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РАСПРЕДЕЛЕНИЕ РАССТОЯНИЯ МЕЖДУ ДВУМЯ СЛУЧАЙНЫМИ ТОЧКАМИ В ТЕЛЕ ИЗ R^3

Н. Г. АГАРОНЯН, В. ХАЛАТЯН

Ереванский государственный университет[1]
E-mails: *narine78@ysu.am*; *khalatyan96@gmail.com*

Аннотация. В настоящей работе получена формула для вычисления функции плотности $f_\rho(x)$ растояния между двумя случайными независимыми точками, случайно и равномерно выбранными в ограниченном выпуклом теле D. Формула позволяет найти явный вид функции плотности $f_\rho(x)$ для тела D с известным распределением длины хорды. В частности, получено явное выражение для $f_\rho(x)$ для случая шара диаметра d в R^3.

MSC2010 numbers: 60D05; 52A22; 53C65.

Ключевые слова: функция распределения длины хорды; кинематическая мера в R^3; ограниченное выпуклое тело.

1. Введение

В прошлом веке немецкий математик В. Бляшке сформулировал проблему исследования ограниченного выпуклого тела вероятностными методами. В частности, проблему распознавания ограниченных выпуклых тел по распределению длины хорды. В этой статье рассматривается задача для трехмерных тел. Результат для плоских областей см. в [11] и [12].

Пусть D - ограниченное выпуклое тело в трехмерном евклидовом пространстве R^3 с объемом $V(D)$ и площадью поверхности $S(D)$. Пусть P_1 и P_2 - две точки, выбранные случайным образом, независимо и с равномерным распределением в D, мы собираемся найти вероятность того, что расстояние $\rho(P_1, P_2)$ между P_1 и P_2 равно или меньше x, то есть мы хотели бы найти функцию распределения $F_\rho(x)$ расстояния $\rho(P_1, P_2)$. По определению имеем

$$F_\rho(x) = P(P_1, P_2 \in D : \rho(P_1, P_2) \le x) = \frac{\iint\limits_{\{P_1, P_2 \in D \,:\, \rho(P_1,P_2) \le x\}} dP_1 dP_2}{\iint\limits_{\{P_1, P_2 \in D\}} dP_1\, dP_2} \tag{1.1}$$

[1]Работа первого автора выполнена при финансовой поддержке ГКН МОН РА в рамках научного проекта 18T-1A252.

где dP_i, $i = 1, 2$ - мера Лебега в пространстве R^3. Так как

$$\iint\limits_{\{P_1,\, P_2 \in D\}} dP_1 dP_2 = V^2(D) \tag{1.2}$$

(здесь мы используем, что точки P_1 и P_2 выбираются независимо в D), мы получаем

$$F_\rho(x) = \frac{1}{V^2(D)} \iint\limits_{\{P_1, P_2 \in D \,:\, \rho(P_1, P_2) \le x\}} dP_1\, dP_2. \tag{1.3}$$

Из выражения элемента площади в сферической системе координат, где в качестве начала координат мы выбираем точку P_1, получаем

$$\begin{cases} x = r\cos\psi\sin\theta \\ y = r\sin\psi\sin\theta \\ z = r\cos\theta \end{cases}$$

где r - расстояние между P_1 и P_2, ψ - угол между проекцией отрезка P_1P_2 на XOY и осью OX. θ - это угол, образованный осью OZ и отрезком P_1P_2. Таким образом, используя преобразование из декартовой системы координат в сферическую систему координат, получаем

$$dP_2 = dx_2 d\,y_2\, dz_2 = r^2\, \sin\theta\, dr\, d\theta\, d\psi.$$

Используя это выражение, мы имеем

$$dP_1 dP_2 = r^2\, dr\, \sin\theta\, d\theta\, d\psi = r^2\, dr \cdot dK, \tag{1.4}$$

где dK - элемент кинематической меры в R^3.

Кинематическая плотность в евклидовом пространстве была впервые введена Пуанкаре. В современной терминологии это мера Хаара группы движений (сдвигов и вращений), которая действует в пространстве. Пусть R^3 - евклидово трехмерное пространство, и dK - кинематическая плотность, нормированная так, что мера всех положений относительно точки равна $8\pi^2$. Другими словами, мера всех положений тела D с объемом $V(D)$, для которого D содержит неподвижную точку, равна $8\pi^2 V(D)$.

Используя (1.4), мы можем переписать (1.3) в следующем виде:

$$F_\rho(x) = \frac{1}{V^2(D)} \int_0^x r^2 K(D, r) dr \tag{1.5}$$

где $K(D,r)$ - кинематическая мера всех ориентированных отрезков длины r, лежащих внутри D. Таким образом, мы получаем связь между функцией плотности $f_\rho(x)$ расстояния $\rho(P_1,P_2)$ и кинематической мерой $K(D,x)$:

$$f_\rho(x) = \frac{x^2 K(D,x)}{[V(D)]^2} \tag{1.6}$$

Следует отметить, что мы можем вычислить кинематическую меру всех неориентированных отрезков, которые лежат внутри D, а затем умножить результат на 2.

Пусть $S_1 = MS$ - образ отрезка S при евклидовом движении. М - группа всех евклидовых движений в пространстве R^3. Для локально компактной группы М существует локально конечная мера Хаара, т. е. локально конечная, не тождественно равная нулю борелевская мера, инвариантная как слева, так и справа. Отрезок S_1 можно определить с помощью двух координат (γ, t), где $\gamma \in J$ (J - пространство всех прямых в R^3) содержит отрезок S_1, а t - одномерная координата центра отрезка S_1 на прямой γ. В пространстве M определим меру по ее элементу следующим образом:

$$m(dS_1) = d\gamma\, dt, \tag{1.7}$$

где $d\gamma$ - локально конечная мера в пространстве J, инвариантная относительно группы M, а dt - одномерная мера Лебега на γ. Мера $m(\cdot)$ называется кинематической мерой на группе M.

2. Основная формула

В этом разделе приведена основная формула для вычисления кинематической меры $K(D,x)$ в терминах функции распределения длины хорды тела D. Как известно (см. [1] - [3] или [10]), решение задачи о нахождении кинематической меры $K(D,x)$ отрезков постоянной длины x, целиком лежащих в D, непростое и существенно зависит от формы D. Очевидно, что

$$K(D,r) = 0, \quad \text{если } r \geq diam(D)$$

где $diam(D)$ - диаметр D, т.е. $diam(D) = max\{\rho(x,y) : x, y \in D\}$, где $\rho(x,y)$ - расстояние между точками x и y. Следовательно, только случай $0 \leq r \leq diam(D)$ рассматривается в статье. Очевидно, что в указанном случае

$$K(D,r) = \int_{[D]} \int_{t \in (0, \chi(\gamma) - r)} d\gamma\, dt = \int (\chi(\gamma) - r)^+ d\gamma, \tag{2.1}$$

где $[D]=\{\gamma\in J:\gamma\cap D\neq\varnothing\}$ - множество прямых в R^3, пересекающих тело D, $\chi(\gamma)=\gamma\cap D$ - хорда в D, а

$$x^+=\begin{cases}0, & \text{если } \ x\leq 0\\ x, & \text{если } \ x\geq 0.\end{cases}$$

Известно, что (см. [1] или [4])

$$\int\chi(\gamma)\,d\gamma=2\pi V(D),\qquad \int d\gamma=\frac{\pi}{2}S(D) \tag{2.2}$$

поэтому,

$$K(D,r)=\int\limits_{\chi(\gamma)>r}\chi(\gamma)\,d\gamma-r\int\limits_{\chi(\gamma)>r}d\gamma=2\pi\,V(D)-G(r)-r\,\frac{\pi}{2}S(D)\,[1-F_D(r)], \tag{2.3}$$

где

$$G(x)=\int_{\chi(\gamma)\leq x}\chi(\gamma)\,d\gamma \tag{2.4}$$

и $F_D(\cdot)$ - функция распределения длины хорды тела D, определяемая как

$$F_D(y)=\frac{2}{\pi S(D)}\cdot\int_{\chi(\gamma)\leq y}d\gamma \tag{2.5}$$

(так как $\int_{[D]}d\gamma=\frac{\pi}{2}\cdot S(D)$). Теперь докажем следующую формулу:

$$G(x)=\frac{\pi}{2}S(D)\int_0^x u\,f_D(u)\,du, \tag{2.6}$$

где $f_D(x)$ - функция плотности длины хорды тела D, т. е. $f_D(x)=F'_D(x)$ - первая производная функция распределения. Теперь вычислим производную функции $G(x)$. Имеем

$$\frac{G(x+\Delta x)-G(x)}{\Delta x}=\frac{1}{\Delta x}\int_{x<\chi(\gamma)\leq x+\Delta x}\chi(\gamma)\,d\gamma$$
$$=(x+\theta\Delta x)\,\frac{\pi}{2}S(D)\,\frac{F_D(x+\Delta x)-F_D(x)}{\Delta x}.$$

Тогда, предполагая, что функция распределения $F_D(x)$ обладает плотностью $f_D(x)$, при $\Delta x\to 0$, получим $G'(x)=\frac{\pi}{2}S(D)\,x\,f_D(x)$, откуда следует

$$G(x)=G(0)+\frac{\pi}{2}S(D)\int_0^x u\,f_D(u)\,du=\frac{\pi}{2}S(D)\int_0^x u\,f_D(u)\,du, \tag{2.7}$$

поскольку $G(0)=\int_{\chi(\gamma)\leq 0}\chi(\gamma)\,d\gamma=0$. Теперь преобразуем формулу (2.7) путем интегрирования по частям:

$$G(x)=\frac{\pi}{2}S(D)\int_0^x uf_D(u)du=-\frac{\pi}{2}S(D)\int_0^x u\,d[1-F_D(u)] \tag{2.8}$$
$$=-\frac{\pi}{2}x\,S(D)\,[1-F_D(x)]+\frac{\pi}{2}S(D)\int_0^x[1-F_D(u)]\,du.$$

Наконец, подставляя (2.8) в формулу (2.3) для $K(D,r)$, приходим к основной формуле:

$$K(D,r) = 2\pi V(D) - \frac{\pi}{2} S(D) \int_0^r [1 - F_D(u)]du. \tag{2.9}$$

Теорема 2.1. *Для любого тела D в R^3*

$$K(D,r) = 2\pi V(D) - \frac{\pi}{2} S(D) \int_0^r [1 - F_D(u)]du.$$

Таким образом, если задана явная форма функции $F_D(u)$ для тела D, то можно вывести явное выражение для кинематической меры $K(D,r)$ с помощью (2.9). Формула (2.9) была получена для неориентированных отрезков. Для ориентированных отрезков эту формулу следует умножить на 2. Подставив (2.9) в (2.3) (и умножив на 2), получим основную формулу этой статьи:

$$f_\rho(r) = \frac{4\pi V(D) r^2 + \pi r^2 S(D) \int_0^r F_D(u)\, du - r^3 \pi\, S(D)}{V(D)^2} \tag{2.10}$$

Полученная формула позволяет рассчитать кинематическую меру $K(D,r)$ с помощью функции распределения длины хорды.

3. Случай шара в R^3

В случае шара $D = B_d$ с диаметром d, $V(B_d) = \frac{1}{6}\pi d^3$, $S(B_d) = \pi d^2$. Поэтому, используя теорему 2.1, получаем

$$K(B_d, r) = \frac{\pi^2 d^3}{3} + \frac{\pi^2 d^2}{2} \int_0^r F_D(u) du - \frac{r \pi^2 d^2}{2}. \tag{3.1}$$

Длина хорды Функция распределения длины хорды для шара B_d имеет следующий вид (см. [6], [7] или [8]):

$$F_{B_d}(y) = \begin{cases} 0, & \text{если } y \le 0 \\ (y/d)^2, & \text{если } 0 \le y \le d \\ 1, & \text{если } y \ge d. \end{cases} \tag{3.2}$$

Следовательно, подставляя (3.2) в (3.1), получаем

$$K(B_d, r) = \frac{\pi^2 d^3}{3} + \frac{\pi^2 r^3}{6} - \frac{r \pi^2 d^2}{2}. \tag{3.3}$$

Подставляя этот результат в (1.6), получим функцию плотности расстояния между двумя точками, выбранные в шаре диаметром d

$$f_\rho(x) = \frac{24x^2}{d^3} + \frac{12x^5}{d^6} - \frac{36x^3}{d^4}.$$

Применение явной формы $f_\rho(x)$ (или $F_D(x)$) для некоторого тела D дает нам возможность использовать эти формы в кристаллографии (см. [9]).

Abstract. In the present paper a formula for calculation of the density function $f_\rho(x)$ of the distance between two independent points randomly and uniformly chosen in a bounded convex body D is given. The formula permits to find an explicit form of density function $f_\rho(x)$ for body D with known chord length distributions. In particular, we obtain an explicit expression for $f_\rho(x)$ in the case of a ball of diameter d in R^3.

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ON HYPERBOLIC DECAY OF PREDICTION ERROR VARIANCE FOR DETERMINISTIC STATIONARY SEQUENCES

N. M. BABAYAN, M. S. GINOVYAN

Russian-Armenian University, Yerevan, Armenia
Boston University, Boston, USA, Institute of Mathematics, Yerevan, Armenia
E-mails: *nmbabayan@gmail.com; ginovyan@math.bu.edu*

Abstract. One of the main problems in prediction theory of second-order stationary processes, called direct prediction problem, is to describe the asymptotic behavior of the best linear mean squared one-step ahead prediction error variance in predicting the value $X(0)$ of a stationary process $X(t)$ by the observed past of finite length n as n goes to infinity, depending on the regularity nature (deterministic or non-deterministic) of the underlying observed process $X(t)$. In this paper, we obtain sufficient conditions for hyperbolic decay of prediction error variance for deterministic stationary sequences, generalizing a result obtained by M. Rosenblatt (Some Purely Deterministic Processes, J. of Math. and Mech., 6(6), 801-810, 1957).

MSC2010 numbers: 60G10, 60G25, 62M15, 62M20.

Keywords: prediction problem; deterministic stationary process; singular spectral density; Rosenblatt's theorem.

1. Introduction

1.1. **The prediction problem.** One of the main problems in prediction theory of second-order stationary processes, called direct prediction problem, is to describe the asymptotic behavior of the best linear mean squared one-step ahead prediction error variance in predicting the value $X(0)$ of the stationary process $X(t)$ by the observed past of finite length n as n goes to infinity, depending on the regularity nature (deterministic or nondeterministic) of the underlying observed process $X(t)$.

Let $X(t)$, $t \in \mathbb{Z} = \{0, \pm 1, \ldots\}$, be a wide sense stationary stochastic sequence with spectral function $F(\lambda)$ and spectral density function $f(\lambda)$, $\lambda \in \Lambda = [-\pi, \pi]$. Denote by $\sigma_n^2(F)$ the best linear mean squared one-step ahead prediction error variance in predicting the random variable $X(0)$ by the past of $X(t)$ of finite length n: $X(t)$, $-n \le t \le -1$, and let $\sigma^2(F) = \sigma_\infty^2(F)$ be the prediction error variance of $X(0)$ by the entire infinite past: $X(t)$, $t \le -1$. Define the relative prediction error $\delta_n(F) = \sigma_n^2(F) - \sigma^2(F)$, and observe that it is nonnegative and tends to zero as $n \to \infty$. The direct prediction problem is to describe the rate of decrease of $\delta_n(F)$ to zero as $n \to \infty$, depending on the regularity nature (deterministic or nondeterministic) and on the dependence structure of the underlying observed process $X(t)$.

Notice that the aforementioned prediction problem goes back to the classical works by A. Kolmogorov, G. Szegö and N. Wiener, and later for different classes of stationary models has been considered by many authors. The problem has been studied most intensively for nondeterministic processes, that is, in the case where the prediction error is known to be positive ($\sigma^2(F) > 0$) (see Baxter [2], Devinatz [9], Doob [10], Golinski [14], Grenander and Rosenblatt [17], Grenander and Szegö [18], Helson and Szegö [19], Hirshman [21], Ibragimov [23], Ibragimov and Solev [25], Kolmogorov [27], [28], Pourahmadi [29], Rozanov [32], Wiener [34] and others (more references can be found in Bingham [5] and Ginovyan [13]). This is not surprising because from application point of view the nondeterministic models are more realistic and represent great interest.

The case of deterministic processes, that is, when $\sigma^2(F) = 0$, represents mostly theoretical interest. However, it is also important from application point of view. For example, as it was pointed out by M. Rosenblatt [31], situations of this type arise in Neumann's theoretical model of storm-generated ocean waves. Also, such models are of interest for meteorology, because the meteorological spectra often have a gap in the mesoscale region (see Fortus [11]).

There are only few works devoted to the study of asymptotic behavior of prediction error for deterministic processes. It goes back to the classical work by M. Rosenblatt [31], where using the technique of orthogonal polynomials and Szegö's results, M. Rosenblatt has investigated the asymptotic behavior of the prediction error variance $\delta_n(F) = \sigma_n^2(F)$ for discrete-time deterministic processes in the following two cases:

(a) the spectral density $f(\lambda)$ is continuous and vanishes on an interval,

(b) the spectral density $f(\lambda)$ has a high order contact with zero.

Later the problem (a) was studied by Babayan [3], [4], Davisson [8], and Fortus [11], where some generalizations and extensions of Rosenblatt's result have been obtained.

In this paper we consider the case (b), that is, when the spectral density $f(\lambda)$ has a high order contact with zero, and obtain sufficient conditions for hyperbolic decay of prediction error variance, generalizing the corresponding result of Rosenblatt [31], obtained in this case.

Throughout the paper we will use the following notation. The letters C, c, M and m with or without indices are used to denote positive constants, the values of which can vary from line to line. For two functions $f(\lambda)$ and $g(\lambda)$, $\lambda \in \Lambda$, we will write

$f(\lambda) \underset{\lambda\to\lambda_0}{\cong} g(\lambda)$ if $\lim_{\lambda\to\lambda_0} \frac{f(\lambda}{g(\lambda)} = c$, $c \neq 0$, and $f(\lambda) \underset{\lambda\to\lambda_0}{\sim} g(\lambda)$ if $c = 1$. A similar notation we will use for sequences: for two sequences $\{a_n > 0, n \in \mathbb{N} = \{1, 2, \ldots\}\}$ and $\{b_n > 0, n \in \mathbb{N}\}$, we will write $a_n \cong b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $c \neq 0$, and $a_n \sim b_n$ if $c = 1$.

The paper is organized as follows. In the remainder of this section we introduce the model of interest - a stationary process, recall some key notions and results from the theory of stationary process, and state the infinite prediction problem. In Section 2 we state the finite prediction problem, present a formula for finite prediction error in terms of orthogonal polynomials on the unit circle, and state the Kolmogorov-Szegö theorem. Section 3 is devoted to the asymptotic behavior of the finite prediction error for nondeterministic processes. Here we briefly review some important known results. Section 4 is devoted to the asymptotic behavior of the finite prediction error for deterministic processes. Here we state and prove a number of new theorems.

1.2. **The Model.** In this subsection we introduce the model of interest - a stationary process, recall some key notions and results from the theory of stationary process (Kolmogorov's isometric isomorphism theorem, spectral representations of the covariance function and the process, etc.)

Let $\{X(t),\ t \in \mathbb{Z}\}$ be a centered, real-valued, discrete-time, second-order stationary random process defined on a probability space $(\Omega, \mathcal{F}, P)$ with covariance function $r(t)$, that is, $\mathbb{E}|\mathrm{X(t)}|^2 < \infty$, $\mathbb{E}[\mathrm{X(t)}] = 0$, $r(t) = \mathbb{E}[\mathrm{X(t)X(0)}]$, $t \in \mathbb{Z}$, where $\mathbb{E}[\cdot]$ stands for the expectation operator with respect to measure P.

By the well-known Herglotz' theorem (see [33], p. 421), there is a finite measure μ on $(\Lambda, \mathfrak{B}(\Lambda))$, where $\Lambda = [-\pi, \pi]$ and $\mathfrak{B}(\Lambda)$ is the Borel σ-algebra on Λ, such that for any $t \in \mathbb{Z}$ the covariance function $r(t)$ admits the following *spectral* representation:

$$r(t) = \int_{-\pi}^{\pi} e^{-it\lambda} d\mu(\lambda). \tag{1.1}$$

The measure μ in (1.1) is called the *spectral measure* of the process $X(t)$. The function $F(\lambda) = \mu[-\pi, \lambda]$, $\lambda \in \Lambda$, is called the *spectral function* of the process $X(t)$. If $F(\lambda)$ is absolutely continuous (with respect to Lebesgue measure), then the function $f(\lambda) = dF(\lambda)/d\lambda$ is called the *spectral density* of the process $X(t)$. Notice that $f(\lambda) \geq 0$ and $f(\lambda) \in L^1(\Lambda)$. The set $E_f = \{e^{i\lambda} : f(\lambda) > 0\}$ is called the *spectrum* of the process $X(t)$.

We assume that $X(t)$ is a *non-degenerate process*, that is, $\mathrm{Var}[X(0)] = E|X(0)|^2 = r(0) > 0$. Also, to avoid the trivial cases, we will assume that the spectral measure

μ is non-trivial, that is, μ has infinite support. We write

$$\mu(\lambda) = \mu_{AC}(\lambda) + \mu_S(\lambda) = \int_{-\pi}^{\lambda} f(u)du + \mu_S(\lambda), \tag{1.2}$$

so $f(\lambda)$ is the spectral density and μ_S is the singular part of μ, that is, $\mu_S = \mu_{SC} + \mu_{PP}$, where $\mu = \mu_{AC} + \mu_{SC} + \mu_{PP}$ is the Lebesgue decomposition of μ into an absolutely continuous (with respect to Lebesgue measure) part (μ_{AC}), a singular continuous part (μ_{SC}), and a pure point part (μ_{PP}). The same representations we have also for spectral function $F(\lambda)$.

By the well-known Cramér theorem (see [33], p. 430), for any stationary process $\{X(t),\, t \in \mathbb{Z}\}$ with spectral measure μ there exists an orthogonal stochastic measure $Z = Z(B)$, $B \in \mathfrak{B}(\Lambda)$, such that for every $t \in \mathbb{Z}$ the process $X(t)$ admits the following *spectral* representation:

$$X(t) = \int_{-\pi}^{\pi} e^{it\lambda} dZ(\lambda). \tag{1.3}$$

Moreover, $\mathbb{E}\left[|\mathrm{Z(B)}|^2\right] = \mu(\mathrm{B})$ for every $B \in \mathfrak{B}(\Lambda)$. For definition and properties of orthogonal stochastic measures and stochastic integral in (1.3) we refer, e.g., [33], Chapter VI.

Given a probability space $(\Omega, \mathcal{F}, P)$, define the L^2-space of random variables $\xi = \xi(\omega)$, $\mathbb{E}[\xi] = 0$:

$$L^2(P) = \left\{ \xi :\, ||\xi||^2 = \int_{\Omega} |\xi(\omega)|^2 dP(\omega) < \infty \right\}. \tag{1.4}$$

Then $L^2(P)$ becomes a Hilbert space with the following inner product: for $\xi, \eta \in L^2(P)$

$$(\xi, \eta) = \mathbb{E}[\xi\eta] = \int_{\Omega} \xi(\omega)\overline{\eta(\omega)} \mathrm{dP}(\omega). \tag{1.5}$$

For $a, b \in \mathbb{Z}$, $-\infty \le a \le b \le \infty$, we define the space $H_a^b(X)$ to be the closed linear subspace of the space $L^2(P)$ spanned by the random variables $X(t, \omega)$, $t \in [a, b]$:

$$H_a^b(X) = \overline{sp}\{X(t),\ a \le t \le b\}_{L^2(P)}. \tag{1.6}$$

Observe that the subspace $H_a^b(X)$ consists of all finite linear combinations, $\sum_{k=1}^n c_k X(t_k)$ ($a \le t_k \le b,\ k, n \in \mathbb{N}$), as well as, their $L^2(P)$-limits.

Definition 1.1. The space $H(X) = H_{-\infty}^{\infty}(X)$ is called the *Hilbert space generated by the process* $X(t)$, or the *time-domain* of $X(t)$.

Consider the weighted L^2-space $L^2(\mu)$ of complex-valued functions $\varphi(\lambda)$, $\lambda \in \Lambda$, defined by

$$L^2(\mu) = \left\{ \varphi(\lambda) :\, ||\varphi||_\mu^2 := \int_{-\pi}^{\pi} |\varphi(\lambda)|^2 d\mu(\lambda) < \infty \right\}. \tag{1.7}$$

Then $L^2(\mu)$ becomes a Hilbert space with the following inner product: for $\varphi, \psi \in L^2(\mu)$

$$(\varphi, \psi)_\mu = \int_{-\pi}^{\pi} \varphi(\lambda)\overline{\psi}(\lambda)d\mu(\lambda). \tag{1.8}$$

For $a, b \in \mathbb{Z}$, $-\infty \le a \le b \le \infty$ define the space $H_a^b(\mu)$ to be the closed linear subspace of the space $L^2(\mu)$ spanned by the exponents $e^{it\lambda}$, $t \in [a, b]$:

$$H_a^b(\mu) = \overline{sp}\{e^{it\lambda},\ a \le t \le b\}_{L^2(\mu)}. \tag{1.9}$$

Definition 1.2. The Hilbert space $H(\mu) := H_{-\infty}^{\infty}(\mu)$ is called the *frequency-domain* of the process $X(t)$.

Kolmogorov's Isometric Isomorphism Theorem states that for any stationary process $X(t)$ with spectral measure μ there exists a unique isometric isomorphism V between the time- and frequency-domains $H(X)$ and $L^2(\mu)$, such that $V[X(t)] = e^{it\lambda}$ for any $t \in \mathbb{Z}$. In particular, we have

1. For any random variable $Y \in H(X)$ there exist a unique function $\varphi(\lambda) \in L^2(\mu)$, such that Y admits the spectral representation

$$Y = \int_{-\pi}^{\pi} \varphi(\lambda)dZ(\lambda), \tag{1.10}$$

where Z is the orthogonal stochastic measure in the spectral representation (1.3) of $X(t)$, and for any function $\varphi(\lambda) \in L^2(\mu)$ the stochastic integral (1.10) defines an element $Y \in H(X)$.

2. For any $Y_i \in H(X)$ and $\varphi_i(\lambda) = V[Y_i] \in L^2(\mu)$, $i = 1, 2$,

$$(Y_1, Y_2) = (\varphi_1, \varphi_2)_\mu. \tag{1.11}$$

3. Any linear problem in the time-domain $H(X)$ can be translated into one in the frequency-domain $L^2(\mu)$, and vice versa. This fact allows to study stationary processes using analytic methods.

1.3. **The infinite prediction problem.** Observe first that since by assumption $X(t)$ is a non-degenerate process, the time-domain $H(X)$ of $X(t)$ is non-trivial, that is, $H(X)$ contains elements different from zero.

Definition 1.3. The space $H_{t-n}^t(X)$ is called the *finite history*, or *past of length n and present* of the process $X(u)$ up to time t. The space $H_t(X) = H_{-\infty}^t(X)$ is called the *entire history*, or *infinite past and present* of the process $X(u)$ up to time t. The space

$$H_{-\infty}(X) = \cap_t H_{-\infty}^t(X) \tag{1.12}$$

is called the *remote past* of the process $X(u)$.

It is clear that

$$H_{-\infty}(X) \subset \cdots \subset H_{-\infty}^{t}(X) \subset H_{-\infty}^{t+\tau}(X) \subset \cdots \subset H(X), \quad \tau \in \mathbb{N}.$$

The Hilbert space setting provides a natural framework for stating and solving the problem of predicting future values of the process $X(u)$ from the observed past values. Assume that a realization of the process $X(u)$ for times $u \le t$ is observed and we want to predict the value $X(t+\tau)$ for some $\tau \ge 1$ from the observed values. Since we will never know what particular realization is being observed, it is reasonable to consider as a predictor $\widehat{X}(t,\tau)$ for $X(t+\tau)$ a function of the observed values, $g(\{X(u), u \le t\})$, which is good "on the average". So, as an optimality criterion for our predictor we take the L^2-distance, that is, the mean squared error, and consider only the linear predictors. With these restrictions, the infinite linear prediction problem can be stated as follows.

The infinite linear prediction problem. Given a "parameter" of the process $X(u)$ (e.g., the covariance function $r(t)$ or the spectral function $F(\lambda)$), the entire history $H_{-\infty}^{t}(X)$ of $X(u)$, and a natural number $\tau \in \mathbb{N}$, find a random variable $\widehat{X}(t,\tau)$ such that

a) $\widehat{X}(t,\tau)$ is *linear*, that is, $\widehat{X}(t,\tau) \in H_{-\infty}^{t}(X)$,

b) $\widehat{X}(t,\tau)$ is *mean-square optimal (best)* among all elements $Y \in H_{-\infty}^{t}(X)$, that is, $\widehat{X}(t,\tau)$ minimizes the mean-squared error $||X(t+\tau) - Y||^2_{L^2(P)}$:

$$||X(t+\tau) - \widehat{X}(t,\tau)||^2_{L^2(P)} = \min_{Y \in H_{-\infty}^{t}(X)} ||X(t+\tau) - Y||^2_{L^2(P)}. \tag{1.13}$$

The solution - the random variable $\widehat{X}(t,\tau)$ satisfying a) and b), is called the *best linear τ-step ahead predictor* for an element $X(t+\tau) \in H(X)$. The quantity

$$\sigma^2(\tau) = ||X(t+\tau) - \widehat{X}(t,\tau)||^2_{L^2(P)} = ||X(t+\tau)||^2_{L^2(P)} - ||\widehat{X}(t,\tau)||^2_{L^2(P)}, \tag{1.14}$$

which is independent of t, is called the *prediction error (variance)*.

The advantage of the Hilbert space setting now becomes apparent. Namely, by the *projection theorem* in Hilbert spaces (see [29], p. 312), such a predictor exists, is unique, and is given by

$$\widehat{X}(t,\tau) = P_t X(t+\tau), \tag{1.15}$$

where $P_t := P_{(-\infty,t]}$ is the orthogonal projection operator in $H(X)$ onto $H_{-\infty}^{t}(X)$.

Remark 1.1. The reason for restricting attention to linear predictors is that the best linear predictor $\widehat{X}(t,\tau)$, in this case, depends only on knowledge of the covariance function $r(t)$ or the spectral function $F(\lambda)$. The prediction problem becomes much more difficult when nonlinear predictors are allowed.

1.4. **Deterministic and nondeterministic processes.** From prediction point of view it is natural to distinguish the class of processes for which we have *error-free prediction*, that is, $\sigma^2(\tau) = 0$ for all $\tau \geq 1$, or equivalently, $\widehat{X}(t, \tau) = X(t + \tau)$ for all $t \in \mathbb{Z}$ and $\tau \geq 1$. In this case, the prediction is called *perfect*. It is clear that a process $X(t)$ possessing perfect prediction represents a singular case of *extremely strong dependence* between the random variables forming the process. Such a process $X(t)$ is called *deterministic* or *singular*. From the physical point of view, singular processes are exceptional. From application point of view it is of interest the class of processes for which we have $\sigma^2(\tau) > 0$ for all $\tau \geq 1$. In this case the prediction is called *imperfect*, and the process $X(t)$ is called *nondeterministic*.

Observe that the time-domain $H(X)$ of any non-degenerate stationary process $\{X(t),\ t \in \mathbb{Z}\}$ can be represented as the orthogonal sum $H(X) = H_1(X) \oplus H_{-\infty}(X)$, where $H_{-\infty}(X)$ is the remote past of $X(t)$ defined by (1.12), and $H_1(X)$ is the orthogonal complement of $H_{-\infty}(X)$. So, we can give the following geometric definition of the deterministic (singular), nondeterministic and purely nondeterministic (regular) processes.

Definition 1.4. A stationary process $\{X(t),\, t \in \mathbb{Z}\}$ is called

- *deterministic or singular* if $H_{-\infty}(X) = H(X)$, that is, $H^t_{-\infty}(X) = H^s_{-\infty}(X)$ for all $t, s \in \mathbb{Z}$,
- *nondeterministic* if $H_{-\infty}(X)$ is a proper subspace of $H(X)$, that is, $H_{-\infty}(X) \subset H(X)$,
- *purely nondeterministic (PND) or regular* if $H_{-\infty}(X) = \{0\}$, that is, the remote past $H_{-\infty}(X)$ of $X(t)$ is the trivial subspace, consisting of the singleton zero.

The next result, known as Wold's decomposition theorem (see [1], p. 65), provides a key step for solution of the infinite prediction problem in the time-domain setting, and essentially says that any stationary process can be represented in the form of a sum of two orthogonal stationary components, one of which is perfectly predictable (singular component), while for the other (regular component) an explicit formula for the predictor can be obtained.

Theorem 1.1 (Wold's decomposition)**.** *Every centered non-degenerate discrete-time stationary process $X(t)$ admits a decomposition*

$$X(t) = X_S(t) + X_R(t), \tag{1.16}$$

where

(a) *the processes $X_R(t)$ and $X_S(t)$ are stationary, centered, mutually uncorrelated (orthogonal), and completely subordinated to $X(t)$, that is, $H^t_{-\infty}(X_R) \subseteq H^t_{-\infty}(X)$ and $H^t_{-\infty}(X_S) \subseteq H^t_{-\infty}(X)$ for all $t \in \mathbb{Z}$.*

(b) *the process $X_S(t)$ is deterministic (singular),*

(c) *the process $X_R(t)$ is purely nondeterministic (regular) and has the infinite moving-average representation:*

$$X_R(t) = \sum_{k=0}^{\infty} b_k \varepsilon_0(t-k), \quad \sum_{k=0}^{\infty} |b_k|^2 < \infty, \tag{1.17}$$

where $\varepsilon_0(t)$ is an innovation of $X_R(t)$, that is, $\varepsilon_0(t)$ is a standard white noise process, such that $H^t_{-\infty}(X_R) = H^t_{-\infty}(\varepsilon_0)$ for all $t \in \mathbb{Z}$.

(d) *the representation (1.16) is unique.*

The next theorem contains spectral characterizations of deterministic, nondeterministic and purely nondeterministic processes (see [24], p. 35-36, [32], p. 58, 64)).

Theorem 1.2. *Let $X(t)$ be a discrete-time non-degenerate stationary process with spectral function $F(\lambda) = F_R(\lambda) + F_S(\lambda) = \int_{-\pi}^{\lambda} f(u)du + F_S(\lambda)$. The following assertions hold.*

(a) (Kolmogorov-Szegö alternative). *Either*

$$H^0_{-\infty}(F_R) = H(F_R) \Leftrightarrow \int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda = -\infty \Leftrightarrow \sigma^2(f) = 0 \Leftrightarrow X(t) \text{ is deterministic},$$

or else

$$H^0_{-\infty}(F_R) \neq H(F_R) \Leftrightarrow \int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda > -\infty \Leftrightarrow \sigma^2(f) > 0 \Leftrightarrow X(t) \text{ is nondeterministic}.$$

(b) *The process $X(t)$ is regular (PND) if and only if it is nondeterministic and $F_S(\lambda) \equiv 0$.*

Remark 1.2. The condition

$$\int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda > -\infty \tag{1.18}$$

is called *Szegö condition*. Observe that (1.18) is satisfied if and only if $\log f \in L^1(\Lambda)$, since $\log f(\lambda) \leq f(\lambda)$ and $f(\lambda) \in L^1(\Lambda)$. Also, the Szegö condition (1.18) is connected with the character of zeros of the spectral density $f(\lambda)$, and does not depend on the differential properties of $f(\lambda)$. For example, for any $\alpha \geq 0$ the function $f(\lambda) = \exp\{-|\lambda|^{-\alpha}\}$ is infinitely differentiable, for $\alpha < 1$ the Szegö condition is satisfied, and hence a stationary process $X(t)$ with this spectral density is nondeterministic, while for $\alpha \geq 1$ the Szegö condition is violated, and $X(t)$ is deterministic (see [30], p. 151, [29], p. 68).

Remark 1.3. A stationary process $X(t)$ is deterministic if either it has pure discrete spectrum, or pure singular spectrum, or the Szegö condition is violated: $\log f \notin L^1(\Lambda)$. Thus, for $X(t)$ to be nondeterministic, its spectral density $f(\lambda)$ cannot be zero too often (see [29], p. 68).

2. The Finite Prediction Problem

In practice we never will have the observed entire infinite past, instead will be available only the finite past.

Suppose we have observed the values $X(-n),\dots,X(-1)$ of a centered, real-valued stationary process $X(t)$ with covariance function $r(t)$ and spectral function $F(\lambda)$, the *one-step ahead finite prediction problem* in predicting a random variable $X(0)$ based on the observed values $X(-n),\dots,X(-1)$ is: find the orthogonal projection $\widehat{X}_n(0) = P_{[-n,-1]}X(0)$ of $X(0)$ onto the space $H_n(X) = H_{-n}^{-1}(X) = sp\{X(t),\ -n \le t \le -1\}$, that is, find constants $\hat{c}_k = \hat{c}_{k,n}$, $k = 1,2,\dots,n$, that minimize the one-step ahead prediction error variance $\sigma_n^2(F) = \sigma_n^2(1,F)$:

$$\begin{aligned}\sigma_n^2(F) &= \min_{\xi\in H_n(X)} \|X(0)-\xi\|^2_{L^2(P)} = \min_{\{c_k\}} \left\| X(0) - \sum_{k=1}^n c_k X(-k)\right\|^2_{L^2(P)} \\ &= \left\| X(0) - \sum_{k=1}^n \hat{c}_k X(-k)\right\|^2_{L^2(P)} = ||X(0)-\widehat{X}_n(0)||^2_{L^2(P)}. \qquad (2.1)\end{aligned}$$

If such constants $\hat{c}_k$ can be found, then the best linear 1-step ahead predictor $\widehat{X}_n(0)$ of a random variable $X(0)$ based on the observed finite past: $X(-n),\dots,X(-1)$ can be computed by

$$\widehat{X}_n(0) = \sum_{k=1}^n \hat{c}_k X(-k), \quad \hat{c}_k = \hat{c}_{k,n}, \qquad (2.2)$$

and the mean-squared prediction error $\sigma_n^2(F)$ can be computed by formula (2.1).

Using Kolmogorov's isometric isomorphism $V:\ X(t) \leftrightarrow e^{it\lambda}$ between the time- and frequency-domains $H(X)$ and $L^2(F)$, in view of (2.1), for $\sigma_n^2(F)$ we can write

$$\begin{aligned}\sigma_n^2(F) &= \min_{\{c_k\}} \left\| X(0) - \sum_{k=1}^n c_k X(-k)\right\|^2_{L^2(P)} = \min_{\{c_k\}} \int_{-\pi}^{\pi} \left|1 - \sum_{k=1}^n c_k e^{-ik\lambda}\right|^2 dF(\lambda) \\ &= \min_{\{c_k\}} \int_{-\pi}^{\pi} \left|e^{in\lambda} - \sum_{k=1}^n c_k e^{i(n-k)\lambda}\right|^2 dF(\lambda) = \min_{\{q_n\in\mathcal{Q}_n\}} \int_{-\pi}^{\pi} \left|q_n(e^{i\lambda})\right|^2 dF(\lambda), \qquad (2.3)\end{aligned}$$

where $\mathcal{Q}_n = \left\{q_n : q_n(z) = \sum_{k=0}^n c_k z^{n-k},\ c_0 = 1\right\}$ stands for the set of polynomials of degree n with coefficient of the leading term equal to 1.

Thus, the problem of finding $\sigma_n^2(F)$ becomes to the solution of the following minimum problem:

$$\int_{-\pi}^{\pi} |q_n(e^{i\lambda}|^2 dF(\lambda) = \min, \quad q_n(z) \in \mathcal{Q}_n. \tag{2.4}$$

The polynomial $p_n(z) = p_n(z, F)$ that solves the minimum problem (2.4) is called the *optimal polynomial* for $F(\lambda)$ in the class $\mathcal{Q}_n$. This minimum problem was solved by G. Szegö (see [18], Section 2.2) by showing that the optimal polynomial $p_n(z, F)$ exists, is unique and can be expressed in terms of orthogonal polynomials $\varphi_n(z)$, $n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with respect to $F(\lambda)$.

Recall that the system of orthogonal polynomials $\{\varphi_n(z) = \varphi_n(z; F),\ z = e^{i\lambda},\ n \in \mathbb{Z}_+\}$ is uniquely determined by the following conditions:

(i) $\varphi_n(z) = \kappa_n(F) z^n +$ lower order terms
is a polynomial of degree n, in which the coefficient $\kappa_n = \kappa_n(F)$ is real and positive;

(ii) for arbitrary nonnegative integers k and j

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_k(z)\overline{\varphi_j(z)} dF(\lambda) = \delta_{kj} = \begin{cases} 1, & \text{for } k = j \\ 0, & \text{for } k \neq j, \end{cases} \quad z = e^{i\lambda}.$$

Theorem 2.1 (Szegö theorem)**.** *The optimal polynomial for $F(\lambda)$ in the class $\mathcal{Q}_n$, that is, the polynomial $p_n(z) = p_n(z, F)$ that solves the minimum problem* (2.4) *is given by $p_n(z) = \kappa_n^{-1}(F)\varphi_n(z)$, and the minimum itself is equal to $\kappa_n^{-2}(F)$. Thus, we have*

$$\sigma_n^2(F) = \min_{\{q_n \in \mathcal{Q}_n\}} \int_{-\pi}^{\pi} \left|q_n(e^{i\lambda})\right|^2 dF(\lambda) = \tag{2.5}$$
$$= \int_{-\pi}^{\pi} \left|p_n(e^{i\lambda}, F)\right|^2 dF(\lambda) = \kappa_n^{-2}(F).$$

Remark 2.1. Denote $\mathcal{Q}_n^* = \left\{q_n : q_n(z) = \sum_{k=0}^n c_k z^{n-k},\ c_n = 1\right\}$. Then we have (see [18], Section 3.1):

$$\sigma_n^2(F) = \min_{\{q_n \in \mathcal{Q}_n^*\}} \int_{-\pi}^{\pi} \left|q_n(e^{i\lambda})\right|^2 dF(\lambda) = \int_{-\pi}^{\pi} \left|p_n^*(e^{i\lambda}, F)\right|^2 dF(\lambda), \tag{2.6}$$

where $p_n^*(z) = p_n(z, F)$ is the optimal polynomial for $F(\lambda)$ in the class $\mathcal{Q}_n^*$.

Remark 2.2. From the obvious embedding $\mathcal{Q}_n^* \subset \mathcal{Q}_{n+1}^*$, it follows that the sequence $\{\sigma_n^2(F),\ n \in \mathbb{N}\}$ is non-increasing in n: $\sigma_{n+1}^2(F) \le \sigma_n^2(F)$. Also, it follows from (2.5) that $\sigma_n^2(F)$ is a non-decreasing functional of $F(\lambda)$:

$$\sigma_n^2(F_1) \le \sigma_n^2(F_2) \quad \text{when} \quad F_1(\lambda) \le F_2(\lambda), \quad \lambda \in \Lambda. \tag{2.7}$$

Indeed, by the definition of optimal polynomials $p_n(z, F_1)$ and $p_n(z, F_2)$, corresponding to spectral functions F_1 and F_2, respectively, we have

$$\begin{aligned}\sigma_n^2(F_1) &= \int_{-\pi}^{\pi} \left|p_n(e^{i\lambda}, F_1)\right|^2 dF_1(\lambda) \le \int_{-\pi}^{\pi} \left|p_n(e^{i\lambda}, F_2)\right|^2 dF_1(\lambda) \\ &\le \int_{-\pi}^{\pi} \left|p_n(e^{i\lambda}, F_2)\right|^2 dF_2(\lambda) = \sigma_n^2(F_2).\end{aligned}$$

The *finite prediction problem* is to describe the asymptotic behavior of $\sigma_n^2(F)$ as the length of the observed past increases ($n \to \infty$). The problem was solved by G. Szegö in 1915 in the special case where $F(\lambda)$ is pure absolute continuous, that is, $F_S(\lambda) = 0$, and by A. Kolmogorov in 1941 in the general case (see, e.g., [18], p. 44 or [22], p. 49). The solution is given in the theorem that follows, known as Kolmogorov-Szegö theorem.

Remark 2.3. If $F(\lambda)$ is purely absolutely continuous, that is, $dF(\lambda) = f(\lambda)d\lambda$, then instead of $\sigma_n^2(F)$ and $\sigma^2(F)$ we will write $\sigma_n^2(f)$ and $\sigma^2(f)$, respectively.

Theorem 2.2 (Kolmogorov-Szegö theorem)**.** *For any non-trivial spectral function $F(\lambda)$ the following limiting relation hold:*

$$\lim_{n\to\infty} \sigma_n^2(F) = \sigma^2(F) = \sigma^2(f) = 2\pi G(f), \tag{2.8}$$

where $f(\lambda)$ is the spectral density, that is, the derivative of the absolutely continuous part of $F(\lambda)$, and $G(f)$ is the geometric mean of $f(\lambda)$, given by

$$G(f) = \begin{cases} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda\right\} & \text{if } \log f \in L^1(\Lambda) \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

Define the *relative prediction error* $\delta_n(F)$ to be

$$\delta_n(F) := \sigma_n^2(F) - \sigma^2(F). \tag{2.10}$$

Observe that $\delta_n(F) \ge 0$ and $\delta_n(F) \to 0$ as $n \to \infty$. Note that if the underlying process $X(t)$ is deterministic, then $\delta_n(F) = \sigma_n^2(F)$.

The problem of interest is to describe the rate of decrease of relative prediction error $\delta_n(F)$ to zero as $n \to \infty$, depending on the regularity nature (deterministic or nondeterministic) and the dependence (memory) structure of the model $X(t)$. This problem we discuss in Section 3 for nondeterministic processes and in Section 4 for deterministic processes.

3. Asymptotic behavior of the prediction error variance for nondeterministic processes

In this section we study the asymptotic behavior of the finite prediction error for nondeterministic processes, and review some important known results.

We assume that the model process $X(t)$ is regular, or equivalently, is purely nondeterministic (PND), that is, $X(t)$ has a non-trivial spectral function $F(\lambda) = \int_{-\pi}^{\lambda} f(u)du + F_S(\lambda)$ with $dF_s(\lambda) = 0$ and $\ln f(\lambda) \in L^1(\Lambda)$, and describe the rate of decrease of relative prediction error $\delta_n(F)$ to zero as $n \to \infty$, depending on the dependence (memory) structure of the model $X(t)$ and the smoothness properties of its spectral density $f(\lambda)$.

3.1. **Asymptotic behavior of $\delta_n(f)$ for short-memory processes.** Recall that a *short memory* processes is a second order stationary processes possessing a spectral density $f(\lambda)$ which is bounded away from zero and infinity, that is, there are constants m and M such that $0 < m \le f(\lambda) \le M < \infty$. A typical short memory model example is the stationary autoregressive moving average $(\mathrm{ARMA})(p, q)$ process $X(t)$ defined to be a stationary solution of the difference equation: $\psi_p(B)X(t) = \theta_q(B)\varepsilon(t)$, $t \in \mathbb{Z}$, where ψ_p and θ_q are polynomials of degrees p and q, respectively, B is the backward shift operator defined by $BX(t) = X(t-1)$, and $\{\varepsilon(t), t \in \mathbb{Z}\}$ is a discrete-time white noise, that is, a sequence of zero-mean, uncorrelated random variables.

We first give a result that contains a necessary and sufficient condition for exponential rate of decrease to zero for $\delta_n(f) = \sigma_n^2(f) - \sigma^2(f)$. Notice that the first result of this type goes back to the paper by Grenander and Rosenblatt [17]. The next theorem was proved by Ibragimov [23] (see also Golinskii and Ibragimov [15]).

Theorem 3.1. *A necessary and sufficient condition for*

$$\delta_n(f) = O(q^n), \quad q = e^{-b}, \quad b > 0, \quad n \to \infty \tag{3.1}$$

is that $f(\lambda)$ is a spectral density of a short-memory process, and $1/f(\lambda) \in A_b$, where A_b is the class of 2π–periodic continuous functions $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, admitting an analytic continuation into the strip $z = \lambda + i\mu$, $-\infty < \lambda < \infty$, $|\mu| \le b$.

Observe that (3.1) will be true for all $b > 0$ if and only if the analytic continuation of $f(\lambda)$ is an entire function of $z = \lambda + i\mu$.

Thus, to have exponential rate of decrease to zero for $\delta_n(f)$ the underlying model should be short-memory process with sufficiently smooth spectral density $f(\lambda)$.

Now we give a result that contains a necessary and sufficient condition for hyperbolic rate of decrease to zero for $\delta_n(f)$:

$$\delta_n(f) = O(n^{-\gamma}), \quad \gamma > 0, \quad n \to \infty. \tag{3.2}$$

Bounds of type (3.2) with $\gamma > 1$ for different classes of spectral densities were obtained by Baxter [2], Devinatz [9], Geronimus [12], Grenander and Rosenblatt

[17], Grenander and Szegö [18], and others (see [13] and references therein). The most general result in this direction has been obtained by Ibragimov [23]. To state Ibragimov's theorem, we first introduce the Hölder class of functions.

For a function $\varphi(\lambda) \in L^p(\Lambda)$, we define its L^p-modulus of continuity by

$$\omega_p(\varphi;\delta) = \sup_{0<|t|\le\delta} ||\varphi(\cdot+t) - \varphi(\cdot)||_p, \quad \delta > 0. \tag{3.3}$$

Given numbers $0 < \alpha < 1$, $r \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, and $p \ge 1$, we put $\gamma := r + \alpha$. The Hölder class of functions, denoted by $H_p(\gamma)$, is defined to be the set of those functions $\varphi(\lambda) \in L^p(\Lambda)$ that have r-th derivative $\varphi^{(r)}(\lambda)$, such that $\varphi^{(r)}(\lambda) \in L^p(\Lambda)$ and $\omega_p(\varphi^{(r)};\delta) = O(\delta^\alpha)$ as $\delta \to 0$.

Theorem 3.2. *A necessary and sufficient condition for*

$$\delta_n(f) = O(n^{-\gamma}), \ \gamma = 2(r+\alpha) > 1; \ 0 < \alpha < 1, \ r \in \mathbb{Z}_+, \text{ as } n \to \infty \tag{3.4}$$

is that $f(\lambda)$ is a spectral density of a short-memory process belonging to $H_2(\gamma)$.

Remark 3.1. It follows from Theorem 3.2 that if $\delta_n(f) = O(n^{-\gamma})$ with $\gamma > 1$, then the underlying model $X(t)$ is necessarily a short-memory process. Moreover, as it was pointed out by Grenander and Rosenblatt [17] (see, also, Devinatz [9], p. 118), if the model is not a short-memory process, that is, the spectral density $f(\lambda)$ has zeros or is unbounded, then, in general, we cannot expect $\delta_n(f)$ to go to zero faster than $1/n$ as $n \to \infty$. This question we discuss in the next subsection.

3.2. **Asymptotic behavior of $\delta_n(f)$ for long memory and antipersistent processes.** Recall that a second order stationary process $X(t)$ is said to be *anti-persistent* if the spectral density $f(\lambda)$ vanishes at frequency zero: $f(0) = 0$. And, we say that $X(t)$ displays *long memory* or *long-range dependence* if the spectral density $f(\lambda)$ has a pole at frequency zero, that is, it is unbounded at the origin.

A well-known example of processes that displays long memory or is anti-persistent is an autoregressive fractionally integrated moving average ARFIMA(p, d, q) process $X(t)$ defined to be a stationary solution of the difference equation:

$$\psi_p(B)(1-B)^d X(t) = \theta_q(B)\varepsilon(t), \quad d < 1/2,$$

where B is the backward shift operator, $\varepsilon(t)$ is a discrete-time white noise, and ψ_p and θ_q are polynomials of degrees p and q, respectively. The spectral density $f(\lambda)$ of $X(t)$ is given by

$$f(\lambda) = |1 - e^{-i\lambda}|^{-2d} h(\lambda), \quad d < 1/2, \tag{3.5}$$

where $h(\lambda)$ is the spectral density of an ARMA(p,q) process. Note that the condition $d<1/2$ ensures that $\int_{-\pi}^{\pi} f(\lambda)d\lambda<\infty$, implying that the process $X(t)$ is well defined because $E|X(t)|^2=\int_{-\pi}^{\pi} f(\lambda)d\lambda$.

Observe that for $0<d<1/2$ the model $X(t)$ specified by (3.5) displays long-memory, for $d<0$ it is anti-persistent, and for $d=0$ it displays short-memory. For $d\geq 1/2$ the function $f(\lambda)$ in (3.5) is not integrable, and thus it cannot represent a spectral density of a stationary process (see Brockwell and Davis [7], Section 13.2). The following theorem was proved by A. Inoue (see [26], Theorem 4.3).

Theorem 3.3. *Let $f(\lambda)$ have the form (3.5) with $0<d<1/2$, where $h(\lambda)$ is the spectral density of an ARMA(p,q) process. Then*

$$\delta_n(f)\sim\frac{d^2}{n}\quad\text{as}\quad n\to\infty. \tag{3.6}$$

Another well-known example of processes that displays long memory or is anti-persistent is the Jacobian model. We say that a stationary process $X(t)$ is a Jacobian process, and the corresponding model is a Jacobian model, if its spectral density $f(\lambda)$ has the following form:

$$f(\lambda)=h(\lambda)\prod_{k=1}^{m}|e^{i\lambda}-e^{i\lambda_k}|^{-2d_k}, \tag{3.7}$$

where $h(\lambda)$ is the spectral density of a short-memory process, the points $\lambda_k\in[-\pi,\pi]$ are distinct, and $d_k<1/2$, $k=1,\dots,m$.

The asymptotic behavior of $\delta_n(f)$ as $n\to\infty$ for Jacobian model (3.7) has been considered in a number of papers (see Golinskii [14], Grenander and Rosenblatt [17], Ibragimov [23], Ibragimov and Solev [25].)

The following theorem was proved in Ibragimov and Solev [25].

Theorem 3.4. *Let $f(\lambda)$ have the form (3.7), where $h(\lambda)$ is the spectral density of a short-memory process, the points $\lambda_k\in[-\pi,\pi]$ are distinct, and $d_k<1/2$ ($d_k\neq 0$), $k=1,\dots,m$. If $f(\lambda)$ satisfies the Lipschitz condition with exponent $\alpha\geq 1/2$, then*

$$\delta_n(f)\sim\frac{1}{n}\quad\text{as}\quad n\to\infty. \tag{3.8}$$

More results for this case can be found in Ginovyan [13] and in the references therein.

4. Asymptotic behavior of the predictor error for deterministic processes

4.1. **Background.** In this section we discuss the asymptotic behavior of the predictor error for deterministic processes. We assume that the process $X(t)$ possesses a spectral density $f(\lambda)$ and the Szegö condition (1.18) is violated. As it was mentioned in Introduction, this problem was first studied by M. Rosenblatt [31], where using the technique of orthogonal polynomials and Szegö's results, M. Rosenblatt has investigated the asymptotic behavior of the prediction error $\delta_n(f) = \sigma_n^2(f)$ in the following two cases:

(a) the spectral density $f(\lambda)$ is continuous and vanishes on an interval,
(b) the spectral density $f(\lambda)$ has a high order contact with zero, so that the Szegö condition is violated.

For the case (a), in [31] M. Rosenblatt proved the following result.

Theorem 4.1. *Let the spectral density $f(\lambda)$ of a discrete-time stationary process $X(t)$ be positive and continuous on the interval $(\pi/2-\alpha, \pi/2+\alpha)$, $0<\alpha<\pi$, and zero elsewhere, then the prediction error $\sigma_n^2(f)$ approaches zero exponentially as $n\to\infty$. More precisely, the following asymptotic relation holds:*

$$\delta_n(f) := \sigma_n^2(f) \cong \Big(\sin\frac{\alpha}{2}\Big)^{2n+1} \quad \text{as} \quad n\to\infty, \tag{4.1}$$

implying that

$$\lim_{n\to\infty} (\sigma_n(f))^{1/n} = \sin\frac{\alpha}{2}. \tag{4.2}$$

Later this result has been generalized by Babayan [3], [4] to the case of several arcs, without having to stipulate continuity of the spectral density $f(\lambda)$ (see also Davisson [8]). To state the corresponding result we first introduce the concept of a transfinite diameter of a set (see, e.g., Goluzin [16], Chapter 7). Let E be a bounded closed set in the complex plane. Denote by $T_n(z,E)$ the Chebyshev polynomial which deviates least from zero on the set E in the uniform metric. We set $C_n(E) = \max_{z\in E}|T_n(z,E)|$. Then $\lim_{n\to\infty}(C_n(E))^{1/n} =: \tau(E)$ exists and is called the *transfinite diameter* (or *Chebyshev constant*, or *capacity*) of the set E.

Remark 4.1. Notice that the transfinite diameter of the unit circle $\mathbb{T}$ is equal to 1 (see Goluzin [16], Section 7.1), and the transfinite diameter of an arc of $\mathbb{T}$ of length 2α $(0<\alpha<\pi)$ is equal to $\sin(\alpha/2)$ (see Rosenblatt [31]). Thus, the right hand side of (4.2) is the transfinite diameter of the closure of the spectrum $E_f = \{e^{i\lambda} : \lambda \in [\pi/2-\alpha, \pi/2+\alpha]\}$ of the process $X(t)$.

Using some results from geometric function theory, in [4] was proved the following theorem, extending Theorem 4.1.

Theorem 4.2. *Let the spectrum $E_f = \{e^{i\lambda} : f(\lambda) > 0\}$ of the process $X(t)$ consist of a finite number of arcs of the unit circle. Then the following asymptotic relation* *holds:*

$$\lim_{n\to\infty}(\sigma_n(f))^{1/n} = \tau(\overline{E}_f), \tag{4.3}$$

where $\overline{E}_f$ is the closure of E_f.

Remark 4.2. It follows from Theorem 4.2 and Remark 4.1 that if the spectral density $f(\lambda)$ vanish on an interval, then the prediction error $\sigma_n(f)$ decreases to zero exponentially, that is, $\sigma_n(f) = O(e^{-bn})$, $b > 0$ as $n \to \infty$. Conversely, a necessary condition for $\sigma_n(f)$ to tend to zero exponentially is that $f(\lambda)$ should vanish on a set of positive Lebesgue measure.

Concerning the case (b), in [31] M. Rosenblatt proved that if the spectral density $f(\lambda)$ of a stationary process $X(t)$ is positive away from zero, and has a very high order contact with zero at $\lambda = 0$, so that the Szegö condition (1.18) is violated, then the prediction error $\delta_n(f) = \sigma_n^2(f)$ decreases to zero hyperbolically as $n \to \infty$. More precisely, in [31] was considered a deterministic process $X(t)$ with spectral density $f_a(\lambda)$ given by formula:

$$f_a(\lambda) = \frac{e^{(2\lambda-\pi)\varphi(\lambda)}}{\cos\lambda(\pi\varphi(\lambda))}, \quad f_a(-\lambda) = f_a(\lambda), \quad 0 \le \lambda \le \pi, \tag{4.4}$$

where $\varphi(\lambda) = \frac{a}{2}\cot\lambda$ and a is a fixed positive parameter.

It is easy to show that

$$f_a(\lambda) \sim \exp\left\{-\frac{a\pi}{2|\lambda|}\right\}|\sin(\lambda)| \quad \text{as} \quad \lambda \to 0, \tag{4.5}$$

so that $f_a(\lambda)$ has a very high order contact with zero only at $\lambda = 0$.

In [31], using the formula (2.5) and the technique of orthogonal polynomials on the unit circle, M. Rosenblatt proved the following result.

Theorem 4.3. *For a process $X(t)$ with spectral density $f_a(\lambda)$ given by (4.4) the following asymptotic formula for prediction error $\delta_n(f) = \sigma_n^2(f)$ holds:*

$$\delta_n(f_a) = \sigma_n^2(f_a) \cong \frac{\Gamma^2\left(\frac{a+1}{2}\right)}{\pi 2^{2-a}}\ n^{-a} \sim n^{-a} \quad \text{as} \quad n \to \infty. \tag{4.6}$$

In the next subsection we extend Theorem 4.3 to more broad class of spectral densities.

4.2. **The main results.** In this subsection, we analyze the asymptotic behavior of the prediction error in the case where the spectral density $f(\lambda)$ of the model has a high order contact with zero, so that the Szegö condition (1.18) is violated.

Based on Rosenblatt's result for this case - Theorem 4.3, we can expect that for any deterministic process with spectral density possessing a zero of type (4.5), the rate of prediction error $\sigma_n^2(f)$ should be the same as in (4.6). However, the method applied in [31] does not allow to prove this assertion. Here, using a different approach, we extend Rosenblatt's theorem to more broad class of spectral densities.

To this end, we first examine the asymptotic behavior of the ratio $\sigma_n(fg)/\sigma_n(f)$ as $n \to \infty$, where $g(\lambda)$ is some nonnegative function, such that the product $f(\lambda)g(\lambda)$ is a spectral density, that is, $fg \in L^1(\Lambda)$.

To make the approach clear, we first assume that $f(\lambda)$ is a spectral density of a nondeterministic process, in which case the geometric mean $G(f)$ is positive (see (2.8) and (2.9)). Then, in this case, we can write

$$\lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = \frac{\sigma_\infty^2(fg)}{\sigma_\infty^2(f)} = \frac{G(fg)}{G(f)} = G(g). \tag{4.7}$$

It turns out that under some additional assumptions imposed on functions f and g, the asymptotic relation (4.7) remains valid also in the case of deterministic process, that is, when $\sigma_\infty^2(f) = 0$, or equivalently, $G(f) = 0$.

To state the corresponding results we need some definitions.

Definition 4.1. *A sequence of numbers $\{a_n,\, n \in \mathbb{N}\}$ is said to be slowly decreasing if*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1. \tag{4.8}$$

It is easy to check that the following simple assertions hold:

1. If $\{a_n,\, n \in \mathbb{N}\}$ is a slowly decreasing sequence, then for any $\nu \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{a_{n+\nu}}{a_n} = 1. \tag{4.9}$$

2. If $\{a_n,\, n \in \mathbb{N}\}$ is a sequence such that $a_n \to a \neq 0$ as $n \to \infty$, then $\{a_n\}$ is a slowly decreasing sequence.

3. If $\{a_n,\, n \in \mathbb{N}\}$ and $\{b_n,\, n \in \mathbb{N},\}$ are non-zero slowly decreasing sequences, then $ca_n, c \neq 0$, $1/a_n$, $a_n^k, k \in \mathbb{N}$, a_nb_n and a_n/b_n also are slowly decreasing sequences.

4. If $\{a_n,\, n \in \mathbb{N}\}$ is a non-zero slowly decreasing sequence, and $\{b_n,\, n \in \mathbb{N}\}$ is a sequence such that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = c \neq 0, \tag{4.10}$$

then $\{b_n,\, n \in \mathbb{N}\}$ is also a slowly decreasing sequence.

5. If $\{a_n,\, n \in \mathbb{N}\}$ is a slowly decreasing sequence of nonnegative numbers, then

$$\lim_{n\to\infty} (a_n)^{1/n} = 1. \tag{4.11}$$

Remark 4.3. It follows from assertion 2 that the notion of slowly decreasing sequence is more significant in the case where $a_n \to 0$ as $n \to \infty$. Also, it follows from assertion 5 that if $\{a_n,\, n \in \mathbb{N}\}$ is a slowly decreasing sequence of nonnegative numbers such that $a_n \to 0$ as $n \to \infty$, then it converges to zero slowly than the geometric progression $\{q^n, n \in \mathbb{N}\}$ for any q, $0 < q < 1$, that is, $q^n = o(a_n)$ as $n \to \infty$.

In what follows we consider the class of processes for which the sequence of prediction errors $\{\sigma_n(f)\}$ is slowly decreasing. Moreover, in view of Remarks 4.2 and 4.3, it is reasonable to consider deterministic processes except those for which the spectral densities vanish on an interval.

Definition 4.2. *We define the class A to be the set of all nonnegative, Riemann integrable functions $h(\lambda)$, $\lambda \in \Lambda$. Also, define $A_+ = \{h \in A : h(\lambda) \geqslant m > 0\}$, $A^- = \{h \in A : h(\lambda) \leqslant M < \infty\}$, and $A_+^- = A_+ \cap A^-$.*

Now we are in position to state the main results of this paper.

The following theorem describes the asymptotic behavior of the ratio $\sigma_n(fg)/\sigma_n(f)$ as $n \to \infty$ for the class of above described processes.

Theorem 4.4. *Let the spectral density $f(\lambda)$ be such that the sequence $\{\sigma_n(f)\}$ is slowly decreasing, and let $g(\lambda) = h(\lambda) \cdot \frac{t_1(\lambda)}{t_2(\lambda)}$, where $h(\lambda) \in A_+^-$ and $t_1(\lambda)$, $t_2(\lambda)$ are nonnegative trigonometric polynomials. If $f(\lambda)g(\lambda) \in A$, then*

$$\lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(g), \tag{4.12}$$

where $G(g)$ is the geometric mean of $g(\lambda)$.

The next theorem extends Rosenblatt's Theorem 4.3.

Theorem 4.5. *Let $f(\lambda) = f_a(\lambda)g(\lambda)$, where $f_a(\lambda)$ is defined by (4.4) and $g(\lambda)$ satisfies the assumptions of Theorem 4.4. Then*

$$\delta_n(f) = \sigma_n^2(f) \cong \frac{\Gamma^2\left(\frac{a+1}{2}\right) G(g)}{\pi 2^{2-a}}\, n^{-a} \sim n^{-a} \quad \text{as} \quad n \to \infty, \tag{4.13}$$

where $G(g)$ is the geometric mean of $g(\lambda)$.

4.3. **Auxiliary lemmas.** To prove the theorems, we first establish a number of lemmas.

Lemma 4.1. *Assume that the sequence $\sigma_n(f)$ is slowly decreasing, that is,*

$$\lim_{n\to\infty} \frac{\sigma_{n+1}(f)}{\sigma_n(f)} = 1. \tag{4.14}$$

Then for any nonnegative trigonometric polynomial $t(\lambda)$ we have

$$\liminf_{n\to\infty} \frac{\sigma_n^2(ft)}{\sigma_n^2(f)} \geqslant G(t), \tag{4.15}$$

where $G(t)$ is the geometric mean of $t(\lambda)$.

Proof. Let the polynomial $t(\lambda)$ be of degree ν. Then by Fejér-Riesz theorem (see [18], Section 1.12), there exists an algebraic polynomial $s_\nu(z)$ of degree ν in $z \in \mathbb{C}$, such that

$$t(\lambda) = |s_\nu(e^{i\lambda})|^2, \quad s_\nu(z) \neq 0, \quad |z| < 1. \tag{4.16}$$

Observing that $\ln |s_\nu(z)|^2$ is a harmonic function, we have

$$\ln |s_\nu(0)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |s_\nu(e^{i\lambda})|^2 d\lambda,$$

implying that

$$|s_\nu(0)|^2 = G(t) > 0. \tag{4.17}$$

Let $p_n^*(z, ft)$ be the optimal polynomial of degree n for $f(\lambda)t(\lambda)$ in the class $\mathcal{Q}_n^*$ (see formula (2.6)). We set

$$r_{n+\nu}(z) = \frac{p_n^*(z, ft)s_\nu(z)}{s_\nu(0)}, \tag{4.18}$$

and observe that $r_{n+\nu}(z) \in \mathcal{Q}_{n+\nu}^*$, and

$$\int_{-\pi}^{\pi} |r_{n+\nu}(e^{i\lambda})|^2 f(\lambda) d\lambda \geqslant \int_{-\pi}^{\pi} |p_{n+\nu}^*(e^{i\lambda}, f)|^2 f(\lambda) d\lambda. \tag{4.19}$$

Therefore, in view of (4.16), (4.18) and (4.19),we can write

$$\sigma_n^2(ft) = \int_{-\pi}^{\pi} |p_n^*(e^{i\lambda}, ft)|^2 f(\lambda) t(\lambda) d\lambda = \int_{-\pi}^{\pi} |p_n^*(e^{i\lambda}, ft) s_\nu(e^{i\lambda})|^2 f(\lambda) d\lambda.$$

$$= |s_\nu(0)|^2 \int_{-\pi}^{\pi} |r_{n+\nu}(e^{i\lambda})|^2 f(\lambda) d\lambda \geqslant |s_\nu(0)|^2 \int_{-\pi}^{\pi} |p_{n+\nu}^*(e^{i\lambda}, f)|^2 f(\lambda) d\lambda = |s_\nu(0)|^2 \sigma_{n+\nu}^2(f),$$

which, in view of (4.17), implies that

$$\liminf_{n\to\infty} \frac{\sigma_n^2(ft)}{\sigma_{n+v}^2(f)} \geqslant |s_\nu(0)|^2 = G(t). \tag{4.20}$$

Now, taking into account (4.14) and (4.9), from (4.20) we obtain (4.15). $\square$

Lemma 4.2. *Let the sequence $\sigma_n(f)$ satisfy (4.14), and let $t(\lambda)$ be a nonnegative trigonometric polynomial such that the function $f(\lambda)/t(\lambda) \in A$. Then the following inequality holds:*

$$\limsup_{n\to\infty} \frac{\sigma_n^2(f/t)}{\sigma_n^2(f)} \leqslant G(1/t). \tag{4.21}$$

Proof. Let the polynomial $s_\nu(z)$ be as in (4.16), and let $p_n^*(z, f/t)$ be the optimal polynomial of degree n for $f(\lambda)/t(\lambda)$ in the class $\mathcal{Q}_n^*$ (see formula (2.6)). For $n > \nu$ we set

$$r_n(z) = \frac{p_{n-\nu}^*(z, f)s_\nu(z)}{s_\nu(0)},$$

and observe that $r_n(z) \in \mathcal{Q}_n^*$. Therefore, we have

$$\begin{aligned}\sigma_n^2(f/t) \quad &= \int_{-\pi}^{\pi} |p_n^*(e^{i\lambda}, f/t)|^2 f(\lambda)/t(\lambda) d\lambda \le \int_{-\pi}^{\pi} |r_n(e^{i\lambda})|^2 f(\lambda)/t(\lambda) d\lambda \\ &= \frac{1}{|s_\nu(0)|^2} \int_{-\pi}^{\pi} |p_{n-\nu}^*(e^{i\lambda}, f)|^2 f(\lambda) d\lambda = \frac{1}{|s_\nu(0)|^2} \sigma_{n-\nu}^2(f),\end{aligned}$$

which, in view of (4.17), implies that

$$\limsup_{n\to\infty} \frac{\sigma_n^2(f/t)}{\sigma_{n-\nu}^2(f)} \leqslant \frac{1}{|s_\nu(0)|^2} = G(1/t). \tag{4.22}$$

Finally, taking into account (4.14) and (4.9), from (4.22) we obtain (4.21). □

Lemma 4.3. *Let $h(\lambda)$ be a function from the class A_+^-. Then for any $\varepsilon > 0$ a trigonometric polynomial $t(\lambda)$ can be found to satisfy the following condition:*

$$|h - t\|_1 = \int_{-\pi}^{\pi} |h(\lambda) - t(\lambda)| d\lambda \leqslant \epsilon. \tag{4.23}$$

Moreover, if m and M are the constants from the Definition 4.2, then the polynomial $t(\lambda)$ can be chosen so that for all $\lambda \in [-\pi, \pi]$ one of the following inequalities is satisfied:

$$m - \varepsilon < t(\lambda) < h(\lambda), \tag{4.24}$$

$$h(\lambda) < t(\lambda) < M + \varepsilon. \tag{4.25}$$

Proof. We first prove the inequalities (4.23) with (4.24). Without loss of generality, we can assume that $h(-\pi) = h(\pi)$. Otherwise by changing one of these values we can make them equal as follows: $h(-\pi) = h(\pi) = \min\{h(-\pi), h(\pi)\}$.

Let $\{\lambda_i\}$ $(-\pi = \lambda_0 < \lambda_1 < \cdots < \lambda_k = \pi)$ be a partition of the segment $[-\pi, \pi]$, and let s be the Darboux lower sum corresponding to this partition:

$$s = \sum_{i=1}^{k} m_i \Delta\lambda_i, \ m_i = \inf_{\lambda\in\Delta_i} h(\lambda), \ \Delta_i = [\lambda_{i-1}, \lambda_i], \ \Delta\lambda_i = \lambda_i - \lambda_{i-1}, \ i = 1, \dots, k.$$

On the segment $[-\pi, \pi]$ we define a step-function $\varphi_k(\lambda)$ corresponding to given partition as follows:

$$\varphi_k(\lambda) = \begin{cases} m_i, & \text{if } \ \lambda \in (\lambda_{i-1}, \lambda_i), \ i = 1, \dots, k-1, \\ \min\{m_{i-1}, m_i\}, & \text{if } \lambda = \lambda_i, \\ m_1 (= m_k), & \text{if } \lambda = \lambda_0 \text{ or } \lambda = \lambda_k. \end{cases}$$

It is clear that such defined function $\varphi_k(\lambda)$ satisfies the following conditions:

$$\varphi_k(\lambda) \le h(\lambda), \ \lambda \in [-\pi, \pi] \quad \text{and} \quad \int_{-\pi}^{\pi} \varphi_k(\lambda) d\lambda = s. \tag{4.26}$$

Since the function $h(\lambda)$ is integrable, for an arbitrary given $\epsilon > 0$ a partition of the segment $[-\pi, \pi]$ can be found so that the corresponding Darboux lower sum satisfies the condition:

$$\int_{-\pi}^{\pi} h(\lambda)d\lambda - s = \int_{-\pi}^{\pi}[h(\lambda)-\varphi_k(\lambda)]d\lambda = \|h-\varphi_k\|_1 < \frac{\epsilon}{3}. \tag{4.27}$$

Now using the function $\varphi_k(\lambda)$ we construct a new continuous function. To this end, we connect all the adjacent segments of the graph of $\varphi_k(\lambda)$ (the steps) by line segments as follows: for each interior point of the partition $\lambda_i, i = 1, \dots, k-1$, the endpoint of the lower step with abscissa λ_i we connect with some interior point of the adjacent (from the left or from the right) upper step, the abscissa λ_i^* of which satisfies the condition:

$$|\lambda_i - \lambda_i^*| < \varepsilon/(3kM). \tag{4.28}$$

Then, we remove the part of the upper step lying under the constructed slanting segment. The obtained polygonal line is a graph of some continuous piecewise linear function, denoted by $h_k(\lambda)$, satisfying the condition:

$$h_k(\lambda) \le \varphi_k(\lambda) \le h(\lambda) \le M, \quad \lambda \in [-\pi, \pi]. \tag{4.29}$$

Taking into account that the functions $h_k(\lambda)$ and $\varphi_k(\lambda)$ coincide outside the segments $[\lambda_i, \lambda_i^*]$ (or $[\lambda_i^*, \lambda_i]$), in view of (4.29) and (4.28), we can write

$$\|\varphi_k - h_k\|_1 = \int_{-\pi}^{\pi}[\varphi_k(\lambda)-h_k(\lambda)]d\lambda = \sum_{i=1}^{k-1}\left|\int_{\lambda_i}^{\lambda_i^*}[\varphi_k(\lambda)-h_k(\lambda)]d\lambda\right| < \frac{\epsilon}{3}. \tag{4.30}$$

Next, according to Weierstrass theorem (see, e.g., [18], Section 1.9), for function $h_k(\lambda)$ a trigonometric polynomial $\widetilde{t}(\lambda)$ can be found so that uniformly for all $\lambda \in [-\pi, \pi]$,

$$-\frac{\epsilon}{12\pi} < h_k(\lambda) - \widetilde{t}(\lambda) < \frac{\epsilon}{12\pi}. \tag{4.31}$$

Setting $t(\lambda) = \widetilde{t}(\lambda) - \frac{\epsilon}{12\pi}$, from (4.31) we get

$$0 < h_k(\lambda) - t(\lambda) < \frac{\epsilon}{6\pi}. \tag{4.32}$$

Therefore

$$\|h_k - t\|_1 = \int_{-\pi}^{\pi}[h_k(\lambda)-t(\lambda)]d\lambda < \frac{\epsilon}{3}. \tag{4.33}$$

Combining the inequalities (4.27), (4.30) and (4.33), we obtain

$$\|h-t\|_1 \le \|h-\varphi_k\|_1 + \|\varphi_k - h_k\|_1 + \|h_k - t\|_1 \le \epsilon,$$

and the inequality (4.23) follows.

Now we proceed to prove the inequality (4.24). Observe first that the second inequality in (4.24) follows from (4.32) and (4.29). To prove the first inequality in (4.24), observe that by construction of function $h_k(\lambda)$, we have

$$h_k(\lambda) \ge \min\{m_1, \dots, m_k\} \ge m. \tag{4.34}$$

Next, in view of (4.32), we get

$$t(\lambda) \geq h_k(\lambda) - \frac{\epsilon}{6\pi} > h_k(\lambda) - \epsilon. \tag{4.35}$$

Combining (4.34) and (4.35), we obtain the first inequality in (4.24).

The proof of inequalities (4.23) with (4.25) is completely similar to that of (4.23) with (4.24). The only difference is that now instead of Darboux lower sum should be used the upper sum and in the construction of function $h_k(\lambda)$, the endpoints of the upper steps of the function $\varphi_k(\lambda)$ should be connected with the interior points of the adjacent lower steps. □

Lemma 4.4. *Let $h(\lambda) \in A_+^-$ and let the sequence $\sigma_n(f)$ satisfy (4.14). Then the following asymptotic relation holds:*

$$\lim_{n\to\infty} \frac{\sigma_n^2(fh)}{\sigma_n^2(f)} = G(h). \tag{4.36}$$

Proof. Observe first that together with $h(\lambda)$ the function $1/h(\lambda)$ also belongs to the class A_+^-:

$$m \leq h(\lambda) \leq M \quad \text{and} \quad 1/M \leq 1/h(\lambda) \leq 1/m. \tag{4.37}$$

By Lemma 4.3, for a given small enough $\epsilon > 0$, we can find two trigonometric polynomials $t_1(\lambda)$ and $t_2(\lambda)$ to satisfy the following conditions:

$$\|h - t_1\|_1 < \epsilon, \qquad \frac{m}{2} < t_1(\lambda) < h(\lambda), \tag{4.38}$$

$$\|1/h - t_2\|_1 < \epsilon, \quad \frac{1}{2M} < t_2(\lambda) < \frac{1}{h(\lambda)}. \tag{4.39}$$

Now in view of (2.7) and Lemmas 4.1, 4.2, to obtain

$$\liminf_{n\to\infty} \frac{\sigma_n^2(fh)}{\sigma_n^2(f)} \geq \liminf_{n\to\infty} \frac{\sigma_n^2(ft_1)}{\sigma_n^2(f)} \geq G(t_1), \tag{4.40}$$

and

$$\limsup_{n\to\infty} \frac{\sigma_n^2(fh)}{\sigma_n^2(f)} \leq \limsup_{n\to\infty} \frac{\sigma_n^2(f/t_2)}{\sigma_n^2(f)} \leq G(1/t_2). \tag{4.41}$$

Next, it follows from (4.37) – (4.39) that

$$\|h - 1/t_2\|_1 = \|h/t_2(t_2 - 1/h)\|_1 \leqslant 2M^2\epsilon,$$

$$\|t_1 - 1/t_2\|_1 \leqslant \|t_1 - h\|_1 + \|h - 1/t_2\|_1 \leqslant \epsilon(1 + 2M^2).$$

Hence, in view of (4.37) and (4.39), we can write

$$\left|\ln \frac{G(t_1)}{G(1/t_2)}\right| = |\ln[G(t_1)G(t_2)]| = \left|\int_{-\pi}^{\pi} \ln[t_1(\lambda)t_2(\lambda)]d\lambda\right| \leqslant \int_{-\pi}^{\pi} |t_1(\lambda)t_2(\lambda) - 1|d\lambda$$

$$= \|t_2(t_1 - 1/t_2)\|_1 \leqslant \frac{1}{m}\|t_1 - 1/t_2\|_1 \leqslant \frac{\epsilon}{m}(1 + 2M^2).$$

Thus, the quantities $G(t_1)$ and $G(1/t_2)$ can be made arbitrarily close. Hence, taking into account that $G(t_1) \leqslant G(h) \leqslant G(1/t_2)$, from (4.40) and (4.41) we obtain (4.36). □

Taking into account that $G(h) > 0$, from (4.10) and (4.36) we obtain the following result.

Corollary 4.1. *If the sequence $\sigma_n(f)$ is slowly decreasing and $h(\lambda) \in A_+^-$, then the sequence $\sigma_n(fh)$ is also slowly decreasing.*

Lemma 4.5. *Let the sequence $\sigma_n(f)$ be slowly decreasing, and let $h(\lambda) \in A^-$. Then*

$$\limsup_{n\to\infty} \frac{\sigma_n^2(fh)}{\sigma_n^2(f)} \leqslant G(h). \tag{4.42}$$

Proof. Observe that the function $h_\epsilon(\lambda) = h(\lambda) + \epsilon$ belongs to the class A_+^-. Then we have the asymptotic relation (see, [18], Section 3.1 (d)):

$$\lim_{\epsilon\to 0} G(h_\epsilon) = G(h). \tag{4.43}$$

Hence, using (2.7) and Lemma 4.4, we obtain

$$\limsup_{n\to\infty} \frac{\sigma_n^2(fh)}{\sigma_n^2(f)} \leq \lim_{n\to\infty} \frac{\sigma_n^2(fh_\epsilon)}{\sigma_n^2(f)} = G(h_\epsilon).$$

Passing to the limit as $\epsilon \to 0$, and taking into account (4.43), we obtain the desired inequality (4.42). □

As an immediate consequence of Lemma 4.5, we have the following result.

Corollary 4.2. *Let the sequence $\sigma_n(f)$ be slowly decreasing, and let $g(\lambda) \in A^-$ with $G(g) = 0$. Then $\sigma_n(fg) = o(\sigma_n(f))$ as $n \to \infty$.*

Lemma 4.6. *Let the sequence $\sigma_n(f)$ be slowly decreasing, and let $h(\lambda) \in A_+$. Then*

$$\liminf_{n\to\infty} \frac{\sigma_n^2(fh)}{\sigma_n^2(f)} \geqslant G(h). \tag{4.44}$$

Proof. Let $h_l(\lambda)$ denote the truncation of $h(\lambda)$ at the level $l \in \mathbb{N}$:

$$h_l(\lambda) = \begin{cases} h(\lambda), & h(\lambda) \leqslant l \\ l, & h(\lambda) > l. \end{cases}$$

Then by Monotone Convergence Theorem of Beppo Levi (see [6], Theorem 2.8.2), we have

$$\lim_{l\to\infty} G(h_l) = G(h). \tag{4.45}$$

Next, by Lemma 4.4 we get

$$\liminf_{n\to\infty} \frac{\sigma_n^2(fh)}{\sigma_n^2(f)} \geqslant \lim_{n\to\infty} \frac{\sigma_n^2(fh_l)}{\sigma_n^2(f)} = G(h_l).$$

Hence passing to the limit as $l \to \infty$, and taking into account (4.45) we obtain the desired inequality (4.44). $\square$

As an immediate consequence of Lemma 4.6, we have the following result.

Corollary 4.3. *Let the sequence $\sigma_n(f)$ be slowly decreasing, $g(\lambda) \in A_+$ with $G(g) = \infty$, and let $fg \in A$. Then $\sigma_n(f) = o(\sigma_n(fg))$ as $n \to \infty$.*

4.4. **Proof of main results.** In this subsection we prove the main results of this paper - Theorems 4.4 and 4.5.

Proof of Theorem 4.4. We have

$$\frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = \frac{\sigma_n^2(fht_1/t_2)}{\sigma_n^2(f)} = \frac{\sigma_n^2(fht_1/t_2)}{\sigma_n^2(fht_1)} \cdot \frac{\sigma_n^2(fht_1)}{\sigma_n^2(fh)} \cdot \frac{\sigma_n^2(fh)}{\sigma_n^2(f)}. \tag{4.46}$$

Next, by Lemma 4.4 we have

$$\lim_{n\to\infty} \frac{\sigma_n^2(fh)}{\sigma_n^2(f)} = G(h) > 0. \tag{4.47}$$

This, in view of Corollary 4.1, implies that the sequence $\sigma_n^2(fh)$ is also slowly decreasing. Therefore, by Lemma 4.1, we have

$$\liminf_{n\to\infty} \frac{\sigma_n^2(fht_1)}{\sigma_n^2(fh)} \geq G(t_1).$$

On the other hand, since $t_1(\lambda) \in A^-$, then according to Lemma 4.5, we get

$$\limsup_{n\to\infty} \frac{\sigma_n^2(fht_1)}{\sigma_n^2(fh)} \leq G(t_1).$$

Therefore

$$\lim_{n\to\infty} \frac{\sigma_n^2(fht_1)}{\sigma_n^2(fh)} = G(t_1) > 0 \tag{4.48}$$

This implies that the sequence $\sigma_n^2(fht_1)$ is also slowly decreasing. Hence we can apply Lemma 4.2, to obtain

$$\limsup_{n\to\infty} \frac{\sigma_n^2(fht_1/t_2)}{\sigma_n^2(fht_1)} \leqslant G(1/t_2).$$

Next, it is easy to see that $1/t_2 \in A_+$. Hence, according to Lemma 4.6, we obtain

$$\liminf_{n\to\infty} \frac{\sigma_n^2(fht_1/t_2)}{\sigma_n^2(fht_1)} \geqslant G(1/t_2).$$

Therefore

$$\lim_{n\to\infty} \frac{\sigma_n^2(fht_1/t_2)}{\sigma_n^2(fht_1)} = G(1/t_2). \tag{4.49}$$

Finally, combining the relations (4.46) - (4.49), we obtain

$$\lim_{n\to\infty} \frac{\sigma_n^2(fg)}{\sigma_n^2(f)} = G(1/t_2)G(t_1)G(h) = G(ht_1/t_2) = G(g).$$

Theorem 4.4 is proved. $\square$

As an immediate consequence of Theorem 4.4, we have the following result.

Corollary 4.4. *If the sequence $\sigma_n(f)$ is slowly decreasing and $g(\lambda)$ satisfies the conditions of Theorem 4.4, then the sequence $\sigma_n(fg)$ is also slowly decreasing.*

The proof of Theorem 4.5 immediately follows from Theorems 4.3 and 4.4.

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SOME ESTIMATES FOR THE SOLUTIONS OF THE FIRST ORDER NON-ALGEBRAIC CLASSES OF EQUATIONS

G. BARSEGIAN, F. MENG, W. YUAN

Guangzhou University, China[1]
Institue of mathematics of NAN Armenii
E-mails: *barsegiangrigor@yahoo.com; mfnfdbx@163.com; wjyuan1957@126.com*

Abstract. For some large classes of differential equations of the first order we give bounds for Ahlfors-Shimizu characteristics of meromorphic solutions in the complex plane of these equations. The considered equations largely generalize algebraic ones for which the obtained results imply the known Goldberg theorem. Characteristics of meromorphic solutions in a given domain weren't studied at all. We consider solutions in a given domain of some (large) equations and give bounds for Ahlfors-Shimizu characteristic for these solutions.

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1. Introduction

In this paper, we consider complex differential equations of the first order in two cases: for meromorphic solutions in the complex plane and in a given domain.

For algebraic equations, there is the classical Goldberg theorem related to meromorphic solutions in the complex plane. We study much larger equations; respectively our result implies as a particular case the mentioned Goldberg theorem.

Characteristics of meromorphic solutions in a given domain weren't studied. Recently G. Barsegian started similar studies; these studies were presented during his lectures in Guangzhou university in 2017. His aprroaches based on some new results related to arbitrary meromorphic functions in a given domain, see [4].

In this paper, we consider solutions in a given domain of some equations and give bounds for Ahlfors-Shimizu characteristic for the solutions.

2. Meromorphic solutions in the complex plane

We consider the following equation

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$$(*) \qquad \phi_0(z,w)\,(w')^m + \phi_1(z,w)\,(w')^{m-1} + \cdots + \phi_m(z,w) = 0,$$

where $\phi_i(z,w) = \sum_{\mu(i)=1}^{n(i)} \eta_{i,\mu(i)}(z)\chi_{i,\mu(i)}(w)$ for $i = 0,1,2,\ldots,m$ and $\mu(i) = 1,2,\ldots,n(i)$. Obviously we should exclude the case $\phi_0(z,w) \equiv 0$, since then the degree m reduces.

We put the following restrictions: all coefficient $\chi_{i,\mu(i)}(w)$ are meromorphic in $\mathbb{C}$, all coefficients $\eta_{i,\mu(i)}(z)$ with $i \neq 0$ are entire functions and all coefficients $\eta_{0,\mu(0)}(z)$ are polynomials. The equations $(*)$ with similar restrictions we will refer as $(F_p^{e,m}(\mathbb{C}))$.

Note that algebraic differential equations of the first order (see related studies in [8]) are particular cases of equations $(F_p^{e,m}(\mathbb{C}))$ when all mentioned above entire and meromorphic functions are polynomials.

For meromorphic function w in $\mathbb{C}$ we make use of classical Ahlfors-Shimizu characteristic

$$A(r) = A(r,w) = \frac{1}{\pi}\int\int_{D(r)} \frac{|w'|^2}{(1+|w|^2)^2}\,d\sigma,$$

where $D(r) = \{z : |z| < r\}$; for entire functions $\eta_{i,\mu(i)}$ we denote $M_i^{\mu(i)}(r) = \max_{z\in\partial D(r)} |\eta_{i,\mu(i)}(z)|$.

Theorem 2.1. *For any equation $(F_p^{e,m}(\mathbb{C}))$ with meromorphic solution $w(z)$ in the complex plane we have*

$$(2.1) \qquad A(r) \leq K_1 r^2 \max_{1\leq i\leq m}\left[\max_{\mu(i)} M_i^{\mu(i)}(r)\right]^{2/i}, \; \textit{for } r\to\infty, \; r\notin E,$$

where $K_1 < \infty$ is a constant independent of w and E is a set of finite logarithmic measure.

Corollary 2.1 ([7]). *Meromorphic solutions (in the complex plane) of algebraic differential equations are of finite order.*

Indeed, in this case all $M_i^{\mu(i)}(r)$ have polynomial growth so that Corollary 2.1 follows from (2.1). Thus Theorem 2.1 generalizes widely this old result in [7].

3. Meromorphic solutions in a given domain

Let D be a simply connected domain with smooth boundary ∂D of finite length.

Consider again equation $(*)$ by assuming that $w(z)$ is its meromorphic solutions in the closure $\bar{D} = D \cup \partial D$. In this case we assume that all $\eta_{i,\mu(i)}(z)$ are regular functions in $z \in \bar{D}$ and all $\chi_{i,\mu(i)}(w)$ are meromorphic functions in $w \in w(\bar{D})$. In addition we assume that $|\phi_0(z,w)| \geq c(D) = const > 0$ for $z \in \bar{D}$ and w with

$|w| < 10$. The equation with similar restrictions we will refer as $(F^{r,m}(D))$. In particular case when all $\chi_{i,\mu(i)}(w)$ are regular functions we will refer the equation as $(F^{r,r}(D))$.

For similar functions $w(z)$ as above, we consider *Ahlfors islands over the disk* $\Delta(\rho, a) = \{w : |w - a| < \rho\}$ which can be defined as those domains $\tilde{g}_k$ for $k = 1, 2, \ldots, n(D, \Delta(\rho, a), w)$, on the Riemann surface $\{w(z) : z \in \bar{D}\}$ which projected one to one and onto $\Delta(\rho, a)$ (see [1] or [9, Chapter 13]).

Defining $M_i^\mu(D) = \max_{z \in \partial D} |\eta_{i,\mu}(z)|$ and denoting by $S(D)$ the area of D, we formulate the following theorem.

Theorem 3.1. *Let $w(z)$ be a meromorphic in $\bar{D}$ solution of the equation $(F^{r,r}(D))$. Then for any set of disks $\Delta(\rho_\nu, a_\nu)$, $\nu = 1, 2, \ldots, q$, with non-intersecting closures we have*

$$\sum_{\nu=1}^{q} n(D, \Delta(\rho_\nu, a_\nu), w) \leq K_2 S(D), \tag{3.1}$$

where $K_2 < \infty$ is a constant independent of w; the constant depends only on the equation and D.

In the next result we make use of Ahlfors-Shimizu characteristic $A(D, w)$ (for arbitrary domain D) and another characteristics in Ahlfors theory of covering surfaces (see [1] and [9, Chapter 13])

$$L(D, w) = \int_{\partial D} \frac{|w'|}{(1 + |w|^2)} ds.$$

Theorem 3.2. *Let $w(z)$ be a meromorphic in $\bar{D}$ solution of the equation $(F^{r,m}(D))$ with $w(\bar{D})$ implying a disk $D(\varrho)$, where $\varrho = const > 0$. Then*

$$A(D, w) \leq K_3 S(D) + h_3 L(D, w), \tag{3.2}$$

where both constants K_3 and h_3 are independent on w; the constant depend only on the equation, D and ϱ.

4. Proofs

4.1. **Proof of Theorem 2.1.** We need some obvious comments.

In the case when the first coefficient $\phi_0(z, w)$ is nonconstant polynomial in z we can decompose $\phi_0(z, w)$ as $\Lambda_0(w) z^T + \Lambda_1(w) z^{T-1} + \cdots + \Lambda_T(w)$, $T \geq 1$, where $\Lambda_0(w)$ is a meromorphic function in w.

In the case when the first coefficient $\phi_0(z, w)$ does not depend on z we denote it by $\phi_0(w)$; obviously it is meromorphic in w.

All coefficients $\chi_{i,\mu(i)}(w)$ are meromorphic in the complex plane. This implies that for a fixed disk, say $D(10)$, any coefficients $\chi_{i,\mu(i)}(w)$, taken for any $i = 0, 1, 2, \ldots, m$, $\mu(i) = 1, 2, \ldots, n(i)$, have only a finite number of zeros in the disk $D(10)$. The same is true for the poles. The same is true for the zeros and poles of functions $\Lambda_0(w)$ and $\phi_0(w)$.

Now we take five non-passing through all these zeros and poles curves $\gamma_1, \ldots, \gamma_5$ in $D(10)$ with the distance between two different similar curves > 2. Then obviously all mentioned above functions do not vanish at any point $w = a \in \gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_5$ and there is a constant M such that the modules $|\cdots|$ of all mentioned above functions $\leq M$. Taking arbitrary five values $a_1, \ldots, a_5$ each belonging respectively to $\gamma_1, \ldots, \gamma_5$ we get the following statement.

Proposition 4.1. *There are five values $a_\nu \in D(10)$, $\nu = 1, \ldots, 5$; with non-intersecting closures of $\Delta(1, a_\nu)$, $\nu = 1, \ldots, 5$, such that*

1. All those functions $\chi_{i,\mu(i)}(w)$ which include variable w does not vanish at any point $w = a \in (a_1, \ldots, a_5)$ consequently

$$\Phi_i = \max_{1 \leq i \leq m} \sum_{\mu(i)=1}^{n(i)} |\chi_{i,\mu(i)}(a_\nu)| \leq n(i)M < \infty,$$

where Φ_i depends only on $a_1, \ldots, a_5$ and the involved coefficients;

2. Function $\phi_0(w)$ do not vanish at any point $w = a \in (a_1, \ldots, a_5)$, respectively we have

$$\Phi_0 = \min_{1 \leq \nu \leq q} |\phi_0(a_\nu)| > 0,$$

where Φ_0 is a constant depending only on $a_1, \ldots, a_5$ and ϕ_0;

3. Function $\Lambda_0(w)$ does not vanish at any point $w = a \in (a_1, \ldots, a_5)$, respectively we have

$$\Phi_\Lambda = \min_{1 \leq \nu \leq 5} |\Lambda_0(a_\nu)| > 0,$$

where Φ_Λ is a constant depending only on $a_1, \ldots, a_5$ and $\Lambda_0(w)$.

We need the following result.

Theorem A ([2, Theorem 1]). *For any meromorphic function w* in $\mathbb{C}$, *any set $a_1, a_2, \ldots, a_q \in \mathbb{C}$, $q > 4$, of distinct values and any monotonically decreasing on $[0, \infty)$ function $\psi(r)$ with $\psi(r) \to 0$ as $r \to \infty$, there is a set $E \subset [0, \infty)$ of finite logarithmic measure and for every $r \notin E$* there is *a subset $\{z_k^*(a_\nu)\} \subset D(r)$, $1 \leq \nu \leq q$, $1 \leq k \leq n^*(r, a_\nu)$, of the a_ν-points of w* for which

$$|w'(z_k^*(a_\nu))| \geq \psi(r) \frac{A^{1/2}(r)}{r}, \; 1 \leq \nu \leq q, \; 1 \leq k \leq n^*(r, a_\nu), \tag{4.1}$$

and

$$\sum_{\nu=1}^{q} n^*(r,a_\nu) \geq (q-4)A(r) - o[A(r)], \quad r \to \infty, \quad r \notin E. \tag{4.2}$$

A slight modification (see [2]) replaces $\psi(r)$ by a positive constant: *given positive* ε, $0 < \varepsilon < 1/2$, *there is a constant* $K = K\{a_1, a_2, \ldots, a_q, \varepsilon) > 0$ *such that* (4.1) *becomes*

$$|w'(z_k^*(a_\nu))| \geq K\frac{A^{1/2}(r)}{r}, \ 1 \leq \nu \leq q, \quad 1 \leq k \leq n^*(r,a_\nu) \tag{4.3}$$

for a set of a_ν*-points which satisfy*

$$\sum_{\nu=1}^{q} n^*(r,a_\nu) \geq (q-4-\varepsilon)A(r) - o[A(r)], \quad r \to \infty, \quad r \notin E. \tag{4.4}$$

Now we apply Theorem A and Proposition 4.1 to solutions w in the complex plane of equations $(F_p^{e,m}(\mathbb{C}))$.

Due to definitions and Theorem A we have the following.

Property 4.1. *Let* $a_1, \ldots, a_5$ *be the points mentioned in Proposition 4.1. Then in any* $D(r)$ *with* $r \notin E$, *there is a set* $Z(r)$ *consisting of* $\sum_{\nu=1}^5 n^*(r,a_\nu)$ *points* $z_k^*(a_\nu)$, $1 \leq \nu \leq 5$, $1 \leq k \leq n^*(r,a_\nu)$, *such that at each similar point we have inequality (4.1) for* $1 \leq \nu \leq 5$, $1 \leq k \leq n^*(r,a_\nu)$, *and we have also*

$$\sum_{\nu=1}^{5} n^*(r,a_\nu) \geq (1-\varepsilon)A(r) - o[A(r)], \quad r \to \infty, \quad r \notin E.$$

Due to the last inequality for $r \to \infty$, $r \notin E$, we have $\sum_{\nu=1}^5 n^*(r,a_\nu) \to \infty$ and since the points $z_k^*(a_\nu)$ cannot have any limit point in any finite disk we obtain the following.

Property 4.2. *For any constant* $H > 1$ *there is a constant* $r(H)$ *such that any disk* $D(r)$ *with* $r \geq r(H)$, $r \notin E$, *implies a point* $z_k^*(a_\nu)$ *occurring in Property 4.1 and satisfying also* $|z_k^*(a_\nu)| > H$.

Now we take any point $z_k^*(a_\nu)$ satisfying Property 4.2 and put it into equation $(*)$. We have

$$\phi_0(z_k^*(a_\nu), w(z_k^*(a_\nu)))\,(w'(z_k^*(a_\nu)))^m + \phi_1(z_k^*(a_\nu), w(z_k^*(a_\nu)))\,(w'(z_k^*(a_\nu)))^{m-1} +$$

$$\cdots + \phi_m(z_k^*(a_\nu), w(z_k^*(a_\nu))) = 0. \tag{4.5}$$

It is well known (see [10, Section III, Problem 21]) that all roots of an algebraic equation

$$z^m + b_1 z^{m-1} + \cdots + b_m = 0$$

are contained in the disk $|z| \le \max_{1\le i\le m}(m|b_i|)^{1/i}$.

Applying this to (4.5) we get

$$|w'(z_k^*(a_\nu))| \le \max_{1\le i\le m}\left[m\frac{\phi_i(z_k^*(a_\nu), w(z_k^*(a_\nu)))}{\phi_0(z_k^*(a_\nu), w(z_k^*(a_\nu)))}\right]^{1/i}. \tag{4.6}$$

Notice that item 1 in Proposition 4.1 is valid for $w = a = w(z_k^*(a_\nu))$ (since $w(z_k^*(a_\nu)) \in (a_1, \dots, a_5)$), so that we have

$$|\phi_i(z_k^*(a_\nu), w(z_k^*(a_\nu)))| \le n(i)\Phi_i \max_{\mu(i)} M_i^{\mu(i)}(r), i = 1, 2, \dots, m. \tag{4.7}$$

Now we need to consider below bounds for ϕ_0 which in our case is polynomial in z and meromorphic in w.

We can have only the following cases for $\phi_0(z, w)$:

(case 1) there are non-constant polynomial coefficients $\eta_{0,\mu(0)}(z)$;

(case 2) all $\eta_{0,\mu(0)}(z)$ are constants however not all $\chi_{0,\mu(0)}(w)$ are constants;

(case 3) all $\eta_{0,\mu(0)}(z)$ are constants and, in addition, all $\chi_{0,\mu(0)}(w)$ are constants.

In the first case we can decompose $\phi_0(z, w)$ (as $\Lambda_0(w)z^T + \Lambda_1(w)z^{T-1} + \dots + \Lambda_T(w)$, $T \ge 1$) and note that at the pair (z, w), where $w \in (a_1, \dots, a_5)$ and $|z|$ is enough large, the term $\Lambda_0(a_\nu)\left(z_k^*(a_\nu)\right)^T$ become dominant in $\phi_0(z_k^*(a_\nu), a_\nu)$, so that we have $|\phi_0(z_k^*(a_\nu), a_\nu)| \ge (1/2)|\Lambda_0(a_\nu)||z_k^*(a_\nu)|^T$ for $|z_k^*(a_\nu)| > r_0$; here obviously r_0 depends on ϕ_0 and values $a_1, \dots, a_5$. Consequently taking $r(H)$ (in Property 4.2) equals to r_0 and taking into account item 3 in Proposition 4.1 (i.e. $\Phi_\Lambda = \min_{1\le\nu\le q}|\Lambda_0((a_\nu)| > 0$) and also Property 4.2 we obtain the following assertion: for any disk $D(r)$ with $r \ge r(H)$, $r \notin E$, there is a value $a_\nu \in (a_1, \dots, a_5)$ and corresponding point $z_k^*(a_\nu)$ with $|z_k^*(a_\nu)| > H$ such that

$$|\phi_0(z_k^*(a_\nu), w(z_k^*(a_\nu))| \ge \frac{1}{2}\Phi_\Lambda|(z_k^*(a_\nu)|^T > \frac{1}{2}\Phi_\Lambda H^T.$$

Making use of (4.6) and (4.7) applied at the same point (where $|z_k^*(a_\nu)| > H$) we have

$$|w'(z_k^*(a_\nu))| \le \max_{1\le i\le m}\left[\frac{2mn(i)\Phi_i}{\Phi_\Lambda H^T}\max_{\mu(i)} M_i^{\mu(i)}(r)\right]^{1/i}.$$

Then applying (4.3) we get

$$A(r) \le \frac{1}{K^2}r^2\max_{1\le i\le m}\left[\max_{\mu(i)} M_i^{\mu(i)}(r)\frac{2mn(i)\Phi_i}{\Phi_\Lambda H^T}\right]^{2/i}, \text{ for } r \to \infty,\ r \notin E,$$

so that obtain Theorem 2.1 with

$$K_1 \le \frac{1}{K^2}\max_{1\le i\le m}\left[\frac{2mn(i)\Phi_i}{\Phi_\Lambda H^T}\right]^{2/i}.$$

In the second case, $\phi_0(z, w)$ become function in w merely, namely become function $\phi_0(w)$ in Proposition 4.1. Due to item 2 in Proposition 4.1, function $\phi_0(w)$ do not vanish at any point $w = a \in (a_1, \dots, a_5)$, respectively, we have $\Phi_0 = \min\limits_{1 \le \nu \le q} |\phi_0(a_\nu)| > 0$. Repeating the above arguments we get Theorem 2.1 with

$$K_1 \le \frac{1}{K^2} \max_{1 \le i \le m} \left[\frac{2mn(i)\Phi_i}{\Phi_0} \right]^{2/i}.$$

In the third case, $\phi_0(z, w)$ is simply a constant: $\phi_0(z, w) = c_0 = const$ which should be non-zero, otherwise the degree m in our equation reduces. With the same arguments we obtain Theorem 2.1 with

$$K_1 \le \frac{1}{K^2} \max_{1 \le i \le m} \left[\frac{2mn(i)\Phi_i}{|c_0|} \right]^{2/i}.$$

The discussed three cases exclude each other so that in any given case we have Theorem 2.1 with one of the mentioned K_1.

4.2. **Proof of Theorem 3.1.** Assume that $e_k(\rho_k, a_k)$, $k = 1, 2, \dots, n$, are some disjoint domains in D which w maps one-to-one onto $\Delta(\rho_k, a_k)$; note that for different (even all) $e_k(\rho_k, a_k)$ the values of a_k and/or ρ_k may coincide. Any set of domains $e_k(\rho_k, a_k)$ contains a subdomain $e_k\left(\frac{\rho_k}{2}, a_k\right)$ such that $w\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right)$ coincides with $\Delta(\frac{\rho_k}{2}, a_k)$. Clearly, each $e_k\left(\frac{\rho_k}{2}, a_k\right)$ is contained in a domain $e_k(\rho_k, a_k)$.

The diameters $d\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right)$ of the domains $e_k\left(\frac{\rho_k}{2}, a_k\right)$ were first given in [2] and were applied to CDE. Later on similar applications were given in [6] and [5] based on the following inequality

$$\sum_{k=1}^{n} d\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right) \le \sqrt{\frac{3\pi}{2}} \sqrt{S(D)} \sqrt{n},$$

where $S(D)$ is the Euclidean area of D. We need a slightly more sharp inequality established in [[3], inequality (6′)]:

$$\sum_{k=1}^{n} d\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right) \le \sqrt{\frac{3\pi}{2}} \sum_{k=1}^{n} S^{1/2}(e_k(\rho_k, a_k)). \tag{4.8}$$

In addition we have also the following.

Lemma 4.1 ([3, Lemma 2])**.** *Let z_k be the point in $e_k(\rho_k, a_k)$ which w maps onto a_k, i.e. $w(z_k) = a_k$. Then*

$$|w'(z_k)| \ge \frac{\rho_k}{2d\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right)}, \quad k = 1, 2, \dots, n. \tag{4.9}$$

Now we take q pairwise different values a_ν, $q = 1, 2, \dots, q$, and consider as $\cup e_k\left(\frac{\rho_k}{2}, a_k\right)$ the union of all domains $e_{\nu,t}$, $\nu = 1, 2, \dots, q$, $t = 1, 2, \dots, n(a_\nu)$, which w maps one-to-one and onto the disk $\Delta(\rho_\nu, a_\nu)$. (Important remark: in this case the disk $\Delta(\rho_\nu, a_\nu)$ remains the same for all $t = 1, 2, \dots, n(a_\nu)$). In other words, function w maps each domain $e_{\nu,t}$ onto an Ahlfors simple island over $\Delta(\rho_\nu, a_\nu)$ so that $n(a_\nu)$ becomes (in this case) the usual number $n(D,\Delta(\rho_\nu,a_\nu),w)$ of simple islands over $\Delta(\rho_\nu,a_\nu)$.

Thus we can rewrite (4.8) and (4.9) as

$$\sum_{\nu=1}^{q}\sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} d\left(e_{\nu,t}\right)\le\sqrt{\frac{3\pi}{2}}\sum_{\nu=1}^{q}\sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right) \tag{4.10}$$

and

$$|w'(z_t(a_\nu))|\ge\frac{\rho_\nu}{2d\left(e_{\nu,t}\right)},\ \nu=1,2,\ldots,q,\ t=1,2,\ldots,n(D,\Delta(\rho_\nu,a_\nu),w), \tag{4.11}$$

where $z_t(a_\nu)\in e_{\nu,t}$ and satisfies $w(z_t(a_\nu))=a_\nu$.

Denote $N=\sum_{\nu=1}^{q} n(D,\Delta(\rho_\nu,a_\nu),w)$. Due to (4.10) we conclude that the set of all domains $e_{\nu,t}$ implies some domains $\tilde{e}_s$, $s=1,2,\ldots,\tilde{n}=\left[\frac{1}{2}N+1\right]'$, (here $[x]'$ means entire part of x), which satisfy

$$d\left(\tilde{e}_s\right)\le\frac{1}{\tilde{n}}\sqrt{\frac{3\pi}{2}}\sum_{\nu=1}^{q}\sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right);$$

indeed assuming contrary we come to contradiction with inequality (4.8). Since $(N/\tilde{n})\le 2$, we have for any $s=1,2,\ldots,\tilde{n}=\left[\frac{1}{2}N+1\right]'$,

$$d\left(\tilde{e}_s\right)\le\sqrt{6\pi}\frac{1}{N}\sum_{\nu=1}^{q}\sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right). \tag{4.12}$$

Since $\tilde{e}_s$ coincides with one of $e_{\nu,t}$, we conclude $\tilde{e}_s$ implies an a_ν-point $z_t(a_\nu)$; to stress that this is namely a point lying in $\tilde{e}_s$ (which satisfies (4.12)) we denote it by $\tilde{z}_t(a_\nu)$. This means that (4.10) is valid also for any given $\tilde{e}_s$ with corresponding point $\tilde{z}_t(a_\nu)$. Now (4.10) and (4.12) yield

$$N\le\frac{\sqrt{24\pi}}{\rho_\nu}\left|w'(\tilde{z}_t(a_\nu))\right|\sum_{\nu=1}^{q}\sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right). \tag{4.13}$$

Applying Cauchy-Schwarz inequality to the last double sum we have

$$\sum_{\nu=1}^{q}\sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right)\le N^{1/2}\left(\sum_{\nu=1}^{q}\sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S\left(e_{\nu,t}\right)\right)^{1/2},$$

so that (4.13) yields

$$N\le\frac{24\pi}{\rho_\nu^2}\left|w'(\tilde{z}_t(a_\nu))\right|^2\sum_{\nu=1}^{q}\sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S\left(e_{\nu,t}\right),$$

and taking into account that the last sum dominated by the area $S(D)$ we obtain

$$\sum_{\nu=1}^{q} n(D,\Delta(\rho_\nu,a_\nu),w)\le\frac{24\pi}{\rho_\nu^2}\left|w'(\tilde{z}_t(a_\nu))\right|^2 S(D). \tag{4.14}$$

Assume now that $w(z)$ is a solution of $(F^{r,r}(D))$ in Theorem 3.1. Considering the equation $(F^{r,r}(D))$ at this point $\tilde{z}_t(a_\nu)$, we notice that all coefficients in $(F^{r,r}(D))$ are defined at this point since we assumed in Theorem 3.1 that $(a_1,\ldots,a_q) \in w(\bar{D})$. Arguing as in (4.6) we obtain

$$|w'(\tilde{z}_t(a_\nu))| \le \max_{1\le i\le m} \left[m \frac{\phi_i(\tilde{z}_t(a_\nu), w(\tilde{z}_t(a_\nu)))}{\phi_0(\tilde{z}_t(a_\nu), w(\tilde{z}_t(a_\nu)))} \right]^{1/i}. \tag{4.15}$$

Since the values $a_1,\ldots,a_q$ are fixed in Theorem 3.1 and functions $\chi_{i,\mu(i)}(w)$ are regular (so that all $\chi_{i,\mu(i)}(a_\nu)$ are finite for any $\nu = 1, 2, \ldots, q$) we have

$$\Phi_i = \max_{1\le\nu\le q} \sum_{\mu(i)=1}^{n(i)} |\chi_{i,\mu(i)}(a_\nu)| < \infty;$$

note that Φ_i depend only on functions $\chi_{i,\mu(i)}(w)$ and values $a_1,\ldots,a_q$.

Applying this to (4.15) we get

$$\begin{aligned} |w'(\tilde{z}_t(a_\nu))| &\le \max_{1\le i\le m} \left[mn(i) \max_{\mu(i)} \frac{M_i^{\mu(i)}(D)}{c(D)} \sum_{\mu(i)=1}^{n(i)} |\chi_{i,\mu(i)}(a_\nu)| \right]^{1/i} \\ &\le \max_{1\le i\le m} \left[mn(i)\Phi_i \max_{\mu(i)} \frac{M_i^{\mu(i)}(D)}{c(D)} \right]^{1/i} \\ &\le \max_{1\le i\le m} [mn(i)\Phi_i]^{1/i} \max_{1\le i\le m} \left[\max_{\mu(i)} \frac{M_i^{\mu(i)}(D)}{c(D)} \right]^{1/i}. \end{aligned}$$

In turn applying the last inequality to (4.14) we obtain the following inequality

$$\sum_{\nu=1}^{q} n(D, \Delta(\rho_\nu, a_\nu), w) \le \frac{24\pi}{\rho^2} \max_{1\le i\le m} [mn(i)\Phi_i]^{2/i} \max_{1\le i\le m} \left[\max_{\mu(i)} \frac{M_i^{\mu(i)}(D)}{c(D)} \right]^{2/i} S(D),$$

i.e. we obtain Theorem 3.1 with

$$K_2 = \frac{24\pi}{\rho^2} \max_{1\le i\le m} [mn(i)\Phi_i]^{2/i} \max_{1\le i\le m} \left[\max_{\mu(i)} \frac{M_i^{\mu(i)}(D)}{c(D)} \right]^{2/i}.$$

4.3. **Proof of Theorem 3.2.** Let $w(z)$ be a meromorphic function in $\bar{D}$ solving equation $(F^{r,m}(D))$. Clearly we should assume that all $\chi_{i,\mu(i)}(w)$ defined on $w(\bar{D})$.

Remember that $D(\varrho) \subset w(\bar{D})$ so that all coefficients $\chi_{i,\mu(i)}(a_\nu)$ defined at any point $w = a \in D(\varrho)$.

Arguing as in the Proposition 4.1 we can fix five points $a_\nu \in D(\varrho)$, $\nu = 1,\ldots,5$, with non-intersecting closures of $\Delta(\varrho/10, a_\nu)$, $\nu = 1,\ldots,5$, such that these points do not pass through zeros and poles of these coefficients. Respectively we have

$$\Phi_i = \max_{1\le i\le m} \sum_{\mu(i)=1}^{n(i)} |\chi_{i,\mu(i)}(a_\nu)| < \infty.$$

Repeating the proof of Theorem 3.2 with similar a_1,. . . ,a_5, we find first the point $\tilde{z}_t(a_\nu)$, (where $a_\nu \in (a_1, \dots, a_5)$) and obtain (instead of (4.14)) the following inequality

$$\sum_{\nu=1}^{5} n\left(D, \Delta\left(1, a_\nu\right), w\right) \leq \left(\frac{10}{\varrho}\right)^2 24\pi \left|w'(\tilde{z}_t(a_\nu))\right|^2 S(D).$$

Then we put this $\tilde{z}_t(a_\nu)$ into equation $(F^{r,m}(D))$ and arguing as above (after (4.15)), we get similarly

$$\begin{aligned} |w'(\tilde{z}_t(a_\nu))| &\leq \max_{1\leq i\leq m}\left[mn(i)\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\sum_{\mu(i)=1}^{n(i)}|\chi_{i,\mu(i)}(a_\nu)|\right]^{1/i} \\ &\leq \max_{1\leq i\leq m}\left[mn(i)\Phi_i\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\right]^{1/i}. \end{aligned}$$

The last two inequalities yield

$$\sum_{\nu=1}^{5} n\left(D, \Delta\left(, a_\nu\right), w\right) \leq K_3 S(D), \tag{4.16}$$

where

$$K_3 = 24\pi\left(\frac{10}{\varrho}\right)^2 \max_{1\leq i\leq m}\left[mn(i)\Phi_i\right]^{2/i} \max_{1\leq i\leq m}\left[\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\right]^{2/i};$$

so that K_3 depends only on equation $(F^{r,m}(D))$ and ϱ.

Finally we need the second fundamental theorem in Ahlfors theory of covering surfaces (see [1] and [9, Chapter 13]): *for any w in $\bar{D}$ and any set of pairwise different points a_ν, $\nu = 1, 2, \dots, q$, $q > 4$,* we have

$$(q-4)A(D,w) \leq \sum_{\nu=1}^{q} n(D, \Delta(\rho_\nu, a_\nu), w) + hL(D,w),$$

where $h < \infty$ is a constant depending on $\Delta(\rho_\nu, a_\nu)$, $\nu = 1, 2, \dots, q$.

Applying this inequality with the above five values a_1,. . . ,a_5, we have

$$A(D,w) \leq \sum_{\nu=1}^{5} n\left(D, \Delta\left(\frac{1}{4}, a_\nu\right), w\right) + h_3 L(D,w),$$

where h_3 depends on these values a_1,. . . ,a_5; in other words depends only on ϱ. From here taking into account (4.16) we obtain Theorem 3.2 with the above defined K_3.

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SOLVABILITY OF FRACTIONAL MULTI-POINT BOUNDARY VALUE PROBLEMS WITH NONLINEAR GROWTH AT RESONANCE

Z. BAITICHE, K. GUERBATI, M. BENCHOHRA AND Y. ZHOU

Ghardaia University, Algeria [1]
University of Sidi Bel-Abbes, Algeria
King Saud University, Saudi Arabia
Xiangtan University, China
King Abdulaziz University, Saudi Arabia
E-mails: *baitichezidane19@gmail.com; guerbati_k@yahoo.com;*
benchohra@yahoo.com; yzhou@xtu.edu.cn

Abstract. This work is concerned with the solvability of multi-point boundary value problems for fractional differential equations with nonlinear growth at the resonance. Existence results are obtained with the use of the coincidence degree theory. As an application, we discuss an example to illustrate the obtained results.

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Keywords: fractional differential equation; fractional Caputo derivative; multi-point boundary value problem; resonance; coincidence degree theory.

1. Introduction

This paper is devoted to the solvability of the following fractional multi-point boundary value problems (BVPs) at the resonance

$$(1.1)\qquad \begin{cases} \left(\phi(t)^C D^\alpha_{0+} u(t)\right)' = f\left(t, u(t), u'(t), u''(t), {}^C D^\alpha_{0+} u(t)\right), \quad t \in I = [0,1], \\ u(0) = 0,\ {}^C D^\alpha_{0+} u(0) = 0,\ u''(0) = \sum_{i=1}^{m} a_i u''(\xi_i),\ u'(1) = \sum_{j=1}^{l} b_j u'(\eta_j), \end{cases}$$

where ${}^C D^\alpha_{0+}$ is the Caputo fractional derivative, $2 < \alpha \le 3$, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \eta_1 < \cdots < \eta_l < 1$, $a_i, b_j \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, l$, $\phi(t) \in C^1([0,1])$, and $\mu = \min_{t \in I} \phi(t) > 0$. The nonlinearity is such that the following conditions are satisfied:

(H_0) $f : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a Carathéodory function, that is,

(i) for each $x \in \mathbb{R}^4$, the function $t \longrightarrow f(t,x)$ is Lebesgue measurable;

(ii) for almost every $t \in [0,1]$, the function $t \longrightarrow f(t,x)$ is continuous on $\mathbb{R}^4$;

(iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1([0,1], \mathbb{R})$ such that for a.e. $t \in [0,1]$ and every $|x| \le r$, we have $|f(t,x)| \le \varphi_r(t)$.

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The resonant conditions of (1.1) are as follows:

(H_1) $\sum_{i=1}^{m} a_i = 1, \quad \sum_{j=1}^{l} b_j = 1, \quad \sum_{j=1}^{l} b_j\eta_j = 1.$

This means that the linear operator $Lu = \left(\phi^C D_{0+}^{\alpha} u\right)'$ corresponding to the problem (1.1) has a nontrivial solution or, in a functional framework, L is not invertible, that is, dim kerL ≥ 1.

In order to be sure that the linear operator Q (to be specified later on) is well defined, we assume, in addition, that

(H_2) There exist $p, q \in \mathbb{Z}^+, q \geq p+1$ such that $\Delta(p,q) = d_{11}d_{22} - d_{12}d_{21}$, where

$$d_{11} = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{s^p(\xi_i - s)^{\alpha-3}}{p\phi(s)} ds, \quad d_{21} = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{s^q(\xi_i - s)^{\alpha-3}}{q\phi(s)} ds,$$

$$d_{12} = \int_0^1 \frac{s^p(1-s)^{\alpha-2}}{p\phi(s)} ds - \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{s^p(\eta_j - s)^{\alpha-2}}{p\phi(s)} ds,$$

$$d_{22} = \int_0^1 \frac{s^q(1-s)^{\alpha-2}}{q\phi(s)} ds - \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{s^q(\eta_j - s)^{\alpha-2}}{q\phi(s)} ds.$$

Note that $\Delta(p,q) \neq 0$ (see [19, 23]).

Fractional calculus is an extension of the ordinary differentiation and integration to arbitrary non-integer order. In particular, time fractional differential equations are used when attempting to describe the transport processes with long memory. Recently, the study of time fractional ordinary and partial differential equations has been received great attention by many researchers, both in theory and in applications. We refer the reader to the monographs [1, 2, 20, 26, 30, 34], the papers [35] – [39], and the references therein. The question of existence of solutions for fractional boundary-value problems at the resonance case has been extensively studied by many authors (see [5] – [8, 10, 12, 13, 14, 17, 18, 21, 22, 32], and the references therein. It is worth to mention that there are a number of papers dealing with the solutions of multi-point boundary value problems of fractional differential equations at the resonance (see [7, 8, 10, 17]).

In [8], Bai and Zhang considered a three-point boundary value problem of fractional differential equations with nonlinear growth given by

$$D_{0+}^{\alpha} u(t) = f\big(t, u(t), D_{0+}^{\alpha-1} u(t)\big), \quad t \in [0,1],$$

$$u(0) = 0, \quad u(1) = \sigma u(\eta),$$

where D_{0+}^{α} is the standard Riemann-Liouville derivative, $1 < \alpha \leq 2$, $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and $\sigma \in (0, \infty)$, $\eta \in (0, 1)$ are given constants such that $\sigma\eta^{\alpha-1} = 1$. The authors applied the coincidence degree theorem to prove existence of solutions. In [10], Chen and Tang have studied the following class of multi-point boundary value problems for fractional differential equations at the resonance by employing the coincidence degree theorem:

$$\Big(a(t)^C D^{\alpha}_{0^+}u(t)\Big)' = f\big(t, u(t), u'(t), {}^C D^{\alpha}_{0^+}u(t)\big), \quad t \in J,$$
$$u(0) = 0, \quad {}^C D^{\alpha}_{0^+}u(0) = 0, \quad u(1) = \sum_{j=1}^{m-1} \sigma_j u(\xi_j),$$

where $1 < \alpha \leq 2$, $f : [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions, $a(t) \in C^1([0,1])$, $\min_{t\in J} a(t) > 0$, $J = [0,1]$, $\sigma_j \in \mathbb{R}^*_+$, $\xi_j \in (0,1)$, $j = 1, \dots, m-1$, $m \in \mathbb{N}$, $m > 1$, and $\sum_{j=1}^{m-1} \sigma_j \xi_j = 1$. The results are obtained under the assumption that

$$\Lambda_0 = \sum_{j=1}^{m-1} \sigma_j \left(\xi_j \int_0^1 s(1-s)^{\alpha-1} \frac{1}{\phi(s)} ds - \int_0^{\xi_j} s(\xi_j - s)^{\alpha-1} \frac{1}{\phi(s)} ds \right) \neq 0.$$

In [7], Bai and Zhang considered the solvability of the following fractional multi-point boundary value problems at the resonance with dim kerL $= 2$ by applying the coincidence degree theorem:

$$D^{\alpha}_{0^+}u(t) = f\big(t, u(t), D^{\alpha-2}_{0^+}u(t), D^{\alpha-1}_{0^+}u(t)\big), \quad t \in (0,1),$$
$$I^{\alpha-1}_{0^+}u(0) = 0, \quad D^{\alpha-1}_{0^+}u(0) = D^{3-\alpha}_{0^+}(\eta), \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i),$$

where $2 < \alpha < 3$, $0 < \eta \leq 1$, $0 < \eta_1 < \eta_2 < \dots < \eta_m < 1$, $m \geq 2$, $\sum_{i=1}^m \alpha_i \eta_i^{\alpha-1} = \sum_{i=1}^m \alpha_i \eta_i^{\alpha-2} = 1$. $D^{\alpha}_{0^+}$ and $I^{\alpha}_{0^+}$ are the standard Riemann-Liouville fractional derivative and the fractional integral, respectively, and $f : [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions. The results are obtained under the assumption that

$$R = \frac{1}{\alpha}\eta^{\alpha} \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} \Big[1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\Big] - \frac{1}{\alpha-1}\eta^{\alpha-1} \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} \Big[1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-1}\Big] \neq 0.$$

Jiang [17], by using the coincidence degree theorem, has obtained an existence result for the boundary value problems of fractional differential equations at the resonance with dim kerL $= 2$:

$$D^{\alpha}_{0^+}u(t) = f\big(t, u(t), D^{\alpha-1}_{0^+}u(t)\big), \quad \forall t \in J = [0,1],$$
$$u(0) = 0, \quad D^{\alpha-1}_{0^+}u(0) = \sum_{i=1}^{m} a_i D^{\alpha-1}_{0^+}(\xi_i), \quad D^{\alpha-2}_{0^+}u(0) = \sum_{j=1}^{n} b_j D^{\alpha-2}_{0^+}(\eta_j),$$

where $2 < \alpha < 3$, $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \dots < \eta_n < 1$, $\sum_{i=1}^m a_i = 1$, $\sum_{j=1}^n b_j = 1$, $\sum_{j=1}^n b_j \eta_j = 1$, and $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions. The results are obtained under the assumption that

$$\frac{1}{3}\Big(1 - \sum_{j=1}^{n} b_j \eta_j^3\Big) \sum_{i=1}^{m} a_i \xi_i - \frac{1}{2}\Big(1 - \sum_{j=1}^{n} b_j \eta_j^2\Big) \sum_{i=1}^{m} a_i \xi_i^2 \neq 0.$$

In this paper, we study problem (1.1), which allow f to have a nonlinear growth.

The rest of the paper is organized as follows. In Section 2, we introduce some notation, definitions and preliminary results, which will be used in the proofs of our main results (see [1, 2, 20, 26, 27, 28, 30, 34]). In Section 3, we state and prove our main results by applying the coincidence degree theorem. In Section 4 we provide an example.

2. Preliminaries

Definition 2.1. *Let $\alpha > 0$. For a function $u : (0,\infty) \longrightarrow \mathbb{R}$, the Riemann-Liouville fractional integral of order α of u is defined by*

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Remark 2.1. *The notation $I_{0^+}^{\alpha} u(t) \mid_{t=0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(0,\varepsilon)(\varepsilon > 0)$ of 0 as follows:*

$$I_{0^+}^{\alpha} u(t) \mid_{t=0} = \lim_{t \to 0+} I_{0^+}^{\alpha} u(t).$$

Generally, $I_{0^+}^{\alpha} u(t) \mid_{t=0}$ is not necessarily equal to zero. For instance, let $\alpha \in (0,1)$ and $u(t) = t^{-\alpha}$. Then we have

$$I_{0^+}^{\alpha} t^{-\alpha}|_{t=0} = \lim_{t \to 0+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds = \Gamma(1-\alpha).$$

Definition 2.2. *Let $\alpha > 0$ and $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. The Caputo fractional derivative of order α of a function $u : (0,\infty) \longrightarrow \mathbb{R}$ is given by*

$${}^{C}D_{0^+}^{\alpha} u(t) = I_{0^+}^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.2. *Let $\alpha, \eta > 0$ and $n = [\alpha] + 1$. Then the following relations hold:*

$${}^{C}D_{0^+}^{\alpha} t^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)} t^{\eta-\alpha}, \quad (\eta > n-1),$$

and ${}^{C}D_{0^+}^{\alpha} t^{k} = 0$, $(k = 0, \ldots, n-1)$.

Lemma 2.3. *Let $\alpha, \beta \geq 0$, and $u \in L^1([0,1])$. Then $I_{0^+}^{\alpha} I_{0^+}^{\beta} u(t) = I_{0^+}^{\alpha+\beta} u(t)$ and ${}^{C}D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$, for all $t \in [0,1]$*

Lemma 2.4. *Let $\alpha > 0$ and $n = [\alpha] + 1$, then*

$$I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} u(t) = u(t) + \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

Lemma 2.5. *Let $\alpha > 0$ and $n = [\alpha] + 1$. If ${}^{C}D_{0^+}^{\alpha} u(t) \in C[0,1]$, then $u(t) \in C^{n-1}([0,1])$.*

Proof. Let $v(t) \in C[0,1]$ be such that ${}^{C}D_{0^+}^{\alpha} u(t) = v(t)$. Then by Lemma 2.3, we have

$$u(t) = I_{0^+}^{\alpha} v(t) + \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

It is easy to check that $u(t) \in C^{n-1}([0,1])$. □

Lemma 2.6. *Let $\alpha > 0$ and $u \in L^1\big([0,1],\mathbb{R}\big)$. Then for all $t \in [0,1]$ we have*

$$I_{0^+}^{\alpha+1} u(t) \leq \|I_{0^+}^{\alpha} u\|_{L^1}.$$

Proof. Let $u \in L^1\big([0,1],\mathbb{R}\big)$, then by Lemma 2.3 we have

$$I_{0^+}^{\alpha+1} u(t) = I_{0^+}^{1} I_{0^+}^{\alpha} u(t) = \int_0^t I_{0^+}^{\alpha} u(s) ds \leq \int_0^1 |I_{0^+}^{\alpha} u(s)| ds = \|I_{0^+}^{\alpha} u\|_{L^1}.$$

□

Lemma 2.7. *The fractional integral $I_{0^+}^{\alpha}$, $\alpha > 0$ is bounded in $L^1\big([0,1],\mathbb{R}\big)$, and*

$$\|I_{0^+}^{\alpha} u\|_{L^1} \leq \frac{\|u\|_{L^1}}{\Gamma(\alpha+1)}.$$

Proof. Let $u \in L^1\big([0,1],\mathbb{R}\big)$, then can write

$$\begin{aligned}
\|I_{0^+}^{\alpha} u\|_{L^1} &= \int_0^1 |I_{0^+}^{\alpha} u(t)| dt \leq \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^t (t-s)^{\alpha-1} |u(s)| ds dt \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 |u(s)| ds \int_s^1 (t-s)^{\alpha-1} dt \leq \frac{1}{\Gamma(\alpha+1)} \int_0^1 |u(s)| ds = \frac{\|u\|_{L^1}}{\Gamma(\alpha+1)}.
\end{aligned}$$

□

Now we recall the coincidence degree continuation theorem and some related notions (for more details see [25]).

Definition 2.3. *Let X and Y be real Banach spaces. A linear operator $L : dom\, L \subset X \longrightarrow Y$ is said to be a Fredholm operator of index zero if*

(1) *$Im\, L$ is a closed subset of Y;*
(2) *$dim\, ker\, L = codim\, Im\, L < \infty$.*

It follows from Definition 2.3 that there exist continuous projectors $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ such that

$$KerL = ImP, \quad ImL = KerQ, \quad X = Ker\, L \oplus KerP, \quad Y = Im\, L \oplus Im\, Q.$$

Also, it follows that

$$L_p = L\,|_{dom\, L \bigcap Ker\, P} : \, dom\, L \bigcap Ker\, P \longrightarrow Im\, L$$

is invertible and its inverse is denoted by K_p.

Definition 2.4. *Let L be a Fredholm operator of index zero, and let Ω be an open bounded subset of X such that $dom\, L \bigcap \Omega \neq \emptyset$. Then the map $N : \overline{\Omega} \longrightarrow X$ will be called $L-$ compact on $\overline{\Omega}$ if*

(1) *$QN(\overline{\Omega})$ is bounded,*
(2) *$K_{P,Q}\, N = K_p(I-Q)N : \overline{\Omega} \longrightarrow X$ is compact.*

Theorem 2.8. *Let $L : dom\, L \subset X \longrightarrow Y$ be a Fredholm operator of index zero, and let $N : X \longrightarrow Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for every $(x,\, \lambda) \in \Big[\Big(dom\, L \backslash Ker\, L\Big) \bigcap \partial\Omega\Big] \times (0,1)$.*
(2) *$Nx \notin Im\, L$ for every $x \in Ker L \bigcap \partial\Omega$.*
(3) *$deg\, (QN\mid_{Ker\, L}, \Omega \bigcap Ker\, L, 0) \neq 0$, where $Q : Y \longrightarrow Y$ is a projection such that $Im\, L = Ker\, Q$.*

Then, the abstract equation $Lx = Nx$ has at least one solution in $dom\, L \bigcap \overline{\Omega}$.

For our purposes, the adequate functional space is:

$$X := \Big\{ u : {}^{C}D^{\alpha}_{0+}u \in C\big([0,1], \mathbb{R}\big),\ u \,\text{satisfies the boundary conditions of (1.1)} \Big\},$$

equipped with the norm:

$$\|u\|_X = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|{}^{C}D^{\alpha}_{0+}u\|_\infty,$$

where

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

By means of the functional analysis theory, we can prove that $(X, \|\cdot\|_X)$ is a Banach space. Let $Y = L^1[0,1]$ be the space of real measurable functions $t \longrightarrow y(t)$ defined on $[0,1]$ such that $t \longrightarrow |y(t)|$ is Lebesgue integrable. Then Y is a Banach space with the norm $\|y\|_{L^1} = \int_0^1 |y(t)|dt$. Define L to be the linear operator from $dom\, L \bigcap X$ to Y :

$$Lu = \big(\phi^{C}D^{\alpha}_{0+}u\big)', \quad u \in dom\, L.$$

where $dom\, L = \Big\{ u \in X \mid {}^{C}D^{\alpha}_{0+}u(t) \,\text{is absolutely continuous on [0, 1]} \Big\}$, and define the operator $N : X \longrightarrow Y$ as follows:

$$Nu(t) = f\big(t, u(t), u'(t), u''(t), {}^{C}D^{\alpha}_{0+}u(t)\big), \quad t \in [0,1].$$

Then the boundary value problem (1.1) can be written in the following form:

$$Lu = Nu, \quad u \in dom\, L.$$

To study the compactness of the operator N, we will need the following lemma.

Lemma 2.9. *A subset $U \subset X$ is a relatively compact set in X if and only if U is uniformly bounded and equicontinuous. Here the uniformly boundedness means that there exists $M > 0$ such that for every $u \in U$*

$$\|u\|_X = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|{}^C D^\alpha_{0^+} u\|_\infty \leq M,$$

and the equicontinuity means that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$|u^{(i)}(t_1) - u^{(i)}(t_2)| < \varepsilon, \quad \forall u \in U,\ \forall t_1, t_2 \in I,\ |t_1 - t_2| < \delta,\ \forall i \in \{0, 1, 2\}.$$

and

$$|{}^C D^\alpha_{0^+} u(t_1) - {}^C D^\alpha_{0^+} u(t_2)| < \varepsilon, \quad \forall u \in U,\ \forall t_1, t_2 \in I,\ |t_1 - t_2| < \delta.$$

3. The main results

In this section we state and prove our main results.

Lemma 3.1. *Let $y \in Y$, $\phi(t) \in C^1[0,1]$, $\mu = \min_{t \in I} \phi(t) > 0$ and (H_1) hold, and let T_1, $T_2 : Y \longrightarrow Y$ be two linear operators defined by*

$$T_1(y) = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r) dr ds,$$

$$T_2(y) = \int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r) dr ds - \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r) dr ds.$$

Then $u \in X$ is a solution of the following linear fractional differential problem:

$$\begin{cases} \left(\phi(t) {}^C D^\alpha_{0^+} u(t)\right)' = y(t), \quad t \in I = [0,1], \\ u(0) = 0,\ {}^C D^\alpha_{0^+} u(0) = 0,\ u''(0) = \sum_{i=1}^{m} a_i u''(\xi_i),\ u'(1) = \sum_{j=1}^{l} b_j u'(\eta_j), \end{cases} \tag{3.1}$$

if and only if

$$u(t) = c_1 t + c_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r) dr ds, \quad c_1, c_2 \in \mathbb{R}, \tag{3.2}$$

and

$$T_1(y) = T_2(y) = 0. \tag{3.3}$$

Proof. Let u be a solution of the problem (3.1). Then we have

$$\phi(t) {}^C D^\alpha_{0^+} u(t) = c + \int_0^t y(s) ds, \quad c \in \mathbb{R}.$$

Since ${}^C D^\alpha_{0^+} u(0) = 0$, we find

$$ {}^C D^\alpha_{0^+} u(t) = \frac{1}{\phi(t)} \int_0^t y(s) ds.$$

By Lemma 2.4, we get

$$u(t) = c_0 + c_1 t + c_2 t^2 + \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds, \quad c_0, c_1, c_2 \in \mathbb{R}.$$

Since $u(0) = 0$, we have

$$u(t) = c_1 t + c_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds, \quad c_1, c_2, \in \mathbb{R}.$$

By $u''(0) = \sum_{i=1}^m a_i u''(\xi_i)$ and $\sum_{i=1}^l a_i = 1$, we obtain

$$\sum_{i=1}^l a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r)drds = 0.$$

From the conditions $u'(1) = \sum_{j=1}^l b_j u'(\eta_j)$ and $\sum_{j=1}^l b_j = \sum_{j=1}^l b_j \eta_j = 1$, we get

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)drds - \sum_{j=1}^l b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)drds = 0.$$

Thus, we have $T_1(y) = T_2(y) = 0$. On the other hand, if c_1, c_2 are arbitrary real constants and

$$u(t) = c_1 t + c_2 t^2 + \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds,$$

then clearly $u(0) = 0$, and by Lemma 2.2 and 2.3, we obtain

$$\begin{cases} & {}^C D_{0^+}^{\alpha} u(0) = 0 \\ \forall\ t \in [0,1], & \Big(\phi(t) {}^C D_{0^+}^{\alpha} u(t)\Big)' = y(t). \end{cases}$$

Taking into account that (3.3) holds, we get the following equations:

$$u''(0) - \sum_{i=1}^m a_i u''(\xi_i) = \frac{T_1(y)}{\Gamma(\alpha-2)} = 0, \quad u'(1) - \sum_{j=1}^l b_j u'(\eta_j) = \frac{T_2(y)}{\Gamma(\alpha-1)} = 0.$$

Thus, u is a solution of the problem (3.1). This completes the proof. □

Lemma 3.2. *Assume that the conditions $(H_0) - (H_2)$ hold.*

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds. \tag{3.4}$$

Furthermore, we have

$$\|K_p y\|_X \leq \rho_1 \|y\|_{L^1}, \tag{3.5}$$

where

$$\rho_1 = \frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + 1\right). \tag{3.6}$$

Proof. It is clear that $Ker\, L = \big\{u \mid u(t) = c_1 t + c_2 t^2,\ c_1, c_2 \in \mathbb{R}\big\}$. Furthermore, Lemma 3.1 implies that

$$Im\, L = \big\{y \in Y \mid T_1(y) = T_2(y) = 0\big\}. \tag{3.7}$$

Consider a continuous linear mapping $Q : Y \longrightarrow Y$ defined by

$$Qy = Q_1(y)t^{p-1} + Q_2(y)t^{q-1}, \tag{3.8}$$

where p, q are given in (H_2), and

$$Q_1(y) = \frac{1}{\Delta(p,q)}\big(d_{22}T_1(y) - d_{21}T_2(y)\big),$$

$$Q_2(y) = \frac{1}{\Delta(p,q)}\big(- d_{12}T_1(y) + d_{11}T_2(y)\big).$$

We prove that $Ker\, Q = Im\, L$. Obviously, $ImL \subset Ker\, Q$. Also, if $y \in Ker\, Q$, then

$$\begin{cases} d_{22}T_1(y) - d_{21}T_2(y) = 0. \\ -d_{12}T_1(y) + d_{11}T_2(y) = 0. \end{cases} \tag{3.9}$$

The determinant of coefficients for (3.9) is $\Delta(p, q) \neq 0$. Therefore $T_1(y) = T_2(y) = 0$, implying that $y \in Im\, L$. Thus, $Ker\, Q \subset Im\, L$. Now, we show that $Q^2y = Qy$, $y \in Y$. For $y \in Y$, we have

$$\begin{aligned} Q_1(Q_1(y)t^{p-1}) &= \frac{1}{\Delta(p,q)}\left[d_{22}T_1\left(Q_1(y)t^{p-1}\right) - d_{21}T_2\left(Q_1(y)t^{p-1}\right)\right] \\ &= \frac{1}{\Delta(p,q)}\big(d_{22}d_{11} - d_{21}d_{12}\big)Q_1y = Q_1y, \end{aligned}$$

and

$$\begin{aligned} Q_1\big(Q_2(y)t^{q-1}\big) &= \frac{1}{\Delta(p,q)}\left[d_{22}T_1\big(Q_2(y)t^{q-1}\big) - d_{21}T_2\big(Q_2(y)t^{q-1}\big)\right] \\ &= \frac{1}{\Delta(p,q)}\big(d_{22}d_{21} - d_{21}d_{22}\big)Q_2y = 0. \end{aligned}$$

Similarly, we obtain

$$Q_2\big(Q_1(y)t^{p-1}\big) = 0, \quad Q_2\big(Q_2(y)t^{q-1}\big) = Q_2y.$$

Therefore, we get

$$\begin{aligned} Q^2y &= Q_1\big(Q_1(y)t^{p-1}\big)t^{p-1} + Q_1\big(Q_2(y)t^{q-1}\big)t^{p-1} \\ &\quad + Q_2\big(Q_1(y)t^{p-1}\big)t^{q-1} + Q_2\big(Q_2(y)t^{q-1}\big)t^{q-1} \\ &= Q_1(y)t^{p-1} + Q_2(y)t^{q-1} = Qy, \end{aligned}$$

showing that the operator Q is a projector.

Take $y \in Y$ of the form $y = \big(y - Qy\big) + Qy$ to obtain $\big(y - Qy\big) \in KerQ = ImL$ and $Qy \in ImQ$. Thus, $Y = Im\, Q + Im\, L$. Also, for any $y \in Im\, Q \cap Im\, L$, from $y \in Im\, Q$ there exist constants c_1, $c_2 \in \mathbb{R}$ such that $y(t) = c_1t^{p-1} + c_2t^{q-1}$, and from $y \in Im\, L$ we obtain

$$\begin{cases} d_{11}c_1 + d_{21}c_2 = 0, \\ d_{12}c_1 + d_{22}c_2 = 0. \end{cases} \tag{3.10}$$

The determinant of coefficients for (3.10) is $\Delta(p,q) \neq 0$. Therefore (3.10) has a unique solution $c_1 = c_2 = 0$, which implies that $Im\, Q \cap Im\, L = 0$. Then, we have

$$Y = Im\, Q \oplus Ker\, Q = Im\, Q \oplus Im\, L. \tag{3.11}$$

Thus, $dim\, Ker\, L = 2 = dim\, Im\, Q = codim\, Ker\, Q = codim\, Im\, L$, showing that L is a Fredholm operator of index zero.

Let a mapping $P : X \longrightarrow X$ be defined by

$$Pu(t) = u'(0)t + \frac{u''(0)}{2}t^2. \tag{3.12}$$

We note that P is a linear continuous projector and $Im\, P = Ker\, L$. It follows from $u = (u - Pu) + Pu$ that $X = Ker\, P + Ker\, L$. By simple calculation, we obtain that $KerL \cap KerP = \{0\}$, and hence

$$X = Ker\, L \oplus Ker\, P. \tag{3.13}$$

Define $K_p : Im\, L \longrightarrow dom\, L \cap Ker\, P$ as follows:

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds.$$

Now, we show that K_p is the inverse of $L\, |_{dom\, L \cap Ker\, P}$. In fact, for $u \in dom\, L \cap Ker\, P$, we have

$$\begin{aligned}(K_p L)u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s \left(\phi\, {}^C D_{0+}^{\alpha} u\right)'(r)drds = I_{0+}^{\alpha}\, {}^C D_{0+}^{\alpha} u(t)\\ &= u(t) + u(0) + u'(0)t + \frac{u''(0)}{2}t^2.\end{aligned}$$

In view of $u \in dom\, L \cap Ker\, P$, we have $u(0) = 0$ and $Pu = 0$. Thus

$$(K_p L)u(t) = u(t), \tag{3.14}$$

and for $y \in Im\ L$, we find

$$(LK_p)y(t) = L(K_p y)(t) = \left[\phi(t)\, {}^C D_{0+}^{\alpha} I_{0+}^{\alpha} \left(\frac{I_{0+}^1 y}{\phi}\right)(t)\right]' = y(t).$$

Thus, $K_p = \left(L\, |_{dom\, L \cap Ker\, P}\right)^{-1}$. Again, for each $y \in Im\ L$, in view of Lemmas 2.3, 2.6 and 2.7, we can write

$$\begin{aligned}\|K_p y\|_X &= \sum_{i=0}^{2} \max_{t \in I} \left|(K_p y)^{(i)}(t)\right| + \max_{t \in I} \left|{}^C D_{0+}^{\alpha}(K_p y)(t)\right|\\ &= \sum_{i=0}^{2} \max_{t \in I} \left|I_{0+}^{\alpha-i} \left(\frac{I_{0+}^1 y}{\phi}\right)(t)\right| + \max_{t \in I} \left|\left(\frac{I_{0+}^1 y}{\phi}\right)(t)\right|\\ &\leq \sum_{i=0}^{2} \max_{t \in I} \left|\frac{I_{0+}^{\alpha+1-i} y(t)}{\mu}\right| + \max_{t \in I} \left|\frac{I_{0+}^1 y(t)}{\mu}\right|\end{aligned}$$

$$\leq \sum_{i=0}^{2} \frac{\|y\|_{L^1}}{\mu\Gamma(\alpha+1-i)} + \frac{\|y\|_{L^1}}{\mu} \leq \rho_1 \|y\|_{L^1},$$

and the result follows. □

Lemma 3.3. *Suppose that Ω is an open bounded subset of X such that dom $L \bigcap \overline{\Omega} \neq \emptyset$. Then N is L-compact on $\overline{\Omega}$.*

Proof. It is clear that $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are bounded, due to the fact that f satisfies the Carathéodory conditions. Using the Lebesgue dominated convergence theorem, we can easily show that QN and $K_{P,Q}N = K_p(I-Q)N : \overline{\Omega} \longrightarrow X$ are continuous. By the hypothesis (iii) on the function f, there exists a constant $M > 0$, such that $|(I-Q)N(u(t))| \leq M$, for all $u \in \Omega$ and $t \in [0,1]$. For $i = 0, 1, 2$, $0 \leq t_1 \leq t_2 \leq 1$, and $u \in \Omega$, we can write

$$\begin{aligned}
&\left|\big(K_{P,Q}Nu\big)^{(i)}(t_2) - \big(K_{P,Q}Nu\big)^{(i)}(t_1)\right| \\
=&\frac{1}{\Gamma(\alpha-i)}\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-i-1}}{\phi(s)}\int_0^s (I-Q)Nu(r)drds\right. \\
&\left. - \int_0^{t_1}\frac{(t_1-s)^{\alpha-i-1}}{\phi(s)}\int_0^s (I-Q)Nu(r)drds\right| \\
\leq&\frac{M}{\mu\Gamma(\alpha-i)}\left\{\int_0^{t_1}(t_2-s)^{\alpha-i-1} - (t_1-s)^{\alpha-i-1}ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-i-1}ds\right\} \\
=&\frac{M}{\mu\Gamma(\alpha+1-i)}(t_2^{\alpha-i} - t_1^{\alpha-i}).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\left|{}^C D^{\alpha}_{0^+} K_{P,Q}Nu(t_2) - {}^C D^{\alpha}_{0^+} K_{P,Q}Nu(t_1)\right| \\
=&\left|\frac{1}{\phi(t_2)}\int_0^{t_2}(I-Q)Nu(s)ds - \frac{1}{\phi(t_1)}\int_0^{t_1}(I-Q)Nu(s)ds\right| \\
=&\left|\left(\frac{1}{\phi(t_2)} - \frac{1}{\phi(t_1)}\right)\int_0^{t_1}(I-Q)Nu(s)ds + \frac{1}{\phi(t_2)}\int_{t_1}^{t_2}(I-Q)Nu(s)ds\right| \\
\leq&\frac{M}{\mu^2}\big|\phi(t_2)-\phi(t_1)\big| + \frac{M}{\mu}\big(t_2-t_1\big).
\end{aligned}$$

Since t^{α}, $t^{\alpha-1}$, $t^{\alpha-2}$ and $\phi(t)$ are uniformly continuous on $[0,1]$, we conclude that $K_p(I-Q)N : \overline{\Omega} \longrightarrow X$ is compact. □

Now we are in position to state the main result of this paper.

Theorem 3.4. *Assume that, in addition to $(H_0)-(H_2)$, the following conditions hold.*

(H_3) *There exists a Carathéodory function $\Phi : [0,1]\times(\mathbb{R}^+)^4 \longrightarrow \mathbb{R}^+$ that is nondecreasing with respect to the last four arguments and satisfies the inequality:*

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big| \leq \Phi\big(t,|x_0|,|x_1|,|x_2|,|x_3|\big).$$

(H_4) $\lim_{r\to\infty}\sup\frac{1}{r}\int_0^1\left|\Phi\big(s,r,r,r,r\big)\right|ds<\frac{1}{\rho_1+\rho_2}$ *where* ρ_1 *is defined by* (3.6), *and*

$$\rho_2=\frac{1}{\mu}\left(\frac{2}{\Gamma(\alpha)}+\frac{5}{\Gamma(\alpha-1)}\right).$$

(H_5) *There exists a constant* $A>0$ *such that for* $u\in dom\,L\backslash Ker\,L$, *if* $|u^{'}(t)|>A$ *or* $|u^{''}(t)|>A$ *for all* $t\in[0,1]$, *then* $T_1(Nu)\neq 0$ *or* $T_2(Nu)\neq 0$.

(H_6) *There exists a constant* $B>0$ *such that for any* $c_1,c_2\in\mathbb{R}$, *if* $|c_1|>B, |c_2|>B$, *then either*

$$T_1N\big(c_1t+c_2t^2\big)+T_2N\big(c_1t+c_2t^2\big)<0,$$

or

$$T_1N\big(c_1t+c_2t^2\big)+T_2N\big(c_1t+c_2t^2\big)>0.$$

Then, the problem (1.1) *has at least one solution.*

Remark 3.5. A sufficient condition for (H_3) to be satisfied is the existence of functions $\theta_i(t)\in Y,\ i=0,\dots,5$ and a constant $\nu\in(0,1)$ such that for all $x_0,x_1,x_2,x_3\in\mathbb{R}$ and $t\in[0,1]$ the nonlinearity f verifies one of the following growth conditions:

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big|\le\sum_{i=0}^{3}\theta_i(t)|x_i|+\theta_4(t)|x_0|^{\nu}+\theta_5(t),$$

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big|\le\sum_{i=0}^{3}\theta_i(t)|x_i|+\theta_4(t)|x_1|^{\nu}+\theta_5(t),$$

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big|\le\sum_{i=0}^{3}\theta_i(t)|x_i|+\theta_4(t)|x_2|^{\nu}+\theta_5(t),$$

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big|\le\sum_{i=0}^{3}\theta_i(t)|x_i|+\theta_4(t)|x_3|^{\nu}+\theta_5(t).$$

In this case, (H_4) reduces to the following:

(H_4^*) $\sum_{i=0}^{3}\|\theta_i\|_{L^1}<\frac{1}{\rho_1+\rho_2}$.

Proof of Theorem 3.4. Consider the set

$$\Omega_1=\Big\{u\in dom\,L\backslash Ker\,L\mid Lu=\lambda Nu,\,\lambda\in[0,1]\Big\},$$

and observe that for $u\in\Omega_1$, we have $Lu=\lambda Nu$. Thus, $\lambda\neq 0$, $Nu\in Im\,L=Ker\,Q\subset Y$, and hence, $Q(Nu)=0$, that is, $T_1(Nu)=T_2(Nu)=0$. It follows from condition (H_5) that there exist $t_1,t_2\in[0,1]$, such that $|u'(t_1)|\le A, |u''(t_2)|\le A$.

If $t_1=t_2=0$, then we have $|u'(0)|\le A, |u''(0)|\le A$. Otherwise, in view of $Lu=\lambda Nu$, we obtain

$$u(t)=u'(0)t+\frac{u''(0)}{2}t^2+\frac{\lambda}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s Nu(r)drds.$$

If $t_2 \neq 0$, then

$$u''(t_2) = u''(0) + \frac{\lambda}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s Nu(r)drds,$$

and, together with $|u''(t_2)| \leq A$, we get

$$|u''(0)| \leq |u''(t_2)| + \frac{1}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s |Nu(r)|drds \leq A + \frac{\|Nu\|_{L^1}}{\mu\Gamma(\alpha-1)}.$$

Consequently, we have

$$|u''(0)| \leq A + \frac{1}{\mu\Gamma(\alpha-1)} \|Nu\|_{L^1}. \tag{3.15}$$

If $t_1 \neq 0$, then

$$u'(t_1) = u'(0) + u''(0)t_1 + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s Nu(r)drds,$$

and, according to (3.15) and $|u'(t_1)| \leq A$, we get

$$\begin{aligned}|u'(0)| &\leq |u'(t_1)| + |u''(0)| + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s |Nu(r)|drds \\ &\leq 2A + \frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right)\|Nu\|_{L^1}.\end{aligned}$$

Therefore

$$|u'(0)| \leq 2A + \frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right)\|Nu\|_{L^1}. \tag{3.16}$$

Next, for $u \in \Omega_1$, we get

$$\begin{aligned}\|Pu\|_X &= \sum_{i=0}^{2} \max_{t\in[0,1]} \left|(Pu)^{(i)}(t)\right| + \max_{t\in[0,1]} \left|{}^C D_{0^+}^{\alpha}(Pu)(t)\right| \\ &\leq 2|u'(0)| + 3|u''(0)|.\end{aligned}$$

From (3.15) and (3.16), we obtain

$$\|Pu\|_X \leq 7A + \rho_2 \|Nu\|_{L^1}. \tag{3.17}$$

Again, for all $u \in \Omega_1$, we have $(I-P)u \in dom\, L \cap Ker\, P$, and hence, by (3.14) and (3.5), we find

(3.18)

$$\|(I-P)u\|_X = \|K_p L(I-P)u\|_X \leq \rho_1 \|L(I-P)u\|_{L^1} = \rho_1 \|Lu\|_{L^1} \leq \rho_1 \|Nu\|_{L^1}.$$

From (3.17) and (3.18), we obtain

$$\|u\|_X \leq \|Pu\|_X + \|(I-P)u\|_X \leq 7A + \big(\rho_1 + \rho_2\big)\|Nu\|_{L^1}. \tag{3.19}$$

On the other hand, from (H_3), we have

$$\begin{aligned}\|Nu\|_{L^1} &= \int_0^1 \left| f\big(s, u(s), u'(s), u''(s), {}^C D_{0^+}^{\alpha} u(s)\big)\right| ds \\ &\leq \int_0^1 \left| \Phi\big(s, u(s), u'(s), u''(s), {}^C D_{0^+}^{\alpha} u(s)\big)\right| ds\end{aligned}$$

$$\leq \int_0^1 \Big|\Phi\big(s, \|u\|_X, \|u\|_X, \|u\|_X, \|u\|_X\big)\Big| ds. \tag{3.20}$$

Because the function Φ is Carathéodory, the function $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, given by $\Psi(r) = \frac{1}{r}\int_0^1 \big|\Phi\big(s, r, r, r, r\big)\big| ds$, is well defined. Let $l = \lim_{r\to\infty} \sup \Psi(r)$. By (H_4) we have $0 < l < \frac{1}{\rho_1+\rho_2}$, and hence, for each $0 < \varepsilon < \frac{1}{\rho_1+\rho_2} - l$, there exists r_ε such that $r \geq r_\varepsilon \Longrightarrow \Psi(r) < l + \varepsilon$. If $\|u\|_X \geq r_\varepsilon$, then $\Psi(\|u\|_X) < \frac{1}{\rho_1+\rho_2}$, and hence, (3.20) implies that

$$\|Nu\|_{L^1} \leq (l+\varepsilon)\|u\|_X. \tag{3.21}$$

Therefore, in view of (3.19) and (3.21), we obtain

$$r_\varepsilon \leq \|u\|_X \leq \frac{7A}{1 - \big(\rho_1+\rho_2\big)(l+\varepsilon)}.$$

Consequently, we have

$$\|u\|_X \leq \max\left\{ r_\varepsilon, \frac{7A}{1-(l+\varepsilon)\big(\rho_1+\rho_2\big)} \right\} = \frac{7A}{1-(l+\varepsilon)\big(\rho_1+\rho_2\big)}. \tag{3.22}$$

Since (3.22) is valid for all $0 < \varepsilon < \frac{1}{\rho_1+\rho_2} - l$, we get

$$\|u\|_X \leq \frac{7A}{1 - l\big(\rho_1+\rho_2\big)}.$$

So, Ω_1 is bounded. Denote

$$\Omega_2 = \Big\{ u \in Ker\, L \mid Nu \in Im\, L \Big\},$$

and observe that for $u \in \Omega_2$, we have $u \in Ker\, L = \big\{ u \mid u(t) = c_1 t + c_2 t^2,\ c_1, c_2 \in \mathbb{R} \big\}$, and $Q(Nu) = 0$, that is,

$$T_1 N\big(c_1 t + c_2 t^2\big) = T_2 N\big(c_1 t + c_2 t^2\big) = 0.$$

From condition (H_6), we get $|c_1| \leq B, |c_2| \leq B$. Hence, Ω_2 is bounded. Define

$$\Omega_3 := \Big\{ u \in Ker\, L \mid -\lambda J u + (1-\lambda) Q N u = 0,\ \lambda \in [0,1] \Big\}$$

provided that the first part of condition (H_6) holds, or

$$\Omega_3 := \Big\{ u \in Ker\, L \mid -\lambda J u + (1-\lambda) Q N u = 0,\ \lambda \in [0,1] \Big\}$$

provided that the second part of (H_6) holds, where $J : Ker\, L \longrightarrow Im\, Q$ is the linear isomorphism given by

$$J\big(c_1 t + c_2 t^2\big) = \omega_1 t^{p-1} + \omega_2 t^{q-1}, \quad c_1, c_2 \in \mathbb{R}, \tag{3.23}$$

with

$$\omega_1 = \frac{1}{\Delta(p,q)}\big(d_{22}|c_1| - d_{21}|c_2|\big), \quad \omega_2 = \frac{1}{\Delta(p,q)}\big(-d_{12}|c_1| + d_{11}|c_2|\big).$$

Without loss of generality, we assume that the first part of (H_6) holds.
In fact $u \in \Omega_3$, means that $u = c_1 t + c_2 t^2$ and $-\lambda Ju + (1-\lambda)QNu = 0$. Then we obtain

$$-\lambda J\big(c_1 t + c_2 t^2\big) + (1-\lambda)QN\big(c_1 t + c_2 t^2\big) = 0. \tag{3.24}$$

If $\lambda = 0$, then $|c_1| \leq B, |c_2| \leq B$. If $\lambda = 1$, then

$$\begin{cases} d_{22}|c_1| - d_{21}|c_2| = 0 \\ -d_{12}|c_1| + d_{11}|c_2| = 0. \end{cases} \tag{3.25}$$

The determinant of coefficients for (3.25) is $\Delta(p,q) \neq 0$. Thus, the system (3.25) has only zero solution, that is, $c_1 = c_2 = 0$.
Otherwise, if $\lambda \neq 0$ and $\lambda \neq 1$, in view of (3.23), the equation (3.24) becomes

$$\lambda\big(\omega_1 t^{p-1} + \omega_2 t^{q-1}\big) = (1-\lambda)\Big(Q_1 N\big(c_1 t + c_2 t^2\big)t^{p-1} + Q_2 N\big(c_1 t + c_2 t^2\big)t^{q-1}\Big).$$

Hence

$$\begin{cases} \lambda\omega_1 = (1-\lambda)Q_1\big(c_1 t + c_2 t^2\big), \\ \lambda\omega_2 = (1-\lambda)Q_2\big(c_1 t + c_2 t^2\big). \end{cases}$$

Thus, we have

$$\begin{cases} \lambda|c_1| = (1-\lambda)T_1 N\big(c_1 t + c_2 t^2\big), \\ \lambda|c_2| = (1-\lambda)T_2 N\big(c_1 t + \delta_2 t^2\big). \end{cases}$$

Then, we get

$$\lambda\big(|\delta_1| + |\delta_2|\big) = (1-\lambda)\Big(T_1 N\big(\delta_1 t + \delta_2 t^2\big) + T_2 N\big(\delta_1 t + \delta_2 t^2\big)\Big) < 0.$$

By the first part of condition (H_6), we have $|\delta_1| \leq B, |\delta_2| \leq B$. Hence, Ω_3 is bounded.

Now, we proceed to show that all the conditions of Theorem 2.8 are satisfied. Let Ω be a bounded open set of X containing $\bigcup_{i=1}^{3} \overline{\Omega}_i$. By Lemma 3.3, N is L-compact on $\overline{\Omega}$. Because Ω_1 and Ω_2 are bounded sets, we have

(1) $Lu \neq \lambda Nu$ for each $(u,\lambda) \in \Big[\Big(domL \backslash KerL\Big) \cap \partial\Omega\Big] \times (0,1)$;
(2) $Nu \notin ImL$ for each $u \in KerL \cap \partial\Omega$.

To show that the condition (3) of Theorem 2.8 is satisfied, we define

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu,$$

and observe that, because Ω_3 is bounded, then we have

$$H(u,\lambda) \neq 0, \quad \forall u \in KerL \bigcap \partial\Omega.$$

Appealing to the homotopy property of the degree, we obtain

$$\begin{aligned} deg\Big(QN\mid_{kerL}, \Omega \bigcap KerL, 0\Big) &= deg\Big(H(\cdot,0), \Omega \bigcap KerL, 0\Big) \\ = deg\Big(H(\cdot,1), \Omega \bigcap KerL, 0\Big) &= deg\Big(\pm J, \Omega \bigcap KerL, 0\Big) \neq 0. \end{aligned}$$

Thus, the condition (3) of Theorem 2.8 is also satisfied.

Finally, we can apply Theorem 2.8, to conclude that the abstract equation $Lu = Nu$ has at least one solution in dom $\mathrm{L} \bigcap \overline{\Omega}$, and hence, the boundary value problem (1.1) has at least one solution in X. Theorem 3.4 is proved. $\square$

4. An example

To illustrate our main result, we discuss an example.

Example 4.1. *Let us consider the following fractional boundary value problem*

$$\begin{gathered}\Big(\phi(t)^C D_{0+}^{\frac{5}{2}} u(t)\Big)' = f\big(t, u(t), u'(t), u''(t), {}^C D_{0+}^{\frac{5}{2}} u(t)\big),\ t \in [0,1] \\ u(0) = {}^C D_{0+}^{\alpha} u(0) = 0,\ u''(0) = -u''\left(\frac{1}{3}\right) + 2u''\left(\frac{1}{6}\right), \\ u'(1) = -2u'\left(\frac{1}{4}\right) + 3u'\left(\frac{1}{2}\right).\end{gathered} \tag{4.1}$$

where $\phi(t) = e^{t-3}$ and

$$f\big(t, x_0, x_1, x_2, x_3\big) = x_2 + \cos x_3\big(1 - \sin x_1\big) + \sqrt{|x_2|}.$$

Now show that the conditions of Theorem 3.4 are fulfilled.

Corresponding to the notation of the problem (1.1), we have that $\alpha = \frac{5}{2},\ l = 2,\ m = 2,\ a_1 = -1,\ a_2 = 2,\ \xi_1 = \frac{1}{3},\ \xi_2 = \frac{1}{6},\ b_1 = -2,\ b_2 = 3,\ \eta_1 = \frac{1}{4},\ \eta_2 = \frac{1}{2},\ \mu = \min_{t\in I} \phi(t) = e^{-3} > 0$. Then we have $a_1 + a_2 = b_1 + b_2 = 1,\ b_1\eta_1 + b_2\eta_2 = 1$. Thus, the condition (H_1) is satisfied.

Also, we find

$$T_1(y) = -\int_0^{\frac{1}{3}} \left(\frac{1}{3} - s\right)^{-\frac{1}{2}} e^{3-s} \int_0^s y(r)drds + 2\int_0^{\frac{1}{6}} \left(\frac{1}{6} - s\right)^{-\frac{1}{2}} e^{3-s} \int_0^s y(r)drds,$$

$$\begin{aligned} T_2(y) = & \int_0^1 (1-s)^{\frac{1}{2}} e^{3-s} \int_0^s y(r)drds - 2\int_0^{\frac{1}{4}} \left(\frac{1}{4} - s\right)^{\frac{1}{2}} e^{3-s} \int_0^s y(r)drds \\ & + 3\int_0^{\frac{1}{2}} \left(\frac{1}{2} - s\right)^{\frac{1}{2}} e^{3-s} \int_0^s y(r)drds. \end{aligned}$$

By simple calculations, we get

$$\Delta(1,2) = \begin{vmatrix} \frac{-761}{993} & \frac{-301}{982} \\ \frac{1545}{311} & \frac{463}{431} \end{vmatrix} = \frac{263}{376} \neq 0,$$

Therefore, the condition (H_2) holds.

On the other hand, we have

$$\Big| f\big(t, x_0, x_1, x_2, x_3\big)\Big| \leq |x_2| + \sqrt{|x_2|} + 2.$$

It is easy to see that the condition (H_3) holds, where

$$\theta_0(t) = \theta_1(t) = \theta_3(t) = 0,\ \theta_2(t) = 1,\ \theta_4(t) = \frac{1}{2},\ \theta_5(t) = 2,\ \nu = \frac{1}{2}.$$

Next, we have

$$(\rho_1 + \rho_2) \sum_{i=0}^{3} \|\theta_i\|_{L^1} = e^{-3} \left(\frac{1}{\Gamma(3.5)} + \frac{3}{\Gamma(2.5)} + \frac{6}{\Gamma(1.5)} + 1 \right) = \frac{833}{1620} < 1.$$

Therefore, the condition (H_4^*) holds.

Let $A = 9$ and assume that $|u''(t)| > 9$ holds for all $t \in [0,1]$. Then, by the continuity of $u''(t)$, we have either $u''(t) > 9$ for all $t \in [0,1]$, or $u''(t) < -9$ for all $t \in [0,1]$. If $u''(t) > 9$, then for all $t \in [0,1]$ we obtain

$$\begin{aligned}
T_2(y) = & \int_0^1 (1-s)^{\frac{1}{2}} e^{3-s} \int_0^s \Big(u''(r) + \cos {}^C D_{0+}^{\alpha} u(r) \big(1 - \sin u'(r)\big) + \sqrt{|u''(r)|} \Big) dr ds \\
& - 2 \int_0^{\frac{1}{4}} \Big(\frac{1}{4} - s\Big)^{\frac{1}{2}} e^{3-s} \int_0^s \Big(u''(r) + \cos {}^C D_{0+}^{\alpha} u(r) \big(1 - \sin u'(r)\big) + \sqrt{|u''(r)|} \Big) dr ds \\
& + 3 \int_0^{\frac{1}{2}} \Big(\frac{1}{2} - s\Big)^{\frac{1}{2}} e^{3-s} \int_0^s \Big(u''(r) + \cos {}^C D_{0+}^{\alpha} u(r) \big(1 - \sin u'(r)\big) + \sqrt{|u''(r)|} \Big) dr ds. \\
& \geq 5 \int_0^1 s(1-s)^{\frac{1}{2}} e^{3-s} ds - 14 \int_0^{\frac{1}{4}} s\Big(\frac{1}{4} - s\Big)^{\frac{1}{2}} e^{3-s} ds + 15 \int_0^{\frac{1}{2}} s\Big(\frac{1}{2} - s\Big)^{\frac{1}{2}} e^{3-s} ds \\
& \geq \frac{7280}{257}.
\end{aligned}$$

If $u''(t) < -9$, then for all $t \in [0,1]$ we obtain

$$\begin{aligned}
T_2(y) = & \int_0^1 (1-s)^{\frac{1}{2}} e^{3-s} \int_0^s \Big(u''(r) + \cos {}^C D_{0+}^{\alpha} u(r) \big(1 - \sin u'(r)\big) + \sqrt{|u''(r)|} \Big) dr ds \\
& - 2 \int_0^{\frac{1}{4}} \Big(\frac{1}{4} - s\Big)^{\frac{1}{2}} e^{3-s} \int_0^s \Big(u''(r) + \cos {}^C D_{0+}^{\alpha} u(r) \big(1 - \sin u'(r)\big) + \sqrt{|u''(r)|} \Big) dr ds \\
& + 3 \int_0^{\frac{1}{2}} \Big(\frac{1}{2} - s\Big)^{\frac{1}{2}} e^{3-s} \int_0^s \Big(u''(r) + \cos {}^C D_{0+}^{\alpha} u(r) \big(1 - \sin u'(r)\big) + \sqrt{|u''(r)|} \Big) dr ds. \\
& \leq -4 \int_0^1 s(1-s)^{\frac{1}{2}} e^{3-s} ds + 14 \int_0^{\frac{1}{4}} s\Big(\frac{1}{4} - s\Big)^{\frac{1}{2}} e^{3-s} ds - 12 \int_0^{\frac{1}{2}} s\Big(\frac{1}{2} - s\Big)^{\frac{1}{2}} e^{3-s} ds \\
& \leq -\frac{12329}{544}.
\end{aligned}$$

So, the condition (H_5) is satisfied.

Let $B = 1$ and $c_1, c_2 \in \mathbb{R}$ be such that $|c_1| > 1$, $|c_2| > 1$. Then we have

$$T_1 N\big(c_1 t + c_2 t^2\big) + T_2 N\big(c_1 t + c_2 t^2\big) = \big(2|c_2| + \sqrt{2|c_2|}\big)(d_{11} + d_{12}) < 0.$$

So, the condition (H_6) is satisfied.

Thus, all the assumptions of Theorem 3.4 are satisfied, and hence, the problem (4.1) has at least one solution.

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SOME RESULTS ON THE PAINLEVÉ III DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

L. LIU, J. ZHANG

Beihang University, P. R. China[1]
E-mails: *buaazhum101@163.com; jlzhang@buaa.edu.cn*

Abstract. In this paper, we investigate the following two Painlevé III equations: $\overline{w}\underline{w}(w^2-1) = w^2+\mu$ and $\overline{w}\underline{w}(w^2-1) = w^2-\lambda w$, where $\overline{w} := w(z+1)$, $\underline{w} := w(z-1)$ and μ ($\mu \neq -1$) and $\lambda \notin \{\pm 1\}$ are constants. We discuss the equations of existence of rational solutions, of Borel exceptional values and the exponents of convergence of zeros, poles and fixed points of transcendental meromorphic solutions of these equations.

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Keywords: meromorphic solution; Painlevé difference equation; finite order.

1. Introduction

After the completion of the differential Nevanlinna theory, the value distribution of solutions of difference equations has received a considerable attention of a number of researchers. Halburd and Korhonen [1] abstracted the difference Painlevé II equation by using the value distribution theory. Chen and Shon [2] dealt with the properties of solutions of complex difference Riccati equations. It is an important discovery that difference Riccati equation plays an important role in the study of difference Painlevé equations.

We assume that the readers are familiar with the fundamental results and the standard notion of Nevanlinna's value distribution theory of meromorphic functions (see [3] – [5]).

Let w be a meromorphic function in the complex plane and let z be an arbitrary element in the complex plane. By $\rho(w)$, $\lambda(w)$ and $\lambda(1/w)$ we denote the order, the exponents of convergence of zeros and poles of w, respectively. The exponent of convergence of fixed points is defined by

$$\tau(w) = \limsup_{r\to\infty} \frac{\log N\left(r, \frac{1}{w-z}\right)}{\log r}.$$

The field of small functions of w is defined by

$$S(w) = \{\alpha \,\text{meromorphic} : T(r,\alpha) = S(r,w)\},$$

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where $S(r,w)$ is any quantity satisfying $S(r,w)=o\,(T(r,w))$ for all r outside a set of finite logarithmic measure. A meromorphic solution w is called *admissible* if all the coefficients of a difference equation are in the field $\mathcal{S}(w)$. For instance, all the non-rational meromorphic solutions of a difference equation which has only rational coefficients, are admissible.

Recently, Halburd and Korhonen [9], developing the Nevanlinna value distribution theory on difference expressions (see [6] – [8]), considered the following difference equation:

$$\overline{w}+\underline{w}=R(z,w), \tag{1.1}$$

where R is rational in w and is meromorphic in z with slow growth of coefficients. They proved that if the equation (1.1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or the equation (1.1) can be transformed to eight simple difference equations. These simple difference equations include the Painlevé I, II difference equations and some linear difference equations. We recall the family including Painlevé III difference equations.

Theorem A ([10]). *Assume that the equation:*

$$\overline{w}\underline{w}=R(z,w), \tag{1.2}$$

has an admissible meromorphic solution w of hyper-order less than one, where $R(z,w)$ is rational and irreducible in w and meromorphic in z. Then either w satisfies the following difference Riccati equation:

$$\overline{w}=\frac{\alpha w+\beta}{w+\gamma},$$

where α, β, $\gamma\in\mathcal{S}(w)$ are algebraic functions, or the equation (1.2) *can be transformed to one of the following equations:*

$$\overline{w}\underline{w}=\frac{\eta w^2-\lambda w+\mu}{(w-1)(w-\nu)}, \tag{1.3a}$$

$$\overline{w}\underline{w}=\frac{\eta w^2-\lambda w}{(w-1)}, \tag{1.3b}$$

$$\overline{w}\underline{w}=\frac{\eta(w-\lambda)}{(w-1)}, \tag{1.3c}$$

$$\overline{w}\underline{w}=hw^m. \tag{1.3d}$$

In (1.3a)*, the coefficients satisfy $\kappa^2\overline{\mu}\underline{\mu}=\mu^2$, $\overline{\lambda}\mu=\kappa\underline{\lambda}\overline{\mu}$, $\kappa\overline{\overline{\lambda}}\underline{\lambda}=\underline{\kappa}\lambda\overline{\lambda}$, and one of the following conditions:*

$$(1)\ \eta\equiv 1,\ \overline{\nu}\underline{\nu}=1,\ \kappa=\nu; \qquad (2)\ \overline{\eta}=\underline{\eta}=\nu,\ \kappa\equiv 1.$$

In (1.3b), $\eta\overline{\eta} = 1$ *and* $\overline{\overline{\lambda}}\underline{\lambda} = \lambda\overline{\lambda}$.

In (1.3c), *the coefficients satisfy one of the following conditions:*

(1) $\eta \equiv 1$, *and either* $\lambda = \overline{\lambda}\underline{\lambda}$ *or* $\overline{\lambda}^{[3]}\underline{\lambda}_{[3]} = \overline{\overline{\lambda}}\underline{\underline{\lambda}}$;

(2) $\overline{\lambda}\underline{\lambda} = \overline{\overline{\lambda}}\underline{\underline{\lambda}}$, $\overline{\eta}\overline{\lambda} = \overline{\overline{\lambda}}\underline{\eta}$, $\eta\underline{\eta} = \overline{\overline{\eta}}\underline{\eta}_{[3]}$;

(3) $\overline{\overline{\eta}}\underline{\underline{\eta}} = \eta\underline{\eta}$, $\lambda = \underline{\eta}$;

(4) $\overline{\lambda}^{[3]}\underline{\lambda}_{[3]} = \overline{\overline{\lambda}}\underline{\underline{\lambda}}\lambda, \eta\lambda = \overline{\overline{\eta}}\underline{\underline{\eta}}$.

In (1.3d), $h \in \mathcal{S}(w)$ *and* $m \in \mathbb{Z}$, $|m| \leq 2$.

The difference Painlevé III equations (1.3a)–(1.3d) have been studied recently by Zhang and Yang [11], and Zhang and Yi [12, 13], where a number of interesting results were obtained. In particular, Zhang and Yi [12] studied the following equation:

$$\overline{w}\underline{w}(w-1)^2 = w^2 - \lambda w + \mu, \tag{1.4}$$

where λ and μ are constants, and obtained the following two results.

Theorem B ([12]). *Let* $w(z) = \frac{P(z)}{Q(z)}$, *where* $P(z)$ *and* $Q(z)$ *are relatively prime polynomials of degrees* p *and* q, *respectively. If* $w(z)$ *is a solution of equation* (1.4), *then one of the following assertions holds:*

(i) $p = q$, $a^2(a-1)^2 = a^2 - \lambda a + \mu$, *where* $a = w(\infty)$;

(ii) $p < q$, $\lambda = \mu = 0$, *and* $P(z)$ *is a constant.*

Example 1.1. The rational function $w(z) = \frac{1}{(z+1)^2}$ is a solution of the difference equation $\overline{w}\underline{w}(w-1)^2 = w^2$. This shows that the conclusion (ii) of Theorem B may occur.

Theorem C ([12]). *If* w *is a transcendental meromorphic solution of equation* (1.4) *of finite order* $\rho(w)$, *then the following assertions hold:*

(i) $\tau(w) = \rho(w)$;

(ii) *If* $\lambda\mu \neq 0$, *then* $\lambda(w) = \rho(w)$.

Example 1.2. The function $w(z) = \sec^2 \frac{\pi z}{2}$ is a solution of the difference equation $\overline{w}\underline{w}(w-1)^2 = w^2$, and 0 is a Picard exceptional value of w. This shows that the condition $\lambda\mu \neq 0$ is necessary in assertion (ii) of Theorem C.

In this paper, motivated by the above theorems and equation (1.3a), we study two difference Painlevé III equations that follow. Observe first that if in equation (1.3a) of Theorem A, $\kappa = \nu = -1$ when both μ and λ are constants, then we have at least one of μ and λ to be 0 from $\overline{\lambda}\mu = \kappa\underline{\lambda}\overline{\mu}$. So, in Section 3, we discuss the question of existence of rational solutions of the following difference Painlevé III equation:

$$\overline{w}\underline{w}(w^2-1)=w^2+\mu, \tag{1.5}$$

where μ ($\mu \neq -1$) is a constant, and investigate the value distribution. In Section 4, we discuss the same questions, that is, the existence of rational solutions and the value distribution, of the following difference Painlevé III equation:

$$\overline{w}\underline{w}(w^2-1)=w^2-\lambda w, \tag{1.6}$$

where $\lambda(\lambda \neq \pm 1)$ is a constant.

The reminder of the paper is organized as follows. In Section 2, we state a number of auxiliary lemmas, which will be used to prove our main results. In Section 3, we study the equation (1.5). Section 4 is devoted to equation (1.6).

2. Auxiliary lemmas

In this section we state a number of auxiliary lemmas, which will be used to prove our main results. We first state the following lemma, which is a difference analogue of the logarithmic derivative lemma, and reads as follows.

Lemma 2.1. *Let f be a meromorphic function of finite order, and let c be a non-zero complex constant. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=S(r,f).$$

In view of Lemma 2.1, we can obtain the following difference analogues of the Clunie and Mohon'ko lemmas (see [7, 8]).

Lemma 2.2 ([8])**.** *Let f be a transcendental meromorphic solution of a finite order ρ for a difference equation of the form:*

$$U(z,f)P(z,f)=Q(z,f),$$

where $U(z,f)$, $P(z,f)$ and $Q(z,f)$ are difference polynomials such that the total degree $\deg_f U(z,f)=n$ in $f(z)$ and its shifts, and $\deg_f Q(z,f)\leq n$. If $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts, then, for each $\varepsilon>0$, we have

$$m\left(r,P(z,f)\right)=O\left(r^{\rho-1+\varepsilon}\right)+S(r,f),$$

possibly outside an exceptional set of a finite logarithmic measure.

Lemma 2.3 ([7, 8])**.** *Let w be a transcendental meromorphic solution of a finite order of the difference equation:*

$$P(z,w)=0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a)\not\equiv 0$ for a meromorphic function $a\in\mathcal{S}(w)$, then

$$m\left(r,\frac{1}{w-a}\right)=S(r,w).$$

Lemma 2.4 (See, e.g., [11, Theorem 3.1]). *Let w be a non-constant meromorphic solution of a finite order of equations (1.3a) – (1.3d) with constant coefficients, and let $m\neq 2$ in equation (1.3d). Then the following equalities hold:*

$$m(r,w)=S(r,w),\quad \lambda\left(\frac{1}{w}\right)=\rho(w).$$

We conclude this section by the following lemma.

Lemma 2.5 (See, e.g., [5, pp. 79–80]). *Let f_j $(j=1,\dots,n)$ $(n\geq 2)$ be meromorphic functions, and let g_j $(j=1,\dots,n)$ be entire functions. Assume that the following conditions are fulfilled:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)}\equiv 0$;

(ii) $g_h(z)-g_k(z)$ *is not a constant for* $1\leq h<k\leq n$;

(iii) $T(r,f_j)=S(r,e^{g_h(z)-g_k(z)})$ *for* $1\leq j\leq n$ *and* $1\leq h<k\leq n$.

Then $f_j(z)\equiv 0$, $j=1,\dots,n$.

3. Equation (1.5)

Theorem 3.1. *There is no any non-constant rational solution of equation (1.5).*

Proof. Assume the opposite that $w(z)=\frac{P(z)}{Q(z)}$ is a non-constant rational solution of equation(1.5), where $P(z)$ and $Q(z)$ are relatively prime polynomials of degrees p and q, respectively. Also, we assume that the leading coefficient of $P(z)$ is a $(a\neq 0)$ and the leading coefficient of $Q(z)$ is 1. Substituting $w(z)=\frac{P(z)}{Q(z)}$ into (1.5), we get

$$\frac{P(z+1)}{Q(z+1)}\frac{P(z-1)}{Q(z-1)}\left(\left(\frac{P(z)}{Q(z)}\right)^2-1\right)=\left(\frac{P(z)}{Q(z)}\right)^2+\mu. \tag{3.1}$$

We set $s=p-q$, and discuss the following three possible cases.

Case 1. Let $s>0$. Then $\frac{P(z)}{Q(z)}=az^s(1+o(1))$ as z tends to infinite and from (3.1), we get

$$a^2(z+1)^s(z-1)^s(1+o(1))\left(a^2z^{2s}(1+o(1))-1\right)=a^2z^{2s}(1+o(1))+\mu,$$

which is a contradiction as z tends to infinite.

Case 2. Let $s<0$. Now we have $\frac{P(z)}{Q(z)}=o(1)$ and $\frac{P(z+1)}{Q(z+1)}=o(1)$ as z tends to infinite. By (3.1), we obtain $\mu=0$. From (1.5), when $\mu=0$, we have

$$\overline{w}\underline{w}=\frac{w^2}{w^2-1}.$$

Let $w(z) = \frac{1}{f(z)}$. Substituting $w = \frac{1}{f}$ into the above equation, we obtain

$$\overline{f}\underline{f} = 1 - f^2.$$

Observing that the coefficients on the left- and right-hand sides of the above equation are $\frac{1}{a^2}$ and $-\frac{1}{a^2}$, respectively, we get $\frac{2}{a^2} = 0$, which is impossible.

Case 3. Let $s = 0$. Then $w(z) = \frac{P(z)}{Q(z)} = a + o(1)$ as z tends to infinity and from (3.1), we get

$$a^2(a^2 - 1) = a^2 + \mu, \tag{3.2}$$

where $a \notin \{0, \pm 1\}$. We rewrite (3.1) as follows:

$$\frac{P(z+1)}{Q(z+1)} \frac{P(z-1)}{Q(z-1)} = \frac{P^2(z) + \mu Q^2(z)}{P^2(z) - Q^2(z)}.$$

We assume that there is a point z_0 such that $P^2(z_0) + \mu Q^2(z_0) = 0$ and $P^2(z_0) - Q^2(z_0) = 0$. Since $\mu \neq -1$, we obtain $P(z_0) = 0$ and $Q(z_0) = 0$, which is a contradiction. Thus, the degrees of $P^2(z) + \mu Q^2(z)$ and $P^2(z) - Q^2(z)$ both are $2p$, and we have

$$(a^2 + \mu)\overline{P}\underline{P} = a^2(P^2 + \mu Q^2), \tag{3.3}$$

$$(a^2 - 1)\overline{Q}\underline{Q} = P^2 - Q^2. \tag{3.4}$$

Next, we assume $P = ar$, $p = n$. Then from (3.3) we have that

$$\mu Q^2 = \overline{r}\underline{r}\left(a^2 + \mu\right) - r^2 a^2, \tag{3.5}$$

where

$$r = z^n + A_{n-1}z^{n-1} + A_{n-2}z^{n-2} + A_{n-3}z^{n-3} + \cdots + A_1 z + A_0, \tag{3.6}$$

$$Q = z^n + B_{n-1}z^{n-1} + B_{n-2}z^{n-2} + B_{n-3}z^{n-3} + \cdots + B_1 z + B_0. \tag{3.7}$$

We rewrite (3.4) as follows:

$$(a^2 - 1)\overline{Q}\underline{Q} + Q^2 = P^2. \tag{3.8}$$

Substituting (3.6) and (3.7) into (3.5) and comparing the coefficients of terms z^{2n}, z^{2n-1}, z^{2n-2}, we obtain the following two equations:

$$\begin{aligned} \mu(B_{n-1} - A_{n-1}) &= 0, \\ \mu(B_{n-1}^2 + 2B_{n-2}) &= \mu(A_{n-1}^2 + 2A_{n-2} - n) - a^2 n. \end{aligned}$$

If $\mu = 0$, then from the last equation we get $a^2 n = 0$, which is a contradiction. If $\mu \neq 0$, then the last two equations become

$$B_{n-1} = A_{n-1}, \quad B_{n-2} = A_{n-2} - \frac{n\left(a^2 + \mu\right)}{2\mu}.$$

By the same way, we substitute (3.6) and (3.7) into (3.8), and compare the coefficients of terms z^{2n}, z^{2n-1}, z^{2n-2}, to obtain

$$B_{n-1} = A_{n-1}, \quad B_{n-2} = A_{n-2} - \frac{n\left(1-a^2\right)}{2a^2}.$$

So, we get $a^4 = \mu(1-2a^2)$. On the other hand, from (3.2) we have $\mu = a^2\left(a^2-2\right)$. It is obvious that $a^2 = 1$, which is a contradiction. □

Theorem 3.2. *If w is a transcendental meromorphic solution of equation* (1.5) *of a finite order $\rho(w) > 0$, then the following assertions hold:*

(i) $\lambda\left(\frac{1}{w}\right) = \tau(w) = \rho(w)$;

(ii) *when $\mu \neq 0$, we have $\lambda(w) = \rho(w)$;*

(iii) *w has at most one non-zero Borel exceptional value.*

Proof. Denote $\phi(z) = w(z) - z$, and observe that $\phi(z)$ is a transcendental meromorphic function and $T(r,\phi) = T(r,w) + S(r,w)$. Substituting $w(z) = \phi(z) + z$ into (1.5), we obtain

$$\left(\overline{\phi} + z + 1\right)\left(\underline{\phi} + z - 1\right)\left((\phi+z)^2 - 1\right) = (\phi+z)^2 + \mu.$$

Denote

$$P(z,\phi) = \left(\overline{\phi} + z + 1\right)\left(\underline{\phi} + z - 1\right)\left((\phi+z)^2 - 1\right) - (\phi+z)^2 - \mu,$$

and observe that $P(z,0) = \left(z^2-1\right)^2 - z^2 - \mu \not\equiv 0$. From Lemma 2.3, we get

$$m\left(r, \frac{1}{w-z}\right) = m\left(r, 1/\phi\right) = S(r,\phi),$$

implying that $N\left(r, \frac{1}{w-z}\right) = T\left(r,w\right) + S(r,w)$, and hence $\tau(w) = \rho(w)$.

In view of Lemma 2.4 we have $m(r,w) = S(r,w)$. Then, the equality $\lambda\left(\frac{1}{w}\right) = \rho(w)$ holds.

To prove the assertion (ii), for $\mu \neq 0$, we denote

$$P_1(z,w) = \overline{w}\underline{w}(w^2-1) - w^2 - \mu,$$

and observe that $P_1(z,0) = -\mu \not\equiv 0$. Then, from Lemma 2.3, we obtain $m(r,1/w) = S(r,w)$, implying that $\lambda(w) = \rho(w)$.

Now we proceed to prove the assertion (iii) of the theorem. To this end, we assume that a and b are two non-zero finite Borel exceptional values of w, and set

$$f(z) = \frac{w(z)-a}{w(z)-b}. \tag{3.9}$$

Then, we have $\rho(f)=\rho(w)$, $\lambda(f)=\lambda(w-a)<\rho(f)$ and $\lambda(1/f)=\lambda(w-b)<\rho(f)$. Since f is of finite order, we suppose that

$$f(z)=g(z)e^{dz^n}, \tag{3.10}$$

where d ($d\neq 0$) is a constant, n ($n\geq 1$) is an integer, and $g(z)$ is a meromorphic function satisfying the condition:

$$\rho(g)<\rho(f)=n. \tag{3.11}$$

Then, we have

$$f(z+1)=g(z+1)g_1(z)e^{dz^n}, \quad f(z-1)=g(z-1)g_2(z)e^{dz^n}, \tag{3.12}$$

where $g_1(z)=e^{ndz^{n-1}+\cdots+d}$ and $g_2(z)=e^{-ndz^{n-1}+\cdots+(-1)^n d}$. From (3.9) we get $w=\dfrac{bf-a}{f-1}$. Next, in view of (1.5), (3.9) to (3.12), we can write

$$A(z)e^{4dz^n}+B(z)e^{3dz^n}+C(z)e^{2dz^n}+D(z)e^{dz^n}+E=0, \tag{3.13}$$

where

$$\begin{aligned}
A(z) &= \left[b^4-2b^2-\mu\right]g^2\overline{g}g_1\underline{g}g_2,\\
B(z) &= \left[-2b^2(ab-1)+2ab+2\mu\right]g\overline{g}g_1\underline{g}g_2\\
&\quad+\left[-ab(b^2-1)+b^2+\mu\right]g^2(\overline{g}g_1+\underline{g}g_2),\\
C(z) &= \left[b^2(a^2-1)-a^2-\mu\right]\overline{g}\underline{g}g_1g_2+\left[a^2b^2-a^2-b^2-\mu\right]g^2\\
&\quad-\left[-2ab(ab-1)+2ab+2\mu\right]g(\overline{g}g_1+\underline{g}g_2),\\
D(z) &= \left[-a^3b+ab+a^2+\mu\right](\overline{g}g_1+\underline{g}g_2)+2(-a^3b+a^2+ab+\mu)g,\\
E &= a^4-2a^2-\mu.
\end{aligned}$$

Applying Lemma 2.5 to (3.13) and taking into account (3.11), we see that all the coefficients vanish. Since a and b are non-zero constants, we deduce from $A(z)=0$ and $E=0$ that

$$a^4-2a^2=\mu,\ \ b^4-2b^2=\mu. \tag{3.14}$$

Then, we have $(a^2-b^2)(a^2+b^2-2)=0$. Now we discuss the following two cases.

Case 1. Let $a^2=b^2$. Due to $a\neq b$, we get $a=-b$. Denote $G=g$, $G_1=\overline{g}g_1$ and $G_2=\underline{g}g_2$. From $B(z)=0$, $D(z)=0$, we have

$$\begin{gathered}
2(b^4+\mu)G_1G_2=(-b^4-\mu)G(G_1+G_2),\\
2(a^4+\mu)G=(-a^4-\mu)(G_1+G_2).
\end{gathered}$$

Noting that $\mu\neq -1$, we get $b^4+\mu\neq 0$ and $a^4+\mu\neq 0$ by (3.14). Thus, we have

$$2G_1G_2=-G(G_1+G_2),\ \ 2G=-(G_1+G_2).$$

From the last two equations, we obtain

$$G^2 = G_1G_2,\ 4G_1G_2 = (G_1 + G_2)^2.$$

So, we have $-G = G_1 = G_2$ and $\overline{f} = \underline{f} = -f$. From (3.9), the equality $a = -b$ and the above equation, we get

$$\overline{w} = \underline{w} = \frac{a^2}{w}.$$

Hence, from (1.5) we get $a^4(w^2 - 1) = w^4 + \mu w^2$. Therefore, w is a constant, which is a contradiction.

Case 2. Let $a^2 + b^2 = 2$. When $B(z) = 0$ and $D(z) = 0$, then using arguments similar to those applied in Case 1, we get

$$2G_1G_2 = -G(G_1 + G_2),\ 2G = -(G_1 + G_2).$$

Noting that $\mu \neq -1$, the above equations also lead to a contradiction by the similar reasoning as in Case 1. This completes the proof of the theorem. □

4. Equation (1.6)

Theorem 4.1. *Let $w(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are relatively prime polynomials of degrees p and q, respectively. If $w(z)$ is a non-constant rational solution of equation (1.6), then*

$$p = q,\ a(a^2 - 1) = a - \lambda,\ \text{where } a = \pm\frac{\sqrt{6}}{3},\ \lambda = \frac{4a}{3}.$$

Proof. For $p \neq q$, the proof of the theorem is similar to that of Cases 1 and 2 in Theorem 3.1, so we only prove the theorem for $p = q$. We assume that the leading coefficient of $P(z)$ is a $(a \neq 0)$, and the leading coefficient of $Q(z)$ is 1. Substituting $w(z) = \frac{P(z)}{Q(z)}$ into (1.6), we get

$$\frac{P(z+1)}{Q(z+1)}\frac{P(z-1)}{Q(z-1)}\left(\left(\frac{P(z)}{Q(z)}\right)^2 - 1\right) = \left(\frac{P(z)}{Q(z)}\right)^2 - \lambda\frac{P(z)}{Q(z)}. \tag{4.1}$$

When $p = q$, we have $\frac{P(z)}{Q(z)} = a + o(1)$ and $\frac{P(z+1)}{Q(z+1)} = a + o(1)$ as z tends to infinite. Then, from (4.1) we get the following equation

$$a(a^2 - 1) = a - \lambda, \tag{4.2}$$

where $a \notin \{0, \pm 1\}$.

We rewrite (4.1) as follows:

$$\frac{P(z+1)}{Q(z+1)}\frac{P(z-1)}{Q(z-1)} = \frac{P^2(z) - \lambda P(z)Q(z)}{P^2(z) - Q^2(z)}.$$

Arguments, similar to those applied in the proof of Theorem 3.1 (Case 3), can be used to conclude that the degrees of $P^2(z) - \lambda P(z)Q(z)$ and $P^2(z) - Q^2(z)$ both are $2p$ for $\lambda \neq \pm 1$. Hence, we have

$$(a^2 - \lambda a)\overline{P}\underline{P} = a^2(P^2 - \lambda PQ), \tag{4.3}$$

$$(a^2 - 1)\overline{Q}\underline{Q} = P^2 - Q^2. \tag{4.4}$$

Next, we assume $P = ar$, $p = n$, and use (4.3) to obtain

$$\lambda rQ = \overline{r}\underline{r}\,(\lambda - a) + ar^2, \tag{4.5}$$

where

$$r = z^n + A_{n-1}z^{n-1} + A_{n-2}z^{n-2} + A_{n-3}z^{n-3} + \cdots + A_1 z + A_0, \tag{4.6}$$

$$Q = z^n + B_{n-1}z^{n-1} + B_{n-2}z^{n-2} + B_{n-3}z^{n-3} + \cdots + B_1 z + B_0. \tag{4.7}$$

We rewrite (4.4) as follows:

$$(a^2 - 1)\overline{Q}\underline{Q} + Q^2 = P^2. \tag{4.8}$$

Substituting (4.6) and (4.7) into (4.5) and comparing the coefficients of terms z^{2n}, z^{2n-1}, z^{2n-2}, we obtain the following two equations:

$$\lambda(B_{n-1} - A_{n-1}) = 0,$$
$$\lambda(B_{n-2} + A_{n-1}B_{n-1} + A_{n-2}) = \lambda(A_{n-1}^2 + 2A_{n-2} - n) + an.$$

For $\lambda = 0$, from the last equation we get $an = 0$, which is a contradiction. For $\lambda \neq 0$, the last two equations become

$$B_{n-1} = A_{n-1}, \quad B_{n-2} = A_{n-2} + \frac{n\,(a - \lambda)}{\lambda}.$$

By the same way, we substitute (4.6) and (4.7) into (4.8), and compare the coefficients of terms z^{2n}, z^{2n-1}, z^{2n-2}, to obtain

$$B_{n-1} = A_{n-1}, \quad B_{n-2} = A_{n-2} + \frac{n\left(a^2 - 1\right)}{2a^2}.$$

So, we get $2a^3 = \lambda(3a^2 - 1)$. And from (4.2), we have $\lambda = 2a - a^3$. By the above equations, we have $\left(3a^2 - 2\right)\left(a^2 - 1\right) = 0$. Since $a^2 \neq 1$, we get $a = \pm\frac{\sqrt{6}}{3}$ and $\lambda = \pm\frac{4\sqrt{6}}{9}$. Therefore $\dfrac{\lambda}{a} = \dfrac{4}{3}$. $\square$

Theorem 4.2. *If w is a transcendental meromorphic solution of equation* (1.6) *of a finite order $\rho(w) > 0$, then the following assertions hold:*

(i) $\lambda\left(\dfrac{1}{w}\right) = \tau(w) = \rho(w)$;

(ii) *when $\lambda \neq 0$, we have $\lambda(w) = \rho(w)$;*

(iii) *w has at most one non-zero Borel exceptional value.*

Proof. Denote $\phi(z) = w(z) - z$, and observe that $\phi(z)$ is a transcendental meromorphic function and $T(r,\phi) = T(r,w) + S(r,w)$. Substituting $w(z) = \phi(z) + z$ into (1.6), we obtain $\left(\overline{\phi} + z + 1\right)\left(\underline{\phi} + z - 1\right)\left((\phi+z)^2 - 1\right) = (\phi+z)^2 - \lambda(\phi+z)$. Denote

$$P(z,\phi) = \left(\overline{\phi} + z + 1\right)\left(\underline{\phi} + z - 1\right)\left((\phi+z)^2 - 1\right) - (\phi+z)^2 + \lambda(\phi+z),$$

and observe that $P(z,0) = \left(z^2-1\right)^2 - z^2 + \lambda z \not\equiv 0$. Then, from Lemma 2.3, we obtain

$$m\left(r, \frac{1}{w-z}\right) = m\left(r, 1/\phi\right) = S(r,\phi),$$

implying that $N\left(r, \dfrac{1}{w-z}\right) = T(r,w) + S(r,w)$, and hence $\tau(w) = \rho(w)$.

We deduce from Lemma 2.4 that $m(r,w) = S(r,w)$. Then, the equality $\lambda\left(\dfrac{1}{w}\right) = \rho(w)$ holds.

To prove the assertion (ii), for $\lambda \neq 0$, we rewrite (1.6) as follows:

$$\overline{w}\underline{w} = \frac{w^2 - \lambda w}{w^2 - 1}.$$

Let $w(z) = \frac{1}{f(z)}$. Substituting $w = \frac{1}{f}$ into the last equality, we get

$$\overline{f}\underline{f}f\lambda = \overline{f}\underline{f} - 1 + f^2.$$

From Lemma 2.2, we obtain $m(r,1/w) = S(r,w)$. Therefore, $\lambda(w) = \rho(w)$.

Now we proceed to prove the assertion (iii) of the theorem. To this end, we assume that a and b are two non-zero finite Borel exceptional values of w, and set

$$f(z) = \frac{w(z) - a}{w(z) - b}. \tag{4.9}$$

Then, we have $\rho(f) = \rho(w)$, $\lambda(f) = \lambda(w-a) < \rho(f)$ and $\lambda(1/f) = \lambda(w-b) < \rho(f)$. Since f is of finite order, we suppose that

$$f(z) = g(z)e^{dz^n}, \tag{4.10}$$

where d ($d \neq 0$) is a constant, n ($n \geq 1$) is an integer, and $g(z)$ is a meromorphic function satisfying the condition:

$$\rho(g) < \rho(f) = n. \tag{4.11}$$

Then, we have

$$f(z+1) = g(z+1)g_1(z)e^{dz^n}, \quad f(z-1) = g(z-1)g_2(z)e^{dz^n}, \tag{4.12}$$

where $g_1(z) = e^{ndz^{n-1} + \cdots + d}$ and $g_2(z) = e^{-ndz^{n-1} + \cdots + (-1)^n d}$. From (4.9), we get $w = \dfrac{bf - a}{f - 1}$. In view of (1.6), (4.9) to (4.12), we can write

$$A(z)e^{4dz^n} + B(z)e^{3dz^n} + C(z)e^{2dz^n} + D(z)e^{dz^n} + E = 0, \tag{4.13}$$

where

$$\begin{aligned}
A(z) &= \left[b^4 - 2b^2 + b\lambda\right] g^2 \overline{g} g_1 \underline{g} g_2,\\
B(z) &= \left[-2b^2(ab-1) + 2ab - \lambda(a+b)\right] g\overline{g} g_1 \underline{g} g_2\\
&\quad + \left[-ab(b^2-1) + b(b-\lambda)\right] g^2(\overline{g} g_1 + \underline{g} g_2),\\
C(z) &= \left[b^2(a^2-1) - a^2 + a\lambda\right] \overline{g}\underline{g} g_1 g_2 + \left[a^2b^2 - a^2 - b^2 + b\lambda\right] g^2\\
&\quad + \left[2ab(ab-1) - 2ab + \lambda(a+b)\right] g(\overline{g} g_1 + \underline{g} g_2),\\
D(z) &= \left[-2a^3b + 2ab + 2a^2 - \lambda(a+b)\right] g + (-a^3b + a^2 + ab - a\lambda)(\overline{g} g_1 + \underline{g} g_2),\\
E &= a^4 - 2a^2 + a\lambda.
\end{aligned}$$

Applying Lemma 2.5 to (4.13) and taking into account (4.11), we see that all the coefficients vanish. Since a and b are non-zero constants, we deduce from $A(z) = 0$ and $E = 0$ that

$$a^3 - 2a = -\lambda,\ b^3 - 2b = -\lambda. \tag{4.14}$$

Then, we have $(a-b)(a^2+ab+b^2-2) = 0$. Since $a \neq b$, it follows that $a^2+b^2+ab = 2$. By (4.14), a and b are distinct zeros of the equation $z^3 - 2z + \lambda = 0$.

According to the algebraic basic theorem, the above equation has three solutions. Denoting by x the third solution, and using the relationship between roots and coefficients, we obtain $abx = -\lambda$, $ab + ax + bx = -2$, $a + b + x = 0$, implying that

$$x = -\frac{\lambda}{ab},\ a + b = -x = \frac{\lambda}{ab},\ ab + (a+b)x = ab - \frac{\lambda^2}{a^2b^2} = -2.$$

So, we have

$$ab(a+b) = \lambda,\ 2ab + a^2b^2 = (a+b)\lambda, a^2 + b^2 + ab = 2.$$

Denote $G = g$, $G_1 = \overline{g} g_1$ and $G_2 = \underline{g} g_2$. From $B(z) = 0$, $D(z) = 0$ and the above equations, we have

$$\begin{aligned}
(2b^2 - 2ab^3 - a^2b^2)G_1G_2 &= (2ab^3 + a^2b^2 - ab - b^2)G(G_1 + G_2),\\
(2a^2 - 2a^3b - a^2b^2)G &= (2a^3b + a^2b^2 - ab - a^2)(G_1 + G_2).
\end{aligned}$$

Because

$$\frac{G_1G_2}{G(G_1+G_2)} = \frac{2ab^3 + a^2b^2 - ab - b^2}{2b^2 - 2ab^3 - a^2b^2} = \frac{b^2 - ab}{2b^2 - 2ab^3 - a^2b^2} - 1.$$

By $a^2 + b^2 + ab = 2$, we gain $2b^2 - 2ab^3 - a^2b^2 = b^3(b-a)$, and hence, we have

$$\frac{G_1G_2}{G(G_1+G_2)} = \frac{1}{b^2} - 1.$$

Thus, we get

$$G_1G_2 = \left(\frac{1}{b^2} - 1\right) G(G_1 + G_2),\ G = \left(\frac{1}{a^2} - 1\right)(G_1 + G_2).$$

Noting that $\lambda \neq \pm 1$, by (4.14), we get $a^2 \neq 1$ and $b^2 \neq 1$. Moreover, since the last two equations are homogeneous, there exist two non-zero constants α and β, such that $G_1 = \alpha G$ and $G_2 = \beta G$. Then, we have

$$\alpha\beta = \frac{a^2 - a^2b^2}{b^2 - a^2b^2}. \tag{4.15}$$

On the other hand, combining (4.10) and (4.12), we get $\overline{f} = \alpha f$, $\underline{f} = \beta f$, which yields $\alpha\beta = 1$. Thus, by (4.15), we have $a^2 = b^2$. When $a = b$, then we get a contradiction. So, we have only to consider the case $a = -b$. From $B(z) = 0$, $D(z) = 0$ and $a = -b$, we have

$$\begin{aligned} 2b^4G_1G_2 &= (-b^4 + b\lambda)G(G_1 + G_2), \\ 2a^4G &= (-a^4 + a\lambda)(G_1 + G_2), \end{aligned} \tag{4.16}$$

implying that

$$(-b^4 - b\lambda)G_1G_2 = (-b^4 + b\lambda)G^2. \tag{4.17}$$

Since the last equation is homogeneous, there exist two non-zero constants α and β, such that $G_1 = \alpha G$ and $G_2 = \beta G$. Then, we have

$$\alpha\beta(b^3 + \lambda) = b^3 - \lambda. \tag{4.18}$$

On the other hand, combining (4.11) and (4.13), we get $\overline{f} = \alpha f$, $\underline{f} = \beta f$, which yields $\alpha\beta = 1$. Thus by (4.18), we have $\lambda = 0$, and, in view of (4.16) and (4.17), we infer that $2G = -(G_1 + G_2)$ and $G_1G_2 = G^2$. Then, $G_1 = G_2 = -G$. Thus, we have $\alpha = \beta = -1$ and $\overline{f} = \underline{f} = -f$, and by the similar reasoning as in Case 1 of the proof of Theorem 3.1, we get a contradiction. □

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SOME INEQUALITIES FOR RATIONAL FUNCTIONS WITH FIXED POLES

A. MIR

University of Kashmir, Srinagar, India
E-mail: *drabmir@yahoo.com*

Abstract. By using lemmas of Dubinin and Osserman some results for rational functions with fixed poles and restricted zeros are proved. The obtained results strengthen some known results for rational functions and, in turn, produce refinements of some polynomial inequalities as well.

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1. Introduction

Let $\mathbb{P}_n$ denote the class of all complex polynomials of degree at most n. If $P \in \mathbb{P}_n$, then concerning the estimate of $|P'(z)|$ on $|z| = 1$, we have

$$|P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality (1.1) is a famous result due to Bernstein [3]. It is worth mentioning that in (1.1) equality holds if and only if $P(z)$ has all its zeros at the origin. So, it is natural to seek improvements under appropriate assumption on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials $P(z)$ having no zeros in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

whereas, if $P(z)$ has no zeros in $|z| > 1$, then by

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

The inequality (1.2) was conjectured by Erdös and later it was verified by Lax [7], whereas the inequality (1.3) is due to Turán [10].

Jain [6] had used a parameter β and proved an interesting generalization of (1.3). More precisely, Jain proved that if $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \le 1$, then for every β with $|\beta| \le 1$, we have

$$\max_{|z|=1} \left|zP'(z) + \frac{n\beta}{2} P(z)\right| \ge \frac{n}{2}\{1 + Re(\beta)\} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Li, Mohapatra and Rodriguez [12] gave a new perspective to inequalities (1.1) – (1.3), and extended them to rational functions with fixed poles. Essentially, in these inequalities they replaced the polynomial $P(z)$ by a rational function $r(z)$ with poles $a_1, a_2, ..., a_n$ all lying in $|z| > 1$, and z^n was replaced by a Blaschke product $B(z)$. Before proceeding towards their results, we first introduce the set of rational functions involved.

For $a_j \in \mathbb{C}$ with $j = 1, 2, ..., n$, we define

$$W(z) = \prod_{j=1}^{n}(z - a_j);\ B(z) = \prod_{j=1}^{n}\left(\frac{1 - \overline{a}_j z}{z - a_j}\right) = \frac{W^*(z)}{W(z)},$$

where

$$W^*(z) = z^n \overline{W\left(\frac{1}{\overline{z}}\right)}$$

and

$$\mathbb{R}_n = \mathbb{R}_n(a_1, a_2, ..., a_n) = \left\{\frac{P(z)}{W(z)} :\ P \in \mathbb{P}_n\right\}.$$

Then $\mathbb{R}_n$ is defined to be the set of rational functions with poles $a_1, a_2, ..., a_n$ at most and with finite limit at ∞. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. Also, for $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$, the conjugate transpose r^* of r is defined by $r^*(z) = B(z)\overline{r(\frac{1}{\overline{z}})}$.

In the past few years several papers pertaining to Bernstein-type inequalities for rational functions have appeared in the study of rational approximations (see [2], [4], [11] – [13]). For $r \in \mathbb{R}_n$, Li, Mohapatra and Rodriguez [12] proved the following, similar to (1.1), inequality for rational functions:

$$|r'(z)| \leq |B'(z)| \max_{|z|=1} |r(z)|. \tag{1.5}$$

As extensions of (1.2) and (1.3) to rational functions, Li, Mohapatra and Rodriguez also showed that if $r \in \mathbb{R}_n$, and $r(z) \neq 0$ in $|z| < 1$, then for $|z| = 1$,

$$|r'(z)| \leq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|, \tag{1.6}$$

whereas, if $r \in \mathbb{R}_n$ has exactly n zeros in $|z| \leq 1$, then for $|z| = 1$,

$$|r'(z)| \geq \frac{|B'(z)|}{2} |r(z)|. \tag{1.7}$$

Very recently, Wali and Shah [13] proved an interesting refinement of (1.7). Namely, they proved that if $r \in \mathbb{R}_n$, and r has exactly n zeros in $|z| \leq 1$, where $r(z) = \frac{P(z)}{W(z)}$, with $P(z) = \sum_{j=0}^{n} c_j z^j$, then for $|z| = 1$,

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{\sqrt{|c_n|} - \sqrt{|c_0|}}{\sqrt{|c_n|}}\right\}|r(z)|. \tag{1.8}$$

In this paper, we establish some results for rational functions $r(z) = \frac{P(z)}{W(z)}$ with restricted zeros, where $P(z) = \sum_{j=0}^{n} c_j z^j$, by involving some coefficients of $P(z)$.

Our results strengthen some known inequalities for rational functions and, in turn, produce refinements of some polynomial inequalities as well.

2. Main results

In what follows we shall always assume that all the poles $a_1, a_2, ..., a_n$ of $r(z)$ lie in $|z| > 1$. In the case where all poles are in $|z| < 1$, we can obtain analogous results with suitable modifications.

Theorem 2.1. *Suppose that $r \in \mathbb{R}_n$, and all the n zeros of r lie in $|z| \le 1$. If $r(z) = \frac{P(z)}{W(z)}$, where $P(z) = \sum_{j=0}^{n} c_j z^j$, then for every β with $|\beta| \le 1$ and $|z| = 1$, we have*

$$\left| zr'(z) + \frac{n\beta}{2} r(z) \right| \ge \frac{1}{2}\left\{ |B'(z)| + nRe(\beta) + \frac{|c_n| - |c_0|}{|c_n| + |c_0|} \right\} |r(z)|. \tag{2.1}$$

The result is best possible in the case $\beta = 0$, and in (2.1) equality holds for $r(z) = B(z) + \lambda$ with $|\lambda| = 1$.

We first discuss some consequences of Theorem 2.1. If we take $\alpha_j = \alpha$, $|\alpha| \ge 1$, for $j = 1, 2, ..., n$, then $W(z) = (z - \alpha)^n$ and $r(z) = \frac{P(z)}{(z-\alpha)^n}$, and hence we have

$$\begin{aligned} r'(z) &= \frac{(z-\alpha)^n P'(z) - n(z-\alpha)^{n-1} P(z)}{(z-\alpha)^{2n}} \\ &= -\left\{ \frac{nP(z) - (z-\alpha)P'(z)}{(z-\alpha)^{n+1}} \right\} = \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}, \end{aligned}$$

where $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ is the polar derivative of $P(z)$ with respect to point α. It generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Also, $W^*(z) = (1 - \overline{\alpha} z)^n$, which gives $B(z) = \left(\frac{1-\overline{\alpha}z}{z-\alpha} \right)^n$, implying that

$$B'(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2 - 1)}{(z-\alpha)^{n+1}}.$$

With this choice, from (2.1) for $|z| = 1$, we get

$$\begin{aligned} &\left| zD_\alpha P(z) + \frac{n\beta}{2}(\alpha - z)P(z) \right| \\ &\qquad \ge \frac{1}{2}\left\{ \frac{n(|\alpha|^2 - 1)}{|z-\alpha|} + nRe(\beta)|z-\alpha| + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}|z-\alpha| \right\} |P(z)| \\ &\qquad \ge \frac{1}{2}\left\{ \frac{n(|\alpha|^2 - 1)}{|\alpha| + 1} + nRe(\beta)(|\alpha| - 1) + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}(|\alpha| - 1) \right\} |P(z)| \\ &\qquad = \frac{|\alpha| - 1}{2}\left\{ n(1 + Re(\beta)) + \frac{|c_n| - |c_0|}{|c_n| + |c_0|} \right\} |P(z)|. \end{aligned}$$

Thus, from Theorem 2.1 we immediately get the following result.

Corollary 2.1. *If $P(z)=\sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then for every $\alpha,\ \beta\in\mathbb{C}$ with $|\alpha|\ge 1$ and $|\beta|\le 1$, we have*

$$\max_{|z|=1}\left|zD_\alpha P(z)+\frac{n\beta}{2}(\alpha-z)P(z)\right|$$
$$\ge\frac{|\alpha|-1}{2}\left\{n(1+Re(\beta))+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}\max_{|z|=1}|P(z)|. \tag{2.2}$$

Remark 2.1. *Since $|c_n|\ge|c_0|$ and hence for $\beta=0$, the above corollary provides an improvement of a result due to Shah* [9].

Remark 2.2. *Dividing both sides of* (2.2) *by $|\alpha|$ and letting $|\alpha|\to\infty$, we obtain the following result, which as a special case, gives a strengthening of the classical Turán inequality* [10].

Corollary 2.2. *If $P(z)=\sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z|\le 1$. Then for every $\beta\in\mathbb{C}$ with $|\beta|\le 1$, we have*

$$\max_{|z|=1}\left|zP'(z)+\frac{n\beta}{2}P(z)\right|\ge\frac{1}{2}\left\{n(1+Re(\beta))+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}\max_{|z|=1}|P(z)|. \tag{2.3}$$

Remark 2.3. *The above inequality for $\beta=0$ was also independently proved by Dubinin* [5]. *Also, it is easy to see that the inequality* (2.3) *improves the inequality* (1.4) *as well.*

Taking $\beta=0$ in Theorem 2.1, we get the following result.

Corollary 2.3. *Suppose $r\in\mathbb{R}_n$, and all the n zeros of r lie in $|z|\le 1$. If $r(z)=\frac{P(z)}{W(z)}$, where $P(z)=\sum_{j=0}^{n} c_j z^j$, then for $|z|=1$ we have*

$$|r'(z)|\ge\frac{1}{2}\left\{|B'(z)|+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}|r(z)|.$$

The result is sharp and equality holds for $r(z)=B(z)+\lambda$ with $|\lambda|=1$.

Remark 2.4. *Again, since $|c_n|\ge|c_0|$, it is easy to verify that*

$$\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\ge\frac{\sqrt{|c_n|}-\sqrt{|c_0|}}{\sqrt{|c_n|}},$$

showing that Corollary 2.3 strengthens the inequality (1.8).

Instead of proving Theorem 2.1, we will prove the following more general result.

Theorem 2.2. *Suppose $r(z)=\frac{P(z)}{W(z)}$, where $P(z)=z^s\left(\sum_{j=0}^{n-s} c_{j+s} z^j\right)$, and all the zeros of r lie in $|z|\le 1$ with a zero of multiplicity s at the origin. Then for every β with $|\beta|\le 1$ and $|z|=1$ we have*

$$\left|zr'(z)+\frac{n\beta}{2}r(z)\right|\ge\frac{1}{2}\left\{|B'(z)|+nRe(\beta)+s+\frac{|c_n|-|c_s|}{|c_n|+|c_s|}\right\}|r(z)|. \tag{2.4}$$

The result is best possible in the case $\beta = s = 0$, *and equality in* (2.4) *holds for* $r(z) = B(z) + \lambda$ *with* $|\lambda| = 1$.

Remark 2.5. *For* $s = 0$, *the inequality* (2.4) *reduces to* (2.1).

The next result generalizes the inequality (1.7).

Theorem 2.3. *Let* $r \in \mathbb{R}_n$, *and assume that* r *has all its zeros in* $|z| \leq 1$. *Then for every* β *with* $|\beta| \leq 1$ *and* $|z| = 1$ *we have*

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \geq \frac{1}{2}(1 - |\beta|)|r(z)|. \tag{2.5}$$

Equality in (2.5) *holds when* $\beta = 0$ *for* $r(z) = aB(z) + b$ *with* $|a| = |b|$.

The above inequality (2.5) will be a consequence of a more fundamental inequality presented by the following theorem.

Theorem 2.4. *Let* $r \in \mathbb{R}_n$, *and assume that* r *has all its zeros in* $|z| \leq 1$. *Then for every* β *with* $|\beta| \leq 1$ *and* $|z| = 1$, *we have*

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \geq \frac{1}{2}\left\{(1 - |\beta|)|r(z)| + \left(\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)\min_{|z|=1}|r(z)|\right\}. \tag{2.6}$$

Equality in (2.6) *holds when* $\beta = 0$ *for* $r(z) = aB(z) + b$ *with* $|a| = |b|$.

Remark 2.6. *Theorem 2.4 is a refinement of Theorem 2.3, this can easily be seen by observing that* $|1 + \frac{\beta}{2}| \geq |\frac{\beta}{2}|$ *for* $|\beta| \leq 1$.

Theorem 2.5. *Suppose* $r \in \mathbb{R}_n$, *and all the* n *zeros of* r *lie in* $|z| \geq 1$. *If* $r(z) = \frac{P(z)}{W(z)}$, *where* $P(z) = \sum_{j=0}^{n} c_j z^j$, *then for* $|z| = 1$, *we have*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\frac{|r(z)|^2}{\| r(z) \|^2}\right\} \| r(z) \|, \tag{2.7}$$

where $\| r(z) \| = \max_{|z|=1} |r(z)|$. *The result is best possible and equality in* (2.7) *holds for* $r(z) = B(z) + \lambda$, $|\lambda| = 1$.

Remark 2.7. *Since all zeros of* $r(z) = \frac{P(z)}{W(z)}$, *and hence of* $P(z) = \sum_{j=0}^{n} c_j z^j$, *lie in* $|z| \geq 1$, *we have* $|c_0| \geq |c_n|$, *showing that Theorem 2.5 is an improvement of* (1.6).

3. Lemmas

In this section we state a number of lemmas, which will be used in the proofs of main results stated in Section 2.

Lemma 3.1. *(see [5]) If $P(z)=\sum_{j=0}^{n}c_jz^j$ is a polynomial of degree n having all its zeros in $|z|\leq 1$, then at each point z of the circle $|z|=1$ at which $P(z)\neq 0$, we have*

$$Re\bigg(\frac{zP'(z)}{P(z)}\bigg)\geq\frac{n-1}{2}+\frac{|c_n|}{|c_n|+|c_0|}.$$

Lemma 3.2. *(see [2]) If $|z|=1$, then*

$$Re\bigg(\frac{zW'(z)}{W(z)}\bigg)=\frac{n-|B'(z)|}{2}.$$

Lemma 3.3. *(see [12]) If $r\in\mathbb{R}_n$, then for $|z|=1$, we have*

$$|r'(z)|+|(r^*(z))'|\leq|B'(z)|\max_{|z|=1}|r(z)|.$$

Lemma 3.4. *Suppose $r\in\mathbb{R}_n$ is such that $r(z)=\frac{P(z)}{W(z)}$, where $P(z)=\sum_{j=0}^{n}c_jz^j$, and all the zeros of r lie in $|z|>1$. Then for $|z|=1$, we have*

$$Re\bigg(\frac{zr'(z)}{r(z)}\bigg)\leq\frac{1}{2}\bigg\{|B'(z)|-\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\bigg\}.$$

Proof. We have $r(z)=\frac{P(z)}{W(z)}$, where

$$P(z)=\sum_{j=0}^{n}c_jz^j=c_n\prod_{j=1}^{n}(z-z_j),$$

with $c_n\neq 0$ and $|z_j|>1$, $j=1,2,...,n$.

By direct calculation, we get

$$Re\left(\frac{zr'(z)}{r(z)}\right)=Re\left(\frac{zP'(z)}{P(z)}\right)-Re\left(\frac{zW'(z)}{W(z)}\right).\tag{3.1}$$

Let $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$, therefore, $P(z)=z^n\overline{Q(\frac{1}{\bar z})}$. Since $P(z)$ has all its zeros in $|z|>1$, it follows that $Q(z)$ has all its zeros in $|z|<1$, and hence

$$G(z)=\frac{Q(z)}{z^{n-1}\overline{Q(\frac{1}{\bar z})}}=\frac{zQ(z)}{P(z)}=\frac{\overline{c_n}}{c_n}z\prod_{j=1}^{n}\left(\frac{1-\bar z_jz}{z-z_j}\right)\tag{3.2}$$

is analytic in $|z|\leq 1$ with $G(0)=0$ and $|G(z)|=1$ for $|z|=1$. Hence by a result of Osserman for the boundary Schwartz lemma [8], we have

$$|G'(z)|\geq\frac{2}{1+|G'(0)|},\quad for\quad |z|=1.\tag{3.3}$$

It easily follows from (3.2) that for $|z|=1$,

$$\frac{zG'(z)}{G(z)}=(n+1)-2Re\left(\frac{zP'(z)}{P(z)}\right).\tag{3.4}$$

Further, using (3.2), it can easy be verified that

$$\frac{zG'(z)}{G(z)}=1+\sum_{j=1}^{n}\frac{|z_j|^2-1}{|z-z_j|^2}.$$

Since $|z_j| > 1$ for $1 \leq j \leq n$, it follows from above that $\frac{zG'(z)}{G(z)}$ is real and positive. Also, taking into account that $|G(z)| = 1$ for $|z| = 1$, we have

$$\frac{zG'(z)}{G(z)} = \left|\frac{zG'(z)}{G(z)}\right| = |G'(z)| \quad \text{and} \quad |G'(0)| = \prod_{j=1}^{n}\left|\frac{1}{z_j}\right| = \left|\frac{c_n}{c_0}\right|.$$

Using these observations, from (3.3) and (3.4), we get for $P(z) \neq 0$ and $|z| = 1$,

$$(n+1) - 2Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{2}{1 + \left|\frac{c_n}{c_0}\right|},$$

implying that

$$Re\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{n+1}{2} - \frac{|c_0|}{|c_0| + |c_n|}. \tag{3.5}$$

Finally, using (3.5), Lemma 3.2 and (3.1), we get

$$Re\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{1}{2}\left\{|B'(z)| - \frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right\},$$

which completes the proof of of the lemma. □

Lemma 3.5. *Let $r, s \in \mathbb{R}_n$, and let all the n zeros of s lie in $|z| \leq 1$ and for $|z| = 1$,*

$$|r(z)| \leq |s(z)|.$$

Then for every $|\beta| \leq 1$ and $|z| = 1$, we have

$$|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)| \leq |B(z)s'(z) + \frac{\beta}{2}B'(z)s(z)|. \tag{3.6}$$

Equality in (3.6) *holds for $r(z) = \mu s(z)$, $|\mu| = 1$.*

Proof. The proof follows on the same lines as those given in the proof of Theorem 3.2 of Li [11]. Hence, we omit the details.

Lemma 3.6. *Let $r \in \mathbb{R}_n$, and let all the n zeros of r lie in $|z| \leq 1$. Then for every $|\beta| \leq 1$ and $|z| = 1$, we have*

$$|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)| \leq |B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)|. \tag{3.7}$$

Proof. Since $r^*(z) = B(z)\overline{r(1/\bar{z})}$, we have

$$|r^*(z)| = |r(z)| \text{ for } |z| = 1.$$

Also, since $r(z)$ has all its zeros in $|z| \leq 1$, we can apply Lemma 3.5 with $r(z)$ and $s(z)$ being replaced by $r^*(z)$ and $r(z)$, respectively, to obtain the result.

4. Proofs of theorems

Proof of Theorem 2.2. Since $r(z)=\frac{P(z)}{W(z)}\in\mathbb{R}_n$, where $P(z)$ has all its zeros in $|z|\le 1$ with a zero of multiplicity s at the origin, we can write

$$P(z)=z^s h(z), \tag{4.1}$$

where $h(z)=\sum_{j=0}^{n-s}c_{j+s}z^j$ is a polynomial of degree $n-s$ having all its zeros in $|z|\le 1$.

From (4.1), we have

$$Re\left(\frac{zP'(z)}{P(z)}\right)=s+Re\left(\frac{zh'(z)}{h(z)}\right).$$

By a direct calculation, we obtain for every β with $|\beta|\le 1$,

$$\frac{zr'(z)}{r(z)}+\frac{n\beta}{2}=\frac{zP'(z)}{P(z)}-\frac{zW'(z)}{W(z)}+\frac{n\beta}{2}.$$

Therefore for $0\le\theta<2\pi$ by Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}
Re\left(\frac{zr'(z)}{r(z)}+\frac{n\beta}{2}\right)\Bigg|_{z=e^{i\theta}} &= Re\left(\frac{zP'(z)}{P(z)}\right)\Bigg|_{z=e^{i\theta}}-Re\left(\frac{zW'(z)}{W(z)}\right)\Bigg|_{z=e^{i\theta}}+\frac{n}{2}Re(\beta)\\
&= s+Re\left(\frac{zh'(z)}{h(z)}\right)\Bigg|_{z=e^{i\theta}}-Re\left(\frac{zW'(z)}{W(z)}\right)\Bigg|_{z=e^{i\theta}}+\frac{n}{2}Re(\beta)\\
&\ge\left(s+\frac{n-s-1}{2}+\frac{|c_n|}{|c_n|+|c_s|}\right)-\left(\frac{n-|B'(e^{i\theta})|}{2}\right)+\frac{n}{2}Re(\beta)\\
&=\frac{1}{2}\left\{|B'(e^{i\theta})|+s+\frac{|c_n|-|c_s|}{|c_n|+|c_s|}+nRe(\beta)\right\},
\end{aligned}$$

for the points $e^{i\theta},0\le\theta<2\pi$, other then the zero of $r(z)$. Hence, we have

$$\left|e^{i\theta}r'(e^{i\theta})+\frac{n}{2}\beta r(e^{i\theta})\right|\ge\frac{1}{2}\left\{|B'(e^{i\theta})|+s+\frac{|c_n|-|c_s|}{|c_n|+|c_s|}+nRe(\beta)\right\}|r(e^{i\theta})|, \tag{4.2}$$

for the points $e^{i\theta},0\le\theta<2\pi$, other then the zeros of $r(z)$.

Since (4.2) is true for the points $e^{i\theta},0\le\theta<2\pi$, which are the zeros of $r(z)$ as well, it follows that

$$\left|zr'(z)+\frac{n}{2}\beta r(z)\right|\ge\frac{1}{2}\left\{|B'(z)|+s+\frac{|c_n|-|c_s|}{|c_n|+|c_s|}+nRe(\beta)\right\}|r(z)|,$$

for $|z|=1$ and for every β with $|\beta|\le 1$. This completes the proof of the theorem.

Proof of Theorem 2.3. By a direct calculation (see, e.g., [12], p. 529), one can obtain

$$|(r^*(z))'|=|B'(z)r(z)-r'(z)B(z)| \text{ for } |z|=1,$$

and hence, using the fact that $|B(z)|=1$ for $|z|=1$, we get

$$|(r^*(z))'|\ge|B'(z)||r(z)|-|r'(z)|.$$

This gives for $|z| = 1$,

$$|r'(z)| + |(r^*(z))'| \geq |B'(z)||r(z)|. \tag{4.3}$$

Next, for any $|\beta| \leq 1$, we have

$$\begin{aligned}
&\left|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\right| + \left|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\right| \\
&\geq |B(z)|\,|r'(z)| + |B(z)||(r^*(z))'| - \left|\frac{\beta}{2}\right||B'(z)||r(z)| - \left|\frac{\beta}{2}\right||B'(z)||r^*(z)|,
\end{aligned}$$

and hence, by using (4.3) and the fact that $|r(z)| = |r^*(z)|$ for $|z| = 1$, we obtain

$$\begin{aligned}
&\left|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\right| + \left|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\right| \\
&\geq |r'(z)| + |(r^*(z))'| - |\beta||B'(z)||r(z)| \geq |B'(z)||r(z)| - |\beta||B'(z)||r(z)|. \qquad (4.4)
\end{aligned}$$

Now, by Lemma 3.6, we have for $|z| = 1$,

$$\left|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\right| \geq \left|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\right|. \tag{4.5}$$

The inequalities (4.4) and (4.5) together yield to

$$\left|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\right| \geq \frac{|B'(z)|}{2}(1 - |\beta|)|r(z)|, \tag{4.6}$$

for $|z| = 1$ and $|\beta| \leq 1$.

Finally, taking into account that $|B'(z)| \neq 0$ and $|B(z)| = 1$ for $|z| = 1$, from (4.6), we get

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \geq \frac{1}{2}(1 - |\beta|)|r(z)|,$$

for $|z| = 1$ and $|\beta| \leq 1$. $\square$

Proof of Theorem 2.4. Observe first that if $r(z)$ has some zeros on $|z| = 1$, then $\min_{|z|=1}|r(z)| = 0$, and in this case, the result follows from Theorem 2.3.

So, henceforth, we assume that all the zeros of $r(z)$ lie in $|z| < 1$. Let $m := \min_{|z|=1}|r(z)|$. Clearly $m > 0$, and we have $|\lambda m| < |r(z)|$ on $|z| = 1$ for any λ with $|\lambda| < 1$. By Rouche's theorem, the rational function $G(z) = r(z) + \lambda m$ has all its zeros in $|z| < 1$. Let $H(z) = B(z)\overline{G(1/\bar{z})} = r^*(z) + \bar{\lambda}mB(z)$, then $|H(z)| = |G(z)|$ for $|z| = 1$. Applying Lemma 3.6, for any β with $|\beta| \leq 1$ and $|z| = 1$, we get

$$\begin{aligned}
&\left|B(z)\Big((r^*(z))' + \bar{\lambda}B'(z)m\Big) + \frac{\beta}{2}B'(z)\Big(r^*(z) + \bar{\lambda}B(z)m\Big)\right| \\
&\leq \left|B(z)r'(z) + \frac{\beta}{2}B'(z)\Big(r(z) + \lambda m\Big)\right|, \qquad (4.7)
\end{aligned}$$

implying that

$$\begin{aligned}
&\Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z) + \bar{\lambda}\Big(1+\frac{\beta}{2}\Big)B(z)B'(z)m\Big| \\
&\le \Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| + \Big|\frac{\beta}{2}\Big|\,|\lambda|m|B'(z)| \qquad (4.8)
\end{aligned}$$

for $|z|=1, |\beta|\le 1$ and $|\lambda|<1$.
Choosing the arguments of λ on the left hand side of (4.8) to satisfy

$$\begin{aligned}
&\Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z) + \bar{\lambda}\Big(1+\frac{\beta}{2}\Big)B(z)B'(z)m\Big| \\
&= \Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\Big| + |\lambda|m\Big|1+\frac{\beta}{2}\Big|\,|B(z)B'(z)|, \qquad (4.9)
\end{aligned}$$

in view of (4.8), (4.9) and the fact that $|B(z)|=1$ for $|z|=1$, we get

$$\begin{aligned}
&\Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| \ge \Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\Big| \\
&+ |\lambda||B'(z)|\Big\{\Big|1+\frac{\beta}{2}\Big| - \Big|\frac{\beta}{2}\Big|\Big\}m. \qquad (4.10)
\end{aligned}$$

Finally, letting $|\lambda|\to 1$ in (4.10) and adding $|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)|$ to both sides, and using (4.4), we get the required assertion. Theorem 2.4 is proved.

Proof of Theorem 2.5. Since $r(z)=\frac{P(z)}{W(z)}$, where $P(z)=\sum_{j=0}^{n}c_jz^j$ and $r(z)$ has all its zeros in $|z|\ge 1$, and also $r^*(z)=B(z)\overline{r(1/\bar{z})}$, we have

$$z(r^*(z))' = zB'(z)\overline{r(\frac{1}{\bar{z}})} - \frac{B(z)}{z}\overline{r'(\frac{1}{\bar{z}})},$$

and therefore, for $|z|=1$ (so that $z=\frac{1}{\bar{z}}$), we get

$$|(r^*(z))'| = \Big|zB'(z)\overline{r(z)} - B(z)\overline{zr'(z)}\Big| = |B(z)|\Big|\frac{zB'(z)}{B(z)}\overline{r(z)} - \overline{zr'(z)}\Big|. \qquad (4.11)$$

Taking into account that (see [12], formula (15))

$$\frac{zB'(z)}{B(z)} = |B'(z)| > 0,$$

from (4.11) for $|z|=1$ with $r(z)\ne 0$, we get

$$\begin{aligned}
\Big|\frac{z(r^*(z))'}{r(z)}\Big|^2 &= \Big||B'(z)| - \frac{zr'(z)}{r(z)}\Big|^2 \\
&= |B'(z)|^2 + \Big|\frac{zr'(z)}{r(z)}\Big|^2 - 2|B'(z)|Re\Big(\frac{zr'(z)}{r(z)}\Big),
\end{aligned}$$

which, in view of Lemma 3.4, for $|z|=1$ with $r(z)\ne 0$, gives

$$\begin{aligned}
\Big|\frac{z(r^*(z))'}{r(z)}\Big|^2 &\ge |B'(z)|^2 + \Big|\frac{zr'(z)}{r(z)}\Big|^2 - |B'(z)|\Big\{|B'(z)| - \frac{|c_0|-|c_n|}{|c_0|+|c_n|}\Big\} \\
&= \Big|\frac{zr'(z)}{r(z)}\Big|^2 + \Big(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\Big)|B'(z)|.
\end{aligned}$$

This implies for $|z| = 1$ that

$$|r'(z)|^2 + \left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)|B'(z)||r(z)|^2 \leq |(r^*(z))'|^2.$$

Combining this with Lemma 3.3, for $|z| = 1$ we get

$$\begin{aligned}|r'(z)| + &\left\{|r'(z)|^2 + \left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)|B'(z)||r(z)|^2\right\}^{\frac{1}{2}} \\ &\leq |r'(z)| + |(r^*(z))'| \leq |B'(z)| \ \| r(z) \|,\end{aligned}$$

or equivalently,

$$\begin{aligned}&|r'(z)|^2 + \left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)|B'(z)||r(z)|^2 \\ &\leq |B'(z)|^2 \ \| r(z) \|^2 - 2|B'(z)||r'(z)| \ \| r(z) \| + |r'(z)|^2,\end{aligned}$$

which, in view of the fact that $|B'(z)| \neq 0$, after simplification, for $|z| = 1$ gives

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\frac{|r(z)|^2}{\| r(z) \|^2}\right\} \| r(z) \| .$$

This completes the proof of the theorem.

Remark 4.1. *From inequality* (4.10), *for* $|z| = 1$ *and for every* $|\beta| \leq 1$, *we have*

$$\begin{aligned}&\left|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\right| - \left|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\right| \\ &\geq |B'(z)|\left\{\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\}\min_{|z|=1}|r(z)|. \qquad (4.12)\end{aligned}$$

Since $|B'(z)| \neq 0$ for $|z| = 1$, from (4.12) we get the following inequality

(4.13)

$$\min_{|z|=1}\left\{\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| - \left|\frac{(r^*(z))'}{B'(z)} + \frac{\beta}{2}\frac{r^*(z)}{B(z)}\right|\right\} \geq \left(\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)\min_{|z|=1}|r(z)|.$$

Taking $\beta = 0$ in (4.13), we get

$$\min_{|z|=1}\left\{\left|\frac{r'(z)}{B'(z)}\right| - \left|\frac{(r^*(z))'}{B'(z)}\right|\right\} \geq \min_{|z|=1}|r(z)|,$$

yielding

$$\min_{|z|=1}\left|\frac{r'(z)}{B'(z)}\right| \geq \min_{|z|=1}|r(z)|. \qquad (4.14)$$

Clearly, the inequality (4.14) gives a generalization of the corresponding result for polynomials (see [1], Theorem 1).

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