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RESULTS ON MEROMORPHIC FUNCTION SHARING TWO SETS WITH ITS LINEAR c-DIFFERENCE OPERATOR

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Abstract. In this paper, the existing results concerning difference operator sharing two sets have been extended up to the most general form, namely linear difference operator. Furthermore, we have been able to find out the specific form of the function. A considerable number of examples have been exhibited throughout the paper pertinent with different issues.

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1. Introduction, definitions and results

Through out the paper, the term "meromorphic (resp. entire)"will always mean meromorphic in the whole complex plane $\mathbb{C}$ which are non-constant, unless specifically stated otherwise. We shall adopt the standard notations of the Nevanlinna's value distribution theory of meromorphic functions from ([9, 16]). For such a meromorphic function f and $a \in \overline{\mathbb{C}} =: \mathbb{C} \cup \{\infty\}$, each z with $f(z) = a$ will be called a-point of f. We denote $\mathbb{C}^*$ by $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

In 1926, *Nevanlinna* first showed that a non-constant meromorphic function on the complex plane $\mathbb{C}$ is uniquely determined by the pre-images, ignoring multiplicities, of five distinct values (including infinity). The beauty of this result lies in the fact that there is no counterpart of this result in the real function theory. A few years latter, he showed that when multiplicities are taken into consideration, four points are enough and in that case either the two functions coincides or one is the bilinear transformation of the other one. Clearly these results initiated the study of uniqueness of two meromorphic functions f and g. The study becomes more interesting if the function g is related with f.

If for $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a *CM* (Counting Multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (Ignoring Multiplicities).

Definition 1.1. For a non-constant meromorphic function f and any set $\mathcal{S} \subset \overline{\mathbb{C}}$, we define

$$E_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \Big\{(z,p) \in \mathbb{C}\times\mathbb{N} : f(z) = a, \text{ with multiplicity } p\Big\},$$

$$\overline{E}_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \Big\{(z,1) \in \mathbb{C}\times\{1\} : f(z) = a\Big\}.$$

If $E_f(\mathcal{S}) = E_g(\mathcal{S})$ ($\overline{E}_f(\mathcal{S}) = \overline{E}_g(\mathcal{S})$) then we simply say f and g share $\mathcal{S}$ Counting Multiplicities(CM) (Ignoring Multiplicities(IM)).

More formally it can be explained as follows.

Definition 1.2. [3] If f is a meromorphic function and $S \subset \overline{\mathbb{C}}$ then if $z_0 \in f^{-1}(S)$, the value of $E_f(S)$ at the point z_0 is denoted by $E_f(S)(z_0) : f^{-1}(S) \to \mathbb{N}$ and is equal to the multiplicity of zero of the function $f(z) - f(z_0)$ at z_0 i.e. the order of the pole of the function $(f(z) - f(z_0))^{-1}$ at z_0 if $f(z_0) \in \mathbb{C}$ (resp. of the function $f(z)$ if z_0 is a pole for f).

Evidently, if $\mathcal{S}$ contains one element only, then it coincides with the usual definition of $CM(IM)$ sharing of values.

In 2001, an idea of gradation of sharing known as weighted sharing has been introduced by *Lahiri* [11, 12] which measure how close a shared value is to being share CM or to being shared IM. So for the purpose of relaxing the nature of sharing the sets, the notion of weighted sharing of values and sets, has become an effective tool.

Recently, the definition have been reorganized by us [3] as follows.

Definition 1.3. [3] For $k \in \overline{\mathbb{N}}$ and $z_0 \in f^{-1}(S)$, let us put that $E_f(S,k)(z_0) = \min\{E_f(S)(z_0), k+1\}$. Given $S \subset \overline{\mathbb{C}}$, we say that meromorphic functions f and g share the set S up to multiplicity k (or share S with weight k, or simply share (S,k)) if $f^{-1}(S) = g^{-1}(S)$ and for each $z_0 \in f^{-1}(S)$ we have $E_f(S,k)(z_0) = E_g(S,k)(z_0)$, which is represented by the notation $E_f(S,k) = E_g(S,k)$.

As we proceed through the literature of the shift and difference operators of a meromorphic function f, we feel that there should be a streamline in the definitions. This is one of the motivations of writing this paper. To this end, below we are providing several definitions in a compact and convenient way.

In what follows, c always means a non-zero constant. We now define the shift and difference operator in the following manner.

Definition 1.4. For a meromorphic function f, let us now denote its shift I_cf and difference operators Δ_cf respectively by $I_cf(z) = f(z+c)$ and $\Delta_cf(z) = (I_c - 1)f(z) = f(z+c) - f(z)$.

Next we define $\Delta_c^s f := \Delta_c^{s-1}(\Delta_c f), \forall s \in \mathbb{N} - \{1\}$.

For the purpose of generalizing the above definitions, we now propose the definition of linear shift operator $\mathcal{L}_p(f, I)$ as follows.

Definition 1.5. For a meromorphic function f and a positive integer p, we define

$$\begin{aligned} \mathcal{L}_p(f,I) &= a_pI_{c_p}f(z) + a_{p-1}I_{c_{p-1}}f(z) + \ldots + a_0I_{c_0}f(z) \\ &= a_pf(z+c_p) + \ldots + a_1f(z+c_1) + a_0f(z+c_0), \end{aligned} \tag{1.1}$$

$a_p(\neq 0), \ldots, a_1, a_0 \in \mathbb{C}$, $c_p, \ldots, c_1, c_0 \in \mathbb{C}$.

In particular, for suitable choice of c_j, say $c_j = jc$, for $j \in \{0, 1, \ldots, p\}$, we call $\mathcal{L}_p(f, I)$ as a linear c-shift operator $\mathcal{L}_p(f, I_c)$ as follows.

Definition 1.6. For $c \in \mathbb{C}^*$ and a positive integer p, we define

$$\begin{aligned} \mathcal{L}_p(f,I_c) &= a_pI_{pc}f(z) + a_{p-1}I_{(p-1)c}f(z) + \ldots + a_0I_0f(z) \\ &= a_pf(z+pc) + a_{p-1}f(z+(p-1)c) + \ldots + a_0f(z). \end{aligned} \tag{1.2}$$

Analogous to the definitions 1.5 and 1.6, we now introduce the definitions of linear difference operator $\mathcal{L}_p(f, \Delta)$ and linear c-difference operator $\mathcal{L}_p(f, \Delta_c)$ in the following manner.

Definition 1.7.

$$\begin{aligned} \mathcal{L}_p(f,\Delta) = a_p\Delta_{c_p}f(z) + a_{p-1}\Delta_{c_{p-1}}f(z) + \ldots + a_0\Delta_{c_0}f(z) = \\ a_pf(z+c_p) + \ldots + a_1f(z+c_1) + a_0f(z+c_0) - \left(\sum_{j=0}^{p} a_j\right) f(z) \\ = \mathcal{L}_p(f,I) - \left(\sum_{j=0}^{p} a_j\right) f(z), \end{aligned} \tag{1.3}$$

Definition 1.8. For $c \in \mathbb{C}^*$, a positive integer p, putting $c_j = jc$, $j \in \{0, 1, \ldots, p\}$, in (1.3) we define

$$\begin{aligned} \mathcal{L}_p(f,\Delta_c) = a_p\Delta_{pc}f(z) + a_{p-1}\Delta_{(p-1)c}f(z) + \ldots + \\ + a_1\Delta_cf(z) + a_0\Delta_0f(z) = \mathcal{L}_p(f,I_c) - \left(\sum_{j=0}^{p} a_j\right) f(z). \end{aligned} \tag{1.4}$$

For the specific choices of the constants as $a_j = (-1)^{p-j}\binom{p}{j}$, where $0 \leqslant j \leqslant p$, in the expression $\mathcal{L}_p(f, \Delta_c)$, one can easily get that $\mathcal{L}_p(f, \Delta_c) = \Delta_c^p f$.

For the sake of convenience, we are now going to introduce linear c-difference odd operator $\mathcal{L}_p^o(f, \Delta_c)$ as follows:

Definition 1.9. For $c \in \mathbb{C}^*$, putting $c_j = (2j+1)c$, $j \in \{0, 1, \ldots, p\}$, in (1.3) we define,

$$(1.5)\mathcal{L}_p^o(f, \Delta_c) = a_p \Delta_{(2p+1)c} f(z) + a_{p-1} \Delta_{(2p-1)c} f(z) + \ldots + a_1 \Delta_1 f(z) + +a_0 \Delta_c f(z).$$

Henceforth unless otherwise stated for $a \neq 0$, throughout the paper, we denote, for $n \in \mathbb{N}$, by $\mathcal{S}_a^n = \{a, a\theta, a\theta^2, \ldots, a\theta^{n-1}\}$, where $\theta = \exp\left(\frac{2\pi i}{n}\right)$, and $S_2 = \{\infty\}$.

Recently a number of papers ([6, 8, 21] etc.) have focused on the value distribution in difference analogues of meromorphic functions.

In this perspective, many researchers have become interested to deal with the uniqueness problem of meromorphic function that share values or sets with its shift or difference operators. Below we are mentioning few of them.

Theorem A. [21] *Let $c \in \mathbb{C}^*$, and suppose that $f(z)$ is a non-constant meromorphic function with finite order such that $E_f(\mathcal{S}_1^n, \infty) = E_{I_c f}(\mathcal{S}_1^n, \infty)$ and $E_f(\mathcal{S}_2, \infty) = E_{I_c f}(\mathcal{S}_2, \infty)$. If $n \geqslant 4$, then $I_c f \equiv t f$, where $t^n = 1$.*

The following example shows that *Theorem A* is not valid for 'infinite ordered' meromorphic function.

Example 1.1. Let $c \in \mathbb{C}^*$ and $f(z) = \exp\left(\sin\left(\frac{\pi z}{c}\right)\right)$. It is clear that $I_c f = \exp\left(-\sin\left(\frac{\pi z}{c}\right)\right)$. It is easy to verify that $E_f(\mathcal{S}_1^n, \infty) = E_{I_c f}(\mathcal{S}_1^n, \infty)$ and $E_f(\mathcal{S}_2, \infty) = E_{I_c f}(\mathcal{S}_2, \infty)$ for any value of $n \in \mathbb{N}$ but the conclusion of *Theorem A* ceases to hold.

Example 1.2. Let $c \in \mathbb{C}^*$ and $f(z) = \exp\left(\exp\left(\frac{\pi i z}{c}\right)\right)$. It is clear that $I_c f = \exp\left(-\exp\left(\frac{\pi i z}{c}\right)\right)$. It is easy to verify that $E_f(\mathcal{S}_1^n, \infty) = E_{I_c f}(\mathcal{S}_1^n, \infty)$ and $E_f(\mathcal{S}_2, \infty) = E_{I_c f}(\mathcal{S}_2, \infty)$ for any value of $n \in \mathbb{N}$ but the conclusion of *Theorem A* ceases to hold.

The next examples show that for $n = 1$ or $n = 2$ *Theorem A* is not true.

Example 1.3. Let $f(z) = \dfrac{e^{\mathcal{B}z} + \sin^2\left(\frac{2\pi z}{c}\right) - 1}{\sin^2\left(\frac{2\pi z}{c}\right) - 1}$, where $e^{\mathcal{B}c} = -1$. It easy to verify that $E_f(\mathcal{S}_1^1, \infty) = E_{I_c f}(\mathcal{S}_1^1, \infty)$ and $E_f(\mathcal{S}_2, \infty) = E_{I_c f}(\mathcal{S}_2, \infty)$ but $I_c f \not\equiv f$.

Example 1.4. Let $f(z) = \dfrac{\exp\left(\frac{\pi i z}{2c}\right) - \exp\left(-\frac{\pi i z}{2c}\right) a^2}{\sqrt{2} i a}$, where a is a non-zero constant. It easy to verify that $E_f(\mathcal{S}_1^2, \infty) = E_{I_c f}(\mathcal{S}_1^2, \infty)$ and $E_f(\mathcal{S}_2, \infty) = E_{I_c f}(\mathcal{S}_2, \infty)$ but $I_c f \not\equiv f$.

By replacing $I_c f$ by $\Delta_c f$ in *Theorem A*, *Chen - Chen* [5] obtained the following result.

Theorem B. [5] *Let $c \in \mathbb{C}^*$ and $\mathcal{S}_a^n$ and $\mathcal{S}_2$ be defined as in* Theorem A. *Suppose that $f(z)$ is a non-constant meromorphic function with finite order such that $E_f(\mathcal{S}_a^n, 2) = E_{\Delta_c f}(\mathcal{S}_a^n, 2)$ and $E_f(\mathcal{S}_2, \infty) = E_{\Delta_c f}(\mathcal{S}_2, \infty)$. If $n \geqslant 7$, then $\Delta_c f \equiv t f$, where $t^n = 1$ with $t \neq -1$.*

In this direction, *Banerjee - Bhattacharyya* [4] successfully reduced the weight of the sets as well as the lower bound of n in *Theorem B*, by obtaining the following two results.

Theorem C. [4] *Suppose that f is a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_b^n, 2) = E_{\Delta_c f}(\mathcal{S}_b^n, 2)$, where $b^n = a \in \mathbb{C}^*$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c f}(\mathcal{S}_2, 0)$, and $n \geqslant 6$. Then there is a constant $t \in \mathbb{C}$ such that $\Delta_c f \equiv t f$, where $t^n = 1$ and $t \neq -1$.*

Theorem D. *Suppose that f is a non-constant meromorphic function of finite order, $\mathcal{S}_b^n$ be defined as in Theorem C, and such that $E_f(\mathcal{S}_b^n, 1) = E_{\Delta_c f}(\mathcal{S}_b^n, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c f}(\mathcal{S}_2, 0)$, and $n \geqslant 7$. Then there is a constant $t \in \mathbb{C}$ such that $\Delta_c f \equiv t f$, where $t^n = 1$ and $t \neq -1$.*

The following examples show that the condition 'finite orderedness' of the function f is not necessary in *Theorems B, C, D.*

Example 1.5. For a complex number t $(\neq -1)$, let

$$f(z) = \frac{\exp\left(\frac{z}{c}\log(t^{\frac{1}{p}} + 1)\right)}{\exp\left(\exp\left(\dfrac{2\pi i z}{c}\right)\right) - 1}$$

It is easy to verify that $\Delta_c^p f \equiv t f$, for all positive integer p. As t is a complex constant satisfying $t^n = 1$, it follows that $(\Delta_c^p f)^n - 1 \equiv f^n - 1$. Hence $E_{\Delta_c^p f}(\mathcal{S}_1^n, \infty) = E_f(\mathcal{S}_1^n, \infty)$ and $E_{\Delta_c^p f}(\mathcal{S}_2, \infty) = E_f(\mathcal{S}_2, \infty)$.

In the same manner more examples can be formed as follows:

Example 1.6. Let $f(z) = \dfrac{\exp\left(\frac{z}{c}\log(t^{\frac{1}{p}} + 1)\right)\sin\left(\frac{2\pi z}{c}\right)}{\exp\left(\sin\left(\frac{2\pi z}{c}\right)\right) - 1}$.

Example 1.7. Let $f(z) = \dfrac{\exp\left(\frac{z}{c}\log(t^{\frac{1}{p}}+1)\right)\cos\left(\frac{2\pi z}{c}\right)}{\exp\left(\cos\left(\frac{2\pi z}{c}\right)\right)-1}$.

Example 1.8. Let $f(z) = \dfrac{\exp\left(\frac{z}{c}\log(t^{\frac{1}{p}}+1)\right)\exp\left(\frac{2k\pi i z}{c}\right)}{\exp\left(\exp\left(\dfrac{2\pi i z}{c}\right)\right)-1}$.

Recently, in this direction *Deng - Liu - Yang* [7] obtained the following result.

Theorem E. [7] *Let $c \in \mathbb{C}^*$ and $\mathcal{S}_a^n$, $\mathcal{S}_2$ be defined as in Theorem A. Suppose that $f(z)$ is a non-constant meromorphic function such that $E_f(\mathcal{S}_a^n, k) = E_{\Delta_c f}(\mathcal{S}_a^n, k)$ and $E_f(\mathcal{S}_2, \infty) = E_{\Delta_c f}(\mathcal{S}_2, \infty)$. If $n \geqslant 7$, when $k = 1$ or $n \geqslant 5$, when $k \geqslant 2$, then $\Delta_c f \equiv tf$, where $t^n = 1$ with $t \neq -1$.*

Remark 1.1. We know that all the lemmas and hence the corresponding results so far obtained based on the lemmas related to a function and its shift $I_c f$ or $\Delta_c f$ are for finite ordered meromorphic functions only, so we have a strong doubt about the validity of *Theorem E* for the case of "infinite ordered"meromorphic function.

For the purpose of further improvements as well as extensions of *Theorems B, C, D, E*, we propose the following questions.

(i). Can we replace the difference operator $\Delta_c f$ by a more general setting $\mathcal{L}_p(f, \Delta_c)$ in *Theorem B, C, D, E* ?

(ii). Is it possible to relax the nature of sharing $(\mathcal{S}_2, \infty)$ in *Theorems B, E* further by $(\mathcal{S}_2, 0)$?

In this paper, we have answered the above questions affirmatively. Followings are the main result of this paper.

Theorem 1.1. *Let n, $p \in \mathbb{N}$, and f be a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_a^n, 1) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_a^n, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_2, 0)$. If $n \geqslant \max\left\{p+4,\ 7\right\}$, then there exists a constant $t \in \mathbb{C}$ such that $\mathcal{L}_p(f, \Delta_c) \equiv tf$, where $t^n = 1$ and $t \neq -1$.*

Theorem 1.2. *Let n, $p \in \mathbb{N}$, and f be a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_a^n, 2) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_a^n, 2)$ and $E_f(\mathcal{S}_2, 0) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_2, 0)$. If $n \geqslant \max\left\{p+3,\ 6\right\}$, then the conclusion of* Theorem 1.1 *holds.*

Remark 1.2. Since $I_c f$, $\Delta_c f$ and $\mathcal{L}_p(f, I_c)$ are the very special forms of $\mathcal{L}_p(f, \Delta_c)$, so it is clear that *Theorem 1.1* and *Theorem 1.2* improved and extended the *Theorems B, C, D* and *E* in a large extent.

Let us denote by $\mathbb{P}_c$ as the field of periods c $(\neq 0)$ of meromorphic functions defined in $\mathbb{C}$. That means

$$\mathbb{P}_c = \{g : g \text{ is meromorphic and } g(z+c) = g(z),\ \forall z \in \mathbb{C}\}.$$

From *Theorem 1.1* and *Theorem 1.2*, we can now easily deduce the following *Corollaries*:

Corollary 1.1. *Let $n,\ s \in \mathbb{N}$, and f be a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_1^n, 1) = E_{\Delta_c^s f}(\mathcal{S}_1^n, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^s f}(\mathcal{S}_2, 0)$. If $n \geqslant \max\left\{s+4, 7\right\}$, then there exists a constant $t \in \mathbb{C}$ such that $\overset{s}{\Delta}_c f \equiv tf$, where $t^n = 1$ and $t \neq -1$.*

Corollary 1.2. *Let $n,\ s \in \mathbb{N}$, and f be a non-constant meromorphic function of finite order such that $E_f(\mathcal{S}_1^n, 2) = E_{\Delta_c^s f}(\mathcal{S}_1^n, 2)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^s f}(\mathcal{S}_2, 0)$. If $n \geqslant \max\left\{s+3, 6\right\}$, then the conclusion of* Corollary 1.1 *holds.*

Remark 1.3. From *Examples 1.1* and *1.2* , we see that *Corollaries 1.1* and *1.2* are not valid for 'infinite ordered' meromorphic functions for the case $s = 1$, $a_1 = 1$, $a_0 = 0$.

Corollary 1.3. *Let s, where $1 \leqslant s \leqslant 3$, be an integer and f be a non-constant meromorphic function of finite order. Suppose $E_f(\mathcal{S}_1^7, 1) = E_{\Delta_c^s f}(\mathcal{S}_1^7, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^s f}(\mathcal{S}_2, 0)$. Then there exists a constant $t \in \mathbb{C}$ such that $\overset{s}{\Delta}_c f \equiv tf$, where $t^7 = 1$ and $t \neq -1$.*

Corollary 1.4. *Let s, where $1 \leqslant s \leqslant 3$, be an integer and f be a non-constant meromorphic function of finite order. Suppose $E_f(\mathcal{S}_1^6, 1) = E_{\Delta_c^s f}(\mathcal{S}_1^6, 1)$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^s f}(\mathcal{S}_2, 0)$. Then there exists a constant $t \in \mathbb{C}$ such that $\overset{s}{\Delta}_c f \equiv tf$, where $t^6 = 1$ and $t \neq -1$.*

From the following three examples we see that the conclusion of *Corollary 1.3* and *Corollary 1.4* actually occurs for the case $s = 1$, $s = 2$ and $s = 3$.

Example 1.9. Let $f(z) = (1+\zeta)^{z/c} \dfrac{\exp\left(\frac{2\pi i z}{c}\right)}{\exp\left(\frac{2\pi i z}{c}\right) - 1}$, where $\zeta = \exp\left(\dfrac{2\pi i}{7}\right)$ $\left(\zeta = \exp\left(\dfrac{2\pi i}{6}\right)\right)$. Clearly $E_f(\mathcal{S}_1^7, 1) = E_{\Delta_c f}(\mathcal{S}_1^7, 1)$ $(E_f(\mathcal{S}_1^6, 2) = E_{\Delta_c f}(\mathcal{S}_1^6, 2))$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c f}(\mathcal{S}_2, 0)$ and $\Delta_c f \equiv \zeta f$.

Example 1.10. Let $f(z) = \left(1+\sqrt{\zeta}\right)^{z/c} \dfrac{\sin\left(\frac{2\pi z}{c}\right)}{\sin\left(\frac{2\pi z}{c}\right) - 1}$, where $\zeta = \exp\left(\dfrac{2\pi i}{7}\right)$ $\left(\zeta = \exp\left(\dfrac{2\pi i}{6}\right)\right)$. Clearly $E_f(\mathcal{S}_1^7, 1) = E_{\Delta_c f}(\mathcal{S}_1^7, 1)$ $(E_f(\mathcal{S}_1^6, 2) = E_{\Delta_c f}(\mathcal{S}_1^6, 2))$ and $E_f(\mathcal{S}_2, 0) = E_{\Delta_c^2 f}(\mathcal{S}_2, 0)$ and $\Delta_c^2 f \equiv \zeta f$.

Example 1.11. Let $f(z)=\left(1+\sqrt[3]{\zeta}\omega\right)^{z/c}\dfrac{\cos\left(\frac{2\pi z}{c}\right)}{\exp\left(\frac{2\pi iz}{c}\right)-1}$, where $\zeta=\exp\left(\dfrac{2\pi i}{7}\right)$ $\left(\zeta=\exp\left(\dfrac{2\pi i}{6}\right)\right)$. Clearly $E_f(\mathcal{S}_1^7,1)=E_{\Delta_c f}(\mathcal{S}_1^7,1)$ $(E_f(\mathcal{S}_1^6,2)=E_{\Delta_c f}(\mathcal{S}_1^6,2))$ and $E_f(\mathcal{S}_2,0)=E_{\Delta_c^3 f}(\mathcal{S}_2,0)$ and $\Delta_c^3 f\equiv\zeta f$.

Remark 1.4. We note that the linear difference equation

$$\Delta_c^s f(z)=\sum_{i=0}^{s}(-1)^{s-i}\binom{n}{i}f(z+ci)=t\,f(z), \tag{1.6}$$

where $t^s=1,\ t\neq-1$, can be solved in terms of linear combinations of exponential functions with coefficients in $\mathbb{P}_c$. In fact, if f be a finite ordered meromorphic function satisfies the relation $\Delta_c^s f\equiv tf$, then $f(z)$ must assume the following form

$$f(z)=\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}}+\ldots+\pi_0(z)\alpha_0^{\frac{z}{c}},$$

where all $\pi_j\in\mathbb{P}_c$, and α_j are roots of the equation $\displaystyle\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}z^j=t$.

Following example shows that in *Theorems 1.1* and *1.2* the term 'finite order meromorphic functions' can not be removed for a special class of linear c-difference odd operator, where $a_j=(-1)^j\dbinom{p}{j}2^{p-j}$. We note that in this case (1.5) takes the form $\mathcal{L}_p^{o}(f,\Delta_c)=\displaystyle\sum_{j=0}^{p}a_jf(z+(2j+1)c)$.

Example 1.12. For $c\in\mathbb{C}^*$, we suppose that $f(z)=\exp\left(\cos\left(\dfrac{\pi z}{c}\right)\right)$. We choose $\mathcal{L}_p(f,\Delta_c)$ as $\mathcal{L}_p^{o}(f,\Delta_c)$. Since $\cos\left(\dfrac{\pi(z+(2j+1)c)}{c}\right)=-\cos\left(\dfrac{\pi z}{c}\right)$ and it follows that $\mathcal{L}_p^{o}(f,\Delta_c)=\exp\left(-\cos\left(\dfrac{\pi z}{c}\right)\right)$; so f satisfies all the conditions of *Theorems 1.1* and *1.2* but $\mathcal{L}_p^{o}(f,\Delta_c)\not\equiv tf$.

However, unfortunately, we were not succeeded to find any counter example for general linear c-difference operator.

The next example shows that the set $\mathcal{S}_1$ in *Corollary 1.3* simply can not be replaced by an arbitrary set.

Example 1.13. Let $\mathcal{S}_a^{\#}=\left\{a,\dfrac{a}{\sqrt{\omega}},\dfrac{a}{\omega},0,\dfrac{a}{\omega\sqrt{\omega}},a\omega,a\sqrt{\omega}\right\}$ and $\mathcal{S}_2=\{\infty\}$, where a is any non-zero complex number, ω is non-real cube root of unity,

$$f(z)=\exp\left(\frac{z}{c}\log(\omega^{\frac{1}{2p}}+1)\right)\frac{1}{\cos^2\left(\frac{2\pi z}{c}\right)-1},.$$

where p $(1\leqslant p\leqslant 4)$ be an integer.

It is easy to verify that $E_f(\mathcal{S}_a^{\#},1)=E_{\Delta_c^p f}(\mathcal{S}_a^{\#},1)$ and $E_f(\mathcal{S}_2,0)=E_{\Delta_c^p f}(\mathcal{S}_2,0)$ but neither $\Delta_c^p f\equiv f$ with $t^7=1$ nor f has the specific form as above.

Though the standard definitions and notations of the value distribution theory are available in [9, 16], we explain here some of them which are used in the paper.

Definition 1.10. [13] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f| = 1)$ the counting function of simple a-points of f. For a positive integer p, we denote $N(r, a; f| \leqslant p)(N(r, a; f| \geqslant p))$ the counting function of those a-points of f whose multiplicities are not greater (less) than p where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f| \leqslant p)(\overline{N}(r, a; f| \geqslant p))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Definition 1.11. [11] We denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f| \geqslant 2)$.

Definition 1.12. [20, 18] Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, each point in this counting function is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$.

Definition 1.13. [6, 9] Let f, g share a value IM. We denote by $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 1.14. [14] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f|g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

2. Some useful lemmas

In this section, we are going to discuss some lemmas which will needed later to prove our main results. We define, for a non-constant meromorphic functions f,

$$\mathcal{F} = \left(\frac{f}{a}\right)^n, \quad \mathcal{G} = \left(\frac{\mathcal{L}_p(f, \Delta_c)}{a}\right)^n. \tag{2.1}$$

Associated to $\mathcal{F}$ and $\mathcal{G}$, we next define $\mathcal{H}$ and Ψ as follows:

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F} - 1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G} - 1}\right), \tag{2.2}$$

and

$$\Psi = \frac{\mathcal{F}'}{\mathcal{F}(\mathcal{F} - 1)} - \frac{\mathcal{G}'}{\mathcal{G}(\mathcal{G} - 1)}. \tag{2.3}$$

Lemma 2.1. [6] *Let g be a meromorphic function of finite order ρ, and let $c \in \mathbb{C}^*$ be fixed. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{g(z+c)}{g(z)}\right) + m\left(r, \frac{g(z)}{g(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 2.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1) satisfying $E_{\mathcal{F}}(1,m) = E_{\mathcal{G}}(1,m)$, $0 \leqslant m < \infty$ with $\mathcal{H} \not\equiv 0$, then*

$$N_E^{1)}\left(r, \frac{1}{\mathcal{F}-1}\right) = N_E^{1)}\left(r, \frac{1}{\mathcal{G}-1}\right) \leqslant N(r,\mathcal{H}) + S(r,\mathcal{F}) + S(r,\mathcal{G}).$$

Proof. Since $E_{\mathcal{F}}(1,q) = E_{\mathcal{G}}(1,q)$, so it is obvious that any simple 1-point of $\mathcal{F}$ and $\mathcal{G}$ is a zero of $\mathcal{H}$. The construction of $\mathcal{H}$ implies that, $m(r,\mathcal{H}) = S(r,\mathcal{F}) + S(r,\mathcal{G})$. By the *First Fundamental Theorem*, we get

$$\begin{aligned} & N_E^{1)}\left(r, \frac{1}{\mathcal{F}-1}\right) = N_E^{1)}\left(r, \frac{1}{\mathcal{G}-1}\right) \\ \leqslant \; & N\left(r, \frac{1}{\mathcal{H}}\right) \leqslant N(r,\mathcal{H}) + S(r,\mathcal{F}) + S(r,\mathcal{G}). \end{aligned}$$

The proof is complete. □

Lemma 2.3. [10] *Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}^*$. Then*

$$\begin{aligned} N(r,0;f(z+c)) &\leqslant N(r,0;f(z)) + S(r,f(z)), \\ N(r,\infty;f(z+c)) &\leqslant N(r,\infty;f(z)) + S(r,f(z)), \\ \overline{N}(r,0;f(z+c)) &\leqslant \overline{N}(r,0;f(z)) + S(r,f(z)), \\ \overline{N}(r,\infty;f(z+c)) &\leqslant \overline{N}(r,\infty;f(z)) + S(r,f(z)) \end{aligned}$$

Lemma 2.4. *Let g be a meromorphic function of finite order ρ, and let $c \in \mathbb{C}^*$ be fixed. Then*

$$T(r,g(z+c)) = T(r,g(z)) + S(r,g).$$

Proof. The lemma can be proof in the line of the proof of [6, Theorem 2.1]. □

Lemma 2.5. *Let f be a transcendental meromorphic function of finite order, then $S(r,\mathcal{L}_p(f,\Delta_c))$ can be replaced by $S(r,f)$.*

Proof. In view of *Lemma 2.4*, we have

$$T(r,\mathcal{L}_p(f,\Delta_c)) \leqslant \sum_{j=1}^{p} T(r,f(z+cj)) + T(r,f) + O(1) \leqslant (p+1)\, T(r,f) + O(1),$$

with this the lemma follows. □

Lemma 2.6. [19] *Let f be a non-constant meromorphic function and $\mathcal{Q}(f) = \sum_{i=0}^{n} a_i f^i$, where $a_i \in \mathbb{C}$ with $a_n \neq 0$. Then $T(r,\mathcal{Q}(f)) = n\, T(r,f) + O(1)$.*

Lemma 2.7. [15] *If $N\left(r,0;f^{(k)}|f\neq 0\right)$ be the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N\left(r,0;f^{(k)}|f\neq 0\right)\leqslant k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geqslant k)+S(r,f).$$

Lemma 2.8. *Let $\mathcal{F}$ and $\mathcal{G}$ share $(1,t)$, $1\leqslant t<\infty$ and $(\infty,0)$, then*

$$\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})\leqslant\frac{1}{t+1}\left\{\overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})\right\}+\frac{2}{t+1}\overline{N}(r,\infty;\mathcal{F})+S(r,f).$$

Proof. In view of *Lemma 2.5* and *2.7*, one must have

$$\begin{aligned}\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G}) &= \overline{N}_{L}(r,1;\mathcal{F})+\overline{N}_{L}(r,1;\mathcal{G})\\ &\leqslant \overline{N}(r,1;\mathcal{F}|\geqslant t+2)+\overline{N}(r,1;\mathcal{G}|\geqslant t+2)\\ &\leqslant \frac{1}{t+1}\left\{N(r,0;\mathcal{F}'|\mathcal{F}\neq 0)+N(r,0;\mathcal{G}'|\mathcal{G}\neq 0)\right\}\\ &\leqslant \frac{1}{t+1}\left\{\overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+2\overline{N}(r,\infty;\mathcal{F})\right\}+S(r,f).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.9. *Let $\mathcal{F}$ and $\mathcal{G}$ share $(1,t)$, $1\leqslant t<\infty$ and $(\infty,0)$, then*

$$\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})\leqslant\frac{1}{t}\left\{\overline{N}(r,0;\mathcal{F})+N(r,\infty;\mathcal{F})\right\}+S(r,\mathcal{F})+S(r,f).$$

Proof. In view of *Lemma 2.7*, we have

$$\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})\leqslant\overline{N}(r,1;\mathcal{F}|\geqslant t+1)\leqslant\frac{1}{t}N(r,0;\mathcal{F}'|\mathcal{F}=1).$$

We omit the details since rest of the proof follows the line of the proof of *Lemma 2.8.* □

Lemma 2.10. *For a meromorphic function f, we suppose that F and G be given as in (2.1) and $\Psi\not\equiv 0$. If f and $\mathcal{L}_p(f,\Delta_c)$ share (∞,k), where $0\leqslant k<\infty$ and $\mathcal{F}$, $\mathcal{G}$ share $(1,t)$, then*

$$\begin{aligned}&\left\{n(k+1)-1\right\}\overline{N}(r,\infty;f|\geqslant k+1)\\ \leqslant\ &\frac{t+2}{t+1}\left\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\right\}+\frac{2}{t+1}\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Proof. It is clear that $\mathcal{F}$ and $\mathcal{G}$ share (∞,nk) since f and $\mathcal{L}_p(f,\Delta_c)$ share (∞,k). Let z_0 be a pole of $\mathcal{F}$ of multiplicity $q(\geqslant nk+1)$, then z_0 must be a pole of $\mathcal{G}$ of multiplicity $r(\geqslant nk+1)$ and conversely. Again one may note that there is no pole of $\mathcal{F}$ and $\mathcal{G}$ of multiplicity q, where $nk<q<n(k+1)$. Next by using *Lemmas 2.5,*

2.6 and *2.8*, we get from the definition of Ψ that

$$\begin{aligned}
&\left\{nk+n-1\right\}\overline{N}(r,\infty;f|\geqslant k+1)\leqslant N(r,0;\Psi)\leqslant N(r,\infty;\Psi)+S(r,\mathcal{F})+S(r,\mathcal{G})\\
\leqslant\ & \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+S(r,\mathcal{F})+S(r,\mathcal{G})\\
\leqslant\ & \overline{N}(r.0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))+\frac{1}{t+1}\Big\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\\
&+2\overline{N}(r,\infty;f)\Big\}+S(r,f)\\
\leqslant\ & \frac{t+2}{t+1}\Big\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\Big\}+\frac{2}{t+1}\overline{N}(r,\infty;f)+S(r,f).
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.11. [17, 20] *If $\mathcal{F}$ and $\mathcal{G}$ share $(\infty,0)$ and $\Psi\equiv 0$, then $\mathcal{F}\equiv\mathcal{G}$.*

Lemma 2.12. [18] *Let $\mathcal{H}\equiv 0$ and $\mathcal{F}$, $\mathcal{G}$ share $(\infty,0)$, then $\mathcal{F}$ and $\mathcal{G}$ share $(1,\infty)$, (∞,∞).*

Lemma 2.13. [1] *Let $\mathcal{F}$, $\mathcal{G}$ be two meromorphic functions sharing $(1,2)$ and (∞,k), where $0\leqslant k\leqslant\infty$. Then one of the following cases holds.*

(i). $T(r,\mathcal{F})+T(r,\mathcal{G})\leqslant 2\{N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+\overline{N}(r,\infty;\mathcal{F})+$

$$+N(r,\infty;\mathcal{G})+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\}+S(r,\mathcal{F})+S(r,\mathcal{G}).$$

(ii). $\mathcal{F}\equiv\mathcal{G}$.

(iii). $\mathcal{F}\mathcal{G}\equiv 1$.

3. Proofs of the main results

Proof of Theorem 1.1. Let $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1).

Now we discuss the following two cases.

Case 1. Let us suppose that $\mathcal{H}\not\equiv 0$. Then in view of *Lemma 2.11*, we have $\Psi\not\equiv 0$. Since $E_f(\mathcal{S}_a^n,1)=E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_a^n,1)$ and $E_f(\mathcal{S}_2,0)=E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_2,0)$, it follows that $\mathcal{F}$ and $\mathcal{G}$ share $(1,1)$ and $(\infty,0)$. By the *Second Fundamental Theorem*, we get

$$\begin{aligned}
&T(r,\mathcal{F})+T(r,\mathcal{G})\\
\leqslant\ & \overline{N}(r,1;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,1;\mathcal{G})+\overline{N}(r,0;\mathcal{G})+\overline{N}(r,\infty;\mathcal{G})\\
&-\overline{N}_0(r,0;\mathcal{F}')-\overline{N}_0(r,0;\mathcal{G}')+S(r,\mathcal{F})+S(r,\mathcal{G}).
\end{aligned}$$

Using *Lemma 2.6* and *Lemmas 2.1, 2.2, 2.3* of [2, p.384], we get

$$\begin{aligned}
(3.1)\quad & n\Big\{T(r,f)+T(r,\mathcal{L}_p(f,\Delta_c))\Big\}\\
\leqslant\ & 4\Big\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\Big\}+6\overline{N}(r,\infty;f)-2\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&+S(r,\mathcal{F})+S(r,\mathcal{G}).
\end{aligned}$$

Applying *Lemma 2.8* with $t=1$ and *Lemma 2.10* with $t=1$, $k=0$, we get from (3.1) that

$$\begin{aligned}
(3.2)\ n&\left\{T(r,f)+T(r,\mathcal{L}_p(f,\Delta_c))\right\}\\
\leqslant\ & \frac{9}{2}\left\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\right\}+7\overline{N}(r,\infty;f)+S(r,f)+S(r,\mathcal{L}_p(f,\Delta_c))\\
\leqslant\ & \left(\frac{9}{2}+\frac{21}{2(n-2)}\right)\left\{\overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\right\}+S(r,f)+S(r,\mathcal{L}_p(f,\Delta_c))\\
\leqslant\ & \left(\frac{9}{2}+\frac{21}{2(n-2)}\right)\left\{T(r,f)+T(r,\mathcal{L}_p(f,\Delta_c))\right\}+S(r,f)+S(r,\mathcal{L}_p(f,\Delta_c)).
\end{aligned}$$

which contradicts $n\geqslant 7$.

Case 2. Let us suppose that $\mathcal{H}\equiv 0$.

On integration twice, we get

$$\mathcal{F}=\frac{\mathcal{A}\mathcal{G}+\mathcal{B}}{\mathcal{C}\mathcal{G}+\mathcal{D}}, \tag{3.3}$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in\mathbb{C}$ such that $\mathcal{A}\mathcal{D}-\mathcal{B}\mathcal{C}\neq 0$.

We now discuss the following two cases.

Case a. Let $\mathcal{A}\mathcal{C}\neq 0$. We thus see that $\mathcal{A}\neq 0$ and $\mathcal{C}\neq 0$.

It follows from (3.3) that

$$\mathcal{F}-\frac{\mathcal{A}}{\mathcal{C}}=\frac{\mathcal{B}\mathcal{C}-\mathcal{A}\mathcal{D}}{\mathcal{C}(\mathcal{C}\mathcal{G}+\mathcal{D})}. \tag{3.4}$$

Clearly it follows from (3.4) that all the zeros of $\mathcal{F}-\dfrac{\mathcal{A}}{\mathcal{C}}$ corresponds from the poles of $\mathcal{G}$. We also see from our hypothesis that $\mathcal{F}$ and $\mathcal{G}$ share (∞,∞), so from (3.3) we see that ∞ is an e.v.P of $\mathcal{G}$. In other words $\mathcal{F}$ omits the value $\frac{\mathcal{A}}{\mathcal{C}}$.

By the *Second Fundamental Theorem*, we get

$$\begin{aligned}
nT(r,f)\ &\leqslant\ \overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+\overline{N}\left(r,\frac{\mathcal{A}}{\mathcal{C}};\mathcal{F}\right)+S(r,\mathcal{F})\\
&=\ \overline{N}(r,0;f)+S(r,f)\leqslant T(r,f)+S(r,f),
\end{aligned}$$

which contradicts $n\geqslant 7$.

Case b. Let $\mathcal{A}\mathcal{C}=0$. This shows that one of $\mathcal{A}$ and $\mathcal{C}$ is zero, otherwise for $\mathcal{A}=0=\mathcal{C}$ leads the function $\mathcal{F}$ to be a constant and which would be a contradiction.

Subcase b.1. Let $\mathcal{A}\neq 0$ and $\mathcal{C}=0$. Then,

$$\mathcal{F}=\alpha\mathcal{G}+\beta, \tag{3.5}$$

where $\alpha=\dfrac{\mathcal{A}}{\mathcal{D}}$ and $\beta=\dfrac{\mathcal{B}}{\mathcal{D}}$.

If $\mathcal{F}$ has no 1-points, then by *Second Fundamental Theorem*, we get

$$nT(r,f)\leqslant\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+S(r,\mathcal{F})\leqslant 2T(r,f)+S(r,f),$$

which contradicts $n\geqslant 7$.

If $\mathcal{F}$ and $\mathcal{G}$ both have some 1-points, then we have $\alpha+\beta=1$.

If $\beta=0$, then $\alpha=1$. So we have $\mathcal{F}\equiv\mathcal{G}$. Thus we have $\mathcal{L}_p\left(f,\Delta_c\right)\equiv tf$, where $t^n=1$ with $t\neq-1$.

Next, we suppose that $\beta\neq0$. So it is clear that $\mathcal{F}-\beta=\alpha\mathcal{G}$. By *Second Fundamental Theorem*, we get

$$\begin{aligned} nT(r,f) &\leqslant \overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,\beta;\mathcal{F})+S(r,\mathcal{F}) \\ &\leqslant \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;\mathcal{G})\leqslant(p+3)T(r,f)+S(r,f), \end{aligned}$$

which contradicts $n\geqslant p+4$.

Subcase b.2. Let $\mathcal{A}=0$ but $\mathcal{C}\neq0$. Then we have

$$\mathcal{F}=\frac{1}{\gamma\mathcal{G}+\delta}, \tag{3.6}$$

where $\gamma=\dfrac{\mathcal{C}}{\mathcal{B}}$ and $\delta=\dfrac{\mathcal{D}}{\mathcal{B}}$.

If $\mathcal{F}$ has no 1-points, then proceeding exactly same way as done in *Subcase b.1*, we arrive at a contradiction.

If $\mathcal{F}$ and $\mathbb{G}$ have some 1-points, then it follows from (3.6) that $\gamma+\delta=1$.

We now see from (3.6) that

$$\mathcal{F}=\frac{1}{\gamma\mathcal{G}+1-\gamma}. \tag{3.7}$$

We note that as $\mathcal{C}\neq0$, $\gamma\neq0$. Suppose $\delta\neq0$. So $\gamma\neq1$. Since $\mathcal{F}$ and $\mathcal{G}$ share (∞,∞), so from (3.7), we see that $\mathcal{F}$ and $\mathcal{G}$ omit ∞.

By the *Second Fundamental Theorem*, we get

$$\begin{aligned} & nT(r,f)=T(r,\mathcal{F})\leqslant\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+\overline{N}\left(r,\frac{1}{1-\gamma};\mathcal{F}\right)+S(r,\mathcal{F}) \\ \leqslant\ & \overline{N}(r,0;f)+\overline{N}(r,0;\mathcal{G})+S(r,f)\leqslant(p+2)\,T(r,f)+S(r,f), \end{aligned}$$

which contradicts $n\geqslant p+4$.

Next we suppose that $\delta=0$. Therefore $\gamma=1$. Then we get $\mathcal{F}\mathcal{G}\equiv1$, i.e., $f\left(\mathcal{L}_p(f,\Delta_c)\right)\equiv\theta a^2$, where $\theta^n=1$.

Next since $\mathcal{F}$ and $\mathcal{G}$ share (∞,∞), so we have $N\left(r,\dfrac{\mathcal{L}_p(f,\Delta_c)}{f}\right)=N(r,0;f)$ and so in view of *Lemma 2.1*, we get

$$\begin{aligned} & 2T(r,f)\leqslant T\left(r,\frac{\theta a^2}{f^2}\right)+S(r,f)\leqslant T\left(r,\frac{\mathcal{L}_p(f,\Delta_c)}{f}\right)+S(r,f) \\ \leqslant\ & N\left(r,\frac{\mathcal{L}_p(f,\Delta_c)}{f}\right)+S(r,f)\leqslant N(r,0;f)+S(r,f)\leqslant T(r,f)+S(r,f), \end{aligned}$$

which is a contradiction.

This completes the proof of *Theorem 1.1*. □

Proof of Theorem 1.2. Let $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1) and $\Psi \not\equiv 0$, since otherwise the proof follows from the *Lemma 2.11.* Again Since $E_f(\mathcal{S}_a^n, 2) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_a^n, 2)$ and $E_f(\mathcal{S}_2, 0) = E_{\mathcal{L}_p(f,\Delta_c)}(\mathcal{S}_2, 0)$, so it follows that $\mathcal{F}$, $\mathcal{G}$ share $(1,2)$ and $(\infty, 0)$. Let if possible (i) of *Lemma 2.13* holds. Then with the help of *Lemma 2.6*, one must have

$$\begin{aligned} (3.8) \qquad & n\left\{T(r,f) + T(r,\mathcal{L}_p(f,\Delta_c))\right\} \\ \leqslant\; & 4\left\{\overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{L}_p(f,\Delta_c))\right\} + 6\overline{N}(r,\infty;f) + S(r,\mathcal{F}) + S(r,\mathcal{G}). \end{aligned}$$

Now with the help of *Lemma 2.10* with $t=2$, $k=0$, we get from (3.8)

$$\begin{aligned} & n\left\{T(r,f) + T(r,\mathcal{L}_p(f,\Delta_c))\right\} \\ \leqslant\; & \left(4 + \frac{24}{3n-5}\right)\left\{T(r,f) + T(r,\mathcal{L}_p(f,\Delta_c))\right\} + S(r,f) + S(r,\mathcal{L}_p(f,\Delta_c)), \end{aligned}$$

which contradicts $n \geqslant 6$.

Now the rest of the proof follows from the line of the proof of *Theorem 1.1.* $\square$

4. Proofs of the corollaries

Proof of Corollary 1.1. Let us suppose that $\mathcal{F} = f^n$ and $\mathcal{G} = \left(\overset{s}{\Delta}_c\right)^n$. Then, following the same procedure as adopted in the proof of *Theorem 1.1*, we obtain

$$(4.1) \qquad \overset{s}{\Delta}_c \equiv tf.$$

Proof of Corollary 1.2. The proof can be carried out exactly the line of the proof of *Theorem 1.2* and that of *Corollary 1.1.*

Proof of Remark 1.4. Since the distinct roots of $\displaystyle\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}z^j = t$ are $\alpha_j = 1 + |t|^{\frac{1}{s}} e^{\frac{\theta + 2\pi i j}{s}}$, where $-\pi < \theta \leqslant \pi$, $j = 0, 1, \ldots, s-1$, therefore the general solution of the relation $\overset{s}{\Delta}_c f \equiv t\, f$ will be of the form

$$f(z) = \pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \pi_0(z)\alpha_0^{\frac{z}{c}}.$$

<u>**Verification**</u>:

$$\begin{aligned}
\Delta_c^s f &= \binom{s}{0} f(z+sc) - \binom{s}{1} f(z+(s-1)c) + \ldots + (-1)^s \binom{s}{s} f(z) \\
&= \binom{s}{0}\left\{\pi_{s-1}(z+sc)\alpha_{s-1}^{\frac{z}{c}}\alpha_{s-1}^{p} + \ldots + \pi_0(z+sc)\alpha_0^{\frac{z}{c}}\alpha_0^{s}\right\} \\
&\quad -\binom{s}{1}\left\{\pi_{s-1}(z+(s-1)c)\alpha_{s-1}^{\frac{z}{c}}\alpha_{s-1}^{s-1} + \ldots + \pi_0(z+(s-1)c)\alpha_0^{\frac{z}{c}}\alpha_0^{s-1}\right\} \\
&\quad + \ldots + \binom{s}{s}(-1)^s\left\{\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \pi_0(z)\alpha_0^{\frac{z}{c}}\right\} \\
&= \left\{\binom{s}{0}\alpha_{s-1}^{s} - \binom{s}{1}\alpha_{s-1}^{s-1} + \ldots + \binom{s}{s}(-1)^s\right\}\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} \\
&\quad + \ldots + \left\{\binom{s}{0}\alpha_0^{s} - \binom{s}{1}\alpha_0^{s-1} + \ldots + \binom{s}{s}(-1)^s\right\}\pi_0(z)\alpha_0^{\frac{z}{c}} \\
&= (\alpha_{s-1}-1)^s\,\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + (\alpha_0-1)^s\,\pi_0(z)\alpha_0^{\frac{z}{c}} \\
&= \left(|t|^{\frac{1}{s}}e^{\frac{\theta+2(s-1)\pi i}{s}}\right)^s \pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \left(|t|^{\frac{1}{s}}e^{\frac{\theta+oi}{s}}\right)^s \pi_0(z)\alpha_0^{\frac{z}{c}} = \\
&= t\left\{\pi_{s-1}(z)\alpha_{s-1}^{\frac{z}{c}} + \ldots + \pi_0(z)\alpha_0^{\frac{z}{c}}\right\} = t\,f(z).
\end{aligned}$$

5. Concluding remarks

In this section, we have the following observation.

Observation 5.1. A non-constant finite ordered meromorphic function satisfying the relation

$$\mathcal{L}_p(f,\Delta_c) \equiv tf \tag{5.1}$$

must assume the following form

$$f(z) = \pi_p(z)\alpha_p^{\frac{z}{c}} + \ldots + \pi_1(z)\alpha_1^{\frac{z}{c}},$$

where $\pi_j(z),\ (j=1,\ldots,p) \in \mathbb{P}_c$, and $\alpha_j\ (j=1,\ldots,p)$ are the roots of the equation

$$a_p w^p + a_{p-1}w^{p-1} + \ldots + a_1 w - \left(\sum_{j=1}^{p} a_j + t\right) = 0.$$

For $p=1$, we have $\mathcal{L}_1(f,\Delta_c) \equiv t\,f$, which implies that $f(z+c) = \left(\dfrac{a_1+t}{a_1}\right) f(z)$. Clearly, in this case the general solution of (5.1) is

$$f(z) = \pi_1(z)\left(\frac{a_1+t}{a_1}\right)^{\frac{z}{c}} = \pi_1(z)\alpha_1^{\frac{z}{c}},$$

where α_1 is a root of the equation $a_1 w - (a_1+t) = 0$.

Verification:

$$\begin{aligned}
\mathcal{L}_1(f,\Delta_c) &= a_1 f(z+c) - (a_1) f(z) \\
&= a_1\left\{\pi_1(z+c)\alpha_1\alpha_1^{\frac{z}{c}}\right\} - a_1\left\{\pi_1(z)\alpha_1^{\frac{z}{c}}\right\} \\
&= \left\{a_1\alpha_1 - a_1\right\}\pi_1(z)\alpha_1^{\frac{z}{c}} = t\,\pi_1(z)\alpha_1^{\frac{z}{c}} = t\,f(z).
\end{aligned}$$

For $p=2$, we have $\mathcal{L}_2(f,\Delta_c) \equiv t\,f$, which in turn implies that

$$a_2 f(z+2c) + a_1 f(z+c) - (a_1+a_2+t) f(z) \equiv 0.$$

Let $\alpha_1,\ \alpha_2$ be the roots of the equation

$$a_2 w^2 + a_1 w - (a_1+a_2+t) = 0.$$

Then

$$\alpha_1,\ \alpha_2 = \frac{-a_1 \pm \sqrt{a_1^2 + 4a_2(a_1+a_2+t)}}{2a_2}.$$

In this case the general solution of (5.1) is

$$\begin{aligned}
f(z) &= \pi_1(z)\left(\frac{-a_1 + \sqrt{a_1^2+4a_2(a_1+a_2+t)}}{2a_2}\right)^{\frac{z}{c}} + \pi_2(z)\left(\frac{-a_1 - \sqrt{a_1^2+4a_2(a_1+a_2+t)}}{2a_2}\right)^{\frac{z}{c}} \\
&= \pi_1(z)\alpha_1^{\frac{z}{c}} + \pi_2(z)\alpha_2^{\frac{z}{c}}.
\end{aligned}$$

Lets verify the above fact.

$$\begin{aligned}
\mathcal{L}_2(f,\Delta_c) &= a_2 f(z+2c) + a_1 f(z+c) - (a_1+a_2) f(z) \\
&= a_2\left\{\pi_1(z+2c)\lambda_1^2\lambda_1^{\frac{z}{c}} + \pi_2(z+2c)\lambda_2^2\lambda_2^{\frac{z}{c}}\right\} \\
&\quad + a_1\left\{\pi_1(z+c)\lambda_1\lambda_1^{\frac{z}{c}} + \pi_2(z+c)\lambda_2\lambda_2^{\frac{z}{c}}\right\} \\
&\quad - (a_1+a_2)\left\{\pi_1(z)\lambda_1^{\frac{z}{c}} + \pi_2(z)\lambda_2^{\frac{z}{c}}\right\} \\
&= \left\{a_2\lambda_1^2 + a_1\lambda_1 - (a_1+a_2)\right\}\pi_1(z)\lambda_1^{\frac{z}{c}} + \left\{a_2\lambda_2^2 + a_1\lambda_2 - (a_1+a_2)\right\}\pi_2(z)\lambda_2^{\frac{z}{c}} \\
&= t\,\pi_1(z)\lambda_1^{\frac{z}{c}} + t\,\pi_2(z)\lambda_2^{\frac{z}{c}} = t\left\{\pi_1(z)\lambda_1^{\frac{z}{c}} + \pi_2(z)\lambda_2^{\frac{z}{c}}\right\} = t\,f(z).
\end{aligned}$$

So we conjecture that, the general solution of the relation (5.1) is

$$f(z) = \pi_p(z)\alpha_p^{\frac{z}{c}} + \pi_{p-1}(z)\alpha_{p-1}^{\frac{z}{c}} + \ldots + \pi_1(z)\alpha_1^{\frac{z}{c}},$$

where $\pi_j(z)\ (j=1,\ldots,p) \in \mathbb{P}_c$, and $\alpha_j\ (j=1,\ldots,p)$ are the roots of the equation

$$a_p w^p + a_{p-1} w^{p-1} + \ldots + a_1 w - \left(\sum_{j=1}^{p} a_j + t\right) = 0.$$

But unfortunately we have not succeeded to prove it.

An open question. What would be the general meromorphic solution of the difference equation $\mathcal{L}_p(f, \Delta) \equiv t\, f$?

Список литературы

[1] A. Banerjee, "Meromorphic functions sharing two sets", Czech. Math. J., **57** (132), 1199 – 1214 (2007).

[2] A. Banerjee, "Uniqueness of meromorphic functions sharing two sets with finite weight II", Tamkang J. Math., **41** (4), 379 – 392 (2010).

[3] A. Banerjee and M. B. Ahamed, "On uniqueness of meromorphic functions sharing three sets with finite weights", Bull. Polish Acad. Sci. Math., **62** (3), 243 – 256 (2014).

[4] A. Banerjee and S. Bhattacharyya, "On the uniqueness of meromorphic functions and its difference operator sharing values or sets", Rend. Circ. Mat. Palermo, II. Ser, DOI 10.1007/s12215-016-0295-1.

[5] B. Chen and Z. Chen, "Meromorphic functions sharing two sets with its difference operator", Bull. Malays. Math. Soc., **35** (3), 765 -– 774 (2012).

[6] Y. M. Chiang and S. J. Feng, "On the Nevanlinna characteristic $f(z+\eta)$ and difference equation in complex plane", Ramanujan J., **16**, 105 – 129 (2008).

[7] B. Deng, D. Liu and D. Yang, "Meromorphic functions and its difference operator share two sets with weight k", Turkish J. Math., **41**, 1155 – 1163 (2017).

[8] R. G. Halburd and R. J. Korhonen, "Nevanlinna theory for the difference operator", Ann. Acad. Sci. Fenn. Math., **31** (2), 463 – 478 (2006).

[9] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford (1964).

[10] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and J. L. Zhang, "Value sharing results for shifts of meromorphic function and sufficient conditions for periodicity", J. Math. Anal. Appl., **355**, 352 -– 363 (2009).

[11] I. Lahiri, "Weighted sharing and uniqueness of meromorphic functions", Nagoya Math. J., **161**, 193 -– 206 (2001).

[12] I. Lahiri, "Weighted value sharing and uniqueness of meromorphic functions", Complex Var. Theory Appl., **46**, 241 -– 253 (2001).

[13] I. Lahiri, "Value distribution of certain differential polynomials", Int. J. Math. Math. Sci., **28** (2), 83 – 91 (2001).

[14] I. Lahiri and A. Banerjee, "Weighted sharing of two sets", Kyungpook Math. J., **46** (1), 79 – 87 (2006).

[15] I. Lahiri and S. Dewan, "Value distribution of the product of a meromorphic function and its derivative", Kodai Math. J., **26**, 95 -– 100 (2003).

[16] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin (1993).

[17] H. X. Yi, "Meromorphic functions that share three sets", Kodai Math. J., **20**, 22 -– 32 (1997).

[18] H. X. Yi, "Meromorphic functions that share one or two values II", Kodai Math. J., **22**, 264 – 272 (1999).

[19] C. C. Yang, "On deficiencies of differential polynomials II", Math. Z., **125**, 107 – 112 (1972).

[20] H. X. Yi and L. Z. Yang, "Meromorphic functions that share two sets", Kodai Math. J., **20**, 127 – 134 (1997).

[21] J. Zhang, "Value distribution and shared sets of differences of meromorphic functions", J. Math. Anal. Appl., **367** (2), 401 -– 408 (2010).

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ON THE DERIVATIVES OF THE HEUN FUNCTIONS

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Abstract. The Heun functions satisfy linear ordinary differential equations of second order with certain singularities in the complex plane. The first order derivatives of the Heun functions satisfy linear second order differential equations with one more singularity. In this paper we compare these equations with linear differential equations isomonodromy deformations of which are described by the Painlevé equations $P_{II} - P_{VI}$.

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Keywords: linear ordinary differential equation; Heun function; isomonodromy deformation.

1. Introduction

The general Heun equation is the most general second order linear Fuchsian ordinary differential equation with four regular singular points in the complex plane [2, 3, 4, 5]. Although it is a genaralization of the well-studied Gauss hypergeometric equation with three regular singularities, it is much more difficult to investigate properties of the Heun functions. The additional singularity causes many complications in comparison with the hypergeometric case (for instance, the solutions in general have no integral representations involving simpler mathematical functions). There also exist confluent Heun equations (see [3, 4]) which have irregular singularities. There are many studies on the properties of solutions of the Heun equations from different perspectives (see, for instance, [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein). The Heun functions (and their confluent cases) appear extensively in many problems of mathematics, mathematical physics, physics and engineering (e.g., [18, 19, 20]). An extensive bibliography can be found at [1].

The general Heun equation is given by the following equation:

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-t}\right)\frac{du}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}u = 0, \tag{1.1}$$

where the parameters satisfy the Fuchsian relation

$$1 + \alpha + \beta = \gamma + \delta + \varepsilon. \tag{1.2}$$

This equation has four regular singular points at $z = 0,\, 1,\, t$ and ∞. Its solutions, the Heun functions, are usually denoted by $u = H(t,\, q;\, \alpha,\, \beta,\, \gamma,\, \delta;\, z)$ assuming that ε is obtained from (1.2). The parameter q is referred to as the accessory parameter.

It is well-known that the derivative of the hypergeometric function ${}_2F_1$ is again a hypergeometric function with different values of the parameters. However, for the Heun function it is generally not the case. The first order derivative of the general Heun function satisfies a second order Fuchsian differential equation with five regular singular points [7, 8, 12]. It can be verified by direct computations that the function $v(z) = du/dz$, where $u = u(z)$ is a solution of (1.1), satisfies the following equation:

(1.3)

$$\frac{d^2v}{dz^2} + \left(\frac{\gamma+1}{z} + \frac{\delta+1}{z-1} + \frac{\varepsilon+1}{z-t} - \frac{\alpha\beta}{\alpha\beta z - q}\right)\frac{dv}{dz} + \frac{f(z)}{z(z-1)(z-t)(\alpha\beta z - q)}v = 0,$$

where $f(z) = z(\alpha\beta z - 2q)(\alpha\beta + \gamma + \delta + \varepsilon) + (q^2 + q(\gamma + t(\gamma+\delta) + \varepsilon) - \alpha\beta\gamma t)$. We see that an additional singularity at $z = q/(\alpha\beta)$ involving the accessory parameter is added.

It is known that in some cases equation (1.3) reduces to a Heun equation (1.1) with altered parameters [8]. Indeed, we can observe that in four cases when $q = 0$, $q = \alpha\beta$, $q = \alpha\beta t$ and $\alpha\beta = 0$ the additional singularity in (1.3) disappears and we obtain the Heun equation (1.1) with different parameters [8]. The equation for the derivatives of the Heun functions allows one to construct several new expansions of solutions of the Heun equations in terms of various special functions (e.g., hypergeometric functions) [7]. Similar results hold for confluent cases [12].

This paper is organized as follows. In Section 2 we give a list of all confluent Heun equations together with linear second order equations for the derivatives of the Heun functions. In Section 3 we briefly describe the theory of isomonodromy deformations of linear equations and show how the famous Painlevé equations appear in this context. Next, in Section 4 we present our main results. In particular, we will compare linear equations for the Heun derivatives with linear differential equations, isomonodromy deformations of which are described by the Painlevé equations.

2. Confluent Heun equations and equations for derivatives of confluent Heun functions

The general Heun equation is given by (1.1) together with (1.2) and the linear equation for the derivative of the Heun functions is (1.3).

The confluent Heun equation is written as

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon\right)\frac{du}{dz} + \frac{\alpha z - q}{z(z-1)}u = 0 \tag{2.1}$$

and the linear equation for the function $v = du/dz$ is given by

$$\frac{d^2v}{dz^2} + \left(\frac{\gamma+1}{z} + \frac{\delta+1}{z-1} + \varepsilon - \frac{\alpha}{\alpha z - q}\right)\frac{dv}{dz} + \frac{g(z)}{z(z-1)(\alpha z - q)}v = 0, \tag{2.2}$$

where $g(z) = (\alpha+\varepsilon)(\alpha z^2 - 2qz) + (q^2 - (\gamma+\delta-\varepsilon)q + \alpha\gamma)$.

The double-confluent Heun equation is

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z^2} + \frac{\delta}{z} + \varepsilon\right)\frac{du}{dz} + \frac{\alpha z - q}{z^2}u = 0 \tag{2.3}$$

and the linear equation for the function $v = du/dz$ is given by

$$\frac{d^2v}{dz^2} + \left(\frac{\gamma}{z^2} + \frac{\delta+2}{z} + \varepsilon - \frac{\alpha}{\alpha z - q}\right)\frac{dv}{dz} + \frac{h(z)}{z^2(\alpha z - q)}v = 0, \tag{2.4}$$

where $h(z) = (\alpha+\varepsilon)(\alpha z^2 - 2qz) + (q^2 - \delta q - \alpha\gamma)$.

The bi-confluent Heun equation is

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \delta + \varepsilon z\right)\frac{du}{dz} + \frac{\alpha z - q}{z}u = 0 \tag{2.5}$$

and the linear equation for the function $v = du/dz$ is given by

$$\frac{d^2v}{dz^2} + \left(\frac{\gamma+1}{z} + \delta + \varepsilon z - \frac{\alpha}{\alpha z - q}\right)\frac{dv}{dz} + \frac{k(z)}{z(\alpha z - q)}v = 0, \tag{2.6}$$

where $k(z) = (\alpha+\varepsilon)z(\alpha z - 2q) + (q^2 - \delta q - \alpha\gamma)$.

The tri-confluent Heun equation is

$$\frac{d^2u}{dz^2} + \left(\gamma + \delta z + \varepsilon z^2\right)\frac{du}{dz} + (\alpha z - q)u = 0 \tag{2.7}$$

and the linear equation for the function $v = du/dz$ is given by

$$\frac{d^2v}{dz^2} + \left(\gamma + \delta z + \varepsilon z^2 - \frac{\alpha}{\alpha z - q}\right)\frac{dv}{dz} + \frac{p(z)}{(\alpha z - q)}v = 0, \tag{2.8}$$

where $p(z) = (\alpha+\varepsilon)(\alpha z^2 - 2qz) + (q^2 - \delta q - \alpha\gamma)$.

3. Isomonodromic deformations of linear equations and the Painlevé equations

In this section we briefly review the theory of isomonodromic deformations of linear second order differential equations following [21, 22, 23]. We shall use notation similar to [22].

The isomonodromic deformations of linear second order differential equations of the form

$$\frac{d^2v}{dz^2} + p_1(z)\frac{dv}{dz} + p_2(z)v = 0, \tag{3.1}$$

with p_1, p_2 being rational functions of z and parameters of deformation $t_1, \dots, t_n$, are governed by a completely integrable Hamiltonian system of partial differential equations with respect to the parameters. When there is one parameter of deformation, t, the Painlevé equations $P_I - P_{VI}$ appear as the compatibility condition of the extended linear system consisting of equation (3.1) and equaton

$$\frac{\partial v}{\partial t} = a(z,t)\frac{\partial v}{\partial z} + b(z,t)v. \tag{3.2}$$

The Painlevé equations $P_I - P_{VI}$ are nonlinear second order differential equations with the so-called Painlevé property. They have many interesting properties and appear in many areas of mathematics. See, for instance, [24, 25, 21] and numerous references therein. The completely integrable Hamiltonian system is then equivalent to a Painlevé equation for one of the variables. Below we shall present necessary formulas for equations $P_{II} - P_{VI}$.

To get the sixth Painlevé equation one chooses

$$p_1(z,t) = \frac{1-\kappa_0}{z} + \frac{1-\kappa_1}{z-1} + \frac{1-\theta}{z-t} - \frac{1}{z-\lambda}, \tag{3.3}$$

$$p_2(z,t) = \frac{\kappa}{z(z-1)} - \frac{t(t-1)H_{VI}}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \tag{3.4}$$

where

$$\begin{aligned} t(t-1)H_{VI} &= \lambda(\lambda-1)(\lambda-t)\mu^2 \\ &\quad -\{\kappa_0(\lambda-1)(\lambda-t) + \kappa_1\lambda(\lambda-t) + (\theta-1)\lambda(\lambda-1)\}\mu + \kappa(\lambda-t). \end{aligned}$$

Then the compatibility between (3.1) and (3.2) with certain $a(z,t)$ and $b(z,t)$ (see [21, 22, 23] for details) leads to the Hamiltonian system

$$\frac{d\lambda}{dt} = \frac{\partial H_{VI}}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H_{VI}}{\partial \lambda}$$

and by eliminating the function μ one can get the sixth Painlevé equation

$$\begin{aligned} \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &\quad + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha_6 + \beta_6\frac{t}{\lambda^2} + \gamma_6\frac{t-1}{(\lambda-1)^2} + \delta_6\frac{t(t-1)}{(\lambda-t)^2}\right), \end{aligned} \tag{3.5}$$

where

$$\alpha_6 = \frac{1}{2}\kappa_\infty^2, \quad \beta_6 = -\frac{1}{2}\kappa_0^2, \quad \gamma_6 = \frac{1}{2}\kappa_1^2, \quad \delta_6 = \frac{1}{2}(1-\theta^2)$$

and

$$\kappa = \frac{1}{4}(\kappa_0 + \kappa_1 + \theta - 1)^2 - \frac{1}{4}\kappa_\infty^2.$$

To get the fifth Painlevé equation one chooses

$$p_1(z,t) = \frac{1-\kappa_0}{z} + \frac{\eta t}{(z-1)^2} + \frac{1-\theta}{z-1} - \frac{1}{z-\lambda}, \tag{3.6}$$

$$p_2(z,t) = \frac{\kappa}{z(z-1)} - \frac{tH_V}{z(z-1)^2} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \tag{3.7}$$

where

$$tH_V = \lambda(\lambda-1)^2\mu^2 - \{\kappa_0(\lambda-1)^2 + \theta\lambda(\lambda-1) - \eta t\lambda\}\mu + \kappa(\lambda-1).$$

Then similarly to the previous case the corresponding Hamiltonian system with the Hamiltonian H_V leads to the fifth Painlevé equation

$$\begin{aligned} \frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(\alpha_5\lambda + \frac{\beta_5}{\lambda}\right) \\ &\quad + \gamma_5\frac{\lambda}{t} + \delta_5\frac{\lambda(\lambda+1)}{\lambda-1}, \end{aligned} \tag{3.8}$$

where

$$\alpha_5 = \frac{1}{2}\kappa_\infty^2, \quad \beta_5 = -\frac{1}{2}\kappa_0^2, \quad \gamma_5 = (1+\theta)\eta, \quad \delta_5 = \frac{1}{2}\eta^2$$

and

$$\kappa = \frac{1}{4}(\kappa_0+\theta)^2 - \frac{1}{4}\kappa_\infty^2.$$

To get the fourth Painlevé equation one chooses

$$p_1(z,t) = \frac{1-\kappa_0}{z} - \frac{z+2t}{2} - \frac{1}{z-\lambda}, \tag{3.9}$$

$$p_2(z,t) = \frac{1}{2}\theta_\infty - \frac{H_{IV}}{2z} + \frac{\lambda\mu}{z(z-\lambda)}, \tag{3.10}$$

where

$$H_{IV} = 2\lambda\mu^2 - (\lambda^2 + 2t\lambda + 2\kappa_0)\mu + \theta_\infty\lambda.$$

Then the corresponding Hamiltonian system with the Hamiltonian H_{IV} leads to the fourth Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2-\alpha_4)\lambda + \frac{\beta_4}{\lambda}, \tag{3.11}$$

where

$$\alpha_4 = -\kappa_0 + 2\theta_\infty + 1, \quad \beta_4 = -2\kappa_0^2.$$

The standard third Painlevé equation is given by

$$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\alpha_3\lambda^2 + \beta_3}{t} + \gamma_3\lambda^3 + \frac{\delta_3}{\lambda}. \tag{3.12}$$

However, for our purpose it is more convenient to consider an equation which can be obtained from (3.12) by changing $\lambda(t) \to \lambda(t^2)/t$ and by renaming the new variable $\tau = t^2$ as t again. This equation is given by

$$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\alpha_3\lambda^2 + \gamma_3\lambda^3}{4t^2} + \frac{\beta_3}{4t} + \frac{\delta_3}{4\lambda}. \tag{3.13}$$

Equation (3.13), which will be denoted by P'_{III}, appears in the result of isomonodromic deformations of the linear equation (3.1) with

$$p_1(z,t) = \frac{\eta_0 t}{z^2} + \frac{1-\theta_0}{z} - \eta_\infty - \frac{1}{z-\lambda}, \tag{3.14}$$

$$p_2(z,t) = \frac{\eta_\infty(\theta_0+\theta_\infty)}{2z} - \frac{tH'_{III}}{z^2} + \frac{\lambda\mu}{z(z-\lambda)}, \tag{3.15}$$

where

$$tH'_{III} = \lambda^2\mu^2 - \{\eta_\infty\lambda^2 + \theta_0\lambda - \eta_0 t\}\mu + \frac{1}{2}\eta_\infty(\theta_0+\theta_\infty)\lambda$$

and the parameters are related by

$$\alpha_3 = -4\eta_\infty\theta_\infty,\ \ \beta_3 = 4\eta_0(1+\theta_0),\ \ \gamma_3 = 4\eta_\infty^2,\ \ \delta_3 = -4\eta_0^2.$$

Finally, the second Painlevé equation

$$\frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \alpha_2 \tag{3.16}$$

appears in the result of isomonodromic deformations of the linear equation (3.1) with

$$p_1(z,t) = -2z^2 - t - \frac{1}{z-\lambda}, \tag{3.17}$$

$$p_2(z,t) = -(2\alpha_2+1)z - 2H_{II} + \frac{\mu}{z-\lambda}, \tag{3.18}$$

where

$$H_{II} = \frac{1}{2}\mu^2 - \left(\lambda^2 + \frac{1}{t}\right)\mu - \left(\alpha_2 + \frac{1}{2}\right)\lambda. \tag{3.19}$$

4. Main results

In this section we compare equations for the derivatives of the Heun functions with the linear differential equations whose isomonodromy deformations are governed by the Painlevé equations $P_{II} - P_{VI}$.

Let us consider the equation for the derivative of the general Heun function (1.3). By choosing parameters

$$\alpha\beta = \kappa_0 + \kappa_1 + \theta + \kappa,\ \ \beta = \frac{1}{2}(\pm\kappa_\infty - 1 - \kappa_0 - \kappa_1 - \theta),$$
$$\gamma = -\kappa_0,\ \ \delta = -\kappa_1,\ \ \varepsilon = -\theta,\ \ q = \alpha\beta\lambda,$$

we can calculate that the resulting equation is the same as equation (3.1) with (3.3), (3.4) and the expression for H_{VI} provided that

$$\mu = \frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta}{\lambda-t}.$$

If now λ and μ are viewed as functions of t, substituting this condition into the Hamiltonian system leading to the sixth Painlevé equation, we find that λ satisfies the Riccati equation

$$\frac{d\lambda}{dt} = \frac{\kappa_0 t - (1 + \kappa_0 + (\kappa_0 + \kappa_1)t + \theta)\lambda + (1 + \kappa_0 + \kappa_1 + \theta)\lambda^2}{t(t-1)}$$

and $\kappa_0 + \kappa_1 + \theta + \kappa = 0$. This gives classical solutions of the sixth Painlevé equation provided that $\kappa_0 = \pm\kappa_\infty - \theta - \kappa_1 - 1$. However, with this additional condition on the parameters we have $\alpha\beta = 0$ and $q = 0$.

In the equation for the derivative of the confluent Heun function (2.2) we first make the change of variables $v(z) \to (1 - z/(z-1))^\sigma v(z/(z-1))$, renaming the new independent variable as z again, then put

$$\gamma = -\kappa_0, \quad \delta = \kappa_0 + \theta + 2\sigma, \quad \varepsilon = -t\eta,$$
$$\sigma = -\frac{1}{2}(\kappa_0 \pm \kappa_\infty + \theta), \quad q = \frac{\alpha\lambda}{\lambda - 1}, \quad \alpha = \frac{1}{2}t\eta(2 + \kappa_0 \pm \kappa_\infty + \theta).$$

The resulting equation is the same as equation (3.1) with (3.6), (3.7) and the expression for H_V provided that

$$\mu = \frac{\kappa_0}{\lambda} - \frac{t\eta}{(\lambda-1)^2} + \frac{\theta - \kappa_0 \pm \kappa_\infty}{2(\lambda - 1)}.$$

Substituting this condition into the Hamiltonian system leading to the fifth Painlevé equation, we see that λ satisfies the Riccati equation

$$t\frac{d\lambda}{dt} \pm \kappa_\infty\lambda^2 - (\pm\kappa_\infty - \kappa_0 - t\eta) - \kappa_0 = 0$$

and $\eta(2 + \kappa_0 \pm \kappa_\infty + \theta) = 0$. Again, with this additional condition on the parameters we have $\alpha = 0$ and $q = 0$.

In the equation for the derivative of the bi-confluent Heun function (2.6) we take

$$\gamma = -\kappa_0, \quad \delta = -t, \quad q = \alpha\lambda, \quad \alpha = \frac{\theta_\infty + 1}{2}, \quad \varepsilon = -\frac{1}{2}.$$

The resulting equation is the same as equation (3.1) with (3.9), (3.10) and the expression for H_{IV} provided that

$$\mu = t + \frac{\kappa_0}{\lambda} + \frac{\lambda}{2}.$$

Substituting this condition into the Hamiltonian system leading to the fourth Painlevé equation, we find that λ satisfies the Riccati equation

$$\frac{d\lambda}{dt} = \lambda^2 + 2t\lambda + 2\kappa_0$$

and $\theta_\infty + 1 = 0$. Again, with this additional condition on the parameters we have $\alpha = 0$ and $q = 0$.

In the equation for the derivative of the double-confluent Heun function (2.6) we take

$$\gamma = t\eta_0, \quad \delta = -1 - \theta_0, \quad q = \alpha\lambda, \quad \alpha = \frac{1}{2}\eta_\infty(\theta_0 + \theta_\infty + 2), \quad \varepsilon = -\eta_\infty.$$

The resulting equation is the same as equation (3.1) with (3.14), (3.15) and the expression for H'_{III} provided that

$$\mu = \eta_\infty - \frac{t\eta_0}{\lambda^2} + \frac{\theta_0 + 1}{\lambda}.$$

Substituting this condition into the Hamiltonian system leading to the modified third Painlevé equation P'_{III}, we find that λ satisfies the Riccati equation

$$t\frac{d\lambda}{dt} = \eta_\infty \lambda^2 + (\theta_0 + 2)\lambda - t\eta_0$$

and $\eta_\infty(\theta_0 + \theta_\infty + 2) = 0$. Again, with this additional condition on the parameters we have $\alpha = 0$ and $q = 0$.

In the equation for the derivative of the tri-confluent Heun function (2.8) we take

$$\gamma = -t, \quad \delta = 0, \quad q = \alpha\lambda, \quad \alpha = 1 - 2\alpha_2, \quad \varepsilon = -2.$$

The resulting equation is the same as equation (3.1) with (3.17), (3.18) and the expression for H_{II} provided that

$$\mu = 2\lambda^2 + t.$$

Substituting this condition into the Hamiltonian system leading to the second Painlevé equation, we see that λ satisfies the Riccati equation

$$2\frac{d\lambda}{dt} = 2\lambda^2 + t$$

and $2\alpha_2 = 1$. Again, with this additional condition on the parameters we have $\alpha = 0$ and $q = 0$.

Hence, we see that in all cases we can reduce equations for the derivatives of the Heun functions to certain linear equations, isomonodromy deformations of which lead to the Painlevé equations with an additional constraint on λ and μ. However, in order to get classical solutions of the Painlevé equations we need an additional constraint on the parameters. Therefore, those linear equations isomonodromy deformations of which are described by classical solutions of the Painlveé equations cannot be obtained from the equations for the derivatives of the Heun functions.

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Список литературы

[1] https://theheunproject.org/bibliography.html
[2] K. Heun, "Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten", Math. Ann., **33**, 161 – 179 (1889).
[3] A. Ronveaux (Ed.), Heun's Differential Equations, Oxford University Press, Oxford (1995).
[4] S. Y. Slavyanov, W. Lay, Special Functions. A Unified Theory Based on Singularities, Oxford University Press, Oxford (2000).
[5] B. D. Sleeman, V. B. Kuznetsov, Heun Functions, https://dlmf.nist.gov/31.
[6] G. Filipuk, "A hypergeometric system of the Heun equation and middle convolution", J. Phys. A: Mathematical and Theoretical, **42**, 175208 (11 pp.) (2009).
[7] A. M. Ishkhanyan, "Appell hypergeometric expansions of the solutions of the general Heun equation", Constr. Approx. **49** (3), 445 – 459 (2019).
[8] A. Ishkhanyan, K. A. Suominen, "New solutions of Heun's general equation", J. Phys. A: Math. Gen. **36**, L81 – L85 (2003).
[9] T. A. Ishkhanyan, T. A. Shahverdyan, A. M. Ishkhanyan, "Expansions of the solutions of the general Heun equation giverned by two-term recurrence relations for coefficients", Advances in High Energy Physics, 4263678 (9pp.) (2018).
[10] M. N. Hounkonnou, A. Ronveaux, "About derivatives of Heun's functions from polynomial transformations of hypergeometric equations", Appl. Math. Comp. **209**, 421 – 424 (2009).
[11] K. Kuiken, "Heun's equation and the hypergeometric equation", SIAM J. Math. Anal. **10** (3), 655 – 657 (1979).
[12] C. Leroy, A. M. Ishkhanyan, "Expansions of the solutions of the confluent Heun equation in terms of the incomplete Beta and the Appell generalized hypergeometric functions", Integral Transforms Spec. Funct. **26**, 451 – 459 (2015).
[13] R. Maier, "The 192 solutions of the Heun equation", Math. Comp. **76** (258), 811 – 843 (2007).
[14] A. Ronveaux, "Factorization of the Heun's differential operator", Applied Mathematics and Computation, **141**, 177 – 184 (2003).
[15] R. Schäfke, D. Schmidt, "The connection problem for general linear ordinary differential equations at two regular singular points with applications to the theory of special functions", SIAM J. Math. Anal., **11** (5), 848 – 862 (1980).
[16] R. Vidunas, G. Filipuk, "A classification of coverings yielding Heun-to-hypergeometric reductions", Osaka Journal of Mathematics, **51**, 867 – 903 (2014).
[17] R. Vidunas, G. Filipuk, "Parametric transformations between the Heun and Gauss hypergeometric functions", Funkcialaj Ekvacioj, **56**, 271 – 321 (2013).
[18] R. V. Craster, V. H. Hoàng, "Applications of Fuchsian differential equations to free boundary problems", Proc. R. Soc. Lond. A, **454**, 1241 – 1252 (1998).
[19] A. J. Guttmann, T. Prellberg, "Staircase polygons, elliptic integrals, Heun functions, and lattice Green functions", Phys. Rev. E **47** (4), R2233 – R2236 (1993).
[20] G. S. Joyce, "On the cubic lattice Green functions", Proc. Roy. Soc. London Ser. A, **445**, 463 – 477 (1994).
[21] K. Iwasaki, H. Kimura, S. Shimomura, M. Yoshida, From Gauss to Painlevé. A modern theory of special functions, Aspects of Mathematics, E16. Friedr. Vieweg & Sohn, Braunschweig (1991).
[22] Y. Ohyama, S. Okumura, "A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations", J. Phys. A **39**, 12129 – 12151 (2006).
[23] K. Okamoto, "Isomonodromic deformation and Painlevé equations, and the Garnier system, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33**, 575 – 618 (1986).
[24] P. A. Clarkson, Painlevé Transcendents, https://dlmf.nist.gov/32.
[25] V. I. Gromak, I. Laine, S. Shimomura, "Painlevé differential equations in the complex plane", De Gruyter Studies in Mathematics, **28**, Walter de Gruyter & Co., Berlin (2002).

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ON THE NOETHER AND THE CAYLEY-BACHARACH THEOREMS WITH PD MULTIPLICITIES

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Abstract. In this paper we prove the Noether theorem with the multiplicities described by PD operators. Despite the known analog versions in this case the provided conditions are necessary and sufficient. We also prove the Cayley-Bacharach theorem with PD multiplicities. As far as we know this is the first generalization of this theorem for multiple intersections.

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1. Introduction

Let Π be the space of all bivariate polynomials. Let also Π_n be the space of bivariate polynomials of total degree at most n :

$$\Pi_n = \left\{ \sum_{i+j\le n} a_{ij}x^iy^j \right\}.$$

We have that

$$N := \dim \Pi_n = \binom{n+2}{2}. \tag{1.1}$$

Consider a set of s linear operators (functionals) on Π_n :

$$\mathcal{L}_s = \{L_1, \ldots, L_s\}.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$L_i p = c_i, \qquad i = 1, 2, \ldots s, \tag{1.2}$$

is called the Lagrange interpolation problem with operators.

In our paper we consider linear operators L which are partial differential operators evaluated at points:

$$Lf = p\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f\big|_{(x_0,y_0)},$$

where $p \in \Pi$. We say that L has degree d, where $d = \deg p$.

Definition 1.1. A set of operators $\mathcal{L}_s$ is called *n-correct* if for any data $\{c_1, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions (1.2).

A necessary condition of n-correctness of $\mathcal{L}_s$ is: $|\mathcal{L}_s| = s = N$.

A polynomial $p \in \Pi_n$ is called an *n-fundamental polynomial* for an operator $L_k \in \mathcal{X}_s$ if

$$L_i p = \delta_{ik},\ \ i = 1, \ldots, s,$$

where δ is the Kronecker symbol.

We denote the n-fundamental polynomial for $L \in \mathcal{L}_s$ by $p_L^\star = p_{L,\mathcal{L}}^\star$. Sometimes we also call fundamental a polynomial at which vanish all operators but one, since it is a nonzero constant times the fundamental polynomial.

The following is a Linear Algebra fact:

Proposition 1.2. *The set of operators $\mathcal{L}_N$, with $|\mathcal{L}_N| = N = \binom{n+2}{2}$, is n-poised if and only if the following implication holds:*

$$p \in \Pi_n \ \textit{and}\ L_i p = 0,\ \ i = 1, \ldots, N \Rightarrow p = 0.$$

1.1. **n-independent and n-dependent sets.** Next we introduce an important concept of n-dependence of sets of operators:

Definition 1.3. A set of operators $\mathcal{L}$ is called *n-independent* if each operator has a fundamental polynomial in Π_n. Otherwise, $\mathcal{L}$ is called *n-dependent.*

Clearly fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence of the set $\mathcal{L}$ is $|\mathcal{L}| \le N$.

Suppose λ is a point in the plane. Consider the operator L_λ defined by $L_\lambda f = f(\lambda)$. We say that a set of points $\mathcal{X}$ is n-independent (n-correct) if the set of operators $\{L_\lambda : \lambda \in \mathcal{X}\}$ is n-independent (n-correct).

Suppose a set of operators $\mathcal{L}$ is n-independent. Then by using the Lagrange formula:

$$p = \sum_{L \in \mathcal{L}} c_L p_{L,\mathcal{L}}^\star, \quad c_L = Lp,$$

we obtain a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.2).

Thus we get a simple characterization of n-independence:
A node set $\mathcal{L}_s$ is n-independent if and only if the interpolation problem (1.2) is *n-solvable,* meaning that for any data $\{c_1, \ldots, c_s\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (1.2).

Now suppose that $\mathcal{L}_s$ is n-dependent. Then some operator $L_{i_0},\ i_0 \in \{1, \ldots, s\}$, does not possess an n-fundamental polynomial. This means that the following implication holds:

$$p \in \Pi_n,\ L_{i_0} p = 0\ \forall i \in \{1, \ldots, s\} \setminus \{i_0\} \Rightarrow L_{i_0} p = 0.$$

Let ℓ be a line. We say that $p \in \Pi$ vanishes at $\lambda \in \ell$ with the multiplicity m if

$$(D_a)^i p\big|_\lambda = 0,\ i = 0, \dots, m-1,$$

where $a||\ell$ and D_a is the directional derivative.

The following proposition is well-known (see, e.g., [6] Proposition 1.3):

Proposition 1.4. *Suppose that ℓ is a line and a polynomial $p \in \Pi_n$ vanishes at some points of ℓ with the sum of multiplicities $n+1$. Then we have*

$$p = \ell r,\ \textit{where}\ r \in \Pi_{n-1}. \tag{1.3}$$

Note that this relation also yields that the mentioned $n+1$ conditions are independent, since $\dim \Pi_n - \dim \Pi_{n-1} = n+1$.

1.2. **Multiple intersections.** Let us start with the following well-known relation for polynomial R and functions g and f (see, e.g., [3], formula (16)):

$$R(D)[gf] = \sum_{i,j\geq 0} \frac{1}{i!j!} g^{(i,j)} R^{(i,j)}(D) f. \tag{1.4}$$

Here we use the following notations

$$R(D) := R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}), \quad R^{(i,j)} := D^{(i,j)} R := \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j R.$$

Notice that to verify (1.4) it suffices to check it for R being a monomial, which reduces (1.4) to Leibniz's rule.

To simplify notation, we shall use the same letter p, say, to denote the polynomial p and the curve given by the equation $p(x,y) = 0$. Thus the notation $\lambda \in p$ means that the point λ belongs to the curve $p(x,y) = 0$. Similarly $p \cap q$ for polynomials p and q stands for the set of intersection points of the curves $p(x,y) = 0$ and $q(x,y) = 0$.

Below we bring the definition of multiplicities described by PD operators (see [8], [4], [7]):

Definition 1.5. The following space is called the multiplicity space of the polynomial $p \in \Pi_n$ at the point $\lambda \in p$:

$$\mathcal{M}_\lambda(p) = \left\{h \in \Pi : D^\alpha h(D) p(\lambda) = 0\ \forall \alpha \in \mathbb{Z}_+^2\right\}.$$

Denote by $\mathcal{Z}_0 = p \cap q$ the set of intersection points of curves (polynomials) p and q.

Definition 1.6. Suppose that $p, q \in \Pi$ and $\lambda \in \mathcal{Z}_0$. Then the following space is called the multiplicity space of the intersection point λ :

$$\mathcal{M}_\lambda(p,q) = \mathcal{M}_\lambda(p) \cap \mathcal{M}_\lambda(q).$$

We have that (see [4]) the spaces $\mathcal{M}_\lambda(p,q)$ are D-invariant, meaning that

$$f \in \mathcal{M}_\lambda(p,q) \Rightarrow \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \in \mathcal{M}_\lambda(p,q). \tag{1.5}$$

The number $\dim \mathcal{M}_\lambda(p,q)$ is called the arithmetical multiplicity of the point λ.

Denote

$$\mathcal{M}(p,q) = \bigcup_{\lambda \in \mathcal{Z}_0} \mathcal{M}_\lambda(p,q).$$

We say that $f \in \Pi_k$ vanishes at $\mathcal{M}_\lambda(p,q)$ if $h(D)f(\lambda) = 0\ \forall h \in \mathcal{M}_\lambda(p,q)$.

We say also that the polynomials p and q have no intersection point at infinity if the leading homogeneous parts of p and q have no common factor.

Theorem 1.7 ([4], Theorem 3)**.** *Suppose that polynomials $p,q \in \Pi$, $\deg p = m$, $\deg q = n$, have no intersection point at infinity. Then the number of the intersection points, counted with the arithmetical multiplicities, equals mn :*

$$\sum_{\lambda \in \mathcal{Z}_0} \dim \mathcal{M}_\lambda(p,q) = mn. \tag{1.6}$$

Let us bring the formulation of this result in the homogeneous case. Let Π_n^0 be the space of trivariate homogeneous polynomials of total degree n. In analog way we are defining the multiplicity space $\mathcal{M}_\lambda^0(p,q)$.

Theorem 1.8 ([4], Corollary 3)**.** *Suppose that polynomials $p \in \Pi_m^0, q \in \Pi_n^0$ have no common component. Then the number of the intersection points, counted with the arithmetical multiplicities, equals mn :*

$$\sum_{\lambda \in \mathcal{Z}_0} \dim \mathcal{M}_\lambda^0(p,q) = mn.$$

2. The Noether theorem

Suppose that $p,q \in \Pi$, $\deg p = m$, $\deg q = n$, and $p \cap q := \{\lambda_1, \ldots, \lambda_s\}$. Let us choose a basis in the space $\mathcal{M}_{\lambda_k}(p,q)$ in the following way. Let $\{L_{m1}^k, \ldots, L_{mi_m}^k\}$ be a maximal independent set of linear operators with the highest degree $m := m_k$. Next we choose $\{L_{m-11}^k, \ldots, L_{m-1i_{m-1}}^k\}$ to be a maximal independent set of linear operators with the degree $m-1$. Continuing similarly for the degree 0 we have only one operator L_{01}^k.

It is easily seen that the above operators $L_{\mu i}^k$, form a basis in the linear space $\mathcal{M}_{\lambda_k}(p,q)$. Denote

$$\mathcal{L}^k(p,q) := \mathcal{L}^{\lambda_k}(p,q) := \bigcup_{i,\mu} L_{\mu i}^k, \qquad \mathcal{L}(p,q) := \bigcup_k \mathcal{L}^k(p,q).$$

Notice that, according to Theorem 1.7, we have that $|\mathcal{L}(p,q)| = mn$, provided that p and q have no intersection point at infinity.

Lemma 2.1. *The set of linear operators $\mathcal{L}(p,q)$ is γ_0-independent for sufficiently large γ_0.*

Proof. Consider the set of the linear operators of fixed node $\lambda_{k_0} = (x_0, y_0)$ of degrees up to ν, i.e.,

$$\mathcal{S}_{\nu,k_0} := \bigcup_{\mu \le \nu} L^{k_0}_{\mu i}.$$

Let us first find a fundamental polynomial p^* for an operator of the highest degree ν, say, for $L^{k_0}_{\nu 1}$ within $\mathcal{S}_{\nu,k_0}$. We seek p^* in the form

$$p^*(x,y) = \sum_{i+j=\nu} a_{ij}(x-x_0)^i(y-y_0)^j.$$

Then we readily get that $L^{k_0}_{\mu i}p^* = 0,$ if $\mu \le \nu - 1$. Now suppose that

$$L^{k_0}_{\nu s} f = p_s\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f\big|_{(x_0,y_0)}, \quad s = 1, \ldots, i_s,$$

where $p_s(x,y) = \sum_{i+j\le\nu} b^s_{ij}(x-x_0)^i(y-y_0)^j$. Then the conditions of the fundamentality of p^* reduce to the following linear system:

$$L^k_{\nu i}p^* = \sum_{i+j=\nu} a_{ij}b^s_{ij}i!j! = \delta_{ij}, \quad s = 1, \ldots, i_s.$$

The linear independence of highest degrees of the operators $L^k_{\nu i}$ means the independence of the vectors $\{b^s_{ij}\}_{i+j=\nu}$. Hence the above system has a solution.

Now notice that to complete the proof it is enough to obtain a fundamental polynomial of $L^k_{\nu i}$ over the set $\mathcal{S}_{\nu,k_0} \cup \bigcup_{k\ne k_0} \mathcal{L}^k(p,q)$. To this purpose for each $k \in \{1, \ldots, s\} \setminus \{k_0\}$ consider m_k lines passing through λ_k, and not passing through λ_{k_0}. Then by multiplying p^* by the product of these lines we obtain, in view of the formula (1.4), a polynomial which is a desired fundamental polynomial. $\square$

Next, we are going to prove the Noether theorem with the multiplicities described by PD operators.

Theorem 2.2. *Suppose that polynomials $p, q \in \Pi,$ $\deg p = m,$ $\deg q = n,$ have no intersection point at infinity. Suppose also that $f \in \Pi_k$ vanishes at $\mathcal{M}_\lambda(p,q)$ for each $\lambda \in p \cap q$. Then we have that*

$$f = Ap + Bq, \tag{2.1}$$

where $A \in \Pi_{k-m},$ $B \in \Pi_{k-n}$.

Note that the inverse theorem is true. Indeed, if (2.1) holds then $f \in \Pi_k$ and, in view of the formula (1.4), we have that and f vanishes at $\mathcal{M}_\lambda(p,q)$ for each $\lambda \in p \cap q$.

Proof. Step 1. Suppose that $k \geq k_0 = \max\{m+n, \gamma_0\}$, where γ_0 is chosen such that the set of linear operators $\mathcal{L}(p,q)$ is γ_0-independent.

Consider two linear spaces

$$\mathcal{V} = \{f \in \Pi_k \ : \ f \text{ vanishes at } \mathcal{M}_\lambda(p,q) \ \forall \lambda \in p \cap q\},$$

$$\mathcal{W} = \{Ap + Bq \ : \ A \in \Pi_{k-m}, \ B \in \Pi_{k-n}\}.$$

In view of the formula (1.4) we have that $\mathcal{W} \subset \mathcal{V}$. To prove the relation (2.1) we need to verify that $\mathcal{W} = \mathcal{V}$. To this end it suffices to show that $\dim \mathcal{W} = \dim \mathcal{V}$.

Since the set of linear operators $\mathcal{L}(p,q)$ is γ_0-independent we obtain readily that the set is also k-independent, where $k \geq \gamma_0$.

Hence, in view of Theorem 1.7, we have that

$$\dim \mathcal{V} = \dim \Pi_k - |\mathcal{L}(p,q)| = \binom{k+2}{2} - mn.$$

Denote

$$\mathcal{W}_1 = \{Ap \ : \ A \in \Pi_{k-m}\}, \quad \mathcal{W}_2 = \{Bq \ : \ B \in \Pi_{k-n}\}.$$

Since p and q have no common component we conclude that

$$\mathcal{W}_1 \cap \mathcal{W}_2 = \{Cpq \ : \ C \in \Pi_{k-m-n}\}.$$

Now we readily obtain that

$$\begin{aligned} \dim \mathcal{W} &= \dim(\mathcal{W}_1 + \mathcal{W}_2) = \dim \mathcal{W}_1 + \dim \mathcal{W}_2 - \dim(\mathcal{W}_1 \cap \mathcal{W}_2) \\ &= \binom{k-m+2}{2} + \binom{k-n+2}{2} - \binom{k-m-n+2}{2} \\ &= \binom{k+2}{2} - mn. \end{aligned} \tag{2.2}$$

The last equality here holds since $k \geq m+n$ (actually it holds for $k \geq m+n-2$).

Step 2. $n+m \leq k \leq k_0$.

Let us apply decreasing induction with respect to k. The first step $k = k_0$ was checked in Step 1. Assume Theorem is true for all f with $\deg f = k$ and let us prove that it is true also for all f with $\deg f = k-1$.

Suppose that f_0 is an arbitrary polynomial with $\deg f_0 = k-1$. Choose a line ℓ_0 such that

(i) $\ell_0 \cap p \cap q = \emptyset$, and

(ii) ℓ_0 intersects q at n points, counted also multiplicities, i.e., it does not intersect q at infinity.

We have that $\deg f_0\ell_0 = k$. Also, in view of the formula (1.4) and (1.5), i.e., the D-invariance of $\mathcal{M}_\lambda(p,q)$, we have that $f_0\ell_0$ vanishes at $\mathcal{M}_\lambda(p,q)$ for each $\lambda \in p \cap q$.

Hence, in view of the induction hypothesis, we get

$$f_0\ell_0 = Ap + Bq, \tag{2.3}$$

where $A \in \Pi_{k-m},\ B \in \Pi_{k-n}$.

We have that ℓ_0 intersects q at n points, counted also multiplicities. In view of (2.3) these (multiple) points are also zeros of A since p differs from zero there.

For every polynomial $C_0 \in \Pi_{k-m-n}$ we have also that

$$f_0\ell_0 = (A - C_0q)p + (B + C_0p)q. \tag{2.4}$$

Consider arbitrary $k-m-n+1$ points $\lambda_1, \ldots, \lambda_{k-m-n}$, in $\ell_0 \setminus q$. Choose $C_0 \in \Pi_{k-m-n}$ such that $A - C_0q$ is zero at these points. For this, according to Proposition 1.4, we just solve an independent interpolation problem

$$C_0(\lambda_i) = \frac{A(\lambda_i)}{q(\lambda_i)}, \quad i = 0, \ldots, k-m-n.$$

Note that the common n (multiple) zeros of ℓ_0 and q also are zeroes of $A - C_0q$. Thus, altogether we have that $A - C_0q$ is zero at $k-m-n+1+n = k-m+1$ points in ℓ_0. Thus, in view of Proposition 1.4, ℓ_0 divides $A - C_0q \in \Pi_{k-m}$. From (2.4) we readily conclude that ℓ_0 divides $B + C_0p$.

Finally by dividing the relation (2.4) by ℓ_0 we get that

$$f_0\ell_0 = A'p + B'q, \tag{2.5}$$

where $A' \in \Pi_{k-m-1},\ B \in \Pi_{k-n-1}$.

Step 3. $k \le n+m-1$.

Let us again apply decreasing induction with respect to k. The first step $k = m+n-1$ was checked in Step 2. Assume Theorem is true for all f with $\deg f = k$ and let us prove that it is true also for all f with $\deg f = k-1$.

Suppose that f_0 is an arbitrary polynomial with $\deg f_0 = k-1$. Choose a line ℓ_0 in the same way as in Step 2. Then we get the relation (2.3) where the polynomial $A \in \Pi_{k-m}$ has n zeros in ℓ_0, counting also the multiplicities. In this case we have that $k-m \le n-1$. Thus, in view of Proposition1.4, ℓ_0 divides A. From (2.4) we readily conclude that ℓ_0 divides also B. Finally by dividing the relation (2.3) by ℓ_0 we complete the proof as in Step 2. □

At the end let us bring the formulation of Theorem 2.2 in the homogeneous case.

Theorem 2.3. *Suppose that $p \in \Pi_m^0$ and $q \in \Pi_n^0$ have no common component. Suppose also that $f \in \Pi_k^0$ vanishes at $\mathcal{M}_\lambda^0(p,q)$ for each $\lambda \in p \cap q$. Then we have that*

$$f = Ap + Bq,$$

where $A \in \Pi_{k-m}^0,\ B \in \Pi_{k-n}^0$.

It is known that the set $\mathcal{Z}_0 := p \cap q$, where p and q are polynomials, of degree m and n, respectively, is $(m+n-2)$-independent, provided that $|\mathcal{Z}_0| = mn$. Below we prove this result without the last restriction (cf. [4], Corollary 1).

Corollary 2.4. *Suppose that polynomials $p, q \in \Pi,\ \deg p = m,\ \deg q = n$, have no common component. Then the set of linear operators $\mathcal{L}(p,q)$ and consequently the set $\mathcal{Z}_0$ are $(m+n-2)$-independent.*

Proof. Let us assume first that p and q have no intersection point at infinity. Then we have that $|\mathcal{L}(p,q)| = mn$. By using the evaluation (2.2) in the case $k = m+n-2$ we obtain

$$\begin{aligned} \dim \mathcal{W} = \dim(\mathcal{W}_1 + \mathcal{W}_2) &= \dim \mathcal{W}_1 + \dim \mathcal{W}_2 - \dim(\mathcal{W}_1 \cap \mathcal{W}_2) \\ &= \binom{n}{2} + \binom{m}{2} - 0 = \binom{m+n}{2} - mn. \end{aligned} \tag{2.6}$$

Thus we have that $\dim \Pi_{m+n-2} - \dim \mathcal{W} = mn$. This means that the set of linear operators $\mathcal{L}(p,q)$ and consequently $\mathcal{Z}_0$ is $(m+n-2)$-independent.

Now assume only that p and q have no common component. Let us use the concept of the associate polynomial (see section 10.2, [9]).

Let $p(x,y) = \sum_{i+j\le m} a_{ij}x^iy^j$ and $\deg p = m$. Then the following trivariate homogeneous polynomial is called associated with $p:$

$$\bar{p}(x,y,z) = \sum_{i+j+k=m} a_{ij}x^iy^jz^k.$$

Evidently we have that

$$p = p_1p_2 \Leftrightarrow \bar{p} = \bar{p}_1\bar{p}_2.$$

It is easily seen from here that polynomials p and q have no common component if and only if $\bar{p}$ and $\bar{q}$ have no common component. By applying Theorem 2.3 to the polynomials $\bar{p}$ and $\bar{q}$ we get that the set of linear operators $\mathcal{L}^0(p,q)$ is $(m+n-2)$-independent. Therefore its subset corresponding to the finite intersection points, i.e., to $\mathcal{Z}_0$, is $(m+n-2)$-independent, which implies the desired result. $\square$

3. The Cayley-Bacharach theorem

The evaluation (2.2) in the case $k = m+n-3$ gives

$$\begin{aligned} \dim \mathcal{W} = \dim(\mathcal{W}_1 + \mathcal{W}_2) &= \dim \mathcal{W}_1 + \dim \mathcal{W}_2 - \dim(\mathcal{W}_1 \cap \mathcal{W}_2) \\ &= \binom{n-1}{2} + \binom{m-1}{2} - 0 = \binom{m+n-1}{2} - (mn-1). \end{aligned} \tag{3.1}$$

Thus we have that $\dim \Pi_{m+n-2} - \dim \mathcal{W} = mn - 1$, i.e., out of mn linear operators in $\mathcal{L}(p,q)$ only $mn-1$ are linearly independent.

According to the Cayley-Bacharach classic theorem (see, e.g., [1], [5]), i.e., in the case $|\mathcal{Z}_0| = mn$, where $\mathcal{Z}_0 := p \cap q$, we have that any subset of $\mathcal{Z}_0$ of cardinality $mn-1$ is $(m+n-3)$-independent. This means that no point from $\mathcal{Z}_0$ has a fundamental polynomial of degree $(m+n-3)$, i.e., for any point $\lambda_0 \in \mathcal{Z}_0$ the following implication holds:

$$p \in \Pi_{m+n-3},\ p(\lambda) = 0\ \forall \lambda \in \mathcal{Z}_0 \setminus \{\lambda_0\} \Rightarrow p(\lambda) = 0\ \forall \lambda \in \mathcal{Z}_0.$$

In this section we are going to study the situation in the general multiple intersection case. Suppose $p \in \Pi_m$,

$$p(x,y) = \sum_{i+j\leq m} a_{ij} x^i y^j.$$

Denote the kth homogeneous part of p by $p^{\{k\}}$, i.e.,

$$p^{\{k\}}(x,y) = \sum_{i+j=k} a_{ij} x^i y^j.$$

We accept a very common restriction from the theory of intersection. Namely, we assume that the two polynomials p and q have no common tangent line at an intersection point $\lambda \in \mathcal{Z}_0$. This means that the lowest homogeneous parts of the polynomials have no common factor at this point.

Theorem 3.1. *Suppose that polynomials $p, q \in \Pi$, $\deg p = m$, $\deg q = n$, have no intersection point at infinity and $\lambda \in \mathcal{Z}_0$. Suppose also that p and q have no common tangent line at λ. Then we have that the set of linear operators $\mathcal{L}^{\lambda}(p,q)$ contains only one operator of the highest degree: $\bar{L}$. Suppose also that $f \in \Pi_{m+n-3}$ vanishes at $\mathcal{L}(p,q) \setminus \{\bar{L}\}$. Then we have that f vanishes at all $\mathcal{L}(p,q)$.*

Proof. Assume, without loss of generality, that $\lambda = \theta := (0,0)$. Suppose that p and q are bivariate polynomials having n_0 and m_0-fold zero at the origin, respectively, $n_0, m_0 \geq 1$:

$$p(x,y) = \sum_{m_0 \leq i+j \leq m} a_{ij} x^i y^j, \quad q(x,y) = \sum_{n_0 \leq i+j \leq n} b_{ij} x^i y^j.$$

Suppose also that p and q have no common tangent line at the origin, i.e., $p^{\{m_0\}}$ and $q^{\{n_0\}}$ have no common factor.

Let $\bar{\mathcal{L}} := \{\bar{L}_1, \ldots, \bar{L}_s\}$ be a maximal independent set of linear operators with the highest degree in the space $\mathcal{M}_\theta(p,q)$.

Assume that $f \in \Pi_{m+n-3}$ vanishes at $\mathcal{L}(p,q) \setminus \bar{\mathcal{L}}$. We are going to prove that f vanishes at $\mathcal{L}(p,q)$.

This shall complete the proof of Theorem. Indeed, as was verified above, there are $mn-1$ linearly independent operators in the set of mn linear operators $\mathcal{L}(p,q)$, which clearly implies here that $s=1$.

Let ℓ be any line passing through θ. By using the formula (1.4) with $g=\ell,\ f=f$ and $R\in\mathcal{L}(p,q)$, we obtain that the polynomial ℓf vanishes at $\mathcal{L}(p,q)$. Therefore, since $\deg \ell f = m+n-2$, we get from Theorem 2.2 that

$$\ell f = A(\ell)p + B(\ell)q, \tag{3.2}$$

where $A(\ell)\in\Pi_{n-2},\ B(\ell)\in\Pi_{m-2}$. Assume, without loss of generality, that $m_0\le n_0$. Assume also that $m_0\ge 2$. If $m_0=1$ we go to the *final part of the proof.* Now we are going to prove that

$$A(\ell)^{\{k\}} = \ell A'_{k-1} \quad k=0,\ldots,n_0-2, \tag{3.3}$$

where $A'_{k-1},\in\Pi^0_{k-1}$, do not depend on ℓ, and

$$B(\ell)^{\{k\}} = \ell B'_{k-1} \quad k=0,\ldots,m_0-2, \tag{3.4}$$

where $B'_{k-1},\in\Pi^0_{k-1}$, do not depend on ℓ.

First let us prove (3.3) for $k\le n_0-m_0-1$. Let us apply induction on k. Consider the case $k=0$. Then we get from the relation (3.2) that $A(\ell)^{\{0\}}p^{\{m_0\}}=\ell f^{\{m_0-1\}}$. Thus we have $xf^{\{m_0-1\}}=c_1p^{\{m_0\}}$ and $yf^{\{m_0-1\}}=c_2p^{\{m_0\}}$, where c_1 and c_2 are constants. Therefore we have that $(c_2x-c_1y)f^{\{m_0-1\}}=0$, i.e., $f^{\{m_0-1\}}=0$. Thus $A(\ell)^{\{0\}}=0=\ell\cdot 0$. Assume that (3.3) is true for all k not exceeding s and let us prove it for $k=s+1$. We readily get from the relation (3.2) that

$$A(\ell)^{\{s+1\}}p^{\{m_0\}} + A(\ell)^{\{s\}}p^{\{m_0+1\}} + \cdots + A(\ell)^{\{0\}}p^{\{m_0+s+1\}} = \ell f^{\{m_0+s+1\}}. \tag{3.5}$$

We have that all terms above except possibly the first have factor ℓ. Hence we get that $A(\ell)^{\{s+1\}}=\ell A'_s$. In fact we have this relation for all ℓ except m_0 tangent lines of p at θ. Then by a continuity argument we get the relation for all ℓ.

Next, by dividing (3.9) by ℓ we see that A'_s does not depend on ℓ.

Now assume that $n_0-m_0\le k\le n_0-2$. Here we are going to prove (3.3) for k and (3.4) for $k-n_0+m_0$. Let us again apply induction on k. Consider the case $k=n_0-m_0$. We get from the relation (3.2) that

(3.6)
$$A(\ell)^{\{n_0-m_0\}}p^{\{m_0\}} + A(\ell)^{\{n_0-m_0-1\}}p^{\{m_0+1\}} + \cdots + A(\ell)^{\{0\}}p^{\{n_0\}} + B(\ell)^{\{0\}}q^{\{n_0\}} = \ell f^{\{n_0-1\}}.$$

Now let us use $\ell=\ell_1$ which is a tangent line of q at θ, i.e., $q^{\{n_0\}}=\ell_1\tilde q$, where $\tilde q\in\Pi_{n_0-1}$.

We have that all terms in (3.6) except possibly the first have factor ℓ_1. Hence we get that $A := A(\ell_1)^{\{n_0-m_0\}} = \ell_1 A'_{n_0-m_0-1}$.

Meanwhile, let us verify also that if $\ell_1 = y - k_1 x$ is a factor of multiplicity μ of $q^{\{n_0\}}$ then it is a factor of multiplicity at least μ in A. Assume that

$$A = C_1 \prod_i (y - a_i x),\ q^{\{n_0\}} = C_2 \prod_i (y - b_i x).$$

Assume also ℓ is given by an equation $y - kx = 0$. By setting in (3.6) $y = kx$, and by using the induction hypothesis, we obtain

$$C_1 p^{\{m_0\}}(x, kx) \prod_i (k - a_i)x = C_2 B(\ell)^{\{0\}}(x, kx) \prod_i (k - b_i)x. \tag{3.7}$$

Consider both sides of (3.7) as polynomials on k. Now k_1 is a root of the right hand side of multiplicity at least μ. On the other hand $k = k_1$ is not a root of $p^{\{m_0\}}(x, kx)$ since p and q have no common factor. Thus we get that $k = k_1$ is a root of multiplicity at least μ in $q^{\{n_0\}}(x, kx)$, i.e., $y - k_1 x$ is a factor of multiplicity at least μ in $q^{\{n_0\}}(x, y)$.

Next, we have that

$$\begin{aligned} A(\ell)^{\{n_0-m_0\}} &= A(\ell_1)^{\{n_0-m_0\}} + A(\ell - \ell_1)^{\{n_0-m_0\}} \\ &= \ell A'_{n_0-m_0-1} + (\ell - \ell_1) A'_{n_0-m_0-1} + A(\ell - \ell_1)^{\{n_0-m_0\}} \\ &= \ell A'_{n_0-m_0-1} - (k - k_1) x A'_{n_0-m_0-1} - (k - k_1) A(x)^{\{n_0-m_0\}} \\ &= \ell A'_{n_0-m_0-1} - (k - k_1) \left[x A'_{n_0-m_0-1} - A(x)^{\{n_0-m_0\}} \right]. \end{aligned} \tag{3.8}$$

We have that $A(\ell)^{\{n_0-m_0\}}$ contains all factors of q_{n_0}. Thus the polynomial of degree $n_0 - m_0$ in the square brackets contains all factors of q_{n_0} except possibly ℓ_1, in all $n_0 - 1$ factors. Hence this polynomial is identically zero and $A(\ell)^{\{n_0-m_0\}} = \ell A'_{n_0-m_0-1}$. As above we readily conclude that $A'_{n_0-m_0-1}$ does not depent on ℓ. Similarly by using tangent lines of p we get that $B(\ell)^{\{0\}} = 0 = \ell \cdot 0$.

Now assume that (3.3) is true for k not exceeding s and (3.4) is true for k not exceeding $s + m_0 - n_0$. Let us prove (3.3) for $k = s+1$ and (3.4) for $k = s + m_0 - n_0 + 1$.

We get from the relation (3.2) that

$$\begin{aligned} &A(\ell)^{\{s+1\}} p^{\{m_0\}} + A(\ell)^{\{s\}} p^{\{m_0+1\}} + \cdots + A(\ell)^{\{0\}} p^{\{m_0+s+1\}} \\ &+ B(\ell)^{\{s+m_0-n_0+1\}} q^{\{n_0\}} + B(\ell)^{\{s+m_0-n_0\}} q^{\{n_0+1\}} + \cdots + B(\ell)^{\{0\}} q^{\{m_0+s+1\}} \\ &\qquad = \ell f^{\{m_0+s+1\}}. \end{aligned} \tag{3.9}$$

Here, in the same way as above, by using tangent lines of p and q at θ, we complete the proof of this part.

Now let us go to the *final part of the proof.* Let us choose a line ℓ_0 whose intersection multiplicity with p at θ equals to m_0. We also require that ℓ_0 intersects $\mathcal{Z}$ only at θ. We have that outside of θ the line ℓ_0 intersects p at $m - m_0$ points, counting also the multiplicities. We deduce from the relation (3.2), with $\ell = \ell_0$, that these $m - m_0$ points are roots for $B(\ell_0)$, since q does not vanish there. Then, in view of the relation (3.4), we have that

$$B(\ell_0) = \sum_{i=0}^{m-2} B^{\{i\}}(\ell_0) = \sum_{i=m_0-1}^{m-2} B^{\{i\}}(\ell_0).$$

Thus, by assuming that $\ell_0 = y - k_0 x$, we see that the trace of the polynomial $B(\ell_0)$ on the line ℓ_0 has the form

$$B(\ell_0)(x, k_0 x) = \sum_{i=m_0-1}^{m-2} b_i x^i = x^{m_0-1} \sum_{i=0}^{m-m_0-1} b_{i+m_0-1} x^i.$$

On the other hand this polynomial vanishes at $m - m_0$ nonzero points, counting also the multiplicities. Hence, in view of Proposition 1.4, we conclude that $B(\ell_0)$ has a factor ℓ_0. Now we readily get from the relation (3.2), with $\ell = \ell_0$, that $A(\ell_0)$ also has a factor ℓ_0. Then by dividing the relation (3.2) by ℓ_0 we get that

$$f = Ap + Bq,$$

where $A \in \Pi_{n-3},\ B \in \Pi_{m-3}$. Finally from this relation we readily conclude that f vanishes at $\mathcal{L}(p, q)$. □

At the end let us consider a simple example. Let $p(x, y) = x^m$ and $q(x, y) = y^n$. Then we have that

$$\mathcal{L}(p, q) = \mathcal{L}_\theta(p, q) = \left\{x^i y^j : i \le m-1,\ j \le n-1\right\}.$$

It is easily seen that in this set there is only one operator of the highest degree:

$$\bar{L} = \left(\frac{\partial}{\partial x}\right)^{m-1} \left(\frac{\partial}{\partial y}\right)^{n-1}.$$

Also for this operator we have that the set of the operators $\mathcal{L}(p, q) \setminus \{\bar{L}\}$ is $(m+n-3)$-independent. Moreover, only the operator $\bar{L} \in \mathcal{L}(p, q)$ has this property.

Список литературы

[1] D. Eisenbud, M. Green and J. Harris, “Cayley-Bacharach theorems and conjectures”, Bull. Amer. Math. Soc. (N.S.), **33** (3), 295 -– 324 (1996).

[2] J. L. Coolidge, A Treatise on Algebraic Plane Curves, Dover Publications, Inc., New York (1959).

[3] H. Hakopian, “A multivariate analog of fundamental theorem of algebra and Hermite interpolation”, Constructive Theory of Functions, Varna 2002 DARBA, Sofia (B. Bojanov, Ed.), 1 – 18, (2003).

[4] H. Hakopian, “The multivariate fundamental theorem of Algebra, Bezout’s theorem and Nullstellensatz”, in: D. K. Dimitrov et al. (eds.) Approximation Theory: a volume dedicated to Borislav Bojanov, 73 -– 97, Marin Drinov Acad. Publ. House, Sofia (2004).

[5] H. Hakopian, K. Jetter and G. Zimmermann, "Vandermonde matrices for intersection points of curves", Jaen J. Approx. **1** (1), 67 -– 81 (2009).
[6] H. Hakopian, K. Jetter and G. Zimmermann, "A new proof of the Gasca-Maeztu conjecture for $n = 4$," J. Approx. Theory **159**, 224 -– 242 (2009).
[7] H. Hakopian, M. Tonoyan, "Partial differential analogs of ordinary differential equations and systems", New York J. Math. **10**, 89 -– 116 (2004).
[8] M. G. Marinari, H. M. Möller and T. Mora, "On multiplicities in polynomial system solving", Trans. Amer. Math. Soc. **348**, no. 8, 3283 – 3321 (1996).
[9] R. Walker, Algebraic Curves, Princeton, New Jersey (1950).

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A UNIQUENESS THEOREM FOR FRANKLIN SERIES

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Abstract. In this paper we obtain, that if the partial sums $\sigma_{q_k}(x)$ of a Franklin series converge in measure to a function f, the ratio $\frac{q_{k+1}}{q_k}$ is bounded and the majorant of partial sums satisfies to a necessary condition, then the coefficients of the series are restored by the function f.

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1. Introduction

It is well known that there are trigonometric series converging almost everywhere to zero and having at least one non-zero coefficient. This also applies to the series in other classical orthogonal systems, for instance, to the series in Haar, Walsh and Franklin systems.

The uniqueness problem and reconstruction of coefficients of series by various orthogonal systems has been considered in a number of papers. Uniqueness theorems for almost everywhere convergent or summable trigonometric series were obtained in the papers [1] and [4], under some additional conditions imposed on the series. Results on uniqueness and restoration of coefficients for series by Haar, Franklin and generalized Haar systems have been obtained, for instance, in the papers [3],[6],[7] and [9]-[12] .

In this paper we will consider series by Franklin system.

The orthonormal Franklin system consists of piecewise linear and continuous functions. This system was constructed by Franklin [2] as the first example of a complete orthonormal system, which is a basis in $\mathbb{C}[0,1]$.

Let $n = 2^\mu + \nu$, $\mu \geq 0$, where $1 \leq \nu \leq 2^\mu$. Denote

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}}, & \text{for} \quad 0 \leq i \leq 2\nu, \\ \frac{i-\nu}{2^\mu}, & \text{for} \quad 2\nu < i \leq n. \end{cases} \tag{1.1}$$

By S_n we denote the space of functions that are continuous and piecewise linear on $[0,1]$ with nodes $\{s_{n,i}\}_{i=0}^n$, that is $f \in S_n$ if $f \in C[0,1]$, and it is linear on each closed interval $[s_{n,i-1}, s_{n,i}], i = 1, 2, \cdots, n$. It is clear, that $\dim S_n = n+1$, and the set $\{s_{n,i}\}_{i=0}^{n}$ is obtained by adding the point $s_{n,2\nu-1}$ to the set $\{s_{n-1,i}\}_{i=0}^{n-1}$. Hence, there exists a unique function $f_n \in S_n$, which is orthogonal to S_{n-1} and $\|f_n\|_2 = 1$. Setting $f_0(x) = 1,\ f_1(x) = \sqrt{3}(2x-1)$ for $x \in [0,1]$, we obtain an orthonormal system $\{f_n(x)\}_{n=0}^{\infty}$, which was defined equivalently by Franklin [2].

Here we quote a result by G. Gevorkyan [3] on restoration of coefficients of series by Franklin system.

Specifically, in [3] it was proved that if the Franklin series $\sum_{n=0}^{\infty} a_n f_n(x)$ converges a.e. to a function $f(x)$ and

$$\lim_{\lambda\to\infty}\left(\lambda \cdot |\{x \in [0,1] : \sup_{k\in\mathbb{N}} |S_k(x)| > \lambda\}|\right) = 0,$$

where

$$S_k(x) = \sum_{j=0}^{k} a_j f_j(x)$$

then the coefficients a_n of the Franklin series can be reconstructed by the following formula,

$$a_n = \lim_{\lambda\to\infty} \int_0^1 \big[f(x)\big]_\lambda f_n(x)dx,$$

where

$$\big[f(x)\big]_\lambda = \begin{cases} f(x), & \text{if} \quad |f(x)| \le \lambda, \\ 0, & \text{if} \quad |f(x)| > \lambda. \end{cases}$$

Similar result on uniqueness is also obtained for the Haar system (see [5]).

Afterwards Gevorkyan's result was extended by V. Kostin [10] to the series by generalized Haar system.

Consider the d-dimensional Franklin series

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}),$$

where $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}_0^d$ is a vector with non-negative integer coordinates, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad \mathbf{x} = (x_1, \cdots, x_d) \in [0,1]^d$ and

$$f_{\mathbf{n}}(\mathbf{x}) = f_{n_1}(x_1) \cdot \ldots \cdot f_{n_d}(x_d).$$

The following theorem for multiple Franklin series was proved in [7].

Theorem A. *If the partial sums*

$$\sigma_{2^k}(\boldsymbol{x}) = \sum_{\boldsymbol{n}: n_i \le 2^k, i=1,\cdots,d} a_{\boldsymbol{n}} f_{\boldsymbol{n}}(\boldsymbol{x})$$

converge in measure to a function f and

$$\lim_{m\to\infty}\left(\lambda_m \cdot |\{\boldsymbol{x} \in [0,1]^d : \sup_k |\sigma_{2^k}(\boldsymbol{x})| > \lambda_m\}|\right) = 0$$

for some sequence $\lambda_m \to +\infty$*, then for any* $\boldsymbol{n} \in \mathbb{N}_0^d$

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_m} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}.$$

In this theorem instead of the partial sums $\sigma_{2^k}(\mathbf{x})$ one can take square partial sums $\sigma_{q_k}(\mathbf{x})$, where $\{q_k\}$ is any increasing sequence of natural numbers, for which the ratio q_{k+1}/q_k is bounded. The following theorem is proved in [11].

Theorem B. *Let* $\{q_k\}$ *be an increasing sequence of natural numbers such that the ratio* q_{k+1}/q_k *is bounded. If the sums* $\sigma_{q_k}(\boldsymbol{x})$ *converge in measure to a function* f *and there exists a sequence* $\lambda_m \to +\infty$ *so that*

$$\lim_{m\to\infty} \left(\lambda_m \cdot |\{\boldsymbol{x} \in [0,1]^d : \sup_k |\sigma_{q_k}(\boldsymbol{x})| > \lambda_m\}|\right) = 0,$$

then for any $\boldsymbol{n} \in \mathbb{N}_0^d$

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_m} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}.$$

2. Lemmas and the main result

Let functions $h_m(x) : [0,1] \to \mathbb{R}$, satisfy the following conditions:

$$0 \le h_1(x) \le h_2(x) \le \cdots \le h_m(x) \le \cdots, \lim_{m\to\infty} h_m(x) = \infty, \tag{2.1}$$

there exists dyadic points

$$0 = t_{m,0} < t_{m,1} < t_{m,2} < \cdots < t_{m,n_m} = 1,$$

so that the intervals

$$I_k^m = [t_{m,k-1}, t_{m,k}), \qquad k = 1, \cdots, n_m,$$

are dyadic as well, i.e. I_k^m is of the form

$$\mathcal{D} = \left\{ \left[\frac{i}{2^j}, \frac{i+1}{2^j}\right), 0 \le i \le 2^j - 1, j \ge 0 \right\}$$

and the function $h_m(x)$ is constant on those intervals,

$$h_m(x) = \lambda_k^m, \qquad x \in I_k^m, \quad k = 1, \cdots, n_m.$$

Moreover

$$\inf_{m,k} \int_{I_k^m} h_m(x) dx = \inf_{m,k} |I_k^m| \lambda_k^m > 0, \tag{2.2}$$

$$\sup_{m,k} \left(\frac{\lambda_k^m}{\lambda_{k-1}^m} + \frac{\lambda_{k-1}^m}{\lambda_k^m} \right) < +\infty \tag{2.3}$$

and

$$\sup_{m,k} \left(\frac{|I_k^m|}{|I_{k-1}^m|} + \frac{|I_{k-1}^m|}{|I_k^m|} \right) < +\infty. \tag{2.4}$$

In other words, for any function h_m the interval $[0,1]$ can be partitioned into dyadic intervals, so that the values of the function on neighbouring intervals are equivalent to each other and so are the lengths of neighbouring intervals. The following theorem is proved in [9].

Theorem C. *Let $h_m(x)$ be sequence of functions satisfying conditions* $(2.1)-(2.3)$. *If the partial sums $\sigma_{2^\nu} = \sum_{n=0}^{2^\nu} a_n f_n$ converge in measure to a function f and*

$$\lim_{m\to\infty} \int_{\{x\in[0,1];\quad \sup_\nu|\sigma_\nu(x)|>h_m(x)\}} h_m(x)dx = 0,$$

then for any $n \in \mathbb{N}_0$

$$a_n = \lim_{m\to\infty} \int_0^1 \left[f(x)\right]_{h_m(x)} f_n(x)dx,$$

where

$$\left[f(x)\right]_{\lambda(x)} = \begin{cases} f(x), & \text{if} \quad |f(x)| \le \lambda(x), \\ 0, & \text{if} \quad |f(x)| > \lambda(x). \end{cases}$$

Now we are in position to state the main result of this paper.

Theorem 2.1. Let $h_m(x)$ be sequence of functions satisfying conditions $(2.1)-(2.3)$, and $\{q_k\}$ be an increasing sequence of natural numbers such that the ratio q_{k+1}/q_k is bounded. If the partial sums $\sigma_{q_k}(x)$ converge in measure to a function f and

$$\lim_{m\to\infty} \int_{\{x\in[0,1];\quad \sup_k|\sigma_{q_k}(x)|>h_m(x)\}} h_m(x)dx = 0, \tag{2.5}$$

then for any $n \in \mathbb{N}_0$

$$a_n = \lim_{m\to\infty} \int_0^1 \left[f(x)\right]_{h_m(x)} f_n(x)dx. \tag{2.6}$$

To prove Theorem 2.1 we will need the following two lemmas.

Lemma 2.1. *Let $0 = t_0 < t_1 < \cdots < t_n = 1$ and $h(x) = \lambda_k$, if $x \in I_k := [t_{k-1}, t_k)$ and $I_k \in \mathcal{D}$, when $k = 1, \cdots, n$. Moreover $\gamma > 0$*

$$\frac{1}{\gamma} \le \frac{\lambda_k}{\lambda_{k+1}} \le \gamma, \text{ when } k = 1, \cdots, n-1, \tag{2.7}$$

then there exists points $0 = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_s = 1$ such that $h(x) = \tilde{\lambda}_l$, $x \in \tilde{I}_l = [\tilde{t}_{l-1}, \tilde{t}_l) \in \mathcal{D}$, $l = 1, \cdots s$. Besides that

$$\frac{1}{2\gamma} \le \frac{|\tilde{I}_l|}{|\tilde{I}_{l+1}|} \le 2\gamma, \tag{2.8}$$

$$\frac{1}{\gamma} \le \frac{\tilde{\lambda}_l}{\tilde{\lambda}_{l+1}} \le \gamma, \text{ when } l = 1, \cdots, s-1, \tag{2.9}$$

$$\min_l \int_{\tilde{I}_l} h_m(x)dx = \min_k \int_{I_k} h_m(x)dx > 0. \tag{2.10}$$

The proof of the Lemma 2.1 can be found in [9], but we present it here for the sake of completeness.

Proof. Denote

$$c = \min_k \int_{I_k} h_m(x)dx = \min_k \lambda_k |I_k|,$$

and let $1 \leq k_0 \leq n$ such that $\lambda_{k_0}|I_{k_0}| = c$. From definition c follows that for any i, $-k_0 + 1 \leq i \leq n - k_0$ there exists $n_i \geq 0$ such that

$$2^{n_i} c \leq \lambda_{k_0+i}|I_{k_0+i}| < 2^{n_i+1}c. \tag{2.11}$$

Suppose that $n_0 = 0$ and denote

$$t_{i,j} = t_{k_0+i-1} + \frac{|I_{k_0+i}|}{2^{n_i}} j, \quad \text{when} \quad j = 0, \cdots, 2^{n_i},$$

$$I_{i,j} = [t_{i,j-1}, t_{i,j}), \quad \text{and} \quad \lambda_{i,j} = \lambda_i, \quad \text{when} \quad j = 1, \cdots, 2^{n_i}.$$

Therefore

$$\int_{I_{0,1}} h_m(x)dx = c \leq \int_{I_{i,j}} h_m(x)dx = \lambda_{i,j}|I_{i,j}| < 2c.$$

From the definition $c, I_{i,j}$, (2.11) and (2.7) follows that

$$|I_{i,j}| = |I_{i,1}| < \frac{2c}{\lambda_{k_0+i}} \leq \frac{2c\gamma}{\lambda_{k_0+i-1}} \leq 2\gamma |I_{i-1,2^{n_{i-1}}}|,$$

similarly we obtain

$$|I_{i,j}| = |I_{i,1}| \geq \frac{c}{\lambda_{k_0+i}} \geq \frac{c}{\gamma\lambda_{k_0+i-1}} \geq \frac{1}{2\gamma}|I_{i-1,2^{n_{i-1}}}|.$$

From the last two inequalities follows that the ratio of the lengths of intervals $I_{i,j}$ with common endpoint is not greater than 2γ. By renumbering the intervals $\{I_{i,j}; -k_0+1 \leq i \leq n-k_0, 1 \leq j \leq 2^{n_i}\}$ in increasing order with respect to the left endpoint, we obtain the intervals $\tilde{I}_l, l = 1, \cdots, \sum_{i=-k_0+1}^{n-k_0} 2^{n_i}$, which satisfy the condition (2.8). From the definition $\tilde{I}_l$ it follows that the function $h_m(x)$ is constant,

$$h_m(x) = \tilde{\lambda}_l, \quad x \in \tilde{I}_l$$

and from (2.7) we get (2.9), so $\frac{\tilde{\lambda}_l}{\tilde{\lambda}_{l+1}} = 1$ or there exists k, such that

$$\frac{\tilde{\lambda}_l}{\tilde{\lambda}_{l+1}} = \frac{\lambda_k}{\lambda_{k+1}}.$$

□

Lemma 2.2. *Let $h_m(x)$ be sequence of functions satisfying conditions (2.1)−(2.3), then there exists dyadic points $0 = \tilde{t}_{m,0} < \tilde{t}_{m,1} < \cdots < \tilde{t}_{m,\tilde{n}_m} = 1$ so that the* *intervals* $\tilde{I}_k^m = [\tilde{t}_{m,k-1}, \tilde{t}_{m,k}) \in \mathcal{D}$, $k = 1, \cdots, \tilde{n}_m$ *are dyadic as well and the function* $h_m(x)$ *is constant on those intervals,*

$$h_m(x) = \tilde{\lambda}_k^m, \text{ if } x \in \tilde{I}_k^m, \; k = 1, \cdots, \tilde{n}_m$$

and the conditions $(2.2) - (2.4)$ *are satisfied.*

3. The proof of the main theorem

Let $\{s_{n,i}\}_{i=0}^n$ be the points given in (1.1), $s_{n,-1} = 0$ and $s_{n,n+1} = 1$. Let us define the function $N_i^n(x)$ as follows. It is linear on intervals $[s_{n,j-1}, s_{n,j}], j = 1, 2, \cdots, n$, and

$$N_i^n(s_{n,j}) = \begin{cases} 1, & \text{if} \quad i = j, \\ & \\ 0, & \text{if} \quad i \neq j, \end{cases} \qquad j = 0, 1, \cdots, n.$$

Let $\{q_k\}$ be an increasing sequence of natural numbers and M be a number satisfying the inequality

$$\frac{q_{k+1}}{q_k} \leq M, \text{ for all } k \in \mathbb{N}.$$

For any $j \in \{0, 1, \cdots, q_\nu\}$ denote

$$\Delta_j^\nu := [s_{q_\nu, j-1}, s_{q_\nu, j+1}],$$

$$M_j^{q_\nu}(x) := \frac{N_j^{q_\nu}(x)}{\|N_j^{q_\nu}(x)\|_1} = \frac{2}{|\Delta_j^\nu|} N_j^{q_\nu}(x).$$

Obviously

$$\frac{1}{2q_\nu} \leq |\Delta_j^\nu| \leq \frac{4}{q_\nu}, \tag{3.1}$$

$$\operatorname{supp} M_j^{q_\nu} = \Delta_j^\nu \qquad \text{and} \qquad \int_0^1 M_j^{q_\nu}(x) dx = 1.$$

Recall that

$$\sigma_{q_\nu}(x) = \sum_{n=0}^{q_\nu} a_n f_n(x).$$

Let's denote

$$\sigma^*(x) = \sup_\nu |\sigma_{q_\nu}(x)|,$$

and prove that for any j_0, ν_0 the following statement is true:

$$\int_0^1 \sigma_{q_{\nu_0}}(x) M_{j_0}^{q_{\nu_0}}(x) dx = \lim_{m \to \infty} \int_0^1 \left[f(x)\right]_{h_m(x)} M_{j_0}^{q_{\nu_0}}(x) dx.$$

For any $m \in \mathbb{N}$ denote

$$E_m := \{x \in \operatorname{supp}(M_{j_0}^{q_{\nu_0}}) = \Delta_{j_0}^{\nu_0} : \; \sigma^*(x) \geq h_m(x)\}.$$

From $(2.3), (2.4)$ it follows that there exists $\gamma > 0$ such that

$$\frac{\lambda_k^m}{\gamma} \leq \lambda_{k+1}^m \leq \gamma \lambda_k^m \quad \text{and} \quad \frac{|I_k^m|}{\gamma} \leq |I_{k+1}^m| \leq \gamma |I_k^m|. \tag{3.2}$$

Denote

$$\varepsilon_0 = \inf_{m,k} \int_{I_k^m} h_m(x)dx = \inf_{m,k} \lambda_k^m |I_k^m| > 0.$$

Let ε be an arbitrary positive number. Under the conditions of the theorem a number m_0 can be chosen to satisfy

$$2^7 M \int_{E_m} h_m(x)dx < \varepsilon, \ m \geq m_0.$$

Take

$$\varepsilon \leq \frac{2^3 \varepsilon_0}{\gamma}.$$

Let M_1 be a number such that

$$h_m(x) \geq M_1, \text{ for all } x \in [0,1], \text{ when } m \geq m_1,$$

then

$$M_1 |E_m| \leq \int_{E_m} h_m(x)dx < \frac{\varepsilon}{2^7 M}, \text{when } m \geq \max(m_0, m_1) =: m_2.$$

Therefore

$$|E_m| \leq \frac{\varepsilon}{2^7 M M_1},$$

let's take

$$M_1 = \frac{q_{\nu_0} \varepsilon}{2^2}.$$

Hence from (3.1) we obtain

$$|E_m| < \frac{2^2 \varepsilon}{2^7 M q_{\nu_0} \varepsilon} = \frac{1}{2^5 M q_{\nu_0}} \leq \frac{|\Delta_{j_0}^{\nu_0}|}{2^4 M}. \tag{3.3}$$

Let's fix a number $m \geq m_2$ and prove that

$$|E_m \cap I_k^m| < \frac{|I_k^m|}{8M}, \qquad k = 1, \cdots, n_m. \tag{3.4}$$

Suppose that there exists k_0, such that

$$|E_m \cap I_{k_0}^m| \geq \frac{|I_{k_0}^m|}{8M},$$

therefore

$$\varepsilon_0 \leq \lambda_{k_0}^m |I_{k_0}^m| \leq 8M \lambda_{k_0}^m |E_m \cap I_{k_0}^m| \leq 8M \int_{E_m} h_m(x)dx \leq \frac{\varepsilon}{2^4} \leq \frac{\varepsilon_0}{2\gamma} < \varepsilon_0,$$

which is a contradiction.

Note that for any $J \in \mathcal{D}$, which can represented in the form $\bigcup_{k=l}^{j} I_k^m$, from (3.4) we get

$$|J \cap E_m| = \sum_{k=l}^{j} |I_k^m \cap E_m| \leq \frac{1}{8M} \sum_{k=l}^{j} |I_k^m| = \frac{1}{8M} |J|,$$

therefore

$$|J \cap E_m| \leq \frac{|J|}{8M}, \quad \text{for any} \quad J = \bigcup_{k=l}^{j} I_k^m \in \mathcal{D}. \tag{3.5}$$

It is clear that if $J \in \mathcal{D}$ and $J \supset I_{k_0}^m$, then $J = \bigcup_{k=l}^{j} I_k^m$, when $l \le k_0 \le j$. Suppose $\nu \ge \nu_0$. We set

$$\Omega_\nu := \{A : A = [s_{q_\nu, j-1}, s_{q_\nu, j}] \quad \text{and} \quad A \subset \Delta_{j_0}^{\nu_0}\}.$$

Obviously

$$\frac{1}{2q_\nu} \le |A| \le \frac{2}{q_\nu}, \quad \text{for all} \quad A \in \Omega_\nu.$$

If $\nu = \nu_0$, then we set

$$\Omega_{\nu_0}^1 = \left\{ A \in \Omega_{\nu_0} : |A \cap E_m| > \frac{1}{8M}|A| \right\}, \quad Q_{\nu_0} = \bigcup_{A \in \Omega_{\nu_0}^1} A,$$

and

$$\Omega_{\nu_0}^2 = \{A \in \Omega_{\nu_0} : A \not\subset Q_{\nu_0}\}, \quad P_{\nu_0} = \bigcup_{A \in \Omega_{\nu_0}^2} A.$$

From (3.3) we have, that

$$Q_{\nu_0} = \emptyset, \quad \text{and} \quad P_{\nu_0} = \operatorname{supp}(M_{j_0}^{q_{\nu_0}}).$$

Now suppose we have defined the sets $\Omega_{\nu'}^1, \ \Omega_{\nu'}^2, \ Q_{\nu'}$ for all $\nu' < \nu$. Let's denote

$$\Omega_\nu^1 = \left\{ A \in \Omega_\nu : |A \cap E_m| > \frac{1}{8M}|A| \ \text{ and } \ A \not\subset \bigcup_{\nu' < \nu} Q_{\nu'} \right\}, \tag{3.6}$$

$$Q_\nu = \bigcup_{A \in \Omega_\nu^1} A, \quad \Omega_\nu^2 = \left\{ A \in \Omega_\nu : A \not\subset \bigcup_{\nu' \le \nu} Q_{\nu'} \right\}, \quad P_\nu = \bigcup_{A \in \Omega_\nu^2} A.$$

Thus we have defined the families $\Omega_\nu^1, \Omega_\nu^2$ and the sets Q_ν, P_ν, satisfying to the following conditions,

$$\Omega_\nu^1 \subset \Omega_\nu, \quad \Omega_\nu^2 \subset \Omega_\nu,$$

$$\operatorname{supp}(M_{j_0}^{q_{\nu_0}}) = P_\nu \cup \left(\bigcup_{\nu' \le \nu} Q_{\nu'} \right), \quad P_\nu \cap \left(\bigcup_{\nu' \le \nu} Q_{\nu'} \right) = \emptyset, \tag{3.7}$$

$$Q_{\nu'} \cap Q_{\nu''} = \emptyset, \quad \text{if} \quad \nu' \ne \nu''. \tag{3.8}$$

From (3.6) and (3.8) we obtain

$$\left| \bigcup_{\nu' \le \nu} Q_{\nu'} \right| < 8M|E_m|, \quad \text{for any} \quad \nu.$$

Now let us prove that for any $A \in \Omega_\nu^1, \ \ \nu \ge \nu_0$, there exists k such that $A \subset I_k^m$. Otherwise, there exists k_0 such that $A \supset I_{k_0}^m$. Since $A = \bigcup_{k=l}^{j} I_k^m, \ l \le k_0 \le j$, therefore from (3.5) we get $|A \cap E_m| \le |A|/8M$, but $A \in \Omega_\nu^1$. For any $\nu > \nu_0$ denote

$$J_\nu = \{j : \Delta_j^\nu \cap Q_\nu \ne \emptyset, \quad \Delta_j^\nu \subset P_{\nu-1}\}.$$

Now let us prove that

$$|\sigma_{q_\nu}(x)| \leq 3h_m(x), \quad \text{if} \quad x \in \Delta_j^\nu, j \in J_\nu. \tag{3.9}$$

Suppose $A \in \Omega_\nu^1$, with $A \subset \Delta_j^\nu$, therefore $A \subset I_l^m$ for some l. Let's prove that

$$\Delta_j^\nu \subset I_k^m \cup I_{k+1}^m, \quad \text{when} \quad k = l-1 \quad \text{or} \quad k = l. \tag{3.10}$$

Without loss of generality suppose that $\Delta_j^\nu \supset I_{l+1}^m$. From (3.2) we get

$$2|A| = |\Delta_j^\nu| > |I_{l+1}^m| \geq \frac{|I_l^m|}{\gamma},$$

therefore

$$\varepsilon_0 \leq |I_l^m|\lambda_l^m < 2\gamma|A|\lambda_l^m \leq 16\gamma M|A \cap E_m|\lambda_l^m \leq 16\gamma M \int_{E_m} h_m(x)dx$$

$$< \frac{16\gamma M\varepsilon}{2^7 M} \leq \frac{16 \cdot 2^3\gamma\varepsilon_0}{2^7\gamma} = \varepsilon_0,$$

which is a contradiction.

Let Δ_1 and Δ_2 be respectively the left and right halves of the interval Δ_j^ν, $\Delta_1 \subset \Delta_j^\nu$, $\Delta_2 \subset \Delta_j^\nu$. From (3.10) we get, that there exists l_1, l_2 such that $\Delta_1 \subset I_{l_1}^m$, $\Delta_2 \subset I_{l_2}^m$, it is clear that $|l_1 - l_2| \leq 1$. Therefore

$$h_m(x) = \lambda_{l_j}^m, \qquad x \in \Delta_j, j = 1, 2. \tag{3.11}$$

Since $\Delta_1, \Delta_2 \subset \Delta_j^\nu \subset P_{\nu-1}, (j \in J_\nu)$, then there exists $\tilde{\Delta}_1, \tilde{\Delta}_2 \in \Omega_{\nu-1}$, so that $\Delta_i \subset \tilde{\Delta}_i \subset P_{\nu-1},\ i = 1, 2$, we get that

$$|\Delta_i \cap E_m| \leq |\tilde{\Delta}_i \cap E_m| \leq \frac{1}{8M}|\tilde{\Delta}_i| \leq \frac{1}{8M} \cdot \frac{2}{q_{\nu-1}} \leq \frac{1}{4q_\nu} \leq \frac{|\Delta_i|}{2}. \tag{3.12}$$

Suppose that $x \in \Delta_1$, (the case $x \in \Delta_2$ is considered similarly). Since $\sigma_{q_\nu}(x)$ is a linear function on $\Delta_1 = [\alpha, \beta]$, we have set

$$I := \{t \in \Delta_1 : |\sigma_{q_\nu}(t)| < \lambda_{l_1}^m\}$$

is an interval. From (3.11) and (3.12) we get

$$|I| = |\{t \in \Delta_1 : |\sigma_{q_\nu}(t)| < h_m(t)\}| \geq |\Delta_1 \cap E_m^c| \geq \frac{1}{2}|\Delta_1|. \tag{3.13}$$

Since $\sigma_{q_\nu}(t)$ is linear, then

$$|\sigma'_{q_\nu}(t)| < \frac{2\lambda_{l_1}^m}{\frac{1}{2}(\beta - \alpha)} = \frac{4\lambda_{l_1}^m}{\beta - \alpha}. \tag{3.14}$$

From (3.14) we get

$$|\sigma_{q_\nu}(\alpha)| < \lambda_{l_1}^m + \frac{4\lambda_{l_1}^m}{\beta - \alpha} \cdot \frac{\beta - \alpha}{2} = 3\lambda_{l_1}^m,$$

similarly we obtain

$$|\sigma_{q_\nu}(\beta)| < 3\lambda_{l_1}^m.$$

Using the last inequalities and (3.10), we get

$$|\sigma_{q_\nu}(t)| < 3h_m(t), \quad t \in [\alpha, \beta] = \Delta_1.$$

Similarly we obtain (according to definition of P_ν), that if $\Delta_j^\nu \subset P_\nu$, then

$$|\sigma_{q_\nu}(x)| \le 3h_m(x), \quad \text{if} \quad x \in \Delta_j^\nu \subset P_\nu.$$

Now let's define by induction expansions ψ_n for $M_{j_0}^{q_{\nu_0}}$,

$$M_{j_0}^{q_{\nu_0}} = \psi_n = \sum_{\nu \le n} \sum_{j \in J_\nu} \alpha_{\nu,j}^n M_j^{q_\nu} + \sum_{j:\Delta_j^n \subset P_n} \alpha_j^n M_j^{q_n}, \tag{3.15}$$

where

$$\sum_{\nu \le n} \sum_{j \in J_\nu} \alpha_{\nu,j}^n + \sum_{j:\Delta_j^n \subset P_n} \alpha_j^n = 1, \quad \alpha_{\nu,j}^n \ge 0, \ \alpha_j^n \ge 0. \tag{3.16}$$

Since $P_{\nu_0} = \operatorname{supp}(M_{j_0}^{q_{\nu_0}})$, then $\psi_{\nu_0} = M_{j_0}^{q_{\nu_0}}$. Suppose we have defined expansions $\psi_{\nu_0}, \cdots, \psi_n$, satisfying (3.15) and (3.16). Clearly for any $\Delta_j^n \subset P_n$ we have

$$M_j^{q_n}(x) = \sum_{\nu:\Delta_\nu^{n+1} \subset \operatorname{supp} M_j^{q_n}} \beta_\nu M_\nu^{q_{n+1}}(x), \quad \beta_\nu \ge 0. \tag{3.17}$$

Note that if $\Delta_j^n \subset P_n$ and $\Delta_\nu^{n+1} \subset \operatorname{supp} M_j^{q_n} = \Delta_j^n$, then either $\Delta_\nu^{n+1} \cap Q_{n+1} \ne \emptyset$ and, therefore $\nu \in J_{n+1}$, or $\Delta_\nu^{n+1} \subset P_{n+1}$. Therefore, inserting the expressions (3.17) in (3.15) and grouping similar terms, we obtain

$$M_{j_0}^{q_{\nu_0}} = \psi_{n+1} = \sum_{\nu \le n+1} \sum_{j \in J_\nu} \alpha_{\nu,j}^{n+1} M_j^{q_\nu} + \sum_{j:\Delta_j^{n+1} \subset P_{n+1}} \alpha_j^{n+1} M_j^{q_{n+1}}.$$

Since the integrals of all functions $M_j^{q_\nu}$ are 1, we get that

$$\sum_{\nu \le n+1} \sum_{j \in J_\nu} \alpha_{\nu,j}^{n+1} + \sum_{j:\Delta_j^{n+1} \subset P_{n+1}} \alpha_j^{n+1} = 1,$$

therefore for any n

$$(\sigma_{q_n}, M_{j_0}^{q_{\nu_0}}) = \sum_{\nu \le n} \sum_{j \in J_\nu} \alpha_{\nu,j}^n (\sigma_{q_n}, M_j^{q_\nu}) + \sum_{j:\Delta_j^n \subset P_n} \alpha_j^n (\sigma_{q_n}, M_j^{q_n}).$$

Note that

$$(f_p, M_j^{q_\nu}) = \int_0^1 f_p(x) M_j^{q_\nu}(x) dx = 0, \quad \text{if} \quad \nu \ge \nu_0 \quad \text{and} \quad p > q_\nu.$$

Therefore

$$(\sigma_{q_n}, M_j^{q_\nu}) = \sum_{p=0}^{q_n} a_p (f_p, M_j^{q_\nu}) = \sum_{p=0}^{q_\nu} a_p (f_p, M_j^{q_\nu}) = (\sigma_{q_\nu}, M_j^{q_\nu}). \tag{3.18}$$

Hence we have

$$\int_0^1 \sigma_{q_{\nu_0}}(t) M_{j_0}^{q_{\nu_0}}(t) dt - \int_0^1 \left[f(t)\right]_{h_m(t)} M_{j_0}^{q_{\nu_0}}(t) dt = (\sigma_{q_n} - \left[f\right]_{h_m}, M_{j_0}^{q_{\nu_0}}) \tag{3.19}$$

$$= \sum_{\nu\le n}\sum_{j\in J_\nu}\alpha^n_{\nu,j}(\sigma_{q_\nu}-\left[f\right]_{h_m}, M_j^{q_\nu}) + \sum_{j:\Delta_j^n\subset P_n}\alpha^n_j(\sigma_{q_n}-\left[f\right]_{h_m}, M_j^{q_n}) =: I_n^1 + I_n^2.$$

Using (3.9) and (3.18), for I_n^1 we will have the estimate

$$|I_n^1| = \left|\sum_{\nu\le n}\sum_{j\in J_\nu}\alpha^n_{\nu,j}(\sigma_{q_\nu}-\left[f\right]_{h_m}, M_j^{q_\nu})\right| \le \sum_{\nu\le n}\sum_{j\in J_\nu}\alpha^n_{\nu,j}(|\sigma_{q_\nu}|+h_m, M_j^{q_\nu})$$

$$\le 4\sum_{\nu\le n}\sum_{j\in J_\nu}\alpha^n_{\nu,j}(h_m, M_j^{q_\nu}) = 4(h_m, \sum_{\nu\le n}\sum_{j\in J_\nu}\alpha^n_{\nu,j}M_j^{q_\nu}).$$

By

$$\sum_{\nu\le n}\sum_{j\in J_\nu}\alpha^n_{\nu,j}M_j^{q_\nu} \le M_{j_0}^{q_{\nu_0}},$$

we have

$$|I_n^1| \le 4\int\limits_{\bigcup\limits_{\nu\le n}\bigcup\limits_{j\in J_\nu}\Delta_j^\nu} h_m(t)M_{j_0}^{q_{\nu_0}}(t)dt.$$

Denote

$$J_\nu^1 := \{j\in J_\nu : \exists k \;\; s.t. \;\; \Delta_j^\nu \subset I_k^m\}, \quad J_\nu^2 := J_\nu \setminus J_\nu^1,$$

$$A_n := \bigcup_{\nu\le n}\bigcup_{j\in J_\nu^1}\Delta_j^\nu, \;\; B_n := \bigcup_{\nu\le n}\bigcup_{j\in J_\nu^2}\Delta_j^\nu.$$

It is easy to notice that

$$|I_n^1| \le \left(\int_{A_n} h_m(t)M_{j_0}^{q_{\nu_0}}(t)dt + \int_{B_n} h_m(t)M_{j_0}^{q_{\nu_0}}(t)dt\right) \le C\left(\int_{A_n} h_m(t)dt + \int_{B_n} h_m(t)dt\right)$$

$$=: C(I_n^3 + I_n^4).$$

From (3.10) we get that for any $j\in J_\nu^2$ there exists k, such that

$$\Delta_j^\nu \subset I_k^m \cup I_{k+1}^m,$$

and the definitions Ω_ν^1 and Q_ν, we obtain that for any k there exists $(\nu(k), j(k))$ pair, such that

$$j(k)\in J_\nu^2 \quad \text{and} \quad \Delta_{j(k)}^{\nu(k)} \subset I_k^m \cup I_{k+1}^m.$$

Applying (3.2) we get

$$I_n^4 \le \sum_{k=1}^{n_m}(\lambda_k^m + \lambda_{k+1}^m)|\Delta_{j(k)}^{\nu(k)}| \le (\gamma+1)\sum_{k=1}^{n_m}\lambda_k^m|\Delta_{j(k)}^{\nu(k)}|,$$

and from (3.6) we get

$$|\Delta_{j(k)}^{\nu(k)}| \le 2|Q_{\nu(k)}\cap(I_k^m\cup I_{k+1}^m)| \le 2|\bigcup_{\nu\le n}Q_\nu\cap(I_k^m\cup I_{k+1}^m)|.$$

Therefore

$$I_n^4 \le 2(\gamma+1)\left(\sum_{k=1}^{n_m}\lambda_k^m\left|\bigcup_{\nu\le n}Q_\nu\cap I_k^m\right| + \gamma\sum_{k=1}^{n_m}\lambda_{k+1}^m\left|\bigcup_{\nu\le n}Q_\nu\cap I_{k+1}^m\right|\right)$$

$$= 2(\gamma+1)^2 \sum_{k=1}^{n_m} \lambda_k^m \left| \bigcup_{\nu \le n} Q_\nu \cap I_k^m \right| =: 2(\gamma+1)^2 I_n^5.$$

Using (3.6) we can estimate I_n^5 as follows,

$$I_n^5 \le 8M \sum_{k=1}^{n_m} \lambda_k^m \left| E_m \cap \left(\bigcup_{\nu \le n} Q_\nu \right) \cap I_k^m \right| \le 8M \sum_{k=1}^{n_m} \lambda_k^m \left| E_m \cap I_k^m \right| = 8M \int_{E_m} h_m(t)dt$$

$$< \frac{8M\varepsilon}{2^7 M} = \frac{\varepsilon}{2^4}.$$

If $j \in J_\nu^1$,then there exists k, such that $\Delta_j^\nu \subset I_k^m$, therefore

$$|\Delta_j^\nu| \le 2|\Delta_j^\nu \cap Q_\nu|,$$

from the last inequality we get,

$$|A_n \cap I_k^m| \le 2| \bigcup_{\nu \le n} Q_\nu \cap I_k^m|.$$

Therefore

$$I_n^3 = \int_{A_n} h_m(t)dt = \sum_{k=1}^{n_m} \lambda_k^m |A_n \cap I_k^m| \le 2 \sum_{k=1}^{n_m} \lambda_k^m | \bigcup_{\nu \le n} Q_\nu \cap I_k^m| = 2I_n^5.$$

So

$$|I_n^1| \le C \left(\frac{(\gamma+1)^2 \varepsilon}{2^3} + \frac{2\varepsilon}{2^4} \right) = \frac{C\varepsilon(1+(\gamma+1)^2)}{2^3} = \varepsilon C_\gamma. \tag{3.20}$$

Now let us estimate I_n^2. Since

$$\sum_{j:\Delta_j^n \subset P_n} \alpha_j^n M_j^{q_n} \le M_{j_0}^{q_{\nu_0}}, \quad \text{then}$$

$$|I_n^2| \le (|\sigma_{q_n} - \left[f\right]_{h_m}|, \sum_{j:\Delta_j^n \subset P_n} \alpha_j^n M_j^{q_n}) \le \int\limits_{\bigcup\limits_{j:\Delta_j^n \subset P_n} \Delta_j^n} |\sigma_{q_n}(t) - \left[f(t)\right]_{h_m(t)}| M_{j_0}^{q_{\nu_0}}(t)dt$$

$$\le C \int\limits_{\bigcup\limits_{j:\Delta_j^n \subset P_n} \Delta_j^n} |\sigma_{q_n}(t) - \left[f(t)\right]_{h_m(t)}|dt.$$

Denote

$$C_n = \bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n \cap E_m, \quad D_n = \bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n \cap E_m^c \cap \{t, |\sigma_{q_n}(t) - f(t)| \le \varepsilon\},$$

$$F_n = \bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n \cap E_m^c \cap \{t, |\sigma_{q_n}(t) - f(t)| > \varepsilon\}.$$

It is clear see that

$$C_n \cup D_n \cup F_n = \bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n \quad \text{and} \quad |f(t)| \le h_m(t) \quad \text{a. e.,} \quad \text{when} \quad t \in D_n \cup F_n \subset E_m^c.$$

Therefore

$$|I_n^2| \le C \left(\int_{C_n} |\sigma_{q_n}(t) - \big[f(t)\big]_{h_m(t)}| dt + \int_{D_n} |\sigma_{q_n}(t) - f(t)| dt + \int_{F_n} |\sigma_{q_n}(t) - f(t)| dt \right)$$

$$=: C(I_n^6 + I_n^7 + I_n^8).$$

If $t \in C_n$, then

$$|\sigma_{q_n}(t) - \big[f(t)\big]_{h_m(t)}| \le |\sigma_{q_n}(t)| + |\big[f(t)\big]_{h_m(t)}| \le 4h_m(t),$$

and

$$I_n^6 \le 4 \int_{C_n} h_m(t) dt \le 4 \int_{E_m} h_m(t) dt \le \frac{2^2 \varepsilon}{2^7 M} = \frac{\varepsilon}{2^5 M} < \varepsilon.$$

From definition of D_n it follows that if $t \in D_n$, then

$$|\sigma_{q_n}(t) - f(t)| \le \varepsilon, \ \text{ therefore } \ I_n^7 \le \int_{D_n} \varepsilon \le \varepsilon.$$

Since $\sigma_{q_n}(x)$ converge in measure to the function f, then there exists n such that

$$|\{t, |\sigma_{q_n}(t) - f(t)| > \varepsilon\}| < \frac{\varepsilon}{\max\{h_m(t), \ t \in [0,1]\}},$$

and

$$|\sigma_{q_n}(t) - f(t)| \le |\sigma_{q_n}(t)| + |f(t)| \le 4h_m(t), \ \text{ for a. e.,} \ \ t \in F_n \subset E_m^c.$$

Therefore

$$I_n^8 \le 4 \int_{F_n} h_m(t) dt \le 4 \max\{h_m(t), \ t \in [0,1]\} \cdot |\{t, |\sigma_{q_n}(t) - f(t)| > \varepsilon\}| < 4\varepsilon.$$

So $|I_n^2| \le 6\varepsilon$, therefore by (3.19) and (3.20), we get

$$\left| (\sigma_{q_{\nu_0}}, M_{j_0}^{q_{\nu_0}}) - \int_0^1 \big[f(t)\big]_{h_m(t)} M_{j_0}^{q_{\nu_0}}(t) dt \right| \le C_\gamma \varepsilon.$$

Now let's prove that for any $n \in \mathbb{N}_0$ the coefficient a_n can be reconstructed by (2.6). Take arbitrary n and choose ν so that $q_\nu \ge n$, then $f_n \in S_{q_\nu}$. Taking into account that the system of functions $\{M_j^{q_\nu}\}_{j \in \{0,1,\cdots,q_\nu\}}$ is a basis in the space S_{q_ν}, one can find number $\beta_j, j \in \{0, 1, \cdots, q_\nu\}$, such that

$$f_n(x) = \sum_{j \in \{0,1,\cdots,q_\nu\}} \beta_j M_j^{q_\nu}(x).$$

Therefore

$$a_n = (\sigma_{q_\nu}, f_n) = \sum_{j=0}^{q_\nu} \beta_j (\sigma_{q_\nu}, M_j^{q_\nu}) = \sum_{j=0}^{q_\nu} \beta_j \lim_{m \to \infty} \int_0^1 \big[f(x)\big]_{h_m(x)} M_j^{q_\nu}(x) dx$$

$$= \lim_{m \to \infty} \int_0^1 \big[f(x)\big]_{h_m(x)} f_n(x) dx.$$

Список литературы

[1] A. B. Aleksandrov, “A-integrability of the boundary values of harmonic functions”, Math. Notes **30**, no. 1, 515 – 523 (1981).

[2] Ph. Franklin, “A set of continuous orthogonal functions”, Math. Ann., 522 – 528 (1928).

[3] G. G. Gevorkyan, “Uniqueness of Franklin series”, Math. Notes of the Academy of Sciences of the USSR, **46**, no. 2, 609 – 615 (1989).

[4] G. G. Gevorkyan, “On the uniqueness of trigonometric series”, Mathematics of the USSR-Sbornik, **182**, no. 2, 325 – 338 (1991).

[5] G. G. Gevorkyan, “On uniqueness of additive functions of dyadic cubes and series by Haar systems”, J. Contemp. Math. Analysis, **30**, no. 5, 2 – 13 (1995).

[6] G. G. Gevorkyan, K. A. Navasardyan, “On Haar series of A-integrable functions”, J. Contemp. Math. Analysis, **52**, no. 3, 149 – 160 (2017).

[7] G. G. Gevorkyan, M. P. Poghosyan, “On recovering of coefficients of a Franklin series with the ”good” majorant of partial sums”, Izv. NAN Armenii. Ser. Math., **52**, no. 5, 254 – 260 (2017).

[8] B. S. Kashin, A. A. Sahakyan, Orthogonal Series, Moscow, AFC (1999).

[9] K. A. Keryan, “Uniqueness theorem for Franklin series”, Izv. NAN Armenii. Ser. Math., **52**, no. 2, 26 – 38 (2017).

[10] V. V. Kostin, “Reconstructing coefficients of series from certain orthogonal systems of functions”, Mathematical Notes, **73**, no. 5, 662 – 679 (2003).

[11] K. A. Navasardyan, “Uniqueness theorems for multiple Franklin series”, Proceedings of the YSU, Physical and Mathematical Sciences, **51**, no. 3, 241 – 249 (2017).

[12] K. A. Navasardyan, “On a uniqueness theorem for the Franklin system”, Proceedings of the YSU, Physical and Mathematical Sciences, **52**, no. 2, 93 – 100 (2018).

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ON THE INTEGRABILITY WITH WEIGHT OF TRIGONOMETRIC SERIES

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Abstract. In this paper we have found the necessary and sufficient conditions for the power integrability with a weight of the sum of the sine and cosine series whose coefficients belong to a subclass of $\gamma RBVS$ class.

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Keywords: trigonometric series; integrability; L_p spaces; rest bounded variation.

1. Introduction

Were Young [15], Boas [1], and then Haywood [3] who have studied the integrability of the formal series

$$g(x) := \sum_{n=1}^{\infty} a_n \sin nx \tag{1.1}$$

and

$$f(x) := \sum_{n=1}^{\infty} b_n \cos nx \tag{1.2}$$

imposing certain conditions on the coefficients a_n and b_n respectively (we denote λ_n either a_n or b_n).

Their results deal with above mentioned trigonometric series whose coefficients are monotone decreasing. Lately, the monotonicity condition on the sequence $\{\lambda_n\}$ was replaced by Leindler [5] to a more general ones $\{\lambda_n\} \in R_0^+BVS$.

A sequence $c := \{c_n\}$ of positive numbers tending to zero is of rest bounded variation, or briefly R_0^+BVS, if it possesses the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c)c_m \tag{1.3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

Later on, Németh [8] considered weight functions more general than power one and obtained some sufficient conditions for the integrability of the sine series with such weights. Namely, he used the so-called almost increasing (decreasing) sequences.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (decreasing) if there exists a constant $C := C(\gamma) \geq 1$ such that

$$C\gamma_n \geq \gamma_m \quad (\gamma_n \leq C\gamma_m)$$

holds for any $n \geq m$.

Here and in the sequel, a function $\gamma(x)$ is defined by the sequence γ in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n$, $n \in \mathbb{N}$ and there exist positive constants C_1 and C_2 such that $C_1\gamma_n \leq \gamma(x) \leq C_2\gamma_{n+1}$ for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$.
In 2005 S. Tikhonov [11] has proved two theorems providing necessary and sufficient conditions for the $p-$th power integrability of the sums of sine and cosine series with weight γ. His results refine the assertions of such results presented earlier by others which show that such conditions depend on the behavior of the sequence γ.

We present Tikhonov's results below.

Theorem 1.1 ([11]). *Suppose that $\{\lambda_n\} \in R_0^+BVS$ and $1 \leq p < \infty$.*

(A) *If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_1 > 0$ such that the sequence $\{\gamma_n n^{-p-1+\varepsilon_1}\}$ is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2}\lambda_n^p < \infty \tag{1.4}$$

is sufficient for the validity of the condition

$$\gamma(x)|g(x)|^p \in L(0,\pi). \tag{1.5}$$

(B) *If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_2 > 0$ such that the sequence $\{\gamma_n n^{p-1-\varepsilon_2}\}$ is almost increasing, then the condition (1.4) is necessary for the validity of condition (1.5).*

Theorem 1.2 ([11]). *Suppose that $\{\lambda_n\} \in R_0^+BVS$ and $1 \leq p < \infty$.*

(A) *If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_3 > 0$ such that the sequence $\{\gamma_n n^{-1+\varepsilon_3}\}$ is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2}\lambda_n^p < \infty \tag{1.6}$$

is sufficient for the validity of the inclusion

$$\gamma(x)|f(x)|^p \in L(0,\pi). \tag{1.7}$$

(B) *If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_4 > 0$ such that the sequence $\{\gamma_n n^{p-1-\varepsilon_4}\}$ is almost increasing, then the condition (1.6) is necessary for the validity of condition (1.7).*

Some new results pertaining to related problems, with those we mentioned above, one can find for example in [12] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=m}^{2m-1} |c_k - c_{k+1}| \leq K\left(c\right) c_n \right\},$$

in [13] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=m}^{2m-1} |c_k - c_{k+1}| \leq K\left(c\right) \sum_{k=[m/a]}^{[am]} \frac{c_k}{k} \right\}$$

for some $a > 1$, in [2] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq K\left(c\right) m^{\theta-1} \sum_{k=[m/a]}^{\infty} \frac{c_k}{k^\theta} \right\}$$

for some $a > 1$ and $\theta \in (0, 1]$, in [10] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=m}^{\infty} |c_k - c_{k+r}| \leq K\left(c\right) m^{\theta-1} \sum_{k=[m/a]}^{\infty} \frac{c_k}{k^\theta} \right\}$$

for some $a > 1$ and $\theta \in (0, 1]$ and $r \in \mathbb{N}$, in [4] when

$$\{\lambda_n\} \in \left\{ \{c_k\} : \sum_{k=2m}^{\infty} k|c_k - c_{k+2}| \leq \frac{K\left(c\right)}{m} \sum_{k=m}^{2m-1} k|c_k - c_{k+2}| \right\},$$

where $K(c)$ is a positive constant depending only on a nonnegative sequence $c = \{c_k\}$.

Now, for further investigations we recall an another class of sequences. Namely, was again Leindler [6] who introduced a new class of sequences which is a wider class than the class R_0^+BVS.

Definition 1.1. *A sequence $c := \{c_k\}$ of nonnegative numbers tending to zero belongs to $RBVS_+^{r,\delta}$, if it has the property*

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq \frac{K(c)}{m^{r+1+\delta}} \sum_{n=1}^{m} n^{r+1} c_n$$

for all natural numbers m, where $r, \delta \in \mathbb{R}$ and $K(c)$ is a positive constant depending only on the sequence c.

As is pointed out by Leindler [6], if $0 < \delta \leq 1$ and $c \in R_0^+BVS$, then $c \in RBVS_+^{r,\delta}$ also holds true. Indeed,

$$c_m \leq m^{1-\delta} c_m \leq K(c) m^{-r-1-\delta} \sum_{n=1}^{m} n^{r+1} c_n.$$

Subsequently, the embedding relation $R_0^+BVS \subset RBVS_+^{r,\delta}$ holds true as well. Moreover, it is clear that for a nonnegative sequence $\{c_k\}$ and $m \in \mathbb{N}$

$$\frac{1}{m^{r+1+\delta_1}}\sum_{k=1}^{m} k^{r+1}c_k \le \frac{1}{m^{r+1+\delta_2}}\left(m\right)^{\delta_2-\delta_1}\sum_{k=1}^{m} k^{r+1}c_k \le \frac{1}{m^{r+1+\delta_2}}\sum_{k=1}^{m} k^{r+1}c_k,$$

when $\delta_2 \le \delta_1$, $r \in \mathbb{R}$ and

$$\frac{1}{m^{r_1+1+\delta}}\sum_{k=1}^{m} k^{r_1+1}c_k \le \frac{1}{m^{r_1+1+\delta}}\left(m\right)^{r_1-r_2}\sum_{k=1}^{m} k^{r_2+1}c_k = \frac{1}{m^{r_2+1+\delta}}\sum_{k=1}^{m} k^{r_2+1}c_k,$$

when $r_2 \le r_1$, $\delta \in \mathbb{R}$. Hence

$$RBVS_+^{r,\delta_1} \subseteq RBVS_+^{r,\delta_2} \quad (\delta_2 \le \delta_1)$$

and

$$RBVS_+^{r_1,\delta} \subseteq RBVS_+^{r_2,\delta} \quad (r_2 \le r_1).$$

Therefore in this paper we are concerned about finding the necessary and sufficient conditions on the sequence $\{\lambda_n\} \in RBVS_+^{r,\delta}$ so that $\gamma(x)|f(x)|^p \in L(0,\pi)$ and $\gamma(x)|g(x)|^p \in L(0,\pi)$, which indeed is the aim of this paper.

To achieve this goal we need some helpful statements given in next section.

2. Auxiliary lemmas

Lemma 2.1 ([7]). *Let $\lambda_n > 0$ and $a_n \ge 0$. Then*

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{\nu=1}^{n}a_\nu\right)^p \le p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\left(\sum_{\nu=n}^{\infty}\lambda_\nu\right)^p, \quad p \ge 1.$$

Lemma 2.2 ([9]). *Let $\lambda_n > 0$ and $a_n \ge 0$. Then*

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{\nu=n}^{\infty}a_\nu\right)^p \le p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\left(\sum_{\nu=1}^{n}\lambda_\nu\right)^p, \quad p \ge 1.$$

3. Main results

At first, we prove the following.

Theorem 3.1. *Suppose that $\{\lambda_n\} \in RBVS_+^{r,\delta}$, $r \ge 0$, $0 < \delta \le 1$ and $1 \le p < \infty$. If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_1 > 0$ such that the sequence $\{\gamma_n n^{\varepsilon_1-1-\delta p}\}$ is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty}\gamma_n n^{p(2-\delta)-2}\lambda_n^p < \infty \tag{3.1}$$

is sufficient for the validity of the condition

$$\gamma\left(x\right)\left|g\left(x\right)\right|^p \in L(0,\pi). \tag{3.2}$$

Proof. First we denote

$$\widetilde{D}_n(x)=\sum_{k=1}^{n}\sin kx,\ n\in\mathbb{N}.$$

Using Abel's transformation, $\lambda_n\to 0$, and the well-known estimate $|\widetilde{D}_n(x)|=\mathcal{O}(1/x)$ we have

$$\begin{aligned}\sum_{n=m+1}^{\infty}\lambda_n\sin nx &= \lim_{N\to\infty}\left(\sum_{n=m+1}^{N-1}(\lambda_n-\lambda_{n+1})\widetilde{D}_n(x)+\lambda_N\widetilde{D}_N(x)-\lambda_{m+1}\widetilde{D}_m(x)\right)\\ &= \sum_{n=m+1}^{\infty}(\lambda_n-\lambda_{n+1})\widetilde{D}_n(x)-\sum_{n=m+1}^{\infty}(\lambda_n-\lambda_{n+1})\widetilde{D}_m(x).\end{aligned}$$

Whence, for $x\in\left(\frac{\pi}{n+1},\frac{\pi}{n}\right]$, since $|\sin nx|\le nx$, $|\widetilde{D}_n(x)|\le\frac{C}{x}$, and $\{\lambda_n\}\in RBVS_+^{r,\delta}$ we obtain

$$\begin{aligned}|g(x)| &\le C\left(x\sum_{k=1}^{n}k\lambda_k+n\sum_{k=n}^{\infty}|\lambda_k-\lambda_{k+1}|\right)\\ &\le C\left(\frac{1}{n}\sum_{k=1}^{n}k\lambda_k+\frac{1}{n^{r+\delta}}\sum_{k=1}^{n}k^{r+1}\lambda_k\right)\\ &\le C\left(\frac{1}{n}\sum_{k=1}^{n}k\lambda_k+\frac{1}{n^{\delta}}\sum_{k=1}^{n}k\lambda_k\right)\le\frac{C}{n^{\delta}}\sum_{k=1}^{n}k\lambda_k.\end{aligned}$$

Here and elsewhere, C denotes positive constant, which may be different in different cases. So, we get

$$\int_0^{\pi}\gamma(x)|g(x)|^p dx=\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\gamma(x)|g(x)|^p dx\le C\sum_{n=1}^{\infty}\frac{\gamma_n}{n^{2+\delta p}}\left(\sum_{k=1}^{n}k\lambda_k\right)^p.$$

The use of Lemma 2.1 implies

$$\int_0^{\pi}\gamma(x)|g(x)|^p dx\le C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^{2+\delta p}}\right)^{1-p}(n\lambda_n)^p\left(\sum_{k=n}^{\infty}\frac{\gamma_k}{k^{2+\delta p}}\right)^p.$$

Since $\{m^{\varepsilon_1-\delta p-1}\gamma_m\}$ is almost decreasing sequence, then we get

$$\sum_{k=n}^{\infty}\frac{\gamma_k}{k^{2+\delta p}}=\sum_{k=n}^{\infty}\frac{\gamma_k}{k^{1+\delta p-\varepsilon_1}k^{1+\varepsilon_1}}\le C\frac{\gamma_n}{n^{1+\delta p-\varepsilon_1}}\sum_{k=n}^{\infty}\frac{1}{k^{1+\varepsilon_1}}\le C\frac{\gamma_n}{n^{1+\delta p}}.$$

Thus, we obtain

$$\int_0^{\pi}\gamma(x)|g(x)|^p dx\le C\sum_{n=1}^{\infty}\gamma_n n^{p(2-\delta)-2}\lambda_n^p.$$

The proof is completed. □

Theorem 3.2. *Suppose that $\{\lambda_n\}\in RBVS_+^{r,\delta}$, $r\ge 0$, $0<\delta\le 1$ and $1\le p<\infty$. If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_2>0$ such that the* *sequence $\{\gamma_n n^{p-1-\varepsilon_2}\}$ is almost increasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p\delta-2} \lambda_n^p < +\infty \tag{3.3}$$

is necessary for the validity of condition (3.2).

Proof. Let us show first that $g(x) \in L(0,\pi)$. Namely, if $1 < p < +\infty$ and $p+q = pq$, then applying Hölder's inequality, we get

$$\int_0^{\pi} |g(x)|dx \leq \left(\int_0^{\pi} \gamma(x)|g(x)|^p dx\right)^{1/p} \left(\int_0^{\pi} (\gamma(x))^{-q/p} dx\right)^{1/q}.$$

Now using the estimation (see [11], page 440)

$$\int_0^{\pi} (\gamma(x))^{-q/p} dx < C,$$

we have

$$\int_0^{\pi} |g(x)|dx \leq C \left(\int_0^{\pi} \gamma(x)|g(x)|^p dx\right)^{1/p} < +\infty.$$

Let $p = 1$. Then we can set up that $\{\gamma_n\}$ is almost increasing, and whence

$$\begin{aligned} \int_0^{\pi} |g(x)|dx &\leq \sum_{n=1}^{\infty} \frac{1}{C\gamma_n} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \gamma(x)|g(x)|dx \\ &\leq \frac{1}{C\gamma_1} \int_0^{\pi} \gamma(x)|g(x)|dx < +\infty. \end{aligned}$$

Therefore, for all $p \in [1,+\infty)$ we showed that $g(x) \in L(0,\pi)$. Using this fact we can integrate the function $g(x)$ so that we have

$$F(x) := \int_0^x g(t)dt = \sum_{n=1}^{\infty} \lambda_n \int_0^x \sin nt dt = 2\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin^2 \frac{nx}{2}.$$

Denoting

$$d_\nu := \int_{\frac{\pi}{\nu+1}}^{\frac{\pi}{\nu}} |g(x)|dx, \quad \nu \in \mathbb{N},$$

and taking into account that $\{\lambda_n\} \in RBVS_+^{r,\delta}$

$$\begin{aligned} F(\pi/m) &\geq C\sum_{n=1}^{m} \frac{\lambda_n}{n} \left(\frac{n}{m}\right)^2 = \frac{C}{m^2} \sum_{n=1}^{m} n\lambda_n \\ &\geq \frac{C}{m^{r+2}} \sum_{n=1}^{m} n^{r+1}\lambda_n = Cm^{\delta-1} \frac{1}{m^{r+\delta+1}} \sum_{n=1}^{m} n^{r+1}\lambda_n \\ &\geq \frac{Cm^{\delta-1}}{K(\lambda)} \sum_{n=m}^{\infty} |\lambda_n - \lambda_{n+1}| \geq \frac{Cm^{\delta-1}\lambda_m}{K(\lambda)} \end{aligned}$$

then

$$\lambda_n \leq Cn^{1-\delta} F(\pi/n) \leq Cn^{1-\delta} \sum_{\nu=n}^{\infty} d_\nu.$$

So we have

$$I := \sum_{n=1}^{\infty} \gamma_n n^{p\delta-2} \lambda_n^p \le C \sum_{n=1}^{\infty} \gamma_n n^{p-2} \left(\sum_{\nu=n}^{\infty} d_\nu \right)^p .$$

The use of Lemma 2.2 gives

$$I \le C \sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2} \right)^{1-p} \left(\sum_{\nu=1}^{n} \gamma_\nu \nu^{p-2} \right)^p .$$

The sequence $\{\gamma_n n^{p-1-\varepsilon_2}\}$ is almost increasing, by assumption, which implies

$$\begin{aligned} I &\le C \sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2} \right)^{1-p} \left(\sum_{\nu=1}^{n} \frac{\gamma_\nu \nu^{p-1-\varepsilon_2}}{\nu^{1-\varepsilon_2}} \right)^p \\ &\le C \sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2} \right)^{1-p} \left(\gamma_n n^{p-1-\varepsilon_2} \sum_{\nu=1}^{n} \frac{1}{\nu^{1-\varepsilon_2}} \right)^p \\ &\le C \sum_{n=1}^{\infty} d_n^p \left(\gamma_n n^{p-2} \right)^{1-p} \left(\gamma_n n^{p-1} \right)^p \le C \sum_{n=1}^{\infty} d_n^p \gamma_n n^{2(p-1)}. \end{aligned}$$

Now, if $1 < p < +\infty$ and $q = \frac{p}{p-1}$, then applying Hölder's inequality, we easily get

$$d_n^p = \left(\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)| dx \right)^p \le C n^{2(1-p)} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)|^p dx.$$

Subsequently, we obtain that

$$\begin{aligned} I &\le C \sum_{n=1}^{\infty} \gamma_n \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |g(x)|^p dx \\ &\le C \sum_{n=1}^{\infty} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \gamma(x) |g(x)|^p dx \le C \int_0^{\pi} \gamma(x) |g(x)|^p dx. \end{aligned}$$

For $p = 1$, we also have

$$I \le C \sum_{n=1}^{\infty} \gamma_n d_n \le C \int_0^{\pi} \gamma(x) |g(x)| dx.$$

The proof is completed. □

Theorem 3.3. *Suppose that $\{\lambda_n\} \in RBVS_+^{r,\delta}$, $r \ge 0$, $0 < \delta \le 1$ and $1 \le p < \infty$. If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_3 > 0$ such that the sequence $\{\gamma_n n^{\varepsilon_3 - 1}\}$ is almost decreasing, then the condition (3.1) is sufficient for the validity of the condition*

$$\gamma(x) |f(x)|^p \in L(0, \pi). \tag{3.4}$$

Proof. Similar as in the proof of Theorem 3.1, we have

$$\begin{aligned} |f(x)| &\le \left| \sum_{k=1}^{n} \lambda_k \cos kx \right| + \left| \sum_{k=n+1}^{\infty} \lambda_k \cos kx \right| \\ &\le \sum_{k=1}^{n} \lambda_k + \sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}| \, |D_k(x)| + \lambda_{n+1} |D_n(x)| , \end{aligned}$$

where

$$D_n(x)=\sum_{k=1}^{n}\cos kx,\quad n\in\mathbb{N}.$$

Hence, for $x\in\left(\frac{\pi}{n+1},\frac{\pi}{n}\right]$ since $|D_n(x)|\leq\frac{C}{x}$ and $\{\lambda_n\}\in RBVS_{+}^{r,\delta}$

$$\begin{aligned}|f(x)| &\leq C\left(\sum_{k=1}^{n}\lambda_k+n\sum_{k=n}^{\infty}|\lambda_k-\lambda_{k+1}|\right)\\ &\leq C\left(\sum_{k=1}^{n}\lambda_k+\frac{1}{n^{r+\delta}}\sum_{k=1}^{n}k^{r+1}\lambda_k\right)\\ &\leq C\left(\sum_{k=1}^{n}\lambda_k+\sum_{k=1}^{n}k^{1-\delta}\lambda_k\right)\leq C\sum_{k=1}^{n}k^{1-\delta}\lambda_k.\end{aligned}$$

Therefore

$$\int_0^{\pi}\gamma(x)|f(x)|^p dx=\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\gamma(x)|f(x)|^p dx\leq C\sum_{n=1}^{\infty}\frac{\gamma_n}{n^2}\left(\sum_{k=1}^{n}k^{1-\delta}\lambda_k\right)^p.$$

Using Lemma 2.1 and the fact that $\{m^{\varepsilon_3-1}\gamma_m\}$ is almost decreasing sequence, we obtain similar as in the proof of Theorem 3.1 that

$$\begin{aligned}\int_0^{\pi}\gamma(x)|f(x)|^p dx &\leq C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^2}\right)^{1-p}\left(n^{1-\delta}\lambda_n\right)^p\left(\sum_{k=n}^{\infty}\frac{\gamma_k}{k^2}\right)^p\\ &\leq C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^2}\right)^{1-p}\left(n^{1-\delta}\lambda_n\right)^p\left(\sum_{k=n}^{\infty}\frac{\gamma_k k^{\varepsilon_3-1}}{k^{1+\varepsilon_3}}\right)^p\\ &\leq C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^2}\right)^{1-p}\left(n^{1-\delta}\lambda_n\right)^p\left(\gamma_n n^{\varepsilon_3-1}\sum_{k=n}^{\infty}\frac{1}{k^{1+\varepsilon_3}}\right)^p\\ &\leq C\sum_{n=1}^{\infty}\left(\frac{\gamma_n}{n^2}\right)^{1-p}\left(n^{1-\delta}\lambda_n\right)^p\left(\gamma_n n^{-1}\right)^p\leq C\sum_{n=1}^{\infty}\gamma_n n^{p(2-\delta)-2}\lambda_n^p.\end{aligned}$$

This ends our proof. □

Theorem 3.4. *Suppose that $\{\lambda_n\}\in RBVS_{+}^{r,\delta}$, $r\geq 0$, $0<\delta\leq 1$ and $1\leq p<\infty$. If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_2>0$ such that the sequence $\{\gamma_n n^{p-1-\varepsilon_2}\}$ is almost increasing, then the condition (3.3) is necessary for the validity of condition (3.4).*

Proof. Similar as in the proof of Theorem 3.2 we can show that the condition (3.4) implies $f(x)\in L(0,\pi)$. Integrating the function f, we write

$$H(x)=\int_0^{x}f(t)\,dt=\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sin nx.$$

Now, we prove if $\{\lambda_n\} \in RBVS_+^{r,\delta}$ then $\{\frac{\lambda_n}{n}\} \in RBVS_+^{r,\delta}$. Suppose $\{\lambda_n\} \in RBVS_+^{r,\delta}$. Then for $m \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=m}^{\infty}\left|\frac{\lambda_k}{k}-\frac{\lambda_{k+1}}{k+1}\right| &\leq \sum_{k=m}^{\infty}\frac{1}{k+1}\left|\lambda_k-\lambda_{k+1}\right|+\sum_{k=m}^{\infty}\frac{1}{k\left(k+1\right)}\lambda_k \\
&\leq \frac{1}{m+1}\sum_{k=m}^{\infty}\left|\lambda_k-\lambda_{k+1}\right|+\sum_{k=m}^{\infty}\frac{1}{k\left(k+1\right)}\sum_{l=k}^{\infty}\left|\lambda_l-\lambda_{l+1}\right| \\
&\leq \frac{K\left(\lambda\right)}{m^{2+r+\delta}}\sum_{n=1}^{m}n^{r+1}\lambda_n+\sum_{l=m}^{\infty}\left|\lambda_l-\lambda_{l+1}\right|\sum_{k=m}^{\infty}\frac{1}{k^2} \\
&\leq \frac{K\left(\lambda\right)}{m^{2+r+\delta}}\sum_{n=1}^{m}n^{r+1}\lambda_n+\frac{C}{m}\sum_{l=m}^{\infty}\left|\lambda_l-\lambda_{l+1}\right| \\
&\leq \frac{\left(1+C\right)K\left(\lambda\right)}{m^{2+r+\delta}}\sum_{n=1}^{m}n^{r+1}\lambda_n \leq \frac{\left(1+C\right)K\left(\lambda\right)}{m^{1+r+\delta}}\sum_{n=1}^{m}n^{r+1}\frac{\lambda_n}{n},
\end{aligned}$$

whence $\{\frac{\lambda_n}{n}\} \in RBVS_+^{r,\delta}$.

Applying Theorem 3.2 to the function H we obtain

$$\sum_{n=1}^{\infty}\gamma_n^* n^{p\delta-2}\lambda_n^p \leq C\int_0^{\pi}\gamma^*\left(x\right)\left|H\left(x\right)\right|^p dx,$$

where $\{\gamma_n^*\}$ satisfies the following condition: there exists $\varepsilon > 0$ such that the sequence $\left\{\gamma_n^* n^{p-1-\varepsilon}\right\}$ is almost increasing. For $\gamma_n^* = \gamma_n n^p$, this condition is obviously satisfied. Then

$$\begin{aligned}
I &:= \sum_{n=1}^{\infty}\gamma_n n^{p\delta-2}\lambda_n^p = \sum_{n=1}^{\infty}\gamma_n n^p n^{p\delta-2}\left(\frac{\lambda_n}{n}\right)^p \\
&= \sum_{n=1}^{\infty}\gamma_n^* n^{p\delta-2}\lambda_n^p \leq C\int_0^{\pi}\frac{\gamma\left(x\right)}{x^p}\left|H\left(x\right)\right|^p dx \\
&\leq C\sum_{n=1}^{\infty}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}}\frac{\gamma\left(x\right)}{x^p}\left(\int_0^x\left|f\left(t\right)\right|dt\right)^p dx \\
&\leq C\sum_{n=1}^{\infty}\gamma_n n^{p-2}\left(\int_0^{\frac{\pi}{n}}\left|f\left(t\right)\right|dt\right)^p = C\sum_{n=1}^{\infty}\gamma_n n^{p-2}\left(\sum_{v=k}^{\infty}\int_{\frac{\pi}{v+1}}^{\frac{\pi}{v}}\left|f\left(t\right)\right|dt\right)^p.
\end{aligned}$$

Denoting

$$f_v = \int_{\frac{\pi}{v+1}}^{\frac{\pi}{v}}\left|f\left(t\right)\right|dt, \quad v \in \mathbb{N}$$

and using Lemma 2.2 we get

$$I \leq C\sum_{n=1}^{\infty}\gamma_n n^{2p-2}\left(f_n\right)^p.$$

Now, if $1 < p < +\infty$ and $q = \frac{p}{p-1}$, then applying Hölder's inequality, we easily get

$$f_n^p = \left(\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |f(x)| dx \right)^p \leq C n^{2(1-p)} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |f(x)|^p dx.$$

Subsequently, we obtain that

$$\begin{aligned} I &\leq C \sum_{n=1}^{\infty} \gamma_n \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} |f(x)|^p dx \\ &\leq C \sum_{n=1}^{\infty} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \gamma(x) |f(x)|^p dx \leq C \int_0^{\pi} \gamma(x) |g(x)|^p dx. \end{aligned}$$

For $p = 1$, we also have

$$I \leq C \sum_{n=1}^{\infty} \gamma_n d_n \leq C \int_0^{\pi} \gamma(x) |g(x)| dx,$$

and the proof is completed. □

Remark 3.1. *Since $R_0^+ BVS \subset RBVS_+^{r,1}$, then Theorem 1.1 and Theorem 1.2 are consequences of our results.*

Remark 3.2. *We know that the class of zero monotone decreasing sequences is a subclass of the class $R_0^+ BVS$. Whence, our results also hold true when condition $\{\lambda_n\} \in RBVS_+^{r,\delta}$ is replaced with condition $\{\lambda_n\} \in M := \{\mathbf{c} : c_n \downarrow 0\}$.*

4. Conclusions

The integrability of functions defined by trigonometric series has been attractive for lots of researchers during last six decades. The questions of integrability with weight of such functions, whose coefficients of their trigonometric series belong to various classes of sequences such as the decreasing sequences [3], the power-monotone sequences [8], the quasi-monotone sequences [14] and the general monotone sequences which is very important class for such questions, see [5], [2], [10], [12] and [13], are of the great interest. Here, in the present paper, we go one step further, finding the necessary and sufficient conditions for the power integrability with a weight of the sum of the sine and cosine series whose coefficients belong to the $RBVS_+^{r,\delta}$, $r \geq 0$, class and in the same time covering the results proved previously by others. In our results, $0 < \delta \leq 1$, we assume that the quantities

$$\sum_{n=1}^{\infty} \gamma_n n^{p(2-\delta)-2} \lambda_n^p \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n n^{p\delta-2} \lambda_n^p$$

are finite, which both coincide, $\delta = 1$, with finiteness of the famous quantity

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p.$$

Among this, we have showed as well the embedding relation

$$RBVS_{+}^{r;\delta_1} \subseteq RBVS_{+}^{r;\delta_2}, \text{ when } 0 < \delta_2 \leq \delta_1 \leq 1.$$

The $RBVS_{+}^{r;\delta}$ class seems to be considered here for the second time since it has been introduced and employing it, especially in the proof of our findings, shows that it could also be useful in other topics similar to this already considered here.

Список литературы

[1] R. P. Boas, Jr., "Integrability of trigonometrical series III", Quart. J. Math. (Oxford), **3** (2), 217 – 221 (1952).

[2] M. Dyachenko and S. Tikhonov, "Integrability and continuity of functions represented by trigonometric series: coefficients criteria", Studia Math. **193** (3),, 285 – 306 (2009).

[3] P. Heywood, "On the integrability of functions defined by trigonometric series", Quart. J. Math. (Oxford), **5** (2), 71 – 76 (1954).

[4] Xh. Z. Krasniqi, "Rest bounded second variation sequences and p-th power integrability of some functions related to sums of formal trigonometric series", Acta Math. Acad. Paedagog. Nyházi. (N.S.) **31**, no. 2, 249 – 257 (2015).

[5] L. Leindler, "A new class of numerical sequences and its applications to sine and cosine series", Analysis Math., **28**, 279 – 286 (2002).

[6] L. Leindler, "Two embedding theorems", Acta Sci. Math. (Szeged), **71**, no. 3 – 4, 619 – 630 (2005).

[7] L. Leindler, "Über verschiedene Konvergenzarten trigonometrischer Reihen. III. Bedingungen in der Metrik von L^p.", (German) Acta Sci. Math. (Szeged) **27**, 205 – 215 (1966).

[8] J. Németh, "Power-monotone sequences and integrability of trigonometric series", JIPAM. J. Inequal. Pure Appl. Math. **4**, no. 1, Article 3, 6 pp. (electronic) (2003).

[9] M. K. Potapov, "A certain imbedding theorem", [in Russian] Mathematica (Cluj), **14** (37), 123 – 146 (1972).

[10] B. Szal, "On weighted L^p integrability of functions defined by trigonometric series", J. Inequal. Appl. **2010**, 1 – 19 (2010).

[11] S. Yu. Tikhonov, "On the integrability of trigonometric series", [in Russian] Mat. Zametki **78**, no. 3, 476 – 480 (2005); translation in Math. Notes **78**, no. 3 – 4, 437 – 442 (2005).

[12] S. Yu. Tikhonov, "Trigonometric series with general monotone coefficients", J. Math. Anal. Appl., **326**, 721 – 735 (2007).

[13] D. S. You, P. Zhou and S. P. Zhou, "On L^p integrability and convergence of trigonometric series", Studia Math. **182** (3), 215 – 226 (2007).

[14] R. Askey and S. Wainger, "Integrability theorems for Fourier series", Duke Math. J. **33**, 223 – 228 (1966).

[15] W. H. Young, "On the Fourier series of bounded variation", Proc. London Math. Soc., **12**, 41 – 70 (1913).

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СОСТОЯТЕЛЬНОСТЬ ИЗБЫТОЧНОГО РИСКА В РОБАСТНОМ ОЦЕНИВАНИИ ГАУССОВСКОГО СРЕДНЕГО

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Аннотация. В данной работе мы вводим понятие избыточного риска для оценки среднего вектора Гауссовского распределения, когда наблюдения искажены выбросами. Известно, что выборочное средство теряет свои хорошие свойства в присутствии выбросов [5, 6]. Кроме того, даже выборочная медиана неоптимальна в миминимаксном смысле скорость оптимальная в многомерном случае. Оценка достигающяя оптимальной минимаксной скорости был установлен в [1]. Однако даже эти результаты оптимальности минимаксной скорости не дают количественной оценки того, насколько быстро загрязненная модель приближается к риску в незагрязненной модели, когда скорость загрязнения стремится к нулю. Данная статья делает первый шаг в заполнении этого пробела, показывая, что существует оценка с избыточным риском, стремящимся к нулю, когда пропорция выбросов приближается к нулю.

MSC2010 number: 62F35.

Ключевые слова: избыточный риск; робастное оценивание; среднее нормального распределения.

1. Введение

В последние годы мы стали свидетелями возрождения интереса к статистическим методам, которые могут эффективно работать с наборами данных, содержащие выбросы. В частности, в модели загрязнения Хубера в задаче оценки среднего вектора нормального распределения [1] установил оптимальную минимаксную скорость и показал, что она достигается в случае многомерной медианы, известная как медиана Тьюки. Более того, [2] разработал общую теорию для получения минимаксной скорости (верхняя и нижняя границы) в широком классе статистических моделей. Эти работы ориентированы на статистическую сложность оценок, не обращая внимания на вычислительную сложность. Вычислительная сложность оценок был адресован в [4] и [3], которые проанализировали риск вычислимых оценок. Интересно, что результаты, доказанные в этих работах, обеспечивают только порядок минимаксной скорости, но ничего не говорят о том, насколько быстро риск в загрязненной выбросами модели стремится к риску без выбросов.

В этой статье мы вводим понятие избыточного риска, которое определяется как разница между рисками в моделях с выбросами и без. Затем мы представляем анализ этого риска для процедуры, которую мы назвали "group hard thresholding". Это также можно рассматривать как версию усеченной средней оценки. Наш главный результат показывает что этот избыточный риск стремится к нулю, когда уровень загрязнения стремится к нулю.

Более формально, давайте предположим, что мы наблюдаем n случайных векторов $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ in $\mathbb{R}^p$, которые удовлетворяют

$$\boldsymbol{Y}_i = \boldsymbol{\mu} + \boldsymbol{\theta}_i + \boldsymbol{\xi}_i, \quad \boldsymbol{\xi}_i \overset{\text{iid}}{\sim} \mathcal{N}(\mathbf{0}, I_p). \tag{1.1}$$

В верхней формуле, $\boldsymbol{\mu}$ неизвестный параметр, которую мы хотим оценить, $\{\boldsymbol{\theta}_i\}$ это произвольные детерминированные векторы, указывающие какие из наблюдений являются выбросами и $\boldsymbol{\xi}_i$ случайный шум. В этой статье, мы предполагаем, что $\boldsymbol{\Theta} = [\boldsymbol{\theta}_1 \ldots \boldsymbol{\theta}_n]$ усеченная по столбцам матрица. Все наблюдения с индексами $i \in \mathcal{O} = \{\ell : \|\boldsymbol{\theta}_\ell\|_2 > 0\}$ являются выбросами, в то время, как остальные приходят из $\mathcal{N}(\boldsymbol{\mu}, I_p)$. Пусть

$$o = \mathrm{Card}(\mathcal{O}), \quad \text{и} \quad \varepsilon = \frac{o}{n}.$$

Предполагается, что параметр ε меньше $1/2$, который играет важную роль в робастном оценивании. В астности, известно что минимаксная скорость оценки в модели (1.1) имеет порядок $\frac{p}{n} + \varepsilon^2$.

В этой статье мы рассмотриваем более точную меру точности оценки, избыточный риск. Напомним, что риск оценки [1] $\widehat{\boldsymbol{\mu}}$ определяется как

$$R[\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}; \boldsymbol{\Theta}] = [\mathbb{E}_{\boldsymbol{\mu},\boldsymbol{\Theta}} \|\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}\|_2^2]^{1/2}.$$

Здесь и далее в статье обозначение $\mathbb{E}_{\boldsymbol{\mu},\boldsymbol{\Theta}}[h]$ означает математическое ожидание по распределению $\{\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n\}$ как определено в (1.1) (мы неявно предполагаем, что h зависит от наблюдений $\{\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n\}$). Это хорошо известный факт, что в случае, где нет выбросов, т.е. когда $\boldsymbol{\Theta} \equiv \mathbf{0}_{p\times n}$ риск удовлетворяет равняется $\sqrt{p/n}$. Формально,

$$\inf_{\widehat{\boldsymbol{\mu}}} \sup_{\boldsymbol{\mu}\in\mathbb{R}^p} R[\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}; \mathbf{0}] = \sup_{\boldsymbol{\mu}\in\mathbb{R}^p} R[\overline{\boldsymbol{Y}}_n, \boldsymbol{\mu}; \mathbf{0}] = \sqrt{\frac{p}{n}}, \tag{1.2}$$

[1] Оценка является любой измеримой функция от $(\mathbb{R}^p)^n$ до $\mathbb{R}^p$

где $\overline{\boldsymbol{Y}}_n = \frac{1}{n}\sum_{i=1}^n \boldsymbol{Y}_i$ является выборочным средним наблюдаемых данных. Определим

$$\|\boldsymbol{\Theta}\|_{0,2} := \sum_{i=1}^n \mathbf{1}(\|\boldsymbol{\theta}_i\|_2 > 0).$$

Базируясь на (1.2), определим избыточный риск оценки $\widehat{\boldsymbol{\mu}}$ в худшем случае следующим образом

$$\mathcal{E}(\widehat{\boldsymbol{\mu}}; n, p, \varepsilon) = \sup_{\boldsymbol{\mu}\in\mathbb{R}^p; \|\boldsymbol{\Theta}\|_{0,2}\le \varepsilon n} R[\widehat{\boldsymbol{\mu}}, \boldsymbol{\mu}; \boldsymbol{\Theta}] - \sqrt{\frac{p}{n}}.$$

Минимаксный избыточный риск определяется таким образом

$$\mathcal{E}(n, p, \varepsilon) = \inf_{\widehat{\boldsymbol{\mu}}} \mathcal{E}(\widehat{\boldsymbol{\mu}}, n, p, \varepsilon),$$

где инфимум берется по всевозможным оценкам $\widehat{\boldsymbol{\mu}}$. Заметим, что согласно определению, рассматривающиеся оценки выше могут зависеть от n, p и $\varepsilon = o/n$. Основным результатом этой статьи является то, что избыточный риск нашей оценки, введенный в следующем разделе стремится к нулю когда $\varepsilon = \varepsilon_n \to 0$ и $n \to \infty$, при таком $p = p_n$, что p_n/n ограничена сверху константой.

2. Групповой жесткий порог

В этом разделе мы определяем оценку $\widehat{\boldsymbol{\mu}}_{\text{GHT}}$, называемую group hard thresholding, и доказываем, что при этой оценке избыточный риск стремится к 0, когда доля выбросов ε стремится к 0. Грубо говоря, $\widehat{\boldsymbol{\mu}}_{\text{GHT}}$ является средним арифмитическим векторов $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ заминяя все вектора с большим расстоянием от покоординатной медианы ею же.

Более формально, пусть $\widehat{\boldsymbol{\mu}}_{\text{Med}} := \texttt{Med}(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n)$ есть покоординатная медиана выборки $\{\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n\}$. Для фиксированного порога $\lambda > 0$ и каждого $i \in \{1, \ldots, n\}$, положим

$$\widehat{\boldsymbol{\theta}}_i = HT_\lambda(\boldsymbol{Y}_i - \widehat{\boldsymbol{\mu}}_{\text{Med}}) := (\boldsymbol{Y}_i - \widehat{\boldsymbol{\mu}}_{\text{Med}})\mathbf{1}(\|\boldsymbol{Y}_i - \widehat{\boldsymbol{\mu}}_{\text{Med}}\|_2 > \lambda) \tag{2.1}$$

$$\widehat{\boldsymbol{\mu}}_{\text{GHT}} = \frac{1}{n}\sum_{i=1}^n (\boldsymbol{Y}_i - \widehat{\boldsymbol{\theta}}_i) := L_n(\boldsymbol{Y} - \widehat{\boldsymbol{\Theta}}). \tag{2.2}$$

Далее мы сформулируем основную теорему статьи, показывающую, что избыточный риск нашей оценки стремится к нулю, если доля выбросов $\varepsilon = \varepsilon_n$ стремится к 0 так, что $\varepsilon_n p_n^{1/4}$ также стремится к 0. Заметим, что это условие выполнено при фиксированном p, но модель также позволяет размерность быть бесконечностью, т.е. $p = p_n \to \infty$ при ограничении $\varepsilon_n p_n^{1/4} \log^{1/2} \varepsilon_n^{-1} = o(1)$ когда размер выборки n стремится к бесконечности.

Теорема 2.1. *Для* $\widehat{\boldsymbol{\mu}}_{\mathrm{GHT}}$ *определенным в* (2.2) *и* $\lambda^2 = p + 8\sqrt{p\log\varepsilon^{-1}} + 16\log\varepsilon^{-1}$ *мы имеем*

$$\overline{\lim_{n\to\infty}}\, \mathcal{E}(\widehat{\boldsymbol{\mu}}_{\mathrm{GHT}}, n, p_n, \varepsilon_n) = 0,$$

если $\varepsilon_n p_n^{1/4}\log^{1/2}\varepsilon_n^{-1} = o(1)$ *и* $p_n = O(n)$ *когда* $n \to \infty$.

Доказательство. Пусть $s_i = \mathbf{1}(\|\boldsymbol{Y}_i - \widehat{\boldsymbol{\mu}}_{\mathrm{Med}}\|_2 \le \lambda)$ и $\boldsymbol{\delta} = \widehat{\boldsymbol{\mu}}_{\mathrm{Med}} - \boldsymbol{\mu}^*$. Используя тот факт, что $\boldsymbol{Y} = \boldsymbol{\mu}\mathbf{1}_n^\top + \boldsymbol{\Theta} + \boldsymbol{\Xi}$, можно написать следующее соотношение

$$\begin{aligned} \widehat{\boldsymbol{\mu}}_{\mathrm{GHT}} - \boldsymbol{\mu} &= L_n(\boldsymbol{\Theta} + \boldsymbol{\Xi} - \widehat{\boldsymbol{\Theta}}) \\ &= \frac{1}{n}\sum_{i=1}^n \boldsymbol{\xi}_i + T_1(n) + T_2(n) + T_3(n), \end{aligned} \tag{2.3}$$

где

$$T_1(n) = \frac{1}{n}\sum_{i=1}^n \boldsymbol{\xi}_i(s_i - 1), \quad T_2(n) = \frac{\boldsymbol{\delta}}{n}\sum_{i=1}^n (1 - s_i),$$

$$T_3(n) = \frac{1}{n}\sum_{i\in\mathcal{O}} \boldsymbol{\theta}_i s_i.$$

Для простоты обозначений обозначим $\mathbb{L}_2$ нормой вектора $\boldsymbol{V}$ следующим образом

$$\|\boldsymbol{V}\|_{\mathbb{L}_2} = \big(\mathbb{E}[\|\boldsymbol{V}\|_2^2]\big)^{1/2}.$$

Заметим, что достаточно показать, что $T_i(n) \to 0$ когда $\varepsilon \to 0$ для $i \in \{1, 2, 3\}$. В самом деле,

$$\bigg\|\frac{1}{n}\sum_{i=1}^n \boldsymbol{\xi}_i\bigg\|_{\mathbb{L}_2}^2 = \frac{p}{n}.$$

Тогда,

$$\begin{aligned} \mathcal{E}(\widehat{\boldsymbol{\mu}}; n, p, \varepsilon) &\le \|T_1(n) + T_2(n) + T_3(n)\|_{\mathbb{L}_2} \\ &\le \|T_1(n)\|_{\mathbb{L}_2} + \|T_2(n)\|_{\mathbb{L}_2} + \|T_3(n)\|_{\mathbb{L}_2}. \end{aligned}$$

Можно проверить, что следующее отношение выполнено с некоторой константой C

$$\|\boldsymbol{\delta}\|_{\mathbb{L}_4} \le C\sqrt{p}\bigg(\frac{1}{\sqrt{n}} \vee \varepsilon\bigg) = o(p^{1/4}) \tag{2.4}$$

и

$$\|\boldsymbol{\xi}_i\|_{\mathbb{L}_4}^4 = \mathbb{E}\bigg(\sum_{j=1}^p \xi_{ij}^2\bigg)^2 = 3p + p(p-1) \le (p+1)^2. \tag{2.5}$$

Сначала мы ограничим сверху $\mathbb{E}[1-s_i]$ для всех $i \in \mathcal{O}^c$. Имея (2.4), мы получаем $\|\boldsymbol{\delta}\|_2 = o_{\mathbb{P}}\big(p^{1/4}\big)$ и, следовательно,

$$\begin{aligned}\mathbb{E}[1-s_i] &= \mathbb{P}(s_i=0) = \mathbb{P}(\|\boldsymbol{Y}_i - \widehat{\boldsymbol{\mu}}_{\mathrm{Med}}\|_2^2 > \lambda^2) = \mathbb{P}(\|\boldsymbol{\delta}+\boldsymbol{\xi}_i\|_2^2 > \lambda^2)\\ &\le \mathbb{P}\big(\|\boldsymbol{\xi}_i\|_2^2 > \lambda^2(1-o(1))\big) \lesssim \varepsilon^8, \qquad \forall i \in \mathcal{O}^c,\end{aligned} \tag{2.6}$$

где последнее неравенство следует от концентрации случайной величины χ_p^2 и выбора λ^2. Для $T_1(n)$, используя (2.5), мы имеем

$$\begin{aligned}\|T_1(n)\|_{\mathbb{L}_2} &\le \Big\|\frac{1}{n}\sum_{i\in\mathcal{O}^c}\boldsymbol{\xi}_i(1-s_i)\Big\|_{\mathbb{L}_2} + \Big\|\frac{1}{n}\sum_{i\in\mathcal{O}}\boldsymbol{\xi}_i(1-s_i)\Big\|_{\mathbb{L}_2}\\ &= O(\sqrt{p}\varepsilon^2) + \Big\|\frac{1}{n}\sum_{i\in\mathcal{O}}\boldsymbol{\xi}_i(1-s_i)\Big\|_{\mathbb{L}_2}.\end{aligned}$$

С другой стороны, неравенство Коши-Шварца-Буняковского дает

$$\Big\|\sum_{i\in\mathcal{O}}\boldsymbol{\xi}_i(1-s_i)\Big\|_2^2 \le \sum_{i\in\mathcal{O}}(1-s_i)\Big\|\sum_{i\in\mathcal{O}}\boldsymbol{\xi}_i\boldsymbol{\xi}_i^\top\Big\|_{\mathrm{op}} \le n\varepsilon\Big\|\sum_{i\in\mathcal{O}}\boldsymbol{\xi}_i\boldsymbol{\xi}_i^\top\Big\|_{\mathrm{op}}.$$

Известная верхняя оценка для операторной нормы матрицы с нормальными элементами (см. Лемма 9 в [4]) дает

$$\Big\|\sum_{i\in\mathcal{O}}\boldsymbol{\xi}_i(1-s_i)\Big\|_{\mathbb{L}_2}^2 \le 3n\varepsilon(p+n\varepsilon+4).$$

Следовательно,

$$\|T_1(n)\|_{\mathbb{L}_2} \lesssim \sqrt{p}\,\varepsilon^2 + \frac{\sqrt{n\varepsilon p}+n\varepsilon}{n} = \sqrt{p}\,\varepsilon^2 + \sqrt{\varepsilon p/n} + \varepsilon = o(1). \tag{2.7}$$

Для ограничения сверху $\|T_2(n)\|_{\mathbb{L}_2}$, мы используем неравенство Коши-Шварца-Буняковского вместе с (2.4) и (2.6), чтобы получить

$$\begin{aligned}\|T_2(n)\|_{\mathbb{L}_2} &= \Big\|\frac{\boldsymbol{\delta}}{n}\sum_{i=1}^n(1-s_i)\Big\|_{\mathbb{L}_2} \le \frac{\|\boldsymbol{\delta}\|_{\mathbb{L}_4}}{n}\cdot\sum_{i=1}^n\|1-s_i\|_{\mathbb{L}_4}\\ &= \frac{o(p^{1/4})}{n}(\varepsilon n + n(1-\varepsilon)\varepsilon^2) = \varepsilon o(p^{1/4}) = o(1),\end{aligned} \tag{2.8}$$

где на третьем шаге мы ограничили $\|1-s_i\|_{\mathbb{L}_4} = \mathbb{E}^{1/4}[1-s_i] \le 1$ для $i \in \mathcal{O}$ и $\|1-s_i\|_{\mathbb{L}_4} \le \varepsilon^2$ для $i \in \mathcal{O}^c$. Для получения верхней оценки $\mathbb{L}_2$ нормы $T_3(n)$ заметим, что когда $s_i = 1$, тогда у нас есть верхняя оценка для $\|\boldsymbol{\theta}_i\|_2$. Формально, неравенство $\|\boldsymbol{\theta}_i + \boldsymbol{\delta} + \boldsymbol{\xi}_i\|_2 \le \lambda$ эквивалентно $s_i = 1$, следовательно

$$\|\boldsymbol{\theta}_i\|_2 \le 2\|\boldsymbol{\delta}\|_2 + 2|\eta_i| + \big(\lambda^2 - \|\boldsymbol{\xi}_i\|_2^2\big)_+^{1/2} + (2|\boldsymbol{\delta}^\top\boldsymbol{\xi}_i|)^{1/2}, \tag{2.9}$$

для стандартных нормальный случайных величин η_i для $i \in \{1,\dots,n\}$.

Используя неравенство Гельдера можно показать, что

$$\sum_{i\in\mathcal{O}}|\boldsymbol{\delta}^\top\boldsymbol{\xi}_i|^{1/2}\le(n\varepsilon)^{3/4}\Big\{\sum_{i\in\mathcal{O}}|\boldsymbol{\delta}^\top\boldsymbol{\xi}_i|^2\Big\}^{1/4}=(n\varepsilon)^{3/4}\Big\{\boldsymbol{\delta}^\top\sum_{i\in\mathcal{O}}\boldsymbol{\xi}_i\boldsymbol{\xi}_i^\top\boldsymbol{\delta}\Big\}^{1/4}$$
$$\le(n\varepsilon)^{3/4}\|\boldsymbol{\delta}\|_2^{1/2}\|\boldsymbol{\xi}_O\|_{\rm op}^{1/2},$$

где $\|\boldsymbol{\xi}_O\|_{\rm op}^{1/2}$ спектральная норма матрицы полученной из векторов $\boldsymbol{\xi}_i$ для $i\in\mathcal{O}$. Отсюда следует, что[2]

$$\Big\|\sum_{i\in\mathcal{O}}|\boldsymbol{\delta}^\top\boldsymbol{\xi}_i|^{1/2}\Big\|_{\mathbb{L}_2}\le(n\varepsilon)^{3/4}\|\boldsymbol{\delta}\|_{\mathbb{L}_2}^{1/2}\|\boldsymbol{\xi}_O\|_{\mathbb{L}_2}^{1/2}$$
$$=O\Big((n\varepsilon)^{3/4}(\varepsilon^{1/2}p^{1/4})((n\varepsilon)^{1/4}+p^{1/4})\Big)$$
$$=O\Big(n\varepsilon p^{1/4}+(n\varepsilon)^{3/4}\sqrt{\varepsilon p}\Big).$$

Можно показать, что

$$\mathbb{E}[(\lambda^2-\|\boldsymbol{\xi}_i\|_2^2)_+])\lesssim\sqrt{p\log\varepsilon^{-1}}+\log\varepsilon^{-1}\lesssim\sqrt{p}\,\log\varepsilon^{-1}$$

Используя неравенство треугольника для $\mathbb{L}_2$ нормы, получаем

$$\|T_3(n)\|_{\mathbb{L}_2}\le\frac{1}{n}\sum_{i\in\mathcal{O}}\|\boldsymbol{\theta}_i\mathbf{1}(s_i=1)\|_{\mathbb{L}_2}$$
$$\lesssim\varepsilon(\|\boldsymbol{\delta}\|_{\mathbb{L}_2}+1)+\varepsilon p^{1/4}\log^{1/2}\varepsilon^{-1}+\varepsilon p^{1/4}+\varepsilon^{5/4}p^{1/2}n^{-1/4}$$
$$\lesssim\varepsilon p^{1/4}\log^{1/2}\varepsilon^{-1}+\varepsilon p^{1/4}+\varepsilon^{5/4}p^{1/4}.$$

Поскольку $\varepsilon_n=o(1)$ и $\varepsilon_n p_n^{1/4}\log^{1/2}\varepsilon_n^{-1}=o(1)$, получаем, что $\|T_3(n)\|_{\mathbb{L}_2}=o(1)$ и доказательство теоремы следует. □

Abstract. In this work we introduce the notion of the excess risk in the setup of estimation of the Gaussian mean when the observations are corrupted by outliers. It is known that the sample mean loses its good properties in the presence of outliers [5, 6]. In addition, even the sample median is not minimax-rate-optimal in the multivariate setting. The optimal rate of the minimax risk in this setting was established by [1]. However, even these minimax-rate-optimality results do not quantify how fast the risk in the contaminated model approaches the risk in the uncontaminated model when the rate of contamination goes to zero. The present paper does a first step in filling this gap by showing that the group hard thresholding estimator has an excess risk that goes to zero when the corruption rate approaches zero.

[2] Для простоты мы рассматриваем случай когда $n^{-1/2}\lesssim\varepsilon_n$.

Список литературы

[1] M. Chen, C. Gao and Z. Ren, “A general decision theory for Huber’s ε-contamination model”, Electron. J. Statist., 10:3752–3774 (2016).
[2] M. Chen, C. Gao and Z. Ren, “Robust covariance and scatter matrix estimation under Huber’s contamination model”, Annals of Statistics, **46**(5), 1932 – 1960 (2018).
[3] Y. Cheng, I. Diakonikolas and R. Ge, “High-dimensional robust mean estimation in nearly-optimal time”, arXiv:1811.09380 (2018).
[4] O. Collier and A. S. Dalalyan, “Rate-optimal estimation of p-dimensional linear functionals in a sparse gaussian model”, Electron. J. Statist., **13**(2), 2830 – 2864 (2019).
[5] P. J. Huber, “Robust estimation of a location parameter”, The annals of mathematical statistics, **35**(1), 73 – 101 (1964).
[6] P. J. Huber, “A robust version of the probability ratio test”, The annals of mathematical statistics, **36** (6), no. 1, 753 – 758 (1965).

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SOME UPPER BOUND ESTIMATES FOR THE MAXIMAL MODULUS OF THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. This paper deals with the problem of finding some upper bound estimates for the maximal modulus of the polar derivative of a complex polynomial on a disk under certain constraints on the zeros and on the functions involved. A variety of interesting results follow as special cases from our results.

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Keywords: complex polynomial; polar derivative; maximum modulus; zeros.

1. Introduction

Let $\mathbb{P}_n$ denote the space of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n and $P'(z)$ is the derivative of $P(z)$. A famous result known as Bernstein's inequality (for reference, see [3]) states that if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1}\left|P'(z)\right| \leq n\max_{|z|=1}\left|P(z)\right|, \tag{1.1}$$

where as concerning the maximum modulus of $P(z)$ on the circle $|z| = R \geq 1$, we have (for reference see [11]),

$$\max_{|z|=R}\left|P(z)\right| \leq R^n\max_{|z|=1}\left|P(z)\right|. \tag{1.2}$$

Both the above inequalities are sharp and equality in each holds only when $P(z)$ is a constant multiple of z^n.

It was observed by Bernstein [3] that (1.1) can be deduced from (1.2), by making use of Gauss - Lucas theorem and the proof of this fact was given by Govil, Qazi and Rahman [4].

If we restrict ourselves to the class of polynomials $P \in \mathbb{P}_n$, with $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced

$$\max_{|z|=1}\left|P'(z)\right| \leq \frac{n}{2}\max_{|z|=1}\left|P(z)\right|, \tag{1.3}$$

and

$$\max_{|z|=R\geq 1}\left|P(z)\right| \leq \frac{R^n+1}{2}\max_{|z|=1}\left|P(z)\right|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later proved by Lax [8], where as inequality (1.4) was proved by Ankeny and Rivlin [1], for which they made use of (1.3).

Inequality (1.1) can be seen as a special case of the following inequality which is also due to Bernstein [3].

Theorem A. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \le 1$ and $f(z)$ be a polynomial of degree at most n. If $|f(z)| \le |F(z)|$ for $|z| = 1$, then for $|z| \ge 1$, we have*

$$\big|f'(z)\big| \le \big|F'(z)\big|. \tag{1.5}$$

Equality holds in (1.5) *for $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.*

Inequality (1.1) can be obtained from inequality (1.5) by taking $F(z) = Mz^n$, where $M = \max\limits_{|z|=1}|f(z)|$. In the same way, inequality (1.2) follows from a result which is a special case of Bernstein-Walsh lemma ([10], Corollary 12.1.3).

Theorem B. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \le 1$ and $f(z)$ be a polynomial of degree at most n. If $|f(z)| \le |F(z)|$ for $|z| = 1$, then*

$$\big|f(z)\big| < \big|F(z)\big|, \quad \textit{for} \quad |z| > 1,$$

unless $f(z) = e^{i\eta}F(z)$ for some $\eta \in \mathbb{R}$.

In 2011, Govil et al. [5] proved a more general result which provides a compact generalization of inequalities (1.1), (1.2), (1.3) and (1.4) and includes Theorem A and Theorem B as special cases. In fact, they proved that if $f(z)$ and $F(z)$ are as in Theorem A, then for any β with $|\beta| \le 1$ and $R \ge r \ge 1$, we have

$$\big|f(Rz) - \beta f(rz)\big| \le \big|F(Rz) - \beta F(rz)\big|, \quad \text{for} \quad |z| \ge 1. \tag{1.6}$$

Further, as a generalization of (1.6), Liman et al. [6] in the same year 2011 and under the same hypothesis as in Theorem A, proved that

$$\begin{aligned} &\Big|f(Rz) - \beta f(rz) + \gamma\Big\{\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big\}f(rz)\Big| \\ &\le \Big|F(Rz) - \beta F(rz) + \gamma\Big\{\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big\}F(rz)\Big|, \end{aligned} \tag{1.7}$$

for every $\beta, \gamma \in \mathbb{C}$ with $|\beta| \le 1, |\gamma| \le 1$ and $R > r \ge 1$.

For $f \in \mathbb{P}_n$, the polar derivative $D_\alpha f(z)$ of $f(z)$ with respect to the point α is defined as

$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z).$$

Note that $D_\alpha f(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $f(z)$ with respect to α (see [9]). It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty}\Big\{\frac{D_\alpha f(z)}{\alpha}\Big\} := f'(z),$$

uniformly with respect to z for $|z| \leq R, R > 0$.

Recently, Liman et al. [7] besides proving some other results also proved the following generalization of (1.6) to the polar derivative $D_\alpha f(z)$ of a polynomial $f(z)$ with respect to $\alpha, |\alpha| \geq 1$.

Theorem C. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that $|f(z)| \leq |F(z)|$ for $|z| = 1$. If $\alpha, \beta, \gamma \in \mathbb{C}$ be such that $|\alpha| \geq 1, |\beta| \leq 1$ and $|\lambda| < 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have*

$$\begin{aligned} &\left| z\Big[(n-m)\Big\{f(Rz) - \beta f(rz)\Big\} + D_\alpha f(Rz) - \beta D_\alpha f(rz)\Big] \right. \\ &\qquad\qquad \left. + \frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz) - \beta f(rz)\Big\} \right| \\ &\leq \left| z\Big\{D_\alpha F(Rz) - \beta D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz) - \beta F(rz)\Big\} \right|. \end{aligned} \tag{1.8}$$

Equality holds in (1.8) for $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

While making an attempt towards the generalization of the above inequalities, the authors found that there is a room for the generalization of (1.6) to the polar derivative of a polynomial which in turn induces inequalities towards more generalized form. The essence in the papers by Liman et al. [7] and Govil et al. [5] is the origin of thought for the new inequalities presented in this paper.

2. Main results

The main aim of this paper is to obtain some more general results for the maximal modulus of the polar derivative of a polynomial under certain constraints on $|z|$ and on the functions considered. We first prove the following generalization of inequalities (1.6) and (1.7) and of Theorem C.

Theorem 2.1. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\big|f(z)\big| \leq |F(z)|, \quad \textit{for} \quad |z| = 1.$$

If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ be such that $|\alpha| \geq 1, |\beta| \leq 1, |\gamma| \leq 1$ and $|\lambda| < 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have

$$\begin{aligned} &\left| z\Big[(n-m)\Big\{f(Rz) + \psi f(rz)\Big\} + D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big] \right. \\ &\qquad\qquad\qquad\qquad \left. + \frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz) + \psi f(rz)\Big\} \right| \\ &\leq \left| z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz) + \psi F(rz)\Big\} \right|, \end{aligned} \tag{2.1}$$

where

$$\psi = \psi(R, r, \beta, \gamma) = \gamma\Big\{\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big\} - \beta.$$

The result is sharp and equality in (2.1) *holds for* $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

The following result immediately follows from Theorem 2.1.

Corollary 2.1. *If* $f \in \mathbb{P}_n$, *and* $f(z)$ *does not vanish in* $|z| < 1$, *then for every* $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ *such that* $|\alpha| \geq 1, |\beta| \leq 1, |\gamma| \leq 1$ *and* $|\lambda| < 1$, *we have for* $R > r \geq 1$ *and* $|z| \geq 1$,

$$\begin{aligned}&\Big|z\Big\{D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big\} + \frac{n\lambda}{2}(|\alpha| - 1)\Big\{f(Rz) + \psi f(rz)\Big\}\Big|\\ &\leq \Big|z\Big\{D_\alpha Q(Rz) + \psi D_\alpha Q(rz)\Big\} + \frac{n\lambda}{2}(|\alpha| - 1)\Big\{Q(Rz) + \psi Q(rz)\Big\}\Big|,\end{aligned}\tag{2.2}$$

where $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$.

Equality holds in (2.2) for $f(z) = e^{i\eta}Q(z), \eta \in \mathbb{R}$. Taking $\lambda = 0$ in Corollary 2.1, we get the following result.

Corollary 2.2. *If* $f \in \mathbb{P}_n$, *and* $f(z) \neq 0$ *in* $|z| < 1$, *then for every* $|\alpha| \geq 1, |\beta| \leq 1, |\gamma| \leq 1, R > r \geq 1$ *and* $|z| \geq 1$,

$$\begin{aligned}&\Big|D_\alpha f(Rz) - \beta D_\alpha f(rz) + \gamma\Big(\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big)D_\alpha f(rz)\Big|\\ &\leq \Big|D_\alpha Q(Rz) - \beta D_\alpha Q(rz) + \gamma\Big(\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big)D_\alpha Q(rz)\Big|,\end{aligned}\tag{2.3}$$

where $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$.

Inequality (2.3) should be compared with a result of Liman, Mohapatra and Shah ([6], Lemma 2.3), where $f(z)$ is replaced by $D_\alpha f(z), |\alpha| \geq 1$.

Taking $r = 1$ in Corollary 2.2, we get the following generalization of a result due to Aziz and Rather [2].

Corollary 2.3. *If* $f \in \mathbb{P}_n$, *and* $f(z)$ *does not vanish in* $|z| < 1$, *then for every* $\alpha, \beta, \gamma \in \mathbb{C}$ *with* $|\alpha| \geq 1, |\beta| \leq 1$ *and* $R > 1$,

$$\begin{aligned}&\Big|D_\alpha f(Rz) - \beta D_\alpha f(z) + \gamma\Big(\Big(\frac{R+1}{2}\Big)^n - |\beta|\Big)D_\alpha f(z)\Big|\\ &\leq \Big|D_\alpha Q(Rz) - \beta D_\alpha Q(z) + \gamma\Big(\Big(\frac{R+1}{2}\Big)^n - |\beta|\Big)D_\alpha Q(z)\Big|, \quad \textit{for} \quad |z| \geq 1,\end{aligned}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

If we take $\beta = 0$ in Theorem 2.1, we get the following.

Corollary 2.4. *Let $F \in \mathbb{P}_n$, having all zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\big|f(z)\big| \leq \big|F(z)\big|, \quad \textit{for} \quad |z| = 1.$$

If $\alpha, \gamma, \lambda \in \mathbb{C}$ be such that $|\alpha| \geq 1, |\gamma| \leq 1$ and $|\lambda| < 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have

$$\begin{aligned}\bigg|z\Big[(n-m)\Big\{f(Rz) + \gamma\Big(\frac{R+1}{r+1}\Big)^n f(rz)\Big\} + D_\alpha f(Rz) + \gamma\Big(\frac{R+1}{r+1}\Big)^n D_\alpha f(rz)\Big] \\ + \frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz) + \gamma\Big(\frac{R+1}{r+1}\Big)^n f(rz)\Big\}\bigg|\end{aligned}$$

$$\leq \bigg|z\Big\{D_\alpha F(Rz) + \gamma\Big(\frac{R+1}{r+1}\Big)^n D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz) + \Big(\frac{R+1}{r+1}\Big)^n F(rz)\Big\}\bigg|. \tag{2.4}$$

Equality holds in (2.4) for $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

Remark 1.1. For $\gamma = 0$, Corollary 2.4 reduces to Theorem C.

Theorem 2.2. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\big|f(z)\big| \leq |F(z)|, \quad \textit{for} \quad |z| = 1.$$

If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ be such that $|\alpha| \geq 1, |\beta| \leq 1$ and $|\gamma| \leq 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have

$$\begin{aligned}\Big|z\Big[(n-m)\Big\{f(Rz) + \psi f(rz)\Big\} + D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big]\Big| \\ + \frac{n}{2}(|\alpha|-1)\big|F(Rz) + \psi F(rz)\big|\end{aligned}$$

$$\leq \Big|z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\}\Big| + \frac{n}{2}(|\alpha|-1)\big|f(Rz) + \psi f(rz)\big|, \tag{2.5}$$

where ψ is defined in Theorem 2.1.

Equality holds in (2.5) for $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

From Theorem 2.2, we have the following:

Corollary 2.5. *If $f \in \mathbb{P}_n$, and $f(z)$ does not vanish in $|z| < 1$, then for every $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ with $|\alpha| \geq 1, |\beta| \leq 1, |\gamma| \leq 1$, we have for $R > r \geq 1$, and $|z| \geq 1$,*

$$\begin{aligned}&\Big|z\Big\{D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big\}\Big| + \frac{n}{2}(|\alpha|-1)\big|Q(Rz) + \psi Q(rz)\big| \\ &\leq \Big|z\Big\{D_\alpha Q(Rz) + \psi D_\alpha Q(rz)\Big\}\Big| + \frac{n}{2}(|\alpha|-1)\big|f(Rz) + \psi f(rz)\big|,\end{aligned}$$

where $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$. and ψ is defined in Theorem 2.1.

Remark 1.2. For $\gamma = 0$, Corollary 2.5 reduces to a result of Liman et al. [7].

3. Lemmas

We need the following lemmas to prove our theorems. The first lemma is due to Liman, Mohapatra and Shah [6].

Lemma 3.1. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$, then for every $R > r \geq 1$,*

$$\big|f(Rz)\big| > \Big(\frac{R+1}{r+1}\Big)^n \big|f(rz)\big|, \quad \textit{for} \quad |z| = 1.$$

Lemma 3.2. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$,*

$$2\big|zD_\alpha f(z)\big| \geq n(|\alpha| - 1)\big|f(z)\big|, \quad \textit{for} \quad |z| = 1.$$

The above lemma is due to Shah [12].

Lemma 3.3. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq k$, then for $|\alpha| \geq k$, the polar derivative*

$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z),$$

of $f(z)$ at the point α also has all its zeros in $|z| \leq k$.

The above lemma is due to Laguerre ([9], p.49).

4. Proofs of theorems

Proof of Theorem 2.1. If $F(z)$ has a zero on $|z| = 1$, then the result is obvious, so we assume that $F(z)$ has no zeros on $|z| = 1$. Since $|f(z)| \leq |F(z)|$ for $|z| = 1$, therefore, for every $\delta \in \mathbb{C}$ with $|\delta| > 1$, we have $|f(z)| < |\delta F(z)|$, for $|z| = 1$. Also all the zeros of $F(z)$ lie in $|z| < 1$, it follows by Rouche's theorem that all the zeros of $g(z) = f(z) - \delta F(z)$ lie in $|z| < 1$. Now by Lemma 3.1, we have in particular

$$|g(rz)| < |g(Rz)|, \quad \text{for} \quad |z| = 1 \quad \text{and} \quad R > r \geq 1.$$

Since $g(Rz)$ has all its zeros in $|z| \leq \frac{1}{R} < 1$, a direct application of Rouche's theorem shows that the polynomial $g(Rz) - \beta g(rz)$ has all its zeros in $|z| < 1$ for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$. Again by using Lemma 3.1, we have

$$\begin{aligned}\big|g(Rz) - \beta g(rz)\big| &\geq \big|g(Rz)\big| - |\beta|\big|g(rz)\big| \\ &> \Big\{\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big\}\big|g(rz)\big|,\end{aligned}$$
$$\text{for} \quad |z| = 1 \quad \text{and} \quad R > r \geq 1.$$

That is

$$\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\big|g(rz)\big|<\big|g(Rz)-\beta g(rz)\big|,$$
$$\text{for}\quad |z|=1\quad\text{and}\quad R>r\geq 1.$$

If γ is any complex number with $|\gamma|\leq 1$, then it follows by Rouche's theorem that all the zeros of $T(z):=g(Rz)-\beta g(rz)+\gamma\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(rz)$ lie in $|z|<1$. Using Lemma 3.2, we get for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq 1$ and $|z|=1$,

$$2\big|zD_\alpha T(z)\big|\geq n(|\alpha|-1)\big|T(z)\big|.$$

Hence for any complex number λ with $|\lambda|<1$, we have for $|z|=1$,

$$2\big|zD_\alpha T(z)\big|>n|\lambda|(|\alpha|-1)\big|T(z)\big|.$$

Therefore, it follows by Lemma 3.3, that all the zeros of

$$\begin{aligned}W(z)&:=2zD_\alpha T(z)+n\lambda(|\alpha|-1)T(z)\\&=2zD_\alpha g(Rz)+2z\psi D_\alpha g(rz)+n\lambda\big(|\alpha|-1\big)\big(g(Rz)+\psi g(rz)\big)\end{aligned}\tag{4.1}$$

lie in $|z|<1$.

Replacing $g(z)$ by $f(z)-\delta F(z)$ and using definition of polar derivative gives

$$\begin{aligned}W(z)=2z&\left[n\Big\{f(Rz)-\delta F(Rz)\Big\}+(\alpha-Rz)\Big\{f(Rz)-\delta F(Rz)\Big\}'\right]\\&+2z\psi\left[n\Big\{f(rz)-\delta F(rz)\Big\}+(\alpha-rz)\Big\{f(rz)-\delta F(rz)\Big\}'\right]\\&+n\lambda(|\alpha|-1)\Big\{f(Rz)-\delta F(Rz)\Big\}+n\lambda\psi(|\alpha|-1)\Big\{f(rz)-\delta F(rz)\Big\},\end{aligned}$$

which on simplification gives

$$\begin{aligned}W(z)&=2z\left[(n-m)f(Rz)+mf(Rz)+(\alpha-Rz)\big(f(Rz)\big)'-\delta\Big\{nF(rz)+(\alpha-rz)\big(F(Rz)\big)'\Big\}\right]\\&\quad+2z\psi\left[(n-m)f(rz)+mf(rz)+(\alpha-rz)\big(f(rz)\big)'-\delta\Big\{nF(rz)+(\alpha-rz)\big(F(rz)\big)'\Big\}\right]\\&\qquad+n\lambda(|\alpha|-1)\Big\{f(Rz)-\delta F(Rz)\Big\}+n\lambda\psi(|\alpha|-1)\Big\{f(rz)-\delta F(rz)\Big\}\\&=2z\Big\{(n-m)f(Rz)+D_\alpha f(Rz)-\delta D_\alpha F(Rz)\Big\}\\&\quad+2z\psi\Big\{(n-m)f(rz)+D_\alpha f(rz)-\delta D_\alpha F(rz)\Big\}\\&\quad+n\lambda(|\alpha|-1)\Big\{f(Rz)-\delta F(Rz)\Big\}+n\lambda\psi(|\alpha|-1)\Big\{f(rz)-\delta F(rz)\Big\}\\&=2z\Big\{(n-m)f(Rz)+\psi(n-m)f(rz)+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big\}\\&\quad+n\lambda(|\alpha|-1)f(Rz)+n\lambda\psi(|\alpha|-1)f(rz)-\delta\Big\{2zD_\alpha F(Rz)+2z\psi D_\alpha F(rz)\\&\qquad+n\lambda\psi(|\alpha|-1)F(Rz)+n\lambda\psi(|\alpha|-1)f(rz)\Big\}.\end{aligned}\tag{4.2}$$

Since by (4.1), $W(z)$ has all its zeros in $|z| < 1$, therefore, by (4.2), we get for $|z| \geq 1$,

$$\begin{aligned} &\left|z\Big[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big]+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\right| \\ &\leq \left|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\right|. \end{aligned} \tag{4.3}$$

To see that the inequality (4.3) holds, note that if the inequality (4.3) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\begin{aligned} &\left|z_0\Big[(n-m)\,A+D_\alpha f(Rz_0)+\psi D_\alpha f(rz_0)\Big]+\frac{n\lambda}{2}(|\alpha|-1)\,A\right| \\ &> \left|z_0\Big\{D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}\right|, \end{aligned} \tag{4.4}$$

where $A = f(Rz_0) + \psi f(rz_0)$. Now, because by hypothesis all the zeros of $F(z)$ lie in $|z| \leq 1$, the polynomial $F(Rz)$ has all its zeros in $|z| \leq \frac{1}{R} < 1$, and therefore, if we use Rouche's theorem and Lemmas 3.1 and 3.3 and argument similar to the above, we will get that all the zeros of

$$z\Big(D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}$$

lie in $|z| < 1$ for every $|\alpha| \geq 1, |\lambda| < 1$ and $R > r \geq 1$, that is

$$z\Big(D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\} \neq 0$$

for every z_0 with $|z_0| \geq 1$.
Therefore, if we take

$$\delta = \frac{z_0\Big[(n-m)\,A+D_\alpha f(Rz_0)+\psi D_\alpha f(rz_0)\Big]+\frac{n\lambda}{2}(|\alpha|-1)\,A}{z_0\Big(D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}},$$

then δ is a well-defined real or complex number, and in view of (4.4) we also have $|\delta| > 1$. Hence, with the choice of δ, we get from (4.2) that $W(z_0) = 0$ for some z_0, satisfying $|z_0| \geq 1$, which is clearly a contradiction to the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus for every $R > r \geq 1, |\alpha| \geq 1, |\lambda| < 1$ and $|z| \geq 1$, inequality (4.3) holds and this completes the proof of
Theorem 1.1.

Proof of Corollary 2.1. Since the polynomial $f(z)$ does not vanish in $|z| < 1$, therefore, all the zeros of the polynomial $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})} \in \mathbb{P}_n$, lie in $|z| \leq 1$ and $|f(z)| = |Q(z)|$ for $|z| = 1$. Applying Theorem 1.1 with $F(z)$ replaced by $Q(z)$, the result follows.

Proof of Theorem 2.2. Since all the zeros of $F(z)$ lie in $|z| \leq 1$, for $R > r \geq 1, |\beta| \leq 1, |\gamma| \leq 1$, it follows as in the proof of Theorem 2.1, that all the zeros of

$$h(z) = F(Rz) - \beta F(rz) + \gamma \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} F(rz) = F(Rz) + \psi F(rz)$$

lie in $|z| \leq 1$. Hence by Lemma 3.2, we get for $|\alpha| \geq 1$,

$$2\big|zD_\alpha h(z)\big| \geq n(|\alpha| - 1)\big|h(z)\big|, \quad \text{for} \quad |z| \geq 1.$$

This gives for every λ with $|\lambda| < 1$ and for $|z| \geq 1$

$$\left| z\Big\{ D_\alpha F(Rz) + \psi D_\alpha F(rz) \Big\} \right| - \frac{n|\lambda|}{2}(|\alpha| - 1)\big|F(Rz) + \psi F(rz)\big| \geq 0. \tag{4.5}$$

Therefore, it is possible to choose the argument of λ in the right hand side of (4.3) such that for $|z| \geq 1$,

$$\begin{aligned} &\left| z\Big\{ D_\alpha F(Rz) + \psi D_\alpha F(rz) \Big\} + \frac{n\lambda}{2}(|\alpha| - 1)\Big\{ F(Rz) + \psi F(rz) \Big\} \right| \\ &= \left| z\Big\{ D_\alpha F(Rz) + \psi D_\alpha F(rz) \Big\} \right| - \frac{n|\lambda|}{2}(|\alpha| - 1)\Big|F(Rz) + \psi F(rz)\Big|. \end{aligned} \tag{4.6}$$

Hence from (4.3), we get by using (4.6) for $|z| \geq 1$,

$$\begin{aligned} &\left| z\left[(n-m)\Big\{ f(Rz) + \psi f(rz) \Big\} + D_\alpha f(Rz) + \psi D_\alpha f(rz) \right] \right| \\ &\qquad\qquad - \frac{n|\lambda|}{2}(|\alpha| - 1)\big|f(Rz) + \psi f(rz)\big| \\ &\leq \left| z\Big\{ D_\alpha F(Rz) + \psi D_\alpha F(rz) \Big\} \right| - \frac{n|\lambda|}{2}(|\alpha| - 1)\Big|F(Rz) + \psi F(rz)\Big|. \end{aligned} \tag{4.7}$$

Letting $|\lambda| \to 1$ in (4.7), we immediately get (2.5) and this completes proof of Theorem 2.2 completely.

Proof of Corollary 2.5. By hypothesis, the polynomial $f(z)$ has all its zeros in $|z| \geq 1$, therefore, all the zeros of the polynomial $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})} \in \mathbb{P}_n$, lie in $|z| \leq 1$ and $|f(z)| = |Q(z)|$ for $|z| = 1$. Applying Theorem 2.2 with $F(z)$ replaced by $Q(z)$, the result follows.

Список литературы

[1] N. C. Ankeny and T. J. Rivlin, “On a theorem of S. Bernstein”, Pacific J. Math., **5**, 849 – 852 (1955).

[2] A. Aziz and N. A. Rather, “Some compact generalizations of Bernstein-Type inequalities for polynpmials”, Math. Ineq. Appl., **7**, 393 – 403 (2004).

[3] S. Bernstein, “Sur la limitation des derivées des polynomes”, C. R. Acad. Sci. Paris, **190**, 338 – 341 (1930).

[4] N. K. Govil, M. A. Qazi and Q. I. Rahman, “Inequalities describing the growth of polynomials not vanishing in a disk of prescribed radius”, Math. Ineq. Appl., **6**, 453 – 466 (2003).

[5] N. K. Govil, A. Liman and W. M. Shah, “Some inequalities concerning derivative and maximum modulus of polynomials, Austr. J. Math. Anal. Appl., **8**, 1 – 8 (2011).

[6] A. Liman, R. N. Mohapatra and W. M. Shah, “Inequalities for polynomials not vanishing in a disk”, Appl. Math. Comp., **218**, 949 – 955 (2011).
[7] A. Liman, I. Q. Peer and W. M. Shah, “On some inequalities concerning the polar derivative of a polynomial”, Ramanujan J., **38**, 349 – 360 (2015).
[8] P. D. Lax, “Proof of a conjecture of P. Erdös on the derivative of a polynomial”, Bull. Amer. Math. Soc., **50**, 509 – 513 (1944).
[9] M. Marden, Geometry of Polynomials, Math., Surveys, no. 3, Amer. Math. Soc., Providence RI (1966).
[10] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press, New York (2002).
[11] M. Riesz, Über einen satz des Herrn Serge Bernstein, Acta Mathematica, **40**, 337 – 347 (1916).
[12] W. M. Shah, “A generalization of a theorem Paul Turán”, J. Ramanujan Math. Soc., **1**, 67 – 72 (1996).

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A NOTE ON RECURSIVE INTERPOLATION FOR THE LIPSCHITZ CLASS

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Abstract. This note is framed in the field of complex analysis and deals with some types of interpolating sequences for Lipschitz functions in the unit disk. We introduce recursion between each point of a sequence and the next. We also add interpolation by the derivative, linking its values to those that the function takes. On the supposition that the sequences are quite contractive and lie in a Stolz angle, we relate the interpolating ones for each type to the uniformly separated sequences.

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1. Introduction

We denote by $\mathbb{D}$ the disk in the complex plane $\mathbb{C}$ and by *Lip* the Lipschitz class, that is, the space of all analytic functions f on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and such that

$$M_f = \sup_{z\neq w} \frac{|f(z)-f(w)|}{|z-w|} < \infty.$$

It is well-known that $f \in Lip$ if and only if $f' \in H^\infty$ (the space of bounded analytic functions on $\mathbb{D}$). We put $\Lambda = (\lambda_n)$ for bounded sequences of complex numbers and l^∞ for their space ($\|\Lambda\|_\infty = \sup_n |\lambda_n|$). We denote by $Z = (z_n)$ any sequence in $\mathbb{D}$ satisfying the Blaschke condition $\sum_n (1-|z_n|) < \infty$. We write $\tau(z,w) = \dfrac{z-w}{1-\overline{z}w}$, so that $|\tau(z,w)|$ is the pseudo-hyperbolic distance between z and w. We put B for the Blaschke product in $\mathbb{D}$ with zeros at Z, that is,

$$B(z) = \prod_n \frac{\overline{z}_n}{z_n}\,\tau(z_n,z),$$

and $B_{i_1,\ldots,i_m}$ for the Blaschke product with zeros at $Z\setminus\{z_{i_1},\ldots,z_{i_m}\}$.

We recall that a sequence Z is called *k-contractive* if there is a constant $0<k<1$ such that

$$|z_{m+1}-z_m| \le k\,|z_m - z_{m-1}|, \quad m \ge 2.$$

We also recall that a sequence Z is called *uniformly separated* (we will abbreviate by writing u.s.) if

$$|B_m(z_m)| \ge \delta > 0, \quad m \in \mathbb{N}.$$

Carleson's theorem ([1]) ensures that the u.s. sequences are the interpolating ones for H^∞ (it means that given any $\Lambda \in l^\infty$, there is $f \in H^\infty$ such that $f(z_n) = \lambda_n$ for all n).

First, we bring up two types of interpolating sequences for *Lip*.

Definition 1.1. *Z is called an interpolating sequence for Lip if given any sequence (ω_n) satisfying*

$$\sup_{i \neq j} \frac{|\omega_i - \omega_j|}{|z_i - z_j|} < \infty, \tag{1.1}$$

there exists $f \in Lip$ such that $f(z_n) = \omega_n$ for all n.

Definition 1.2. *Z is called a double interpolating sequence for Lip if given any sequences (ω_n) satisfying* (1.1) *and $(\lambda_n) \in l^\infty$, there exists $f \in Lip$ such that $f(z_n) = \omega_n$ and $f'(z_n) = \lambda_n$ for all n.*

Both types are characterized in the following two Theorems, but only when the sequence Z is in a *Stolz angle*, that is, when for some $\zeta \in \partial\mathbb{D}$ and $1 < \mu < \infty$,

$$|z_n - \zeta| < \mu\,(1 - |z_n|), \quad n \in \mathbb{N}.$$

For example, the radial sequence $(1-2^{-n})$ satisfies the Blaschke condition, is $(1/2)$-contractive and lies in a Stolz angle ($\zeta = 1$).

Theorem 1.1. ([2], [3]). *A sequence Z in a Stolz angle is interpolating for Lip if and only if Z is the union of two u.s. sequences.*

Theorem 1.2. ([3]). *A sequence Z in a Stolz angle is double interpolating for Lip if and only if Z is u.s.*

The interpolation by Lipschitz functions for a closed set in $\overline{\mathbb{D}}$ has also been studied (see [4]).

Our purpose is to introduce some new types of interpolating sequences for *Lip*. For that, we modify the above Definitions for the case that a recursive relationship of the interpolating function in two consecutive points of the sequence is required. On the other hand, we impose a rather natural ligature between the interpolating function and its derivative, also adding a recursive relationship for the derivative. We are interested in knowing if doing this, we have to restrict ourselves to some sort of sequences to obtain the same results as if recursion is not considered. Recursive interpolating sequences for the space H^∞ have already been addressed in [5], and in this note, we check the effect of introducing recursion in a space of functions that are regular up to the boundary of the disk.

Specifically, we introduce the following sequences.

Definition 1.3. *We say that Z is a recursive interpolating sequence for Lip if given any $\alpha \in \mathbb{C}$ and $\Lambda = (\lambda_n) \in l^\infty$, there exists $f \in Lip$ such that $f(z_1) = \alpha$ and recursively, for each $n \in \mathbb{N}$,*

$$\frac{f(z_{n+1}) - f(z_n)}{z_{n+1} - z_n} = \lambda_n. \tag{1.2}$$

Note that all quotients in (1.2) are bounded, because $f \in Lip$.

Definition 1.4. *If we include $f'(z_n) = \lambda_n$ in Definition 1.3, we say that Z is a double and recursive interpolating sequence for Lip.*

In this Definition, the requirement for the derivative is added to relate its value in a point to a difference quotient of the function in that point. Finally, taking into account that if $g \in H^\infty$, then

$$|g(z) - g(w)| \leq c\,|\tau(z, w)|$$

for a constant $c > 0$, we can state:

Definition 1.5. *We say that Z is an interpolating sequence in a general sense for Lip if given any α, β, $\eta \in \mathbb{C}$, there exists $f \in Lip$ such that $f(z_1) = \alpha$, $f'(z_1) = \beta$ and, recursively, for each $n \in \mathbb{N}$,*

$$\begin{cases} f'(z_n) = \dfrac{f(z_{n+1}) - f(z_n)}{z_{n+1} - z_n} \\ f'(z_{n+1}) = f'(z_n) + \eta\,\tau(z_n, z_{n+1}) \end{cases} \tag{1.3}$$

Note that these two equalities can be interpreted as a certain system of recurrence equations. The next section is devoted to examining these types of interpolating sequences.

2. Statement and proof of results

We will use the following two Lemmas.

Lemma 2.1. ([3]). *If a function $f \in Lip$ vanishes on a sequence Z, then for each $m \in \mathbb{N}$,*

$$|f(z)| \leq M_f\,|z - z_m|\,|B_m(z)|.$$

Lemma 2.2. *If Z is a k-contractive sequence and $k \leq k_0 < 1/2$, then given any integer $p \geq 2$,*

$$|z_{m+1} - z_m| \leq K_0\,|z_{m+p} - z_m|, \quad m \in \mathbb{N},$$

where $K_0 = (1 - k_0)/(1 - 2k_0 + k_0^p)$.

Proof. By the triangle inequality and since Z is k-contractive,

$$\begin{aligned}|z_{m+1}-z_m| &\le |z_{m+p}-z_m| + |z_{m+p}-z_{m+p-1}| + \cdots + |z_{m+2}-z_{m+1}|\\ &\le |z_{m+p}-z_m| + (k^{p-1}+\cdots+k)|z_{m+1}-z_m|.\end{aligned}$$

Since $k \le k_0 < 1/2$, then $k^{p-1}+\cdots+k \le k_0^{p-1}+\cdots+k_0 < 1$, and

$$|z_{m+1}-z_m| \le \frac{1}{1-(k_0^{p-1}+\cdots+k_0)}|z_{m+p}-z_m| = K_0\,|z_{m+p}-z_m|.$$

The proof is complete. □

Our results are the following ones.

Theorem 2.1. *Let Z be a sequence in a Stolz angle and k-contractive for some $k \le k_0 < 1/2$. Then, Z is recursive interpolating for Lip if and only if Z is the union of two u.s. sequences.*

Proof. Suppose that Z is recursive interpolating for *Lip*. Take $\alpha = 0$ and for a fixed $m \in \mathbb{N}$, let Λ be defined by: $\lambda_m = \dfrac{z_{m+2}-z_{m+1}}{z_{m+1}-z_m}$, $\lambda_{m+1} = -1$ and $\lambda_n = 0$, otherwise. Because Z is k-contractive, we have $|\lambda_m| \le k$ and then, $\|\Lambda\|_\infty = 1$. Since the operator given by the quotient on the left in (1.2) is linear and surjective, by the open mapping theorem there is a function $f_m \in Lip$ and a constant $c > 0$ such that $M_{f_m} \le c\,\|\Lambda\|_\infty = c$. We have $f_m(z_m) = z_{m+1}-z_m$ and $f_m(z_n) = 0$, if $n \ne m$. Applying Lemma 2.1 to $Z \setminus \{z_m\}$,

$$|f_m(z)| \le c\,|z-z_{m+1}|\,|B_{m,m+1}(z)|,$$

and evaluating at z_m,

$$|z_{m+1}-z_m| = |f_m(z_m)| \le c\,|z_m-z_{m+1}|\,|B_{m,m+1}(z_m)|,$$

that is,

$$|B_{m,m+1}(z_m)| \ge c. \tag{2.1}$$

This condition (2.1) implies that Z is the union of two u.s. sequences (see [6], p. 1202).

Reciprocally, to meet the requirement in (1.2) we look for f verifying $f(z_n) = \gamma_n$, where $\gamma_1 = \alpha$ and for each $n \ge 2$,

$$\gamma_n = \alpha + \lambda_1(z_2-z_1) + \cdots + \lambda_{n-1}(z_n-z_{n-1}).$$

Suppose $i > j$. Taking into account that Z is k-contractive,

$$\begin{aligned}|\gamma_i-\gamma_j| &= |\lambda_j(z_{j+1}-z_j) + \cdots + \lambda_{i-1}(z_i-z_{i-1})|\\ &\le \|\Lambda\|_\infty\,(|z_{j+1}-z_j| + \cdots + |z_i-z_{i-1}|)\\ &\le \|\Lambda\|_\infty\,(1+k+\cdots+k^{i-j-1})\,|z_{j+1}-z_j|.\end{aligned}$$

If $i > j+1$, then by Lemma 2.2,

$$|\gamma_i - \gamma_j| \le \|\Lambda\|_\infty \frac{1-k_0^{i-j}}{1-2k_0+k_0^{i-j}} |z_i - z_j|.$$

The existence of the desired interpolating function f follows from Theorem 1.1. □

Theorem 2.2. *Let Z be a sequence in a Stolz angle and k-contractive for some $k \le k_0 < 1/2$. Then, Z is double and recursive interpolating for Lip if and only if Z is u.s.*

Proof. The necessity for the sequence Z to be u.s. is a consequence of the requirement that the function f' in H^∞ must interpolate the sequence Λ in l^∞ (Carleson's theorem). As for sufficiency, take $(\lambda_n) \in l^\infty$. By Theorem 2.1, there is $g \in Lip$ verifying (1.2). It is proved in [3] that if Z in a Stolz angle is u.s., then given any sequence $(\alpha_n) \in l^\infty$, there is a function $h \in Lip$ such that $h(z_n) = 0$ and $h'(z_n) = \alpha_n$ for all n. Taking $\alpha_n = \lambda_n - g'(z_n)$, it follows that the function $f = g + h$ performs (1.2) and $f'(z_n) = \lambda_n$ for all n. □

Theorem 2.3. *If Z is a sequence in a Stolz angle and k-contractive for some $k \le k_0 < 1/2$, it verifies the condition*

$$\sum_n |\tau(z_n, z_{n+1})| < \infty \tag{2.2}$$

and is u.s., then Z is interpolating in a general sense for Lip.

Proof. Equivalently, instead of looking for a function $f \in Lip$ that verifies (1.3), we look for it so that $f(z_n) = \gamma_n$ and $f'(z_n) = \gamma'_n$, where

$$\gamma_1 = \alpha, \quad \gamma_2 = \alpha + \beta(z_2 - z_1),$$

$$\gamma_n = \alpha + \beta(z_n - z_1) + \eta \sum_{l=3}^{n} \left(\sum_{m=1}^{l-2} \tau(z_m, z_{m+1}) \right) (z_l - z_{l-1}), \quad n \ge 3;$$

$$\gamma'_1 = \beta, \quad \gamma'_n = \beta + \eta \sum_{l=1}^{n-1} \tau(z_l, z_{l+1}), \quad n \ge 2.$$

Suppose $i > j$. By (2.2), there is a constant $c > 0$ such that

$$\begin{aligned} |\gamma_i - \gamma_j| &\le |\beta|\,|z_i - z_j| + \eta \sum_{l=j+1}^{i} \left(\sum_{m=1}^{l-2} |\tau(z_m, z_{m+1})| \right) |z_l - z_{l-1}| \\ &\le |\beta|\,|z_i - z_j| + c\,\eta \sum_{l=j+1}^{i} |z_l - z_{l-1}| \end{aligned}$$

As in the proof of Theorem 2.1, if $i > j+1$, then

$$|\gamma_i - \gamma_j| \le \left(|\beta| + c\,\eta \frac{1-k_0^{i-j}}{1-2k_0+k_0^{i-j}} \right) |z_i - z_j|.$$

On the other hand, $(\gamma'_n) \in l^\infty$ by (2.2). So, the existence of the interpolating function f follows now from Theorem 1.2. $\square$

The sequence $(1 - 2^{-n})$ is also an example for the condition (2.2).

Список литературы

[1] L. Carleson, "An interpolation problem for bounded analytic functions," Amer. J. Math., **80**, 921 – 930 (1958).

[2] A. M. Kotochigov, "Free interpolation in the spaces of analytic functions with derivative of order s from the Hardy space," Journal of Math. Sciences., **129** (4), 4022 – 4039 (2005).

[3] E. P. Kronstadt, "Interpolating sequences for functions satisfying a Lipschitz condition," Pacific J. Math., **63** (1), 169 – 177 (1976).

[4] J. Bruna, F. Tugores, "Free interpolation for holomorphic functions regular to the boundary," Pacific J. Math., **108** (1), 31 – 49 (1983).

[5] F. Tugores, "Recursive interpolating sequences," Open Math., **16**, 461 – 468 (2018).

[6] J. Bruna, A. Nicolau, K. Øyma, "A note on interpolation in the Hardy spaces of the unit disc," Proc. Am. Math. Soc., **124** (4), 1197 – 1204 (1996).

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