръективное отображение.

Из равенства (2.2) теперь получаем:

$$\begin{cases} A(x, y) = \phi_A x \cdot \gamma_A y \\ B(x, y) = \phi_B x \cdot \gamma_B y \\ C(x, y) = \phi_C x \cdot \gamma_C y \\ D(x, y) = \phi_D x \cdot \gamma_D y \end{cases},$$

где $\phi_A, \phi_B, \phi_C, \phi_D$ - сюръективные эндоморфизмы, а $\gamma_A, \gamma_B, \gamma_C, \gamma_D$ - сюръективные отображения.

Сделав замены в тождестве (1.3), получим:

$$\phi_A x \cdot \gamma_A(\phi_B y \cdot \gamma_B z) = \phi_C(\phi_D x \cdot \gamma_D y) \cdot \gamma_C z.$$

Сделаем замены $x \to h_{\phi_A} 1$, $y \to h_{\phi_B} y$ и $z \to h_{\gamma_B} y$, где $\phi_A h_{\phi_A} = \phi_B h_{\phi_B} = \gamma_B h_{\gamma_B} = id_Q$, и 1 является единицей группы $(Q, \cdot)$. Тогда из последнего равенства получим:

$$\gamma_A(y \cdot z) = \phi_C(\phi_D h_{\phi_A} 1 \cdot \gamma_D h_{\phi_B} y) \cdot \gamma_C h_{\gamma_B} z,$$

или

$$\gamma_A(y \cdot z) = \theta y \cdot \nu z,$$

где $\theta = \phi_C \widetilde{L}_f \gamma_D h_{\phi_B}$ и $\nu = \gamma_C h_{\gamma_B}$, где $f = \phi_D h_{\phi_A} 1$.

Согласно лемме 1.2, γ_A будет квазиэндоморфизмом $(Q, \cdot)$ и из леммы 1.1 получим, что $\gamma_A = \widetilde{R}_g \psi_A$, где $\widetilde{R}_g$ - правое умножение в группе $(Q, \cdot)$, и ψ_A является эндоморфизмом группы $(Q, \cdot)$. Следовательно, для любого $A \in \Sigma$ имеет место равенство

$$A(x, y) = \phi_A x \cdot \gamma_C y = \phi_A x \cdot \widetilde{R}_g \psi_A y = \phi_A x \cdot \psi_A y \cdot g_A,$$

где ϕ_A, ψ_A - сюръективные эндоморфизмы группы $(Q, \cdot)$, а g_A - элемент из Q. □

Abstract. In [1] is proved that the invertible algebra (Q, Σ) satisfying the following second-order formula

$$\forall X, Y, \exists X', Y' \forall x, y, z (X(Y'(x, y), z) = Y(x, X'(y, z))),$$

is linear over the group. In this paper, we prove a more General result, namely that the regular and divisible algebra (Q, Σ) with the specified second-order formula is endo-linear over the group.

Список литературы

[1] Yu. Movsisyan, A. Gevorgyan, "Invertible algebras satisfying associative identities with functional variables", Asian-European Journal of Mathematics, 2050155, https://doi.org/10.1142/s1793557120501557 (2019).

[2] A. A. Albert, "Quasigroup I", Trans. Amer. Math. Soc. **54**, 507 – 519 (1943).

[3] A. A. Albert, "Quasigroup II", Trans. Amer. Math. Soc. **55**, 401 – 419 (1944).

[4] Yu. M. Movsisyan, "Hyperidentities and related concepts, II", Armenian Journal of Mathematics, **2018**, no. 4, 1 – 85 (2018).

[5] S. Davidov, A. Krapez, Yu. Movsisyan, "Functional equations with division and regular operations", Asian-European Journal of Mathematics, **11**, no. 03, 1850033 (2018).

Поступила 24 апреля 2020

После доработки 24 апреля 2020

Принята к публикации 16 сентября 2020

Известия НАН Армении, Математика, том 56, н. 1, 2021, стр. 10 – 15

ON A SCALE OF CRITERIA ON n-DEPENDENCE

D. VOSKANYAN

Yerevan State University
E-mail: *ysudav@gmail.com*

Abstract. In this paper we prove that a planar set $\mathcal{X}$ of at most $mn-1$ points, where $m \le n$, is κ-dependent, if and only if there exists a number r, $1 \le r \le m-1$, and an essentially κ-dependent subset $\mathcal{Y} \subset \mathcal{X}$, $\#\mathcal{Y} \ge rs$, where $r+s-3=\kappa$, belonging to an algebraic curve of degree r, and not belonging to any curve of degree less than r. Moreover, if $\#\mathcal{Y} = rs$ then the set $\mathcal{Y}$ coincides with the set of intersection points of some two curves of degrees r and s, respectively. Let us mention that the first three criteria of the scale, for $m = 1, 2, 3$, are well-known results.

MSC2010 number: 14H50; 41A05; 41A63.

Keywords: plane algebraic curve; intersection point; n-poised set; n-independent set.

1. Introduction, n-independence

Denote by Π_n the space of bivariate algebraic polynomials of total degree less than or equal to n. Its dimension is given by

$$N := \dim \Pi_n = \binom{n+2}{2}.$$

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter p, say, to denote the polynomial p and the curve given by the equation $p(x,y)=0$. More precisely, suppose p is a polynomial without multiple factors. Then the plane curve defined by the equation $p(x,y)=0$ shall also be denoted by p. So lines, conics, and cubics are equivalent to polynomials of degree 1, 2, and 3, respectively.

Suppose a set of k distinct points is given:

$$\mathcal{X}_k = \{(x_i, y_i) : i = 1, 2, \ldots, k\} \subset \mathbb{C}^2.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(x_i, y_i) = c_i, \quad i = 1, \ldots, k, \tag{1.1}$$

is called *interpolation problem*. We denote this problem by $(\Pi_n, \mathcal{X})$. The polynomial p is called *interpolating polynomial*.

Definition 1.1. The set of points $\mathcal{X}_k$ is called *n-poised*, if for any data $(c_1, \ldots, c_k)$, there is a *unique* polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

By a Linear Algebra argument a necessary condition for n-poisedness is

$$k = \#\mathcal{X}_k = \dim \Pi_n = N.$$

Definition 1.2. The interpolating problem $(\Pi_n, \mathcal{X}_k)$ is called n-solvable, if for any data $(c_1, \ldots c_k)$, there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

A polynomial $p \in \Pi_n$ is called *n-fundamental polynomial* of a point $A \in \mathcal{X}$, if $p(A) = 1$ and $p\big|_{\mathcal{X}\setminus\{A\}} = 0$, where $p\big|_{\mathcal{X}}$ means the restriction of p to $\mathcal{X}$. We shall denote such a polynomial by $p^\star_{A,\mathcal{X}}$.
Sometimes we call n-fundamental also a polynomial from Π_n that just vanishes at all the points of $\mathcal{X}$ but A, since such a polynomial is a nonzero constant multiple of $p^\star_A$. A fundamental polynomial can be described as a plane curve containing all but one point of $\mathcal{X}$.

Next we consider an important concept of n-independence and n-dependence of point sets (see [1], [2], [4]).

Definition 1.3. A set of points $\mathcal{X}$ is called *n-independent*, if each its point has an n-fundamental polynomial. Otherwise, it is called *n-dependent*.

Since the fundamental polynomials are linearly independent, we get that $\#\mathcal{X} \leq N$ is a necessary condition for n-independence.

Proposition 1.1. *A set $\mathcal{X}$ is n-independent if and only if the interpolation problem $(\Pi_n, \mathcal{X})$ is n-solvable.*

Proof. Suppose $\mathcal{X} := \mathcal{X}_k$. In the case of n-independence we have the following Lagrange formula for a polynomial $p \in \Pi_n$ satisfying interpolating conditions (1.1):

$$p = \sum_{i=1}^{k} c_i p_i^\star.$$

On the other hand if the interpolation problem is n-solvable then for each point $(x_i, y_i),\ i = 1, \ldots, k$, there exists an n-fundamental polynomial. Indeed, it is the solution of the interpolation problem (1.1), where $c_i = 1$, and $c_j = 0\ \forall j \neq i$. □

Definition 1.4. A set of points $\mathcal{X}$ is called *essentially n-dependent*, if none of its points has an n-fundamental polynomial.

If a point set $\mathcal{X}$ is n-dependent, then for some $A \in \mathcal{X}$, there is no n-fundamental polynomial, which means that for any polynomial $p \in \Pi_n$ we have that

$$p\big|_{\mathcal{X}\setminus\{A\}} = 0 \quad \Longrightarrow \quad p(A) = 0.$$

Thus a set $\mathcal{X}$ is essentially n-dependent means that any plane curve of degree n containing all but one point of $\mathcal{X}$, contains all of $\mathcal{X}$.

In the proof of the main result we will need the following

Proposition 1.2 ([5], Cor. 2.2)**.** *Suppose a set $\mathcal{X}$ is given. Denote by $\mathcal{Y}$ the subset of $\mathcal{X}$ that have n-fundamental polynomials with respect to $\mathcal{X}$. Then the set $\mathcal{X} \setminus \mathcal{Y}$ is essentially n-dependent.*

Corollary 1.1. *Any n-dependent point set has essentially n-dependent subset.*

Set $d(n,k) := \dim \Pi_n - \dim \Pi_{n-k}$. It is easily seen that $d(n,k) = (n+1) + n + \cdots + (n-k+2) = \frac{1}{2}k(2n-k+3)$, if $k \leq n$.

In the sequel we will need the following well-known proposition (see, e.g., [7], Proposition 3.1).

Proposition 1.3. *Let q be a curve of degree k without multiple components and $k \leq n$. Then the following assertions hold:*

(i) *Any set of more than $d(n,k)$ points located on the curve q is n-dependent;*
(ii) *Any set $\mathcal{X}$ of $d(n,k)$ points located on the curve q is n-independent if and only if*

$$p \in \Pi_n, \ p\big|_{\mathcal{X}} = 0 \Rightarrow p = fq, \ \textit{where } f \in \Pi_{n-k}.$$

Corollary 1.2. *The following assertions hold:*

(i) *Any set of at least $n+2$ points located on a line is n-dependent;*
(ii) *Any set of at least $2n+2$ points located on a conic is n-dependent;*
(iii) *Any set of at least $3n+1$ points located on a cubic is n-dependent.*

2. Some known results

Let us start with the three known results which coincide with the first three items of the scale established in this paper, respectively.

Theorem 2.1 ([8])**.** *Any set $\mathcal{X}$ consisting of at most $n+1$ points is n-independent.*

Theorem 2.2 ([1], Prop. 1)**.** *A set $\mathcal{X}$ with no more than $2n+2$ points on the plane is n-dependent if and only if either $n+2$ of them are collinear or $\#\mathcal{X} = 2n+2$ and all the $2n+2$ points belong to a conic.*

Theorem 2.3 ([3], Thm. 5.1)**.** *A set $\mathcal{X}$ consisting of at most $3n$ points is n-dependent if and only if at least one of the following conditions hold:*

(i) *$n+2$ points are collinear;*
(ii) *$2n+2$ points belong to a (possibly reducible) conic;*

(iii) $\#\mathcal{X} = 3n$, *and there exist* $\sigma_3 \in \Pi_3$ *and* $\sigma_n \in \Pi_n$ *such that* $\mathcal{X} = \sigma_3 \cap \sigma_n$.

The following two results describe some properties of essentially dependent point sets laying in a curves of certain degrees.

Proposition 2.1 ([6], Prop. 3.3)**.** *Suppose that* $m \le n$. *If a set* $\mathcal{X}$ *of at most* mn *points is essentially* κ*-dependent then all the points of* $\mathcal{X}$ *lay in a curve of degree* m.

We say that a curve σ is not empty with respect to a set $\mathcal{X}$ if $X \cap \sigma \neq \varnothing$.

Theorem 2.4 ([6], Thm. 3.4)**.** *Assume that* σ_m *is a curve of degree* m, *which is either irreducible or is reducible such that all its irreducible components are not empty with respect to a set* $\mathcal{X} \subset \sigma_m$, *where* $\mathcal{X}$ *is essentially* κ*-dependent and* $m \le n+2$. *Then we have that* $\#\mathcal{X} \ge mn$.

The next result states a necessary and sufficient conditions for a set of mn points to coincide with the set of the intersection points of some two plane algebraic curves of degrees m and n, respectively.

Theorem 2.5 ([6], Thm. 3.1)**.** *A set* $\mathcal{X}$ *with* $\#\mathcal{X} = mn$, $m \le n$, *is the set of intersection points of some two plane curves of degrees* m *and* n, *respectively, if and only if the following two conditions are satisfied:*

(i) *The set* $\mathcal{X}$ *is essentially* $(m+n-3)$*-dependent;*
(ii) *No curve of degree less than* m *contains all of* $\mathcal{X}$.

3. Main result

By combining Proposition 2.1 and Theorem 2.4 we readily get the following

Proposition 3.1. *Suppose that* $\mathcal{X}$ *is an essentially* κ*-dependent point set with* $\#\mathcal{X} \le mn-1$, *where* $m \le n$, *and* $\kappa = m+n-3$. *Then there exists a number* r, $1 \le r \le m-1$, *and a curve* σ_r *of degree* r, *such that the following conditions hold:*

(i) $\#\mathcal{X} \ge rs$, *where* $r+s-3 = \kappa$;
(ii) σ_r *contains all of* $\mathcal{X}$
(iii) *There is no curve of degree less than* r *containing all of* $\mathcal{X}$.

Proof. We obtain from Proposition 2.1 that the set of points $\mathcal{X}$ lies in a curve σ of degree at most m. Without loss of generality we may assume that σ is either irreducible or is reducible such that all its irreducible components are not empty with respect to the set $\mathcal{X}$. Then, notice that the degree of the curve does not equal m, since in that case, in view of Theorem 2.4, we would have that $\#X \ge mn$. Finally, consider such a curve σ_r of the smallest possible degree $1 \le r \le m-1$.

Note that $r < \kappa - r + 3$ since $r < m \leq \kappa - m + 3$. Now, Theorem 2.4 implies that $\#\mathcal{X} \geq rs$, where $r + s - 3 = \kappa$. □

Now we are in a position to formulate the main result of the paper:

Theorem 3.1. *Suppose that $\mathcal{X}$ is a set of points such that $\#\mathcal{X} \leq mn - 1$, where $m \leq n$. Then $\mathcal{X}$ is κ-dependent, where $\kappa = m + n - 3$, if and only if there exist a number r, $1 \leq r \leq m-1$, and an essentially κ-dependent subset $\mathcal{Y} \subset \mathcal{X}$, $\#\mathcal{Y} \geq rs$, where $r + s - 3 = \kappa$, belonging to a curve of degree r, and not belonging to any curve of degree less than r.*
Moreover, if $\#\mathcal{Y} = rs$ then we have that $\mathcal{Y}$ coincides with the set of intersection points of some two plane curves of degrees r and s respectively.

Proof. The sufficiency part is obvious. If some set has a κ-dependent subset then the set itself is κ-dependent. Now let us prove the part of necessity.

We have that the set $\mathcal{X}$ is κ-dependent. By the Corollary 1.1 there exists some essentially κ-dependent subset $\mathcal{Y} \subset \mathcal{X}$. Now, applying Proposition 3.1 to the set of points $\mathcal{Y}$ we get that there exists a curve σ_r of degree r, $1 \leq r \leq m-1$, containing all of $\mathcal{Y}$, such that $\#\mathcal{Y} \geq rs$, where $r + s - 3 = \kappa$, and there is no curve of lower degree containing all of $\mathcal{Y}$.

Finally, let us prove the "moreover"part of the theorem. Here we have that $\mathcal{Y}$ is essentially κ-dependent, $\#\mathcal{Y} = rs$ and there is no curve of degree less than r passing through all the points of $\mathcal{Y}$. Notice also that $r < s$ since $r < m < \kappa - r + 3$. Hence we get from Theorem 2.5 that $\mathcal{Y}$ is the set of intersection points of some two plane curves of degrees r and s, respectively. □

Now, let us mention some necessary conditions for the set $\mathcal{X}$ to be able to apply Theorem 3.1. Suppose that we have a κ-dependent set $\mathcal{X}$. We need to find such numbers m and n, for which $\#\mathcal{X} \leq mn$, where $m + n - 3 = \kappa$ and $m \leq n$. Since the the expression mn achieves its maximum when $m = n = (\kappa + 3)/2$, then $\#\mathcal{X}$ must be not more than $\lfloor (\kappa + 3)^2/4 \rfloor$.

4. Some special cases of Theorem 3.1

In this section we verify that Theorem 3.1 is a generalization of Theorems 2.1, 2.2 and 2.3. For this purpose let us formulate Theorem 3.1 in the special cases with $m = 1, 2, 3, 4$.

Case $m = 1$. A set $\mathcal{X}$ of at most $\kappa+1$ points is never κ-dependent. This is equivalent to Theorem 2.1.

Case $m = 2$. A set $\mathcal{X}$ of at most $2\kappa + 1$ points is κ-dependent if and only if $\kappa + 2$ points of $\mathcal{X}$ are collinear.

Case $m = 3$. A set $\mathcal{X}$ of at most $3\kappa - 1$ points is κ-dependent, if and only if one of the following conditions hold:

(i) $\kappa + 2$ points of $\mathcal{X}$ belong to a line,

(ii) $2\kappa + 2$ points of $\mathcal{X}$ belong to a conic.

Clearly the statement in Case $m = 3$ is a generalization of Theorem 2.2.

Case $m = 4$. A set $\mathcal{X}$ of at most $4\kappa - 5$ points is κ-dependent, if and only if one of the following conditions hold:

(i) $\kappa + 2$ points of $\mathcal{X}$ belong to a line,

(ii) $2\kappa + 2$ points of $\mathcal{X}$ belong to a conic,

(iii) $\#\mathcal{X} = 3\kappa$ and $\mathcal{X}$ coincides with an intersection points of some two algebraic curves of degrees 3 and κ,

(iv) more than 3κ points of $\mathcal{X}$ belong to a cubic.

Finally, this statement generalizes Theorem 2.3.

Note that in above statements we used Corollary 1.2.

Список литературы

[1] D. Eisenbud, M. Green, J. Harris, "Cayley-Bacharach theorems and conjectures", Bull. Amer. Math. Soc. (N.S.), **33** (3), 295 – 324 (1996).

[2] H. Hakopian, K. Jetter, G. Zimmermann, "Vandermonde matrices for intersection points of curves", Jaén J. Approx., **1**, 67 – 81 (2009).

[3] H. Hakopian and A. Malinyan, "Characterization of n-independent sets with no more than $3n$ points", Jaén J. Approx., **4**, 119 – 134 (2012).

[4] H. Hakopian and A. Malinyan, "On n-independent sets located on quartics", Proceedings of YSU, Phys. and Math. Sci., **1**, 6 – 12 (2013).

[5] H. Hakopian and G. Mushyan, "On multivariate segmental interpolation problem", J. Comp. Sci. Appl. Math., **1**, 19 – 29 (2015).

[6] H. Hakopian, D. Voskanyan, "On the intersection points of two plane algebraic curves", J. Contemp. Mathemat. Anal., **54** 54, 90 – 97 (2019).

[7] L. Rafayelyan, "Poised nodes set constructions on algebraic curves", East J. Approx. **17** (3), 285 – 298 (2011).

[8] F. Severi, Vorlesungen über Algebraische Geometrie, Teubner, Berlin (1921). (Translation into German - E. Löffler).

Поступила 24 мая 2020

После доработки 24 мая 2020

Принята к публикации 16 сентября 2020

Известия НАН Армении, Математика, том 56, н. 1, 2021, стр. 16 – 34.

CONJUGATE TRANSFORMS ON DYADIC GROUP

U. GOGINAVA, A. SAATASHVILI

United Arab Emirates University, Al Ain, Abu Dhabi, UAE
Massachusetts Institute of Technology, Cambridge, United States
E-mails: *zazagoginava@gmail.com; saata@mit.edu*

Abstract. In this paper we study the properties of the Lebesgue constant of the conjugate transforms. For conjugate Fejér means we will find necessary and sufficient condition on t for which the estimation $E\left|\widetilde{\sigma}_n^{(t)}f\right|\leq cE\left|f\right|$ holds. We also prove that for dyadic irrational t, $L\log L$ is the maximal Orlicz space for which the estimation $E\left|\widetilde{\sigma}_n^{(t)}f\right|\leq c_1+c_2E\left(\left|f\right|\log^+\left|f\right|\right)$ is valid.

MSC2010 numbers: 42C10; 60G42.
Keywords: conjugate Walsh transform; Martingale transform; convergence in norm.

1. Dyadic Hardy spaces and conjugate transforms

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N}{:=}\mathbb{P}\cup\{0\}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I}:=[0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p,n\in\mathbb{N},\ \ 0\leq p<2^n$.

Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x=(x_0,x_1,...,x_k,...)$ with $x_k\in\{0,1\}\,(k\in\mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the dyadic group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &:= G,\ \ I_n(x):=\ I_n(x_0,...,x_{n-1}) \\ &:= \{y\in G:\ y=(x_0,...,x_{n-1},y_n,y_{n+1},...)\}\,,\ (x\in G,n\in\mathbb{N})\,. \end{aligned}$$

These sets are called the dyadic intervals. Let $0=(0:i\in\mathbb{N})\in G$ denote the null element of G, $\ I_n:=I_n(0)\ (n\in\mathbb{N})$. For every finite set E the number of elents in E we denote by $|E|$, i. e. $|E|:=(E)^{\#}$.

For $k\in\mathbb{N}$ and $x\in G$ denote $r_k(x):=(-1)^{x_k}$, the k-th Rademacher function on dyadic group G.

Denote the dyadic expansion of $t \in \mathbb{I}$ by

$$t = \sum_{j=0}^{\infty} \frac{t_j}{2^{j+1}}, \ t_j = 0, 1.$$

In the case of $t \in \mathbb{Q}$ choose the expansion which terminates in zeros. For $t \in \mathbb{I}$ we denote $\rho(t) := (t_0, t_1, ...) \in G$.

The σ-algebra generated by the dyadic intervals $\{I_n(x) : x \in G\}$ is denoted by A^n, more precisely, $A^n := \sigma\{I_n(x) : x \in G\}$. The expectation and the conditional expectation operators relative to A^n $(n \in \mathbb{N})$ are denoted by E and E_n, respectively.

The norm (or quasinorm) of the space L_p is defined by

$$\|f\|_p := (E|f|^p)^{1/p} \quad (0 < p < +\infty).$$

Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ martingale with respect to $(A^n, n \in \mathbb{N})$ (for details see, e. g. [18, 19]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|.$$

In case of $f \in L_1$, the maximal function can also be given by

$$f^* = \sup_{n \in \mathbb{N}} |E_n f|.$$

For $0 < p < \infty$ the Hardy martingale space H_p consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

For a martingale

$$f \sim \sum_{n=0}^{\infty} \left(f^{(n)} - f^{(n-1)}\right), \ f^{(-1)} = 0,$$

the conjugate transforms are defined by

$$\widetilde{f}^{(t)} \sim \sum_{n=0}^{\infty} r_n(\rho(t)) \left(f^{(n)} - f^{(n-1)}\right),$$

where $t \in \mathbb{I}$ is fixed.

Note that $\widetilde{f}^{(0)} = f$. As is well known, if f is an integrable function, then conjugate transforms $\widetilde{f}^{(t)}$ do exist almost everywhere, but they are not integrable in general.

The following equation holds ([18, 19])

$$\left\| \widetilde{f}^{(t)} \right\|_{H_p} = \|f\|_{H_p} \quad (0 < p < \infty, t \in \mathbb{I}).$$

Furthermore, Khintchin's inequality implies that

$$\|f\|_{H_p}^p \sim \int_{\mathbb{I}} \left\| \widetilde{f}^{(t)} \right\|_{H_p}^p dt \qquad (0 < p < \infty).$$

Let $Q(L) = Q(L)(\mathbb{I})$ be the Orlicz space [9] generated by the Young function Q, i.e. Q is convex continuous even function such that $Q(0) = 0$ and

$$\lim_{u\to+\infty}\frac{Q(u)}{u} = +\infty, \quad \lim_{u\to 0}\frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{Q(L)(\mathbb{I})} = \inf\{k > 0 : \int_{\mathbb{I}} Q(|f|/k) \le 1\}.$$

In particular, if $Q(u) = u\log(1+u), u > 0$ then the corresponding space will be denoted by $L\log^+ L(\mathbb{I})$.

2. Walsh system and conjugate Fejйr means

Let $m \in \mathbb{N}$, then $m = \sum\limits_{i=0}^{\infty} m_i 2^i$, where $m_i \in \{0,1\}$ $(i \in \mathbb{N})$, i.e. m is expressed in the number system of base 2. Denote $|m| := \max\{j \in \mathbb{N}: m_j \neq 0\}$, that is, $2^{|m|} \le m < 2^{|m|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_m(x) := \prod_{k=0}^{\infty}(r_k(x))^{m_k} = r_{|m|}(x)(-1)^{\sum\limits_{k=0}^{|m|-1} m_k x_k} \quad (x \in G, m \in \mathbb{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that ([14], [8])

$$D_{2^n}(x) = \begin{cases} 2^n, \text{if } x \in I_n, \\ 0, \ \text{ if } x \in G\backslash I_n, \end{cases} \tag{2.1}$$

and

$$D_n(x) = w_n(x)\sum_{k=0}^{\infty} n_k\left(D_{2^{k+1}}(x) - D_{2^k}(x)\right). \tag{2.2}$$

Let $x \in I_j\backslash I_{j+1}$. Then from (2.1) and (2.2) we have

$$D_n(x) = w_n(x)\left(\sum_{k=0}^{j-1} n_k 2^k - n_j 2^j\right).$$

Hence,

$$|D_n(x)| = \alpha_j(n),$$

where

$$\alpha_j(n) = \left|\sum_{k=0}^{j-1} n_k 2^k - n_j 2^j\right|.$$

The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M f := \sum_{i=0}^{M-1} \widehat{f}(i)\, w_i,$$

where the number $\widehat{f}(i) = E(f w_i)$ is said to be the ith Walsh-Fourier coefficient of the function f. It is easy to see that $E_n(f) = S_{2^n}(f)$.

For any given $n \in \mathbb{N}$ it is possible to write n uniquely as $n = \sum_{k=0}^{\infty} n_k 2^k$, where $n_k = 0$ or 1 for $k \in \mathbb{N}$. This expression will be called the binary expansion of n and the numbers n_k will be called the binary coefficients of n.

Define the variation of an $n \in \mathbb{N}$ with binary coefficients $(n_k : k \in \mathbb{N})$ by

$$V(n) := \sum_{k=1}^{\infty} |n_k - n_{k-1}| + n_0.$$

The Fejér means of Walsh-Fourier series is defined as follows

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f \qquad (n \in \mathbb{P}).$$

If $f \in L_1$ then it is easy to show that the sequence $(E_n(f) : n \in \mathbb{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n)} : n \in \mathbb{N})$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k \to \infty} E\left(f^{(k)} w_i\right). \tag{2.3}$$

The Walsh-Fourier coefficients of $f \in L_1$ are the same as the ones of the martingale $(E_n(f) : n \in \mathbb{N})$ obtained from f.

Let

$$\beta_0(t) := r_0(\rho(t)), \beta_k(t) := r_n(\rho(t)) \text{ if } 2^{n-1} \leq k < 2^n.$$

Then the nth partial sums of the conjugate transforms is given by

$$\widetilde{S}_n^{(t)} f := \sum_{k=0}^{n-1} \beta_k(t)\, \widehat{f}(k)\, w_k \quad (t \in \mathbb{I}, n \in \mathbb{P}).$$

Let $2^N \leq n < 2^{N+1}$. Then we have

$$\begin{aligned}
\widetilde{S}_n^{(t)} f &= r_0(\rho(t))\, \widehat{f}(0)\, w_0 + \sum_{l=1}^{N} r_l(\rho(t))\,(E_l f - E_{l-1} f) \\
&\quad + r_{N+1}(\rho(t))\,(S_n f - E_N f) \\
&= (-1)^{t_0} E f + \sum_{l=1}^{N} r_l(\rho(t))\,(E_l f - E_{l-1} f) \\
&\quad + r_{N+1}(\rho(t))\,(S_n f - E_N f) = f * \widetilde{D}_n^{(t)},
\end{aligned}$$

where

$$\widetilde{D}_n^{(t)} = \sum_{i=-1}^{N-1} (-1)^{t_{i+1}} \left(D_{2^{i+1}} - D_{2^i}\right) + (-1)^{t_{N+1}} \left(D_n - D_{2^N}\right), D_{2^{-1}} = 0.$$

It is easy to see that $(-1)^{t_l} = 1 - 2t_l$. Then for $\widetilde{D}_n^{(t)}$ we can write

$$\begin{aligned} \widetilde{D}_n^{(t)} &= \sum_{i=-1}^{N-1} (1 - 2t_{i+1}) \left(D_{2^{i+1}} - D_{2^i}\right) + (1 - 2t_{N+1}) \left(D_n - D_{2^N}\right) \\ &= D_n - 2 \sum_{i=-1}^{N-1} t_{i+1} \left(D_{2^{i+1}} - D_{2^i}\right) - 2t_{N+1} \left(D_n - D_{2^N}\right). \end{aligned}$$

Set

$$m := \sum_{i=0}^{N-1} t_{i+1} 2^i < 2^N.$$

Then from (2.2) we get

$$\widetilde{D}_n^{(t)} = D_n - 2w_m D_m - 2t_{N+1} \left(D_n - D_{2^N}\right) - 2t_0.$$

The conjugate $(C, 1)$-means of a martingale f are introduced by

$$\widetilde{\sigma}_n^{(t)} f := \frac{1}{n} \sum_{k=0}^{n-1} \widetilde{S}_k^{(t)} f \qquad (t \in \mathbb{I}, n \in \mathbb{P}).$$

The notiation $a \lesssim b$ in the proofs stands for $a < c \cdot b$, where c is an absolute constant.

3. Two sides estimation of Lebesgue constant of conjugate Walsh-Fourier series

Denote by L_n the lebesgue constants of the Walsh system:

$$L_n := \int_G |D_n| \, d\mu.$$

These constants were studied by many authors. For the trigonometric system it is important to note that results of Fejŭr and Szegx, the letter gave in [16] an explicit formula for Lebesgue constants, namely,

$$L_n = \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k(2n+1) - 1}\right).$$

Along with the trigonometric system the Walsh-Paley system is also severely studied for its importance in applications. The most properties of Lebesgue constants with respect to the Walsh-Paley system were obtained by Fine in [2]. In ([14], p. 34), the two-sided estimate

$$\frac{V(n)}{8} \leq L_n \leq V(n) \tag{3.1}$$

is proved. In [10], Lukomskii presented the estimate $L_n \geq V(n)/4$. Malykhin, Telyakovskii and Kholshchevnikova [11] improved the estimation (3.1) and proved the following

Theorem MTK. *For any positive integer n, the two-sided inequality*

$$\frac{V(n)+1}{3} \leq L_n < V(n)$$

is valid. Here the factors $1/3$ and 1 are sharp.

We would like to mention the work of Astashkin and Semenov [1] in which the sharp two-sided estimate for Lebesgue constants with respect to Walsh-Paley system are obtained.

The Walsh- Fejér kernels were studied by Toledo [17], in particular the following is proved

$$\sup\left\{\|K_n\|_1 : n \in \mathbb{P}\right\} = \frac{17}{15}.$$

Denote by $L_n^{(t)}$ the lebesgue constants of the conjugate transforms:

$$L_n^{(t)} := \int_G \left|\widetilde{D}_n^{(t)} + 2t_0\right| d\mu.$$

For $n, m \in \mathbb{N}$ and $2^N \leq n < 2^{N+1}$ we define

$$T(n,m) := \left\{i : n_i \neq n_{i-1}, m_i = m_{i-1}, i = 0, 1, ..., N-1\right\}.$$

In this section we study the properties of the Lebesgue constant of the conjugate transforms.

Theorem 3.1. *Let n is positive integer and*

$$m = \sum_{i=0}^{N-1} t_{i+1} 2^i, \quad 2^N \leq n < 2^{N+1}$$

and $t \in \mathbb{I}$. The two-sides inequality

$$\begin{aligned} &\max\left(\frac{1}{2}\left|T(n,m)\right| + \frac{1}{3}V(n) - 1, \frac{1}{4}\left|T(n,m)\right| + \frac{2}{3}V(m)) - 1\right) \\ \leq \quad & L_n^{(t)} \leq 2V(m) + \left|T(n,m)\right| + 2 \end{aligned}$$

is valid.

Proof. We have

$$\text{(3.2)} \qquad \begin{aligned} L_n^{(t)} \quad = \quad & \sum_{i=0}^{N-1} \int_{I_i \setminus I_{i+1}} \left|\widetilde{D}_n^{(t)} + 2t_0\right| d\mu + \int_{I_N \setminus I_{N+1}} \left|\widetilde{D}_n^{(t)} + 2t_0\right| d\mu + \\ & + \int_{I_{N+1}} \left|\widetilde{D}_n^{(t)} + 2t_0\right| d\mu := J_1 + J_2 + J_3. \end{aligned}$$

From (2.1), it is easy to see that

$$\int\limits_{I_i\setminus I_{i+1}} \left|\widetilde{D}_n^{(t)}+2t_0\right| d\mu$$

$$=\sum_{x_{i+1}=0}^{1}\cdots\sum_{x_N=0}^{1}\int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots,x_N)} \left|\widetilde{D}_n^{(t)}+2t_0\right| d\mu$$

$$=\sum_{x_{i+1}=0}^{1}\cdots\sum_{x_N=0}^{1}\int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots,x_{N-1},x_N)} \left|\left(1-2t_{N+1}\right)D_n-2w_mD_m\right| d\mu$$

$$=\sum_{x_{i+1}=0}^{1}\cdots\sum_{x_{N-1}=0}^{1}\left(\int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots,x_{N-1},0)} \left|\left(1-2t_{N+1}\right)D_n-2w_mD_m\right| d\mu\right)$$

$$+\sum_{x_{i+1}=0}^{1}\cdots\sum_{x_{N-1}=0}^{1}\left(\int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots,x_{N-1},1)} \left|\left(1-2t_{N+1}\right)D_n-2w_mD_m\right| d\mu\right)$$

$$=\sum_{x_{i+1}=0}^{1}\cdots\sum_{x_{N-1}=0}^{1}\left(\int\limits_{I_{N+1}(0,\ldots,0,x_i=1,x_{i+1},\ldots x_{N-1},0)} \left(\left|\left(1-2t_{N+1}\right)D_n-2w_mD_m\right|+\right.\right.$$

$$\left.\left.\left|\left(1-2t_{N+1}\right)D_n+2w_mD_m\right|\right)d\mu\right).$$

Since

$$\frac{|a-b|+|a+b|}{2}=\max\left\{|a|,|b|\right\}$$

we have

$$\begin{aligned}\int\limits_{I_i\setminus I_{i+1}} \left|\widetilde{D}_n^{(t)}+2t_0\right| d\mu &= \int\limits_{I_i\setminus I_{i+1}} \max\left\{\alpha_i(n),2\alpha_i(m)\right\} d\mu\\ &= \frac{\max\left\{\alpha_i(n),2\alpha_i(m)\right\}}{2^{i+1}}.\end{aligned}$$

Hence

$$J_1=\sum_{i=0}^{N-1}\frac{\max\left\{\alpha_i(n),2\alpha_i(m)\right\}}{2^{i+1}}. \tag{3.3}$$

For J_2 and J_3 we can write

$$\begin{aligned}J_2 &= \int\limits_{I_N\setminus I_{N+1}} \left|D_n-2w_mD_m-2t_{N+1}\left(D_n-D_{2^N}\right)\right| d\mu\\ &= \frac{\left|2^N-n'-2m-2t_{N+1}\left(2^N-n'-2^N\right)\right|}{2^{N+1}}\\ &= \frac{\left|2^N-n'-2m+2t_{N+1}n'\right|}{2^{N+1}},\end{aligned} \tag{3.4}$$

where $n=2^{N}+n', n^{'}<2^{N}$ and

$$J_{3} = \int\limits_{I_{N+1}} \left| D_{n}-2w_{m}D_{m}-2t_{N+1}\left(D_{n}-D_{2^{N}}\right) \right| d\mu = \frac{\left| n-2m-2t_{N+1}\left(n-2^{N}\right) \right| }{2^{N+1}}. \tag{3.5}$$

Combining (3.2)-(3.5) we conclude that

$$\left\| \widetilde{D}_{n}^{(t)}+2t_{0}\right\|_{1} = \sum_{i=0}^{N-1}\frac{\max \left\{ \alpha_{i}\left(n\right) ,2\alpha_{i}\left(m\right) \right\} }{2^{i+1}} + \frac{\left| 2^{N+1}-n-2m+2t_{N+1}\left(n-2^{N}\right) \right| }{2^{N+1}}+\frac{\left| n-2m-2t_{N+1}\left(n-2^{N}\right) \right| }{2^{N+1}}.$$

Set $A(n)=\{i:|n_{i}-n_{i-1}|=1)$ and

$$S\left(n\right) :=\sum_{i\in A\left(n\right) }\frac{\alpha_{i}\left(n\right) }{2^{i+1}}.$$

First, we prove that $S(n)\geq S(n(e))$, where

$$n(e)=n_{N}2^{N}+...+n_{e+1}2^{e+1}+n_{e-1}2^{e}+....+n_{0}2^{1}$$

and $n_{e}=n_{e-1}\neq n_{e+1}$. Set $A_{e}(n)=\{i:i\in A(n), i>e\}$. Then,

$$\begin{aligned} S(n)-S(n(e)) &= \sum_{i\in A_{e}\left(n\right) }\left(\frac{\alpha_{i}\left(n\right) }{2^{i+1}}-\frac{\alpha_{i}\left(n(e)\right) }{2^{i+1}}\right) \\ &\quad +\sum_{i\in A\left(n\right) \backslash A_{e}\left(n\right) }\left(\frac{\alpha_{i}\left(n\right) }{2^{i+1}}-\frac{\alpha_{i+1}\left(n(e)\right) }{2^{i+2}}\right) .\end{aligned}$$

Since

$$\frac{\alpha_{i}\left(n\right) }{2^{i+1}}-\frac{\alpha_{i+1}\left(n(e)\right) }{2^{i+2}}=0,\quad i\in A(n)\backslash A_{e}(n),$$

we get

$$S(n)-S(n(e))=\sum_{i\in A_{e}\left(n\right) }\left(\frac{\alpha_{i}\left(n\right) }{2^{i+1}}-\frac{\alpha_{i}\left(n(e)\right) }{2^{i+1}}\right) .$$

For $i>e$ we can write

$$\begin{aligned} &\alpha_{i}\left(n\right) -\alpha_{i}\left(n(e)\right) =\left| n_{0}2^{0}+\cdots +n_{i-1}2^{i-1}-n_{i}2^{i}\right| \\ &-\left| n\left(e\right)_{0}2^{0}+\cdots +n\left(e\right)_{i-1}2^{i-1}-n\left(e\right)_{i}2^{i}\right| \\ = &\left| n_{0}2^{0}+\cdots +n_{i-1}2^{i-1}-n_{i}2^{i}\right| \\ &-\left| n_{0}2^{1}+\cdots +n_{e-1}2^{e}+n_{e+1}2^{e+1}+\cdots +n_{i-1}2^{i-1}-n_{i}2^{i}\right| .\end{aligned}$$

Suppose that $n_i = 1$. Then we can write

$$\begin{aligned} &\alpha_i(n) - \alpha_i(n(e)) = 2^i - n_0 2^0 - \cdots - n_{i-1} 2^{i-1} \\ &-(2^i - n_0 2^1 - \cdots - n_{e-1} 2^e - n_{e+1} 2^{e+1} - \cdots - n_{i-1} 2^{i-1}) \\ = &\sum_{j=0}^{e-1} n_j 2^j - n_e 2^e. \end{aligned}$$

Let now consider the case when $n_i = 0$. Then we have

$$\alpha_i(n) - \alpha_i(n(e)) = -\left(\sum_{j=0}^{e-1} n_j 2^j - n_e 2^e\right).$$

So, we get following

$$\alpha_i(n) - \alpha_i(n(e)) = (2|n_i - n_e| - 1)\left|\sum_{j=0}^{e-1} n_j 2^j - 2^e n_e\right|.$$

And finally, since $n_{e+1} \neq n_e$, we get

$$\begin{aligned} S(n) - S(n(e)) = \left|\sum_{j=0}^{e-1} n_j 2^j - 2^e n_e\right| \sum_{i \in A_e(n)} \frac{(2|n_i - n_e| - 1)}{2^{i+1}} \\ > \frac{1}{2^{e+2}} - \sum_{i=e+2}^{\infty} \frac{1}{2^{i+1}} \geq 0. \end{aligned}$$

From the definition of function $V(n)$ it is easy to see that$V(n) = V(n(e))$. If we continue this process it is easy to see that we will get

$$n' = \sum_{i=0}^{|n'|-1} n'_i 2^i,$$

where, for $0 \leq i < |n'|$ $n'_i + n'_{i+1} = 1$. First, we suppose that $n'_0 = 0$. Set $n' = (010101...)$. Then we can write $V(n) = V(n')$ and $S(n) \geq S(n')$. Now, we calculate $S(n')$. We suppose that $V(n') = 2s$. It is easy to see that

$$\alpha_{2m}(n') = 2^1 + 2^3 + \cdots + 2^{2m-1} = \frac{2^{2m+1} - 2}{3}$$

and

$$\alpha_{2m-1}(n') = \left|2^1 + 2^3 + \cdots + 2^{2m-3} - 2^{2m-1}\right| = \frac{2^{2m} + 2}{3}.$$

Then we can write

$$\begin{aligned} S\left(n'\right) &= \sum_{m=1}^{s} \frac{\alpha_{2m}\left(n'\right)}{2^{2m+1}} + \sum_{m=1}^{s} \frac{\alpha_{2m-1}\left(n'\right)}{2^{2m}} \\ &= \sum_{m=1}^{s} \frac{1}{2^{2m+1}} \frac{2^{2m+1}-2}{3} + \sum_{m=1}^{s} \frac{1}{2^{2m}} \frac{2^{2m}+2}{3} \\ &= \frac{2}{3}s - \frac{2}{3}\sum_{m=1}^{s} \frac{1}{2^{2m+1}} + \frac{2}{3}\sum_{m=1}^{s} \frac{1}{2^{2m}} \\ &= \frac{2}{3}s + \frac{1}{9}\left(1-\frac{1}{2^{2s}}\right). \end{aligned} \tag{3.6}$$

Now, we suppose that $n_0' = 1$. Set $n' = (101010...)$. Then we can write

$$\alpha_{2m}\left(n'\right) = \left|2^0 + 2^2 + \cdots + 2^{2m-2} - 2^{2m}\right| = \frac{2^{2m+1}+1}{3}$$

and

$$\alpha_{2m-1}\left(n'\right) = 2^0 + 2^2 + \cdots + 2^{2m-2} = \frac{2^{2m}-1}{3}.$$

Hence, we have

$$\begin{aligned} S\left(n'\right) &= \sum_{m=0}^{s} \frac{\alpha_{2m}\left(n'\right)}{2^{2m+1}} + \sum_{m=1}^{s} \frac{\alpha_{2m-1}\left(n'\right)}{2^{2m}} \\ &= \frac{2}{3}s + \frac{1}{2} - \frac{1}{18}\left(1-\frac{1}{2^{2s}}\right) > \frac{2}{3}s. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we conclude

$$\frac{S\left(n\right)}{V\left(n\right)} \geq \frac{S\left(n'\right)}{V\left(n'\right)} > \frac{1}{3}. \tag{3.8}$$

Since $T(n,m) \cap A(m) = \varnothing$, we can write

$$\begin{aligned} &\sum_{i=0}^{N-1} \frac{\max\left\{\alpha_i\left(n\right), 2\alpha_i\left(m\right)\right\}}{2^{i+1}} \\ &\geqslant \sum_{i\in A(m)\backslash\{N\}} \frac{2\alpha_i\left(m\right)}{2^{i+1}} + \sum_{i\in T(n,m)} \frac{\alpha_i\left(n\right)}{2^{i+1}} \\ &= \sum_{i\in A(m)} \frac{2\alpha_i\left(m\right)}{2^{i+1}} + \sum_{i\in T(n,m)} \frac{\alpha_i\left(n\right)}{2^{i+1}} - \frac{2\alpha_N\left(m\right)}{2^{N+1}}. \end{aligned} \tag{3.9}$$

From (3.8) we have

$$\sum_{i\in A(m)} \frac{2\alpha_i\left(m\right)}{2^{i+1}} = 2S\left(m\right) > \frac{2}{3}V(m). \tag{3.10}$$

It is easy to see that if $n_i \neq n_{i-1}$, then $\alpha_i\left(n\right) \geq 2^{i-1}$. So, we have

$$\sum_{i\in T(n,m)} \frac{\alpha_i\left(n\right)}{2^{i+1}} \geq \frac{1}{4}\left|T(n,m)\right|. \tag{3.11}$$

Combining (3.9)-(3.11) we have

$$\sum_{i=0}^{N-1}\frac{\max\{\alpha_i(n),2\alpha_i(m)\}}{2^{i+1}}\geq\frac{2}{3}V(m)+\frac{1}{4}|T(n,m)|-1. \tag{3.12}$$

On the other hand, we can write

$$\begin{aligned}&\sum_{i=0}^{N-1}\frac{\max\{\alpha_i(n),2\alpha_i(m)\}}{2^{i+1}} \\ \geqslant&\sum_{i\in A(n)\setminus\{N,N+1\}}\frac{\alpha_i(n)}{2^{i+1}}+\sum_{i\in T(m,n)}\frac{2\alpha_i(m)}{2^{i+1}} \\ \geq&\frac{1}{3}V(n)+\frac{1}{2}|T(m,n)|-\frac{\alpha_N(N)}{2^{N+1}}-\frac{\alpha_{N+1}(N)}{2^{N+2}} \\ \geq&\frac{1}{3}V(n)+\frac{1}{2}|T(m,n)|-1.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) we have

$$\begin{aligned}&\max\left(\frac{1}{2}|T(n,m)|+\frac{1}{3}V(n)-1,\frac{1}{4}|T(n,m)|+\frac{2}{3}V(m))-1\right) \\ \leqslant&\sum_{i=0}^{N-1}\frac{\max\{\alpha_i(n),2\alpha_i(m)\}}{2^{i+1}}\leq L_n^{(t)}.\end{aligned} \tag{3.14}$$

Now, we prove upper estimation. First, we prove

$$\begin{aligned}&\sum_{i=0}^{N-1}\frac{\max\{\alpha_i(n),2\alpha_i(m)\}}{2^{i+1}} \\ \leq&\ 2\sum_{i\in A(m)\cup T(n,m)}\frac{\max\{\alpha_i(n),2\alpha_i(m)\}}{2^{i+1}}.\end{aligned} \tag{3.15}$$

Suppose that $A(m)\cup T(n,m)=\{r_{1<}r_2......<r_s\}$. Then

$$\sum_{i=0}^{N-1}\frac{\max\{\alpha_i(n),2\alpha_i(m)\}}{2^{i+1}}=\sum_{i=1}^{s-1}\sum_{j=r_i}^{r_{i+1}-1}\frac{\max\{\alpha_j(n),2\alpha_j(m)\}}{2^{j+1}}.$$

Let $n_i=n_{i+1}$ for some i. Then it is easy to see that

$$\alpha_{i+1}(n)=\left|2^{i+1}n_{i+1}-2^in_i..-2^0n_0\right|=\left|2^in_i-2^{i-1}n_{i-1}..-2^0n_0\right|=\alpha_i(n).$$

Consequently,

$$\sum_{j=r_i}^{r_{i+1}-1}\frac{\max\{\alpha_j(n),2\alpha_j(m)\}}{2^{j+1}}\leq\frac{\max\{\alpha_{r_i}(n),2\alpha_{r_i}(m)\}}{2^{r_i}}.$$

Hence (3.15) is proved. Since $\alpha_i(n) \leq 2^i \quad (i \in \mathbb{N})$ and $\alpha_i(n) \leq 2^{i-1} \quad (i \notin A(n))$ from (3.15) we have

$$\sum_{i=0}^{N-1} \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}} \leq 2 \sum_{i \in A(m)} \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}} \tag{3.16}$$

$$+2 \sum_{i \in T(n,m)} \frac{\max\{\alpha_i(n), 2\alpha_i(m)\}}{2^{i+1}} \leq 2|A(m)| + |T(n,m)|.$$

It is easy to see that

$$\frac{\left|2^{N+1} - n - 2m + 2t_{N+1}\left(n - 2^N\right)\right|}{2^{N+1}} \tag{3.17}$$

$$+ \frac{\left|n - 2m - 2t_{N+1}\left(n - 2^N\right)\right|}{2^{N+1}} \leq 2.$$

From (3.16) and (3.17)

$$L_n^{(t)} \leq 2V(m) + |T(n,m)| + 2. \tag{3.18}$$

Combining (3.14) and (3.18) we complete the proof of Theorem 3.1. □

4. Uniformly boundedness of conjugate Fejér means

The first result with respect to the a.e. convergence of the Walsh- Fejér means $\sigma_n f$ is due to Fine [3]. Later, Schipp [12] showed that the maximal operator $\sigma^* f := \sup_n |\sigma_n f|$ is of weak type (1, 1), from which the a. e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$ but Fujii [4] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 (see also Simon [13]). Fujii's theorem was extended by Weisz [20]. Namely, he proved that the maximal operator $\sigma^* f$ and the conjugate maximal operator $\widetilde{\sigma}_*^{(t)}$ are bounded from the martingale Hardy space H_p to the space L_p for $p > 1/2$. Simon [15] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In [7] (see also [6], [5]) the first author proved that the maximal operator $\widetilde{\sigma}_*^{(t)}$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

Weisz [20], [21] considered the norm convergence of conjugate Fejӗr means. In particular, the following is true

Theorem A (Weisz). *If $t \in \mathbb{I}$ then*

$$\left\|\widetilde{\sigma}_n^{(t)} f\right\|_{H_p} \leq c_p \|f\|_{H_p}, \quad (f \in H_p),$$

whenever $p > 1/2$.

Since $\|f\|_{H_1} \lesssim 1 + E\left(|f| \log^+ |f|\right)$ and $E|f| \leq \|f\|_{H_1}$, Theorem A implies that the following is true.

Theorem B. *Let $f \in L\log L$ and $t \in \mathbb{I}$. Then*

$$E\left|\widetilde{\sigma}_n^{(t)} f\right| \lesssim 1 + E\left(|f| \log^+ |f|\right). \tag{4.1}$$

On the other hand, for $t = 0$ we have following estimation.

Theorem C. *Let $f \in L_1$. Then*

$$E\left|\widetilde{\sigma}_n^{(t)} f\right| = E\left|\sigma_n f\right| \lesssim E\left|f\right|. \tag{4.2}$$

In this paper we find necessary and sufficient condition on t for which the estimation (4.2) holds for conjugate Fejér means. We also prove that for dyadic irrational t, $L\log L$ is maximal Orlicz space for which the estimation (4.1) is valid.

Theorem 4.1. *Let $t \in \mathbb{Q}$ and $f \in L_1$. Then*$E\left|\widetilde{\sigma}_n^{(t)} f\right| \lesssim E\left|f\right|$.

Theorem 4.2. *Let $t \notin \mathbb{Q}$ and $Q(L)$ be an Orlicz space for which*$Q(L) \varsubsetneq L\log L$. *Then*

$$\sup_A \left\|\widetilde{\sigma}_{2^A}^{(t)}\right\|_{Q(L)\to L_1} = \infty.$$

Proof of Theorem 4.1. Let $2^A \le n < 2^{A+1}$. Then we can write

$$\widetilde{\sigma}_n^{(t)} f = \frac{1}{n}\sum_{m=1}^{A} \sum_{k=2^{m-1}}^{2^m-1} \widetilde{S}_k^{(t)} f + \frac{1}{n}\sum_{k=2^A}^{n-1} \widetilde{S}_k^{(t)} f. \tag{4.3}$$

Since for $2^{m-1} \le k < 2^m$ $\quad (E_{-1} f = 0)$

$$\begin{aligned} \widetilde{S}_k^{(t)} f &= r_0(\rho(t)) \widehat{f}(0) w_0 + \sum_{l=1}^{m-1} r_l(\rho(t)) (E_l f - E_{l-1} f) \\ &\quad + r_m(\rho(t)) (S_k f - E_{m-1} f) \\ &= \sum_{l=0}^{m-1} r_l(\rho(t)) (E_l f - E_{l-1} f) + r_m(\rho(t)) (S_k f - E_{m-1} f) \end{aligned} \tag{4.4}$$

from (4.3) we have

$$\begin{aligned} \widetilde{\sigma}_n^{(t)} f &= \frac{1}{n}\sum_{m=1}^{A} 2^{m-1} \sum_{l=0}^{m-1} r_l(\rho(t)) (E_l f - E_{l-1} f) \\ &\quad + \frac{1}{n}\sum_{m=1}^{A} r_m(\rho(t)) \sum_{k=2^{m-1}}^{2^m-1} (S_k f - E_{m-1} f) \\ &\quad + \frac{n-2^A}{n} \sum_{l=0}^{A} r_l(\rho(t)) (E_l f - E_{l-1} f) \\ &\quad + \frac{r_{A+1}(\rho(t))}{n} \sum_{k=2^A}^{n-1} (S_k f - E_{m-1} f) \\ &= \frac{1}{n}\sum_{m=1}^{A} 2^{m-1} \sum_{l=0}^{m-1} r_l(\rho(t)) (E_l f - E_{l-1} f) \end{aligned}$$

$$+\frac{1}{n}\sum_{m=1}^{A} r_m\left(\rho\left(t\right)\right)\left(2^m\sigma_{2^m}f-2^{m-1}\sigma_{2^{m-1}}f\right)$$
$$-\frac{1}{n}\sum_{m=1}^{A} r_m\left(\rho\left(t\right)\right)2^{m-1}E_{m-1}f$$
$$+\frac{n-2^A}{n}\sum_{l=0}^{A} r_l\left(\rho\left(t\right)\right)\left(E_lf-E_{l-1}f\right)$$
$$+\frac{r_{A+1}\left(\rho\left(t\right)\right)}{n}\left(n\sigma_nf-2^A\sigma_{2^A}f\right)$$
$$-\frac{r_{A+1}\left(\rho\left(t\right)\right)}{n}\left(n-2^A\right)E_{m-1}f=:\sum_{j=1}^{6}J_jf. \tag{4.5}$$

Since

$$E\left|E_mf\right|\leq E\left|f\right| \tag{4.6}$$

and

$$E\left|\sigma_nf\right|\lesssim E\left|f\right| \tag{4.7}$$

we can write

$$E\left|J_jf\right|\lesssim E\left|f\right|, j=2,3,5,6. \tag{4.8}$$

For J_1f we can write

$$J_1f=\frac{1}{n}\sum_{m=1}^{A}2^{m-1}\widetilde{E}_m^{(t)}f,$$

where

$$\widetilde{E}_m^{(t)}f:=\sum_{l=0}^{m-1}r_l\left(\rho\left(t\right)\right)\left(E_lf-E_{l-1}f\right). \tag{4.9}$$

Since $E_lf=f*D_{2^l}$ we have

$$\widetilde{E}_m^{(t)}f=f*\sum_{l=0}^{m-1}r_l\left(\rho\left(t\right)\right)\left(D_{2^l}-D_{2^{l-1}}\right):=f*\widetilde{D}_{2m}^{(t)}, \tag{4.10}$$

where

$$\widetilde{D}_{2^m}^{(t)}:=\sum_{l=0}^{m-1}r_l\left(\rho\left(t\right)\right)\left(D_{2^l}-D_{2^{l-1}}\right).$$

It is easy to see that $r_l(t) = (-1)^{t_l} = 1 - 2t_l$. Then for $\widetilde{D}_{2^l}^{(t)}$ we can write

$$\begin{aligned}\widetilde{D}_{2^l}^{(t)} &= \sum_{l=0}^{m-1} (1-2t_l)\left(D_{2^l} - D_{2^{l-1}}\right) \\ &= D_{2^{m-1}} - 2\sum_{l=0}^{m-1} t_l \left(D_{2^l} - D_{2^{l-1}}\right) \\ &= (1-2t_{m-1}) D_{2^{m-1}} - 2\sum_{l=0}^{m-2} (t_l - t_{l+1}) D_{2^l}.\end{aligned}$$

Consequently,

$$f * \widetilde{D}_{2^m}^{(t)} = (1-2t_{m-1}) E_{m-1} f - 2\sum_{l=0}^{m-2} (t_l - t_{l+1}) E_l f. \tag{4.11}$$

$t \in \mathbb{Q}$ imply that

$$\sum_{l=0}^{\infty} |t_l - t_{l+1}| < \infty.$$

From (4.6) we get

$$E\left|\widetilde{E}_m^{(t)} f\right| \lesssim \left(|2t_{m-1} - 1| + \sum_{l=0}^{\infty} |t_l - t_{l+1}|\right) E|f| \lesssim E|f|.$$

Consequently,

$$E|J_1 f| = \frac{1}{n}\sum_{m=1}^{A} 2^{m-1} E\left|\widetilde{E}_m^{(t)} f\right| \lesssim E|f|. \tag{4.12}$$

Combining (4.5)-(4.12) we complete the proof of Theorem 4.1. □

Proof of Theorem 4.2. Since $t \notin \mathbb{Q}$ there exists a sequences $\{p_i : i \in \mathbb{P}\}$ and $\{q_i : i \in \mathbb{P}\}$ such that

$$0 \le q_1 \le p_1 < q_2 \le p_2 < \cdots < q_A \le p_A < \cdots$$

and

$$t_j = \begin{cases} 1, q_i \le j \le p_i \\ 0, p_i < j < q_{i+1} \end{cases}, i = 1, 2, ...$$

Set

$$\begin{aligned}\Delta_A \ :&= \bigcup_{t_j = 0 \text{ or } 1, j \in \{0,1,\dots,p_A\} \setminus \{q_1,p_1,\dots,q_A,p_A\}} \\ & I_{p_A+1}\left(t_0, ..., t_{q_1-1}, 0, t_{q_1+1}, ..., t_{p_1-1}, 0, t_{p_1+1}, ..., t_{q_A-1}, 0, t_{q_A+1}, ..., t_{p_A-1}, 0\right).\end{aligned}$$

Define the function

$$f_A(x) := 2^{2A} \mathbb{I}_{\Delta_A}(x),$$

where $\mathbb{I}_E$ is characteristic function of the set E. It is easy to see that

$$\mu(\Delta_A) = \frac{2^{p_A+1-2A}}{2^{p_A+1}} = \frac{1}{2^{2A}}. \tag{4.13}$$

We can write (see (4.5) and (4.9))

$$\widetilde{\sigma}^{(t)}_{2^{2p_A+1}} f_A = \frac{1}{2^{2p_A+1}} \sum_{m=1}^{2p_A+1} 2^{m-1} \widetilde{E}^{(t)}_m f_A \tag{4.14}$$
$$+\frac{1}{2^{2p_A+1}} \sum_{m=1}^{2p_A+1} r_m\left(\rho\left(t\right)\right)\left(2^m \sigma_{2^m} f_A - 2^{m-1}\sigma_{2^{m-1}} f_A\right)$$
$$-\frac{1}{2^{2p_A+1}} \sum_{m=1}^{2p_A+1} r_m\left(\rho\left(t\right)\right) 2^{m-1} E_{m-1} f_A =: F_1 f_A + F_2 f_A + F_3 f_A.$$

From (4.6), (4.7) and (4.13) we have

$$E\left|F_j f_A\right| \lesssim E\left|f_A\right| \lesssim 1, j = 2, 3. \tag{4.15}$$

Set

$$\widetilde{\Delta}_i := I_{p_i+1}\left(x_0, ..., x_{q_1-1}, 0, x_{q_1+1}, ..., x_{p_1-1}, 0, x_{p_1+1}, ..., x_{q_i-1}, 0, x_{q_i+1}, ..., x_{p_i-1}, 1\right).$$

Suppose that $x \in \widetilde{\Delta}_i$ for some $i = 1, 2, ..., A$. Then

$$E_a f_A\left(x\right) = 2^a \int\limits_{I_a(x)} f_A\left(s\right) d\mu\left(s\right) = 0, a > p_i.$$

Therefore, for $m > p_A$ we obtain (see 4.11)

$$\widetilde{E}^{(t)}_m f_A\left(x\right) = -2\sum_{a=0}^{p_i}\left(t_a - t_{a+1}\right) E_a f_A\left(x\right) \tag{4.16}$$
$$= \sum_{k=1}^{i}\left[2E_{q_k-1} f_A\left(x\right) - 2E_{p_k} f_A\left(x\right)\right]$$
$$= \sum_{k=1}^{i}\left[2^{q_k+2A}\mu\left(I_{q_k-1}\left(x\right)\cap\Delta_A\right) - 2^{p_k+1+2A}\mu\left(I_{p_k}\left(x\right)\cap\Delta_A\right)\right].$$

it is easy to calculate that

$$\mu\left(I_{q_k-1}\left(x\right)\cap\Delta_A\right) = \frac{2^{p_A-(q_k-1)-2(A-k+1)}}{2^{p_A+1}} = 2^{-q_k-2(A-k)-2}$$

and

$$\mu\left(I_{p_k}\left(x\right)\cap\Delta_A\right) = \frac{2^{p_A-p_k-[2(A-k)+1]}}{2^{p_A+1}} = 2^{-p_k-2(A-k)-2}.$$

Hence, from (4.16) for $m \geq p_A + 1$ and $x \in \widetilde{\Delta}_i, i = 1, 2, ..., A$ we have

$$\widetilde{E}^{(t)}_m f_A \tag{4.17}$$
$$= \sum_{k=1}^{i}\left[2^{q_k+2A}\cdot 2^{-q_k-2(A-k)-2} - 2^{p_k+1+2A}\cdot 2^{-p_k-2(A-k)-2}\right]$$
$$= \sum_{k=1}^{i}\left[2^{2k-2} - 2^{2k-1}\right] = -\frac{2^{2i}-4}{3}.$$

Consequently, for $x \in \widetilde{\Delta}_i$ we get

$$\begin{aligned} \frac{1}{2^{2p_A+1}} \sum_{m=p_A+1}^{2p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A &= -\frac{1}{2^{2p_A+1}} \sum_{m=p_A+2}^{2p_A+1} 2^{m-1} \frac{2^{2i}-4}{3} \\ = \quad & -\frac{1}{2^{2p_A+1}} \left(2^{2p_A+1} - 2^{p_A+1}\right) \frac{2^{2i}-4}{3}. \end{aligned}$$

Since

$$E\left|\widetilde{E}_m^{(t)} f_A\right| \quad \leq \quad E\left|E_m f_A\right| + 2\sum_{a=0}^{m-1} E\left|E_a f_A\right| \lesssim mE\left|f_A\right| \lesssim m$$

and

$$F_1 f_A = \frac{1}{2^{2p_A+1}} \sum_{m=1}^{p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A + \frac{1}{2^{2p_A+1}} \sum_{m=p_A+2}^{2p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A,$$

we have

$$\begin{aligned} E\left|F_1 f_A\right| \quad &\geq \quad E\left|\frac{1}{2^{2p_A+1}} \sum_{m=p_A+2}^{2p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A\right| \\ &\quad -\frac{1}{2^{2p_A+1}} \sum_{m=1}^{p_A+1} 2^{m-1} E\left|\widetilde{E}_m^{(t)} f_A\right| \\ &\geq \quad \sum_{i=1}^{A} \int_{\widetilde{\Delta}_i} \left|\frac{1}{2^{2p_A+1}} \sum_{m=p_A+1}^{2p_A+1} 2^{m-1} \widetilde{E}_m^{(t)} f_A\right| d\mu \\ &\quad -\frac{1}{2^{2p_A+1}} \sum_{m=1}^{p_A+1} 2^{m-1} m \\ &\gtrsim \quad \sum_{i=1}^{A} 2^{2i} \mu\left(\widetilde{\Delta}_i\right) - \frac{p_A 2^{p_A}}{2^{2p_A+1}} \gtrsim A. \end{aligned} \tag{4.18}$$

Combining (4.14), (4.15) and (4.18) we have

$$E\left|\widetilde{\sigma}_{2^{2p_A+1}}^{(t)} f_A\right| \gtrsim A. \tag{4.19}$$

Let

$$Q\left(2^{2A}\right) \geq 2^{2A}, \qquad A \geq A_0. \tag{4.20}$$

By virtue of estimate (see ([9]))$\|f_A\|_{Q(L)} \leq 1 + E\left|Q\left(f_A\right)\right|$, from (4.13) and (4.20) we can write

$$\begin{aligned} E\left|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}} f_A\right| &\leq \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \|f_A\|_{Q(L)} \\ &\leq \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \left(1 + E\left|Q\left(f_A\right)\right|\right) \\ &= \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \left(1 + Q\left(2^{2A}\right)\mu\left(\Delta_A\right)\right) \\ &= \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \left(1 + \frac{Q\left(2^{2A}\right)}{2^{2A}}\right) \\ &\lesssim \left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \frac{Q\left(2^{2A}\right)}{2^{2A}}, A \geq A_0. \end{aligned}$$

Consequently, by (4.19) we have

$$\left\|\widetilde{\sigma}^{(t)}_{2^{2p_A+1}}\right\|_{Q(L)\to L_1} \gtrsim \frac{A2^{2A}}{Q\left(2^{2A}\right)}, \quad A \geq A_0. \tag{4.21}$$

The fact that$Q\left(L\right) \nsubseteq L\log L$ is equalent to the condition

$$\overline{\lim_{u\to\infty}} \frac{u\log u}{Q\left(u\right)} = \infty.$$

Then there exists $\{u_k : k \in \mathbb{P}\}$ such that

$$\lim_{k\to\infty} \frac{u_k \log u_k}{Q\left(u_k\right)} = \infty, \quad u_{k+1} > u_k, k = 1, 2, ...$$

and a monotonically increasing sequence of positive integers $\{A_k : k \in \mathbb{P}\}$ such that

$$2^{2A_k} \leq u_k < 2^{2(A_k+1)}.$$

Then we have

$$\frac{2^{2A_k} A_k}{Q\left(2^{2A_k}\right)} \gtrsim \frac{u_k \log u_k}{Q\left(u_k\right)} \to \infty \quad \text{as} \quad k \to \infty.$$

Then, from (4.21) we conclude that

$$\sup_k \left\|\widetilde{\sigma}^{(t)}_{2^{2p_{A_k}+1}}\right\|_{Q(L)\to L_1} = \infty.$$

Theorem 4.2 is proved. □

Acknowledgements

The authors would like to thank the referee for valuable remarks and suggestions.

Список литературы

[1] S. V. Astashkin, E. M. Semenov, "Lebesgue constants of a Walsh system and Banach limits" [in Russian]; translated from Sibirsk. Mat. Zh. **57**, no. 3, 512 – 526 (2016), Sib. Math. J. **57**, no. 3, 398 – 410 (2016).

[2] N. J. Fine, "On the Walsh functions", Trans. Am. Math. Soc. **65** (3), 372 – 414 (1949).

[3] N. J. Fine, "Cesàro summability of Walsh-Fourier series", Proc. Nat. Acad. Sci. USA **41**, 558 – 591 (1955).

[4] N. J. Fujii, “Cesàro summability of Walsh-Fourier series”, Proc. Amer. Math. Soc. **77**, 111 – 116 (1979).
[5] U. Goginava, “The maximal operator of the Marcinkiewicz- Fejér means of the d-dimensional Walsh-Fourier series”, East J. Approx. **3**, no. 12, 295 – 302 (2006).
[6] U. Goginava, “The martingale Hardy type inequality for Marcinkiewicz- Fejér means of two-dimensional conjugate Walsh-Fourier series”, Acta Math. Sin. (Engl. Ser.), **27**, no. 10, 1949 – 1958 (2011).
[7] U. Goginava, “The maximal operator of the (C, α) means of the Walsh-Fourier series”, Ann. Univ. Sci. Budapest. Sect. Comput. **26**, 127 – 135 (2006).
[8] B. I. Golubov, A. V. Efimov and V. A. Skvortsov, Series and Transformations of Walsh [in Russian], Nauka, Moscow, (1987); English transl.: Kluwer Acad. publ. (1991).
[9] M. A. Krasnoselski, Ya. B. Rutitski, Convex Functions and Orlicz Spaces [in Russian], Moscow (1958).
[10] S. F. Lukomskii, “Convergence of multiple Walsh series in measure and in L”, East J. Approx. **3** (3), 101 – 116 (1997).
[11] Yu. V. Malykhin, S. A. Telyakovskii and N. N. Kholshchevnikova, “Integrability of the sum of absolute values of blocks of the Fourier-Walsh series for functions of bounded variation”, Translation of Tr. Mat. Inst. Steklova **290**, 323 – 334 (2015). Proc. Steklov Inst. Math. **290**, no. 1, 306 – 317 (2015).
[12] F. Schipp, “Certain rearrangements of series in the Walsh series”, Mat. Zametki **18**, 193 – 201 (1975).
[13] P. Simon, “Investigation with respect to the Vilenkin system”, Ann. Univ. Sci. Sect. Math. (Budapest) **27**, 87 – 101 (1985).
[14] F. Schipp, W.R. Wade, P. Simon and J. Pol, Walsh Series, an Introduction to Dyadic Harmonic Analysis. Adam Hilger, Bristol, New York (1990).
[15] P. Simon, “Cesaro summability with respect to two-parameter Walsh system”, Monatsh. Math. **131**, 321 – 334 (2000).
[16] G. Szegx, “Uber die Lebesgueschen Konstanten bei den Fourierschen Reihen”, (German) Math. Z. **9**, no. 1 - 2, 163 – 166 (1921).
[17] R. Toledo, “On the boundedness of the L_1-norm of Walsh- Fejér kernels”, J. Math. Anal. Appl. **457**, no. 1, 153 – 178 (2018).
[18] F. Weisz, Martingale Hardy Spaces and Their Applications in Fourier Analysis, Springer, Berlin - Heidelberg - New York (1994).
[19] F. Weisz, Summability of Multi-Dimensional Fourier Series and Hardy Space, Kluwer Academic, Dordrecht (2002).
[20] F. Weisz, “Cesaro summability of one and two-dimensional Walsh-Fourier series”, Anal. Math. **22**, 229 – 242 (1996).
[21] F. Weisz, “(C, α) summability of Walsh-Fourier series”, Anal. Math. **27**, no. 2, 141 – 155 (2001).

Поступила 25 октября 2019

После доработки 05 апреля 2020

Принята к публикации 10 июля 2020

Известия НАН Армении, Математика, том 56, н. 1, 2021, стр. 35 – 44.

ON THE PROBLEM OF CONVERGENCE OF DOUBLE FUNCTIONAL SERIES TO $+\infty$

L. GOGOLADZE

I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: *lgogoladze1@hotmail.com*

Abstract. The problem, posed by N. N. Luzin in 1915 on the convergence of trigonometric series to $+\infty$ on the set of positive measure, was solved by S. V. Konyagin in 1988. There naturally arises the question of N. N. Luzin for double trigonometric series. In the present paper the theorem is proved which contains the answer to this problem.

MSC2010 numbers: 40B05.
Keywords: double series; convergence to $+\infty$.

The question whether the trigonometric series converges to $+\infty$ on the set of positive measure was posed by N. N. Luzin (see [1, p. 239]) in 1915. Yu. B. Germier [2] showed that the trigonometric series cannot be summable by the Riemmann method to $+\infty$ on the set of positive measure. At the same time N. N. Luzin and I. I. Privalov [3] have constructed an example of the trigonometric series, summable almost everywhere to $+\infty$ by the Abel method. P. L. Ulyanov [4] has proved that if E is a subset of the second category of the non-degenerate interval $(a, b) \subset [-\pi, \pi]$ such that $(E \cap (c, d)) > 0$ for every non-degenerate interval $(c, d) \subset (a, b)$, then there does not exist the trigonometric series, converging to $+\infty$ or to $-\infty$ when $x \in E$.

D. E. Menshov [5] has shown that there exists the trigonometric series, converging in measure to $+\infty$ on $[-\pi, \pi]$. A. A. Talalyan and F. G. Arutyunyan [6] have shown that the series with respect to the Haar system cannot converge to $+\infty$ on the set of positive measure. V. A. Skvortsov [7] has given a simpler proof of this result. At the same time P. L. Ulyanov [8], R. I. Ovsepyan and A. A. Talalyan [9] have shown that there exist uniformly bounded orthonormal systems of functions such that the series with respect to them converge to $+\infty$ on the set of positive measure.

In 1988, S. V. Konyagin [10] proved

Theorem (S. V. Konyagin)**.** *For every trigonometric series the equality*

$$\operatorname{mes}\{x : x \in [-\pi, \pi],\ -\infty < \varliminf_{m\to\infty} S_m(x) \le \varlimsup_{m\to\infty} S_m(x) = +\infty\} = 0$$

is true, where $S_m(x)$ are partial sums of this series.

It is clear that this gives a negative answer to N. N. Luzin's problem.

There naturally arises N. N. Luzin's problem for double trigonometric series. Below we will prove the theorem, containing an answer to this problem.

In the sequel we use the following simple

Lemma 1. *Let S_{m_1,m_2} be a double sequence of numbers and*

$$\lim_{\substack{m_1\to\infty \\ m_2\to\infty}} S_{m_1,m_2} = +\infty.$$

If there exists

$$\lim_{m_2\to\infty} S_{m_1,m_2},$$

then

$$\lim_{m_1\to\infty}\Big(\lim_{m_2\to\infty} S_{m_1,m_2}\Big) = +\infty.$$

Proof. Let $M > 0$ be an arbitrary number. Then there exists a number N such that

$$S_{m_1,m_2} \geq M \quad \text{when } \ m_1 > N, \ \ m_2 > N.$$

Going to the limit in this inequality when $m_2 \to \infty$, we get

$$\lim_{m_2\to\infty} S_{m_1,m_2} \geq M \quad \text{when } \ m_1 > N.$$

This yields the validity of the lemma. □

Consider the systems of functions

$$\big\{\varphi^{(j)}_{n_j}(x_j)\big\}^{\infty}_{n_j=1}, \quad x_j \in [0,1], \ \ j = 1,2, \tag{1}$$

where every function $\varphi^{(j)}_{n_j}(x_j)$, $j = 1,2$, is measurable and takes finite values, and the series

$$\sum_{n_j=1}^{\infty} a^{(j)}_{n_j}\varphi^{(j)}_{n_j}(x_j), \ \ j = 1,2, \tag{2}$$

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} c_{n_1,n_2}\varphi^{(1)}_{n_1}(x_1)\varphi^{(2)}_{n_2}(x_2). \tag{3}$$

Let

$$S^{(j)}_{m_j} = \sum_{n_j=1}^{m_j} a^{(j)}_{n_j}\varphi^{(j)}_{n_j}(x_j),$$

$$S_{m_1,m_2}(x_1,x_2) = \sum_{n_1=1}^{m_1}\sum_{n_2=1}^{m_2} c_{n_1,n_2}\varphi^{(1)}_{n_1}(x_1)\varphi^{(2)}_{n_2}(x_2)$$

be, respectively, partial sums of series (2) and (3).

Definition. It is said that system $\{\psi_n(x)\}$ has the property K if the following equality holds

$$\operatorname{mes}\Big\{x \in [0,1]: \ -\infty < \underline{\lim}\, S_m(x) \le \overline{\lim}\, S_m(x) = +\infty\Big\} = 0$$

for each series $\sum\limits_{n=1}^{\infty} a_n \psi_n(x)$, where

$$S_m(x) = \sum_{n=1}^{m} a_n \psi_n(x).$$

Theorem 1. *Let the systems* (1) *have the property K, then every series* (3) *will not be convergent to $+\infty$ on the set $E \subset [0,1]^2$, $\mu_2 E > 0$.*

Proof. Assume the contrary, i.e. there exists a series (3) such that

$$\lim_{\substack{m_1\to\infty\\ m_2\to\infty}} S_{m_1,m_2}(x_1,x_2) = +\infty \tag{4}$$

on some set E, $\mu_2 E > 0$. Then there exists a natural number N and a set $E_1 \subset E$, $\mu_2 E_1 > 0$, such that

$$S_{m_1,m_2}(x_1,x_2) \ge 0, \quad m_j \ge N, \ \ j = 1,2, \ \ (x_1,x_2) \in E_1. \tag{5}$$

Hence for $(x_1,x_2) \in E_1$ and fixed $m_2 \ge N$

$$\underline{\lim}_{m_1\to\infty} S_{m_1,m_2}(x_1,x_2) > -\infty. \tag{6}$$

Let

$$\overline{\lim}_{m_1\to\infty} S_{m_1,m_2}(x_1,x_2) = f_{m_2}(x_1,x_2), \ \ (x_1,x_2) \in E_1, \ \ m_2 \ge N. \tag{7}$$

Since $S_{m_1,m_2}(x_1,x_2)$ is measurable, the function $f_{m_2}(x_1,x_2)$ are also measurable and, consequently, the sets

$$A_{m_2} = \{(x_1,x_2) : (x_1,x_2) \in E_1, \ \ f_{m_2}(x_1,x_2) < +\infty\},$$
$$B_{m_2} = \{(x_1,x_2) : (x_1,x_2) \in E_1, \ \ f_{m_2}(x_1,x_2) = +\infty\}$$

are measurable. Besides, let for $x_2 \in [0,1]$ and $m_2 \ge N$

$$C_{m_2}(x_2) = \{x_1 : (x_1,x_2) \in B_{m_2}\}.$$

As far as the system

$$\big\{\varphi^{(1)}_{m_1}(x_1)\big\}_{m_1=1}^{\infty}$$

has the property K, from (6) for almost every $x_2 \in [0,1]$ we will have

$$\mu_1 C_{m_2}(x_2) = 0.$$

It follows that

$$\mu_2 B_{m_2} = 0.$$

Let

$$E_2 = \bigcap_{m_2=N}^{\infty} A_{m_2}.$$

Then $E_2 \subset E_1$ and

$$\mu_2 E_2 = \mu_2 E_1 > 0.$$

Thus at every point of the set E_2

$$0 \le \overline{\lim_{m_1\to\infty}} S_{m_1,m_2}(x_1,x_2) < +\infty, \quad m_2 \ge N. \tag{8}$$

Now let

$$\overline{\lim_{m_2\to\infty}} S_{m_1,m_2}(x_1,x_2) = h_{m_1}(x_1,x_2), \quad (x_1,x_2)\in E_2, \quad m_1 \ge N.$$

By the same reasoning as was made from equality (6) to equality (8) it comes out that there exists a set $E_3 \subset E_2$, $\mu_2 E_3 = \mu_2 E_2 > 0$, such that the inequality (8) for $(x_1,x_2)\in E_3$ and the following inequality holds

$$0 \le \overline{\lim_{m_2\to\infty}} S_{m_1,m_2}(x_1,x_2) < +\infty, \quad (x_1,x_2)\in E_3, \quad m_1 \ge N. \tag{9}$$

It can be easily seen that when $m_1 \ge N$,

$$\begin{aligned} &S_{m_1,m_2}(x_1,x_2) - S_{N,m_2}(x_1,x_2) \\ &= \sum_{n_1=N+1}^{m_1}\sum_{n_2=1}^{m_2} c_{n_1,n_2}\varphi_{n_1}^{(1)}\varphi_{n_2}^{(2)} = S^*_{m_1,m_2}(x_1,x_2), \end{aligned} \tag{10}$$

where $S^*_{m_1,m_2}(x_1,x_2)$ are partial sums of the following series

$$\sum_{n_1=N+1}^{\infty}\sum_{n_2=1}^{\infty} c_{n_1,n_2}\varphi_{n_1}^{(1)}(x_1)\varphi_{n_2}^{(2)}(x_2). \tag{11}$$

From (4), (9) when $m_1 = N$ and from (10) we get

$$\lim_{\substack{m_1\to\infty\\ m_2\to\infty}} S^*_{m_1,m_2}(x_1,x_2) = +\infty, \quad (x_1,x_2)\in E_3, \tag{12}$$

i.e. the series (11) converges to $+\infty$ on the set E_3, $\mu_2 E_3 > 0$.

Let

$$E_4(x_2^0) = \{x_1 : (x_1,x_2^0)\in E_3\},$$

where x_2^0 is chosen so that

$$\mu_1 E_4(x_2^0) > 0.$$

From (12) and $E_4(x_2^0) \subset E_3$ it follows that

$$\lim_{\substack{m_1\to\infty\\ m_2\to\infty}} S^*_{m_1,m_2}(x_1,x_2^0) = \sum_{n_1=N+1}^{\infty}\sum_{n_2=1}^{\infty} c_{n_1,n_2}\varphi_{n_1}^{(1)}(x_1)\varphi_{n_2}^{(2)}(x_2^0) = +\infty. \tag{13}$$

Since $E_4(x_2^0) \subset E_3$, from (9) we have

$$0 \le \overline{\lim_{m_2\to\infty}} S_{m_1,m_2}(x_1,x_2^0) < +\infty, \quad m_1 \ge N. \tag{14}$$

Besides (see (5), (14)), we will have

$$\begin{aligned}
&\overline{\lim_{m_2\to\infty}}\left|\varphi^{(1)}_{m_1}(x_1)\sum_{n_2=1}^{m_2} c_{m_1,n_2}\varphi^{(2)}_{n_2}(x_1,x_2^0)\right| \\
&= \overline{\lim_{m_2\to\infty}}\left|S_{m_1,m_2}(x_1,x_2^0)-S_{m_1-1,m_2}(x_1,x_2^0)\right| \\
&\le \overline{\lim_{m_2\to\infty}}\left|S_{m_1,m_2}(x_1,x_2^0)\right| + \overline{\lim_{m_2\to\infty}}\left|S_{m_1-1,m_2}(x_1,x_2^0)\right| \\
&= \overline{\lim_{m_2\to\infty}} S_{m_1,m_2}(x_1,x_2^0) + \overline{\lim_{m_2\to\infty}} S_{m_1-1,m_2}(x_1,x_2^0) < +\infty, \quad m_1 > N. \qquad (15)
\end{aligned}$$

Without loss of generality, we will assume that for every $m_1 \ge N+1$ there exists $x_1(m_1)$ depending on m_1 such that $\varphi^{(1)}_{m_1}(x_1(m_1)) \ne 0$ (in fact, otherwise, the member with this index m_1 in series (13) will be equal to zero on the set $E_4(x_2^0)$ and if we remove it, the sum of series (13) will not be changed). Therefore from (15) we get

$$-\infty < \overline{\lim_{m_2\to\infty}}\sum_{n_2=1}^{m_2} c_{m_1,n_2}\varphi^{(2)}_{n_2}(x_2^0) < +\infty, \quad m_1 \ge N+1.$$

Since the sequence

$$\sum_{n_2=1}^{m_2} c_{m_1,n_2}\varphi^{(2)}_{n_2}(x_2^0)$$

is bounded for every fixed $m_1 \ge N+1$, we can select for $m_1 = N+1$ a convergent subsequence $m_2(1,p)$. Let

$$\lim_{m_2(1,p)\to\infty}\sum_{n_2=1}^{m_2(1,p)} c_{m_1,n_2}\varphi^{(2)}_{n_2}(x_2^0) = d_{m_1}, \quad m_1 = N+1.$$

Further, from the subsequence $m_2(1,p)$ we select a convergent subsequence $m_2(2,p)$. Let

$$\lim_{m_2(2,p)\to\infty}\sum_{n_2=1}^{m_2(2,p)} c_{m_1,n_2}\varphi^{(2)}_{n_2}(x_2^0) = d_{m_1}, \quad m_1 = N+1, N+2.$$

Proceeding in this way and making use of the Cantor diagonal method, we will have

$$\lim_{m_2(p,p)\to\infty}\sum_{n_2=1}^{m_2(p,p)} c_{m_1,n_2}\varphi^{(2)}_{n_2}(x_2^0) = d_{m_1}, \quad m_1 \ge N+1. \qquad (16)$$

From (13) we get

$$\lim_{\substack{m_1\to\infty\\ m_2(p,p)\to\infty}} S^*_{m_1,m_2(p,p)}(x_1,x_2^0) = +\infty.$$

From here, by using Lemma 1 on the double sequence of numbers and (16), we will have

$$\lim_{\substack{m_1\to\infty\\ m_2(p,p)\to\infty}} S^*_{m_1,m_2(p,p)}(x_1,x_2^0) = \lim_{\substack{m_1\to\infty\\ m_2(p,p)\to\infty}}\sum_{n_1=N+1}^{m_1}\sum_{n_2=1}^{m_2(p,p)} c_{n_1,n_2}\varphi^{(1)}_{n_1}(x_1)\varphi^{(2)}_{n_2}(x_2^0)$$

$$= \lim_{m_1\to\infty}\Big(\lim_{m_2(p,p)\to\infty}\sum_{n_1=N+1}^{m_1}\sum_{n_2=1}^{m_2(p,p)} c_{n_1,n_2}\varphi_{n_1}^{(1)}(x_1)\varphi_{n_2}^{(2)}(x_2^0)\Big)$$

$$= \lim_{m_1\to\infty}\sum_{n_1=N+1}^{m_1}\varphi_{n_1}^{(1)}(x_1)\lim_{m_2(p,p)\to\infty}\sum_{n_2=1}^{m_2(p,p)} c_{n_1,n_2}\varphi_{n_2}^{(2)}(x_2^0)$$

$$= \lim_{m_1\to\infty}\sum_{n_1=N+1}^{m_1} d_{n_1}\varphi_{n_1}^{(1)}(x_1) = +\infty,\quad x_1\in E_4(x_2^0),$$

i.e. the series

$$\sum_{n_1=N+1}^{\infty} d_{n_1}\varphi_{n_1}^{(1)}(x_1)$$

converges to $+\infty$ on the set $E_4(x_2^0)$, $\mu_1 E_4(x_2^0) > 0$. This contradicts the fact that the system

$$\{\varphi_{n_1}^{(1)}(x_1)\}$$

has the property K. □

Lemma 2. *Assume that for some natural* m_1 *and for the set* $E\subset[0,1]$, $\mu E>0$, *the condition*

$$\overline{\lim_{m_2\to\infty}}\,\big|A_{m_1,m_2}\cos m_1 2\pi x_1 + B_{m_1,m_2}\sin m_1 2\pi x_1\big| < \infty,\quad x_1\in E,$$

holds. Then

$$\overline{\lim_{m_2\to\infty}}\,\big|A_{m_1,m_2}\big|<\infty,\quad \lim_{m_2\to\infty}\big|B_{m_1,m_2}\big|<\infty.$$

Proof. Assume the opposite. Then there exists a subsequence $m_2(k)$ such that at every point $x_1\in E$,

$$\lim_{m_2(k)\to\infty}\frac{A_{m_1,m_2(k)}\cos m_1 2\pi x_1 + B_{m_1,m_2(k)}\sin m_1 2\pi x_1}{C_{m_1,m_2(k)}} = 0, \tag{17}$$

where

$$C_{m_1,m_2(k)} = \max\big(|A_{m_1,m_2(k)}|,|B_{m_1,m_2(k)}|\big). \tag{18}$$

From (18) it follows that

$$\begin{gathered}\frac{|A_{m_1,m_2(k)}|}{C_{m_1,m_2(k)}}\le 1,\quad \frac{|B_{m_1,m_2(k)}|}{C_{m_1,m_2(k)}}\le 1,\\ \frac{1}{C_{m_1,m_2(k)}}\big(|A_{m_1,m_2(k)}|+|B_{m_1,m_2(k)}|\ge 1.\end{gathered} \tag{19}$$

Without loss of generality we will assume that there exist the limits

$$\lim_{m_2(k)\to\infty}\frac{A_{m_1,m_2(k)}}{C_{m_1,m_2(k)}} = a_{m_1},\quad \lim_{m_2(k)\to\infty}\frac{B_{m_1,m_2(k)}}{C_{m_1,m_2(k)}} = b_{m_1}. \tag{20}$$

Then from (17)–(20) it follows that when $x_1\in E$, $\mu E>0$,

$$a_{m_1}\cos m_1 2\pi x_1 + b_{m_1}\sin m_1 2\pi x_1 = 0,$$

$$|a_{m_1}| + |b_{m_1}| \geq 1.$$

We have obtained the contradiction. □

The general double trigonometric series has the following form:

$$\sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \big(a_{n_1,n_2} \cos n_1 2\pi x_1 \cos n_2 2\pi x_2 + b_{n_1,n_2} \sin n_1 2\pi x_1 \cos n_2 2\pi x_2$$
$$+ c_{n_1,n_2} \cos n_1 2\pi x_1 \sin n_2 2\pi x_2 + d_{n_1,n_2} \sin n_2 2\pi x_1 \sin n_2 2\pi x_2\big), \quad (x_1, x_2) \in [0,1]^2.$$

Let $S_{m_1,m_2}(x_1, x_2)$ be partial sums of this series, i.e.,

(21)
$$S_{m_1,m_2}(x_1,x_2) = \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \big(a_{n_1,n_2} \cos n_1 2\pi x_1 \cos n_2 2\pi x_2 + b_{n_1,n_2} \sin n_1 2\pi x_1 \cos n_2 2\pi x_2$$
$$+ c_{n_1,n_2} \cos n_1 2\pi x_1 \sin n_2 2\pi x_2 + d_{n_1,n_2} \sin n_1 2\pi x_1 \sin n_2 2\pi x_2\big)$$
$$= \sum_{n_1=0}^{m_1} \Big[\cos n_1 2\pi x_1 \sum_{n_2=0}^{m_2} \big(a_{n_1,n_2} \cos n_2 2\pi x_2 + c_{n_1,n_2} \sin n_2 2\pi x_2\big)$$
$$+ \sin n_1 2\pi x_1 \sum_{n_2=0}^{m_2} \big(b_{n_1,n_2} \cos n_2 2\pi x_2 + d_{n_1,n_2} \sin n_2 2\pi x_2\big)\Big]$$
$$= \sum_{n_2=0}^{m_2} \Big[\cos n_2 2\pi x_2 \sum_{n_1=0}^{m_1} \big(a_{n_1,n_2} \cos n_1 2\pi x_1 + b_{n_1,n_2} \sin n_1 2\pi x_1\big)$$
$$+ \sin n_2 2\pi x_2 \sum_{n_1=0}^{m_2} \big(c_{n_1,n_2} \cos n_1 2\pi x_1 + d_{n_1,n_2} \sin n_1 2\pi x_1\big)\Big].$$

Theorem 2. *Every double trigonometric series cannot converge to $+\infty$ on the set E, $\mu_2 E > 0$.*

Proof. Assume the contrary. Then following the same reasoning as for proving Theorem 1 and using the fact that the one-dimensional trigonometric systems have property K, we get that (see (10), (13), (14), (21)) there exists a natural number N and a set $E_4(x_2^0)$, $\mu_1 E_4(x_2^0) > 0$, such that

(22)
$$\lim_{\substack{m_1\to\infty \\ m_2\to\infty}} S^*_{m_1,m_2}(x_1, x_2^0) = \lim_{\substack{m_1\to\infty \\ m_2\to\infty}} \big(S_{m_1,m_2}(x_1,x_2^0) - S_{N,m_2}(x_1,x_2^0)\big)$$
$$= \sum_{n_1=N+1}^{\infty} \sum_{n_2=0}^{\infty} \big(a_{n_1,n_2} \cos n_1 2\pi x_1 \cos n_2 2\pi x_2^0 + b_{n_1,n_2} \sin n_1 2\pi x_1 \cos n_2 2\pi x_2^0$$
$$+ c_{n_1,n_2} \cos n_1 2\pi x_1 \sin n_2 2\pi x_2^0 + d_{n_1,n_2} \sin n_1 2\pi x_1 \sin n_2 2\pi x_2^0\big) = +\infty$$

and

(23)
$$0 \leq \varlimsup_{m_2\to\infty} S_{m_1,m_2}(x_1, x_2^0) < \infty, \quad m_1 \geq N.$$

Besides (see (15), (21), (23)),

$$\begin{aligned}
&\overline{\lim_{m_2\to\infty}}\Big|\cos m_1 2\pi x_1 \sum_{n_2=0}^{m_2}\big(a_{m_1,n_2}\cos n_2 2\pi x_2^0 + c_{m_1,n_2}\sin n_2 2\pi x_2^0\big)\\
&\qquad + \sin m_1 2\pi x_1 \sum_{n_2=0}^{m_2}\big(b_{m_1,n_2}\cos n_2 2\pi x_2^0 + d_{m_1,n_2}\sin n_2 2\pi x_2^0\big)\Big|\\
&\qquad = \overline{\lim_{m_2\to\infty}}\big|S_{m_1,m_2}(x_1,x_2^0) - S_{m_1-1,m_2}(x_1,x_2^0)\big|\\
&\qquad \le \overline{\lim_{m_2\to\infty}}\big|S_{m_1,m_2}(x_1,x_2^0)\big| + \overline{\lim_{m_2\to\infty}}\big|S_{m_1-1,m_2}(x_1,x_2^0)\big|\\
&\qquad = \overline{\lim_{m_2\to\infty}} S_{m_1,m_1}(x_1,x_2^0) + \overline{\lim_{m_2\to\infty}} S_{m_1-1,m_2}(x_1,x_2^0) < \infty, \quad m_1 > N.
\end{aligned}$$

Hence, in view of Lemma 2, we have

$$-\infty < \overline{\lim_{m_2\to\infty}} \sum_{n_2=0}^{m_2}\big(a_{m_1,n_2}\cos n_2 2\pi x_2^0 + c_{m_1,n_2}\sin n_2 2\pi x_2^0\big) < +\infty, \ \ m_1 > N,$$

$$-\infty < \overline{\lim_{m_2\to\infty}} \sum_{n_2=0}^{m_2}\big(b_{m_1,n_2}\cos n_2 2\pi x_2^0 + d_{m_1,n_2}\sin n_2 2\pi x_2^0\big) < +\infty, \ \ m_1 > N.$$

Besides, acting in the same way as in Theorem 1 and using the diagonal Cantor method, we get that there exists a subsequence of numbers $m_2(p,p)$ such that

$$\lim_{m_2(p,p)\to\infty} \sum_{n_2=0}^{m_2(p,p)}\big(a_{m_1,n_2}\cos n_2 2\pi x_2^0 + c_{m_1,n_2}\sin n_2 2\pi x_2^0\big) = \alpha_{m_1}, \ \ m_1 > N,$$

$$\lim_{m_2(p,p)\to\infty} \sum_{n_2=0}^{m_2(p,p)}\big(b_{m_1,n_2}\cos n_2 2\pi x_2^0 + d_{m_1,n_2}\sin n_2 2\pi x_2^0\big) = \beta_{m_1}, \ \ m_1 > N.$$

From this and (22), applying Lemma 1, we will get

$$\begin{aligned}
&\lim_{\substack{m_1\to\infty\\ m_2(p,p)\to\infty}} S^*_{m_1,m_2(p,p)}(x_1,x_2^0)\\
&= \lim_{m_1\to\infty}\sum_{n_1=N+1}^{m_1}\Big(\lim_{m_2(p,p)\to\infty}\sum_{n_2=0}^{m_2(p,p)}\big[\cos n_1 2\pi x_1\big(a_{n_1,n_2}\cos n_2 2\pi x_2^0 + c_{n_1,n_2}\sin n_2 2\pi x_2^0\big)\\
&\qquad + \sin n_1 2\pi x_1\big(b_{n_1,n_2}\cos n_2 2\pi x_2^0 + d_{n_1,n_2}\sin n_2 2\pi x_2^0\big)\big]\Big)\\
&\qquad = \lim_{m_1\to\infty}\sum_{n_1=N+1}^{m_1}\big(\alpha_{n_1}\cos n_1 2\pi x_1 + \beta_{n_1}\sin n_1 2\pi x_1\big) = +\infty, \quad x_1 \in E_4(x_2^0),
\end{aligned}$$

i.e., the series

$$\sum_{n_1=N+1}^{m_1}\big(\alpha_{n_1}\cos n_1 2\pi x_1 + \beta_{n_1}\sin n_1 2\pi x_1\big)$$

converges to ∞ on the set $E_4(x_2^0)$, $\mu_1 E_4(x_2^0) > 0$. This is contradiction to the fact that the trigonometric system has the property K. □

It is interesting to know that if the systems (1) have the property K, then will the following equality

$$\mu_2 \Bigg((x_1, x_2) \in [0,1]^2 : -\infty < \underline{\lim}_{\substack{m_1 \to \infty \\ m_2 \to \infty}} S_{m_1, m_2}(x_1, x_2) \leq \leq \overline{\lim}_{\substack{m_1 \to \infty \\ m_2 \to \infty}} S_{m_1, m_2}(x_1, x_2) = +\infty \Bigg) = 0$$

hold for every series (3)?

Addition to corrections

Recently some interesting papers have been published along these lines. G. G. Gevorkyan [11] has proved that every Franklin series cannot converge to $+\infty$ on the set of positive measure. The similar result, in particular, for multiple Franklin series was obtained by G. G. Gevorkyan and M. G. Grigoryan [12].

G. G. Gevorkyan [13] has obtained the criterion for almost everywhere convergence of Franklin series on a set. G. G. Gevorkyan, K. A. Keryan, M. P. Poghosyan [14] have proved that there does not exist a series with respect to orthogonal splines and Ciesielskie series, converging to $+\infty$ on the set of positive measure.

Список литературы

[1] N. N. Luzin, Integral and Trigonometric Series [in Russian], Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad (1951).

[2] Yu. B. Germeier, “Riemann and Vallee–Poussin derivatives and their applications to some questions from the theory of trigonometric series” [in Russian], Diss. Cand. Phys.-Math. Sci., Moscow (1946).

[3] I. I. Privalov, Boundary Properties of Analytic Functions [in Russian], 2d ed., Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad (1950).

[4] P. L. Ulyanov, “Solved and unsolved problems in the theory of trigonometric and orthogonal series” [in Russian], Uspehi Mat. Nauk, **19** (1 (115)), 3 – 69 (1964).

[5] D. E. Menshov, “Certain questions from the theory of trigonometric series” [in Russian], Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk, **1950** (8), 3 – 10 (1950).

[6] A. A. Talalyan, F. G. Arutyunyan, “On the convergence to $+\infty$ of the Haar system [in Russian]” Mat. Sb. (N.S.), **66 (108)** (2), 240 – 247 (1965).

[7] V. A. Skvortsov, “Differentiation with respect to nets, and Haar series [in Russian]”, Mat. Zametki, **4** (2), 33 – 40 (1968).

[8] P. L. Ulyanov, “On the divergence of orthogonal series to $+\infty$” [in Russian], Naučn. Dokl. Vysš. Skoly Fiz.-Mat. Nauki, **1958** (4), 63 – 67 (1958).

[9] R. I. Ovsepyan, A. A. Talalyan, “The convergence of orthogonal series to $+\infty$” [in Russian], Mat. Zametki, **8** (2), 129 – 136 (1970).

[10] S. V. Konyagin, “Limits of indeterminacy of trigonometric series”, Mat. Zametki, **44** (6), 770 – 784, (1988) (in Russian); translation in Math. Notes, **44** (5-6), 910 – 920 (1988).

[11] G. G. Gevorkyan, “On the convergence of Franklin series to $+\infty$”, Math. Notes, **100**, 334 – 341 (2019).

[12] G. G. Gevorkyan, M. G. Grigoryan, “On convergence of square partial sums of multiple Franklin series to $+\infty$”, J. Contemp. Math. Anal., **55**, 9 – 18 (2020).

[13] G. G. Gevorkyan, “On ‘Martingale property’ of Franklin series”, Anal. Math., **45**, 809 – 815 (2019).

[14] G. G. Gevorkyan, K. A. Keryan, M. P. Poghosyan, "Convergence to infinity for orthonormal spline series", Acta. Math. Hungar., **162**:2, 604 – 617 (2020).

Поступила 26 июня 2020

После доработки 22 сентября 2020

Принята к публикации 25 октября 2020

Известия НАН Армении, Математика, том 56, н. 1, 2021, стр. 45 – 53

ПРЕОБРАЗОВАНИЕ ФУРЬЕ АССОЦИИРОВАННОЕ С ДЗЕТА ФУНКЦИЕЙ РИМАНА

Г. В. Микаелян

Ереванский государственный университет
E-mail: *gagik.mikaelyan@ysu.am*

Аннотация. Рассматривается преобразование Фурье ассоциированное с нормированном логарифмом модуля дзета функции Римана. Установливаются формулы связи преобразования Фурье и нулей дзета функции Римана, которые приводят к необходимому и достаточному условию выполнении гипотезы Римана.

MSC2010 number: 11M26; 42A38.

Ключевые слова: преобразование Фурье; дзета функции Римана; гипотеза Римана.

1. Введение

В [1] М. Балазар, Е. Сайас и М. Йор получили формулу суммирования логарифма модуля дзета функции Римана с ядром $\frac{1}{|s|^2}$ на критической линии через нетривиальные нули $\zeta(s)$. В [2] А. А. Кондратюк и А. М. Брудин исследовали свойства коэффициентов Фурье функции $\log|\zeta(s)|$. В [3] А. А. Кондратюк и Р. А. Ятсулька применяя метод рядов Фурье для мероморфных функций Л. А. Рубеля и Б. А Тейлора получили формулу суммирования с ядром $\frac{1}{|s|^4}$ на критической линии через нетривиальные нули $\zeta(s)$. В [4] У. В. Басюк и С. И. Тарасюк доказали аналогичный результат с ядром $\frac{1}{|s|^6}$. В [5] А. А. Кондратюк, в [6] А. М. Брудин и Р. А. Ятсулька получили формулы суммирования на критической линии соответственно с ядрами $e^{-s\varepsilon}$ $(\varepsilon > 0)$ и e^{is} через нетривиальные нули $\zeta(s)$.

Все перечисленные результаты позволяют установить новые утверждения, эквивалентные гипотезе Римана.

Метод исследования настоящей работы основан на следующей теореме [7] (см. также [8]).

Теорема 1.1. *Пусть отличная от постоянной функция f мероморфна в нижней полуплоскости $G = \{w : \operatorname{Im} w < 0\}$, аналитична в окрестности бесконечно* *удаленной точки и* $f(\infty)=1$. *Пусть* $\{u_k+iv_k\}_{k=1}^{\infty}$ *последовательность нулей а* $\{p_k+iq_k\}_{k=1}^{\infty}$ *последовательность полюсов функции* f *и*

$$\frac{e^{xv_0}}{i\sqrt{2\pi}x}\int\limits_{-\infty}^{+\infty}e^{-ixu}\frac{f'(u+iv_0)}{f(u+iv_0)}du=h(x),\quad v_0<\min_k v_k, v_0<\min_k q_k, x\neq 0$$

Тогда $h(x)$ *не зависит от* v_0, *равен нулю при* $x>0$ *и при любом* $v<0$ *справедлива формула*

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{-ixu}\log|f(u+iv)|du=\frac{1}{2}\left(e^{-xv}h(x)+e^{xv}\overline{h(-x)}\right)-$$
$$-\frac{\sqrt{2\pi}}{x}\sum_{v_k<v}e^{-ixu_k}\operatorname{sh}(x(v_k-v))+\frac{\sqrt{2\pi}}{x}\sum_{q_k<v}e^{-ixp_k}\operatorname{sh}(x(q_k-v)).\tag{1.1}$$

ζ - функция Римана при $\operatorname{Re}s>1$ определяется формулой

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

$\zeta(s)$ мероморфно продолжается на всю комплексную плоскость с единственным полюсом $s=1$. Гипотеза Римана гласит, что все нетривиальные нули (нереальные) ζ лежат на «критической линии» $\operatorname{Re}s=\frac{1}{2}$.

Наряду с дзета-функцией рассматривается кси-функция Римана

$$\xi(s)=\frac{s-1}{2}\pi^{-\frac{s}{2}}\Gamma\left(1+\frac{s}{2}\right)\zeta(s),\tag{1.2}$$

где Γ гамма-функция Эйлера. ξ-целая функция первого порядка, нули которой совпадают с нетривиальными нулями дзета-функции Римана, находятся в полосе $0\leq\operatorname{Re}s\leq 1$ и симметричны относительно прямой Re $s=\frac{1}{2}$. Рассматривается также четная целая функция первого порядка $\Xi(z)=\xi\left(\frac{1}{2}+iz\right)$, нули которой на действительной оси соответствуют нулям функции $\zeta(s)$ на прямой Re $s=\frac{1}{2}$.

Замена переменной $s=\frac{1}{2}\left(1-\frac{i}{w}\right)$ переводит полуплоскость $\operatorname{Re}s>\frac{1}{2}$ в полуплоскость $\operatorname{Im}w<0$. В полуплоскости $\operatorname{Im}w<0$ рассмотрим функцию

$$\xi_0(w)=\frac{\xi\left(\frac{1}{2}-\frac{i}{2w}\right)}{\xi\left(\frac{1}{2}\right)}.$$

2. Основные результаты

Сформулируем основные результаты настоящей работы. Обозначим

$$\Omega_{\xi_0}(x,v)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{-ixu}\log\left|\xi_0\left(\frac{1}{2}-\frac{i}{2(u+iv)}\right)\right|du,\ v<0,\ x\neq 0.$$

Теорема 2.1. *При любом $v<0$ справедливы формулы*

(2.1)
$$\Omega_{\xi_0}(x,v)=\frac{\sqrt{2\pi}}{2x}e^{-xv}\sum_{n=1}^{\infty}\left(e^{-\mathrm{i}xw_k}-1\right)-\frac{\sqrt{2\pi}}{x}\sum_{v_k<v}e^{-ixu_k}\operatorname{sh}\left(x\left(v_k-v\right)\right),\ x<0,$$

(2.2)
$$\Omega_{\xi_0}(x,v)=\overline{\Omega_{\xi_0}(-x,v)},\ x>0.$$

Теорема 2.2. *Гипотеза Римана выполняется тогда и только тогда когда для любого v $(-\infty<v<0)$*
$$\lim_{x\to 0}\int_{-\infty}^{+\infty}\cos(xu)\log\left|\frac{\xi\left(\frac{1}{2}-\frac{i}{2(u+iv)}\right)}{\xi\left(\frac{1}{2}\right)}\right|du=0.$$

Аналогично выше изложенному, обозначим
$$\zeta_0(w)=\frac{\zeta\left(\frac{1}{2}-\frac{i}{2w}\right)}{\zeta\left(\frac{1}{2}\right)},\quad \operatorname{Im}w<0,$$
$$\Omega_{\zeta_0}(x,v)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-ixu}\log\left|\zeta_0\left(\frac{1}{2}-\frac{i}{2(u+iv)}\right)\right|du,\quad v<0,x\neq 0.$$

Следующие теоремы являются перефразировками теорем 2.1 и 2.2.

Теорема 2.3. *При любом $v<0$ справедливы формулы*
$$\Omega_{\zeta_0}(x,v)=\frac{\sqrt{2\pi}}{4}(\gamma+\log\pi)-\frac{\sqrt{2\pi}}{x}\operatorname{sh}(x(v+1))+$$
$$+\frac{\sqrt{2\pi}}{2x}\left(1-e^{-x}\right)e^{-xv}+\frac{\sqrt{2\pi}}{2x}e^{-xv}\sum_{n=1}^{\infty}\left(e^{-ixw_n}-1\right)+$$
$$+\frac{\sqrt{2\pi}}{4x}e^{-xv}\sum_{n=1}^{\infty}\left(2\left(e^{\frac{x}{1+4n}}-1\right)-\frac{x}{2n}\right)-\frac{\sqrt{2\pi}}{x}\sum_{v_k<v}e^{-ixu_k}\operatorname{sh}\left(x\left(v_k-v\right)\right),x<0,$$
$$\Omega_{\zeta_0}(x,v)=\overline{\Omega_{\zeta_0}(-x,v)},x>0,$$
где γ постоянная Эйлера.

Теорема 2.4. *Гипотеза Римана выполняется тогда и только тогда когда для любых v $(-\infty<v<0)$*
$$\lim_{x\to 0}\int_{-\infty}^{+\infty}\cos(xu)\log\left|\frac{\zeta\left(\frac{1}{2}-\frac{i}{2(u+iv)}\right)}{\zeta\left(\frac{1}{2}\right)}\right|du=\sqrt{2\pi}\left(\frac{\gamma+\log\pi}{4}+\frac{\pi-13}{16}+\frac{3\log 2}{8}-v\right).$$

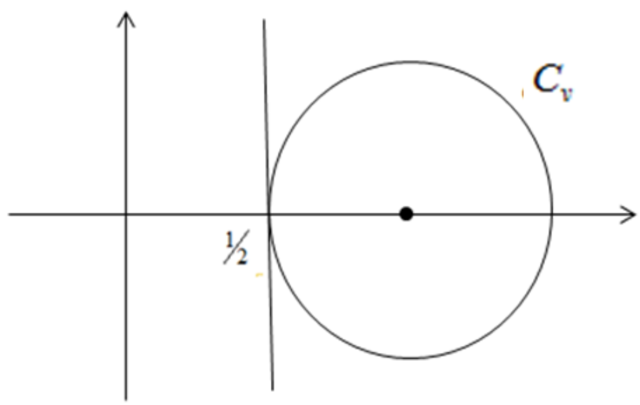
В терминах функции $\Xi(z) = \xi\left(\frac{1}{2} + iz\right)$ интегральное преобразование имеет вид

$$\Omega_\Xi(x,v) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ixu} \log \left| \frac{\Xi\left(-\frac{1}{2(u+iv)}\right)}{\Xi(0)} \right| du$$

или

$$e^{xv}\Omega_\Xi(x,v) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty+iv}^{+\infty+iv} e^{-ixs} \log \left| \frac{\Xi\left(-\frac{1}{2s}\right)}{\Xi(0)} \right| ds.$$

Горизонтальные линии $\operatorname{Im} w = v < 0$ в нижней полуплоскости при отображении $s = \frac{1}{2}\left(1 - \frac{i}{w}\right)$ переводятся в окружности $C_v = \left\{ s : \left| s - \frac{1}{2} - \frac{1}{4|v|} \right| = \frac{1}{4|v|} \right\}$.



Тогда преобразование Фурье можно представить в видах

$$e^{xv}\Omega_\zeta(x,v) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty+iv}^{+\infty+iv} e^{-ixz} \log \left| \frac{\zeta\left(\frac{1}{2} - \frac{i}{2z}\right)}{\zeta\left(\frac{1}{2}\right)} \right| dz$$

или

$$e^{xv}\Omega_\zeta(x,v) = -\frac{i}{\sqrt{2\pi}} \int\limits_{C_v} e^{-\frac{x}{2s-1}} \log \left| \frac{\zeta(s)}{\zeta(\frac{1}{2})} \right| \frac{ds}{(s-\frac{1}{2})^2}.$$

А формулы преобразования Фурье превращаются в следующие формулы

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ix(u+iv)} \log \left| \frac{\xi\left(\frac{1}{2} - \frac{i}{2(u+iv)}\right)}{\xi\left(\frac{1}{2}\right)} \right| du =$$

$$= \frac{\sqrt{2\pi}}{2x} \sum_{k=1}^{\infty} \left(e^{-ixw_k} - 1\right) - \frac{\sqrt{2\pi}}{2x} e^{xv} \sum_{v_k < v} \left(e^{-xv} e^{-ixw_k} - e^{xv} e^{-ix\bar{w}_k}\right),$$

$$\frac{2i}{\sqrt{2\pi}}\int\limits_{C_v} e^{-\frac{x}{2z-1}}\log\left|\frac{\xi(z)}{\xi(\frac{1}{2})}\right|\frac{ds}{(z-\frac{1}{2})^2}=$$

$$=\frac{\sqrt{2\pi}}{2x}\sum_{k=1}^{\infty}\left(e^{\frac{x}{1-2\rho_k}}-1\right)-\frac{\sqrt{2\pi}}{2x}\sum_{\rho_k\in D_v}\left(e^{\frac{x}{1-2\rho_k}}-e^{-2xRe(1-2s)^{-1}}e^{\frac{x}{1-2\bar{\rho}_k}}\right),$$

где D_v круг с границей C_v.

3. Доказательства теорем

Из (1.2) имеем

$$\frac{\xi'(s)}{\xi(s)}=\frac{1}{s-1}-\frac{\log\pi}{2}+\frac{1}{2}\frac{\Gamma'\left(1+\frac{s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}+\frac{\zeta'(s)}{\zeta(s)}. \tag{3.1}$$

Для логарифмической производной $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ гамма функции при $z\neq n$ ($n=0,1,2,\ldots$) справедливо разложение

$$\psi(z)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right).$$

Следовательно при $s\neq -2n$

$$\frac{\Gamma'\left(1+\frac{s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}=-\frac{\gamma}{2}+\sum_{n=0}^{\infty}\left(\frac{1}{2n+2}-\frac{1}{2n+2+s}\right)=-\frac{\gamma}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+s}\right).$$

В силу формулы (см. например [9])

$$\frac{\zeta'(s)}{\zeta(s)}=-\frac{s}{s-1}+\sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n}+\frac{1}{\rho_n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right)+\log 2\pi, \tag{3.2}$$

где $\{\rho_n\}_{n=1}^{\infty}$ комплексные нули функции ζ и (3.1) будем иметь

$$\frac{\xi'(s)}{\xi(s)}=-\frac{\gamma}{2}+\log 2\pi-\frac{\log\pi}{2}-1+\sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n}+\frac{1}{\rho_n}\right). \tag{3.3}$$

Применяя (3.3) и формулу Римана (см. например [9])

$$\sum_{n=1}^{\infty}\frac{1}{\rho_n}=\frac{\gamma}{2}-\log 2\pi+\frac{\log\pi}{2}+1$$

получаем разложение

$$\frac{\xi'(s)}{\xi(s)}=\sum_{n=1}^{\infty}\frac{1}{s-\rho_n}. \tag{3.4}$$

Функция ξ_0 удовлетворяет условиям теоремы 1. Для этой функции вычислим значения $h\left(x\right)$, т.е. интеграл

$$h\left(x\right)=\frac{e^{xv_0}}{2\sqrt{2\pi}x}\int\limits_{-\infty}^{+\infty}e^{-ixu}\frac{\xi_0'\left(\frac{1}{2}-\frac{i}{2(u+iv_0)}\right)}{\xi_0\left(\frac{1}{2}-\frac{i}{2(u+iv_0)}\right)}\frac{du}{\left(u+iv_0\right)^2}$$

при $x < 0$.

При вычислении интеграла мы используем следующую формулу (см. [10])

$$\int\limits_{-\infty}^{+\infty} \frac{e^{i\lambda x}}{y+ix}dx = 2\pi e^{-\lambda y} \quad (\lambda > 0, y > 0). \tag{3.5}$$

Ввиду (3.4)

$$h(x) = \frac{e^{xv_0}}{2\sqrt{2\pi}x}\sum_{n=1}^{\infty}\int\limits_{-\infty}^{+\infty}\frac{e^{-ixu}}{\frac{1}{2}-\frac{i}{2(u+iv_0)}-\rho_n}\frac{du}{(u+iv_0)^2},$$

следовательно в силу (3.5)

$$\int\limits_{-\infty}^{+\infty}\frac{e^{-ixu}}{\frac{1}{2}-\frac{i}{2(u+iv_0)}-\rho_n}\frac{du}{(u+iv_0)^2} = \int\limits_{-\infty}^{+\infty}\frac{e^{-ixu}}{\left(\frac{1}{2}-\rho_n\right)(u+iv_0)-\frac{i}{2}}\frac{du}{u+iv_0} =$$

$$= -2i\int\limits_{-\infty}^{+\infty} e^{-ixu}\left(\frac{\frac{1}{2}-\rho_n}{\left(\frac{1}{2}-\rho_n\right)(u+iv_0)-\frac{i}{2}} - \frac{1}{u+iv_0}\right)du =$$

$$= -2i\int\limits_{-\infty}^{+\infty} e^{-ixu}\frac{du}{u+iv_0-\frac{i}{2\left(\frac{1}{2}-\rho_n\right)}} + 2i\int\limits_{-\infty}^{+\infty}\frac{e^{-ixu}du}{u+iv_0} =$$

$$= 2\int\limits_{-\infty}^{+\infty}\frac{e^{-ixu}du}{\frac{1}{1-2\rho_n}+|v_0|+iu} - 2\int\limits_{-\infty}^{+\infty}\frac{e^{-ixu}du}{|v_0|+iu} =$$

$$= 4\pi e^{x\left(\frac{1}{1-2\rho_n}-v_0\right)} - 4\pi e^{-xv_0} = 4\pi e^{-xv_0}\left(e^{\frac{x}{1-2\rho_n}}-1\right).$$

Формула (3.5) применима, поскольку известно, что для всех n $|\operatorname{Im}\rho_n| > 14$. При $\operatorname{Re}\rho_n > \frac{1}{2}$ и $|v_0| > \frac{1}{56}$

$$\operatorname{Re}\left(|v_0| + \frac{1}{1-2\rho_n}\right) = |v_0| + \frac{1-2\operatorname{Re}\rho_n}{|1-2\rho_n|^2} \geq |v_0| + \frac{1-2\operatorname{Re}\rho_n}{4\,|\operatorname{Im}\rho_n|\,(2\operatorname{Re}\rho_n-1)} =$$

$$= |v_0| - \frac{1}{4\,|\operatorname{Im}\rho_n|} > |v_0| - \frac{1}{56} > 0,$$

а при $\operatorname{Re}\rho_n \leq \frac{1}{2}$

$$\operatorname{Re}\left(|v_0| + \frac{1}{1-2\rho_n}\right) = |v_0| + \frac{1-2\operatorname{Re}\rho_n}{|1-2\rho_n|^2} \geq |v_0| > 0.$$

Таким образом

$$h(x) = \frac{\sqrt{2\pi}}{x}\sum_{n=1}^{\infty}\left(e^{\frac{x}{1-2\rho_n}}-1\right), \quad x < 0. \tag{3.6}$$

Обозначим $\frac{i}{1-2\rho_k} = w_k = u_k + iv_k$ и заметим, что условию $\operatorname{Re}\rho_k > \frac{1}{2}$ соответствует условие $v_k < 0$.

Тогда из (1.1) и (3.6) следуют формулы (2.1) и (2.2).

Поскольку вместе с ρ_k нулем функции является и $1-\rho_k$, то $\sum_{k=1}^{\infty}\frac{1}{1-2\rho_k}=0$, т. е. $\sum_{k=1}^{\infty} w_k=0$. Следовательно, при $x\to 0$

$$\frac{1}{x}\sum_{k=1}^{\infty}\left(e^{-ixw_k}-1\right)=\frac{1}{x}\sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\frac{(-ix)^n w_k^n}{n!}\to 0,$$

поскольку показатель сходимости последовательности $\{\rho_n\}$ равен единице. Таким образом

$$\lim_{x\to 0}\Omega_{\xi_0}(x,v)=\sqrt{2\pi}\sum_{v_k<v}(v-v_k)\,.$$

При $v<0$ введем следующие обозначения $n(v)=\sum\limits_{v_k<v}1$, $N(v)=\int\limits_{-\infty}^{v}n(t)dt$.

Заметим, что

$$\sum_{v_k<v}(v-v_k)=\int_{-\infty}^{v}(v-t)dn(t)=N(v).$$

Следовательно в силу (2.1) и (2.2)

$$\lim_{x\to 0}\Omega_{\xi_0}(x,v)=N(v),$$

и справедливы формулы

$$\lim_{x\to 0}\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\cos(xu)\log\left|\xi_0\left(\frac{1}{2}-\frac{i}{2(u+iv)}\right)\right|du=N(v),$$

$$\lim_{x\to 0}\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\sin(xu)\log\left|\xi_0\left(\frac{1}{2}-\frac{i}{2(u+iv)}\right)\right|du=0.$$

Теоремы 2.1 и 2.2 доказаны.

Доказательство теоремы 2.3 следует из формулы

$$h_{\zeta_0}(x)=\frac{\sqrt{2\pi}}{2x}\left(2\left(1-e^{-x}\right)+2\sum_{n=1}^{\infty}\left(e^{-ixw_n}-1\right)+\right.$$

$$\left.+\sum_{n=1}^{\infty}\left(2\left(e^{\frac{x}{1+4n}}-1\right)-\frac{x}{2n}\right)+\frac{\gamma+\log\pi}{2}x\right)$$

при $x<0$ и теоремы 1.1, применением представления (3.2), формул (3.6) и

$$\int\limits_{-\infty}^{+\infty}\frac{e^{i\lambda x}}{(y+ix)^2}dx=2\pi\lambda e^{-\lambda y},\ \lambda>0, y>0 \text{ (см. [10]).}$$

Теорема 2.4 получается из теоремы 2.3 предельным переходом при $x\to 0$

$$\lim_{x\to 0}\Omega_{\zeta_0}(x,v)=\frac{\sqrt{2\pi}}{4}(\gamma+\log\pi)-\sqrt{2\pi}(v+1)+\frac{\sqrt{2\pi}}{2}+$$

$$+\frac{\sqrt{2\pi}}{4}\sum_{n=1}^{\infty}\left(\frac{2}{1+4n}-\frac{1}{2n}\right)-\sqrt{2\pi}\sum_{n_k<v}(v_k-v)$$

с учетом, что

$$\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{2}{1+4n}-\frac{1}{2n}\right)=-\frac{1}{32}\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{1}{4}\right)}=-\frac{1}{8}\left(\psi\left(\frac{1}{4}+1\right)+\gamma\right)=$$

$$=-\frac{1}{8}\left(\psi\left(\frac{1}{4}\right)+4+\gamma\right)=\frac{1}{8}\left(-\frac{5}{2}+\frac{\pi}{2}+3\log 2\right).$$

Замечание 3.1. *Следует отметить, что функция* $\log|\zeta_0(u+iv)|$ *при любом* v $(-\infty<v<0)$ *не принадлежит пространству* $L_1(-\infty,+\infty)$*, но принадлежит пространству* $L_2(-\infty,+\infty)$*.*

Во первых, поскольку при $w\to\infty$

$$\log\left|\frac{\zeta\left(\frac{1}{2}-\frac{i}{2w}\right)}{\zeta\left(\frac{1}{2}\right)}\right|\sim\frac{1}{|w|},\quad w\to\infty,$$

то функция $\log|\zeta_0(u+iv)|$ не принадлежит пространству $L_1(-\infty,+\infty)$. Известно, что (см. [11]) $\zeta(s)=O(|s|),\ s\to\infty\ \left(\operatorname{Re}s\geq\frac{1}{2}\right)$, т.е.

$$\left|\zeta\left(\frac{1}{2}-\frac{i}{2w}\right)\right|=O\left(\left|\frac{1}{2}-\frac{i}{2w}\right|\right),\quad w\to 0.$$

Так как

$$\left|\frac{1}{2}-\frac{i}{2w}\right|=\sqrt{\left(\frac{1}{2}-\frac{v}{2\left(u^2+v^2\right)}\right)^2+\left(\frac{1}{2}\frac{u}{u^2+v^2}\right)^2}=\frac{1}{2}\sqrt{\frac{u^2+(v-1)^2}{u^2+v^2}}$$

и при $w\to 0$

$$\log\left|\frac{1}{2}-\frac{i}{2w}\right|\leq\log\frac{1}{2}+\frac{1}{2}\log\frac{u^2+(|v|+1)^2}{u^2+v^2}\leq\frac{1}{2}\log\left(1+\frac{1+2|v|}{u^2+v^2}\right)\leq\frac{1+2|v|}{u^2+v^2},$$

$$\log\left|\frac{\zeta\left(\frac{1}{2}-\frac{i}{2w}\right)}{\zeta\left(\frac{1}{2}\right)}\right|=O\left(\frac{1}{|w|}\right)=O\left(\frac{1}{\sqrt{u^2+v^2}}\right),\quad w\to\infty.$$

Следовательно при любом v $(-\infty<v<0)$ $\log|\zeta_0(u+iv)|\in L_2(-\infty,+\infty)$.

Abstract. The Fourier transform associated with the normalized logarithm of the Riemann Zeta function module is considered. Formulas for the relationship between the Fourier transform and the zeros of the Riemann Zeta function are established which lead to a necessary and sufficient condition for the fulfillment of the Riemann hypothesis.

Список литературы

[1] M. Balazard, E. Saias, M. Yor, "Notes sur la fonction ζ de Riemann", **2**, Adv. Math., 143 (2), 284 – 287 (1999).

[2] A. A. Kondratyuk, A. M. Brudin, "On the Fourier series of the zeta-function logarithm on the vertical lines", Mat. Stud. **22**, no. 1, 97 – 104 (2004).

[3] A. A. Kondratyuk, P. A. Yatsulka, "Summation of the Riemann zeta-function logarithm on the critical line", Voronoy's impact on modern science, Book 4, **1**, Proceedings of the Fourth International Conferenceon Analytic Number Theory and Spatial Fesselations, Kyiv, 59 – 62 (2008).

[4] Y. Y. Basiuk, S. I. Tarasyuk, "Fourier coefficients associated with the Rieman zeta-function", Carpatian Math. Publ., **8** (1), 16 – 20 (2016).

[5] A. A. Kondratyuk, "A Carleman-Nevanlinna theorem and summation of the Riemann zeta-function logarithm", Comput. Methods Funct. Theoryl., **4**, 391 – 403 (2004).

[6] A. M. Brudin, P. A. Yatsulka, "A version of Carleman's formula summation of the Riemann-function logarithm on the critical line", Mat. Stud. **35**, 3 – 9 (2011).

[7] Г. В. Микаелян, "Преобразование Фурье, ассоциированное с функциями, мероморфными в полуплоскости", Изв. АН Арм ССР, Сер. Матем., **XIX**, no. 5, 361 – 376 (1984).

[8] Г. В. Микаелян, "О росте функций, мероморфных в полуплоскости", Известия вузов, **4**, 79 – 82 (1988).

[9] H. M. Edwards, Riemanns Zeta Function, New York (1974).

[10] А. П. Прудников, Ю. А. Брычков, О. И. Маричев, Интегралы и Ряды, Элементарные функции, М. Наука (1981).

[11] E. C. Titchmarsh, The theory of the Riemann zeta-function [in Russian], Moscow: Izdat. inost. liter., 408 p. (1953).

Поступила 29 марта 2020

После доработки 22 августа 2020

Принята к публикации 16 сентября 2020

Известия НАН Армении, Математика, том 56, н. 1, 2021, стр. 54 – 65.

NEW INTEGRAL REPRESENTATIONS FOR THE FOX-WRIGHT FUNCTIONS AND ITS APPLICATIONS II

K. MEHREZ

University of Tunis El Manar, Tunisia
University of Kairouan, Tunisia
E-mail: *k.mehrez@yahoo.fr*

Abstract. In this paper our aim is to establish new integral representations for the Fox–Wright function ${}_p\Psi_q[^{(\alpha_p,A_p)}_{(\beta_q,B_q)}|z]$ when $\mu=\sum_{j=1}^q\beta_j-\sum_{k=1}^p\alpha_k+\frac{p-q}{2}=-m,\ \ m\in\mathbb{N}_0$. In particular, closed-form integral expressions are derived for the four parameter Wright function under a special restriction on parameters. Exponential bounding inequalities are derived for a class of the Fox-Wright function. Moreover, complete monotonicity property is presented for these functions.

MSC2010 numbers: 33C20; 33E20; 26D07; 26A42; 44A10.

Keywords: Fox-Wright function; Fox's H-function; complete monotonicity; Log–convexity; Turán type inequalities; reneralized Stieltjes function.

1. Introduction

We use a definition of the Fox-Wright (generalized hypergeometric) function by its series

$$(1.1)\qquad {}_p\Psi_q\left[^{(\alpha_1,A_1),...,(\alpha_p,A_p)}_{(\beta_1,B_1),...,(\beta_q,B_q)}\Big|z\right]={}_p\Psi_q\left[^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|z\right]=\sum_{k=0}^\infty\frac{\prod_{i=1}^p\Gamma(\alpha_l+kA_l)}{\prod_{j=1}^q\Gamma(\beta_l+kB_l)}\frac{z^k}{k!},$$

$$\left(\alpha_i,\beta_j\in\mathbb{C},\ \text{and}\ \ A_i,B_j\in\mathbb{R}^+\ (i=1,...,p,j=1,...,q)\right),$$

where, as usual,

$$\mathbb{N}=\{1,2,3,...\},\ \mathbb{N}_0=\mathbb{N}\cup\{0\},$$

$\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand for the sets of real, positive real and complex numbers, respectively. This function was first introduced by Wright [7] in 1935, who also derived some of its important properties including asymptotic behavior.

The convergence conditions and convergence radius of the series at the right-hand side of (1.1) immediately follow from the known asymptotic of the Euler Gamma-function. To formulate the results, let us first introduce the following notations:

(1.2)
$$\Delta=\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i,\ \rho=\left(\prod_{i=1}^{p}A_i^{-A_i}\right)\left(\prod_{j=1}^{q}B_j^{B_j}\right),$$
$$\mu=\sum_{j=1}^{q}\beta_j-\sum_{k=1}^{p}\alpha_k+\frac{p-q}{2}$$

The defining series in (1.1) converges in the whole complex z-plane if $\Delta>-1$. If $\Delta=-1$, then the series in (1.1) converges for $|z|<\rho$, and $|z|=\rho$ under the condition $\Re(\mu)>\frac{1}{2}$, see [15] for details. If, in the definition (1.1), we set

$$A_1=...=A_p=1\quad\text{and}\quad B_1=...=B_q=1,$$

we get the relatively more familiar generalized hypergeometric function ${}_pF_q[.]$ given by

(1.3)
$${}_pF_q\left[{}^{\alpha_1,...,\alpha_p}_{\beta_1,...,\beta_q}\Big|z\right]=\frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}{}_p\Psi_q\left[{}^{(\alpha_1,1),...,(\alpha_p,1)}_{(\beta_1,1),...,(\beta_q,1)}\Big|z\right],\ \ (\alpha_j>0,\ \beta_j\notin\mathbb{Z}_0^-).$$

Moreover, both the Wright function $W_{\alpha,\beta}(.)$ and the Mittag-Leffler function $E_{\alpha,\beta}(z)$, are particular cases of the Fox-Wright function (1.1):

$$W_{\alpha,\beta}(z)={}_0\Psi_1\left[{}^{-}_{(\beta,\alpha)}\Big|z\right],\ E_{\alpha,\beta}(z)={}_1\Psi_1\left[{}^{(1,1)}_{(\beta,\alpha)}\Big|z\right]$$

Note important properties for this functions including its Turán, Lazarević and Wilker type inequalities, was proved by Mehrez [11] and Mehrez et al in [9],[10].

In a recent papers [12],[13],[14], the authors have studied certain advanced properties of the Fox-Wright function including its new integral representations, the Laplace and Stieltjes transforms, Luke inequalities, Turán type inequalities and completely monotonicity property are derived. In particular, it was shown there that the following Fox-Wright functions are completely monotone:

$${}_p\Psi_q\left[{}^{(\alpha_p,A)}_{(\beta_q,A)}\Big|-z\right],\quad z>0,$$
$${}_{p+1}\Psi_q\left[{}^{(\lambda,1),(\alpha_p,A_p)}_{(\beta_q,1)}\Big|\frac{1}{z}\right],\quad z>0,$$

and was proved that the Fox's H-function $H_{q,p}^{p,0}[.]$ constitutes the representing measure for the Fox-Wright function ${}_p\Psi_q[.]$, if $\mu>0$, i.e., [12, Theorem 1]

(1.4)
$${}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|z\right]=\int_0^{\rho}e^{zt}H_{q,p}^{p,0}\left(t\Big|{}^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\frac{dt}{t}.$$

when $\mu>0$. Here, and in what follows, we use $H_{q,p}^{p,0}[.]$ to denote the Fox's H-function, defined by

(1.5)
$$H_{q,p}^{p,0}\left(z\Big|{}^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)=\frac{1}{2i\pi}\int_{\mathcal{L}}\frac{\prod_{j=1}^{p}\Gamma(A_js+\alpha_j)}{\prod_{k=1}^{q}\Gamma(B_ks+\beta_k)}z^{-s}ds,$$

where $A_j, B_k > 0$ and $\alpha_j, \beta_k > 0$. The contour $\mathcal{L}$ has one of the following forms :

- $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < \infty$;
- $\mathcal{L} = \mathcal{L}_{\infty}$ is a right loop in a horizontal strip starting at the point $\infty + i\varphi_1$ and terminating at the point $\infty + i\varphi_2$ with $\infty < \varphi_1 < \varphi_2 < \infty$;
- $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty + i\varphi_2$, where $\gamma \in \mathbb{R}$.

Details regarding the contour and conditions for convergence of the integral in (1.5) can be found in [16, Sections 1.1,1.2] and [8].

In the course of our investigation, the first main tools is extended some results proved in [12] in the case when $\mu = -m,\ m \in \mathbb{N}_0$. Secondly, we establish the monotonicity of ratios involving the Fox-Wright functions.

2. Main results

The following theorem leads to an extension of the integral equation for the H-function obtained in (1.4) to the case $\mu = -m,\ m \in \mathbb{N}_0$.

Theorem 2.1. *Suppose that* $\mu = -m,\ m \in \mathbb{N}_0$ *and*

$$\sum_{i=1}^{p} A_i = \sum_{j=1}^{q} B_j.$$

If $\gamma \geq 1$, *then the Fox-Wright function* ${}_p\Psi_q[.]$ *possesses the following integral representation*

$$ {}_p\Psi_q\left[\left.{}^{(\alpha_p, A_p)}_{(\beta_q, B_q)}\right| z\right] = \int_0^\rho e^{zt} H^{p,0}_{q,p}\left(t\left|{}^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right.\right)\frac{dt}{t} + \eta \sum_{k=0}^{\infty}\sum_{j=0}^{m} \frac{l_{m-j} k^j \rho^k z^k}{k!}, \tag{2.1}$$

where the coefficients η *and* γ *are defined by*

$$\eta = (2\pi)^{\frac{p-q}{2}} \prod_{i=1}^{p} A_i^{\alpha_i - \frac{1}{2}} \prod_{j=1}^{q} B_j^{\frac{1}{2} - \beta_j},\ \ \gamma = -\min_{1 \leq j \leq p} (\alpha_j / A_j), \tag{2.2}$$

and the coefficients l_r *satisfy the recurrence relation:*

$$l_r = \frac{1}{r} \sum_{n=1}^{r} q_n l_{r-n}, \quad l_0 = 1, \tag{2.3}$$

with

$$q_n = \frac{(-1)^{n+1}}{n+1}\left[\sum_{i=1}^{p} \frac{\mathcal{B}_{n+1}(\alpha_i)}{A_i^n} - \sum_{j=1}^{q} \frac{\mathcal{B}_{n+1}(\beta_j)}{B_j^n}\right],$$

where $\mathcal{B}_n$ is the Bernoulli polynomial defined via generating function [2, p. 588]

$$\frac{te^{at}}{e^t-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(a)\frac{t^n}{n!},\quad |t|<2\pi.$$

Proof. In [1, Theorem 2], the authors found the Mellin transform of the H-function when $\mu=-m,\ m\in\mathbb{N}_0$, that is

$$\begin{aligned}&\frac{\prod_{i=1}^{p}\Gamma(A_ik+\alpha_i)}{\prod_{j=1}^{q}\Gamma(B_jk+\beta_j)}=\\ &=\int_0^\rho H_{q,p}^{p,0}\left(t\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right)t^{k-1}dt+\eta\rho^k\sum_{j=0}^{m}l_{m-j}k^j,\quad \Re(k)>\gamma.\end{aligned}\tag{2.4}$$

This implies that

$$\begin{aligned}{}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|z\right]&=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(A_ik+\alpha_i)z^k}{k!\prod_{j=1}^{q}\Gamma(B_jk+\beta_j)}\\ &=\sum_{k=0}^{\infty}\int_0^\rho H_{q,p}^{p,0}\left(t\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right)\frac{(zt)^k}{k!}\frac{dt}{t}+\sum_{k=0}^{\infty}\left(\eta\rho^k\sum_{j=0}^{m}l_{m-j}\frac{k^jz^k}{k!}\right)\\ &=\sum_{k=0}^{\infty}\int_0^\rho H_{q,p}^{p,0}\left(t\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right)\frac{(zt)^k}{k!}\frac{dt}{t}+\eta\sum_{k=0}^{\infty}\sum_{j=0}^{m}\frac{l_{m-j}k^j\rho^kz^k}{k!}\\ &=\int_0^\rho H_{q,p}^{p,0}\left(t\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right)\left(\sum_{k=0}^{\infty}\frac{(zt)^k}{k!}\right)\frac{dt}{t}+\eta\sum_{k=0}^{\infty}\sum_{j=0}^{m}\frac{l_{m-j}k^j\rho^kz^k}{k!}\\ &=\int_0^\rho e^{zt}H_{q,p}^{p,0}\left(t\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right)\frac{dt}{t}+\eta\sum_{k=0}^{\infty}\sum_{j=0}^{m}\frac{l_{m-j}k^j\rho^kz^k}{k!}.\end{aligned}$$

For the exchange of the summation and integration, we use the asymptotic behavior of the H-function as $z\to0$ [8, Theorem 1.2, Eq. 1.94]

$$H_{q,p}^{m,n}(z)=\mathcal{O}(z^{-\gamma}),\ |z|\longrightarrow0,\tag{2.5}$$

and the asymptotic behavior of the H-function as $z\to\rho$ [1, Theorem 1], we obtain

$$\int_0^\rho t^{k-1}\left|H_{q,p}^{p,0}\left(t\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right)\right|dt\le\int_0^\rho t^{-1}\left|H_{q,p}^{p,0}\left(t\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right)\right|dt<M<\infty.$$

Then, we are in position to apply the Lebesgue dominated convergence theorem. This completes the proof of Theorem 2.1. □

Recall that a function $f:(0,\infty)\longrightarrow(0,\infty)$ is called completely monotonic if $(-1)^nf^{(n)}(x)\ge0$ for $x>0$ and $n\in\mathbb{N}_0$. The celebrated Bernstein theorem asserts that completely monotonic functions are precisely those that can be expressed by the Laplace transform of a non-negative measure.

Corollary 2.1. *Suppose that* $\mu = 0$, $\gamma \geq 1$ *and* $\sum_{i=1}^{p} A_i = \sum_{j=1}^{q} B_j$. *Then, the Fox-Wright function* ${}_p\Psi_q[.]$ *possesses the following integral representation*

$$ {}_p\Psi_q\left[{}^{(\alpha_p, A_p)}_{(\beta_q, B_q)}\Big| -z\right] - \eta e^{-\rho z} = \int_0^\rho e^{-zt} H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\frac{dt}{t}, \quad z \in \mathbb{R}. \tag{2.6}$$

Moreover, if the function $H^{p,0}_{q,p}[.]$ *is non-negative, then the function*

$$ z \mapsto {}_p\Psi_p\left[{}^{(\alpha_p, A)}_{(\beta_p, A)}\Big| -z\right] - \eta e^{-\rho z}, $$

is completely monotonic on $(0, \infty)$.

Proof. The application of Theorem 2.1 for $m = 0$ yields

$$ {}_p\Psi_q\left[{}^{(\alpha_p, A_p)}_{(\beta_q, B_q)}\Big| z\right] = \int_0^\rho e^{zt} H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\frac{dt}{t} + \eta e^{\rho z}, \tag{2.7}$$

which implies that (2.6) holds. Now, suppose that the H-function $H^{p,0}_{p,p}[.]$, then by means of (2.6), we deduce that all prerequisites of the Bernstein Characterization Theorem for the complete monotone functions are fulfilled. □

Example 2.1. *The four parameters Wright function is defined by the series*

$$ \phi\left((\mu_1, a), (\nu_1, b); z\right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a + k\mu_1)\Gamma(b + k\nu_1)}, \quad \mu_1, \nu_1 \in \mathbb{R}, \ a, b \in \mathbb{C}. \tag{2.8}$$

The series on the right-hand side of (2.8) is absolutely convergent for all $z \in \mathbb{C}$ *if* $\mu_1 + \nu_1 > 0$. *If* $\mu_1 + \nu_1 = 0$, *the series is absolutely convergent for* $|z| < |\mu_1|^{\mu_1}|\nu_1|^{\nu_1}$ *and* $|z| = |\mu_1|^{\mu_1}|\nu_1|^{\nu_1}$ *under the condition* $\Re(a+b) > 2$. *Some of the basic properties of the four parameters Wright function was proved in* [4]. *So, by means of formula (2.6) we deduce that the four parameters Wright function* $\phi\left((\mu_1, a), (\nu_1, b); z\right)$ *admits the following integral representation:*

$$ \begin{aligned} &\phi\left((\mu_1, a), (\nu_1, b); z\right) = \\ &= \int_0^{\mu_1^{\mu_1}\nu_1^{\nu_1}} e^{zt} H^{1,0}_{2,1}\left[t\Big|^{(\mu_1,a),(\nu_1,b)}_{(1,1)}\right]\frac{dt}{t} + \frac{\mu_1^{\frac{1}{2}-a}\nu_1^{\frac{1}{2}-b}}{\sqrt{2\pi}} e^{\mu_1^{\mu_1}\nu_1^{\nu_1} z}, \end{aligned} \tag{2.9}$$

where $a, b, \mu_1, \nu_1 \in \mathbb{R}$ *for which* $\mu_1 + \nu_1 = 1$ *and* $a + b = 3/2$.

As a consequence, we derive the finite Laplace Transform for the function

$$ t \mapsto t^{-1} H^{1,0}_{2,1}\left[t\Big|^{(1/2,1/2),(1/2,1)}_{(1,1)}\right] $$

in $(0, 1/2)$. Recall that the finite Laplace Transform of a continuous (or an almost piecewise continuous) function $f(t)$ in $(0, T)$ is denoted by

$$\mathcal{L}_T f(t) = \bar{f}(s,T) = \int_0^T e^{-st} f(t) dt.$$

We note that $\mathcal{L}_T f$ is actually the Laplace transform of the function f which vanishes outside of the interval $(0, T)$.

Example 2.2. *Letting in (2.8) and (2.9), the values $\nu_1 = \mu_1 = a = 1/2$ and $b = 1$ and using the Legendre Duplication Formula*

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z),$$

we get the following curious integral evaluation:

$$\frac{e^{-2z} - e^{-z}}{\sqrt{\pi}} = \int_0^{\frac{1}{2}} e^{-zt} H_{2,1}^{1,0}\left[t \Big|_{(1,1)}^{(1/2,1/2),(1/2,1)} \right] \frac{dt}{t}, \; z \in \mathbb{R}. \tag{2.10}$$

In the next result we show that the function ${}_{p+1}\Psi_q[.] - \eta_1 F_0[.]$ is a Stieltjes transform.

Corollary 2.2. *Let $\sigma > 0$ and $z \in \mathbb{C}$, such that $|\arg(1+z)| < \pi$ and $|z| < 1$. In addition, assume that the hypotheses of Corollary 2.1 are satisfied. If $H_{q,p}^{p,0}[.]$ is non-negative, then, the following representation holds true:*

$$\begin{aligned} f(z) &:= {}_{p+1}\Psi_q\left[{}^{(\sigma,1),(\alpha_p,A_p)}_{(\beta_q,B_q)} \Big| -z \right] - \eta\, {}_1F_0(\sigma; -; -\rho z) = \\ &= \Gamma(\sigma) \int_0^\rho H_{q,p}^{p,0}\left(t \Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)} \right) \frac{dt}{t(1+tz)^\sigma}. \end{aligned} \tag{2.11}$$

In particular, $f(z)$ is completely monotonic on $(0, \infty)$.

Proof. Employing the generalized binomial expansion

$$(1+z)^{-\sigma} = \sum_{k=0}^\infty (\sigma)_k \frac{(-1)^k z^k}{k!}, \; z \in \mathbb{C}, \quad |z| < 1,$$

with the formula (2.4) and the right hand side of (2.11) we obtain

$$\begin{aligned} &\Gamma(\sigma) \int_0^\rho H_{q,p}^{p,0}\left(t \Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)} \right) \frac{dt}{t(1+tz)^\sigma} = \\ &= \Gamma(\sigma) \sum_{k=0}^\infty (\sigma)_k \frac{(-1)^k z^k}{k!} \left[\int_0^\rho t^{k-1} H_{q,p}^{p,0}\left(t \Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)} \right) dt \right] \\ &= \Gamma(\sigma) \sum_{k=0}^\infty (\sigma)_k \frac{(-1)^k z^k}{k!} \left[\frac{\prod_{i=1}^p \Gamma(A_i k + \alpha_i)}{\prod_{j=1}^q \Gamma(B_j k + \beta_j)} - \eta \rho^k \right] \\ &= \sum_{k=0}^\infty \frac{\Gamma(\sigma+k) \prod_{i=1}^p \Gamma(\alpha_i + kA_i)}{\prod_{j=1}^q \Gamma(\beta_j + kB_j)} \frac{(-z^k}{k!} - \eta \sum_{k=0}^\infty \frac{\Gamma(\sigma+k)(-\rho z)^k}{k!} \\ &= {}_{p+1}\Psi_q\left[{}^{(\sigma,1),(\alpha_p,A_p)}_{(\beta_q,B_q)} \Big| -z \right] - \eta\, {}_1F_0(\sigma; -; -\rho z). \end{aligned} \tag{2.12}$$

This completes the proof. □

Corollary 2.3. *Suppose that* $\mu=-1$ *and*

$$\sum_{i=1}^{p} A_i=\sum_{j=1}^{q} B_j.$$

Then the following integral representation

$$\begin{aligned} g(z):&={}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|-z\right]-\eta(l_1-\rho z)e^{-\rho z}= \\ &=\int_0^\rho e^{-zt}H_{q,p}^{p,0}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)dt,\;\; z\in\mathbb{R} \end{aligned} \tag{2.13}$$

holds true. Moreover, if $H_{q,p}^{p,0}[.]$ *is non-negative, then the function* $g(z)$ *is completely monotonic on* $(0,\infty)$.

Proof. From Theorem 2.1, when $\mu=-1$, we have

$${}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|-z\right]=\int_0^\rho e^{-zt}H_{q,p}^{p,0}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\frac{dt}{t}+\eta(l_1-\rho z)e^{-\rho z}. \tag{2.14}$$

Example 2.3. *The four parameters Wright function* $\phi\left((\mu,a),(\nu,b);z\right)$ *possesses the following integral representation*

$$\begin{aligned} \phi\left((\mu,a),(\nu,b);z\right)=&\int_0^{\mu^\mu\nu^\nu} e^{zt}H_{2,1}^{1,0}\left[t\Big|^{(\mu,a),(\nu,b)}_{(1,1)}\right]\frac{dt}{t}+ \\ &+\frac{\mu^{\frac{1}{2}-a}\nu^{\frac{1}{2}-b}}{\sqrt{2\pi}}\left(l_1+\mu^\mu\nu^\nu z\right)e^{\mu^\mu\nu^\nu z}, \end{aligned} \tag{2.15}$$

where

$$l_1=\frac{1}{12}-\frac{6a^2-6a+1}{12\mu}-\frac{6b^2-6b+1}{12\nu},$$

and a,b,μ,ν *be a real number such that* $\mu+\nu=1$ *and* $a+b=1/2$.

The following lemma is called the Jensen's integral inequality, for more details, one may see [5, Chap. I, Eq. (7.15)].

Lemma 2.1. *Let* ω *be a non-negative measure and let* $\varphi\geq 0$ *be a convex function. Then for all* f *be a integrable function we have*

$$\varphi\left(\int f d\nu\Big/\int d\nu\right)\leq\int\varphi\circ f d\nu\Big/\int d\nu. \tag{2.16}$$

In the next theorem we present a new Luke type inequality when $\mu=0$.

Theorem 2.2. *Keep the notations and constraints of hypotheses of Corollary 2.1. Assume that the function* $H_{q,p}^{p,0}[.]$ *is non-negative, then the following two-sided bounding inequality holds true:*

$$\Psi_0\, e^{-(\Psi_1/\Psi_0)z} + \eta\, e^{-\rho z} \le {}_p\Psi_q\left[\begin{matrix}(\alpha_p, A_p)\\(\beta_q, B_q)\end{matrix}\middle| -z\right] \le \tag{2.17}$$
$$\le \left(\Psi_0 - \frac{\Psi_1}{\rho}\right) + \left(\eta + \frac{\Psi_1}{\rho}\right) e^{-\rho z},$$

where

$$\Psi_0 := \frac{\prod_{i=1}^p \Gamma(\alpha_i)}{\prod_{j=1}^q \Gamma(\beta_j)} - \eta, \quad \Psi_1 := \frac{\prod_{i=1}^p \Gamma(\alpha_i + A_i)}{\prod_{j=1}^q \Gamma(\beta_j + B_j)} - \eta\rho.$$

Proof. Letting $\varphi_z(t) = e^{-zt}$, $f(t) = t$, and

$$d\nu(t) = H_{p,p}^{p,0}\left(t\middle|\begin{matrix}(A,\beta_p)\\(A,\alpha_p)\end{matrix}\right)\frac{dt}{t}.$$

From (2.4) we get

$$\int_0^\rho d\nu(t) = \frac{\prod_{i=1}^p \Gamma(\alpha_i)}{\prod_{j=1}^q \Gamma(\beta_j)} - \eta = \Psi_0,$$

and

$$\int_0^\rho f(t)d\nu(t) = \frac{\prod_{i=1}^p \Gamma(\alpha_i + A_i)}{\prod_{j=1}^p \Gamma(\beta_j + B_j)} - \eta\rho = \Psi_1,$$

and using (2.1) when $m = 0$ we find

$$\int_0^\rho \phi_z(f(t))d\nu(t) = {}_p\Psi_p\left[\begin{matrix}(\alpha_p, A)\\(\beta_q, A)\end{matrix}\middle| -z\right] - \eta e^{-\rho z}.$$

Hence, Lemma 2.1 completes the proof of the lower bound of inequalities (2.17). In order to demonstrate the upper bound, we will apply the converse Jensen inequality, due to Lah and Ribarić, which reads as follows. Set

$$A(f) = \int_m^M f(s)d\sigma(s)\Big/\int_m^M d\sigma(s),$$

where σ is a non-negative measure and f is a continuous function. If $-\infty < m < M < \infty$ and φ is convex on $[m, M]$, then according to [3, Theorem 3.37]

$$(M - m)A(\varphi(f)) \le (M - A(f))\varphi(m) + (A(f) - m)\varphi(M). \tag{2.18}$$

Setting

$$\varphi_z(t) = e^{-zt}, \quad d\sigma(t) = d\nu(t), \quad f(s) = s \quad \text{and} \quad [m, M] = [0, \rho],$$

we complete the proof of the upper bound in (2.17). □

In view of inequalities (2.17) and the Laplace transform of the function $x^{\lambda-1}{}_p\Psi_q[x]$ [18, Eq. (7)]

$$\int_0^\infty e^{-t}t^{\lambda-1}{}_p\Psi_q\left[\begin{matrix}(\alpha_p,A_p)\\(\beta_q,B_q)\end{matrix}\middle| zt\right]dt = {}_{p+1}\Psi_q\left[\begin{matrix}(\lambda,1),(\alpha_p,A_p)\\(\beta_q,B_q)\end{matrix}\middle| z\right] \tag{2.19}$$

and make use of the following known formula

$$\int_0^\infty t^\lambda e^{-\sigma t}dt = \frac{\Gamma(\lambda+1)}{\sigma^{\lambda+1}},\ (\lambda > -1,\ \sigma > 0),$$

we can deduce the new following inequalities for the function ${}_{p+1}\Psi_q[.]$:

Corollary 2.4. *Let $\lambda > 0$ and suppose the hypotheses of Corollary 2.1 are satisfied. If the function $H^{p,0}_{q,p}[.]$ is non-negative, then the following two–sided bounding inequality holds true:*

$$\begin{aligned}\frac{\eta\Gamma(\lambda)}{(1+\rho z)^\lambda} + \frac{\Psi_0\Gamma(\lambda)}{(1+(\Psi_1/\Psi_0)z)^\lambda} &\leq {}_{p+1}\Psi_q\left[{}^{(\lambda,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right] \leq \\ &\left(\Gamma(\lambda)\Psi_0 - \frac{\Gamma(\lambda)\Psi_1}{\rho}\right) + \frac{\Gamma(\lambda)\left(\eta+\frac{\Psi_1}{\rho}\right)}{(1+\rho z)^\lambda}.\end{aligned} \tag{2.20}$$

Theorem 2.3. *Keep the notations and constraints of hypotheses of Corollary 2.1. Assume that the function $H^{p,0}_{q,p}[.]$ is non-negative, then the following inequality*

$$\frac{\Gamma(\sigma)\Psi_0}{(1+(\Psi_1/\psi_0)z)^\sigma} + \eta\,{}_1F_0(\sigma;-;-\rho z) \leq {}_{p+1}\Psi_q\left[{}^{(\sigma,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right], \tag{2.21}$$

is valid for all $\sigma > 0$ and $|z| < 1$.

Proof. We set $\varphi(u) = u^\sigma,\ \sigma > 0,\ f(t) = 1/(1+tz)$ and

$$d\nu(t) = t^{-1}\Gamma(\sigma)H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)dt.$$

By (2.4) we have

$$\begin{aligned}\int_0^\rho d\nu(t) &= \Gamma(\sigma)\int_0^\rho H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\frac{dt}{t} \\ &= \frac{\Gamma(\sigma)\prod_{i=1}^p\Gamma(\alpha_i)}{\prod_{j=1}^q\Gamma(\beta_j)} - \eta\Gamma(\sigma).\end{aligned} \tag{2.22}$$

Moreover (2.11), reads

$$\int_0^\rho f(t)d\nu(t) = {}_{p+1}\Psi_q\left[{}^{(1,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right] - \frac{\eta}{1+\rho z}, \tag{2.23}$$

and

$$\int_0^\rho \varphi(f(t))d\nu(t) = {}_{p+1}\Psi_q\left[{}^{(\sigma,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right] - \eta\,{}_1F_0(\sigma;-;-\rho z). \tag{2.24}$$

By means of Lemma 2.1 we obtain

$$\begin{aligned}\Gamma(\sigma)\Psi_0^{1-\sigma}\left({}_{p+1}\Psi_q\left[{}^{(1,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right] - \frac{\eta}{1+\rho z}\right)^\sigma &\leq \\ {}_{p+1}\Psi_q\left[{}^{(\sigma,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right] - \eta\,{}_1F_0(\sigma;-;-\rho z)&\end{aligned} \tag{2.25}$$

By virtue of the left-hand side of inequality (2.20) and (2.25) we conclude the inequality (2.21). □

The next lemma is in fact the so-called the Chebyshev integral inequality, see [5, p. 40].

Lemma 2.2. *If $f, g : [a, b] \longrightarrow \mathbb{R}$ are synchoronous (both increasing or decreasing) integrable functions, and $p : [a, b] \longrightarrow \mathbb{R}$ is a positive integrable function, then*

$$\int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt. \tag{2.26}$$

Note that if f and g are asynchronous (one is decreasing and the other is increasing), then (2.26) is reversed.

Theorem 2.4. *Assume that the hypotheses of Corollary 2.1 are satisfied. Suppose that $\delta, \sigma > 0$. If $H^{p,0}_{q,p}[.]$ is non-negative, then the function*

$$\begin{aligned} F :&= F\left[\begin{smallmatrix}(\sigma,1),(\alpha_p,A_p)\\(\beta_q,B_q)\end{smallmatrix}\middle|\delta; z\right] = \\ &= \frac{{}_{p+1}\Psi_q\left[\begin{smallmatrix}(\sigma,1),(\alpha_p+\delta A_p,A_p)\\ \\(\beta_q+\delta B_q,B_q)\end{smallmatrix}\middle| -z\right] - \eta\ {}_1F_0(\sigma;-;-\rho z)}{{}_{p+1}\Psi_q\left[\begin{smallmatrix}(\sigma,1),(\alpha_p,A_p)\\ \\(\beta_q,B_q)\end{smallmatrix}\middle| -z\right] - \eta\ {}_1F_0(\sigma;-;-\rho z)}, \end{aligned} \tag{2.27}$$

is increasing on $(0, 1)$. In addition, the function $F(z)$ is decreasing on $(0, 1)$ for each $\delta < 0$ and $\sigma > 0$.

Proof. In view of (2.11), using the following property of the Fox H-function [8, Property 1.5, p. 12]

$$H^{n,m}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p+\delta A_p)\\(B_q,\beta_q+\delta B_q)\end{smallmatrix}\middle| z\right] = z^{\delta} H^{n,m}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| z\right],$$

we can rewrite the function F as follows:

$$\begin{aligned} F\left[\begin{smallmatrix}(\sigma,1),(\alpha_p,A_p)\\(\beta_q,B_q)\end{smallmatrix}\middle|\delta; z\right] &= \frac{\int_0^\rho H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p+\delta A_p)\\(B_q,\beta_q+\delta B_q)\end{smallmatrix}\middle| t\right]\frac{dt}{t(1+tz)^\sigma}}{\int_0^\rho H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| t\right]\frac{dt}{t(1+tz)^\sigma}} \\ &= \frac{\int_0^\rho t^{\delta-1} H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| t\right]\frac{dt}{(1+tz)^\sigma}}{\int_0^\rho H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| t\right]\frac{dt}{t(1+tz)^\sigma}}. \end{aligned}$$

Now, we consider the functions $p, f, g : [0, \rho] \longrightarrow \mathbb{R}$, defined by

$$p(t) = t^{-1}(1+tz)^{-\sigma} H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| t\right],\ f(t) = t^{\delta},\ g(t) = \frac{t}{1+tz}.$$

Observe that the functions f and g are increasing, thus, by means of Lemma 2.2, we infer

$$\begin{aligned} &\left(\int_0^\rho t^{\sigma-1} H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| t\right]\frac{dt}{(1+tz)^\sigma}\right)\left(\int_0^\rho H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| t\right]\frac{dt}{(1+tz)^{\sigma+1}}\right) \\ &\leq \left(\int_0^\rho H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| t\right]\frac{dt}{t(1+tz)^\sigma}\right)\left(\int_0^\rho t^{\sigma} H^{p,0}_{q,p}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle| t\right]\frac{dt}{(1+tz)^{\sigma+1}}\right). \end{aligned} \tag{2.28}$$

On the other hand, we have

$$\frac{1}{\sigma}\left[\int_0^\rho H_{q,p}^{p,0}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle|t\right]\frac{dt}{t(1+tz)^\sigma}\right]^2\frac{\partial}{\partial z}F\left[\begin{smallmatrix}(\sigma,1),(\alpha_p,A_p)\\(\beta_q,B_q)\end{smallmatrix}\middle|\delta;z\right]= \tag{2.29}$$

$$=\left(\int_0^\rho H_{q,p}^{p,0}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle|t\right]\frac{dt}{t(1+tz)^\sigma}\right)\left(\int_0^\rho t^\sigma H_{q,p}^{p,0}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle|t\right]\frac{dt}{(1+tz)^{\sigma+1}}\right)$$
$$-\left(\int_0^\rho t^{\sigma-1} H_{q,p}^{p,0}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle|t\right]\frac{dt}{(1+tz)^\sigma}\right)\left(\int_0^\rho H_{q,p}^{p,0}\left[\begin{smallmatrix}(A_p,\alpha_p)\\(B_q,\beta_q)\end{smallmatrix}\middle|t\right]\frac{dt}{(1+tz)^{\sigma+1}}\right).$$

By (2.28) and (2.29) we deduce that the function $z\mapsto F(z)$ is increasing on $(0,1)$ for all $\sigma>0$ and $\delta>0$. Moreover, if $\delta<0$ then the inequality (2.28) is reversed and consequently the function $z\mapsto F(z)$ is decreasing on $(0,1)$ for all $\sigma>0$ and $\delta<0$. Now, the proof of this theorem is completed. □

Acknowledgements: The author is grateful to the reviewers for the suggestions that help to improve the paper.

Список литературы

[1] D. E. Karp, E. Prilepkina, "Some new facts concerning the delta neutral case of Fox's H-Function", Comput. Methods Funct. Theory, **17** (2), 343 – 367 (2017).
[2] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge (2010).
[3] J. E. Pecarić, F. Proschan, Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Math. Science Eng., **187**, Academic Press (1992).
[4] Yu. Luchko, R. Gorenflo, "Scale-invariant solutions of a partial differential equation of fractional order", Fract. Calc. Appl. Anal., **1**, 63 – 78 (1998).
[5] D. S. Mitrinović, J. E. Pecarić, A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers (1993).
[6] A. M. Mathai, A Handbook of Generalized Special Functions for Statistical and physical Sciences, Clarendon Press, Oxford (1993).
[7] E. M. Wright, "The asymptotic expansion of the generalized hypergeometric function", Journal London Math. Soc., **10**, 287 – 293 (1935).
[8] A. M. Mathai, R. K. Saxena H. J. Haubold, The H-functions: Theory and Applications, Springer (2010).
[9] K. Mehrez, S. M. Sitnik, "Functional inequalities for the Mittag–Lefler functions", Results Math., **72** (1), 703 – 714 (2017).
[10] K. Mehrez, S. M. Sitnik, "Turán type inequalities for classical and generalized Mittag-Leffler functions", Anal Math., **44**, 521 – 541 (2018).
[11] K. Mehrez, "Functional inequalities for the Wright functions", Integral Trans. Special Funct., **28** (2), 130 – 144 (2017).
[12] K. Mehrez, "New integral representations for the Fox-Wright functions and its applications", J. Math. Anal. Appl., **468** (2), 650 – 673 (2018).
[13] K. Mehrez, "Monotonicity patterns and functional inequalities for classical and generalized Wright functions", Math. Inequal. Appl., **22** (3), 901 – 916 (2019).
[14] K. Mehrez, S. M. Sitnik, "Functional inequalities for Fox-Wright functions", Ramanujan J., **50** (2), 263 – 287 (2019).
[15] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam (2006).
[16] A. A. Kilbas, M. Saigo, H-transforms and applications, Analytical Methods and Special Functions, Volume 9, Chapman Hall/CRC (2004).

[17] E. M. Wright, "The asymptotic expansion of the generalized hypergeometric function", Journal London Math. Soc., **10**, 287 – 293 (1935).
[18] T. K. Pogány, H. M. Srivastava, "Some Mathieu-type series associated with the Fox–Wright function", Comput. Math. Appl., **57**, 127 – 140 (2009).

Поступила 27 ноября 2019

После доработки 28 марта 2020

Принята к публикации 10 июля 2020

Известия НАН Армении, Математика, том 56, н. 1, 2021, стр. 66 – 80.

QUASI SURE STRASSEN'S LAW OF THE ITERATED LOGARITHM FOR INCREMENTS OF FBM IN HÖLDER NORM

Y. MO AND Q. LIU

Guilin University of Electronic Technology, Guilin, China.[1]
Guilin University of Electronic Technology, Guilin, China
E-mails: *175048247@qq.com* *qihliu@guet.edu.cn*

Abstract. In this paper, we present functional Strassen's law of the iterated logarithm for Csörgő-Révész (C-R) increments of a fractional Brownian motion in Hölder norm with respect to (r, p)-capacity. The method of the proof for our main results is based on the large deviation for the fractional Brownian motion.

MSC2010 numbers: 60F15; 60F17; 60G18; 60G22.
Keywords: Strassen's LIL; C-R increments; FBM; $(r, p)-$capacity; Hölder norm.

1. Introduction and main results

The functional limit theorems for fractional Brownian motion (FBM) have been investigated in various directions. For example, Wang [5] studied functional limit theorems for increments of a fractional Brownian motion under the Sup-norm. At the same time, Lin, Wang and Hwang [3] obtained functional limit theorems for d-dimensional FBM under Hölder norm.

The capacity is a set function on B with the property that it sometimes takes positive values even for $\mu-$null sets, while a set of capacity zero has always $\mu-$measure zero. Therefore, an interesting problem is to find out what property holds not only almost sure but also quasi sure. In this paper, we shall discuss this topic. Liu [4] established quasi sure Strassen-type law of the iterated logarithm for Csörgő-Révész (C-R) increments of Brownian Motion (BM) in Hölder norm with respect to $(r, p)-$capacity. In this paper, we present Strassen's law of the iterated logarithm for C-R increments of

[1]This work was partially supported by NSFC(No.11771105), Guangxi Natural Science Foundation (No. 2017GXNSFFA198012) and Guangxi Distinguished Expert Project.

FBM in Hölder norm with respect to $(r,p)-$capacity. We generalize Strassen-type law of the iterated logarithm of FBM in terms of $(r,p)-$capacities under Hölder norm, the corresponding results in [3], [4] and [7] are extended.

Let (B,H,μ) be an abstract Wiener space and $D^{r,p}$ denotes the Sobolev space, i.e.,

$$D^{r,p}=(1-\mathcal{L})^{-\frac{r}{2}}\boldsymbol{L}^p,\ \|F\|_{r,p}=\|(1-\mathcal{L})^{-\frac{r}{2}}F\|_p,\ F\in L^p,\ r\ge 0,\ 1\le p<\infty,$$

where $\boldsymbol{L}^p$ denotes L^p-space of real-valued functions on (B,μ) and $\mathcal{L}$ is the Ornstein-Uhlenbeck operator on (B,H,μ). For $r\ge 0,\ p>1$, $(r,p)-$capacity is defined by

$$C_{r,p}(O)=\inf\{\|F\|^p_{r,p};F\in D_{r,p},F\ge 1,\mu-a.s.\quad on\quad O\}\text{ for open set }O\subset B,$$

and for any set $A\subset B$, we define $C_{r,p}(A)$ by

$$C_{r,p}(A)=\inf\{C_{r,p}(O);A\subset O\subset\ B,O\text{ is open}\}.$$

Let $\{X(t);t\ge 0\}$ be a standard γ-fractional Brownian motion with $0<\gamma<1$ and $X(0)=0$. The $\{X(t);t\ge 0\}$ has a covariance function

$$R(s,t)=E(X(s)X(t))=\frac{1}{2}(s^{2\gamma}+t^{2\gamma}-|s-t|^{2\gamma}),\quad s,t\ge 0,$$

and a representation

$$X(t)=\int_{R^1}\frac{1}{k_\gamma}\left\{|x-t|^{(2\gamma-1)/2}-|x|^{(2\gamma-1)/2}\right\}dB(x),$$

where $k^2_\gamma=\displaystyle\int_{R^1}\left\{|x-1|^{(2\gamma-1)/2}-|x|^{(2\gamma-1)/2}\right\}^2dx$, $\{B(t);-\infty<t<+\infty\}$ is a Brownian motion and $\dfrac{1}{k_\gamma}\{|x-t|^{(2\gamma-1)/2}-|x|^{(2\gamma-1)/2}\}$ is interpreted to be $I_{(0,t]}$ when $\gamma=\frac{1}{2}$. $\{X(t);t\ge 0\}$ has stationary increments with $E(X(s+t)-X(s))^2=t^{2\gamma},\ s,t\ge 0$ and when $\gamma=\frac{1}{2}$, it is a standard Brownian motion. Let $C_0[0,1]$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$ with value zero at the origin, endowed with usual norm $\|f\|=\sup_{0\le t\le 1}|f(t)|$ and we denote by

$$\mathcal{H}=\{f\in C_0[0,1]:f\text{ is an absolutely continuous function},$$
$$\|f\|^2_{\mathcal{H}}=\int_0^1(\dot f(s))^2ds<\infty\}.$$

Then $\mathcal{H}$ is a Hilbert space with respect to the scalar product

$$\langle f,g\rangle=\int_0^1\dot f(x)\dot g(x)dx,\qquad\text{for}\quad f,g\in\mathcal{H}.$$

Let μ be the Wiener measure on $C_0[0,1]$, then $(C_0, \mathcal{H}, \mu)$ is an abstract Wienner space. Let us consider two Banach spaces as follows

$$\mathcal{C}^\alpha = \left\{ f \in C_0[0,1] : \|f(\cdot)\|_\alpha = \sup_{s,t\in[0,1],s\neq t} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty \right\},$$

$$\mathcal{C}^{\alpha,0} = \left\{ f \in \mathcal{C}^\alpha : \lim_{\delta\to 0} \sup_{s,t\in[0,1],0<|t-s|<\delta} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = 0 \right\},$$

where $0<\alpha<\frac{1}{2}$. In this paper, we assume $0<\alpha<\gamma<\frac{1}{2}$. Then $(C^{\alpha,0}, \mathcal{H}, \mu)$ is also an abstract Wiener space, see [1, Theorem 2.4] for details. Define a mapping $I : \mathcal{C}^{\alpha,0} \to [0,\infty]$ by

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{f}(t)|^2 dt, & f \in \mathcal{H}, \\ +\infty, & \text{otherwise.} \end{cases}$$

The limit set associated with functional laws of the iterated logarithm for $\{X(t); t \geq 0\}$ is K_γ and it is a subset of functions in $\mathcal{C}^{\alpha,0}$ with the form

$$f(t) = \int_{R^1} \frac{1}{k_\gamma}\{|x-t|^{(2\gamma-1)/2} - |x|^{(2\gamma-1)/2}\} g(x)dx, \quad 0 \leq t \leq 1.$$

Here the function $g(x)$ ranges over the unit ball of $L^2(R^1)$ and $\int_{R^1} g^2(s)ds \leq 1$. The subset K of $\mathcal{C}^{\alpha,0}$ is defined by

$$K = \{f \in \mathcal{H} : f \in K_\gamma, 2I(f) \leq 1\}.$$

Let a_u be a non-decreasing function from $(0,+\infty)$ to $(0,+\infty)$ such that

(H$_1$) $a_u \leq u$, for any $u \in (0,+\infty)$;

(H$_2$) The function $\dfrac{u}{a_u}$ is non-decreasing;

(H$_3$) $\lim_{u\to+\infty} \dfrac{a_u}{u} = \rho$ with the constant $\rho < 1$;

(H$_4$) $\lim_{u\to+\infty} \dfrac{\log \frac{u}{a_u}}{\log\log u} = +\infty.$

We set $\ell_u = \log \frac{u \log u}{a_u}$, $\beta_u = (2a_u^{2\gamma}\ell_u)^{-\frac{1}{2}}$ and let $\Delta(t,u)$ denote the path $s \to X(ut + a_u s) - X(ut)$, $s \in [0,1]$.

In the following, we state our main results.

Theorem 1.1. *Assume that* (H$_1$) *and* (H$_2$) *hold, then we have*

$$\lim_{u\to\infty} \sup_{t\in[0,1-\frac{a_u}{u}]} \inf_{f\in K} \|\beta_u \Delta(t,u)(\cdot) - f(\cdot)\|_\alpha = 0, \quad C_{r,p} - q.s. \tag{1.1}$$

Theorem 1.2. *Assume that* (H_1), (H_2) *and* (H_3) *hold, then for any* $f \in K$, *we have*

$$\liminf_{u\to\infty} \inf_{t\in[0,1-\frac{a_u}{u}]} \|\beta_u \Delta(t,u)(\cdot) - f(\cdot)\|_\alpha = 0, \quad C_{r,p} - q.s. \tag{1.2}$$

Moreover, assume that (H_4) *also holds, then for any* $f \in K$, *we have*

$$\lim_{u\to\infty} \inf_{t\in[0,1-\frac{a_u}{u}]} \|\beta_u \Delta(t,u)(\cdot) - f(\cdot)\|_\alpha = 0, \quad C_{r,p} - q.s. \tag{1.3}$$

Remark 1.1. *In* [3], *authors proved the equations* (1.1)-(1.3) *in the sense of probability. Theorem* 1.1 *and Theorem* 1.2 *generalized them from probability to* (r,p)*-capacity. At the same time, in* [4], *Liu established quasi sure Strassen-type law of the iterated logarithm for C-R increments of BM in Hölder norm with respect to* $(r,p)-$*capacity. Theorem* 1.1 *and Theorem* 1.2 *generalized them from BM to FBM. From the proof below, we can see that the method we use is different from that in* [3] *and* [4], *and it is more complicated.*

This paper is organized as follows. In Section 2, we introduce some basic lemmas which will be used in this paper. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

2. Some lemmas

Our proofs are based on the following lemmas. We followed Theorem 3.3 in [7], we have following lemma.

Lemma 2.1. *Assume* $\{X(t); t \geq 0\}$ *is FBM with* $X(0) = 0$. *Then for any closed set* $A \subset \mathcal{C}^{\alpha,0}$, *we have*

$$\limsup_{\varepsilon\to 0} \varepsilon \left(\log C_{r,p} \left(\bigcup_{0\leq t\leq T-h} \left\{ \sqrt{\frac{\varepsilon}{h^{2\gamma}}} \left(X(t+h\cdot) - X(t)\right) \in A \right\} \right) + \log \frac{h}{T} \right) \leq$$
$$\leq - \inf_{f\in A} I(f),$$

where $0 < h < 1$, $0 \leq \alpha < \gamma < 1$.

We followed the method in [6], we have following lemma.

Lemma 2.2. *Let* $1 \leq k < Z$, $q_1, q_2 \in (1,\infty)$ *be given so that* $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}$. *For any* $f \in K$, *put*

$$F_\varepsilon^{(i)} = \left\| \varepsilon \frac{X(t_i + h_i\cdot) - X(t_i)}{\sqrt{h_i^{2\gamma}}} - f \right\|_\alpha, i = 1, 2, ..., n,$$

where $0 \leq t_i < \infty, h_i > 0$. *Then there exits a constant* $C = C(r, p, q_1, f)$, *such that for any* $\delta \in (0,1]$ *and* $\varepsilon \in (0,1]$, *we have*

$$C_{r,p}\left(\bigcap_{i=1}^{n}\left\{Z; a_i < F_\varepsilon^{(i)} < b_i\right\}\right)^{1/p} \leq$$

$$\leq C\delta^{-2r^2-r}n^r\mu\left(\bigcap_{i=1}^{n}\left\{Z; a_i-\delta < F_\varepsilon^{(i)} < b_i+\delta\right\}\right)^{1/q_2}.$$

The following lemmas can be found in the corresponding literature, so we list them directly without proof.

Lemma 2.3. ([3]) *Let* $0 < \gamma < 1$ *and fix* $0 < \alpha < q < \gamma$. *Let* $d_k = k^{k+(1-\iota)}$, $s_k = k^{-k}$ *for* $k \geq 1$ *and* $0 < \iota < 1$. *Let*

$$Y_k(s_k, t) = \int_{|x|\notin I_k} \frac{1}{k_\gamma}\left\{|x - s_k t|^{(2\gamma-1)/2} - |x|^{(2\gamma-1)/2}\right\} dB(x), \qquad 0 \leq t \leq 1,$$

where $B(x)$ *is a standard Brownian motion,* $I_k = (s_k d_{k-1}, s_k d_k]$. *Let* $0 < \beta < \iota$. *Then, for* $\delta = \min\{2\beta(\gamma-q), \iota-\beta, (1-\iota)(2-2\gamma), (2\gamma-2q)\iota\}$, *there is a constant* $C > 0$ *depending only on* γ *such that uniformly in* t, h, k, *we have*

$$\sigma_k^2(t,h) = E\{[Y_k(s_k, t+h) - Y_k(s_k, t)]^2\} \leq Ch^{2q}s_k^{2\gamma}k^{-\delta}.$$

Lemma 2.4. ([3]) *Let* $\{X(t); t \geq 0\}$ *be FBM with* $X(0) = 0$ *and* $\sigma^2(t-s) = E(X(t) - X(s))^2$, *For any* $\epsilon > 0$, *there exists a positive constant* $C_1 = C(\epsilon)$, $\forall x \geq x_0 > 0$ *such that*

$$\mu\left(\sup_{0\leq s<t\leq 1}|X(t+s) - X(t)| \geq x\sigma(t-s)\right) \leq C\exp\left(-\frac{x^2}{2+\epsilon}\right).$$

Lemma 2.5. ([2]) *Consider a separable Banach space* E *with dual* E^* *and a centered Gaussian measure* μ *on* E. *Let* V *be a convex, symmetric, measurable subset of* E. *For all* $f \in H$,

$$\mu(f+V) \geq \mu(V)\exp\left\{-\frac{1}{2}\|f\|_\mu^2\right\}.$$

3. Proof of Theorem 1.1

Put $u_n = \theta^n$ with $1 < \theta < 2$. For any u, there exists an integer n such that $u_n \leq u \leq u_{n+1}$. Let $\Psi_{t,u}(s) = \beta_u[X(t+a_u s) - X(t)]$, $s \in [0,1]$, $t \in [0, u-a_u]$, then $\Psi_{t,u}(s) = \frac{\beta_u}{\beta_{u_{n+1}}}\Psi_{t,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}s)$. We have

$$\begin{aligned}
&\sup_{t\in[0,1-\frac{a_u}{u}]}\inf_{f\in K}\|\beta_u\Delta(t,u)(\cdot)-f(\cdot)\|_\alpha\\
&=\sup_{t\in[0,1-\frac{a_u}{u}]}\inf_{f\in K}\|\beta_u[X(ut+a_us)-X(ut)]-f(s)\|_\alpha\\
&=\sup_{x\in[0,u-a_u]}\inf_{f\in K}\|\beta_u[X(x+a_us)-X(x)]-f(s)\|_\alpha\\
&=\sup_{x\in[0,u-a_u]}\inf_{f\in K}\|\Psi_{x,u}(s)-f(s)\|_\alpha\\
&=\sup_{x\in[0,u-a_u]}\inf_{f\in K}\|\frac{\beta_u}{\beta_{u_{n+1}}}\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}s)-f(s)\|_\alpha \qquad (3.1)\\
&\le \sup_{x\in[0,u_{n+1}-a_{u_n}]}\inf_{f\in K}\left\|\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot)-f(\frac{a_u}{a_{u_{n+1}}}\cdot)\right\|_\alpha\\
&+\left(\frac{\beta_u}{\beta_{u_{n+1}}}-1\right)\sup_{x\in[0,u_{n+1}-a_{u_n}]}\left\|\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot)\right\|_\alpha\\
&+\inf_{f\in K}\left\|f(\frac{a_u}{a_{u_{n+1}}}\cdot)-f(\cdot)\right\|_\alpha := I_1+I_2+I_3.
\end{aligned}$$

For any $\varepsilon > 0$, let $A = \{f \in \mathcal{C}^{\alpha,0}; \|f-K\|_\alpha \ge \varepsilon\}$. If $f \in A$, $f \notin K$, thus there exits $\delta > 0$, $2\inf_{f\in K} I(f) > 1+\delta := \eta > 1$. By Lemma *2.1*, *(H1)* and *(H2)*, we have

$$\begin{aligned}
&C_{r,p}(I_1\ge\varepsilon) =\\
&= C_{r,p}\left(\sup_{x\in[0,u_{n+1}-a_{u_n}]}\inf_{f\in K}\left\|\beta_{u_{n+1}}[X(x+a_{n+1}s)-X(x)]-f(s)\right\|_\alpha\ge\varepsilon\right)\\
&= C_{r,p}\left(\sup_{x\in[0,u_{n+1}-a_{u_n}]}\left\|\sqrt{\frac{1}{2\ell_{n+1}}}\frac{X(x+a_{n+1}s)-X(x)}{\sqrt{a_{u_{n+1}}^{2\gamma}}}-K\right\|_\alpha\ge\varepsilon\right)\\
&\le C_{r,p}\left(\bigcup_{0\le x\le u_{n+1}-a_{u_n}}\left\{\sqrt{\frac{1}{2\ell_{n+1}}}\frac{X(x+a_{n+1}s)-X(x)}{\sqrt{a_{u_{n+1}}^{2\gamma}}}\in A\right\}\right)\\
&\le \frac{u_{n+1}}{a_{u_n}}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^\eta \le \theta\left(\frac{a_{u_n}}{u_n}\right)^{\eta-1}\left(\frac{1}{\log u_{n+1}}\right)^\eta \le \frac{\theta}{[(n+1)\log\theta]^\eta}.
\end{aligned}$$

It is clear that

$$\sum_{n=1}^{\infty}\frac{\theta}{[(n+1)\log\theta]^\eta}<\infty.$$

By the Borel-Cantelli lemma, we get

$$\lim_{n\to\infty} I_1 = 0,\ C_{r,p}-q.s. \qquad (3.2)$$

For any $f \in K \subset \mathcal{C}^{\alpha,0}$, $\|f\|_\alpha \le 1$ and large n, by (3.2), we have

$$\sup_{x\in[0,u_{n+1}-a_{u_n}]} \left\| \Psi_{x,u_{n+1}}\left(\frac{a_u}{a_{u_{n+1}}}\cdot\right)\right\|_\alpha \le 2,\ C_{r,p}-q.s.$$

Obviously,

$$\frac{\beta_u}{\beta_{u_{n+1}}} - 1 \le \theta^\gamma - 1.$$

By (3.4) in [3], we have

$$I_3 = \sup_{f\in K} \left\| f\left(\frac{a_u}{a_{u_{n+1}}}\cdot\right) - f(\cdot)\right\|_\alpha \le 2(\theta-1)^{\gamma-\alpha}.$$

Letting $\theta \to 1$, we get

$$\lim_{n\to\infty} (I_2 + I_3) = 0, \qquad C_{r,p} - q.s. \tag{3.3}$$

Combining (3.1) with (3.2) and (3.3), we obtain (1.1).

4. Proof of Theorem 1.2

First of all, let's prove the following two lemmas.

Lemma 4.1. *If conditions* (H$_1$)-(H$_3$) *hold, for any* $f \in K$, *we have*

$$\liminf_{u\to\infty} \|\beta_u \Delta(1 - \frac{a_u}{u}, u) - f(s)\|_\alpha = 0, \qquad C_{r,p} - q.s.$$

Proof. Case (I). $\limsup_{u\to\infty} \frac{\log u a_u^{-1}}{\log\log u} < \infty$. We take u_m, such that $u_1 = u_0,\ u_{m+1} - a_{u_{m+1}} = u_m,\ m \ge 1$. For $k = 1, 2, \cdots$, we define

$$Z_k(t) = \int_{|x|\in(d_{k-1},d_k]} \frac{1}{K_\gamma}\left\{|x-t|^{(2\gamma-1)/2} - |x|^{(2\gamma-1)/2}\right\} dB(x)$$

and

$$X_k(t) = X(t) - Z_k(t),$$

for $0 \le t \le 1,\ d_k = k^{k+(1-\iota)},\ s_k = k^{-k},\ 0 < \gamma < 1$. Then $\{Z_k(\cdot)\},\ k = 1, 2, \cdots$, are independent and

$$\{s_k^\gamma X_k(\cdot)\} \overset{\mathcal{D}}{=} \{Y_k(s_k, \cdot)\}.$$

where $Y_k(s_k,\cdot)$ is as in Lemma 2.2. For any $\varepsilon>0$, we have

$$
\begin{aligned}
& C_{r,p}\left\{\|\beta_{u_m}\Delta(1-\frac{a_{u_m}}{u_m},u_m)(\cdot)-f(\cdot)\|_\alpha\geq 4\varepsilon\right\} \\
& = C_{r,p}\left\{\|\beta_{u_m}[X(u_{m-1}+a_{u_m}s)-X(u_{m-1})]-f(s)\|_\alpha\geq 4\varepsilon\right\} \\
& \leq C_{r,p}\left\{\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}s)-Z_m(u_{m-1})]-f(s)\|_\alpha\geq 2\varepsilon\right\} \qquad (4.1)\\
& + C_{r,p}\left\{\|\beta_{u_m}[X_m(u_{m-1}+a_{u_m}s)-X_m(u_{m-1})]\|_\alpha\geq 2\varepsilon\right\} \\
& := I_{41}+I_{42}.
\end{aligned}
$$

By Lemmas 2.2-2.4, we have

$$
\begin{aligned}
& I_{42}=C_{r,p}\left\{\|\beta_{u_m}[X_m(u_{m-1}+a_{u_m}s)-X_m(u_{m-1})]\|_\alpha\geq 2\varepsilon\right\} \\
& = C_{r,p}\left\{\left\|\frac{1}{\sqrt{2s_m^{2\gamma}\ell_{u_m}}}\frac{Y_m(s_m,u_{m-1}+a_{u_m}\cdot)-Y_m(s_m,u_{m-1})}{\sqrt{a_{u_m}^{2\gamma}}}\right\|_\alpha\geq 2\varepsilon\right\} \\
& \leq C\varepsilon^{(-2k^2-k)p}\mu\left(\left\|\frac{1}{\sqrt{2s_m^{2\gamma}\ell_{u_m}}}\frac{Y_m(s_m,u_{m-1}+a_{u_m}\cdot)-Y_m(s_m,u_{m-1})}{\sqrt{a_{u_m}^{2\gamma}}}\right\|_\alpha\geq \varepsilon\right)^{\frac{p}{q_2}} \\
& \leq C_1\mu\left(\left\|\frac{1}{\sqrt{2s_m^{2\gamma}\ell_{u_m}}}[Y_m(s_m,\frac{u_{m-1}}{a_{u_m}}+\cdot)-Y_m(s_m,\frac{u_{m-1}}{a_{u_m}})]\right\|_\alpha\geq \varepsilon\right)^{\frac{p}{q_2}} \\
& \leq C_1\mu\left(\sup_{0\leq x<y\leq 1}\left|\frac{Y_m(s_m,y)-Y_m(s_m,x)}{(y-x)^q\sqrt{Cs_m^{2\gamma}m^{-\delta}}}\right|\geq\varepsilon\sqrt{\frac{n^\delta}{C}2\ell_{u_m}}\right)^{\frac{p}{q_2}} \\
& \leq C_1\exp\left(-\frac{p}{q_2}\frac{2\varepsilon^2m^\delta}{(2+\epsilon)C}\log\frac{u_m\log u_m}{a_{u_m}}\right)\leq C_1\left(\frac{a_{u_m}}{u_m\log u_m}\right)^{C_2m^\delta},
\end{aligned}
$$

where

$$
C_1=C\varepsilon^{(-2k^2-k)p}>0,\quad C_2=\frac{p}{q_2}\frac{2\varepsilon^2}{(2+\epsilon)C}>0.
$$

For large m, we have $C_2m^\delta>1$, thus $\sum_{m=1}^\infty I_{42}<\infty$. By the Borel-Cantelli lemma, we get

$$
\lim_{m\to\infty}\{\|\beta_{u_m}[X_m(u_{m-1}+a_{u_m}s)-X_m(u_{m-1})]\|_\alpha\}=0, C_{r,p}-q.s. \qquad (4.2)
$$

Since $\{Z_k(\cdot)\},\ k=1,2,\cdots$ are independent, by Lemma 2.2, we have

(4.3)

$$C_{r,p}\left\{\bigcap_{m=l}^{n}\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq 2\varepsilon\right\}^{\frac{q_2}{p}}$$
$$\leq Cn^{rq_2}\varepsilon^{(-2r^2-r)q_2}\prod_{k=l}^{n}\mu(\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq\varepsilon)$$
$$:=C_3n^{rq_2}\prod_{m=l}^{n}J_m\left(C_3=C\varepsilon^{(-2r^2-r)q_2}>0\right).$$

Moreover,

$$\begin{aligned}J_m&=\mu(\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq\varepsilon)\\&\leq\mu\bigg(\|\beta_{u_m}[X(u_{m-1}+a_{u_m}\cdot)-X(u_{m-1})]-f(\cdot)\|_\alpha\geq\frac{1}{2}\varepsilon\bigg)\\&+\mu\bigg(\|\beta_{u_m}[X_m(u_{m-1}+a_{u_m}\cdot)-X_m(u_{m-1})]\|_\alpha\geq\frac{1}{2}\varepsilon\bigg):=J_m^1+J_m^2.\end{aligned}\tag{4.4}$$

By the same method of the estimate of I_{42}, exist $C_{21}>0,\ C_{22}>0$, such that

$$J_m^2\leq C_{21}\left(\frac{a_{u_m}}{u_m\log u_m}\right)^{C_{22}m^\delta}.\tag{4.5}$$

Let $f^{(\epsilon)}=(1-\frac{\epsilon}{2})f$, for $f\in K$ and $0<\epsilon<1$. Then $f^{(\epsilon)}\in K$ and $\|f-f^{(\epsilon)}\|_\alpha<\frac{\epsilon}{2}$. For large n, by Lemma 2.5 and Lemma 2.3, we have

$$\mu\bigg(\|\beta_{u_m}[X(u_{m-1}+a_{u_m}\cdot)-X(u_{m-1})]-f(\cdot)\|_\alpha<\frac{1}{2}\varepsilon\bigg)$$
$$\geq\mu\bigg(\bigg\|\frac{X(u_{m-1}+a_{u_m}\cdot)-X(u_{m-1})}{a_{u_m}^\gamma}-f^{(\epsilon)}(\cdot)\sqrt{2\ell_{u_m}}\bigg\|_\alpha<\frac{\varepsilon}{4}\sqrt{2\ell_{u_m}}\bigg)$$
$$\geq\exp\left(-\|f^{(\epsilon)}\|_\gamma^2\ell_{u_m}\right)\mu\bigg(\bigg\|\frac{X(u_{m-1}+a_{u_m}\cdot)-X(u_{m-1})}{a_{u_m}^\gamma}\bigg\|_\alpha<\frac{\varepsilon}{4}\sqrt{2\ell_{u_m}}\bigg)$$
$$\geq\exp\left(-(1-\frac{\epsilon}{2})^2\|f\|_\gamma^2\log\frac{u_m\log u_m}{a_{u_m}}\right)\left(1-C\exp\left(-\frac{\varepsilon^2}{8(2+\epsilon)}\log\frac{u_m\log u_m}{a_{u_m}}\right)\right)$$
$$\geq\exp\left(-(1-\frac{\epsilon}{2})^2\log\frac{u_m\log u_m}{a_{u_m}}\right)\left(1-C\exp\left(-\frac{\varepsilon^2}{8(2+\epsilon)}\log\frac{u_m\log u_m}{a_{u_m}}\right)\right)$$
$$=\left(\frac{a_{u_m}}{u_m\log u_m}\right)^{C_{11}}\left(1-C\left(\frac{a_{u_m}}{u_m\log u_m}\right)^{C_{12}}\right),$$

where

$$C>0,\quad 0<C_{11}=(1-\frac{\epsilon}{2})^2<1,\quad C_{12}=\frac{\varepsilon^2}{8(2+\epsilon)}>0.$$

Therefore, for large m, $1 - C\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{12}} > \frac{1}{2}$, we have

$$\begin{aligned} J_m^1 &\leq \exp\left\{-\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\left(1 - C\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{12}}\right)\right\} \\ &\leq \exp\left\{-\frac{1}{2}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\}. \end{aligned} \tag{4.6}$$

By (4.3)-(4.6), for large n, we have

$$\begin{aligned} &C_{r,p}\left\{\bigcap_{m=l}^{n} \|\beta_{u_m}[Z_m(u_{m-1} + a_{u_m}\cdot) - Z_m(u_{m-1})] - f(\cdot)\|_\alpha \geq 2\varepsilon\right\}^{\frac{q_2}{p}} \\ &= C_1 n^{rq_2} \prod_{m=l}^{n}\left(C_{21}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{22}m^\delta} + \exp\left\{-\frac{1}{2}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\}\right). \\ &\leq C_1 n^{rq_2} \prod_{m=l}^{n} \exp\left\{-\frac{1}{4}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\} \\ &= C_1 n^{rq_2} \exp\left\{-\frac{1}{4}\sum_{m=l}^{n}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &C_{r,p}\left\{\bigcap_{m=l}^{n} \|\beta_{u_m}[Z_m(u_{m-1} + a_{u_m}\cdot) - Z_m(u_{m-1})] - f(\cdot)\|_\alpha \geq 2\varepsilon\right\} \\ &\leq C_1 n^{rp} \exp\left\{-\frac{p}{q_2}\frac{1}{4}\sum_{m=l}^{n}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}}\right\}. \end{aligned} \tag{4.7}$$

By *(H_3)* and definition of u_m, we have

$$\sum_{m=l}^{n}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}} \geq A(\log u_n)^{\eta_0},$$

where $A = A(l) > 0, \eta_0 = 1 - C_{11} > 0$. Since $\limsup_{u\to\infty} \frac{\log u a_u^{-1}}{\log\log u} < \infty$, we take $\theta > \frac{2}{\eta_0}$ and $u_0 = e^{(\log n_0)^\theta}$, for large n, we can proof

$$\log u_n \geq (\log n)^\theta, \ n \geq n_0.$$

Thus, for $n_0 < l < n$, we get

$$\sum_{m=l}^{n}\left(\frac{a_{u_m}}{u_m \log u_m}\right)^{C_{11}} \geq A(\log u_n)^{\eta_0} \geq A(\log n)^{\eta_0\theta} \geq (\log n)^2. \tag{4.8}$$

By (4.7) and (4.8), we obtain

$$C_{r,p}\left\{\bigcap_{m=l}^{n}\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq 2\varepsilon\right\}$$
$$\leq C_1 n^{rp}\exp\left\{-\frac{p}{4q_2}(\log n)^2\right\}\to 0,\qquad n\to\infty.$$

So,

$$C_{r,p}\left\{\bigcup_{l=1}^{\infty}\bigcap_{m=l}^{\infty}\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}\cdot)-Z_m(u_{m-1})]-f(\cdot)\|_\alpha\geq 2\varepsilon\right\}=0.$$

Thus,

$$\liminf_{u\to\infty}\{\|\beta_{u_m}[Z_m(u_{m-1}+a_{u_m}s)-Z_m(u_{m-1})]-f(s)\|_\alpha\}=0, \tag{4.9}$$

$C_{r,p}$ - q.s. By (4.1), (4.2) and (4.9), Lemma 4.1 holds.

Case (II) $\limsup_{u\to\infty}\frac{\log ua_u^{-1}}{\log\log u}=\infty$, see Lemma 4.2. Hence, the proof is completed.

Lemma 4.2. *If conditions* (H$_1$)-(H$_4$) *hold, for any* $f\in K$, *we have*

$$\lim_{u\to\infty}\inf_{t\in[0,1-\frac{a_u}{u}]}\|\beta_u\Delta(t,u)-f(s)\|_\alpha=0,\qquad C_{r,p}-q.s.$$

Proof. Since $\lim_{u\to\infty}\frac{\log ua_u^{-1}}{\log\log u}=\infty$, we choose u_n, such that $\frac{u_n}{a_{u_n}}=n^d,\ d>1$. Let $t_i=ia_{u_{n+1}},\ i=0,1,...,k_n=\left[\frac{u_n}{a_{u_{n+1}}}\right]-1,\ \ g(n)=\frac{\log\frac{u_n}{a_{u_n}}}{\log\log u_n}=\frac{\log n^d}{\log\log u_n}$. We have $u_n=\exp(n^{\frac{d}{g(n)}})$, $g(n)$ is non-decreasing and $g(n)\to\infty, n\to\infty$. Moreover, for any $b>0$, $\frac{n^b}{\log u_n}\to\infty$ and

$$1\leq\frac{u_{n+1}}{u_n}=\exp\left\{(n+1)^{\frac{d}{g(n+1)}}-n^{\frac{d}{g(n)}}\right\}\leq\exp\left(n^{\frac{d}{g(n)}-1}\right)\to 1,\ n\to\infty.$$

Therefore, we have

$$\begin{aligned}
&\inf_{t\in[0,1-\frac{a_u}{u}]}\|\beta_u\Delta(t,u)(\cdot)-f(\cdot)\|_\alpha\\
&=\inf_{x\in[0,u-a_u]}\|\beta_u[X(x+a_us)-X(x)]-f(s)\|_\alpha\\
&=\inf_{x\in[0,u-a_u]}\|\frac{\beta_u}{\beta_{u_{n+1}}}\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}s)-f(s)\|_\alpha\\
&\leq\inf_{x\in[0,u_n-a_{u_{n+1}}]}\left\|\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot)-f(\frac{a_u}{a_{u_{n+1}}}\cdot)\right\|_\alpha\\
&+\left(\frac{\beta_u}{\beta_{u_{n+1}}}-1\right)\sup_{x\in[0,u_n-a_{u_{n+1}}]}\left\|\Psi_{x,u_{n+1}}(\frac{a_u}{a_{u_{n+1}}}\cdot)\right\|_\alpha\\
&+\left\|f(\frac{a_u}{a_{u_{n+1}}}\cdot)-f(\cdot)\right\|_\alpha:=I_5+I_2'+I_3'.
\end{aligned}\tag{4.10}$$

Similarly to the proof of (3.3), we have

$$\lim_{n\to\infty}(I_2'+I_3')=0, \qquad C_{r,p}-q.s. \tag{4.11}$$

On the other hand,

(4.12)

$$\begin{aligned}
&C_{r,p}\{I_5\geq 4\varepsilon\}=C_{r,p}\{\inf_{x\in[0,u_n-a_{u_{n+1}}]}\left\|\Psi_{x,u_{n+1}}(\cdot)-f(\cdot)\right\|_\alpha\geq 4\varepsilon\}\\
&=C_{r,p}\{\inf_{x\in[0,u_n-a_{u_{n+1}}]}\left\|\beta_{u_{n+1}}[X(x+a_{u_{n+1}}\cdot)-X(x)]-f(\cdot)\right\|_\alpha\geq 4\varepsilon\}\\
&\leq C_{r,p}\{\min_{x_i\in[0,u_n-a_{u_{n+1}}]}\left\|\beta_{u_{n+1}}[Z_i(x_i+a_{u_{n+1}}\cdot)-Z_i(x_i)]-f(\cdot)\right\|_\alpha\geq 2\varepsilon\}\\
&+\leq C_{r,p}\{\max_{x_i\in[0,u_n-a_{u_{n+1}}]}\left\|\beta_{u_{n+1}}[X_i(x_i+a_{u_{n+1}}\cdot)-X_i(x_i)]-f(\cdot)\right\|_\alpha\geq 2\varepsilon\}\\
&:=I_{51}+I_{52}.
\end{aligned}$$

By Lemmas 2.2-2.4, we have

$$\begin{aligned}
I_{52}&\leq\sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1}C_{r,p}\left\{\|\beta_{u_n+1}[X_i(x_i+a_{u_{n+1}}s)-X_i(x_i)]\|_\alpha\geq 2\varepsilon\right\}\\
&=\sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1}C_{r,p}\left\{\left\|\frac{1}{\sqrt{2s_i^{2\gamma}\ell_{u_{n+1}}}}\frac{Y_i(s_i,x_i+a_{u_{n+1}}\cdot)-Y_i(s_i,x_i)}{\sqrt{a_{u_{n+1}}^{2\gamma}}}\right\|_\alpha\geq 2\varepsilon\right\}\\
&\leq C\varepsilon^{(-2k^2-k)p}\sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1}\mu\left(\left\|\frac{1}{\sqrt{2s_i^{2\gamma}\ell_{u_{n+1}}}}\frac{Y_i(s_i,x_i+a_{u_{n+1}}\cdot)-Y_i(s_i,x_i)}{\sqrt{a_{u_{n+1}}^{2\gamma}}}\right\|_\alpha\geq\varepsilon\right)^{\frac{p}{q_2}}\\
&\leq C_1\sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1}\mu\left(\left\|\frac{1}{\sqrt{2s_i^{2\gamma}\ell_{u_{n+1}}}}[Y_i(s_i,i+\cdot)-Y_i(s_i,i)]\right\|_\alpha\geq\varepsilon\right)^{\frac{p}{q_2}}\\
&\leq C_1\sum_{i=0}^{[\frac{u_n}{a_{u_{n+1}}}]-1}\mu\left(\sup_{0\leq x<y\leq 1}\left|\frac{Y_i(s_i,y)-Y_i(s_i,x)}{(y-x)^q\sqrt{Cs_i^{2\gamma}i^{-\delta}}}\right|\geq\varepsilon\sqrt{\frac{i^\delta}{C}2\ell_{u_{n+1}}}\right)^{\frac{p}{q_2}}\\
&\leq C_1\sum_{i=0}^{[\frac{u_n}{a_{u_n}}]-1}\exp\left(-\frac{p}{q_2}\frac{2\varepsilon^2 i^\delta}{(2+\epsilon)C}\log\frac{u_{n+1}\log u_{n+1}}{a_{u_{n+1}}}\right)\\
&\leq C_1\sum_{i=n_0}^{[\frac{u_n}{a_{u_{n+1}}}]-1}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_2 i^\delta}\leq C_1\frac{u_{n+1}}{a_{u_{n+1}}}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_2{n_0}^\delta}\\
&\leq C_1\,(n+1)^{-d(C_2n_0^\delta-1)-C_2{n_0}^\delta}.
\end{aligned}$$

where

$$C_1 = C\varepsilon^{(-2k^2-k)p} > 0, \quad C_2 = \frac{p}{q_2}\frac{\varepsilon^2}{(2+\epsilon)C} > 0.$$

Taking n_0 to be large enough such that $d(C_2 n_0^\delta - 1) + C_2 {n_0}^\delta > 1$, thus

$$\sum_{n=1}^{\infty} I_{52} < \infty.$$

By the Borel-Cantelli Lemma, we have

$$\limsup_{n\to\infty} \max_{x_i\in[0,u_n-a_{u_{n+1}}]} \left\|\beta_{u_{n+1}}[X_i(x_i+a_{u_{n+1}}\cdot) - X_i(x_i)]\right\|_\alpha = 0, \tag{4.13}$$

$C_{r,p}$ - q.s. Since $\{Z_k(\cdot)\}, k = 1, 2, \cdots$ are independent, by Lemma 2.2, we have

(4.14)

$$\begin{aligned}
I_{51} &= C_{r,p}\{\min_{x_i\in[0,u_n-a_{u_{n+1}}]} \left\|\beta_{u_{n+1}}[Z_i(x_i+a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\right\|_\alpha \ge 2\varepsilon\}\\
&= C_{r,p}\{\bigcap_{i=0}^{k_n} \left\|\beta_{u_{n+1}}[Z_i(x_i+a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\right\|_\alpha \ge 2\varepsilon\}\\
&\le C(k_n+1)^{rp}\varepsilon^{(-2r^2-r)p}\prod_{i=0}^{k_n}\mu\big(\|\beta_{u_{n+1}}[Z_i(x_i+a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\|_\alpha \ge \varepsilon\big)^{\frac{p}{q_2}}\\
&:= C_1(k_n+1)^{rp}\prod_{i=0}^{k_n} L_i^{\frac{p}{q_2}}. \left(C_1 = C\varepsilon^{(-2r^2-r)p} > 0\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
L_i &= \mu\big(\|\beta_{u_{n+1}}[Z_i(x_i+a_{u_{n+1}}\cdot) - Z_i(x_i)] - f(\cdot)\|_\alpha \ge \varepsilon\big)\\
&\le \mu\left(\|\beta_{u_{n+1}}[X(x_i+a_{u_{n+1}}\cdot) - X(x_i)] - f(\cdot)\|_\alpha \ge \frac{1}{2}\varepsilon\right)\\
&+ \mu\left(\|\beta_{u_{n+1}}[X_i(x_i+a_{u_{n+1}}\cdot) - X_i(x_i)] - f(\cdot)\|_\alpha \ge \frac{1}{2}\varepsilon\right)\\
&:= L_i^1 + L_i^2.
\end{aligned} \tag{4.15}$$

By the same method of the estimate of I_{52}, there exist $C_{21} > 0,\ C_{22} > 0$ such that

$$L_i^2 \le C_{21}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{22}i^\delta}. \tag{4.16}$$

By the same method of the estimate of J_m^1, there exist $C>0,\ C_{11}>0,\ C_{12}>0$ such that

$$\begin{aligned} L_i^1 &\le \exp\left\{-\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\left(1-C\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{12}}\right)\right\} \\ &\le \exp\left\{-\frac{1}{2}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}. \end{aligned} \tag{4.17}$$

Therefore, by (4.14)-(4.17), we have

$$\begin{aligned} I_{51} &= C_1(k_n+1)^{rp}\prod_{i=0}^{k_n} L_i^{\frac{p}{q_2}} \le C_1(k_n+1)^{rp}\times \\ &\times \prod_{i=0}^{k_n}\left[\exp\left\{-\frac{1}{2}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}+C_{21}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{22}i^{\delta}}\right]^{\frac{p}{q_2}} \\ &\le C_1(k_n+1)^{rp}\prod_{i=0}^{k_n}\left[\exp\left\{-\frac{1}{4}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}\right]^{\frac{p}{q_2}} \\ &\le C_1 n^{drp}\exp\left\{-\frac{p}{4q_2}\frac{u_n}{a_{u_{n+1}}}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}. \end{aligned}$$

Take a appropriate d, such that

$$\sum_{n=1}^{\infty} I_{51} \le \sum_{n=1}^{\infty} C_1 n^{drp}\exp\left\{-\frac{p}{4q_2}\frac{u_n}{a_{u_{n+1}}}\left(\frac{a_{u_{n+1}}}{u_{n+1}\log u_{n+1}}\right)^{C_{11}}\right\}<\infty.$$

By Borel-Cantelli Lemma, we have

$$\limsup_{n\to\infty}\ \min_{x_i\in[0,u_n-a_{u_{n+1}}]}\left\|\beta_{u_{n+1}}[Z_i(x_i+a_{u_{n+1}}\cdot)-Z_i(x_i)]-f(\cdot)\right\|_{\alpha}=0, \tag{4.18}$$

$C_{r,p}$-q.s. By (4.10)-(4.13) and (4.18), the proof of Lemma *4.2* is completed.

Below, we prove Theorem 1.2. Obviously, by Lemmas 4.1-4.2, (1.2) holds. Moreover, by Lemma 4.2, (1.3) also holds.

Список литературы

[1] P. Baldi, A. G. Ben, G. Kerkyacharian, "Large deviations and the Strassen theorem in Hölder norm", Stoch.Proce. Appl., **42**, 171 – 180 (1992).

[2] D. Monrad and H. Rootzén, "Small values of Gaussian processes and functional laws of the iterated logarithm", Prob.Theor.Rel.Fields., **101**, 173 – 192 (1995).

[3] Z. Y. Lin, W. S. Wang, K. S. Hwang, "Functional limit theorems for d-dimensional FBM in Hölder norm", Acta. Math. Sin., **22** (6), 1767 – 1780 (2006).

[4] Y. H. Liu, "The functional limit theorem for increament of a Brownian motion in Hölder norm with respect to $(r,p)-$capacities", J. Syst. Sci. Math. Sci. (Chinese), **30** (5), 611 – 618 (2010).

[5] W. S. Wang, "Functional limit theorems for the increments of Gaussian samples", J. Theor. Probab., **18** (2), 327 – 343 (2005).

[6] W. S. Wang, "A generalization of functional law of the iterated logarithm for $(r, p)-$capacities on the Wiener space", Stoch. Proce. Appl., **96**, 1 – 16 (2001).

[7] J. Xu, Y. M. Zhu, and J. C. Liu, "Quasi sure large deviation increments of fractional Brownian motion in Hölder norm", Acta.Math.Sin., **31** (6), 913 – 920 (2015).

Поступила 24 октября 2019

После доработки 17 мая 2020

Принята к публикации 10 июля 2020

Индекс 77735

ИЗВЕСТИЯ НАН АРМЕНИИ: МАТЕМАТИКА

том 56, номер 1, 2021

Содержание

IZVESTIYA NAN ARMENII: MATEMATIKA

Vol. 56, No. 1, 2021

Contents