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**DISTRIBUTION OF THE DISTANCE BETWEEN TWO
RANDOM POINTS IN A BODY FROM R^n**

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Abstract. In the present paper a formula for calculation of the density function $f_\rho(x)$ of the distance between two independent points randomly and uniformly chosen in a bounded convex body D is given. The formula permits to find an explicit form of density function $f_\rho(x)$ for body with known chord length distributions. In particular, we obtain an explicit expression for $f_\rho(x)$ in the case of a ball of diameter d . A simulation model is suggested to calculate empirically the cumulative distribution function of the distance between two points in a body from R^n , where explicit form of the function is hard to obtain. In particular, simulation is performed for balls and ellipsoids in R^n .

MSC2010 numbers: 60D05; 52A22; 53C65.

Keywords: chord length distribution; bounded convex body.

1. INTRODUCTION

In the present paper we continue investigations of [10] and [11]. In the previous two papers a similar problem was considered. In this paper we generalize those results for the bodies in n -dimensional Euclidean space.

Let D be a bounded, convex body in n -dimensional Euclidean space, with the volume $V(D)$ and the surface area $S(D)$. Let P_1 and P_2 be two points chosen at random, independently and with uniform distribution in D . Firstly, we are going to find the probability that the distance $\rho(P_1, P_2)$ between P_1 and P_2 is less or equal to x , that is we would like to find the distribution function $F_\rho(x)$ of $\rho(P_1, P_2)$. By definition, we have

$$(1.1) \quad F_\rho(x) = P(P_1, P_2 \in D : \rho(P_1, P_2) \leq x) = \frac{\iint_{\{(P_1, P_2) : \rho(P_1, P_2) \leq x\}} dP_1 dP_2}{\iint_{\{P_1, P_2 \in D\}} dP_1 dP_2},$$

where dP_i , $i = 1, 2$ is an element of Lebesgue measure in R^n . As

$$\iint_{\{P_1, P_2 \in D\}} dP_1 dP_2 = V^2(D)$$

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(here we use that points P_1 and P_2 select independently in D) we get

$$(1.2) \quad F_\rho(x) = \frac{1}{V^2(D)} \iint_{\{P_1, P_2\} : \rho(P_1, P_2) \leq x} dP_1 dP_2.$$

From the expression of the volume element in spherical coordinate where as the origin we select the point P_1 , we have

$$\begin{cases} x_1 = r \cos \psi_1 \\ x_2 = r \sin \psi_1 \cos \psi_2 \\ x_3 = r \sin \psi_1 \sin \psi_2 \cos \psi_3 \\ x_{n-1} = r \sin \psi_1 \dots \sin \psi_{n-2} \cos \psi_{n-1} \\ x_n = r \sin \psi_1 \dots \sin \psi_{n-2} \sin \psi_{n-1}, \end{cases}$$

where r is the distance between P_1 and P_2 . Thus, using transformation from the Cartesian coordinate system to spherical coordinate system, we obtain

$$(1.3) \quad dP_2 = dx_{21} dx_{22} \dots dx_{2n} = r^{n-1} \sin^{n-2} \psi_1 \sin^{n-3} \psi_2 \dots \sin \psi_{n-2} dr d\psi_1 d\psi_2 \dots d\psi_{n-1}.$$

Using (1.3) expression we have

$$(1.4) \quad dP_1 dP_2 = r^{n-1} \sin^{n-2} \psi_1 \sin^{n-3} \psi_2 \dots \sin \psi_{n-2} dP_1 dr d\psi_1 d\psi_2 \dots d\psi_{n-1},$$

where dK is an element of kinematic measure in R^n .

The kinematic density in Euclidean space was first introduced by Poincare. In modern terminology it is the Haar measure of the group of motions (translations and rotations) which acts on R^n . Let R^n be the Euclidean n-space, and let dK be the kinematic density. We know that

$$(1.5) \quad dK = \sin^{n-2} \psi_1 \sin^{n-3} \psi_2 \dots \sin \psi_{n-2} dP_1 d\psi_1 d\psi_2 \dots d\psi_{n-1}.$$

Using (1.4) and (1.5) we can rewrite (1.2) in the following form:

$$F_\rho(x) = \frac{1}{V^2(D)} \int_0^x r^{n-1} K(D, r) dr,$$

where $K(D, r)$ is the kinematic measure of all oriented segments of length r that lie inside D . Therefore, we obtain a relationship between the density function $f_\rho(x)$ of $\rho(P_1, P_2)$ and the kinematic measure $K(D, x)$:

$$(1.6) \quad f_\rho(x) = \frac{x^{n-1} K(D, x)}{V^2(D)}.$$

It should be noted that we can calculate the kinematic measure of all the unoriented segments that lie inside D and then multiply the result by 2.

Let $S_1 = MS$ be the image of segment S under an Euclidean motion. M is the group of all Euclidean motions in the space R^n . For the locally compact group M , there is a locally finite Haar measure, i.e. a locally finite, non identically zero Borel measure, invariant both from the left and the right. Segment S_1 can be defined by

means of the two coordinates (γ, t) , where $\gamma \in J$ (J is the space of all straight lines in R^n) contains segment S_1 , and t is the one dimensional coordinate of the center of the segment S_1 on the line γ . In the space M , we define a measure by its element in the following way:

$$K(dS_1) = d\gamma dt,$$

where $d\gamma$ is an element in a locally finite measure in the space J , which is invariant with respect to the group M and dt is the one-dimensional Lebesgue measure on γ . The measure $K(\cdot, \cdot)$ is said to be a kinematic measure on the group M .

2. THE MAIN FORMULA

This section gives a main formula for calculating the kinematic measure $K(D, r)$ in terms of chord length distribution function of body D . Obviously,

$$K(D, r) = 0, \quad \text{if } r \geq \text{diam}(D)$$

where $\text{diam}(D)$ is the diameter of D , i.e. $\text{diam}(D) = \max\{\rho(x, y) : x, y \in D\}$, where $\rho(x, y)$ is the distance between the points x and y . Therefore, only the case $0 \leq r \leq \text{diam}(D)$ is considered in the paper. It is evident that in the mentioned case

$$(2.1) \quad K(D, r) = \int_{[D]} \int_{t \in (\chi(\gamma) - r)} d\gamma dt = \int_{[D]} (\chi(\gamma) - r)^+ d\gamma,$$

where $[D] = \{\gamma \in J : \gamma \cap D \neq \emptyset\}$ is the set of lines in R^n intersecting body D , $\chi(\gamma) = \gamma \cap D$ is a chord in D , while

$$x^+ = \begin{cases} 0, & \text{if } x \leq 0 \\ x, & \text{if } x \geq 0. \end{cases}$$

Let O_n be the surface area of the n -dimensional unit sphere. O_n is defined [1]

$$O_n = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})},$$

where Γ is the gamma function which satisfies the recursion formula,

$$\Gamma(n+1) = n\Gamma(n)$$

especially, $\Gamma(n+1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$.

Consequently,

$$(2.2) \quad \begin{aligned} K(D, r) &= \int_{\chi(\gamma) > r} \chi(\gamma) d\gamma - r \int_{\chi(\gamma) > r} d\gamma = \\ &= \frac{O_{n-1}}{2} V(D) - G(r) - r \frac{O_{n-2}}{2(n-1)} S(D) [1 - F_D(r)], \end{aligned}$$

where

$$G(x) = \int_{\chi(\gamma) \leq x} \chi(\gamma) d\gamma$$

and $F_D(\cdot)$ is the chord length distribution function of body D , defined as

$$F_D(y) = \frac{2(n-1)}{O_{n-2}} \int_{\chi(\gamma) \leq y} d\gamma$$

(since $\int_{[D]} d\gamma = \frac{O_{n-2}}{2(n-1)} \cdot S(D)$).

Now we will prove the following formula:

$$G(x) = \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u f_D(u) du,$$

where $f_D(x)$ is the chord length density function of body D , i.e. $f_D(x) = F'_D(x)$ is the first derivative of the distribution function. Afterwards, for calculating the derivative of the function $G(x)$ we observe that

$$\begin{aligned} \frac{G(x + \Delta x) - G(x)}{\Delta x} &= \frac{1}{\Delta x} \int_{x < \chi(\gamma) \leq x + \Delta x} \chi(\gamma) d\gamma = \\ &= (x + \theta \Delta x) \frac{O_{n-2}}{2(n-1)} S(D) \frac{F_D(x + \Delta x) - F_D(x)}{\Delta x}. \end{aligned}$$

Then, assuming that the distribution function $F_D(x)$ possesses the density $f_D(x)$, when $\Delta x \rightarrow 0$, we get $G'(x) = \frac{O_{n-2}}{2(n-1)} S(D) x f_D(x)$ which implies

$$(2.3) \quad G(x) = G(0) + \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u f_D(u) du = \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u f_D(u) du,$$

since $G(0) = \int_{\chi(\gamma) \leq 0} \chi(\gamma) d\gamma = 0$. Now, we transform formula (2.3) by means of integration by parts:

$$\begin{aligned} G(x) &= \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u f_D(u) du = -\frac{O_{n-2}}{2(n-1)} S(D) \int_0^x u d[1 - F_D(u)] = \\ (2.4) \quad &= -x \frac{O_{n-2}}{2(n-1)} S(D) [1 - F_D(x)] + \frac{O_{n-2}}{2(n-1)} S(D) \int_0^x [1 - F_D(u)] du. \end{aligned}$$

At last, substituting (2.4) into formula (2.2) we come to the main formula of this section:

$$K(D, r) = \frac{O_{n-1}}{2} V(D) - \frac{O_{n-2}}{2(n-1)} S(D) \int_0^r [1 - F_D(u)] du.$$

Theorem 2.1. *For any body D in R^n*

$$(2.5) \quad K(D, r) = \frac{O_{n-1}}{2} V(D) - \frac{O_{n-2}}{2(n-1)} S(D) \int_0^r [1 - F_D(u)] du.$$

Thus, if the explicit form of the function $K(D, r)$ is given, then we can derive the explicit expression for the density function by means of (2.5). Formula (2.5) has been obtained for unoriented segments. For oriented segments this formula should

be multiplied by 2. Substituting (2.5) into (1.6) (and multiplying by 2) we obtain the main formula of the present paper:

$$(2.6) \quad f_\rho(x) = \frac{1}{V^2(D)} \left(x^{n-1} O_{n-1} V(D) - \frac{x^{n-1} O_{n-2}}{(n-1)} S(D) \int_0^x [1 - F_D(u)] du \right).$$

If $n=2$ then

$$f_\rho(x) = \frac{1}{V^2(D)} \left(O_1 x V(D) - x O_0 S(D) \int_0^x [1 - F_D(u)] du \right),$$

where $O_0 = 2$ and $O_1 = 2\pi$. This result is proved in [9].

If $n=3$ then

$$f_\rho(x) = \frac{1}{V^2(D)} \left(x^2 O_2 V(D) - \frac{x^2 O_1 S(D)}{2} \int_0^x [1 - F_D(u)] du \right),$$

where $O_1 = 2\pi$ and $O_2 = 4\pi$. This result is proved in [11].

3. THE CASE OF A BALL IN R^n

In case of the ball $D = B_d$ with diameter d , the chord length distribution function has the following form

$$(3.1) \quad F_{B_d}(y) = \begin{cases} 0, & \text{if } y \leq 0 \\ 1 - \left[1 - \left(\frac{y}{d} \right)^2 \right]^{\frac{n-1}{2}}, & \text{if } 0 \leq y \leq d \\ 1, & \text{if } y \geq d. \end{cases}$$

Consequently, substituting (3.1) into (2.5) we obtain

$$K(D, r) = \frac{O_{n-1}}{2} V(D) - \frac{O_{n-2}}{2(n-1)} S(D) \int_0^r \left[1 - \left(\frac{u}{d} \right)^2 \right]^{\frac{n-1}{2}} du.$$

Substituting this result in (1.6) or (2.6) we obtain the density function of the distance between two points chosen independently in the ball of diameter d

$$f_\rho(x) = \frac{1}{V^2(D)} \left(x^{n-1} O_{n-1} V(D) - \frac{x^{n-1} O_{n-2}}{(n-1)} S(D) \int_0^x \left[1 - \left(\frac{u}{d} \right)^2 \right]^{\frac{n-1}{2}} du \right).$$

This formula for $n=2$ and $n=3$ was obtained in [10] and [11].

4. MOMENTS OF DISTANCE BETWEEN TWO POINTS IN R^n

One of the simplest applications of the formulae (2.6) is the calculation of the k -th moment between two points randomly and independently distributed on the bounded convex domain. To find the k -th moment between points (we denote it by M_k^n , where n is the dimension of space) we need to calculate the following integral

$$(4.1) \quad M_k^n = \int_0^d x^k f_\rho^n(x) dx.$$

Using (2.6) we rewrite the last equation in the following form:

$$M_k^n = \int_0^d x^k f_\rho^n(x) dx = \frac{O_{n-1}}{V(D)} \int_0^d x^{n+k-1} dx -$$

$$\begin{aligned}
 (4.2) \quad & -\frac{O_{n-2}S(D)}{(n-1)V^2(D)} \int_0^d x^{n+k-1} dx \int_0^x [1 - F_D(u)] du = \\
 & = \frac{O_{n-1}d^{n+k}}{V(D)(n+k)} - \frac{O_{n-2}S(D)}{(n-1)(n+k)V^2(D)} \int_0^d dx^{n+k} \int_0^x [1 - F_D(u)] du = \\
 & = \frac{O_{n-1}d^{n+k}}{V(D)(n+k)} - \frac{O_{n-2}S(D)}{(n-1)(n+k)V^2(D)} \int_0^d dx^{n+k} \int_0^x [1 - F_D(u)] du = \\
 & = \frac{O_{n-1}d^{n+k}}{V(D)(n+k)} - \frac{O_{n-2}S(D)}{(n-1)(n+k)V^2(D)} \times \\
 & \quad \left[d^{n+k} \int_0^d [1 - F_D(u)] du - \int_0^d x^{n+k} (1 - F_D(u)) du \right].
 \end{aligned}$$

In (4.2) we can calculate the integral $\int_0^d [1 - F_D(u)] du$. Consider the value of $G(x)$ function at point $x = d$. Since $G(d) = \frac{O_{n-1}V(D)}{2}$, we get

$$\begin{aligned}
 G(d) &= \frac{O_{n-2}S(D)}{2(n-1)} \int_0^d u f_D(u) du = -\frac{O_{n-2}S(D)}{2(n-1)} \int_0^d u d(1 - F_D(u)) = \\
 &= -\frac{O_{n-2}S(D)}{2(n-1)} (d(1 - F_D(d)) - \int_0^d (1 - F_D(u)) du) = \frac{O_{n-2}S(D)}{2(n-1)} \int_0^d (1 - F_D(u)) du
 \end{aligned}$$

therefore

$$(4.3) \quad \int_0^d [1 - F_D(u)] du = \frac{O_{n-1}V(D)(n-1)}{O_{n-2}S(D)}$$

Putting (4.3) in (4.2) we obtain

$$(4.4) \quad M_k^n = \frac{O_{n-2}S(D)}{(n-1)(n+k)V^2(D)} \int_0^d x^{n+k} (1 - F_D(u)) du.$$

5. MEAN DISTANCE BETWEEN TWO POINTS IN A DOMAIN AND THE CASE OF BALL IN R^n

Using (4.4) for $k=1$ we obtain a formula for calculating the mean distance between two points uniformly and independently distributed in a bounded convex domain :

$$(5.1) \quad M_1^n = \frac{O_{n-2}S(D)}{(n-1)(n+1)V^2(D)} \int_0^d x^{n+1} (1 - F_D(u)) du.$$

In case of the ball $D = B_d$ with diameter d , putting (3.1) in (5.1) we obtain

$$M_1^n = \frac{O_{n-2}S(D)}{(n-1)(n+1)V^2(D)} \int_0^d x^{n+1} \left[1 - \left(\frac{x}{d} \right)^2 \right]^{\frac{n-1}{2}} dx.$$

If $n=2$ then

$$M_1^2 = \frac{4\pi r}{3\pi^2 r^4} \int_0^d x^3 \left[1 - \left(\frac{x}{d} \right)^2 \right]^{\frac{1}{2}} dx = \frac{64}{45} d.$$

This is the result from [9].

If $n = 3$ then

$$M_1^3 = \frac{9}{16r^4} \int_0^d x^4 \left[1 - \left(\frac{x}{d} \right)^2 \right] dx = \frac{18}{35} d.$$

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ON CONTINUITY OF BUFFON FUNCTIONALS IN THE SPACE
OF PLANES IN $\mathbb{R}^3$

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ABSTRACT. The paper considers measures in the space $\mathbf{IE}$ of planes in $\mathbb{R}^3$, and combinatorial decompositions for their values on "Buffon sets" in $\mathbf{IE}$. These decompositions, written in terms of a "wedge function" depending on the measure, have been known since long in Combinatorial Integral Geometry, yet their explicit expressions have been well established only for "non-degenerate" Buffon sets. Theorem 1 removes this gap and presents a decomposition algorithm valid with no similar restriction. Theorem 2 presents a result in a direction converse to Theorem 1. Starting from the decomposition algorithm, a combinatorial valuation Ψ_F is defined that depends on "general" continuous additive wedge function $F(W)$. The question is: when Ψ_F becomes a measure in the space $\mathbf{IE}$? Theorem 2 points at special "tetrahedral inequalities", the analogues of triangular inequalities of the planar theory. If Ψ_F satisfies these "tetrahedral inequalities", then Ψ_F becomes a measure and the corresponding $F(W)$ is called a "wedge metric" (to stress the connection of the paper's topic with Hilbert's Fourth Problem).

MSC2010 numbers: 53C65; 53C60; 31A10.

Keywords: integral geometry; Buffon functional; wedge metric; Hilbert's Fourth Problem.

1. INTRODUCTION

The paper considers measures in the space $\mathbf{IE}$ of planes in $\mathbb{R}^3$, and combinatorial decompositions for their values on Buffon sets in $\mathbf{IE}$ (i.e. members of Buffon rings in $\mathbf{IE}$). The existence of similar decompositions in the spaces of Integral Geometry was first discovered in [5], they together with first applications have been discussed in the books [6] and [8] (see also [18]). Although later on further applications have been found (in convexity theory, see [7], [11]-[13], [15], Hilbert's Fourth problem, see [4], [13], [14], [17], [19], [20], tomography, see [9]), in the basic theory the initial effort left many gaps. The present paper fills some of the gaps by presenting new results, Theorems 1, 2.

One of the basics of the combinatorial theory for $\mathbf{IE}$ known already in [1], [2], [3] was the so-called "four indicator rule" valid for "non-degenerate" Buffon sets. In Theorem 1 we give its extension for quite general Buffon sets in $\mathbf{IE}$. (The

corresponding result for the space of lines in the plane have been recently announced in [10].)

The general combinatorial algorithm given in Theorem 1 permits construction of the combinatorial valuation Ψ_F that depends on a continuous additive "wedge function" $F(W)$. The valuation Ψ_F lives on subsets of $\mathbf{IE}$ that make up a special set ring $\mathbf{U}_3$ of Buffon sets, including the degenerate cases. After brief discussion of the key properties of Ψ_F , follows demonstration of the role of special "tetrahedral inequalities" in the generation of measures by Ψ_F (Theorem 2). In the author's earlier paper [4] a similar theorem was proved about generation of measures in the space of lines on the plane by linearly additive pseudo-metrics. That theorem was called in [17] "the most elegant and natural solution" of Hilbert's Fourth Problem. By analogy, an additive $F(W)$ satisfying the tetrahedral inequalities we call a "wedge metric". The construction of Ψ_F on the basis of Theorem 1 permits to essentially simplify the proof of Theorem 2, as compared with the proof of the corresponding planar theorem given in [4] that was based on the planar version of the "four indicator rule".

2. WEDGE COMBINATORICS

2.1. Wedges in $\mathbf{IR}^3$. The tool of wedges in $\mathbf{IR}^3$ have been shown in [5] and the books [6], [8] to be a rather effective in the theory of measures in the space

$\mathbf{IE}$ = the space of planes in Euclidean 3-space $\mathbf{IR}^3$, $e \in \mathbf{IE}$.

We consider the spaces

$\mathbf{S}$ = the space of directions in $\mathbf{IR}^3$, (elliptical plane), $\omega \in \mathbf{S}$,

$\mathbf{s}_\Omega$ = the circle of spatial directions $\omega \in \mathbf{S}$ orthogonal to some $\Omega \in \mathbf{S}$,

and use the notation

γ = a line in $\mathbf{IR}^3$,

ν = a segment of a line $\gamma \subset \mathbf{IR}^3$.

Given a line $\gamma \subset \mathbf{IR}^3$ of direction $\Omega \in \mathbf{S}$ or a needle $\nu \subset \gamma$, instead of $\mathbf{s}_\Omega$ we may write $\mathbf{s}_\gamma$ or $\mathbf{s}_\nu$.

A flag f in $\mathbf{IR}^3$ is a triad

$$f = (P, \gamma, \Phi)$$

consisting of a point $P \in \mathbf{IR}^3$, a line γ containing P , and $\Phi \in \mathbf{s}_\gamma$.

Denote by $\mathcal{C}_\gamma$ the following family of flags depending on a line $\gamma \subset \mathbf{IR}^3$:

$$\mathcal{C}_\gamma = \{f = (P, \gamma, \Phi) : P \in \gamma \text{ and } \Phi \in \mathbf{s}_\gamma\},$$

$\mathcal{C}_\gamma$ can be identified with unit radius circular cylinder with axis γ . A wedge is defined to be a subset of $\mathcal{C}_\gamma$ having the product form

$$W = \{P \in \nu\} \times \{\Phi \in \lambda\} = \nu \times \lambda \subset \mathcal{C}_\gamma,$$

where $\nu \subset \gamma$ is a needle (= finite open segment of a line in $\mathbf{R}^3$) and $\lambda \subset \mathbf{s}_\gamma$ is an arc (of length not exceeding π). By $e_\nu(\Phi)$ we denote the plane containing the needle ν and the direction $\Phi \in \mathbf{s}_\nu$. For every edge $W = \nu \times \lambda$ the dihedral region V is defined to be

$$(2.1) \quad V = \cup_{\Phi \in \lambda} e_\nu(\Phi).$$

2.2. Wedges associated with $\{P_i\}$. Let a finite set of points $\{P_i\}$ be given in $\mathbf{R}^3$. For a 2-subset $\{P_i, P_j\}$ from that set, by $e_{ij}(\Phi)$ we denote the plane containing the needle $\nu = \{P_i P_j\}$ and the direction $\Phi \in \mathbf{s}_\nu$. The values of Φ for which the plane $e_{ij}(\Phi)$ contains points from $\{P_i\}$ outside the line carrying P_i and P_j , split $\mathbf{s}_\nu$ into pairwise disjoint open arcs

$$\lambda_1, \dots, \lambda_l \subset \mathbf{s}_\nu.$$

(they “belong” to $\{P_i P_j\}$). Each $\{P_i, P_j\} = \nu$ together with one of the belonging arcs $\lambda_r = \lambda$ determines a wedge $W_s = (\nu, \lambda)$. In this writing, the index s “codes” $(\{ij\}, r)$, i.e. there is a one-to-one correspondence

$$(2.2) \quad s \mapsto (\{ij\}, r).$$

All the wedges W_s obtained in this way form the system of wedges *associated* with our finite set $\{P_i\}$. By the definition of λ_r , for every associated wedge W_s its dihedral region

$$V_s = \cup_{\Phi \in \lambda_r} e_{ij}(\Phi)$$

does not contain points from $\{P_i\}$ in its interior, while each of the two planes bounding V_s necessarily contain points from $\{P_i\}$ other than those on their intersection line.

2.3. Buffon rings and sets. Let a finite set of points $\{P_i\}$ be given in $\mathbf{R}^3$. Two planes which avoid any of the points P_i we call equivalent if they induce the same partition of the set $\{P_i\}$. An equivalence class Υ (a maximal set of equivalent planes) is always a connected set in the topology of $\mathbf{IE}$, but its closure will not be compact if for each plane $e \in \Upsilon$ the total $\{P_i\}$ lies in one of the two half-spaces separated by e . All other equivalence classes have compact closures: these we call *atoms*. By $Br\{P_i\}$ we denote the minimal ring of subsets of $\mathbf{IE}$ which contains all atoms (they become atoms of the ring $Br\{P_i\}$ in the usual sense). We call $Br\{P_i\}$ the Buffon ring that corresponds to the set $\{P_i\}$. A set $A \subset \mathbf{IE}$ we call Buffon if

$A \in Br\{P_i\}$ for some point set $\{P_i\} \subset \mathbf{R}^3$. An element $\mathbf{A} \in Br\{P_i\}$ necessarily has the form $\mathbf{A} = \bigcup a_i$, where a_i are some of the atoms of $Br\{P_i\}$.

For the time being, we assume that the number of points in the finite point set $\{P_i\}$ exceeds 2.

Let W_s be a wedge form the system of wedges *associated* with $\{P_i\}$, $s = (\{ij\}, r)$ in the sense of the map of (2.2), γ_{ij} = the line through P_i and P_j . Below always $\Phi \in \lambda_r$.

If γ_{ij} contains no points from $\{P_i\}$ except P_i and P_j , then there exist exactly four different equivalence classes Υ for which we have $e_{ij}(\Phi) \in \partial\Upsilon$. They do not depend on the choice of $\Phi \in \lambda_r$ and we denote them as $\Upsilon_s(++), \Upsilon_s(--), \Upsilon_s(+-)$ and $\Upsilon_s(-+)$. The sign rule is as follows (see Figure 1):

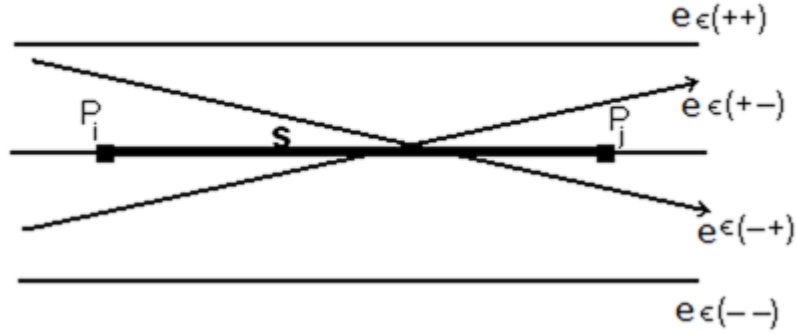


Figure 1

every plane $e \in \Upsilon_s(++)$ or $e \in \Upsilon_s(--)$ leaves P_i and P_j in one half-space,
every plane $e \in \Upsilon_s(+-)$ or $e \in \Upsilon_s(-+)$ leaves P_i and P_j in different half-spaces.

Given $\mathbf{A} \in Br\{P_i\}$, the values of the indicator function

$$I_{\mathbf{A}}(e) = \begin{cases} 1, & \text{if } e \in \mathbf{A}, \\ 0, & \text{otherwise} \end{cases}$$

on the planes from the above four sets we denote correspondingly as

$$I_{\mathbf{A}}(s, +-) \equiv I_{\mathbf{A}}(e) \text{ for } e \in \Upsilon_s(+-), \quad I_{\mathbf{A}}(s, -+) \equiv I_{\mathbf{A}}(e) \text{ for } e \in \Upsilon_s(-+),$$

$$I_{\mathbf{A}}(s, ++) \equiv I_{\mathbf{A}}(e) \text{ for } e \in \Upsilon_s(++), \quad I_{\mathbf{A}}(s, --) \equiv I_{\mathbf{A}}(e) \text{ for } e \in \Upsilon_s(--).$$

2.4. No collinear triads case. This subsection describes the state of the art in [5], [6] and [8].

Let M be some locally finite measure in $\mathbf{IE}$ that vanishes on every bundle of planes (bundle= the set of planes that contain some point $P \in \mathbf{R}^3$). Let a point set $\{P_i\}$

with no three points on a line be given. Then for every Buffon set $A \in Br\{P_i\}$ the following decomposition is valid:

$$(2.3) \quad M(A) = \frac{1}{2} \sum u_s(A) F(W_s),$$

where the summation is over the system of wedges W_s associated with $\{P_i\}$. The coefficients $u_s(\mathbf{A})$ are integers that do not depend on the choice of measure m and are given by “four indicator formula”

$$(2.4) \quad u_s(\mathbf{A}) = I_{\mathbf{A}}(s, +-) + I_{\mathbf{A}}(s, -+) - I_{\mathbf{A}}(s, ++) - I_{\mathbf{A}}(s, --).$$

For the “wedge function” $F(W)$ the following representation was proposed in [6]:

$$(2.5) \quad F(W) = (2\pi)^{-1} \int_{e \text{ hits } \nu} |W \cap e| M(de),$$

where $W \cap e$ is the angular trace left by the wedge on the plane e , that is, see (2.1)

$$W \cap e = e \bigcap V,$$

while $|\dots|$ stands for the usual angular measure on $\mathbf{s}_\nu$.

2.5. Euclidean motions invariant measure. In the space $\mathbf{IE}$ there exists [18] unique up to a constant factor Euclidean motions invariant locally finite measure; we denote it as μ . We assume that the constant factor is chosen in a way to ensure

$$\mu(\text{planes that hit the unit ball in } \mathbf{R}^3) = 2\pi.$$

For $M = \mu$ the wedge function $F(W)$ given by (2.5) reduces to the product of length of ν and the angular measure of λ :

$$(2.6) \quad F(W) = |\nu| |\lambda|.$$

2.6. More remarks. If the number of points in $\{P_i\}$ equals 2, i.e. $\{P_i\} = \{P_1, P_2\}$ then $Br\{P_i\}$ contains only one element $\mathbf{A}$ = the planes that separate P_1 from P_2 , and there is only one wedge $W_1 = \nu \times \mathbf{s}_\nu$ with ν = the needle joining P_1 and P_2 . Yet (2.3) remains valid since formally $I_{\mathbf{A}}(1, ++) = I_{\mathbf{A}}(1, --) = 0$ and we get $u_1(\mathbf{A}) = 2$. If the number of points in $\{P_i\}$ equals 1, then the corresponding Buffon ring is empty.

In case the point set $\{P_i\}$ is confined to some plane in $\mathbf{R}^3$, every wedge associated with $\{P_i\}$ gets the form $W = \nu \times \mathbf{s}_\nu$, hence always $F(W_s) = \pi |\nu_s|$. The equation (2.3) reduces to the four indicator rule for lines in the plane.

The book [6] starts with derivation of decomposition (2.3) for $M = \mu$ by direct analytical “Invariant Imbedding” method. For general M (2.3) was derived in [2] basing on the planar decomposition for projection of $\{P_i\}$ on the plane.

2.7. Collinear triads permitted. We are going to formulate Theorem 1, which removes the restrictions on the point set $\{P_i\}$ present in the formulation of the four indicators rule. Theorem 1 is instrumental in the construction of the functional Ψ in the Section 4 below.

For the case where the lines γ_{ij} may contain points from $\{P_i\}$ other from P_i and P_j , a decomposition similar to (2.3) survives. However the sets $\Upsilon_s(+ -)$, $\Upsilon_s(- +)$, $\Upsilon_s(++)$ and $\Upsilon_s(--)$ are now no longer well defined, hence the algorithm (2.4) requires modification.

Let W_s be a wedge form the system of wedges *associated* with $\{P_i\}$, and $s = (\{ij\}, r)$ in the sense of (2.2).

The class (+): We say that W_s belongs to the class (+) if the interior of the needle with endpoints P_i, P_j does not contain any points from $\{P_i\}$. For every $W_s \in (+)$, we define the equivalence classes $\Upsilon_s(+ -)$ and $\Upsilon_s(- +)$ in the same way as above. (That definition is no longer consistent for W_s outside (+)).

The class (-): We say that W_s belongs to the class (-) if the interior of complement of the needle with endpoints P_i, P_j contains no other points from $\{P_i\}$. For every $W_s \in (-)$ we define the equivalence classes $\Upsilon_s(++)$ and $\Upsilon_s(--)$ in the same way as above. (That definition is no longer consistent for W_s outside (-)).

Let $\mathbf{A} \in Br\{P_i\}$ be a Buffon set. For W_s from the class (+) we denote by

$$u_s^+(\mathbf{A}) = I_{\mathbf{A}}(s, + -) + I_{\mathbf{A}}(s, - +),$$

and for W_s from the class (-) we denote by

$$u_s^-(\mathbf{A}) = I_{\mathbf{A}}(s, ++) + I_{\mathbf{A}}(s, --).$$

Theorem 1. *Let M be some locally finite measure in $\mathbf{IE}$ that vanishes on every bundle of planes. For any point set $\{P_i\}$ and every Buffon set $A \in Br\{P_i\}$ the following decomposition is valid:*

$$(2.7) \quad M(\mathbf{A}) = \frac{1}{2} \sum_{W_s \in (+)} u_s^+(\mathbf{A}) F(W_s) - \frac{1}{2} \sum_{W_s \in (-)} u_s^-(\mathbf{A}) F(W_s),$$

where the wedge function $F(W)$ is given by the integral (2.5).

The proof follows by a simple check of (2.7).

2.8. An example. Let Π be a bounded convex polygon in some plane $e_0 \subset \mathbf{R}^3$ with vertices $v_1, \dots, v_n$. Let Q be a point outside e_0 . The pair (Q, Π) corresponds

to a *pyramid* K with *apex* Q and *base* Π . We put $\{P_i\} = \{Q, v_1, \dots, v_n\}$; then the set

$$B = \{e \in \mathbf{IE} \mid e \text{ separates } Q \text{ from } \Pi\}$$

belongs to $Br\{P_i\}$.

For any $W_s = (\nu, V)$ from the system of associated wedges, see (2.1), the needle ν is always an edge of K . An edge of K we call *lateral* if it is of Q, v_i type and *basal* if it is of $v_i v_j$ type. A wedge W_s we call a *support* wedge if $V_s \cap \text{int } K = \emptyset$, and a *covering* wedge if $\text{int } K \subset V_s$. We write $W_s \in I$ if W_s is a support wedge on a lateral edge and $W_s \in II$ if W_s is a covering wedge on a basal edge. We have

$$\begin{aligned} u_s(B) &= 1 \text{ if } W_s \in I, \\ u_s(B) &= -1 \text{ if } W_s \in II, \text{ and} \\ u_s(B) &= 0 \text{ for all other cases.} \end{aligned}$$

We remark, that if we assume that position of the apex Q changes so that Q tends to some limiting position $Q_0 \in \text{interior of } \tau$, then the ratio $M(B)[\mu(B)]^{-1}$ would tend to the value of the density of the measure M on the plane containing τ .

3. THE VALUATION Ψ_F

Below, we use both (equivalent) definitions of a wedge:

$$W = \nu \times \lambda \text{ a product set on the unit cylinder } \mathcal{C}_\gamma \text{ and}$$

$$W = (\nu, V), \text{ definition of } V \text{ is given in (2.1).}$$

The wedges from the family $\{W : W \subset \mathcal{C}_\gamma\}$ can be described as $(P_1, P_2, \omega_1, \omega_2)$, where $P_1, P_2 \in \gamma$ are the endpoints of ν , while ω_1, ω_2 are the spatial directions normal to $\mathcal{C}_\gamma$ at the endpoints of the arc λ . This provides a topology in the space of wedges; so we can speak about continuous "wedge functions" $F(W)$ (an $F(W)$ maps the space of wedges onto the numerical axis). Within each class $\{W : W \subset \mathcal{C}_\gamma\}$ the notion of additivity of an $F(W)$ in both ν and in λ is well defined as usual. In fact the functions $F(W)$ generated by means of (2.5) actually generate measures on the cylinders $\mathcal{C}_\gamma$.

Let $\{P_i\}_1$ and $\{P_i\}_2$ be two finite point sets in $\mathbf{R}^3$. Two sets $B_1 \in Br\{P_i\}_1$ and $B_2 \in Br\{P_i\}_2$ we call equivalent if their closures coincide.

We define $\mathbf{U}_3$ to be the set of equivalence classes within the $\bigcup Br\{P_i\}$, where the union is taken over all possible choices of finite sets $\{P_i\} \subset \mathbf{R}^3$. For elements $A, B \in \mathbf{U}_3$ usual set theoretic operations $\cup$ and $\cap$ can be defined. For A , a finite point set $\{P_i\} \subset \mathbf{R}^3$ can be found, such that (up to equivalence) $A \in Br\{P_i\}$.

Similarly, $B \in Br\{Q_i\}$ for some finite set of points $\{Q_i\} \subset \mathbf{R}^3$. Then up to equivalence

$$A \cup B \in Br[\{P_i\} \cup \{Q_i\}] \text{ and } A \cap B \in Br[\{P_i\} \cup \{Q_i\}].$$

Given a continuous wedge function $F(W)$, for any $\{P_i\}$ and any $A \in Br\{P_i\}$ we define a functional

$$(3.1) \quad \Psi_F(A; \{P_i\}) = \frac{1}{2} \sum u_s^+(A) F(W_s) - \frac{1}{2} \sum u_s^-(A) F(W_s),$$

where $u_s(A)$ are calculated according to the rules of (2.7), both sums are over the system of wedges associated with $\{P_i\}$.

Lemma 1. *If $F(W)$ is additive (both in ν and λ) on every cylinder $\mathcal{C}_\gamma$, then the value $\Psi_F(A; \{P_i\})$ does not depend on the choice of $\{P_i\}$, as long as $A \in Br\{P_i\}$ holds. This means that*

$$\Psi_F(A) \equiv \Psi_F(A; \{P_i\}),$$

consistently defines an additive functional Ψ_F that lives on $\mathbf{U}_3$.

The proof of Lemma 1 follows from the "stability" of the sums (see (2.7))

$$\sum_{\nu \subset \gamma} u_s^+(A) F(W_s) - \sum_{\nu \subset \gamma} u_s^-(A) F(W_s),$$

where each sum is over all wedges w_s that have ν on the same line γ (the later contains at least two points from $\{P_i\}$). Stability means no dependence on the presence of "non-essential" points in $\{P_i\}$: a point P_i is "non-essential" for A if the bundle of planes through P_i is disjoint from ∂A .

Next we formulate a continuity property of Ψ_F to be used in the measure construction below.

Let s_1, s_2, s_3 be three linear segments in $\mathbf{R}^3$, while

$$s_i^{(n)} \subset s_i, \quad i = 1, 2, 3$$

be a sequence of needles which approximates ν_i in the sense of endpoint convergence:

$$\lim \nu_i^{(n)} = \nu_i, \quad i = 1, 2, 3.$$

In $\mathbf{IE}$ we consider the sets

$$A = \cap [s_i] \text{ and } A_n = \cap [s_i^{(n)}].$$

Lemma 2. *If a wedge function F is continuous and additive and the functional Ψ_F is nonnegative, i.e.*

$$\Psi_F \geq 0 \quad \text{on } \mathbf{U}_3,$$

then the limit of $\Psi_F(A_n)$ exists and

$$\lim \Psi_F(A_n) = \Psi_F(A).$$

Proof. By additivity of Ψ_F

$$\Psi_F(A) - \Psi_F(A_n) = \sum_r \Psi_F(B_r),$$

where each of the sets B_r is necessarily of the form $[\tau] \cap C$, with τ = a needle component of some set differences $s_k \setminus s_k^{(n)}$, while $C \in \mathbf{U}_3$. Hence by assumed nonnegativity of Ψ_F we have

$$\Psi_F([\tau_1] \cap C) \leq \Psi_F([\tau_1] \cap C) + \Psi_F([\tau_1] \cap C^c) = \Psi_F([\tau_1]) \rightarrow 0, \quad \text{where } C = [\nu_1^{(n)}] \cap [\nu_2^{(n)}].$$

Hence for each r , $\lim \Psi_F(B_r) = 0$ as $n \rightarrow \infty$ and the lemma is proved.

3.1. Tiling in the elliptical 3-space. The space $\mathbf{IE}$ of planes in $\mathbf{IR}^3$ is homeomorphic to $\mathcal{E}_3 \setminus N$, the three dimensional elliptical space with a point N deleted. Recall that $\mathcal{E}_3$ has the interpretation of the space of diameters of the unit sphere in $\mathbf{IR}^4$. We consider a standard map $\mathbf{IE} \Rightarrow \mathcal{E}_3 \setminus N$ under which the images of bundles and pencils (a pencil is the set of planes through a line in R^3) are the "planes" and "lines" in the elliptical geometry of $\mathcal{E}_3$

Assume a finite set $\{P_i\}$ is given in $\mathbf{IR}^3$. The corresponding "planes" produce a partition of $\mathcal{E}_3$ into convex polyhedrons. Except for the cell that contains the point N , these cells are the images of the atoms of the ring $Br\{P_i\}$. To a general $A \in Br\{P_i\}$ corresponds a union of cells.

Let in $\mathbf{IR}^3$ we have a tetrahedron Θ with vertices P_1, P_2, P_3, P_4 . The number of atoms in

$$Br\{P_1, P_2, P_3, P_4\} = Br(\Theta)$$

is seven: an atom of $Br(\Theta)$ can be either of 1-3 type (separation of one vertex from three others) or of 2-2 type (separation of two vertices from two others). The four bundles

$$[P_i] = \text{planes through the point } P_i, [P_i] \subset \mathbf{IE},$$

split $\mathcal{E}_3$ in eight components; each of the eight is a tetrahedron $\theta \subset \mathcal{E}_3$ (each θ is bounded by four "planes" in $\mathcal{E}_3$). Except for the one which contains N , these θ -s are images of the atoms of $Br(\Theta)$. Those four θ -s which correspond to the atoms of 1-3 type have a two-dimensional face in common with the cell containing N , while those three θ -s which correspond to the atoms of 2-2 type, each have two one-dimensional edges in common with the letter cell.

Given a tetrahedron $\theta \subset \mathcal{E}_3 \setminus N$, we write $\theta \in \mathbf{U}_3$ if the $\mathbf{IE}$ -image of θ belongs to $\mathbf{U}_3$. In fact $\theta \in \mathbf{U}_3$ occurs whenever among the four planes in $\mathbf{R}^3$ that correspond to the vertices of θ there are no parallel pairs. So $\theta \in \mathbf{U}_3$ determines a tetrahedron $\Theta \subset \mathbf{R}^3$, while θ itself is identified with an atom of $Br(\Theta)$.

Assume $\sigma \subset \mathcal{E}_3 \setminus N$ corresponds to an atom of some $Br\{P_i\}$. It is always possible (“tiling”) to represent σ as a union of pairwise disjoint tetrahedral cells $\theta_s \subset \mathcal{E}_3 \setminus N$ from the class $\mathbf{U}_3$:

$$\sigma = \bigcup \theta_s.$$

It follows that for every $A \in \mathbf{U}_3$ a representation

$$(3.2) \quad A = \bigcup A_s$$

is valid, where each $A_s \in \mathbf{U}_3$ is an atom of some $Br(\Theta_s)$. In terms of the functional Ψ_F this rewrites as $\Psi_F(A) = \sum \Psi_F(A_s)$. Hence the condition

$$(3.3) \quad \Psi_F(A) \geq 0 \quad \text{for any tetrahedron } \Theta \text{ and every atom } A \in Br(\Theta)$$

guarantees that $\Psi_F(A) \geq 0$ for any $A \in \mathbf{U}_3$. The actual expression of $\Psi_F(A)$ in (3.3) depends on the type (1-3 or 2-2) of the atom A .

3.2. Tetrahedral inequalities. Given a tetrahedron Θ with vertices $P_1, P_2, P_3, P_4 \subset \mathbf{R}^3$, we denote

$]P_1[$ = the set of planes that separate P_1 from P_2, P_3, P_4

$]P_1, P_2[$ = the set of planes that separate P_1, P_2 from P_3, P_4

In order to explicitly put down $\Psi_F(]P_1[)$ and $\Psi_F(]P_1, P_2[)$, we define the following groups of wedges associated with P_1, P_2, P_3, P_4 :

$$I = \{w = (\nu, V) : \nu \text{ is "lateral"; } \Theta \cap V = \emptyset\},$$

$$II = \{w = (\nu, V) : \nu \text{ is "basal"; } \Theta \subset V\},$$

$$III = \{w = (\nu, V) : \nu \text{ is "pure"; } \Theta \cap V = \emptyset\},$$

$$IV = \{w = (\nu, V) : \nu \text{ is "mixed"; } \Theta \subset V\},$$

where

ν is lateral means that $\nu = P_1, P_2, P_1, P_3$ or P_1, P_4 ;

ν is basal means that $\nu = P_2, P_3, P_3, P_4$ or P_4, P_1 ;

ν is pure means that $\nu = P_1, P_2$ or P_3, P_4

ν is mixed means that $\nu = P_1, P_3$ or P_1, P_4, P_2, P_3 or P_2, P_4 .

The first kind tetrahedral inequality writes:

$$(3.4) \quad \Psi_F(]P_1[) = \sum_I F(w_s) - \sum_{II} F(w_s) \geq 0.$$

The second kind tetrahedral inequality writes:

$$(3.5) \quad \Psi_F(]P_1, P_2]) = \sum_{III} F(w_s) - \sum_{IV} F(w_s) \geq 0.$$

We came to the following result.

Lemma 3. *If a continuous and additive wedge function F satisfies the tetrahedral inequalities (3.4) and (3.5) for any tetrahedron $P_1, P_2, P_3, P_4 \subset \mathbf{R}^3$ and any numeration of its vertices, then Ψ_F the combinatorial valuation is nonnegative on $\mathbf{U}_3$.*

3.3. Measure generation. We are now ready to outline the proof of a theorem, whose role in $\mathbf{IE}$ compares with that of the theorem on planar pseudo-metrics proved in [4]. The continuous and additive wedge functions we consider are “general”, i.e. they are not supposed to possess any special representation like (2.5).

Theorem 2. *Let F be a continuous and additive wedge function that satisfies the tetrahedral inequalities (3.4) and (3.5). Then there exists a unique (nonnegative) measure M in $\mathbf{IE}$ whose value on any set $A \in \mathbf{U}_3$ can be calculated as*

$$M(A) = \Psi_F(A).$$

Let F_1 be another wedge function possessing the same properties as F , and M_1 let be the corresponding measure in $\mathbf{IE}$. If for some tetrahedron $\Theta \subset \mathbf{R}^3$ one has

$$F_1(W) = F(W) \quad \text{on wedges } W = (\nu, V) \text{ with endpoints of } \nu \text{ on the edges of } \Theta,$$

then the restrictions of M and M_1 to the set $[\Theta] = \text{planes that hit } \Theta$ coincide.

Proof. Let $\Theta = \{P_1, P_2, P_3, P_4\}$ be a tetrahedron in $\mathbf{R}^3$ with (open) edges $\nu_k, k = 1, \dots, 6$. Given an atom $]P_i, P_j]$ of $Br\{P_i\}$ (a 2-2 tetrahedral set), we choose from the corresponding collection of “mixed” edges a triad ν_k, ν_m, ν_r . Also, there is a natural correspondence $]P_i[\rightarrow \nu_k, \nu_l, \nu_r$ where ν_k, ν_l, ν_r are the three edges of Θ that meet at P_i . So for atoms $A \in Br(\Theta)$ we get a map

$$(3.6) \quad A \rightarrow (\nu_k, \nu_m, \nu_r).$$

Now each plane e that hits Θ but avoids any P_i can be described by the points l_k, l_m, l_r of intersection of e with corresponding ν_k, ν_m, ν_r . For each atom we consider the usual semi-algebra of subsets of the corresponding product $\nu_k \times \nu_l \times \nu_r$ consisting of the products

$$I_1 \times I_2 \times I_3 \quad \text{with} \quad I_1 \subseteq \nu_k, I_2 \subseteq \nu_m, I_3 \subseteq \nu_r,$$

where I_1, I_2, I_3 can be open, semi-open or closed intervals. The sets of the type

$[I_1] \cap [I_2] \cap [I_3]$ = the image of $I_1 \times I_2 \times I_3$ form a semi-algebra in $[\Theta]$. By (3.6) and Lemma 3, the valuation Ψ_F is nonnegative, and so is each value $\Psi_F([I_1] \cap [I_2] \cap [I_3])$. By Lemma 2, the latter value can be obtained as a limit of values of Ψ_F on the sets belonging to the "compact class"

$$\{[I_1] \cap [I_2] \cap [I_3] : I_1, I_2, I_3 \text{ are closed intervals}\}.$$

By a standard criterion of measure theory this implies that Ψ_F is (can be extended to) a measure M_Θ on $[\Theta]$, after we additionally put $M_\Theta([P_i]) = 0$, $i = 1, 2, 3, 4$. The next (and final) step consists in proving that the family of measures M_Θ is consistent: for any tetrahedron $\Theta_1 \subset \Theta$

$$(3.7) \quad M_{\Theta_1} \text{ is the restriction of } M_\Theta \text{ on the set } [\Theta_1] = \{e \in \mathbf{IE} : e \text{ hits } \Theta_1\}.$$

To prove (3.7)) we take three intervals I_1, I_2, I_3 from ν_k, ν_l, ν_r = edges of Θ , and three intervals J_1, J_2, J_3 from a triad ν'_k, ν'_l, ν'_r = edges of Θ_1 . It is enough to show that for the sets

$$A_1 = [I_1] \cap [I_2] \cap [I_3] \quad \text{and} \quad A_2 = [J_1] \cap [J_2] \cap [J_3]$$

we have

$$A_1 \cap A_2 = \emptyset \quad \text{implies} \quad \Psi_F(A_1 \cap A_2) = 0 \quad \text{and}$$

$$A_1 \subset A_2 \quad \text{implies} \quad \Psi_F(A_1 \cap A_2) = \Psi_F(A_1).$$

The last two implications can be seen directly from the algorithm (3.1) as applied to $A_1, A_2 \in Br\{P_i\}$, where $\{P_i\}$ is the set of endpoints of the intervals I_1, I_2, I_3 and J_1, J_2, J_3 .

We note, that this consistency proof implies, that the measure M_Θ does not depend on the map (3.6). Consistency of the measures M_Θ implies the existence of some (unique) measure μ on $\mathbf{IE}$ such that M_Θ is the restriction of M on the set $[\Theta]$. The second assertion of the theorem follows from our construction of the measure M_Θ and the uniqueness of the measure extension. The proof is complete.

3.4. A uniqueness problem. Assume that we have some wedge function $F_0(W)$ that satisfies the conditions of Theorem 3. Let $M(de)$ be the measure in the space of planes guaranteed by the Theorem. Using that $M(de)$, we construct the function $F(W)$ as given by the integral (2.5). Is it true, that always

$$F(W) = F_0(W) ?$$

In other words, can a measure in the space $\mathbf{IE}$ be generated, according to Theorem 2, by two different wedge functions? This seems to be the basic unsolved problem in the theory of wedge metrics.

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ON THE DE RHAM THEOREM AND AN APPLICATION TO
THE MAXWELL-STOKES TYPE PROBLEM

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Abstract. In this paper, we derive an L^p version of the de Rham theorem. The key is an L^p version of the Nečas inequality. Using this result and the variational method, we show the existence of a solution to the Maxwell-Stokes type system.

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1. INTRODUCTION

The final purpose of this paper is to derive the existence of a solution to the Maxwell-Stokes type system.

First, we consider the following quasilinear magneto-static problem:

$$(1.1a) \quad \operatorname{curl} [\mathbf{G}(x, \operatorname{curl} \mathbf{u})] = \mathbf{f} \text{ in } \Omega,$$

$$(1.1b) \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,$$

$$(1.1c) \quad \mathbf{u}_T = \mathbf{u}_T^0 \text{ on } \Gamma,$$

where Ω is a bounded domain in $\mathbb{R}^3$ with a boundary $\Gamma = \partial\Omega$, $\mathbf{u}_T$ denotes the tangent component of $\mathbf{u}$, namely, if we write the unit outer normal vector of the boundary by $\mathbf{n}$, then $\mathbf{u}_T = (\mathbf{n} \times \mathbf{u}) \times \mathbf{n}$, and $\mathbf{u}_T^0$ is a given tangential vector field, that is, $\mathbf{n} \cdot \mathbf{u}_T^0 = 0$ on Γ .

This system is interesting in physics, and may be viewed as the stationary version of the eddy current model, where the relation between the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ is defined by the nonlinear B - H -curve. For the physical nature of the nonlinear B - H -curve, see Kaltenbacher et al. [14] and Pechstein and Jütter [19]. The eddy-current problem is a quasi-static approximation at very low frequency of the Maxwell equation, and the approximation is obtained by neglecting the displacement current in the Maxwell-Ampère law. Here we want to say that the solvability of (1.1a)-(1.1b) depends on the nonlinearity of a vector function $\mathbf{G}(x, \mathbf{z})$, the boundary conditions and the shape of the domain Ω with special emphasis, Such system are investigated by many authors, for example, Pan [18], Miranda

et al. [15, 16], Yin [21, 23], Yin et al. [22]. If a given function $\mathbf{f}$ does not satisfy $\operatorname{div} \mathbf{f} = 0$ in Ω , or Ω has holes, then the system (1.1a)-(1.1c) are not nicely posed problem, so we may introduce an unknown scalar function π to the system, which may be called a potential.

To overcome such difficulty, we consider the following Maxwell-Stokes type system:

$$(1.2a) \quad \operatorname{curl} [S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u}] + \nabla \pi = \mathbf{f} \text{ in } \Omega,$$

$$(1.2b) \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,$$

$$(1.2c) \quad \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma,$$

$$(1.2d) \quad \mathbf{u} \cdot \mathbf{n} = g \text{ on } \Gamma,$$

where $\mathbf{f}$ and g are given functions, and $S(x, t)$ is a Carathéodory function on $\Omega \times [0, \infty)$ satisfying some structure conditions (see section 3). According to the conditions on a function $S(x, t)$, we can see that the equation (1.2a) contains a p -curlcurl equation:

$$\operatorname{curl} [|\operatorname{curl} \mathbf{u}|^{p-2} \operatorname{curl} \mathbf{u}] + \nabla \pi = \mathbf{f} \text{ in } \Omega \quad (1 < p < \infty).$$

If we impose the Dirichlet boundary condition to π , then we derived the solvability of the system (1.2a)-(1.2c) in a multi-connected domain without holes in the author's previous paper Aramaki [8]. The de Rham theorem used there was rather restrictive (cf. Aramaki [6]).

However, in the case where Ω has holes, it is necessary to impose the boundary condition (1.2d) for g satisfying some conditions. For this purpose, we have to derive a more general de Rham theorem.

In this paper, we do not impose any boundary condition to the potential, and we derive the existence of solution to the system (1.2a)-(1.2d). To do so, it is necessary to derive an L^p version of de Rham theorem.

The paper is organized as follows. In section 2, we give an L^p version of the de Rham theorem which is ushered by an L^p version of the celebrated Nečas inequality. In section 3, we give some preliminaries for the Maxwell-Stokes type system. Section 4 is devoted to the existence theory of a solution to the Maxwell-Stokes system, using the de Rham theorem given in section 2.

2. A COARSE VERSION OF THE DE RHAM THEOREM

In this section, let Ω be a bounded domain which means a bounded, connected open subset of $\mathbb{R}^d$ ($d \geq 2$) with a Lipschitz boundary Γ , $1 < q < \infty$, and let q' be the conjugate exponent i.e., $(1/q) + (1/q') = 1$.

From now on we use $L^q(\Omega)$, $W^{m,q}(\Omega)$ ($m \geq 0$, integer), $W^{s,q}(\Gamma)$ ($s \in \mathbb{R}$), and so on, for the standard real L^q and Sobolev spaces of real valued functions. For any real Banach space B , we denote B^d by boldface character $\mathbf{B}$. Hereafter, we use this character to denote vector and vector-valued functions, and we denote the standard inner product of vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^d$ by $\mathbf{a} \cdot \mathbf{b}$. Moreover, for the dual space $\mathbf{B}'$, we denote the duality bracket between $\mathbf{B}'$ and $\mathbf{B}$ by $\langle \cdot, \cdot \rangle_{\mathbf{B}', \mathbf{B}}$.

We consider a coarse version of the de Rham theorem. In order to do so, we first state the Nečas inequality which takes an important role for the proof of a coarse version of the de Rham theorem.

Theorem 2.1 (Nečas inequality). *Let Ω is a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary Γ and $1 < q < \infty$. Then the set*

$$\{\pi \in W^{-1,q}(\Omega); \nabla \pi \in \mathbf{W}^{-1,q}(\Omega)\}$$

is equal to $L^q(\Omega)$, and there exists a constant $C > 0$ depending only on q and Ω such that

$$\|\pi\|_{L^q(\Omega)} \leq C(\|\pi\|_{W^{-1,q}(\Omega)} + \|\nabla \pi\|_{\mathbf{W}^{-1,q}(\Omega)}).$$

For the proof, see Theorem IV.1.1 for $q = 2$ and Remark IV.1.1 for general $1 < q < \infty$ in Boyer and Fabrie [10].

The Nečas inequality now allow the following Poincaré type inequality for the function of $L^q(\Omega)$.

Proposition 2.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary Γ and let $1 < q < \infty$. Then there exists a constant $C > 0$ depending only on q and Ω such that*

$$\|\pi\|_{W^{-1,q}(\Omega)} \leq C \left(\frac{1}{|\Omega|} \left| \int_{\Omega} \pi dx \right| + \|\nabla \pi\|_{\mathbf{W}^{-1,q}(\Omega)} \right) \text{ for all } \pi \in L^q(\Omega),$$

where $|\Omega|$ denotes the volume of Ω .

Proof. Assume that the conclusion is false. Then there exists $\{\pi_n\}_{n=1}^{\infty} \subset L^q(\Omega)$ such that

$$\|\pi_n\|_{W^{-1,q}(\Omega)} \geq n \left(\frac{1}{|\Omega|} \left| \int_{\Omega} \pi_n dx \right| + \|\nabla \pi_n\|_{\mathbf{W}^{-1,q}(\Omega)} \right).$$

By homogeneity, we may assume that $\|\pi_n\|_{W^{-1,q}(\Omega)} = 1$. From Nečas inequality (Theorem 2.1), we can deduce that $\{\pi_n\}$ is bounded in $L^q(\Omega)$. Passing to a subsequence, we may assume that $\pi_n \rightarrow \pi$ weakly in $L^q(\Omega)$. Since the embedding $W_0^{1,q'}(\Omega) \hookrightarrow L^{q'}(\Omega)$ is compact and dense, we can see the embedding $L^q(\Omega) \hookrightarrow W^{-1,q}(\Omega)$ is also compact. Therefore, $\pi_n \rightarrow \pi$ strongly in $W^{-1,q}(\Omega)$. Since $\|\nabla \pi_n\|_{\mathbf{W}^{-1,q}(\Omega)} \rightarrow 0$ as

$n \rightarrow \infty$, we obtain $\nabla \pi = \mathbf{0}$ in the distribution sense and therefore $\pi = c = \text{const.}$. However, we also have

$$\frac{1}{|\Omega|} \left| \int_{\Omega} \pi_n dx \right| \leq \frac{1}{n}.$$

Since $\pi_n \rightarrow \pi = c$ weakly in $L^q(\Omega)$, we obtain $c = 0$, so $\pi = 0$. On the other hand, since $\|\pi\|_{W^{-1,q}(\Omega)} = \lim_{n \rightarrow \infty} \|\pi_n\|_{W^{-1,q}(\Omega)} = 1$, this leads to a contradiction. $\square$

Next we derive that the gradient operator from $L^q(\Omega)$ to $\mathbf{W}^{-1,q}(\Omega)$ has a closed range.

Proposition 2.2. *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary Γ and $1 < q < \infty$. Then the gradient operator $\text{grad} = \nabla : L^q(\Omega) \rightarrow \mathbf{W}^{-1,q}(\Omega)$ has a closed range in $\mathbf{W}^{-1,q}(\Omega)$.*

Proof. Let $\pi_n \in L^q(\Omega)$ and $\nabla \pi_n \rightarrow \mathbf{f}$ in $\mathbf{W}^{-1,q}(\Omega)$ as $n \rightarrow \infty$. Then we may assume that $\int_{\Omega} \pi_n dx = 0$ for all $n \in \mathbb{N}$. By Nečas inequality, we have

$$\|\pi_n - \pi_m\|_{L^q(\Omega)} \leq C(\|\pi_n - \pi_m\|_{\mathbf{W}^{-1,q}(\Omega)} + \|\nabla(\pi_n - \pi_m)\|_{\mathbf{W}^{-1,q}(\Omega)}).$$

However, it follows from Proposition 2.1 that we have

$$\|\pi_n - \pi_m\|_{\mathbf{W}^{-1,q}(\Omega)} \leq C\|\nabla(\pi_n - \pi_m)\|_{\mathbf{W}^{-1,q}(\Omega)}.$$

Thus we obtain

$$\|\pi_n - \pi_m\|_{L^q(\Omega)} \leq C_1\|\nabla(\pi_n - \pi_m)\|_{\mathbf{W}^{-1,q}(\Omega)}.$$

Since $\nabla \pi_n \rightarrow \mathbf{f}$ in $\mathbf{W}^{-1,q}(\Omega)$, $\{\pi_n\}$ is a Cauchy sequence in $L^q(\Omega)$. Therefore, there exists $\pi \in L^q(\Omega)$ such that $\pi_n \rightarrow \pi$ in $L^q(\Omega)$. So we have $\mathbf{f} = \nabla \pi \in \nabla(L^q(\Omega))$. $\square$

We are in a position to state a coarse version of the de Rham theorem.

Theorem 2.2 (A coarse version of the de Rham theorem). *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz boundary Γ , $1 < q < \infty$ and let $\mathbf{h} \in \mathbf{W}^{-1,q'}(\Omega)$. If $\mathbf{h}$ satisfies*

$$(2.1) \quad \langle \mathbf{h}, \mathbf{v} \rangle_{\mathbf{W}^{-1,q'}(\Omega), \mathbf{W}_0^{1,q}(\Omega)} = 0$$

for all $\mathbf{v} \in \mathbf{W}_0^{1,q}(\Omega)$ satisfying $\text{div } \mathbf{v} = 0$ in Ω ,

then there exists a function $\pi \in L_0^{q'}(\Omega) := \left\{ q \in L^{q'}(\Omega); \int_{\Omega} q dx = 0 \right\}$ such that

$$(2.2) \quad \mathbf{h} = \nabla \pi \text{ in } \Omega.$$

Conversely, if (2.2) holds, then clearly (2.1) holds.

Proof. In general, for any subset A of a normed linear space X , define

$$A^{\perp} = \{f \in X'; \langle f, x \rangle_{X', X} = 0 \text{ for all } x \in A\},$$

and for any subset A' of the dual space X' , define

$${}^\perp(A') = \{x \in X; \langle f, x \rangle_{X', X} = 0 \text{ for all } f \in A'\}.$$

It is well known that if A is a closed subspace of X , then it holds that ${}^\perp(A^\perp) = A$ (cf. Taylor and Lay [20, p. 164]).

Define $X = \mathbf{W}^{-1, q'}(\Omega)$ and

$$\mathbf{Y}^{q'}(\Omega) = \{\nabla \pi : \pi \in L^{q'}(\Omega)\}.$$

Then it follows from Proposition 2.2 that $\mathbf{Y}^{q'}(\Omega)$ is a closed subspace of $X = \mathbf{W}^{-1, q'}(\Omega)$, so

$$(2.3) \quad {}^\perp(\mathbf{Y}^{q'}(\Omega)^\perp) = \mathbf{Y}^{q'}(\Omega).$$

If we define

$$\mathbf{Z}^q(\Omega) = \{\mathbf{v} \in \mathbf{W}_0^{1, q}(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\},$$

then $\mathbf{Z}^q(\Omega)$ is a closed subspace of $X' = \mathbf{W}^{-1, q'}(\Omega)' = \mathbf{W}_0^{1, q}(\Omega)$. The theorem clearly means that

$$(2.4) \quad {}^\perp \mathbf{Z}^q(\Omega) \subset \mathbf{Y}^{q'}(\Omega).$$

To derive (2.4), it suffices to show that $\mathbf{Y}^{q'}(\Omega)^\perp \subset \mathbf{Z}^q(\Omega)$ since (2.3) holds. Assume that $\mathbf{v} \in \mathbf{Y}^{q'}(\Omega)^\perp \subset X' = \mathbf{W}_0^{1, q}(\Omega)$. Then for any $\pi \in L^{q'}(\Omega)$,

$$-\int_{\Omega} \pi \operatorname{div} \mathbf{v} dx = \langle \nabla \pi, \mathbf{v} \rangle_{\mathbf{W}^{-1, q'}(\Omega), \mathbf{W}_0^{1, q}(\Omega)} = 0.$$

This implies that

$$\langle \operatorname{div} \mathbf{v}, \pi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = 0 \text{ for all } \pi \in \mathcal{D}(\Omega),$$

where $\mathcal{D}(\Omega)$ is the space of C^∞ functions with compact supports in Ω , and $\mathcal{D}'(\Omega)$ is the space of distributions in Ω . Therefore, $\operatorname{div} \mathbf{v} = 0$ in $\mathcal{D}'(\Omega)$. Since $\operatorname{div} \mathbf{v} \in L^q(\Omega)$, we obtain that $\operatorname{div} \mathbf{v} = 0$ in $L^q(\Omega)$, so $\mathbf{v} \in \mathbf{Z}^q(\Omega)$. $\square$

Remark 2.1. *To tell the truth, the Nečas inequality (Theorem 2.1), Proposition 2.1 and a coarse version of the de Rham theorem (Theorem 2.2) are equivalent. For this facts, see Amrouche et al. [4] for $q = 2$, and see [6] for general $1 < q < \infty$*

3. PRELIMINARIES FOR THE MAXWELL-STOKES TYPE PROBLEM

In this section, we give preliminaries for the Maxwell-Stokes type problem as an application of a coarse version of the de Rham theorem (Theorem 2.2). To do so, we assume that Ω is a bounded domain (connected open subset) of $\mathbb{R}^3$ with a Lipschitz boundary Γ satisfying the following conditions as in Amrouche and Seloula [2] (cf. Amrouche and Seloula [3] and Girault and Raviart [13]).

Assume that Ω is locally situated on one side of Γ . In addition,

- (O1) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_I$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.
- (O2) There exist J connected open surfaces Σ_j , ($j = 1, \dots, J$), called cuts, contained in Ω such that
 - (a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.
 - (b) $\partial\Sigma_j \subset \Gamma$ ($j = 1, \dots, J$), where $\partial\Sigma_j$ denotes the boundary of Σ_j , and Σ_j is non-tangential to Γ .
 - (c) $\overline{\Sigma_j} \cap \overline{\Sigma_k} = \emptyset$ ($j \neq k$).
 - (d) The open set $\dot{\Omega} = \Omega \setminus (\cup_{j=1}^J \Sigma_j)$ is simply connected and of Lipschitz class.

The number J is called the first Betti number which is equal to the number of handles of Ω , and I is called the second Betti number which is equal to the number of holes. We say that if $J = 0$, Ω is simply connected, and if $I = 0$, Ω has no holes.

Define two spaces by

$$\mathbb{K}_N^p(\Omega) = \{\mathbf{v} \in \mathbf{L}^p(\Omega); \operatorname{div} \mathbf{v} = 0, \operatorname{curl} \mathbf{v} = \mathbf{0} \text{ in } \Omega, \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\},$$

$$\mathbb{K}_T^p(\Omega) = \{\mathbf{v} \in \mathbf{L}^p(\Omega); \operatorname{div} \mathbf{v} = 0, \operatorname{curl} \mathbf{v} = \mathbf{0} \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

Then it is well known that $\dim \mathbb{K}_T^p(\Omega) = J$ and $\dim \mathbb{K}_N^p(\Omega) = I$.

We assume that a Carathéodory function $S(x, t)$ satisfies the following conditions: There exist $1 < p < \infty$ and positive constants $0 < \lambda \leq \Lambda < \infty$ such that for a.e. $x \in \Omega$, $S(x, \cdot) \in C^2((0, \infty)) \cap C^0([0, \infty))$ as a function of t , and $S(x, t)$ satisfies that

$$(3.1a) \quad S(x, 0) = 0 \text{ and } \lambda t^{(p-2)/2} \leq S_t(x, t) \leq \Lambda t^{(p-2)/2} \text{ for } t > 0.$$

$$(3.1b) \quad \lambda t^{(p-2)/2} \leq S_t(x, t) + 2tS_{tt}(x, t) \leq \Lambda t^{(p-2)/2} \text{ for } t > 0.$$

$$(3.1c) \quad \text{If } 1 < p < 2, S_{tt}(x, t) < 0, \text{ and if } p \geq 2, S_{tt}(x, t) \geq 0 \text{ for } t > 0.$$

Here $S_t = \partial S / \partial t$, $S_{tt} = \partial^2 S / \partial t^2$. We note that from (3.1a), we have

$$(3.2) \quad \frac{2}{p} \lambda t^{p/2} \leq S(x, t) \leq \frac{2}{p} \Lambda t^{p/2} \text{ for } t \geq 0 \text{ and a.e. } x \in \Omega.$$

Example 3.1. If $S(x, t) = \nu(x)t^{p/2}$, where ν is a measurable function in Ω and satisfies $0 < \nu_* \leq \nu(x) \leq \nu^* < \infty$, then it follows from elementary calculations that (3.1a)-(3.1c) hold.

We have the following lemma with respect to the monotonicity of S_t .

Lemma 3.1. *There exists a constant $c > 0$ such that for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,*

$$\begin{aligned} & (S_t(x, |\mathbf{a}|^2)\mathbf{a} - S_t(x, |\mathbf{b}|^2)\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ & \geq \begin{cases} c|\mathbf{a} - \mathbf{b}|^p & \text{if } p \geq 2, \\ c(|\mathbf{a}| + |\mathbf{b}|)^{p-2}|\mathbf{a} - \mathbf{b}|^2 & \text{if } 1 < p < 2. \end{cases} \end{aligned}$$

In particular, S_t is strictly monotone, that is,

$$(S_t(x, |\mathbf{a}|^2)\mathbf{a} - S_t(x, |\mathbf{b}|^2)\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) > 0 \text{ if } \mathbf{a} \neq \mathbf{b}.$$

For the proof, see Aramaki [7, Lemma 3.5].

Next, we show that $S(x, |\mathbf{a}|^2)$ is strictly convex with respect to $\mathbf{a} \in \mathbb{R}^3$.

Lemma 3.2. *For a.e. $x \in \Omega$, $S(x, |\mathbf{a}|^2)$ is strictly convex with respect to $\mathbf{a} \in \mathbb{R}^3$, that is, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and $0 \leq \theta \leq 1$,*

$$(3.3) \quad S(x, |\theta\mathbf{a} + (1-\theta)\mathbf{b}|^2) \leq \theta S(x, |\mathbf{a}|^2) + (1-\theta)S(x, |\mathbf{b}|^2),$$

and in particular, if $\mathbf{a} \neq \mathbf{b}$ and $0 < \theta < 1$, then we have

$$(3.4) \quad S(x, |\theta\mathbf{a} + (1-\theta)\mathbf{b}|^2) < \theta S(x, |\mathbf{a}|^2) + (1-\theta)S(x, |\mathbf{b}|^2).$$

Proof. For brevity of notation, we put $F(x, t) = S(x, t^2)$. Since

$$F_t(x, t) = 2tS_t(x, t^2) \geq 2\lambda t^{p-1} > 0 \text{ a.e. } x \in \Omega \text{ and } t > 0$$

from (3.1a), and

$$F_{tt}(x, t) = 2(S_t(x, t^2) + 2t^2S_{tt}(x, t^2)) \geq 2\lambda t^{p-2} > 0 \text{ a.e. } x \in \Omega \text{ and } t > 0$$

from (3.1b), we see that for a.e. $x \in \Omega$, $F(x, t)$ is strictly increasing and strictly convex as a function of $t \in [0, \infty)$. Therefore, for a.e. $x \in \Omega$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and $0 \leq \theta \leq 1$, we have

$$F(x, |\theta\mathbf{a} + (1-\theta)\mathbf{b}|) \leq F(x, \theta|\mathbf{a}| + (1-\theta)|\mathbf{b}|) \leq \theta F(x, |\mathbf{a}|) + (1-\theta)F(x, |\mathbf{b}|).$$

Thus $F(x, |\mathbf{a}|) = S(x, |\mathbf{a}|^2)$ is a convex function of $\mathbf{a} \in \mathbb{R}^3$.

Let $\mathbf{a} \neq \mathbf{b}$ and $0 < \theta < 1$. Without loss of generality, we may assume $\mathbf{a} \neq \mathbf{0}$. If $|\theta\mathbf{a} + (1-\theta)\mathbf{b}| < \theta|\mathbf{a}| + (1-\theta)|\mathbf{b}|$, then we have

$$F(x, |\theta\mathbf{a} + (1-\theta)\mathbf{b}|) < F(x, \theta|\mathbf{a}| + (1-\theta)|\mathbf{b}|) \leq \theta F(x, |\mathbf{a}|) + (1-\theta)F(x, |\mathbf{b}|).$$

If $|\theta\mathbf{a} + (1-\theta)\mathbf{b}| = \theta|\mathbf{a}| + (1-\theta)|\mathbf{b}|$, then this implies that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$. By the Schwarz inequality, $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent, so we can write $\mathbf{b} = c\mathbf{a}$, where $c \geq 0$ and $c \neq 1$. This implies $|\mathbf{a}| \neq |\mathbf{b}|$. Thus it follows from the strict convexity of $F(x, t)$ as a function of t that

$$F(x, |\theta\mathbf{a} + (1-\theta)\mathbf{b}|) = F(x, \theta|\mathbf{a}| + (1-\theta)|\mathbf{b}|) < \theta F(x, |\mathbf{a}|) + (1-\theta)F(x, |\mathbf{b}|).$$

□

4. EXISTENCE OF A SOLUTION TO THE MAXWELL-STOKES TYPE SYSTEM

In this section, we derive the existence of a solution to the Maxwell-Stokes type system. Let Ω be a bounded domain in $\mathbb{R}^3$ satisfying (O1) and (O2), $1 < p < \infty$ and let a Carathéodory function $S(x, t)$ satisfy (3.1a)-(3.1c).

We consider the following Maxwell-Stokes type problem: for given $\mathbf{f}$ and g , find $(\mathbf{u}, \pi)$ such that

$$(4.1a) \quad \operatorname{curl} [S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u}] + \nabla \pi = \mathbf{f} \text{ in } \Omega,$$

$$(4.1b) \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,$$

$$(4.1c) \quad \mathbf{u} = g\mathbf{n} \text{ on } \Gamma.$$

Define a space

$$\mathbb{V}^p(\Omega) = \{\mathbf{v} \in \mathbf{W}^{1,p}(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \\ \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0 \text{ for } i = 1, \dots, I\}.$$

Then we can see that $\mathbb{V}^p(\Omega)$ is a separable, reflexive Banach space equipped with the semi-norm

$$(4.2) \quad \|\mathbf{v}\|_{\mathbb{V}^p(\Omega)} = \|\operatorname{curl} \mathbf{v}\|_{\mathbf{L}^p(\Omega)}.$$

By [3, p. 40], since we have

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(p, \Omega) \|\operatorname{curl} \mathbf{v}\|_{\mathbf{L}^p(\Omega)} \text{ for all } \mathbf{v} \in \mathbb{V}^p(\Omega),$$

the definition (4.2) is, in fact, the norm and $\|\mathbf{v}\|_{\mathbb{V}^p(\Omega)}$ and $\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)}$ are equivalent (cf. [15, 16]).

Assume that a given function $\mathbf{f}$ satisfies that $\mathbf{f} \in \mathbb{V}^p(\Omega)' \cap \mathbf{W}^{-1,p'}(\Omega)$, where p' denotes the conjugate exponent of p , and

$$\mathbf{f}|_{\mathbb{V}^p(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega)} \text{ in } \mathbf{W}^{-1,p'}(\Omega) = \mathbf{f}|_{\mathbb{V}^p(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega)} \text{ in } \mathbb{V}^p(\Omega)',$$

that is,

$$(4.3) \quad \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{W}^{-1,p'}(\Omega)', \mathbf{W}_0^{1,p}(\Omega)}$$

for all $\mathbf{v} \in \mathbb{V}^p(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega)$.

Moreover, we assume that $g \in W^{1-1/p,p}(\Gamma)$ and satisfies that

$$(4.4) \quad \int_{\Gamma_i} g d\sigma = 0 \text{ for every } i = 0, 1, \dots, I,$$

where $d\sigma$ denotes the surface area of Γ . We define a space

$$U_g^p(\Omega) = \{\mathbf{v} \in \mathbf{W}^{1,p}(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} = g\mathbf{n} \text{ on } \Gamma\}.$$

If $\mathbf{v} \in \mathbf{U}_g^p(\Omega)$, then $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ and $\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = \int_{\Gamma_i} g d\sigma = 0$, and so we can easily see that $\mathbf{U}_g^p(\Omega)$ is a closed, convex subset of $\mathbb{V}^p(\Omega)$. If we put $\mathbf{g} = g\mathbf{n}$ on Γ , then from (4.4), we have

$$\int_{\Gamma} \mathbf{g} \cdot \mathbf{n} d\sigma = \int_{\Gamma} g d\sigma = \sum_{i=0}^I \int_{\Gamma_i} g d\sigma = 0.$$

Therefore, it follows from Amrouche and Girault [1, Lemma 3.3] that there exists $\mathbf{v} \in \mathbf{W}^{1,p}(\Omega)$ such that $\operatorname{div} \mathbf{v} = 0$ in Ω and $\mathbf{v} = \mathbf{g} = g\mathbf{n}$ on Γ . Thus $\mathbf{v} \in \mathbf{U}_g^p(\Omega)$, so $\mathbf{U}_g^p(\Omega)$ is a non-empty set. Since $\mathbf{U}_g^p(\Omega)$ is a closed and convex subset of $\mathbb{V}^p(\Omega)$, we can see that $\mathbf{U}_g^p(\Omega)$ is sequentially weakly closed subset of $\mathbb{V}^p(\Omega)$ (cf. Ciarlet [12, Theorem 5.13-1]).

We are in a position to state a main theorem.

Theorem 4.1. *Assume that Ω is a bounded domain of $\mathbb{R}^3$ with a Lipschitz boundary Γ and satisfies (O1) and (O2), $1 < p < \infty$ and a Carathéodory function $S(x, t)$ satisfies (3.1a)-(3.1c). Moreover, assume that $\mathbf{f} \in \mathbb{V}^p(\Omega)' \cap \mathbf{W}^{-1,p'}(\Omega)$ satisfies (4.3) and $g \in W^{1-1/p,p}(\Gamma)$ satisfies (4.4). Then the Maxwell-Stokes type system (4.1a)-(4.1c) has a unique weak solution $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$, and there exists a constant $C > 0$ dependent only on p and Ω such that*

$$(4.5) \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}^p + \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}}^{p'} \leq C(\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}^{p'} + \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} + \|g\|_{W^{1-1/p,p}(\Gamma)}^p).$$

Proof. We prove this theorem using the direct method of calculus of variation and using a coarse version of the de Rham theorem. In order to do so, we consider a functional

$$(4.6) \quad J[\mathbf{v}] = \frac{1}{2} \int_{\Omega} S(x, |\operatorname{curl} \mathbf{v}|^2) dx - \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \text{ for } \mathbf{v} \in \mathbf{U}_g^p(\Omega).$$

Step 1. J has a unique minimizer $\mathbf{u} \in \mathbf{U}_g^p(\Omega)$, that is,

$$J(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{U}_g^p(\Omega)} J(\mathbf{v}).$$

From (3.2), the duality and the Young inequality, for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that for all $\mathbf{v} \in \mathbf{U}_g^p(\Omega)$,

$$\begin{aligned} J[\mathbf{v}] &\geq \frac{\lambda}{p} \|\operatorname{curl} \mathbf{v}\|_{L^p(\Omega)}^p - \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'} \|\mathbf{v}\|_{\mathbb{V}^p(\Omega)} \\ &\geq \frac{\lambda}{p} \|\mathbf{v}\|_{\mathbb{V}^p(\Omega)}^p - C(\varepsilon) \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} - \varepsilon \|\mathbf{v}\|_{\mathbb{V}^p(\Omega)}^p. \end{aligned}$$

If we put $\varepsilon = \lambda/2p$, we have

$$(4.7) \quad J[\mathbf{v}] \geq \frac{\lambda}{2p} \|\mathbf{v}\|_{\mathbb{V}^p(\Omega)}^p - C \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} > -\infty \text{ for } \mathbf{v} \in \mathbf{U}_g^p(\Omega).$$

Thus we see that J is coercive on $\mathbf{U}_g^p(\Omega)$.

Next we show that $J : \mathbf{U}_g^p(\Omega) \rightarrow \mathbb{R}$ is sequentially weakly lower semi-continuous. Let $\mathbf{v}_j \rightarrow \mathbf{v}$ weakly in $\mathbb{V}^p(\Omega)$. Since $\text{curl} : \mathbb{V}^p(\Omega) \rightarrow \mathbf{L}^p(\Omega)$ is linear and bounded, it is clear that $\text{curl } \mathbf{v}_j \rightarrow \text{curl } \mathbf{v}$ weakly in $\mathbf{L}^p(\Omega)$.

Then it follows from Aramaki [5] that

$$\int_{\Omega} S(x, |\text{curl } \mathbf{v}|^2) dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} S(x, |\text{curl } \mathbf{v}_j|^2) dx.$$

On the other hand, since $\mathbf{f} \in \mathbb{V}^p(\Omega)'$ and $\mathbf{v}_j \rightarrow \mathbf{v}$ weakly in $\mathbb{V}^p(\Omega)$, we have

$$\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} = \lim_{j \rightarrow \infty} \langle \mathbf{f}, \mathbf{v}_j \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)}.$$

Thus we have

$$J(\mathbf{v}) \leq \liminf_{j \rightarrow \infty} J(\mathbf{v}_j),$$

that is, $J : \mathbf{U}_g^p(\Omega) \rightarrow \mathbb{R}$ is sequentially weakly lower semi-continuous. Therefore, J has a minimizer $\mathbf{u} \in \mathbf{U}_g^p(\Omega)$. See, for example, Ciarlet [12, Theorem 9.3-1].

Using (3.3), we can easily see that J is a convex functional on $\mathbf{U}_g^p(\Omega)$. Moreover, since $\|\mathbf{v}\|_{\mathbb{V}^p(\Omega)}$ and $\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)}$ are equivalent for $\mathbf{v} \in \mathbb{V}^p(\Omega)$, it follows that if $\mathbf{u} \neq \mathbf{v}$ in $\mathbf{U}_g^p(\Omega)$, then $\text{curl } \mathbf{u} \neq \text{curl } \mathbf{v}$ in $\mathbf{L}^p(\Omega)$. From Lemma 3.2, we see that J is strictly convex on $\mathbf{U}_g^p(\Omega)$. Thus the minimizer is unique.

Step 2. Let $\mathbf{u} \in \mathbf{U}_g^p(\Omega)$ be a unique minimizer of J on $\mathbf{U}_g^p(\Omega)$. For any

$$\mathbf{w} \in \mathbf{Z}^p(\Omega) = \{\mathbf{v} \in \mathbf{W}_0^{1,p}(\Omega); \text{div } \mathbf{v} = 0 \text{ in } \Omega\},$$

we have $\mathbf{u} + \tau \mathbf{w} \in \mathbf{U}_g^p(\Omega)$ for all $\tau \in \mathbb{R}$, and so $J(\mathbf{u}) \leq J(\mathbf{u} + \tau \mathbf{w})$. By the Euler-Lagrange equation and assumption (4.3), we have

$$\begin{aligned} 0 &= \left. \frac{d}{d\tau} J[\mathbf{u} + \tau \mathbf{w}] \right|_{\tau=0} \\ &= \int_{\Omega} S_t(x, |\text{curl } \mathbf{u}|^2) \text{curl } \mathbf{u} \cdot \text{curl } \mathbf{w} dx - \langle \mathbf{f}, \mathbf{w} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \\ &= \int_{\Omega} S_t(x, |\text{curl } \mathbf{u}|^2) \text{curl } \mathbf{u} \cdot \text{curl } \mathbf{w} dx - \langle \mathbf{f}, \mathbf{w} \rangle_{\mathbf{W}^{-1,p'}(\Omega)', \mathbf{W}_0^{1,p}(\Omega)}. \end{aligned}$$

From a coarse version of the de Rham theorem (Theorem 2.2), we can derive that there exists a function $\pi \in L_0^{p'}(\Omega)$ such that

$$(4.8) \quad \text{curl} [S_t(x, |\text{curl } \mathbf{u}|^2) \text{curl } \mathbf{u}] + \nabla \pi = \mathbf{f} \text{ in } \mathbf{W}^{-1,p'}(\Omega).$$

Thus $(\mathbf{u}, \pi)$ is a solution of the system (4.1a)-(4.1c).

Step 3. We show the uniqueness of a solution of (4.1a)-(4.1c). Let $(\mathbf{u}_1, \pi_1), (\mathbf{u}_2, \pi_2) \in \mathbf{W}^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be two solutions of (4.1a)-(4.1c). Since $\mathbf{u}_i \cdot \mathbf{n} = g$ and $\mathbf{u}_i \times \mathbf{n} = \mathbf{0}$ for $i = 1, 2$, we see that $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{0}$ on Γ , so we have $\mathbf{u}_1 - \mathbf{u}_2 \in \mathbf{Z}^p(\Omega)$. Thus we

have

$$\langle \nabla \pi_i, \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\mathbf{W}^{-1,p'}(\Omega), \mathbf{W}_0^{1,p}(\Omega)} = 0 \text{ for } i = 1, 2.$$

Therefore, if we take the inner product of (4.1a) and $\mathbf{u}_1 - \mathbf{u}_2$, and then integrate over Ω , we have

$$\begin{aligned} \int_{\Omega} (S_t(x, |\operatorname{curl} \mathbf{u}_1|^2) \operatorname{curl} \mathbf{u}_1 \\ - S_t(x, |\operatorname{curl} \mathbf{u}_2|^2) \operatorname{curl} \mathbf{u}_2) \cdot \operatorname{curl} (\mathbf{u}_1 - \mathbf{u}_2) dx = 0. \end{aligned}$$

From this equality and the strictly monotonicity of S_t (Lemma 3.1), we have $\operatorname{curl} \mathbf{u}_1 = \operatorname{curl} \mathbf{u}_2$ in Ω . This implies $\mathbf{u}_1 = \mathbf{u}_2$. From (4.1a), we have $\nabla \pi_1 = \nabla \pi_2$ in $\mathbf{W}^{-1,p'}(\Omega)$, so in the distribution sense. Since Ω is connected, we have $\pi_1 - \pi_2$ is equal to a constant, so $\pi_1 = \pi_2$ in $L^{p'}(\Omega)/\mathbb{R}$.

Step 4. We derive the estimate (4.5). Let $\mathbf{u} \in \mathbf{U}_g^p(\Omega)$ be the minimizer of J . Then for any $\mathbf{v} \in \mathbf{U}_g^p(\Omega)$ and for $0 < \theta < 1$, since

$$J(\mathbf{u}) \leq J((1 - \theta)\mathbf{u} + \theta\mathbf{v}) = J(\mathbf{u} + \theta(\mathbf{v} - \mathbf{u})),$$

we have

$$\begin{aligned} 0 &\leq \frac{d}{d\theta} J(\mathbf{u} + \theta(\mathbf{v} - \mathbf{u}))|_{\theta=0+} \\ &= \int_{\Omega} S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} (\mathbf{v} - \mathbf{u}) dx - \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (4.9) \quad \int_{\Omega} S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{u} dx - \langle \mathbf{f}, \mathbf{u} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \\ \leq \int_{\Omega} S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} dx - \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \end{aligned}$$

for all $\mathbf{v} \in \mathbf{U}_g^p(\Omega)$. If we put $\mathbf{g} = g\mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma)$, it follows from [1, Lemma 3.3] that there exists $\mathbf{v} \in \mathbf{W}^{1,p}(\Omega)$ such that $\operatorname{div} \mathbf{v} = 0$ in Ω and $\mathbf{v} = \mathbf{g} = g\mathbf{n}$ on Γ , so $\mathbf{v} \in \mathbf{U}_g^p(\Omega)$, and there exists a constant $C > 0$ dependent only on p and Ω such that

$$\inf_{\mathbf{w} \in \mathbf{Z}^p(\Omega)} \|\mathbf{v} + \mathbf{w}\|_{\mathbf{W}^{1,p}(\Omega)} \leq \|g\|_{W^{1-1/p,p}(\Gamma)}.$$

Here we can easily show that $\inf_{\mathbf{w} \in \mathbf{Z}^p(\Omega)} \|\mathbf{v} + \mathbf{w}\|_{\mathbf{W}^{1,p}(\Omega)}$ is achieved. Hence there exists $\mathbf{v}_0 \in \mathbf{U}_g^p(\Omega)$, and there exists a constant $C > 0$ depending only on p and Ω such that

$$(4.10) \quad \|\mathbf{v}_0\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|g\|_{W^{1-1/p,p}(\Omega)}.$$

We estimate (4.9) with $\mathbf{v} = \mathbf{v}_0$. From (3.1a), for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} & \int_{\Omega} S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{u} dx - \langle \mathbf{f}, \mathbf{u} \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \\ & \geq \lambda \int_{\Omega} |\operatorname{curl} \mathbf{u}|^p dx - \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'} \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)} \\ & \geq \lambda \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^p - C(\varepsilon) \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} - \varepsilon \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^p. \end{aligned}$$

On the other hand, using (3.1a), Hölder inequality and (4.10), we have

$$\begin{aligned} & \int_{\Omega} S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v}_0 dx - \langle \mathbf{f}, \mathbf{v}_0 \rangle_{\mathbb{V}^p(\Omega)', \mathbb{V}^p(\Omega)} \\ & \leq \Lambda \int_{\Omega} |\operatorname{curl} \mathbf{u}|^{p-1} |\operatorname{curl} \mathbf{v}_0| dx + C \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'} \|\mathbf{v}_0\|_{\mathbf{W}^{1,p}(\Omega)} \\ & \leq \Lambda \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^{p-1} \|\operatorname{curl} \mathbf{v}_0\|_{L^p(\Omega)} + C \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} + C \|\mathbf{v}_0\|_{\mathbb{V}^p(\Omega)}^p \\ & \leq \varepsilon \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^p + C(\varepsilon) \|\mathbf{v}_0\|_{\mathbf{W}^{1,p}(\Omega)}^p + C \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} \\ & \leq \varepsilon \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^p + C_1(\varepsilon) \|g\|_{W^{1-1/p,p}(\Gamma)}^p + C \|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'}^{p'}. \end{aligned}$$

Therefore, if we choose $\varepsilon > 0$ small enough, then there exists a constant $C > 0$ depending only on p and Ω such that

$$(4.11) \quad \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^p \leq C(\|\mathbf{f}\|_{\mathbb{V}^p(\Omega)'}^{p'} + \|g\|_{W^{1-1/p,p}(\Gamma)}^p).$$

Since $L^{p'}(\Omega) \subset W^{-1,p'}(\Omega)$, taking the Nečas inequality (Theorem 2.1 and the Poincaré inequality (Proposition 2.1) into consideration, we have

$$\begin{aligned} \|\pi\|_{L^{p'}(\Omega)} & \leq C(\|\pi\|_{W^{-1,p'}(\Omega)} + \|\nabla \pi\|_{\mathbf{W}^{-1,p'}(\Omega)}) \\ & \leq C_1 \left(\frac{1}{|\Omega|} \left| \int_{\Omega} \pi dx \right| + \|\nabla \pi\|_{\mathbf{W}^{-1,p'}(\Omega)} \right) \end{aligned}$$

for all $\pi \in L^{p'}(\Omega)$. Since $\operatorname{curl} [S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u}] + \nabla \pi = \mathbf{f}$ in $\mathbf{W}^{-1,p'}(\Omega)$, we have

$$\begin{aligned} \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}} & \leq \left\| \pi - \int_{\Omega} \pi dx \right\|_{L^{p'}(\Omega)} \\ & \leq C \|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)} + \|\operatorname{curl} [S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u}]\|_{\mathbf{W}^{-1,p'}(\Omega)}. \end{aligned}$$

Here, if we note that for all $\mathbf{v} \in \mathbf{W}_0^{1,p}(\Omega)$,

$$\begin{aligned} & |\langle \operatorname{curl} [S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u}], \mathbf{v} \rangle_{\mathbf{W}^{-1,p'}(\Omega), \mathbf{W}_0^{1,p}(\Omega)}| \\ & = \left| \int_{\Omega} S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} dx \right| \\ & \leq C \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^{p-1} \|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)}, \end{aligned}$$

then we obtain

$$\|\operatorname{curl}[S_t(x, |\operatorname{curl} \mathbf{u}|^2) \operatorname{curl} \mathbf{u}]\|_{\mathbf{W}^{-1,p'}(\Omega)} \leq C \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^{p-1}.$$

Thus we have

$$\begin{aligned} (4.12) \quad \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}}^{p'} &\leq C(\|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}^{p'} + \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^p) \\ &\leq C(\|\mathbf{f}\|_{\mathbb{V}^p(\Omega)}^{p'} + \|\mathbf{f}\|_{\mathbf{W}^{-1,p'}(\Omega)}^{p'} + \|\mathbf{u}\|_{\mathbb{V}^p(\Omega)}^p). \end{aligned}$$

Summing (4.11) and (4.12), we get the estimate (4.5). $\square$

Remark 4.1. When $p = 2$ and $S(x, t) = t$, the equation (4.1a) reduces to the Stokes equation

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{F}.$$

For such the Stokes system, there exist many articles, for example, see Cattabriga [11] and Amrouche and Girault [1] and the references therein.

As a concluding remark, we are sure of a potential application of the developed theory of this paper for solving more general Maxwell problem in L^p setting, in particular for inclusion and evolutionary variational inequalities (obstacle problems), and mention recent results in this direction: Azevedo et al. [9], Yousept [24, 25] and Miranda et al. [17].

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О ЕДИНСТВЕННОСТИ ТРИГОНОМЕТРИЧЕСКИХ РЯДОВ

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Аннотация. В работе доказывается, что если суммы Римана тригонометрического ряда ограничены всюду, кроме, быть может, некоторого счетного множества, и по мере сходятся к ограниченной функции f , то этот ряд является рядом Фурье функции f . Отсюда получаются обобщения теорем Кантора и Юнга.

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Ключевые слова: тригонометрическая система; теорема единственности; суммируемость по Риману.

1. ВВЕДЕНИЕ

Если дважды формально интегрировать тригонометрический ряд

$$(1.1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) =: \sum_{n=0}^{\infty} A_n(x),$$

с коэффициентами стремящимися к нулю, то получится равномерно сходящийся ряд с суммой

$$(1.2) \quad F(x) := Ax + B + \frac{a_0 x^2}{4} - \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}.$$

Обозначим

$$(1.3) \quad S(x, h) := \frac{F(x+h) + F(x-h) - 2F(x)}{h^2},$$

и

$$S^*(x) := \sup_{h>0} |S(x, h)|.$$

Выражения $S(x, h)$, $h > 0$, называют суммами Римана ряда (1.1), а $S^*(x)$ -мажорантой Римана этого ряда. Известно, что

$$(1.4) \quad S(x, h) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n(x) \left(\frac{\sin \frac{nh}{2}}{\frac{nh}{2}} \right)^2$$

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и если ряд (1.1) в точке x сходится (см. [1] стр. 187), то

$$\lim_{h \rightarrow 0} S(x, h) = \sum_{n=0}^{\infty} A_n(x),$$

а вообще, если существует $\lim_{h \rightarrow 0} S(x, h) = S$, то говорят, что ряд (1.1) в точке x методом Римана суммируется к значению S .

В теории тригонометрических рядов хорошо известна (см. [2], а также [1] стр. 191)

ТЕОРЕМА КАНТОРА. *Если ряд (1.1) всюду сходится к нулю, то все коэффициенты этого ряда равны нулю.*

На самом деле доказывается, что если ряд (1.1) методом Римана всюду суммируется к нулю, то все коэффициенты этого ряда равны нулю. В дальнейших усилениях и обобщениях теоремы Кантора присутствовала сходимостъ или суммируемость всюду или всюду, кроме, быть может, некоторого счетного множества. Это оправдывается тем, что имеет место (см. [3], а также [1] стр. 804)

ТЕОРЕМА МЕНЬШОВА. *Существует тригонометрический ряд, который почти всюду сходится к нулю, однако не все коэффициенты этого ряда равны нулю.*

Следовательно, для получения теорем о почти всюду сходящихся рядах, нужны дополнительные условия на ряд. В этом направлении первые результаты получены в работах [4]-[6]. В этих работах в качестве дополнительных условий выступают условия на функции распределения мажорант Абеля и Римана.

Здесь мы докажем следующие теоремы.

Теорема 1.1. *Пусть для ряда (1.1) всюду, кроме, быть может, некоторого счетного множества B выполняется*

$$(1.5) \quad S^*(x) < \infty, \quad \text{когда } x \notin B,$$

и суммы $S(x, h)$ по мере сходятся к нулю, когда $h \rightarrow 0$. Тогда все коэффициенты этого ряда равны нулю.

Теорема 1.2. *Пусть для ряда (1.1) всюду, кроме, быть может, некоторого счетного множества B выполняется (1.5) и суммы $S(x, h)$ по мере сходятся к некоторой ограниченной функции f , когда $h \rightarrow 0$. Тогда ряд (1.1) является рядом Фурье функции f .*

Учитывая регулярность метода суммирования Римана, из теорем 1.1, 1.2 получим следующие теоремы.

Теорема 1.3. Пусть для ряда (1.1) всюду, кроме, быть может, некоторого счетного множества B выполняется

$$(1.6) \quad \sup_N \left| \sum_{n=0}^N A_n(x) \right| < \infty, \quad \text{когда } x \notin B,$$

и $\lim_{N \rightarrow \infty} \sum_{n=0}^N A_n(x) = 0$ по мере. Тогда все коэффициенты этого ряда равны нулю.

Теорема 1.4. Пусть для ряда (1.1) всюду, кроме, быть может, некоторого счетного множества B выполняется (1.6) и

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N A_n(x) = f(x) \quad \text{по мере,}$$

где f -некоторая ограниченная функция. Тогда ряд (1.1) является рядом Фурье функции f .

2. НЕКОТОРЫЕ ВСПОМОГАТЕЛЬНЫЕ ЛЕММЫ

Пусть

$$(2.1) \quad \varphi(x) := (1 - |x|)_+, \quad \text{где } t_+ = \max(t, 0), \quad \text{и } \varphi_h(x) := \frac{1}{h} \varphi\left(\frac{x}{h}\right), \quad h > 0.$$

Нетрудно проверить, что

$$(2.2) \quad S(x, h) = \sum_{n=0}^{\infty} (A_n * \varphi_h)(x) = \sum_{n=0}^{\infty} \int_{x-h}^{x+h} A_n(t) \varphi_h(x-t) dt.$$

Напомним, что разделенная разность второго порядка функции g по различным между собой узлам x_1, x_2, x_3 определяется формулой

$$[x_1, x_2, x_3]g := \frac{g(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{g(x_2)}{(x_2 - x_1)(x_2 - x_3)} + \frac{g(x_3)}{(x_3 - x_1)(x_3 - x_2)}.$$

Нетрудно проверить, что

$$2[x - h, x, x + h]F = S(x, h).$$

Простыми вычислениями получается следующая лемма (см. [5] лемма 1)

Лемма 2.1. При любых $x_1 < x_2 < x_3$ имеет место

$$2[x_1, x_2, x_3]F = \sum_{n=0}^{\infty} \int_{x_1}^{x_3} A_n(t) \varphi_{x_1 x_2 x_3}(t) dt,$$

где

$$\varphi_{x_1 x_2 x_3}(t) := \begin{cases} \frac{2}{x_3 - x_1}, & \text{когда } t = x_2, \\ 0, & \text{когда } t \notin (x_1, x_3), \\ \text{линейная на отрезках } [x_1, x_2] \text{ и } [x_2, x_3]. \end{cases}$$

Нам пригодится следующая лемма, анонсированная в работе [7] и доказанная в работе [6]. Частный случай этой леммы был доказан в работе [5].

Пусть точки $x_j^{(1)} < x_j^{(2)} < x_j^{(3)}$, $j = 1, 2, \dots, n$, принадлежат отрезку $[a, b]$ и

$$\varphi_j(x) = \begin{cases} 1, & \text{когда } x = x_j^{(2)}, \\ 0, & \text{когда } x \notin (x_j^{(1)}, x_j^{(3)}), \\ \text{выпуклая на отрезках } [x_j^{(1)}, x_j^{(2)}] \text{ и } [x_j^{(2)}, x_j^{(3)}]. \end{cases}$$

Верна следующая лемма

Лемма 2.2. *Для любой неотрицательной и вогнутой на отрезке $[a, b]$ функции $\phi(x)$ существуют неотрицательные числа α_k , $k = 1, 2, \dots, n$ такие, что*

$$\sum_{k=1}^n \alpha_k \varphi_k(x_j^{(2)}) = \phi(x_j^{(2)}), \quad j = 1, 2, \dots, n,$$

и

$$\sum_{k=1}^n \alpha_k \varphi_k(x) \leq \phi(x), \quad x \in [a, b].$$

Лемма 2.3. *Для функций $F(x)$ и $S(x, h)$, определенные формулами (1.2) и (1.3) имеют место*

$$(2.3) \quad S(x, h) = o(h^{-1}), \quad \text{равномерно по } x,$$

$$(2.4) \quad |F(x+h) + F(x-h) - 2F(x)| = o(h), \quad \text{равномерно по } x.$$

Доказательство. Не ограничивая общности, можем считать, что $|A_n(x)| \leq 1$, для всех x и n . Для произвольного $\varepsilon > 0$ найдем n_1 такое, что $|A_n(x)| < \varepsilon$, $n \geq n_1$. Тогда (см. (1.4)) для $h < \frac{\varepsilon}{n_1}$ получим

$$|S(x, h)| \leq \sum_{n=0}^{n_1-1} 1 + \sum_{n=n_1}^{\lfloor \frac{2}{h} \rfloor} \varepsilon + \sum_{n=\lfloor \frac{2}{h} \rfloor + 1}^{\infty} \varepsilon \frac{4}{n^2 h^2} \leq n_1 + \frac{2\varepsilon}{h} + \frac{4\varepsilon}{h} < \frac{7\varepsilon}{h}.$$

Соотношение (2.3) доказано. Соотношение (2.4) следует из (2.3) и (1.3).

Для простоты формулировки леммы 2.4–2.8 сформулированы для отрезка $[0, 1]$. Однако очевидно, что они верны для любого отрезка $[a, b]$.

Лемма 2.4. *Если*

$$(2.5) \quad S\left(\frac{i}{2^k}, \frac{1}{2^k}\right) = 0, \quad \text{для любых } k = 1, 2, \dots, \quad 1 \leq i \leq 2^k - 1,$$

то $F(x)$ -линейная функция на $[0, 1]$.

Доказательство. Обозначим $x_{k,i} = \frac{i}{2^k}$. Тогда условие (2.5) равносильно

$$F(x_{k,i-1}) + F(x_{k,i+1}) - 2F(x_{k,i}) = 0, \quad k = 1, 2, \dots, \quad 1 \leq i \leq 2^k - 1.$$

Следовательно точки $(x_{k,i-1}, F(x_{k,i-1}))$, $(x_{k,i}, F(x_{k,i}))$ и $(x_{k,i+1}, F(x_{k,i+1}))$, $k = 1, 2, \dots, 1 \leq i \leq 2^k - 1$, находятся на одной прямой. Отсюда, учитывая непрерывность функции F получим утверждение леммы.

Из леммы 2.4 получается следующая лемма.

Лемма 2.5. Если $S(x, h) = 0$, когда $[x - h, x + h] \subset [0, 1]$, то $F(x)$ -линейная функция на $[0, 1]$.

Лемма 2.6. Допустим

$$(2.6) \quad \lim_{h \rightarrow 0} S(x, h) = 0 \quad \text{по мере на } [0, 1],$$

и

$$(2.7) \quad S^*(x) \leq M, \quad \text{для некоторого } M \text{ и всех } x \in [0, 1].$$

Тогда $F(x)$ -линейная функция на $[0, 1]$.

Доказательство. Пусть ε -произвольное положительное число. Тогда существует такое натуральное число k , что

$$(2.8) \quad |F(x + h) - F(x)| < \varepsilon \quad \text{когда } |h| \leq 2^{-k} \text{ и } x \in [0, 1],$$

и

$$\text{mes}(A) < \varepsilon, \quad \text{где } A := \{x \in [0, 1] : |S(x, 2^{-k})| > \varepsilon\}.$$

Обозначим

$$(2.9) \quad \psi(t) := \sum_{m=0}^{2^k-1} \chi_A(t + m2^{-k}), \quad t \in [0, 2^{-k}),$$

где $\chi_A(t)$ -характеристическая функция множества A . Ясно, что

$$(2.10) \quad \int_0^{2^{-k}} \psi(t) dt = \text{mes}(A).$$

Следовательно существует такое $t_0 \in [0, 2^{-k})$, что $\psi(t_0) \leq 2^k \varepsilon$. Поэтому, из (2.9),

$$(2.10) \quad \text{следует существование такого } t_0 \in [0, 2^{-k}), \text{ что}$$

$$(2.11) \quad \text{card}\{m : |S(x_m, 2^{-k})| > \varepsilon\} \leq 2^k \varepsilon, \quad \text{где } x_m = t_0 + m2^{-k},$$

а $\text{card}(G)$ -количество точек множества G .

Обозначим

$$\phi_0(x) := \begin{cases} 2, & \text{когда } x = \frac{1}{2} + t_0, \\ 0, & \text{когда } x \notin (t_0, 1 + t_0), \\ \text{линейная на отрезках } [t_0, \frac{1}{2} + t_0] \text{ и } [\frac{1}{2} + t_0, 1 + t_0]. \end{cases}$$

А для $m = 0, \dots, 2^k - 1$ положим

$$\phi_m(x) := \begin{cases} 2^k, & \text{когда } x = x_m, \\ 0, & \text{когда } x \notin (x_m - 2^{-k}, x_m + 2^{-k}), \\ \text{линейная на отрезках } [x_m - 2^{-k}, x_m] \text{ и } [x_m, x_m + 2^{-k}]. \end{cases}$$

Из этих обозначений и формул (2.1), (2.2) следуют

$$(2.12) \quad S\left(\frac{1}{2} + t_0, \frac{1}{2}\right) = \sum_{n=0}^{\infty} \int_{t_0}^{1+t_0} A_n(x) \phi_0(x) dx$$

и

$$(2.13) \quad S(x_m, 2^{-k}) = \sum_{n=0}^{\infty} \int_{t_0}^{1+t_0} A_n(x) \phi_m(x) dx, \quad m = 0, 1, \dots, 2^k - 1.$$

Нетрудно заметить, что

$$(2.14) \quad \phi_0(x) = \sum_{m=1}^{2^k-1} \alpha_m \phi_m(x), \quad \text{где } \alpha_m = \frac{\phi_0(x_m)}{\phi_m(x_m)}, \quad m = 1, 2, \dots, 2^k - 1.$$

Из (2.12)–(2.14) получим

$$(2.15) \quad S\left(\frac{1}{2} + t_0, \frac{1}{2}\right) = \sum_{m=1}^{2^k-1} \alpha_m S(x_m, 2^{-k}).$$

Очевидно, что интегралы функций ϕ_m , $m = 0, 1, 2, \dots, 2^k - 1$, равны единице и поэтому из (2.14) имеем

$$(2.16) \quad \sum_{m=1}^{2^k-1} \alpha_m = 1 \quad \text{и} \quad 0 \leq \alpha_m \leq 2^{1-k}, \quad m = 1, \dots, 2^k - 1.$$

Обозначим $\Lambda_1 := \{m : |S(x_m, 2^{-k})| \leq \varepsilon\}$ и $\Lambda_2 := \{m : |S(x_m, 2^{-k})| > \varepsilon\}$. Отсюда, с последовательным применением (2.15), (2.16) (2.7) и (2.11) получим

$$\begin{aligned} \left| S\left(\frac{1}{2} + t_0, \frac{1}{2}\right) \right| &\leq \sum_{m \in \Lambda_1} \alpha_m |S(x_m, 2^{-k})| + \sum_{m \in \Lambda_2} \alpha_m |S(x_m, 2^{-k})| \\ &\leq \varepsilon + M \sum_{m \in \Lambda_2} 2^{1-k} < \varepsilon + 2M\varepsilon = \varepsilon(2M + 1). \end{aligned}$$

Отсюда, с учетом (2.8), имеем

$$\left| S\left(\frac{1}{2}, \frac{1}{2}\right) \right| < \varepsilon(2M + 17).$$

Поэтому, с учетом произвольности ε , получим $S\left(\frac{1}{2}, \frac{1}{2}\right) = 0$. Повторяя вышеприведенные рассуждения для отрезка $[x - h, x + h] \subset [0, 1]$, получим $S(x, h) = 0$, когда $[x - h, x + h] \subset [0, 1]$. Отсюда, в силу леммы 2.5, получим, что $F(x)$ -линейная функция на $[0, 1]$. Лемма доказана.

Лемма 2.7. *Если*

$$S\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$$

$t_0 \in [0, 1]$, то найдутся x_0 и $h_0 > 0$, такие что

$$(2.17) \quad S(x_0, h_0) \neq 0 \quad \text{и} \quad t_0 \notin [x_0 - h_0, x_0 + h_0] \subset [0, 1].$$

Доказательство. Как мы заметили при доказательстве леммы 2.6, для любого k имеет место (см. (2.15), (2.16))

$$(2.18) \quad S\left(\frac{1}{2}, \frac{1}{2}\right) = \sum_{m=1}^{2^k-1} \alpha_m^{(k)} S(x_m^{(k)}, 2^{-k}), \quad \text{где} \quad x_m^{(k)} = \frac{m}{2^k}, \quad \text{и} \quad 0 \leq \alpha_m^{(k)} \leq 2^{1-k}.$$

Допустим $S\left(\frac{1}{2}, \frac{1}{2}\right) = d \neq 0$. Тогда, если бы не было таких $x \in [0, 1]$ и $h > 0$, которые удовлетворяли бы (2.17), то среди отрезков $[x_m^{(k)} - 2^{-k}, x_m^{(k)} + 2^{-k}]$, $m = 1, 2, \dots, 2^k - 1$, нашлись бы не более три отрезка, которые содержали бы точку t_0 , для которых выполнялось бы $S(x_m^{(k)}, 2^{-k}) \neq 0$. Тогда из (2.18) следовало бы, что для некоторого m_k выполняется $\alpha_{m_k}^{(k)} |S(x_{m_k}^{(k)}, 2^{-k})| \geq |d|/3$, $k = 1, 2, \dots$. Отсюда получилось бы $|S(x_{m_k}^{(k)}, 2^{-k})| \geq 2^k |d|/6$. Но это противоречит (2.3). Лемма доказана.

Нетрудно заметить, что из непрерывности функции F следует, что при любом $M > 0$ множество

$$E_M := \{x \in (0, 1) : S^*(x) > M\}$$

является открытым множеством и как любое открытое множество является объединением своего составляющих интервалов, т.е. существуют открытые интервалы $I_k^{(M)}$, такие что

$$E_M = \bigcup_k I_k^{(M)}, \quad \text{и} \quad I_k^{(M)} \cap I_m^{(M)} = \emptyset, \quad \text{если} \quad k \neq m.$$

Лемма 2.8. *Допустим $S\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$ и выполняется (2.6). Тогда существует $[x_0 - h_0, x_0 + h_0] \subset E_M$, такое что $S(x_0, h_0) \neq 0$.*

Доказательство. Допустим обратное:

$$(2.19) \quad S\left(\frac{1}{2}, \frac{1}{2}\right) = d \neq 0,$$

выполняется (2.6), но $S(x, h) = 0$ для любого $[x - h, x + h] \subset E_M$. Тогда, в силу леммы 2.5, функция $F(x)$ является линейной функцией на каждом интервале $I_k^{(M)}$.

Пусть $\varepsilon \in (0, 0.25)$ такое, что

$$(2.20) \quad \frac{4M\varepsilon}{1 - 2\varepsilon/3} < \frac{|d|}{3}$$

и

$$(2.21) \quad \left| S\left(\frac{1}{2}, \frac{1}{2}\right) - 2F[z_1, z_2, z_3] \right| < \frac{|d|}{3}, \quad \text{когда } |z_1| < \varepsilon, \quad |z_2 - \frac{1}{2}| < \varepsilon, \quad |z_3 - 1| < \varepsilon.$$

Найдется натуральное число k_0 , такое что

$$(2.22) \quad \text{mes}(D) < \frac{\varepsilon}{6}, \quad \text{где } D := \bigcup_{k > k_0} I_k^{(M)}.$$

Из сходимости по мере к нулю сумм $S(x, h)$ следует существование такого натурального числа N , что для $h_0 := \frac{1}{2N}$ выполняются

$$(2.23) \quad \text{mes}\{x \in [0, 1] : |S(x, h_0)| < \varepsilon\} > 1 - \frac{\varepsilon}{6}$$

и (см. (2.3))

$$(2.24) \quad |S(x, h)| \leq \frac{\varepsilon}{k_0 h}, \quad \text{когда } |h| \leq h_0 \text{ и } x - \text{любое.}$$

Обозначим

$$(2.25) \quad B := D \bigcup G, \quad \text{где } G := \{x \in [0, 1] : |S(x, h_0)| > \varepsilon\}.$$

Из (2.23) и (2.22) следует, что

$$(2.26) \quad \text{mes}(B) < \frac{\varepsilon}{3}.$$

Положим

$$\varphi(t) := \sum_{i=1}^{2N} \chi_B((i-1)h_0 + t), \quad t \in [0, h_0).$$

Очевидно, что $\int_0^{h_0} \varphi(t) dt = \text{mes}(B)$. Поэтому существует такое $t_0 \in [0, h_0)$, что $\varphi(t_0) \leq 2N \text{mes}(B)$, т.е.

$$(2.27) \quad \text{card}\{i : (i-1)h_0 + t_0 \in B\} \leq 2N \text{mes}(B).$$

Обозначим $x_i := (i-1)h_0 + t_0$, $i = 1, 2, \dots, 2N$, и положим

$$(2.28) \quad \Lambda_1 := \{i : x_i \notin B\}, \quad \Lambda_2 := \{i : x_i \in B\} \quad \text{и} \quad H := \bigcup_{i \in \Lambda_2} [x_i - h_0/2, x_i + h_0/2].$$

Из (2.27), (2.28) и (2.26) следует, что

$$(2.29) \quad N_2 := \text{card}\{\Lambda_2\} \leq \frac{\varepsilon}{3h_0}, \quad \text{mes}(H) \leq N_2 h_0 \leq \frac{\varepsilon}{3}.$$

Для $i \in (N_2, 2N - N_2) \cap \Lambda_1$ обозначим

$$(2.30) \quad m_i := \min\{j \geq 1 : i \pm j \in \Lambda_1\}.$$

Убедимся, что такое m_i существует. Действительно, если $i - j \notin \Lambda_1$ или $i + j \notin \Lambda_1$ для $j \leq k$, то из чисел $i - k, i - k + 1, \dots, i + k$ хотя бы k штук принадлежат Λ_2 . Тогда из (2.29) следует, что $k \leq N_2$, т.е. $m_i \leq N_2$.

Положим

$$(2.31) \quad \Lambda_3 := \{i \in (N_2, 2N - N_2) \cap \Lambda_1 : m_i > 1\}, \quad B_3 := \bigcup_{i \in \Lambda_3} [x_{i-m_i}, x_{i+m_i}]$$

и

$$[z_1, z_3] := [x_{N_2}, x_{2N-N_2}] \bigcup B_3.$$

Из (2.29) имеем

$$|z_1| < \frac{\varepsilon}{3} \quad \text{и} \quad |1 - z_3| < \frac{\varepsilon}{3}.$$

Из определения чисел m_i и множеств B_3 и H следует, что

$$B_3 \subset \left\{ x : M(\chi_H, x) \geq \frac{1}{3} \right\},$$

где $M(\chi_H, x)$ -максимальная функция Харди-Литтлвуда характеристической функции множества H . Следовательно

$$(2.32) \quad \text{mes}(B_3) < \varepsilon$$

и поэтому существует $x_i =: z_2$, со свойствами

$$|z_2 - \frac{1}{2}| < \varepsilon \quad z_2 \notin B_3.$$

Обозначим

$$(2.33) \quad \varphi_0(x) := \begin{cases} \frac{2}{z_3 - z_1}, & \text{когда } x = z_2, \\ 0, & \text{когда } x \notin (z_1, z_3), \\ \text{линейная на отрезках } [z_1, z_2] \text{ и } [z_2, z_3]. \end{cases}$$

А для $i \in \Lambda_4 := \{i \in \Lambda_1 : x_i \in (z_1, z_3)\}$ положим

$$\varphi_i(x) := \begin{cases} \frac{2}{x_{i+m_i} - x_{i-m_i}}, & \text{когда } x = x_i, \\ 0, & \text{когда } x \notin (x_{i-m_i}, x_{i+m_i}), \\ \text{линейная на отрезках } [x_{i-m_i}, x_i] \text{ и } [x_i, x_{i+m_i}]. \end{cases}$$

Ясно, что функции φ_0 и φ_i , удовлетворяют условиям леммы 2.2. Применяя эту лемму, получим сумму

$$\sum_{i \in \Lambda_4} \alpha_i \varphi_i(x), \quad \text{с } \alpha_i \geq 0,$$

со свойствами

$$(2.34) \quad \varphi_0(x_i) = \sum_{i \in \Lambda_4} \alpha_i \varphi_i(x_i), \quad \text{когда } i \in \Lambda_4$$

и

$$\sum_{i \in \Lambda_4} \alpha_i \varphi_i(x) \leq \varphi_0(x), \quad \text{когда } x \in [z_1, z_3].$$

Пусть $i, j \in \Lambda_4$ и $i < j$ такие, что если $m \in (i, j)$, то $m \notin \Lambda_4$. Тогда все функции $\varphi_i(x)$, $i \in \Lambda_4$, в силу определения чисел m_i (см. (2.30)), являются линейными

функциями на $[x_i, x_j]$. На том же отрезке функция $\varphi_0(x)$ также линейна. Поэтому, из (2.34) имеем

$$(2.35) \quad \sum_{i \in \Lambda_4} \alpha_i \varphi_i(x) = \varphi_0(x), \quad \text{когда } x \in [z_1, z_3].$$

Учитывая, что интегралы всех функций $\varphi_i(x)$, $i \in \Lambda_4$, и $\varphi_0(x)$ равны единице, из (2.35) получим

$$(2.36) \quad \sum_{i \in \Lambda_4} \alpha_i = 1.$$

Обозначим $\Lambda_5 := \{i \in \Lambda_4 : m_i > 1\}$, $\Lambda_6 := \{i \in \Lambda_4 : m_i = 1\}$. Тогда

$$(2.37) \quad 2[z_1, z_2, z_3]F = \sum_{i \in \Lambda_5} \alpha_i S(x_i, m_i h_0) + \sum_{i \in \Lambda_6} \alpha_i S(x_i, h_0) =: \sigma_5 + \sigma_6.$$

Учитывая, что интегралы всех функций φ равны единице, из (2.35), с применением (2.32), (2.33) и (2.31), получим

$$(2.38) \quad \sum_{i \in \Lambda_5} \alpha_i = \int_{z_1}^{z_3} \sum_{i \in \Lambda_5} \alpha_i \varphi_i(x) dx \leq \int_{B_3} \varphi_0(x) dx \leq \|\varphi_0\|_\infty \text{mes}(B_3) \leq \frac{2\varepsilon}{1 - 2\varepsilon/3}.$$

Из (2.28), (2.25), (2.22) следует, что $|S(x_i, m_i h_0)| < M$, когда $i \in \Lambda_5$. Поэтому из (2.38) получим

$$(2.39) \quad |\sigma_5| \leq \frac{2M\varepsilon}{1 - 2\varepsilon/3}.$$

Для $i \in \Lambda_6$ возможны три случая:

- $(x_{i-1}, x_{i+1}) \subset \bigcup_{k \leq k_0} I_k^{(M)}$,
- $(x_{i-1}, x_{i+1}) \cap E_M = \emptyset$ и $x_i \notin B$,
- для некоторого $k \leq k_0$ имеют место $(x_{i-1}, x_{i+1}) \cap I_k^{(M)} \neq \emptyset$ и $(x_{i-1}, x_{i+1}) \not\subset I_k^{(M)}$.

По нашему предположению, в первом случае имеем $S(x_i, h_0) = 0$.

Во втором случае (см. (2.25) и (2.28)) выполняется $|S(x_i, h_0)| < \varepsilon$. Поэтому, с учетом (2.36) получим

$$(2.40) \quad \sum_{\Lambda_7} \alpha_i |S(x_i, h_0)| \leq \varepsilon,$$

где Λ_7 -множество индексов, удовлетворяющих второму случаю.

Множество индексов i , удовлетворяющих третьему случаю обозначим через Λ_8 . Второму случаю могут удовлетворять не более $2k_0$ штук x_i и для них, в силу

леммы 2.3, имеем $|S(x_i, h_0)| \leq \varepsilon h_0^{-1} k_0^{-1}$. Следовательно

(2.41)

$$\begin{aligned} |\sigma_6| &\leq \varepsilon + \left| \sum_{i \in \Lambda_8} \alpha_i S(x_i, h_0) \right| \leq \varepsilon + \varepsilon h_0^{-1} k_0^{-1} \sum_{i \in \Lambda_8} \alpha_i = \varepsilon + \varepsilon h_0^{-1} k_0^{-1} \int \sum_{i \in \Lambda_8} \alpha_i \varphi_i(x) dx \\ &\leq \varepsilon + \varepsilon h_0^{-1} k_0^{-1} 4k_0 h_0 \|\varphi_0\|_\infty < \frac{9\varepsilon}{1 - 2\varepsilon/3}. \end{aligned}$$

Из (2.41), (2.39) (2.40) и (2.37), имеем (можем предположить, что $M > 5$)

$$|2[z_1, z_2, z_3]F| \leq \frac{4M\varepsilon}{1 - 2\varepsilon/3}.$$

Отсюда, с применением (2.21), получим (см. также (2.20), (2.19))

$$\left| S\left(\frac{1}{2}, \frac{1}{2}\right) \right| < \frac{|d|}{3}.$$

Но это противоречит (2.19). Полученное противоречие доказывает лемму.

3. ДОКАЗАТЕЛЬСТВО ОСНОВНЫХ РЕЗУЛЬТАТОВ

Доказательство теоремы 1.1. Пусть суммы Римана $S(x, h)$ тригонометрического ряда (1.1) по мере сходятся к нулю и мажоранта Римана $S^*(x)$ этого ряда для некоторого счетного множества $B := \{y_p\}_{p=1}^\infty$ удовлетворяет условию

$$(3.1) \quad S^*(x) < \infty, \quad \text{когда } x \notin B.$$

Докажем, что в этом случае функция F -линейная. Допустим обратное. Тогда существуют x_0 и $h_0 > 0$, такие что $S(x_0, h_0) \neq 0$. Применяя лемму 2.7, найдем x'_0 и h'_0 такие, что

$$y_1 \notin [x'_0 - h'_0, x'_0 + h'_0] \subset [x_0 - h_0, x_0 + h_0]$$

и

$$(3.2) \quad S(x'_0, h'_0) \neq 0.$$

Обозначим

$$E_1 := \{x \in (x'_0 - h'_0, x'_0 + h'_0) : S^*(x) > 1\}.$$

Если бы множество E_1 было бы пусто, то в силу леммы 2.6 функция F была бы линейной на $[x'_0 - h'_0, x'_0 + h'_0]$, что противоречит (3.2). Следовательно $E_1 \neq \emptyset$. Тогда в силу леммы 2.8 существует $[x_1 - h_1, x_1 + h_1] \subset E_1$ такое, что $S(x_1, h_1) \neq 0$. Итак, мы нашли $[x_1 - h_1, x_1 + h_1]$, со свойствами

$$y_1 \notin [x_1 - h_1, x_1 + h_1] \subset [x_0 - h_0, x_0 + h_0]$$

и

$$S^*(x) > 1, \quad \text{когда } x \in [x_1 - h_1, x_1 + h_1].$$

Допустим для $i \leq p$ мы нашли числа $x_i, h_i > 0$ такие, что

$$(3.3) \quad \begin{aligned} y_i &\notin [x_i - h_i, x_i + h_i] \subset [x_{i-1} - h_{i-1}, x_{i-1} + h_{i-1}], \quad i \leq p, \\ S(x_i, h_i) &\neq 0, \quad i \leq p, \end{aligned}$$

$$S^*(x) > i \quad \text{когда} \quad x \in [x_i - h_i, x_i + h_i], \quad i \leq p.$$

Применяя лемму 2.7, найдем отрезок $[x'_p - h'_p, x'_p + h'_p]$ со свойствами

$$y_{p+1} \notin [x'_p - h'_p, x'_p + h'_p] \subset [x_p - h_p, x_p + h_p].$$

Обозначим

$$E_{p+1} := \{x \in (x'_p - h'_p, x'_p + h'_p) : S^*(x) > p+1 \quad \text{и} \quad S(x'_p, h'_p) \neq 0\}.$$

Если бы множество E_{p+1} было бы пусто, то в силу леммы 2.6 функция F была бы линейной на $[x'_p - h'_p, x'_p + h'_p]$, что противоречит $S(x'_p, h'_p) \neq 0$. Следовательно $E_{p+1} \neq \emptyset$. Тогда, в силу леммы 2.8 существует $[x_{p+1} - h_{p+1}, x_{p+1} + h_{p+1}] \subset E_{p+1}$ такое, что $S(x_{p+1}, h_{p+1}) \neq 0$. Итак, мы нашли $[x_{p+1} - h_{p+1}, x_{p+1} + h_{p+1}]$, со свойствами

$$(3.4) \quad y_{p+1} \notin [x_{p+1} - h_{p+1}, x_{p+1} + h_{p+1}] \subset [x_p - h_p, x_p + h_p]$$

и

$$(3.5) \quad S^*(x) > p+1, \quad \text{когда} \quad x \in [x_{p+1} - h_{p+1}, x_{p+1} + h_{p+1}].$$

Из (3.4), (3.5) следует, что существует точка z , которая не принадлежит B и $S^*(z) = \infty$. Полученное противоречит (3.1). Следовательно функция F – линейная и из (1.2) имеем

$$(3.6) \quad A_1 + B_1 x + \frac{a_0 x^2}{4} = \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}.$$

Из ограниченности правой части (3.6) на R следует, что $B_1 = a_0 = 0$. Отсюда, учитывая равномерную сходимость правой части (3.6), получим $A_1 = a_n = b_n = 0$ для всех n . Теорема доказана.

Доказательство теоремы 1.2. Пусть ряд

$$(3.7) \quad \frac{c_0}{2} + \sum_{n=0}^{\infty} (c_n \cos nx + d_n \sin nx)$$

является рядом Фурье функции f , $S_f(x, h)$ –суммы Римана ряда (3.7) и ряд (1.1) удовлетворяет условиям теоремы.

Легко видеть, что

$$(3.8) \quad S_f(x, h) = \int_{x-h}^{x+h} f(t) \varphi_h(x-t) dt.$$

Из (3.8) легко получается, что $\lim_{h \rightarrow 0} S_f(x, h) = f(x)$, во всех точках Лебега функции f . Из (3.8) также следует, что $|S_f(x, h)| \leq \|f\|_\infty$. Следовательно, для разности рядов (1.1) и (3.7) имеют место

$$\sup_{h>0} |S(x, h) - S_f(x, h)| < \infty, \quad \text{когда } x \notin B,$$

$$\lim_{h \rightarrow 0} (S(x, h) - S_f(x, h)) = 0 \quad \text{по мере.}$$

В силу теоремы 1.1 разность рядов (1.1) и (3.7) нулевой ряд, т.е. $a_n = c_n$, $b_n = d_n$, для любого n . Теорема доказана.

4. НЕКОТОРЫЕ КОММЕНТАРИИ

Известно, что из условия $S^*(x) < \infty$ почти всюду следует существование предела $\lim_{h \rightarrow 0} S(x, h)$ почти всюду (см. [8], а также [9] стр. 120). Это означает, что в теоремах 1.1 и 1.2 из условия (1.5) следует почти всюду суммируемость методом Римана ряда (1.1), и тем более сходимость по мере сумм $S(x, h)$, когда $h \rightarrow 0$.

Из теоремы 1.1 следует также (см. [1] стр. 792)

ТЕОРЕМА ЮНГА. *Если ряд (1.1) всюду, кроме, быть может, некоторого счетного множества, сходится к нулю, то все коэффициенты этого ряда равны нулю.*

А из теоремы 1.2 получается (см. [1] стр. 200)

ТЕОРЕМА ДЮ БУА-РЕЙМОНА-ЛЕБЕГА. *Если ряд (1.1) всюду сходится к ограниченной функции, то является рядом Фурье этой функции.*

Отметим еще одну важную теорему из теории тригонометрических рядов (см. [12], а также [1] стр. 789)

ТЕОРЕМА ВАЛЛЕ-ПУССЕНА. *Пусть пределы неопределенности ряда (1.1), т.е. функции*

$$(4.1) \quad \underline{G}(x) := \liminf_{N \rightarrow \infty} \sum_{n=0}^N A_n(x) \quad \text{и} \quad \overline{G}(x) := \limsup_{N \rightarrow \infty} \sum_{n=0}^N A_n(x),$$

конечны всюду, кроме, быть может, некоторого счетного множества, и обе функции интегрируемы на $[0, 2\pi]$, то ряд (1.1) суммируется методом Римана почти всюду и является рядом Фурье этой суммы.

Из этой теоремы следует другая теорема

ТЕОРЕМА ВАЛЛЕ-ПУССЕНА. *Если (1.1), всюду, кроме, быть может, некоторого счетного множества сходится к всюду конечной интегрируемой функции, то является рядом Фурье этой функции.*

Очевидно, что условие интегрируемости функций (4.1) сильнее чем условие (1.5) или (1.6). Поэтому из теоремы Валле–Пуссена не следуют теоремы 1.1–1.4. С другой стороны из теорем 1.2 или 1.4 не следует теорема Валле–Пуссена.

В связи с этим интересно было бы выяснить: имеет ли место теорема 1.4, если в нем вместо ограниченности функции f потребовать ее интегрируемость.

Abstract. In paper is proved, that if Riemann’s majorant of trigonometric series is bounded everywhere, except, maybe, of some countable set, and converge in measure to bounded function f , then the series is Fourier series of function f . From this are obtained Cantor’s and Young’s theorems generalization.

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ФУНКЦИИ УНИВЕРСАЛЬНЫЕ ОТНОСИТЕЛЬНО СИСТЕМЫ УОЛША

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Аннотация. В данной работе построена интегрируемая функция универсальна для $L^p[0, 1]$, $p \in (0, 1)$ относительно системы Уолша в смысле знаков, ряд Фурье - Уолша, которой сходится всюду на $(0, 1)$ и в метрике $L^p[0, 1]$, $p \in (0, 1)$, ее коэффициенты Фурье по системе Уолша расположены в убывающем порядке и после выбора подходящих знаков для ее коэффициентов Фурье вновь полученный ряд будет универсальным относительно перестановок в $L^p[0, 1]$, $p \in (0, 1)$.

MSC2010 number: 42C10; 43A15.

Ключевые слова: универсальная функция; ряд Фурье; сходимости; система Уолша.

1. ВВЕДЕНИЕ

Работа продолжает исследования автора связанных с существованием и описанием структуры функций, ряды Фурье которых по заданной классической системе универсальны тем или иным смыслом в различных функциональных классах. Заметим, что единого определения понятия «универсальная функция» не существует. Обычно под этим термином понимается функция, с помощью которой можно «представить» «все» функции. При этом способ представления, а также класс представимых функций может трактоваться различным образом. Примеры универсальных функций привлекали внимание многих математиков и публикации по этой тематике регулярно появляются в математической печати. В этих публикациях изучаются те или иные универсальные функции, а также их свойства. Причем развитие происходит как в действительных, так и в комплексных рамках. По сути первый тип универсальной функции был рассмотрен в 1929 г. Дж. Биркхофом в работе[1]: существует целая функция $g(z)$, которая универсальна относительно сдвигов, т.е. для любой целой функции $f(z)$

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и числа $r > 0$ существует возрастающая последовательность натуральных чисел $\{n_k\}_{k=1}^{\infty}$ такая, что последовательность сдвигов $\{g(z + n_k)\}_{k=1}^{\infty}$ равномерно сходится к $f(z)$ на круге $|z| \leq r$.

В 1935 году Й. Марцинкевич [2], опираясь на некоторые идеи Н. Н. Лузина, доказал существование непрерывной функций, которая универсальна относительно производных чисел (производной отношения). Если $h_n \rightarrow 0$ заранее фиксированная последовательность, то существует функция $F \in C[0, 1]$ такая, что для любой измеримой функций g определенной на $[0, 1]$ существует возрастающая подпоследовательность натуральных чисел n_k такая, что почти всюду на $[0, 1]$

$$\lim_{n \rightarrow \infty} \frac{F(x + h_{n_k}) - F(x)}{h_{n_k}} = g(x)$$

Более того, он в этой статье доказал, что множество таких универсальных функций является множеством 2-ой категории в $C[0, 1]$. В этом направлении интересные результаты были получены также в работах [3]-[4].

В 1952г. Дж. Маклейн в работе [5] доказал, что существует целая функция $g(z)$, которая универсальна относительно производных, а именно: для любой целой функции $f(z)$ и числа $r > 0$ можно найти возрастающую последовательность натуральных чисел $\{n_k\}_{k=1}^{\infty}$ так, чтобы последовательность производных $\{g^{(n_k)}(z)\}_{k=1}^{\infty}$ равномерно сходилась к $f(z)$ на круге $|z| \leq R$.

В 1987г. К. Гроссе — Эрдман [6] доказал существование вещественной действительной функции с универсальным рядом Тейлора: существует функция $g(x) \in C^{\infty}(R)$ с $g(0) = 0$, ряд Тейлора которой в точке $x = 0$ локально — равномерно универсален в $C(R)$, т.е. для любой функции $f(x) \in C(R)$ с $f(0) = 0$ и числа $r > 0$, существует подпоследовательность

$$S_{n_k}(g, 0) = \sum_{m=1}^{n_k} \frac{g^{(m)}(0)}{m!} x^m$$

частичных сумм ряда Тейлора функции $g(x)$, которая равномерно сходится к $f(x)$ на отрезке $|x| \leq r$.

В работах [7],[8] В. Лью был построен комплексные степенные ряды $\sum_{k=1}^{\infty} c_k z^k$ с ненулевыми радиусами сходимости $R > 0$, подпоследовательность частичных сумм которых аппроксимирует любые многочлены на произвольных компактах $\{z \in K : |z| > R\}$ со связанными дополнениями.

Следует отметить, что вопросам существования рядов как по классическим, так и по общим ортонормальным системам, универсальных тем или иным смыслом в различных классах функций, посвящено много работ (см. [9–16]). Для формулировки некоторых результатов существования универсальных рядов напомним следующие обозначения.

Пусть $M[0, 1]$ – совокупность всех (не обязательно конечных) измеримых функций (соотв. $L^0[0, 1]$) – класс всех (соотв. почти везде конечных) измеримых на $[0, 1]$ функций. Под сходимостью в $M[0, 1]$ или в $L^0[0, 1]$ мы будем подразумевать сходимость почти всюду. Пусть $E \subseteq [0, 1]$ некоторое измеримое множество и $|E|$ –

мера Лебега измеримого множества $E \subseteq [0, 1]$, $L^p(E)$ ($p > 0$) – класс всех тех измеримых на E функций, для которых $\int_E |f(x)|^p dx < \infty$. Пусть $\Phi = \{\varphi_k(x)\}_{k=1}^\infty$ – полная ортонормированная система на $[0, 1]$ и пусть $c_k(U) = \int_0^1 U(x)\varphi_k(x)dx$, $k \in \mathbf{N}$ – коэффициенты Фурье по системе $\{\varphi_k(x)\}_{k=0}^\infty$ функции $U \in L^1[0, 1]$, ($\mathbf{N}$ – совокупность всех натуральных чисел). Пусть

$$(1, 1) \quad S_m(x, U, \Phi) = \sum_{k=0}^m c_k(U)\varphi_k(x), \quad \text{spec}(U) = \{k \in \mathbf{N} \cup \{0\}; c_k(U) \neq 0\}$$

Пусть метрическое пространство S – какое-нибудь из пространств $M[0, 1]$ и $L^p[0, 1]$, $p \geq 0$.

Определение 1.1. Будем говорить, что функция $U \in L^1[0, 1]$ универсальна для класса S относительно системы $\{\varphi_k(x)\}_{k=0}^\infty$:

а) в обычном смысле, если ряд Фурье функции $U(x)$ по этой системе универсален в S в обычном смысле, т.е. существует подпоследовательность $\{m_k\}_{k=1}^\infty \subset \mathbf{N}$, $m_k \nearrow^\infty$ такая, что $\{S_{m_k}(x, U, \Phi)\}_{k=1}^\infty$ сходится к $f(x)$ в S .

б) в квази-обычном смысле, если существует последовательность знаков $\{\delta_k = \pm 1\}_{k=0}^\infty$ такая, что ряд $\sum_{k=0}^\infty \delta_k c_k(U)W_k(x)$ был бы универсальным в обычном смысле в S ,

в) в смысле перестановок, если ряд Фурье функции $U(x)$ по этой системе универсален в S в смысле перестановок т.е. для каждой функции $f \in S$ члены ряда $\sum_{k=0}^\infty c_k(U)W_k(x)$ можно переставить так, чтобы вновь полученный ряд $\sum_{k=1}^\infty c_{\sigma(k)}(U)W_{\sigma(k)}(x)$ сходил к функции $f(x)$ в S ,

г) в смысле знаков, если для каждой функции $f \in S$ можно найти последовательность знаков $\{\delta_k = \pm 1\}_{k=0}^\infty$, для которой ряд $\sum_{k=1}^\infty \delta_k c_k(U)W_k(x)$ сходится к функции $f(x)$ в S .

Первой работой, где построены универсальные в обычном смысле тригонометрические ряды в $M[0, 1]$ в смысле сходимости почти всюду, является работа [9] Д.Е.Меньшова. Ряды $\sum_{k=1}^{\infty} a_k \varphi_k(x)$ по любой ортонормированной полной системе $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in [0, 1]$, универсальные в M в обычном смысле (в случае сходимости почти всюду) были построены в работе [10] А.А.Талаляном.(см. также работы [11]-[16]).

В последние годы нами (мною и моими соавторами -учениками) были получены некоторые результаты (см. [17] -[22]), связанные с существованием и описанием структуры функций, ряды Фурье которых по заданной классической системе универсальны тем или иным смыслом в различных функциональных классах. Такие функции мы назовем универсальными относительно классических систем.

Замечание 1.1. *Не существует функции $U \in L^1[0, 1]$, ряд Фурье которой по тригонометрической системе (по системе Уолша) был бы универсальным в классе $M[0, 1]$ в обычном смысле (даже в случае, сходимости по мере).*

В самом деле, если существовала бы функция $U \in L^1[0, 1]$, которая универсальна для класса $M[0, 1]$ относительно тригонометрической системы в обычном смысле, тогда для функции $f(x) = 2U(x)$ нашлась бы подпоследовательность натуральных чисел $\{m_k\}$ такая, что по мере на $[0, 1]$

$$(1, 2) \quad \lim_{k \rightarrow \infty} \sum_{|n|=0}^{m_k} d_n(U) e^{2\pi n i x} = 2U(x), \quad d_n(U) = \int_0^1 U(x) e^{-2\pi n i x} dx$$

С другой стороны, из известной теоремы Колмогорова [23] (ряд Фурье каждой интегрируемой функции по тригонометрической системе сходится в $L^p[0, 1]$, $p \in (0, 1)$) вытекает, что $\sum_{|n|=0}^{m_k} d_n(U) e^{2\pi n i x}$ -сходится к $U(x)$ по мере на $[0, 1]$. Отсюда и из (1.2) вытекает, что $U(x) = 2U(x)$ почти всюду на $[0, 1]$. Пришли к противоречию.

Значит, не существует функции $U \in L^1[0, 1]$ универсальная для классов $M[0, 1]$ и $L^p[0, 1]$, $p \in (0, 1)$ относительно тригонометрической системы (соотв. относительно системы Уолша) в обычном смысле.

Сразу же возникает следующий вопрос, ответ на который нам не известен.

Вопрос 1. Существует ли ограниченная ортонормированная система $\{\varphi_n(x)\}$ такая, что можно было бы построить функцию $U \in L^1[0, 1]$ универсальную для класса $L^p[0, 1]$, $p \in [0, 1)$ или для класса $M[0, 1]$ относительно системы $\{\varphi_n(x)\}$ в обычном смысле?

Отметим, что в работах [18], [20] автором и А.Саргсяном построены интегрируемые функций $U(x)$ универсальные для классов $L^p[0, 1]$, $p \in (0, 1)$ относительно системы Уолша **в смысле знаков**. Отметим также, что нам не известен вопрос о том: существует ли функция $U \in L^1[0, 1]$ универсальна для классов $L^p[0, 1]$, $p \in (0, 1)$ относительно системы Уолша **в смысле перестановок** ?

В этой статье мы построим функцию $U \in L^1[0, 1]$ такую, что после выбора подходящих знаков ($\varepsilon_k = \pm 1$, $k = 0, 1, 2, \dots$) для ее коэффициентов Фурье- Уолша $\{c_k(U)\}_{k=0}^{\infty}$ можно достичь того, чтобы вновь полученный ряд $\sum_{k=0}^{\infty} \varepsilon_k c_k(U) W_k(x)$ будет универсальным в $L^p[0, 1]$, $p \in (0, 1)$ в смысле перестановок.

Верны следующие теоремы

Теорема 1.1. *Существует функция $U \in L^1[0, 1]$ ($\text{supp} U \subset [0, \varepsilon]$, здесь $\varepsilon \in (0, 1)$ — наперед заданное число) со сходящимся по $L^1[0, 1]$ норме и всюду на $(0, 1)$ рядом Фурье - Уолша с монотонно убывающими коэффициентами, и которая является универсальной для класса $M[0, 1]$ относительно системы Уолша в квази- обычном смысле.*

Теорема 1.2. *Существуют числа $\{\varepsilon_k = \pm 1, k \geq 0\}$ и функция $U \in L^1[0, 1]$ ($\text{supp} U \subset [0, \varepsilon]$, здесь $\varepsilon \in (0, 1)$ — наперед заданное число) со сходящимся по $L^1[0, 1]$ норме и всюду на $(0, 1)$ с рядом Фурье - Уолша с монотонно убывающими коэффициентами, такие, что ряд $\sum_{k=0}^{\infty} \varepsilon_k c_k(U) W_k(x)$ -универсален в $L^p[0, 1]$, $p \in (0, 1)$ в смысле перестановок.*

Замечание 1.2. *Нетрудно видеть, что теорема 1.2 окончательна в некотором смысле (неулучшаема), она не верна при $p \geq 1$.*

Теоремы 1.1 и 1.2 следуют из более сильной Теоремы 1.3.

Теорема 1.3. *Существует функция $U \in L^1[0, 1]$ ($\text{supp} U \subset [0, \varepsilon]$, здесь $\varepsilon \in (0, 1)$ — наперед заданное число) со сходящимся по $L^1[0, 1]$ норме и всюду на $(0, 1)$ рядом Фурье-Уолша с монотонно убывающими коэффициентами, которая обладает следующими свойствами:*

1) *она является универсальной для класса $M[0, 1]$ относительно системы*

Уолша в квази- обычном смысле,

2) *для любого $p \in (0, 1)$ функция $U(x)$ универсальна для $L^p[0, 1]$ относительно системы Уолша в смысле знаков,*

3) можно найти последовательность знаков $\{\varepsilon_k = \pm 1\}_{k=0}^{\infty}$ такую, что ряд $\sum_{k=0}^{\infty} \varepsilon_k c_k(U) W_k(x)$ будет универсальным в $L^p[0, 1]$, $p \in (0, 1)$ в смысле перестановок.

2. ВСПОМОГАТЕЛЬНЫЕ ФАКТЫ

Напомним определение системы Уолша-Пэли $\{W_k(x)\}$ (см. [24], [25]).

$$W_0(x) = 1, \quad W_n(x) = \prod_{s=1}^k r_{m_s}(x), \quad n = \sum_{s=1}^k 2^{m_s}, \quad m_1 > m_2 > \dots > m_s \geq 0,$$

где $\{r_k(x)\}_{k=0}^{\infty}$ - система Радемахера:

$$r_0(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}); \\ -1, & x \in [\frac{1}{2}, 1). \end{cases}$$

$$r_0(x+1) = r_0(x), \quad r_k(x) = r_0(2^k x), \quad k = 1, 2, \dots$$

Введем некоторые обозначения. Разобьем полуинтервал $[0, 1)$ на 2^m равных частей и обозначим эти полуинтервалы через $\Delta_m^{(j)}$, которые будем в дальнейшем называть двоичными интервалами или просто интервалами вида $\Delta_m^{(j)}$, $(\Delta_m^{(j)} = [\frac{j-1}{2^m}, \frac{j}{2^m})$, $1 \leq j \leq 2^m$). Пусть

$$(2, 1) \quad S_m(x, g) = \sum_{k=0}^m c_k(g) W_k(x),$$

где $c_k(g) = \int_0^1 g(x) W_k(x) dx$ — коэффициенты Фурье-Уолша функции $g \in L^1(0, 1)$.

Мы будем использовать следующие неравенства (см. [26])

$$(2, 2) \quad |S_n(x, g)| < \frac{2}{\delta} \int_c^d |g(t)| dt \quad \forall x \notin [c - \delta, d + \delta],$$

где $g(t)$ — произвольная интегрируемая и вне (c, d) обращающаяся в ноль функция, δ — произвольное положительное число.

$$(2, 3) \quad \int_0^1 |S_n(x, g)|^p dx < \frac{A}{1-p} \left(\int_0^1 |g(x)| dx \right)^p, \quad \forall p \in (0, 1),$$

где $g(t)$ — произвольная интегрируемая на $(0, 1)$ функция и A — постоянная. В дальнейшем нам полезна будет следующая очевидная лемма.

Лемма 2.1. Для каждой функции $g(x) \in$ существует последовательность $\{P_k(x)\}_{k=1}^{\infty}$ полиномов по системе Уолша с рациональными коэффициентами, которая почти всюду на $[0, 1]$ сходится к $g(x)$.

Мы также будем использовать следующую лемму, доказанную в [20] (см. Лемма 3 в [20]).

Лемма 2.2. Пусть $0 < p_0 < 1$, $n_0 \in N$, $\eta \in (0, 1)$ и $f(x) = \sum_{m=1}^{\tilde{\nu}_0} \tilde{\gamma}_m \chi_{\tilde{\Delta}_m}(x)$ есть такая ступенчатая функция, что $\tilde{\gamma}_m \neq 0$ и $\{\tilde{\Delta}_m\}_{m=1}^{\tilde{\nu}_0}$ — суть непересекающийся двоичные интервалы с $\sum_{m=1}^{\tilde{\nu}_0} |\tilde{\Delta}_m| = 1$. Тогда можно найти полиномы

$$H(x) = \sum_{k=2^{n_0}}^{2^n-1} a_k W_k(x), \quad Q(x) = \sum_{k=2^{n_0}}^{2^n-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1,$$

по системе Уолша, удовлетворяющие следующим условиям:

$$0 < a_{k+1} \leq a_k < \eta, \quad \forall k \in [2^{n_0}, 2^n - 1), \quad H(x) \cdot \chi_{[2^{-n_0}, 1]}(x) = 0,$$

$$\int_0^1 |f(x) - H(x)|^{p_0} dx < \eta, \quad \max_{2^{n_0} \leq M < 2^n} \int_0^1 \left| \sum_{k=2^{n_0}}^M a_k W_k(x) \right| dx < \eta,$$

$$\max_{2^{n_0} \leq M < 2^n} \int_0^1 \left| \sum_{k=2^{n_0}}^M \delta_k a_k W_k(x) \right|^{p_0} dx < 5 \int_0^1 |f(x)|^{p_0} dx.$$

3. ДОКАЗАТЕЛЬСТВО ТЕОРЕМ 1.1 – 1.3

Пусть $0 < \varepsilon < 1$ и $p \in (0, 1)$. Положим

$$(3.1) \quad m_0 = 2 + \left\lfloor \log_2 \frac{1}{\varepsilon} \right\rfloor, \quad M_1 = 2^{m_0}.$$

Рассмотрим счетное множество всех ступенчатых функций вида

$$(3.2) \quad \sum_{j=1}^{2^n} \gamma_j \chi_{\Delta_n^{(j)}}(x),$$

где $\gamma_j \neq 0$ -рациональные числа и пусть $\{f_m(x)\}_{m=1}^\infty$ — есть последовательность этих функций в произвольной нумерацией.

Применим Лемму 2, полагая в ее формулировке

$$n_0 = m_0, \quad \eta = 2^{-4}, \quad p_0 = p, \quad f(x) = f_1(x)$$

Тогда определяются полиномы по системе Уолша вида

$$H_1^{(1)}(x) = \sum_{k=M_1}^{M_2-1} a_k^{(1)} W_k(x), \quad Q_1^{(1)}(x) = \sum_{k=M_1}^{M_2-1} \delta_k^{(1)} a_k^{(1)} W_k(x), \quad M_1 = 2^{m_0}, \quad M_2 = 2^{m_1}$$

удовлетворяющие условиям:

$$0 < a_{k+1}^{(1)} \leq a_k^{(1)} < \frac{1}{2}, \quad \delta_k^{(1)} = \pm 1, \quad \int_0^1 \left| f_1(x) - Q_1^{(1)}(x) \right|^p dx < 2^{-4}, \quad k \in [M_1, M_2),$$

$$\begin{aligned}
 H_1^{(1)}(x) \cdot \chi_{[I_2, 1]}(x) &= 0, \quad I_1 = \frac{1}{M_1}, \\
 \int_0^1 |H_1^{(1)}(x)| dx &< \max_{m \in [M_1, M_2]} \int_0^1 \left| \sum_{k=M_1}^m a_k^{(1)} W_k(x) \right| dx < 2^{-4}, \\
 H_1^{(1)}(x) \cdot \chi_{[I_2, 1]}(x) &= 0, \quad I_1 = \frac{1}{M_1}, \quad \int_0^1 \left| f_1(x) - Q_1^{(1)}(x) \right|^p dx < 2^{-4}, \\
 \max_{m \in [M_1, M_2]} \int_0^1 \left| \sum_{k=M_{2n-1}}^m \delta_k^{(1)} a_k^{(1)} W_k(x) \right|^p dx &< 5 \int_0^1 |f_1(x)|^p dx.
 \end{aligned}$$

Снова применяя лемму 2.2, полагая в ее формулировке

$$n_0 = [\log_2 M_2], \quad \eta = \min\{a_{M_2-1}^{(1)}; 2^{-8}\}, \quad p_0 = p, \quad f(x) = \{f_1(x) - H_1^{(1)}(x)\}.$$

Тогда определяются полиномы по системе Уолша вида

$$H_1^{(2)}(x) = \sum_{k=M_2}^{M_3-1} a_k^{(1)} W_k(x), \quad Q_1^{(2)}(x) = \sum_{k=M_2}^{M_3-1} \delta_k^{(1)} a_k^{(1)} W_k(x), \quad k \in [M_2, M_3) = [2^{m_1}, 2^{m_2})$$

удовлетворяющие условиям:

$$\begin{aligned}
 0 < a_{k+1}^{(1)} &\leq a_k^{(1)} < a_{M_2-1}^{(1)}, \quad \delta_k^{(1)} = \pm 1, \quad \forall k \in [M_2, M_3), \\
 \int_0^1 |\{f_1(x) - H_1^{(1)}(x)\} - Q_1^{(2)}(x)|^p dx &< 2^{-8}, \quad H_1^{(2)}(x) \cdot \chi_{[I_2, 1]}(x) = 0, \quad I_2 = \frac{1}{M_2}, \\
 \int_0^1 |H_1^{(2)}(x)| dx &< \max_{m \in [M_2, M_3]} \int_0^1 \left| \sum_{k=M_2}^m a_k^{(1)} W_k(x) \right| dx < 2^{-8}.
 \end{aligned}$$

Нетрудно видеть, что продолжая эти рассуждения, мы можем по индукции определить последовательности полиномов

$$\{H_n^{(1)}(x)\}_{n=1}^\infty, \quad \{H_n^{(2)}(x)\}_{n=1}^\infty, \quad \{Q_n^{(1)}(x)\}_{n=1}^\infty, \quad \{Q_n^{(2)}(x)\}_{n=1}^\infty$$

вида

$$(3.3) \quad H_n^{(1)}(x) = \sum_{k=M_{2n-1}}^{M_{2n}-1} a_k^{(n)} W_k(x), \quad H_n^{(2)}(x) = \sum_{k=M_{2n}}^{M_{2n+1}-1} a_k^{(n)} W_k(x), \quad M_1 = 2^{m_0},$$

$$(3.4) \quad Q_n^{(1)}(x) = \sum_{k=M_{2n-1}}^{M_{2n}-1} \delta_k^{(n)} a_k^{(n)} W_k(x), \quad Q_n^{(2)}(x) = \sum_{k=M_{2n}}^{M_{2n+1}-1} \delta_k^{(n)} a_k^{(n)} W_k(x),$$

где

$$(3.5) \quad M_k \nearrow^\infty, \quad \delta_k^{(n)} = \pm 1, \quad k \in [M_{2n-1}, M_{2n+1}), n = 0, 1, 2, \dots,$$

которые для каждого $n = 1, 2, \dots$ удовлетворяют условиям:

$$(3.6) \quad 0 < a_{k+1}^{(n)} \leq a_k^{(n)} < a_{M_{2n-1}-1}^{(n-1)}, \quad k \in [M_{2n-1}, M_{2n+1}),$$

$$(3.7) \quad \int_0^1 \left| f_n(x) - \sum_{j=1}^n \left(H_j^{(1)}(x) + Q_j^{(2)}(x) \right) \right|^p dx < 2^{-3n},$$

$$(3.8) \quad H_n^{(1)}(x) \chi_{[I_{2n-1}, 1]}(x) = H_n^{(2)}(x) \chi_{[I_{2n}, 1]}(x) = 0, \quad I_j = \frac{1}{M_j},$$

$$(3.9) \quad \int_0^1 |H_n^{(1)}(x)| dx \leq \max_{m \in [M_{2n-1}, M_{2n})} \int_0^1 \left| \sum_{k=M_{2n-1}}^m a_k^{(n)} W_k(x) \right| dx < 2^{-\frac{2}{p}(n+4)},$$

$$(3.10) \quad \int_0^1 |H_n^{(2)}(x)| dx \leq \max_{m \in [M_{2n}, M_{2n+1})} \int_0^1 \left| \sum_{k=M_{2n}}^m a_k^{(n)} W_k(x) \right| dx < 2^{-\frac{2}{p}(n+4)},$$

$$(3.11) \quad \int_0^1 \left| f_n(x) - Q_n^{(1)}(x) \right|^p dx < 2^{-4n-2},$$

$$(3.12) \quad \max_{m \in [M_{2n-1}, M_{2n})} \int_0^1 \left| \sum_{k=M_{2n-1}}^m \delta_k^{(n)} a_k^{(n)} W_k(x) \right|^p dx < 5 \int_0^1 |f_n(x)|^p dx.$$

Учитывая соотношения (3.3) и (3.8)–(3.10), в силу (2.1) и (2.2) для всех $x \in [I_{2n-1} + 2^{-n}, 1]$ и для каждого $m \in [M_{2n-1}, M_{2n})$, $\forall n \geq 1$ будем иметь

$$(3.13) \quad \left| \sum_{k=M_{2n-1}}^m a_k^{(n)} W_k(x) \right| = \left| S_m(x, H_n^{(1)}) \right| < \frac{2}{2^{-n}} \int_0^1 \left| H_n^{(1)}(t) \right| dt \leq 2^{-n-1},$$

Аналогично для всех $x \in [I_{2n} + 2^{-n}, 1]$ и для каждого $m \in [M_{2n}, M_{2n+1})$, $n \geq 1$ получим

$$(3.14) \quad \left| \sum_{k=M_{2n}}^m a_k^{(n)} W_k(x) \right| < \frac{2}{2^{-n}} \int_0^1 \left| H_n^{(2)}(t) \right| dt \leq 2^{-n-1}.$$

Из (3.9), (3.10) следует

$$(3.15) \quad \sum_{n=1}^{\infty} \sum_{j=1}^2 \left(\int_0^1 \left| H_n^{(j)}(x) \right| dx \right) < \infty.$$

Определим функцию $U(x)$ и последовательность чисел $\{a_k\}$ следующим образом

$$(3.16) \quad a_k = 1, \quad k \in [0, 2^{m_0}), \quad a_k = a_k^{(n)}, \quad k \in [M_{2n-1}, M_{2n+1}), \quad n = 1, 2, \dots$$

$$(3.17) \quad U(x) = H_0(x) + \sum_{n=1}^{\infty} \left(H_n^{(1)}(x) + H_n^{(2)}(x) \right) = \sum_{k=0}^{\infty} a_k W_k(x),$$

где (см. (3.1))

$$(3.18) \quad H_0(x) = \sum_{k=0}^{2^{m_0}-1} W_k(x),$$

Отсюда и из (3.15) следует $U(x) \in L^1[0, 1]$. В силу (3.1), (3.3), (3.8), (3.17) и (3.18) имеем $U(x) = 0$ при $x \in [2^{-m_0}, 1] \subset [\varepsilon, 1]$.

Пусть $x \in (0, 1)$, ясно, что для некоторого натурального n_x , $x \in [I_{2n-1} + 2^{-n}, 1]$ для всех $n \geq n_x$. Тогда в силу (3.13), (3.14) и (3.16) – (3.18) для любых $N \in [M_{2n-1}, M_{2n+1})$ при $n \geq n_x$ будем иметь

$$\begin{aligned} \left| U(x) - \sum_{k=0}^N a_k W_k(x) \right| &\leq \max_{m \in [M_{2n-1}, M_{2n})} \left| \sum_{k=M_{2n-1}}^m a_k^{(n)} W_k(x) \right| + \\ &+ \max_{m \in [M_{2n}, M_{2n+1})} \left| \sum_{k=M_{2n}}^m a_k^{(n)} W_k(x) \right| \leq 2^{-n} \end{aligned}$$

Откуда вытекает, что ряд $\sum_{k=0}^{\infty} a_k W_k(x)$ сходится к $U(x)$ всюду на $(0, 1)$. Учитывая соотношения (3.9), (3.10), (3.16)–(3.18) для любых $N \in [M_{2n-1}, M_{2n+1})$, $n \geq 1$ получим

$$\begin{aligned} \int_0^1 \left| U(x) - \sum_{k=0}^N a_k W_k(x) \right| dx &\leq \max_{m \in [M_{2n-1}, M_{2n})} \int_0^1 \left| \sum_{k=M_{2n-1}}^m a_k^{(n)} W_k(x) \right| dx + \\ &+ \max_{m \in [M_{2n}, M_{2n+1})} \int_0^1 \left| \sum_{k=M_{2n}}^m a_k^{(n)} W_k(x) \right| dx \leq 2^{-2n} \end{aligned}$$

Значит, ряд $\sum_{k=0}^{\infty} a_k W_k(x)$ сходится к $U(x)$ по $L^1[0, 1]$ –норме и следовательно

$$(3.19) \quad a_k = c_k(U) = \int_0^1 U(x) W_k(x) dx, \quad k = 0, 1, 2, \dots$$

В силу (3.6), (3.16) и (3.19) будем иметь $c_k(U) \searrow$. Из (2.3), (3.3), (3.9) и (3.10) следует

$$\begin{aligned} \max_{m \in [M_{2n-1}, M_{2n})} \int_0^1 \left| \sum_{k=M_{2n-1}}^m c_k(U) W_k(x) \right|^p dx &+ \max_{m \in [M_{2n}, M_{2n+1})} \int_0^1 \left| \sum_{k=M_{2n}}^m c_k(U) W_k(x) \right|^p dx \\ (3.20) \quad &\leq \frac{A}{1-p} 2^{-2(n+4)}. \end{aligned}$$

Положим

$$(3.21) \quad \delta_k = 1, \quad k \in [0, 2^{m_0}) \cup \bigcup_{n=1}^{\infty} [M_{2n-1}, M_{2n}); \quad \delta_k = \delta_k^{(n)}, \quad k \in [M_{2n}, M_{2n+1}),$$

$$(3.22) \quad \varepsilon_k = \delta_k^{(n)}, \quad k \in [M_{2n-1}, M_{2n}); \quad \varepsilon_k = 1, \quad k \in [0, 2^{m_0}) \cup \bigcup_{n=1}^{\infty} [M_{2n}, M_{2n+1}).$$

Покажем, что ряд

$$(3.23) \quad \sum_{k=0}^{\infty} \delta_k c_k(U) W_k(x) = H_0(x) + \sum_{j=1}^{\infty} \left(H_j^{(1)}(x) + Q_j^{(2)}(x) \right),$$

универсален как в $M[0, 1)$ относительно системы Уолша в обычном смысле, а ряд

$$(3.24) \quad \sum_{k=0}^{\infty} \varepsilon_k c_k(U) W_k(x) = H_0(x) + \sum_{j=1}^{\infty} \left(Q_j^{(1)}(x) + H_j^{(2)}(x) \right).$$

универсален в $L^p[0, 1)$ относительно перестановок.

Пусть $g(x) \in M[0, 1)$. Применяя лемму 1 мы можем из последовательности $\{f_m(x)\}_{m=1}^{\infty}$ (см. (3.2)) выбрать подпоследовательность $\{f_{l_k}(x)\}_{k=1}^{\infty}$ так, чтобы почти всюду на $[0, 1]$

$$(3.25) \quad \lim_{k \rightarrow \infty} f_{l_k}(x) = (g(x) - H_0(x)).$$

В силу (3.7) имеем

$$\int_0^1 \left| f_{l_k}(x) - \sum_{j=1}^{l_k} \left(H_j^{(1)}(x) + Q_j^{(2)}(x) \right) \right|^p dx < 2^{-8(l_k+2)}.$$

Отсюда следует, что для некоторой подпоследовательности $\{l_{k_q}\}_{q=1}^{\infty}$ выбранной из последовательности $\{l_k\}_{k=1}^{\infty}$ почти всюду на $[0, 1]$ имеет место

$$(3.26) \quad \lim_{q \rightarrow \infty} \left\{ f_{l_{k_q}}(x) - \sum_{j=1}^{l_{k_q}} \left(H_j^{(1)}(x) + Q_j^{(2)}(x) \right) \right\} = 0.$$

Положим $N_q = M_{2l_{k_q}+1} - 1$. Учитывая соотношения (3.3), (3.4), (3.16), (3.19), (3.21), (3.23), (3.25) и (3.26), почти всюду на $[0, 1]$ имеет место

$$\lim_{q \rightarrow \infty} \sum_{s=0}^{N_q} \delta_s c_s(U) W_s(x) = g(x).$$

Итак (см. (3.23)), первая часть теоремы 3 доказана. Теперь покажем, что ряд (3.24) универсален в $L^p[0, 1)$ в смысле перестановок.

Пусть $f(x) \in L^p[0, 1]$. Положим

$$(3.27) \quad f^*(x) = f(x) - H_0(x).$$

По индукции построим последовательности функции $\{f_{\nu_q}(x)\}_{q=1}^{\infty}$ чисел $\{\nu_q\}_{q=1}^{\infty}$, $\{s(q)\}_{q=1}^{\infty}$ и полиномов $\{Q_{\nu_q}^{(1)}(x)\}_{q=1}^{\infty}$, $\{R_q(x)\}_{q=1}^{\infty}$, $\{\beta_q(x)\}_{q=1}^{\infty}$ удовлетворяющих

следующим условиям:

$$(3.28) \quad \int_0^1 |f_{\nu_q}(x)|^p dx \leq 2^{-2q} + \int_0^1 |R_{q-1}(x) + \beta_{q-1}(x)|^p dx, \quad \nu_q > \nu_{q-1} + 1, \nu_0 = 1,$$

(3.29)

$$R_q(x) = \left(\sum_{n=\nu_{q-1}}^{\nu_q-1} H_n^{(2)}(x) \right), \quad \beta_q(x) = \varepsilon_{s(q)} c_{s(q)}(U) W_{s(q)}(x), \quad (R_0(x) = f^*(x), \beta_0(x) = 0),$$

(3.30)

$$s(q) = \min \{ k \in \mathbf{N}, k \in \text{spec} \left(\sum_{j=1}^q \sum_{n=\nu_{j-1}+1}^{\nu_j-1} Q_n^{(1)}(x) \right) \setminus \{s(n)\}_{n=1}^{q-1} \}, s(1) = M_1,$$

(3.31)

$$\int_0^1 \left| f^*(x) - \left\{ \sum_{j=1}^q [\beta_j(x) + R_j(x) + Q_{\nu_j}^{(1)}(x)] \right\} \right|^p dx \leq 2^{-2(q+2)} + \int_0^1 |R_q(x) + \beta_q(x)|^p dx.$$

На первом шаге возьмем функцию $f_{\nu_1}(x)$, $(\nu_1 > \nu_0 + 1)$ из последовательности (3.2) таким образом, чтобы

$$\int_0^1 |f^*(x) - f_{\nu_1}(x)|^p dx \leq 2^{-10}.$$

Отсюда и из (3.11) имеем

$$\begin{aligned} \int_0^1 |f_{\nu_1}(x)|^p dx &\leq 2^{-2(1+4)} + \int_0^1 |f^*(x)|^p dx = 2^{-10} + \int_0^1 |R_0(x) + \beta_0(x)|^p dx \\ \int_0^1 \left| f^*(x) - \{R_1(x) + \beta_1(x) + Q_{\nu_1}^{(1)}(x)\} \right|^p dx &\leq 2^{-10} + \int_0^1 |R_1(x) + \beta_1(x)|^p dx, \end{aligned}$$

где

$$\beta_0(x) = 0, \quad R_0(x) = f^*(x),$$

$$R_1(x) = \left(\sum_{n=1}^{\nu_1-1} H_n^{(2)}(x) \right), \quad s(1) = M_1 = 2^{m_0}, \quad \beta_1(x) = \varepsilon_{s(1)} c_{s(1)}(U) W_{s(1)}(x).$$

Находя функцию $f_{\nu_1}(x)$, числа ν_1 , $s(1)$ и полиномы $R_1(x)$, $\beta_1(x)$ и $Q_{\nu_1}^{(1)}(x)$ таким образом, чтобы при $q = 1$ выполнялись условия (3.28)–(3.31). Предположим, что уже определены функции $f_{\nu_1}(x), \dots, f_{\nu_q}(x)$ числа $\nu_0 < \nu_1 < \dots < \nu_q$; $s(1) < \dots < s(q)$ и полиномы $\{Q_{\nu_j}^{(1)}(x)\}_{j=1}^q$, $\{R_j(x)\}_{j=1}^q$, $\{\beta_j(x)\}_{j=1}^q$ уже построены. Покажем теперь, как следует строить функцию $f_{\nu_{q+1}}(x)$ числа ν_{q+1} , $s(q+1)$, и полиномы $Q_{\nu_{q+1}}^{(1)}(x)$, $R_{q+1}(x)$, $\beta_{q+1}(x)$.

Сначала подберем натуральное число $\nu_{q+1} > \nu_q + q$ следовательно и функцию $f_{\nu_{q+1}}(x)$ из последовательности $\{f_m(x)\}_{m=1}^{\infty}$ (см. (3.2)) так, чтобы

$$(3.32) \quad \int_0^1 \left| f^*(x) - \left\{ \sum_{j=1}^q [R_j(x) + \beta_j(x) + Q_{\nu_j}^{(1)}(x)] \right\} - f_{\nu_{q+1}}(x) \right|^p dx \leq 2^{-2(q+4)}.$$

Из (3.11) и (3.34) следует

$$(3.33) \quad \int_0^1 \left| f^*(x) - \left\{ \sum_{j=1}^q [R_j(x) + \beta_j(x) + Q_{\nu_j}^{(1)}(x)] \right\} - Q_{\nu_{q+1}}^{(1)}(x) \right|^p dx \leq 2^{-2(q+3)}.$$

Покажем, что условия (3.28)–(3.31) выполнены с $q+1$ на месте q . Сначала заметим, что из (3.31), (3.32) вытекает

$$\int_0^1 |f_{\nu_{q+1}}(x)|^p dx \leq 2^{-2(q+1)} + \int_0^1 |R_q(x) + \beta_q(x)|^p dx,$$

т.е. (3.28) выполнено для $q+1$. Положим

$$(3.34) \quad R_{q+1}(x) = \left(\sum_{n=\nu_q}^{\nu_{q+1}-1} H_n^{(2)}(x) \right), \quad \beta_{q+1}(x) = \varepsilon_{s(q+1)} c_{s(q+1)}(U) W_{s(q+1)}(x),$$

$$(3.35) \quad s(q+1) = \min \{ k \in \mathbf{N}, k \in \text{spec} \left(\sum_{j=1}^{q+1} \sum_{n=\nu_{j-1}+1}^{\nu_j-1} Q_n^{(1)}(x) \right) \setminus \{s(n)\}_{n=1}^q \}.$$

С помощью (3.33) - (3.35) проверяем (3.31) для $q+1$:

$$\int_0^1 \left| f^*(x) - \left\{ \sum_{j=1}^{q+1} [R_j(x) + \beta_j(x) + Q_{\nu_j}^{(1)}(x)] \right\} \right|^p dx \leq 2^{-2(q+3)} + \int_0^1 |R_{q+1}(x) + \beta_{q+1}(x)|^p dx.$$

Учитывая соотношения (3.10), (3.12) и (3.29) для всех $q > 1$ получим

$$(3.36) \quad \int_0^1 |R_q(x) + \beta_q(x)|^p dx \leq 2^{-2(q+3)} + \sum_{n=\nu_q}^{\nu_{q+1}-1} \int_0^1 |H_n^{(2)}(x)|^p dx + |c_{s(q+1)}(U)|^p \leq 2^{-2(q+1)} + |c_{s(q)}(U)|^p.$$

В силу (3.12), (3.16), (3.19) и (3.22) имеем

$$(3.37) \quad \begin{aligned} & \max_{m \in [M_{2\nu_{q-1}}, M_{2\nu_q})} \int_0^1 \left| \sum_{k=M_{2\nu_{q-1}}}^m \varepsilon_k c_k(U) W_k(x) \right|^p dx = \\ & = \max_{m \in [M_{2\nu_{q-1}}, M_{2\nu_q})} \int_0^1 \left| \sum_{k=M_{2\nu_{q-1}}}^m \delta_k^{(\nu_q)} a_k^{(\nu_q)} W_k(x) \right|^p dx \leq \end{aligned}$$

$$\leq 5 \int_0^1 |f_{\nu_q}(x)|^p dx \leq 5 \cdot 2^{-2(q-1)} + 5 |c_{s(q-1)}(U)|^p.$$

Ясно, что последовательность чисел (см. (3.5) (3.30))

$$0, 1, \dots, M_1 - 1; M_2, \dots, M_3; M_5, \dots, M_7, \dots; M_{2\nu_1-2}, \dots, M_{2\nu_1-1} - 1;$$

$$s(1); M_{2\nu_1-1}, \dots, M_{2\nu_1} - 1; M_{2\nu_1}, \dots, M_{2\nu_1+1} - 1; \dots;$$

$$M_{2n}, \dots, M_{2n+1} - 1; \dots; M_{2\nu_q-2}, \dots, M_{2\nu_q-1} - 1; s(q); M_{2\nu_q-1} \dots M_{2\nu_q} - 1; M_{2\nu_q} \dots M_{2\nu_q+1}; \dots$$

-есть некоторая перестановка $\{\sigma(k)\}_{k=1}^\infty$. Следовательно ряд

$$(3.38) \quad \sum_{k=1}^\infty \varepsilon_{\sigma(k)} c_{\sigma(k)}(U) W_{\sigma(k)}(x) = H_0(x) + \sum_{q=1}^\infty [R_q(x) + \beta_q(x) + Q_{\nu_q}^{(1)}(x)]$$

-есть перестановленный ряд определяемый в силу (3.3), (3.4), (3.18), (3.22), (3.28), (3.31) и (3.30) ряда (3.24). Основываясь (3.20), (3.27), (3.36), (3.37) и (3.38) для любых $N \in [M_{2\nu_q-1}, M_{2\nu_{q+1}+1})$ и $q > 1$ будем иметь

$$\begin{aligned} & \int_0^1 \left| f(x) - \sum_{k=1}^N \varepsilon_{\sigma(k)} c_{\sigma(k)}(U) W_{\sigma(k)}(x) \right|^p dx \leq \\ & \leq \int_0^1 \left| f^*(x) - \left\{ \sum_{j=1}^q [R_j(x) + \beta_j(x) + Q_{\nu_j}^{(1)}(x)] \right\} \right|^p dx + \\ & + \max_{m \in [M_{2\nu_q-1}, M_{2\nu_q})} \int_0^1 \left| \sum_{k=M_{2\nu_q-1}}^m \varepsilon_k c_k(U) W_k(x) \right|^p dx + \\ & + \sum_{n=\nu_q+1}^{\nu_{q+1}} \max_{m \in [M_{2n}, M_{2n+1})} \int_0^1 \left| \sum_{k=M_{2n}}^m c_k(U) W_k(x) \right|^p dx + \\ & + |c_{s(q+1)}(U)| \leq (2^{-2q} + |c_{s(q)}(U)|^p) + 5(2^{-2(q-1)} + |c_{s(q-1)}(U)|^p) + \\ & + \frac{A}{1-p} 2^{-2(q+1)} + |c_{s(q+1)}(U)|^p. \end{aligned}$$

Откуда, принимая во внимание, что $c_{s(q)}(U) \rightarrow 0$ при $q \rightarrow \infty$ и что $q \rightarrow \infty$ когда $N \rightarrow \infty$ заключаем, что ряд (3.38) сходится к $f(x)$ в метрике $L^p[0, 1]$. Для завершения доказательства теоремы 1.3 осталось проверить, что функция $U(x)$ универсальна для $L^p[0, 1]$ относительно системы Уолша в смысле знаков.

Нетрудно убедиться, что по тем же соображениям, что и при получении последовательности функции $\{f_{\nu_q}(x)\}_{q=1}^\infty$ чисел $\{\nu_q\}_{q=0}^\infty$, $\{s(q)\}_{q=0}^\infty$ и полиномов $\{Q_{\nu_q}^{(1)}(x)\}_{q=1}^\infty$, $\{R_q(x)\}_{q=1}^\infty$, $\{\beta_q(x)\}_{q=1}^\infty$ удовлетворяющих условиям (3.28)-(3.31), по индукции можно выбрать последовательности функции $\{f_{\lambda_q}(x)\}_{q=1}^\infty$, чисел

$\{\lambda_q\}_{q=0}^\infty$ и полиномы $\{Q_{\lambda_q}^{(1)}(x)\}_{q=1}^\infty$, $\{A_q(x)\}_{q=1}^\infty$, $\{B_q(x)\}_{q=1}^\infty$, которые удовлетворяют следующим условиям:

$$(3.39) \quad A_q(x) = \sum_{n=\lambda_{q-1}+1}^{\lambda_q} \left(H_n^{(1)}(x) + H_n^{(2)}(x) \right), \quad A_0(x) = f^*(x), \quad H_0^{(1)}(x) = 0 = \lambda_0,$$

$$(3.40) \quad \int_0^1 |f_{\lambda_q}(x)|^p dx \leq 2^{-p(q+2)} + \int_0^1 |A_{q-1}(x) - H_{\lambda_{q-1}}^{(1)}(x)|^p dx \leq 2^{-2q}, \quad q > 1,$$

$$(3.41) \quad \int_0^1 \left| f^*(x) - \left\{ \sum_{j=1}^q [A_j(x) - H_{\lambda_j}^{(1)}(x) + Q_{\lambda_j}^{(1)}(x)] \right\} \right|^p dx \leq \\ \leq 2^{-2(q+3)} + \int_0^1 |A_q(x) - H_{\lambda_q}^{(1)}(x)|^p dx \leq 2^{-2(q+1)}.$$

Положим

$$(3.42) \quad \eta_k = \delta_k^{(n)}, \quad k \in [M_{2\lambda_{q-1}}, M_{2\lambda_q}); \quad \eta_k = 1, \quad k \notin \bigcup_{q=1}^\infty [M_{2\lambda_{q-1}}, M_{2\lambda_q}).$$

В силу (3.12), (3.16), (3.19), (3.40) и (3.42) имеем

$$(3.43) \quad \max_{m \in [M_{2\lambda_{q-1}}, M_{2\lambda_q})} \int_0^1 \left| \sum_{k=M_{2\lambda_{q-1}}}^m \eta_k c_k(U) W_k(x) \right|^p dx = \\ = \max_{m \in [M_{2\lambda_{q-1}}, M_{2\lambda_q})} \int_0^1 \left| \sum_{k=M_{2\lambda_{q-1}}}^m \delta_k^{(\lambda_q)} a_k^{(\lambda_q)} W_k(x) \right|^p dx \leq 5 \int_0^1 |f_{\lambda_q}(x)|^p dx \leq 5 \cdot 2^{-2q}$$

Из (3.39)–(3.43) для любых $N \in [M_{2\lambda_{q-1}}, M_{2\lambda_{q+1}-1})$ и $q > 1$ будем иметь

$$\int_0^1 \left| f(x) - \sum_{k=0}^N \eta_k c_k(U) W_k(x) \right|^p dx \leq \int_0^1 \left| f^*(x) - \left\{ \sum_{j=1}^q [Q_{\lambda_j}^{(1)}(x) + A_j(x) + B_j(x)] \right\} \right|^p dx + \\ + \sum_{n=\lambda_q+1}^{\lambda_{q+1}} \left(\max_{m \in [M_{2n-1}, M_{2n})} \int_0^1 \left| \sum_{k=M_{2n-1}}^m c_k(U) W_k(x) \right|^p dx + \max_{l \in [M_{2n}, M_{2n+1})} \int_0^1 \left| \sum_{k=M_{2n}}^l c_k(U) W_k(x) \right|^p dx \right) + \\ + \max_{m \in [M_{2\lambda_{q-1}}, M_{2\lambda_q})} \int_0^1 \left| \sum_{k=M_{2\lambda_{q-1}}}^m \delta_k^{(\lambda_q)} a_k^{(\lambda_q)} W_k(x) \right|^p dx \leq 2^{-q+3}.$$

Отсюда и из того, что $q \rightarrow \infty$ когда $N \rightarrow \infty$ заключаем, что ряд $\sum_{k=1}^\infty \eta_k c_k(U) W_k(x)$ сходится к $f(x)$ в метрике $L^p[0, 1)$.

Теорема 1.3 доказана.

Abstract. In this work it is constructed an integrable function $U(x)$ is universal for class $L^p[0, 1]$, $p \in (0, 1)$ with respect to the Walsh system in sense of signs, with strictly decreasing Fourier-Walsh coefficients $\{c_k(U)\}_{k=0}^{\infty} \searrow$, and converging everywhere on $(0, 1)$ and in the norm of $L^1[0, 1]$ Fourier-Walsh series, which has the following property: one can find a numbers $\varepsilon_k = \pm 1$ such that the series $\sum_{k=0}^{\infty} \varepsilon_k c_k(U) W_k(x)$ is universal in $L^p[0, 1]$, $p \in (0, 1)$ in sense of rearrangements.

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**ORIENTATION-DEPENDENT CHORD LENGTH DISTRIBUTION
FUNCTION FOR RIGHT PRISMS WITH RECTANGULAR OR
RIGHT TRAPEZOIDAL BASES**

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Abstract. The paper continues the research to reconstruct a convex body in $\mathbb{R}^n$ from the distribution of characteristics of its k -dimensional sections ($k < n$). In this paper we obtain explicit expressions for the covariogram and the orientation dependent chord length distribution of right prisms with rectangular or right trapezoidal bases.

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Keywords: Stochastic Geometry; convex body; covariogram; random chord length distribution.

1. INTRODUCTION

Reconstruction of convex bodies using random cross-sections makes it possible to simplify the calculation because the estimates of probability characteristics can be obtained using the methods of statistics. These type of problems are fundamental in the theory of geometric tomography and stereology and particularly can be applied in medicine (see [1] – [3]). Quantities characterizing random sections of body D carry some information about D . If there is a connection between geometric characteristics of D and probabilistic characteristics of a random cross-section then by a sample (of experiments) we can estimate the geometric characteristics of D . Let $\mathbb{R}^n$ be the n -dimensional Euclidean space, $D \subset \mathbb{R}^n$ be a bounded convex body with inner points, S^{n-1} be the $(n - 1)$ -dimensional unit sphere centered at the origin, and $L_n(\cdot)$ be the n -dimensional Lebesgue measure in $\mathbb{R}^n$. The function

$$C_D(x) = L_n(D \cap \{D + x\}), \quad x \in \mathbb{R}^n,$$

where $D + x = \{\mathcal{P} + x : \mathcal{P} \in D\}$, is called the covariogram of the body D .

There is a one-to-one correspondence between planar convex bodies and the covariogram (see [9]). Earlier in [2], a conjecture has been formulated by Matheron claiming that such correspondence exists in n -dimensional Euclidean spaces for any

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n . However, in the case $n \geq 4$ Matheron's hypothesis has received a negative answer (see [1] and [3]). The general 3-dimensional case is still open (see [3]).

This paper continues the research to reconstruct convex bodies in $\mathbb{R}^3$ using covariogram (see [6] – [8], [12]) and the orientation-dependent chord length distribution (see [4], [5], [10], [11], [13], and [14]).

Although the general 3-dimensional case is still open, Matheron's conjecture has been confirmed in the case of bounded convex polyhedrons in $\mathbb{R}^3$. Actually, the covariogram problem was found to be equivalent to the problem of rebuilding a convex domain from the length distribution of its orientation-dependent chords (see [2], [9]).

In the current paper we found explicit expressions for the covariogram and the orientation-dependent chord length distribution of a right parallelepiped with rectangular base. Further, the base is transformed into a right trapezoid with the given acute angle, and the mentioned expressions are obtained for right prisms with right trapezoidal bases.

2. CHORD LENGTH DISTRIBUTION IN A RECTANGLE

Let E be a bounded convex subset of $\mathbb{R}^2$. Consider the vector

$$\phi = (\cos \varphi, \sin \varphi) \in S^1,$$

and let l_φ be the subspace of $\mathbb{R}^2$ spanned by ϕ . By $\phi^\perp$ we denote the orthogonal complement of l_φ . For any $y \in \phi^\perp$, let $l_\varphi + y$ be the line which is parallel to ϕ and passes through y . Denote

$$\chi(l_\varphi + y) = L_1((l_\varphi + y) \cap E).$$

If the line $l_\varphi + y$ has a common segment with E , then we will say that it makes a chord in E of length $\chi(l_\varphi + y)$.

Let $\Pi_E(\varphi)$ be the orthogonal projection of E onto $\phi^\perp$. Assuming that y is uniformly distributed over $\Pi_E(\varphi)$, the chord length distribution function in direction ϕ for E is defined by

$$F_E(x, \varphi) = \frac{L_1\{y \in \Pi_E(\varphi) : \chi(l_\varphi + y) \leq x\}}{b_E(\varphi)},$$

where $b_E(\varphi) = L_1(\Pi_E(\varphi))$.

When E is a parallelogram, the distribution function $F_E(x, \varphi)$ and the covariogram $C_E(t, \varphi)$ (which is an alternative notation for $C_E(t\phi)$), are explicitly found in [15]. In particular, the following results can be extracted from [15], section 2.

Lemma 2.1. *Let R be the rectangle $[0, b] \times [0, a] \subset \mathbb{R}^2$, where $a \leq b$, and let $\pi k - \arctan \frac{a}{b} \leq \varphi < \pi(k+1) - \arctan \frac{a}{b}$ for some integer k . Then*

$$(2.1) \quad F_R(x, \varphi) = \begin{cases} 0, & \text{if } x \leq 0 \\ \frac{2x|\sin \varphi| \cdot |\cos \varphi|}{a|\cos \varphi| + b|\sin \varphi|}, & \text{if } 0 < x < x_{\max}(\varphi) \\ 1, & \text{if } x \geq x_{\max}(\varphi) \end{cases}$$

and

$$(2.2) \quad C_R(t, \varphi) = \begin{cases} ab - t(a|\cos \varphi| + b|\sin \varphi|) + t^2|\sin \varphi \cos \varphi|, & \text{if } 0 \leq t \leq x_{\max}(\varphi) \\ 0, & \text{if } t > x_{\max}(\varphi) \end{cases},$$

where

$$(2.3) \quad x_{\max}(\varphi) = \begin{cases} \frac{b}{|\cos \varphi|}, & \text{if } -\arctan \frac{a}{b} + \pi k \leq \varphi < \arctan \frac{a}{b} + \pi k \\ \frac{a}{|\sin \varphi|}, & \text{if } \arctan \frac{a}{b} + \pi k \leq \varphi < -\arctan \frac{a}{b} + \pi(k+1) \end{cases}.$$

Remark 2.1. $x_{\max}(\varphi)$ represents the length of the maximal chord in R in direction ϕ , that is

$$x_{\max}(\varphi) = \max_{y \in \Pi_R(\varphi)} \chi(l_\varphi + y).$$

Remark 2.2. The formula

$$(2.4) \quad b_R(\varphi) = a|\cos \varphi| + b|\sin \varphi|$$

holds for any real φ .

3. CHORD LENGTH DISTRIBUTION IN A RIGHT RECTANGULAR PARALLELEPIPED.

For $\omega \in S^2$, we denote by $\omega^\perp$ the orthogonal complement of $\{t\omega : t \in \mathbb{R}\}$ in $\mathbb{R}^3$. For a bounded convex body $D \subset \mathbb{R}^3$, let $\Pi_D(\omega)$ be the orthogonal projection of D onto the plane $\omega^\perp$.

Let $l_\omega + y$ be the line passing through $y \in \omega^\perp$ with direction vector ω , and $\chi(l_\omega + y) = L_1((l_\omega + y) \cap D)$. Assuming y is uniformly distributed in $\Pi_D(\omega)$, we define the chord length distribution function in direction ω for D by

$$(3.1) \quad F_D(t, \omega) = \frac{L_2\{y \in \Pi_D(\omega) : \chi(l_\omega + y) \leq t\}}{b_D(\omega)},$$

where $b_D(\omega) = L_2(\Pi_D(\omega))$.

Let D be a cylinder with base B (not necessarily convex) placed on the OXY plane, and height h . If ω is given by its spherical coordinates $(1, \varphi, \theta)$, where 1 is the radius, $\varphi \in [0, 2\pi)$ is the azimuthal angle, and $\theta \in [0, \frac{\pi}{2}]$ is the elevation angle, then

$$(3.2) \quad b_D(\omega) = \|B\| \sin \theta + b_B(\varphi)h \cos \theta,$$

where $\|B\|$ is the area of the base. A relation between orientation-dependent chord length distribution functions F_D and F_B is found in [7]:

$$(3.3) \quad F_D(t, \omega) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{b_B(\varphi) \cos \theta}{b_D(\omega)} [(h - t \sin \theta) F_B(t \cos \theta, \varphi) + \\ + t \sin \theta + \sin \theta \int_0^t (1 - F_B(u \cos \theta, \varphi)) du], & \text{if } 0 \leq t < x_{\max}(\omega) \\ 1, & \text{if } t \geq x_{\max}(\omega) \end{cases}$$

where $x_{\max}(\omega)$ is the length of the maximal chord in D in direction ω .

Theorem 3.1. *Let D be the parallelepiped $[0, b] \times [0, a] \times [0, h] \subset \mathbb{R}^3$ and $F_D(t, \omega)$ be the orientation-dependent chord length distribution function of D in direction $\omega = (\cos \varphi \cos \theta, \sin \varphi \cos \theta, \sin \theta) \in S^2$, where $0 \leq \theta \leq \frac{\pi}{2}$. Then*

$$F_D(t, \omega) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{\cos \theta}{ab \sin \theta + b_R(\varphi) \cdot h \cos \theta} \cdot \left((h \cos \theta |\sin 2\varphi| + \right. \\ \left. + 2b_R(\varphi) \sin \theta) \cdot t - \frac{3}{4} \sin 2\theta |\sin 2\varphi| \cdot t^2 \right), & \text{if } 0 < t < x_{\max}(\omega) \\ 1, & \text{if } t \geq x_{\max}(\omega) \end{cases},$$

where $R = [0, b] \times [0, a]$.

Proof. The validity of the formula is obvious when $t \leq 0$ or $t \geq x_{\max}(\omega)$, so we assume $0 < t < x_{\max}(\omega)$ hereinafter. Since

$$(3.4) \quad x_{\max}(\omega) = \begin{cases} \frac{x_{\max}(\varphi)}{\cos \theta}, & \text{if } 0 \leq \theta \leq \arctan \frac{h}{x_{\max}(\varphi)} \\ \frac{h}{\sin \theta}, & \text{if } \arctan \frac{h}{x_{\max}(\varphi)} < \theta \leq \frac{\pi}{2} \end{cases},$$

the inequality

$$x_{\max}(\omega) \cos \theta \leq x_{\max}(\varphi)$$

holds for any $\theta \in [0, \frac{\pi}{2}]$. Thus, taking into account (2.1), (3.2), and (3.3) we conclude that

$$\begin{aligned} F_D(t, \omega) &= \frac{b_R(\varphi) \cos \theta}{ab \sin \theta + b_R(\varphi) h \cos \theta} \cdot \\ &\cdot \left[(h - t \sin \theta) \frac{t \cos \theta |\sin 2\varphi|}{b_R(\varphi)} + 2t \sin \theta - \sin \theta \int_0^t \frac{u \cos \theta |\sin 2\varphi|}{b_R(\varphi)} du \right] = \\ &= \frac{\cos \theta}{ab \sin \theta + b_R(\varphi) \cdot h \cos \theta} \cdot \left[(h \cos \theta |\sin 2\varphi| + 2b_R(\varphi) \sin \theta) \cdot t - \frac{3}{4} \sin 2\theta |\sin 2\varphi| \cdot t^2 \right] \end{aligned}$$

□

Remark 3.1. *When $\theta = 0$ then functions F_D and F_R coincide. If $\theta = \frac{\pi}{2}$ then $x_{\max}(\omega) = h$. In this case F_D coincides with the indicator function of $(-\infty, h]$. For some other special cases the result of Theorem 3.1 is visualized by Figure 1.*

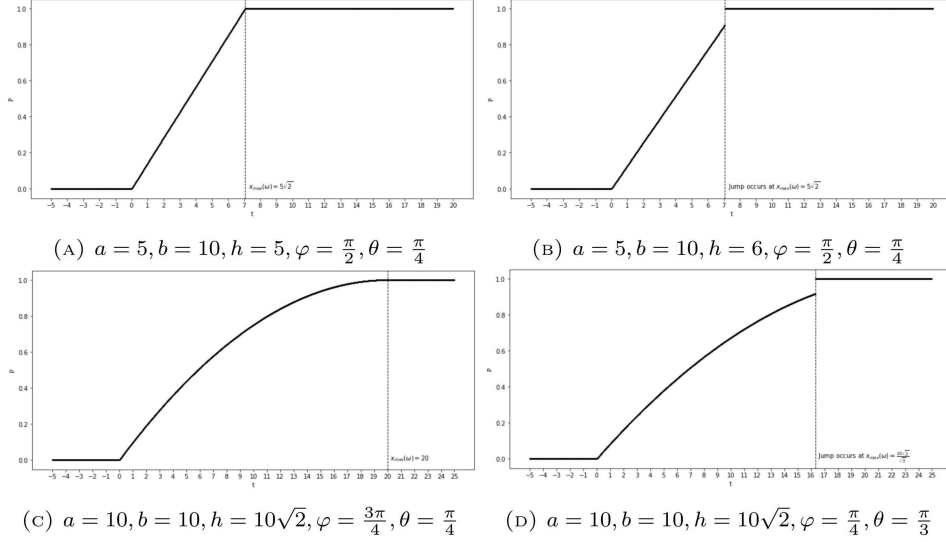


FIGURE 1. Orientation dependent chord length distribution function F_D for different cases

Remark 3.2. *Direct use of (3.3) in the proof of the theorem avoided computation of the covariogram of D . The function $C_D(t\omega)$ could be found explicitly.*

Indeed, if $0 \leq t \leq x_{\max}(\omega)$ then $C_D(t\omega) = L_3(D \cap \{D + t\omega\}) =$

$$= L_2(R \cap \{R + (t \cos \theta)\phi\}) \cdot (h - t \sin \theta) = (h - t \sin \theta) \cdot C_R(t \cos \theta, \varphi).$$

Taking into account (3.4) and (2.2) we obtain

$$(3.5) \quad C_D(t\omega) = \begin{cases} (h - t \sin \theta)(ab - t \cos \theta \cdot b_R(\varphi) + \frac{t^2}{4} \cos^2 \theta |\sin 2\varphi|), & \text{if } 0 \leq t \leq x_{\max}(\varphi) \\ 0, & \text{if } t > x_{\max}(\varphi) \end{cases}.$$

4. CHORD LENGTH DISTRIBUTION IN A RIGHT TRAPEZOID

Let $T \subset \mathbb{R}^2$ be the right trapezoid with the vertices at $O(0,0)$, $A(0,a)$, $C(b - a \cot \psi, a)$, and $B(b,0)$, where $\arctan \frac{a}{b} < \psi < \frac{\pi}{2}$. For every right trapezoid one can choose the parameters a, b , and ψ such that it becomes congruent to $OACB$.

In this section we maintain the notations and terminology introduced earlier in Section 2 for any bounded convex set E .

Proposition 4.1. *Let $\pi k \leq \varphi < \pi(k+1)$ for some integer k . Then*

$$b_T(\varphi) = \begin{cases} a |\cos \varphi| + b |\sin \varphi|, & \text{if } \pi k \leq \varphi < \frac{\pi}{2} + \pi k \\ b |\sin \varphi|, & \text{if } \frac{\pi}{2} + \pi k \leq \varphi < \pi(k+1) - \psi \\ a |\cos \varphi| + (b - a \cot \psi) |\sin \varphi|, & \text{if } \pi(k+1) - \psi \leq \varphi < \pi(k+1) \end{cases}.$$

Proof. To reduce the computational burden, from now on we'll use b_1 for the shorter base of T , that is $b_1 = b - a \cot \psi$.

If $\pi k \leq \varphi < \frac{\pi}{2} + \pi k$, then $\Pi_T(\varphi) = \Pi_R(\varphi)$. Therefore, due to (2.4), we have

$$b_T(\varphi) = b_R(\varphi) = a|\cos \varphi| + b|\sin \varphi|.$$

Similarly, if $\pi(k+1) - \psi \leq \varphi < \pi(k+1)$, then $\Pi_T(\varphi) = \Pi_{[0, b_1] \times [0, a]}(\varphi)$, which implies

$$b_T(\varphi) = L_1(\Pi_{[0, b_1] \times [0, a]}(\varphi)) = a|\cos \varphi| + b_1|\sin \varphi|.$$

Finally, if $\frac{\pi}{2} + \pi k \leq \varphi < \pi(k+1) - \psi$, then

$$b_T(\varphi) = L_1(\Pi_{[0, b] \times \{0\}}(\varphi)) = b \cos(\varphi - \pi k - \frac{\pi}{2}) = (-1)^k b \sin \varphi = b|\sin \varphi|.$$

□

Let $\phi_v^\perp$ be the set of vectors $y \in \phi^\perp$ so that the line $l_\varphi + y$ passes through a vertex of trapezoid T and makes a chord of positive Lebesgue measure there. The two quantities introduced below,

$$x_0(\varphi) = \min_{y \in \phi_v^\perp} \chi(l_\varphi + y) \text{ and } x_1(\varphi) = \max_{y \in \phi_v^\perp} \chi(l_\varphi + y),$$

will play a crucial role in determination of distribution function F_T . The diagrams shown in Figure 2 facilitate case-by-case computations (see Proposition 4.2) of the above mentioned quantities.

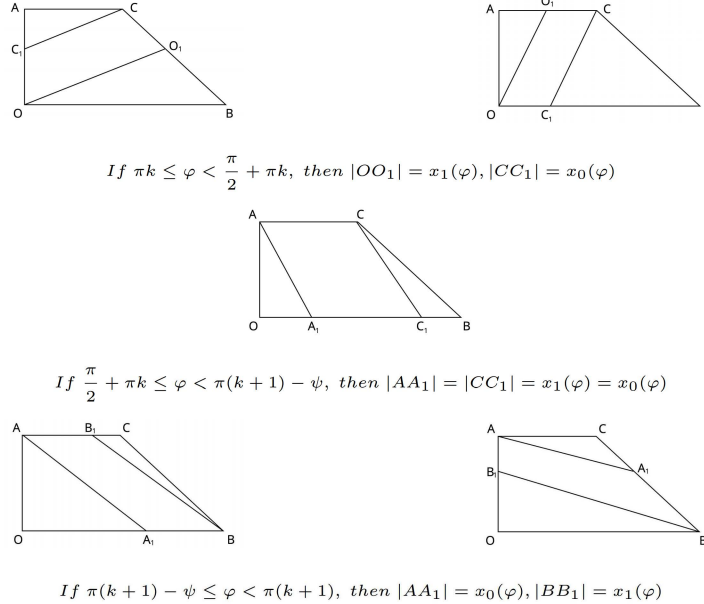


FIGURE 2. Possible dispositions of $x_0(\varphi)$ and $x_1(\varphi)$

Proposition 4.2. $x_1(\varphi) = x_{\max}(\varphi)$ for any angle φ . Furthermore, if for some $k \in \mathbb{Z}$

(i) $\pi k \leq \varphi < \frac{\pi}{2} + \pi k$, then

$$x_0(\varphi) = \begin{cases} \frac{b_1}{|\cos \varphi|}, & \text{if } \pi k \leq \varphi < \pi k + \arctan \frac{a}{b_1} \\ \frac{a}{|\sin \varphi|}, & \text{if } \pi k + \arctan \frac{a}{b_1} \leq \varphi < \frac{\pi}{2} + \pi k \end{cases}$$

$$x_{\max}(\varphi) = \begin{cases} \frac{b \sin \psi}{|\sin(\varphi + \psi)|}, & \text{if } \pi k \leq \varphi < \pi k + \arctan \frac{a}{b_1} \\ \frac{a}{|\sin \varphi|}, & \text{if } \pi k + \arctan \frac{a}{b_1} \leq \varphi < \frac{\pi}{2} + \pi k \end{cases}$$

(ii) $\frac{\pi}{2} + \pi k \leq \varphi < \pi(k+1) - \psi$, then

$$x_0(\varphi) = x_{\max}(\varphi) = \frac{a}{|\sin \varphi|}$$

(iii) $\pi(k+1) - \psi \leq \varphi < \pi(k+1)$, then

$$x_0(\varphi) = \begin{cases} \frac{a}{|\sin \varphi|}, & \text{if } \pi(k+1) - \psi \leq \varphi < \pi(k+1) - \arctan \frac{a}{b} \\ \frac{b_1 \sin \psi}{|\sin(\varphi + \psi)|}, & \text{if } \pi(k+1) - \arctan \frac{a}{b} \leq \varphi < \pi(k+1) \end{cases}$$

$$x_{\max}(\varphi) = \begin{cases} \frac{a}{|\sin \varphi|}, & \text{if } \pi(k+1) - \psi \leq \varphi < \pi(k+1) - \arctan \frac{a}{b} \\ \frac{a}{|\cos \varphi|}, & \text{if } \pi(k+1) - \arctan \frac{a}{b} \leq \varphi < \pi(k+1) \end{cases}$$

Proof. A chord of maximal length in a convex polygon with direction φ , also known as φ -diameter of the polygon, is not necessarily unique (this is also seen in Figure 2: the second, third, and fourth cases) but for any given φ there exists a φ -diameter such that at least one endpoint of the chord coincides with a vertex of the given polygon ([16]). Thus,

$$x_{\max}(\varphi) = \max_{y \in \Pi_R(\varphi)} \chi(l_\varphi + y) = \max_{y \in \phi_\varphi^\perp} \chi(l_\varphi + y) = x_1(\varphi).$$

Case (i), sub-case 1 ($\pi k \leq \varphi < \pi k + \arctan \frac{a}{b_1}$). Observe the first trapezoid in Figure 2. Here we are given $\angle O_1OB = \varphi - \pi k$. By Sine Rule,

$$x_0(\varphi) = |CC_1| = \frac{b_1}{\cos(\varphi - \pi k)} = \frac{b_1}{|\cos \varphi|};$$

$$x_{\max}(\varphi) = |OO_1| = \frac{b \sin \psi}{\sin(\varphi - \pi k + \psi)} = \frac{b \sin \psi}{|\sin(\varphi + \psi)|}.$$

Case (i), sub-case 2 ($\pi k + \arctan \frac{a}{b_1} \leq \varphi < \frac{\pi}{2} + \pi k$). Observe the second trapezoid in Figure 2. Since $\angle AO_1O = \angle O_1OB = \varphi - \pi k$, from the right triangle AO_1O we obtain

$$x_0(\varphi) = x_{\max}(\varphi) = |OO_1| = \frac{a}{\sin(\varphi - \pi k)} = \frac{a}{|\sin \varphi|}.$$

Case (i) is completely proved. The proof of the case (iii) is similar to (i), hence omitted. It remains to discuss the case (ii), where $\frac{\pi}{2} + \pi k \leq \varphi < \pi(k+1) - \psi$ and the third (in the middle) trapezoid in Figure 2 is relevant. In this case we have $\angle AA_1B = \varphi - \pi k \Rightarrow \angle AA_1O = \pi(k+1) - \varphi$, which implies

$$x_0(\varphi) = x_{\max}(\varphi) = |AA_1| = \frac{a}{\sin(\pi(k+1) - \varphi)} = \frac{a}{|\sin \varphi|}.$$

□

By definition, for any angle φ , if $x < 0$ then $F_T(x, \varphi) = 0$, and if $x \geq x_{\max}(\varphi)$ then $F_T(x, \varphi) = 1$. The non-trivial case $0 \leq x < x_{\max}(\varphi)$ is explored below.

Lemma 4.1. $F_T(x, \varphi) = 0$ if $x < 0$ and $F_T(x, \varphi) = 1$ if $x \geq x_{\max}(\varphi)$. For $0 \leq x < x_{\max}(\varphi)$, $F_T(x, \varphi)$ is represented as follows ($k \in \mathbb{Z}$):

(i) For $\pi k \leq \varphi < \frac{\pi}{2} + \pi k$,

$$(4.1) \quad F_T(x, \varphi) = \begin{cases} \frac{x \sin \varphi [\sin(\varphi + \psi) + \cos \varphi \sin \psi]}{b_T(\varphi) \sin \psi}, & \text{if } 0 \leq x < x_0(\varphi) \\ \frac{x \sin^2(\varphi + \psi) - x_0(\varphi) \sin^2 \psi \cos^2 \varphi}{b_T(\varphi) \sin \psi \cos \psi}, & \text{if } x_0(\varphi) \leq x < x_{\max}(\varphi) \end{cases};$$

(ii) if $\frac{\pi}{2} + \pi k \leq \varphi < \pi(k+1) - \psi$,

$$(4.2) \quad F_T(x, \varphi) = \frac{x \sin^2 \varphi \cot \psi}{b_T(\varphi)};$$

(iii) if $\pi(k+1) - \psi \leq \varphi < \pi(k+1)$,

$$(4.3) \quad F_T(x, \varphi) = \begin{cases} \frac{-x \sin \varphi [\sin(\varphi + \psi) + \cos \varphi \sin \psi]}{b_T(\varphi) \sin \psi}, & \text{if } 0 \leq x < x_0(\varphi) \\ \frac{x \cos^2 \varphi \sin^2 \psi - x_0(\varphi) \sin^2(\varphi + \psi)}{b_T(\varphi) \sin \psi \cos \psi}, & \text{if } x_0(\varphi) \leq x < x_{\max}(\varphi) \end{cases}.$$

Proof. Let's note that the function $F_T(x, \cdot)$ is π -periodic. This guarantees that generality will not be lost if we restrict the proof to the case $k = 0$.

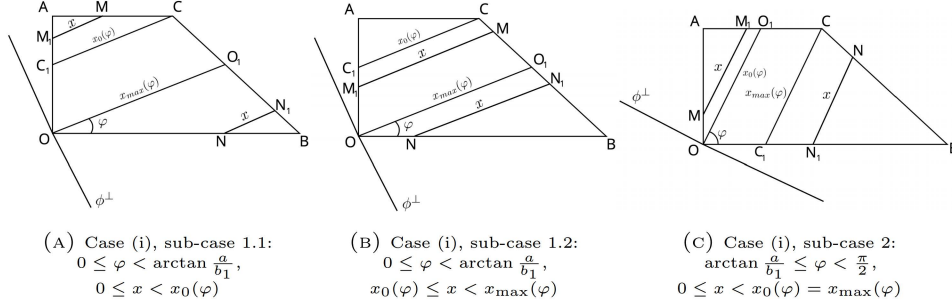
Case (i), sub-case 1.1: let $0 \leq \varphi < \arctan \frac{a}{b_1}$ and $0 \leq x < x_0(\varphi)$.

This case is displayed below in Figure 3(A) which shows $|MM_1| = |NN_1| = x < x_0(\varphi) = |CC_1| < |OO_1| = x_{\max}(\varphi)$.

In this case we have $F_T(x, \varphi) = \frac{1}{b_T(\varphi)} (b_{\Delta AMM_1}(\varphi) + b_{\Delta BNN_1}(\varphi))$. The quantities $b_{\Delta AMM_1}(\varphi)$ and $b_{\Delta BNN_1}(\varphi)$ are equal to the heights of triangles AMM_1 (with base MM_1) and BNN_1 (with base NN_1), respectively. Those are computed below:

$$(4.4) \quad b_{\Delta AMM_1}(\varphi) = x \sin \varphi \cos \varphi.$$

$$(4.5) \quad b_{\Delta BNN_1}(\varphi) = \frac{x \sin \varphi \sin(\varphi + \psi)}{\sin \psi}.$$


 FIGURE 3. Chords in trapezoid in direction ϕ , where $0 \leq \varphi < \frac{\pi}{2}$.

Case (i), sub-case 1.2: let $0 \leq \varphi < \arctan \frac{a}{b_1}$ and $x_0(\varphi) \leq x < x_{\max}(\varphi)$.

This case is displayed in Figure 3(B), where $x_0(\varphi) = |CC_1| \leq x = |MM_1| = |NN_1| < |OO_1| = x_{\max}(\varphi)$. In this case we have $F_T(x, \varphi) = \frac{1}{b_T(\varphi)} (b_{ACMM_1}(\varphi) + b_{\Delta BNN_1}(\varphi)) = \frac{1}{b_T(\varphi)} (b_{\Delta ACC_1}(\varphi) + b_{\Delta BNN_1}(\varphi) + b_{CC_1M_1M}(\varphi)) = \frac{1}{b_T(\varphi)} (x_0(\varphi) \sin \varphi \cos \varphi + \frac{x \sin \varphi \sin(\varphi + \psi)}{\sin \psi} + b_{CC_1M_1M}(\varphi))$. The quantity

$$(4.6) \quad b_{CC_1M_1M}(\varphi) = \frac{(x - x_0(\varphi)) \cos \varphi \sin(\varphi + \psi)}{\cos \psi}$$

is computed from the trapezoid CC_1M_1M with bases CC_1 and MM_1 , as a height.

Case (i), sub-case 2: let $\arctan \frac{a}{b_1} \leq \varphi < \frac{\pi}{2}$ and $0 \leq x < x_0(\varphi) = x_{\max}(\varphi)$.

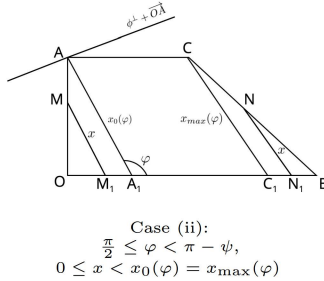
This case is displayed in Figure 3(C), where $x = |MM_1| = |NN_1| < |OO_1| = |CC_1| = x_0(\varphi) = x_{\max}(\varphi)$. Computation of the chord length distribution function in this case is absolutely identical to the case (i), sub-case 1.1.

Combining (4.4), (4.5), and (4.6), for any $\varphi \in [0, \frac{\pi}{2})$ we obtain $F_T(x, \varphi) =$

$$\begin{cases} \frac{x \sin \varphi \cos \varphi + \frac{x \sin \varphi \sin(\varphi + \psi)}{\sin \psi}}{b_T(\varphi)}, & 0 \leq x < x_0(\varphi) \\ \frac{x_0(\varphi) \sin \varphi \cos \varphi + \frac{x \sin \varphi \sin(\varphi + \psi)}{\sin \psi} + \frac{(x - x_0(\varphi)) \cos \varphi \sin(\varphi + \psi)}{\cos \psi}}{b_T(\varphi)}, & x_0(\varphi) \leq x < x_{\max}(\varphi) \end{cases},$$

which is equivalent to (4.1).

Case (ii): let $\frac{\pi}{2} \leq \varphi < \pi - \psi$ and see Figure 4 below.


 FIGURE 4. Chords in trapezoid in direction ϕ , where $\frac{\pi}{2} \leq \varphi < \pi - \psi$.

For $x = |MM_1| = |NN_1| < |AA_1| = |CC_1| = x_0(\varphi) = x_{\max}(\varphi)$ we have

$$\begin{aligned} F_T(x, \varphi) &= \frac{1}{b_T(\varphi)} (b_{\Delta OMM_1}(\varphi) + b_{\Delta BNN_1}(\varphi)) = \\ &= \frac{x \sin(\pi - \varphi) \cos(\pi - \varphi) + \frac{x \sin \varphi \sin(\varphi + \psi)}{\sin \psi}}{b_T(\varphi)} = \frac{x \sin^2 \varphi \cos \psi}{b_T(\varphi) \sin \psi} = \frac{x \sin^2 \varphi \cot \psi}{b_T(\varphi)}. \end{aligned}$$

Case (iii): let $\pi - \psi \leq \varphi < \pi$. Similar to case (i), the distribution function can be computed by three sub-cases, as shown in Figure 5.

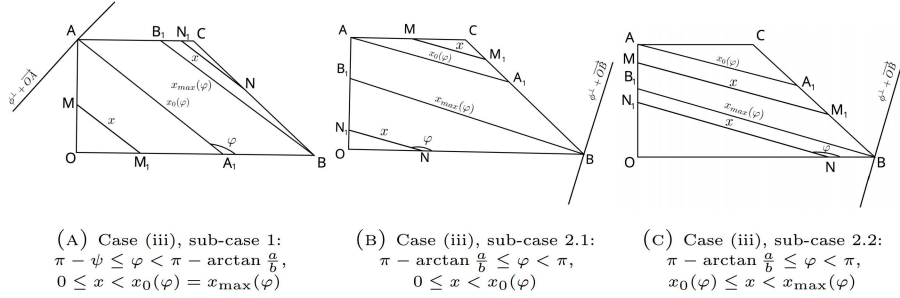


FIGURE 5. Chords in trapezoid in direction ϕ , where $\pi - \psi \leq \varphi < \pi$.

If $\pi - \psi \leq \varphi < \pi$ and $0 \leq x < x_0(\varphi) \leq x_{\max}(\varphi)$, then (sub-cases 1 and 2.1)

$$(4.7) \quad F_T(x, \varphi) = \frac{x \sin(\pi - \varphi) \cos(\pi - \varphi) - \frac{x \sin \varphi \sin(\varphi + \psi)}{\sin \psi}}{b_T(\varphi)}.$$

Inequality $x_0(\varphi) \leq x < x_{\max}(\varphi)$ is possible if only $\pi - \arctan \frac{a}{b} \leq \varphi < \pi$ (sub-case 2.2). The distribution function in this last case is equal to

$$(4.8) \quad F_T(x, \varphi) = \frac{x \sin(\pi - \varphi) \cos(\pi - \varphi) - \frac{x_0(\varphi) \sin \varphi \sin(\varphi + \psi)}{\sin \psi} + \frac{(x - x_0(\varphi)) \cos \varphi \sin(\varphi + \psi)}{\cos \psi}}{b_T(\varphi)}.$$

By simplifying (4.7) and (4.8), we establish (4.3). $\square$

5. COVARIOGRAM OF A RIGHT PRISM WITH RIGHT TRAPEZOIDAL BASE.

COMPUTATION OF CHORD LENGTH DISTRIBUTION FUNCTION

Denote by D_T the right prism $\{(x, y, z) : (x, y) \in T, 0 \leq z \leq h\}$.

Due to the Matheron's formula (see [2]) we have

$$\frac{\partial C_T(t, \varphi)}{\partial t} = -L_1(\{y \in \phi^\perp : L_1(T \cap (l_\varphi + y)) \geq t\}),$$

which can be rewritten in terms of the orientation-dependent chord length distribution function as

$$\frac{\partial C_T(t, \varphi)}{\partial t} = -b_T(\varphi) \cdot [1 - F_T(t, \varphi)].$$

Integration of both parts of the last formula yields

$$(5.1) \quad C_T(t, \varphi) = C_T(0, \varphi) - b_T(\varphi) \cdot \int_0^t [1 - F_T(u, \varphi)] du, \quad t \geq 0.$$

Using the formula (5.1) and Lemma 4.1 we immediately come to an explicit formula for $C_T(t, \varphi)$, the covariogram of T . Since $C_T(t, \cdot)$ has period equal to π , it is enough to have it computed for $\varphi \in [0, \pi)$.

$$C_T(t, \varphi) = \frac{a(b+b_1)}{2} - tb_T(\varphi) + b_T(\varphi) \cdot \int_0^t F_T(u, \varphi) du = \frac{a(b+b_1)}{2} - tb_T(\varphi) +$$

$$\begin{cases} \frac{t^2 \sin \varphi [\sin(\varphi + \psi) + \cos \varphi \sin \psi]}{2 \sin \psi}, & \text{if } 0 \leq \varphi < \frac{\pi}{2}, \quad 0 \leq t < x_0(\varphi) \\ \frac{t^2 \sin^2(\varphi + \psi)}{2 \sin \psi \cos \psi} - \frac{tx_0(\varphi) \sin \psi \cos^2 \varphi}{\cos \psi}, & \text{if } 0 \leq \varphi < \frac{\pi}{2}, \quad x_0(\varphi) \leq t < x_{\max}(\varphi) \\ \frac{t^2}{2} \sin^2 \varphi \cot \psi, & \text{if } \frac{\pi}{2} \leq \varphi < \pi - \psi, \quad 0 \leq t < x_{\max}(\varphi) \\ -\frac{t^2 \sin \varphi [\sin(\varphi + \psi) + \cos \varphi \sin \psi]}{2 \sin \psi}, & \text{if } \pi - \psi \leq \varphi < \pi, \quad 0 \leq t < x_0(\varphi) \\ \frac{t^2 \cos^2 \varphi \sin \psi}{2 \cos \psi} - \frac{tx_0(\varphi) \sin^2(\varphi + \psi)}{\sin \psi \cos \psi}, & \text{if } \pi - \psi \leq \varphi < \pi, \quad x_0(\varphi) \leq t \leq x_{\max}(\varphi) \end{cases}.$$

As a result, for any $t \in [0, x_{\max}(\omega)]$, any $\varphi \in [\pi k, \pi(k+1))$, $k \in \mathbb{Z}$ and any $\theta \in [0, \frac{\pi}{2}]$ the covariogram of D_T is equal to

$$C_{D_T}(t\omega) = L_2(T \cap \{T + (t \cos \theta)\phi\}) \cdot (h - t \sin \theta) = (h - t \sin \theta) \cdot C_T(t \cos \theta, \varphi - \pi k),$$

where $\omega = (\cos \varphi \cos \theta, \sin \varphi \cos \theta, \sin \theta)$ and $x_{\max}(\varphi)$ satisfies (3.4).

Computation of $F_{D_T}(x, \omega)$ requires more workload. Lemma 4.1 shows that if $\pi k \leq \varphi < \frac{\pi}{2} + \pi k$ or $\pi(k+1) - \psi \leq \varphi < \pi(k+1)$ then the function $F_T(x, \varphi)$ is piecewise linear. To have those pieces written in slope-intercept form let's denote

$$m_0(\varphi) = \frac{\sin \varphi [\sin(\varphi + \psi) + \cos \varphi \sin \psi]}{b_T(\varphi) \sin \psi}, \quad m_1(\varphi) = \frac{\sin^2(\varphi + \psi)}{b_T(\varphi) \sin \psi \cos \psi},$$

$$c(\varphi) = \frac{x_0(\varphi) \tan \psi \cos^2 \varphi}{b_T(\varphi)}.$$

Then (4.1) and (4.3) can be represented as

$$(5.2) \quad F_T(x, \varphi) = \begin{cases} m_0(\varphi)x, & \text{if } 0 \leq x < x_0(\varphi) \\ m_1(\varphi)x - c(\varphi), & \text{if } x_0(\varphi) \leq x < x_{\max}(\varphi) \end{cases}$$

and

$$(5.3) \quad F_T(x, \varphi) = \begin{cases} -m_0(\varphi)x, & \text{if } 0 \leq x < x_0(\varphi) \\ c(\varphi)x - m_1(\varphi), & \text{if } x_0(\varphi) \leq x < x_{\max}(\varphi) \end{cases},$$

respectively.

If $t \leq 0$ then $F_{D_T}(t, \omega) = 0$, and if $t \geq x_{\max}(\omega)$ then $F_{D_T}(t, \omega) = 1$. If $0 < t < x_{\max}(\omega)$, then F_{D_T} (unlike F_D) requires several pieces to be written explicitly. The classification of cases is based on two factors:

- 1) the choice of either of the intervals $[\pi k, \frac{\pi}{2} + \pi k)$, $[\frac{\pi}{2} + \pi k, \pi(k+1) - \psi)$, or $[\pi(k+1) - \psi, \pi(k+1))$ for φ to be taken from;
- 2) the magnitude of the orthogonal projection of $x_{\max}(\omega)$ onto the base T compared to $x_0(\varphi)$ for the given φ and θ .

The result is formulated below.

Theorem 5.1.

(i) If $\pi k \leq \varphi < \frac{\pi}{2} + \pi k$ and $x_{\max}(\omega) \cos \theta \leq x_0(\varphi)$, then

$$F_{D_T}(t, \omega) = \frac{b_T(\varphi)}{b_{D_T}(\omega)} \left((hm_0(\varphi) \cos^2 \theta + \sin 2\theta) \cdot t - \frac{3}{4} m_0(\varphi) \sin 2\theta \cos \theta \cdot t^2 \right),$$

$$0 < t < x_{\max}(\omega).$$

(ii) If $\pi k \leq \varphi < \frac{\pi}{2} + \pi k$ and $x_{\max}(\omega) \cos \theta > x_0(\varphi)$, then

$$F_{D_T}(t, \omega) = \frac{b_T(\varphi)}{b_{D_T}(\omega)} \times$$

$$\begin{cases} (hm_0(\varphi) \cos^2 \theta + \sin 2\theta) \cdot t - \frac{3}{4} m_0(\varphi) \sin 2\theta \cos \theta \cdot t^2, & \text{if } 0 < t < \frac{x_0(\varphi)}{\cos \theta} \\ -c(\varphi)(x_0(\varphi) \sin \theta + h \cos \theta) - \frac{m_0(\varphi) - m_1(\varphi)}{2} x_0^2(\varphi) \sin \theta + \\ + (hm_1(\varphi) \cos^2 \theta + [c(\varphi) + 1] \sin 2\theta) \cdot t - \\ - \frac{3}{4} m_1(\varphi) \sin 2\theta \cos \theta \cdot t^2, & \text{if } \frac{x_0(\varphi)}{\cos \theta} \leq t < x_{\max}(\omega) \end{cases}.$$

(iii) If $\frac{\pi}{2} + \pi k \leq \varphi < \pi(k+1) - \psi$, then

$$F_{D_T}(t, \omega) = \frac{h \cos^2 \theta \sin^2 \varphi \cot \psi + b_T(\varphi) \sin 2\theta}{b_{D_T}(\omega)} \cdot t - \frac{\sin 2\theta \sin^2 \varphi \cot \psi [2 \cos \theta + 1]}{4b_{D_T}(\omega)} \cdot t^2,$$

$$0 < t < x_{\max}(\omega).$$

(iv) If $\pi(k+1) - \psi \leq \varphi < \pi(k+1)$ and $x_{\max}(\omega) \cos \theta \leq x_0(\varphi)$, then

$$F_{D_T}(t, \omega) = \frac{b_T(\varphi)}{b_{D_T}(\omega)} \left((\sin 2\theta - hm_0(\varphi) \cos^2 \theta) \cdot t + \frac{3}{4} m_0(\varphi) \sin 2\theta \cos \theta \cdot t^2 \right),$$

$$0 < t < x_{\max}(\omega).$$

(v) If $\pi(k+1) - \psi \leq \varphi < \pi(k+1)$ and $x_{\max}(\omega) \cos \theta > x_0(\varphi)$, then

$$F_{D_T}(t, \omega) = \frac{b_T(\varphi)}{b_{D_T}(\omega)} \times$$

$$\begin{cases} (\sin 2\theta - hm_0(\varphi) \cos^2 \theta) \cdot t + \frac{3}{4} m_0(\varphi) \sin 2\theta \cos \theta \cdot t^2, & \text{if } 0 < t < \frac{x_0(\varphi)}{\cos \theta} \\ -m_1(\varphi)(x_0(\varphi) \sin \theta + h \cos \theta) - \frac{m_0(\varphi) - c(\varphi)}{2} x_0^2(\varphi) \sin \theta + \\ + (hc(\varphi) \cos^2 \theta + [m_1(\varphi) + 1] \sin 2\theta) \cdot t - \\ - \frac{3}{4} c(\varphi) \sin 2\theta \cos \theta \cdot t^2, & \text{if } \frac{x_0(\varphi)}{\cos \theta} \leq t < x_{\max}(\omega) \end{cases}.$$

Proof. Case (i): Since $t \cos \theta \leq x_0(\varphi)$, (5.2) yields $F_T(\varphi) = tm_0(\varphi) \cos \theta$. Then (3.3) outputs

$$(5.4) \quad \begin{aligned} F_{D_T}(t, \omega) &= \frac{b_T(\varphi) \cos \theta}{b_{D_T}(\omega)} \times \\ &\times \left((h - t \sin \theta) \cdot tm_0(\varphi) \cos \theta + 2t \sin \theta - \sin \theta \int_0^t u \cdot m_0(\varphi) \cos \theta du \right) = \\ &= \frac{b_T(\varphi)}{b_{D_T}(\omega)} \left((hm_0(\varphi) \cos^2 \theta + \sin 2\theta) \cdot t - \frac{3}{4} m_0(\varphi) \sin 2\theta \cos \theta \cdot t^2 \right). \end{aligned}$$

Case (ii): If $0 < t < \frac{x_0(\varphi)}{\cos \theta}$, then the formula (5.4) still works. But if $\frac{x_0(\varphi)}{\cos \theta} \leq t < x_{\max}(\omega)$, then there are two expressions for $F_T(u \cos \theta, \varphi)$ to be used under the integral in (3.3). Due to (5.2), those pieces are

$$(5.5) \quad F_T(u \cos \theta, \varphi) = \begin{cases} u \cdot m_0(\varphi) \cos \theta, & \text{if } u < \frac{x_0(\varphi)}{\cos \theta} \\ u \cdot m_1(\varphi) \cos \theta - c(\varphi), & \text{if } \frac{x_0(\varphi)}{\cos \theta} \leq u. \end{cases}$$

Therefore we get

$$\begin{aligned} F_{D_T}(t, \omega) &= \frac{b_T(\varphi) \cos \theta}{b_{D_T}(\omega)} \left[(h - t \sin \theta)(m_1(\varphi)t \cos \theta - c(\varphi)) + 2t \sin \theta - \right. \\ &\quad \left. - \sin \theta \int_0^{\frac{x_0(\varphi)}{\cos \theta}} m_0(\varphi) \cos \theta u du - \sin \theta \int_{\frac{x_0(\varphi)}{\cos \theta}}^t [m_1(\varphi) \cos \theta u - c(\varphi)] du \right] = \\ &= \frac{b_T(\varphi) \cos \theta}{b_{D_T}(\omega)} \left[-hc(\varphi) - \frac{m_0(\varphi) \tan \theta}{2} x_0^2(\varphi) + hm_1(\varphi) \cos \theta \cdot t + \sin \theta c(\varphi)t - \right. \\ &\quad \left. - \sin \theta \cos \theta m_1(\varphi)t^2 + 2t \sin \theta - \sin \theta \cdot \left(\frac{m_1(\varphi) \cos \theta t^2}{2} - c(\varphi) \cdot t - m_1(\varphi) \frac{x_0^2(\varphi)}{2 \cos \theta} + \right. \right. \\ &\quad \left. \left. + c(\varphi) \frac{x_0(\varphi)}{\cos \theta} \right) \right], \end{aligned}$$

and finally

$$(5.6) \quad \begin{aligned} F_{D_T}(t, \omega) &= -c(\varphi)(x_0(\varphi) \sin \theta + h \cos \theta) - \frac{m_0(\varphi) - m_1(\varphi)}{2} x_0^2(\varphi) \sin \theta + \\ &\quad (hm_1(\varphi) \cos^2 \theta + [c(\varphi) + 1] \sin 2\theta) \cdot t - \frac{3}{4} m_1(\varphi) \sin 2\theta \cos \theta \cdot t^2. \end{aligned}$$

Case (iii): Using (4.2) in (3.3), we obtain

$$\begin{aligned} F_{D_T}(t, \omega) &= \frac{b_T(\varphi) \cos \theta}{b_{D_T}(\omega)} \left[(h - t \sin \theta) \cdot \frac{t \cos \theta \sin^2(\varphi) \cot \psi}{b_T(\varphi)} + 2t \sin \theta - \sin \theta \cdot \frac{t^2}{2} \times \right. \\ &\quad \left. \times \frac{\sin^2 \varphi \cot \psi}{b_T(\varphi)} \right] = \frac{b_T(\varphi) \cos \theta}{b_{D_T}(\omega)} \left[\left(\frac{h \cos \theta \sin^2 \varphi \cot \psi}{b_T(\varphi)} + 2 \sin \theta \right) \cdot t - \right. \\ &\quad \left. - \left(\frac{\sin \theta \cos \theta \sin^2 \varphi \cot \psi}{b_T(\varphi)} + \frac{\sin \theta \sin^2 \varphi \cot \psi}{2b_T(\varphi)} \right) \cdot t^2 \right] = \\ &= \frac{h \cos^2 \theta \sin^2 \varphi \cot \psi + b_T(\varphi) \sin 2\theta}{b_{D_T}(\omega)} \cdot t - \frac{\sin 2\theta \sin^2 \varphi \cot \psi [2 \cos \theta + 1]}{4b_{D_T}(\omega)} \cdot t^2. \end{aligned}$$

Cases (iv) and (v): Let's notice that replacing $m_0(\varphi)$ by $-m_0(\varphi)$ and interchanging $c(\varphi)$ with $m_1(\varphi)$ in (5.2) will produce (5.3). Applying the mentioned changes to (5.4) and (5.6), we complete the proof of the theorem in cases (iv) and (v), respectively. $\square$

Remark 5.1. It follows from (3.2) that $b_{D_T}(\omega) = \frac{a(b+b_1)}{2} + b_T(\varphi)h \cos \theta$. This expansion has not been used in the theorem.

Remark 5.2. When $\theta = 0$, one can check that $F_{D_T} \equiv F_T$. If $\theta = \frac{\pi}{2}$, then F_{D_T} is a step function, which can be seen in Figure 6, example (C). The case (D) illustrates an example where the graph of the distribution function comprises of 4 pieces, 2 horizontal lines and 2 arcs of parabolas.

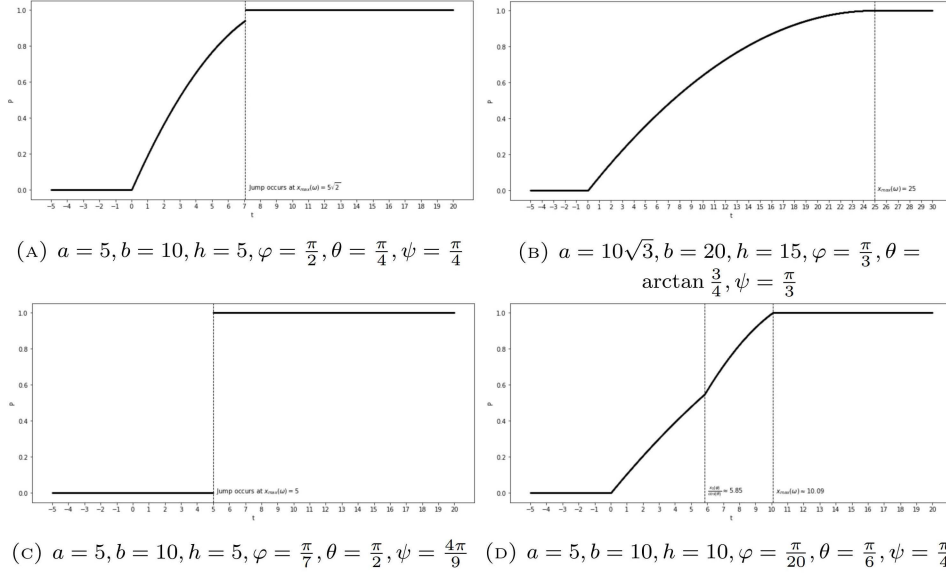


FIGURE 6. Orientation dependent chord length distribution function F_{D_T} for different cases

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СОДЕРЖАНИЕ

N. G. AHARONYAN, V. KHALATYAN, Distribution of the distance between two random points in a body from R^n	3
R. V. AMBARTZUMIAN, On continuity of Buffon functionals in the space of planes in $\mathbb{R}^3$	10
J. ARAMAKI, On the de Rham theorem and an application to the Maxwell-Stokes type problem	23
Г. Г. ГЕВОРГЯН, О единственности тригонометрических рядов	37
М. Г. ГРИГОРЯН, Функции универсальные относительно системы Уолша	51
V. K. OHANYAN, D. M. MARTIROSYAN, Orientation-dependent chord length distribution function for right prisms with rectangular or right trapezoidal bases	68 – 82

IZVESTIYA NAN ARMENII: MATEMATIKA

Vol. 55, No. 6, 2020

CONTENTS

N. G. AHARONYAN, V. KHALATYAN, Distribution of the distance between two random points in a body from R^n	3
R. V. AMBARTZUMIAN, On continuity of Buffon functionals in the space of planes in $\mathbb{R}^3$	10
J. ARAMAKI, On the de Rham theorem and an application to the Maxwell-Stokes type problem	23
G. G. GEVORGYAN, On a uniqueness of trigonometric series	37
M. G. GRIGORYAN, Functions universal with respect to Walsh's system	51
V. K. OHANYAN, D. M. MARTIROSYAN, Orientation-dependent chord length distribution function for right prisms with rectangular or right trapezoidal bases	68 – 82