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CONVERGENCE OF A SUBSEQUENCE OF TRIANGULAR PARTIAL SUMS OF DOUBLE WALSH-FOURIER SERIES

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Abstract. In 1987 Harris proved-among others that for each $1 \le p < 2$ there exists a two-dimensional function $f \in L_p$ such that its triangular Walsh-Fourier series does not converge almost everywhere. In this paper we prove that the set of functions from the space $L_p(\mathbb{I}^2)$, $1 \le p < 2$, with subsequence of triangular partial means $S_{2^A}^{\triangle}(f)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$. We also prove that for each function $f \in L_2(\mathbb{I}^2)$ a.e. convergence $S_{a(n)}^{\triangle}(f) \to f$ holds, where $a(n)$ is a lacunary sequence of positive integers.

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1. Introduction

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} = [0, 1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \le p \le 2^n$.

Denote the dyadic expension of $n \in \mathbb{N}$ and $x \in \mathbb{I}$ by

$$n = \sum_{j=0}^{\infty} n_j 2^j, n_j = 0, 1 \quad \text{and} \quad x = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}, x_j = 0, 1.$$

In the case of $x \in \mathbb{Q}$ chose the expension which terminates in zeros. n_i, x_i are the i-th coordinates of n, x, respectively. Define the dyadic addition $\dot{+}$ as

$$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

Denote by $\oplus$ the dyadic (or logical) addition. That is,

$$k \oplus n = \sum_{i=0}^{\infty} |k_i - n_i| \, 2^i,$$

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where k_i, n_i are the ith coordinate of natural numbers k, n with respect to number system based 2.

The sets $I_n(x) = \{y \in \mathbb{I} : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ for $x \in \mathbb{I}, I_n = I_n(0)$ for $0 < n \in \mathbb{N}$ and $I_0(x) = \mathbb{I}$ are the dyadic intervals of $\mathbb{I}$. For $0 < n \in \mathbb{N}$ denote by $|n| = \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. Set $e_j = 1/2^{j+1}$, the i-th coordinate of e_i is 1, the rest are zeros ($i \in \mathbb{N}$).

The Rademacher system is defined by

$$r_n(x) = (-1)^{x_n}, \quad x \in \mathbb{I}, n \in \mathbb{N}.$$

The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:

$$w_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum\limits_{k=0}^{|n|} n_k x_k}, \; x \in \mathbb{I}, n \in \mathbb{N}.$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [13])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}) \\ 0, & \text{if } x \in [2^{-n}, 1) \end{cases}, \tag{1.1}$$

We consider the double system $\{w_n(x^1) \times w_m(x^2) : n, m \in \mathbb{N}\}$ on the unit square $\mathbb{I}^2 = [0,1) \times [0,1)$.

We denote by $L_0(\mathbb{I}^2)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{I}^2$. $\mu(A)$ is the Lebesgue measure of $A \subset \mathbb{I}^d$.

We denote by $L_p(\mathbb{I}^2)$ the class of all measurable functions f that are 1-periodic with respect to all variable and satisfy

$$\|f\|_p = \left(\int_{\mathbb{I}^2} |f(y^1, y^2)|^p dy^1 dy^2 \right)^{1/p} < \infty.$$

If $f \in L_1(\mathbb{I}^2)$, then

$$\hat{f}(n^1, n^2) = \int_{\mathbb{I}^2} f(y^1, y^2) w_{n^1}(y^1) w_{n^2}(y^2) dy^1 dy^2$$

is the (n^1, n^2)-th Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{N^1,N^2}\left(x^1,x^2;f\right)=\sum_{n^1=0}^{N^1-1}\sum_{n^2=0}^{N^2-1}\hat{f}\left(n^1,n^2\right)w_{n^1}(x^1)w_{n^2}(x^2).$$

The triangular partial sums defined as

$$S_k^{\triangle}\left(x^1,x^2;f\right)=\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\hat{f}\left(i,j\right)w_i(x^1)w_j(x^2).$$

Let $a=(a(n))$ be a lacunary sequence of positive integers with quotient q. That is, $a(n+1)/a(n)\geq q>1$ for any $n\in\mathbb{N}$. Now, set the maximal function

$$S_{a,*}^{\triangle}f=\sup_n\left|S_{a(n)}^{\triangle}\left(f\right)\right|.$$

In 1971 Fefferman proved [2] the following result with respect to the trigonometric system. Let P be an open polygonal region in $\mathbb{R}^2$, containing the origin. Set

$$\lambda P=\left\{\left(\lambda x^1,\lambda x^2\right):\left(x^1,x^2\right)\in P\right\}$$

for $\lambda>0$. Then for every $p>1, f\in L_p\left(\left[-\pi,\pi\right]^2\right)$ it holds the relation

$$\sum_{(n^1,n^2)\in\lambda P}\widehat{f}\left(n^1,n^2\right)\exp\left(i\left(n^1y^1+n^2y^2\right)\right)\to f\left(y^1,y^2\right)\ \text{ as }\lambda\to\infty$$

for a. e. $\left(y^1,y^2\right)\in\left[-\pi,\pi\right]^2$. That is, $S_{\lambda P}f\to f$ a.e. Sjulin gave [14] a better result in the case when P is a rectangle. He proved a.e. convergence for the class $f\in L\left(\log^+L\right)^3\log\log L$ and for functions $f\in L\left(\log^+L\right)^2\log\log L$ when P is a square. This result for squares is improved by Antonov [1]. There is a sharp constrant between the trigonometric and the Walsh case. In 1987 Harris proved [8] for the Walsh system that if S is a region in $[0,\infty)\times[0,\infty)$ with piecewise C^1 boundary not always paralled to the axes and $1\leq p<2$, then there exists an $f\in L_p\left(\mathbb{I}^2\right)$ such that $S_{\lambda P}f$ does not converges a. e. and in L_p norms as $\lambda\to\infty$. In particular, from theorem of Harris follows that for any $1\leq p<2$ there exists an $f\in L_p\left(\mathbb{I}^2\right)$ such that $S_{2^A}^{\triangle}f$ does not converges a. e. as $A\to\infty$.

In this paper we improve this result of Harris for tringular partial sums ($P=\triangle$), In particular, let $1\leq p<2$, then we prove that the set of the functions from the space $L_p(\mathbb{I}^2)$ with subsequence of triangular partial means $S_{2^A}^{\triangle}\left(f\right)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$. We also prove that for each function $f\in L_2(\mathbb{I}^2)$ a.e. convergence $S_{a(n)}^{\triangle}\left(f\right)\to f$ holds, where $a(n)$ is a lacunary sequence of positive integers.

For results with respect to convergence of rectangular and triangular partial sums of Walsh-Fourier series see [6, 12, 15, 9, 10, 11, 7].

2. The main results

The following results are the main statements of the paper.

Theorem 2.1. *Let $1 \leq p < 2$. The set of the functions from the space $L_p(\mathbb{I}^2)$ with subsequence of triangular partial sums $S_{2^A}^{\triangle}(f)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$.*

Theorem 2.2. *The operator $S_{a,*}^{\triangle}$ is of strong type (L_2, L_2). More precisely,*

$$\|S_{a,*}^{\triangle} f\|_2 \leq C_q \|f\|_2.$$

By Theorem 2.2 and by the usual density argument we obtain the following result.

Corollary 2.1. *As $n \to \infty$ we have $S_{a(n)}^{\triangle}(f) \to f$ a.e. for every $f \in L_2(\mathbb{I}^2)$, where $a(n)$ is a lacunary sequence of positive integers.*

The following theorem is proved in [4, 5].

Theorem GGT. *Let $\{T_m\}_{m=1}^{\infty}$ be a sequence of linear continues operators, acting from space $L_p(\mathbb{I}^2)$ in to the space $L_0(\mathbb{I}^2)$. Suppose that there exists the sequence of functions $\{\xi_k\}_{k=1}^{\infty}$ from unit bull $S_p(0,1)$ of space $L_p(\mathbb{I}^2)$, sequences of integers $\{m_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ increasing to infinity such that*

$$\varepsilon_0 = \inf_k \mu\{\left(x^1, x^2\right) \in \mathbb{I}^2 : |T_{m_k} \xi_k \left(x^1, x^2\right)| > \lambda_k\} > 0.$$

Then the set of functions f from space $L_p(\mathbb{I}^2))$, for which the sequence $\{T_m f\}$ converges in measure to an a. e. finite function is of first Baire category in space $L_p(\mathbb{I}^2)$.

Proof of Theorem 2.1. First we prove that there exists a function h_A for which

$$\|h_A\|_p \leq 1 \tag{2.1}$$

and

$$\mu\left\{\left(x^1, x^2\right) \in \mathbb{I}^2 : \left|S_{2^A}^{\triangle}\left(x^1, x^2; h_A\right)\right| > \frac{2^{A/p}}{\sqrt{A}}\right\} \geq \frac{A}{2^{A+3}}. \tag{2.2}$$

Let

$$f_A\left(x^1, x^2\right) = \sum_{k=0}^{A-1} \sum_{l=0}^{2^A-1} w_{2^k \oplus l}\left(x^1\right) w_l\left(x^2\right)$$

and

$$h_A\left(x^1,x^2\right)=\frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}}f_A\left(x^1,x^2\right).$$

We can write

$$\begin{aligned}
\left\|f_A\right\|_p &= \left(\int\limits_{\mathbb{I}^2}\left|\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)D_{2^A}\left(x^1\dotplus x^2\right)\right|^p dx^1dx^2\right)^{1/p}\\
&= \left(\int\limits_{\mathbb{I}}\left|\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)\right|^p\left(\int\limits_{\mathbb{I}}D_{2^A}^p\left(x^1\dotplus x^2\right)dx^2\right)dx^1\right)^{1/p}\\
&= \left(\int\limits_{\mathbb{I}}\left|\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)\right|^p dx^1\left(\int\limits_{\mathbb{I}}D_{2^A}^p\left(x^2\right)dx^2\right)\right)^{1/p}\\
&\leq \left(\int\limits_{\mathbb{I}}\left(\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)\right)^2 dx^1\right)^{1/2}2^{A(1-1/p)}=\sqrt{A}2^{A(1-1/p)}.
\end{aligned}$$

Hence (2.1) is proved.

From simple calculation we obtain that

$$\begin{aligned}
\widehat{h}_A\left(i,j\right) &= \int\limits_{\mathbb{I}^2}h_A\left(y^1,y^2\right)w_i\left(y^1\right)w_j\left(y^2\right)dy^1dy^2\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\int\limits_{\mathbb{I}^2}f_A\left(y^1,y^2\right)w_{2^A-1}\left(y^1\right)w_i\left(y^1\right)w_j\left(y^2\right)dy^1dy^2\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\int\limits_{\mathbb{I}^2}f_A\left(y^1,y^2\right)w_{2^A-1-i}\left(y^1\right)w_j\left(y^2\right)dy^1dy^2\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\widehat{f}_A\left(2^A-1-i,j\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)=\sum_{i+j<2^A}\widehat{h}_A\left(i,j\right)w_i\left(x^1\right)w_j\left(x^2\right)\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i+j<2^A}\widehat{f}_A\left(2^A-1-i,j\right)w_i\left(x^1\right)w_j\left(x^2\right)\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i=0}^{2^A-1}\sum_{j=0}^{2^A-i-1}\widehat{f}_A\left(2^A-1-i,j\right)w_i\left(x^1\right)w_j\left(x^2\right)\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i=0}^{2^A-1}\sum_{j=0}^{i}\widehat{f}_A\left(i,j\right)w_{2^A-1-i}\left(x^1\right)w_j\left(x^2\right).
\end{aligned}$$

Consequently,

$$S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)=\frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}}\sum_{k=0}^{A-1}\sum_{l\leq 2^k\oplus l}w_{2^k\oplus l}\left(x^1\right)w_l\left(x^2\right).$$

We see that $l\leq 2^k\oplus l$ holds if and only if $l_k=0$. Hence, we have

$$S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)=\frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}}\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)\sum_{l\in\{l=0,1,\ldots,2^A-1:l_k=0\}}w_l\left(x^1\dotplus x^2\right).$$

Let

$$\left(x^1,x^2\right)\in G_{A,s}=I_A\left(t_0,...,t_{s-1},1,t_{s+1},...,t_{A-1}\right)\times I_A\left(t_0,...,t_{s-1},0,t_{s+1},...,t_{A-1}\right).$$

Since $x^1\dotplus x^2=I_A\left(e_s\right)$, we can write

$$\begin{aligned}\sum_{l\in\{l=0,1,\ldots,2^A-1:l_k=0\}}w_l\left(x^1\dotplus x^2\right) &= \sum_{l_0=0}^{1}\cdots\sum_{l_{k-1}=0}^{1}\sum_{l_{k+1}=0}^{1}\cdots\sum_{l_{A-1}=0}^{1}(-1)^{l_s}\\ &= \begin{cases}2^{A-1}, & \text{if } k=s\\ 0, & k\neq s\end{cases}.\end{aligned}$$

Hence,

$$\left|S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)\right|\geq\frac{2^{A-1}}{2^{A(1-1/p)}\sqrt{A}}\sum_{s=0}^{A-1}\mathbb{I}_{G_{A,s}}\left(x^1,x^2\right)=\frac{2^{A/p}}{2\sqrt{A}}\sum_{s=0}^{A-1}\mathbb{I}_{G_{A,s}}\left(x^1,x^2\right).$$

Set

$$\Omega_A=\bigcup_{s=0}^{A-1}\bigcup_{t_0=0}^{1}\cdots\bigcup_{t_{s-1}=0}^{1}\bigcup_{t_{s+1}=0}^{1}\cdots\bigcup_{t_{A-1}=0}^{1}G_{A,s}$$

From estimation (2.3) we get

$$\begin{aligned}&\mu\left\{\left(x^1,x^2\right)\in\mathbb{I}^2:\left|S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)\right|>\frac{2^{A/p}}{2\sqrt{A}}\right\}\\ \geq\;&\mu\left(\Omega_A\right)=\frac{1}{2^{2A}}\sum_{s=0}^{A-1}\sum_{x_0=0}^{1}\cdots\sum_{x_{s-1}=0}^{1}\sum_{x_{s+1}=0}^{1}\cdots\sum_{x_{A-1}=0}^{1}=\frac{A}{2^{A+1}}.\end{aligned}$$

Now, we prove that there exists $\left(x_1^1,x_1^2\right),...,\left(x_{p(A)}^1,x_{p(A)}^2\right)\in\mathbb{I}^2$, $p\left(A\right):=\left[2^{A+3}/A\right]+1$, such that

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A\dotplus\left(x_j^1,x_j^2\right)\right)\right)\geq\frac{1}{2}. \tag{2.3}$$

Indeed,

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A \dotplus \left(x_j^1, x_j^2\right)\right)\right) = 1-\mu\left(\bigcap_{j=1}^{p(A)}\left(\overline{\Omega_A \dotplus \left(x_j^1, x_j^2\right)}\right)\right)$$

$$= \quad 1-\int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega}_A}\left(t^1 \dotplus x_1^1, t^2 \dotplus x_1^2\right)\cdots\mathbb{I}_{\overline{\Omega}_A}\left(t^1 \dotplus x_{p(A)}^1, t^2 \dotplus x_{p(A)}^2\right) dt^1 dt^2.$$

Interpreting $\mathbb{I}_{\overline{\Omega_A}}\left(t^1 \dotplus x_1^1, t^2 \dotplus x_1^2\right)\cdots\mathbb{I}_{\overline{\Omega_A}}\left(t^1 \dotplus x_{p(A)}^1, t^2 \dotplus x_{p(A)}^2\right)$ as a function of the $2p\left(A\right)+2$ variables t^1, t^2, $\left(x_1^1, x_1^2\right), ..., \left(x_{p(A)}^1, x_{p(A)}^2\right)$ and integrating over all variables, each over $\mathbb{I}^2$, we note that

$$\int_{\mathbb{I}^2}\cdots\int_{\mathbb{I}^2}\int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega}_A}\left(t^1 \dotplus x_1^1, t^2 \dotplus x_1^2\right)\cdots\mathbb{I}_{\overline{\Omega}_A}\left(t^1 \dotplus x_{p(A)}^1, t^2 \dotplus x_{p(1)}^2\right)$$
$$dt^1 dt^2 dx_1^1 dx_1^2 \cdots dx_{p(A)}^1 dx_{p(A)}^2$$
$$= \int_{\mathbb{I}^2}\left(\int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega}_A}\left(t^1 \dotplus x_1^1, t^2 \dotplus x_1^2\right) dx_1^1 dx_1^2\right)\cdots$$
$$\left(\int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega}_A}\left(t^1 \dotplus x_{p(A)}^1, t^2 \dotplus x_{p(A)}^2\right) dx_{p(A)}^1 dx_{p(A)}^2\right) dt^1 dt^2$$
$$= \left(\mu\left(\overline{\Omega}_A\right)\right)^{p(A)} = \left(1-\mu\left(\Omega_A\right)\right)^{p(A)} \le \left(1-\frac{1}{p\left(A\right)}\right)^{p(A)} \le \frac{1}{2}.$$

Consequently, there exists $\left(x_1^1, x_1^2\right), ..., \left(x_{p(A)}^1, x_{p(A)}^2\right) \in \mathbb{I}^2$ such that

$$\int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega_A}}\left(t^1 \dotplus x_1^1, t^2 \dotplus x_1^2\right)\cdots\mathbb{I}_{\overline{\Omega_A}}\left(t^1 \dotplus x_{p(A)}^1, t^2 \dotplus x_{p(A)}^2\right) dt^1 dt^2 \le \frac{1}{2}. \tag{2.4}$$

Combining (2.4) and (2.4) we conclude that

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A \dotplus \left(x_j^1, x_j^2\right)\right)\right) \ge 1-\frac{1}{2} = \frac{1}{2}.$$

Hence (2.3) is proved. Let $\left(t := t^1 \dotplus t^2 \in \mathbb{I}\right)$

$$F_A\left(x^1, x^2, t\right) = \frac{1}{\left(4p\left(A\right)\right)^{1/p}}\sum_{j=1}^{p(A)} r_j\left(t^1 \dotplus t^2\right) h_A\left(x^1 \dotplus x_j^1, x^2 \dotplus x_j^2\right)$$
$$= \frac{1}{\left(4p\left(A\right)\right)^{1/p}}\sum_{j=1}^{p(A)} r_j\left(t\right) h_A\left(x^1 \dotplus x_j^1, x^2 \dotplus x_j^2\right).$$

Then it is proved in ([3], pp. 7-12) that there exists $t_0 \in \mathbb{I}$, such that

$$\int_{\mathbb{I}} \left|F_A\left(x^1, x^2, t_0\right)\right|^p dx^1 dx^2 \leq 1 \tag{2.5}$$

and

$$\mu\left\{\left(x^1, x^2\right) \in \mathbb{I}^2 : \left|S_{2^A}^{\triangle}\left(x^1, x^2; F_A\right)\right| > \frac{2^{A/p}/\left(2\sqrt{A}\right)}{\left(p\left(A\right)\right)^{1/p}}\right\} \geq \frac{1}{8}. \tag{2.6}$$

Set $\xi_A\left(x^1, x^2\right) := F_A\left(x^1, x^2, t_0\right)$. Then from (2.5) and (2.6) we have $\|\xi_A\|_p \leq 1$ and

$$\mu\left\{\left(x^1, x^2\right) \in \mathbb{I}^2 : \left|S_{2^A}^{\triangle}\left(x^1, x^2; \xi_A\right)\right| > 2^{1-3/p} A^{1/p-1/2}\right\} \geq \frac{1}{8}$$

and using Theorem GGT we complete the proof of Theorem 2.1. □

Proof of Theorem 2.2. First, we suppose that $q \geq 2$. Let $S_n^{\square}(f)$ be n-th square partial sums of the two-dimensional Walsh-Fourier series. It is easy to see that the spectrums of the polynomials

$$S_{a(n)}^{\square}(f) - S_{a(n)}^{\triangle}(f), \quad n = 1, 2, ...$$

are pairwise disjoint that implies

$$\begin{aligned}
\left\|\sup_n \left|S_{a(n)}^{\triangle}(f)\right|\right\|_2^2 &\leq 2\left\|\sup_n \left|S_{a(n)}^{\square}(f)\right|\right\|_2^2 + 2\left\|\sup_n \left|S_{a(n)}^{\triangle}(f) - S_{a(n)}^{\square}(f)\right|\right\|_2^2 \\
&\leq 2\left\|\sup_n \left|S_{a(n)}^{\square}(f)\right|\right\|_2^2 + 2\sum_n \left\|S_{a(n)}^{\triangle}(f) - S_{a(n)}^{\square}(f)\right\|_2^2 \\
&\leq 2\left\|\sup_n \left|S_{a(n)}^{\square}(f)\right|\right\|_2^2 + 2\|f\|_2^2 \leq c\|f\|_2^2,
\end{aligned}$$

where the last inequality is obtained from the L_2 boundedness of the square partial sums majorant operator (see [13]). This completes the proof of Theorem 2.2 in the case of $q \geq 2$. If $2 > q > 1$, then let Q the least natural number for which $q^Q \geq 2$. For any fixed $j = 0, \ldots, Q-1$ we have that the quotient of lacunary sequence n integers $(a(Qn+j))$ is at least 2 since $a(Q(n+1)+j) \geq q^Q a(Qn+j)$. From the above written we have

$$\left\|\sup_n \left|S_{a(Qn+j)}^{\triangle} f\right|\right\|_2^2 \leq C\|f\|_2^2$$

and consequently we also have $\left\|S_{a,*}^{\triangle} f\right\|_2^2 \leq C_q \|f\|_2^2$. □

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LIMIT THEOREMS FOR TAPERED TOEPLITZ QUADRATIC FUNCTIONALS OF CONTINUOUS-TIME GAUSSIAN STATIONARY PROCESSES

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Abstract. Let $\{X(t),\, t \in \mathbb{R}\}$ be a centered real-valued stationary Gaussian process with spectral density f. The paper considers a question concerning asymptotic distribution of tapered Toeplitz type quadratic functional Q_T^h of the process $X(t)$, generated by an integrable even function g and a taper function h. Sufficient conditions in terms of functions f, g and h ensuring central limit theorems for standard normalized quadratic functionals Q_T^h are obtained, extending the results of Ginovyan and Sahakyan (Probab. Theory Relat. Fields 138 (2007), 551–579) to the tapered case and sharpening the results of Ginovyan and Sahakyan (Electronic Journal of Statistics 13 (2019), 255–283) for the Gaussian case.

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1. Introduction

1.1. **The problem.** Let $\{X(t),\, t \in \mathbb{R}\}$ be a centered real-valued stationary Gaussian process with spectral density $f(\lambda)$ and covariance function $r(t)$. The functions $r(t)$ and $f(\lambda)$ are connected by the Fourier integral:

$$r(t) = \int_{\mathbb{R}} e^{i\lambda t} f(\lambda)\, d\lambda. \tag{1.1}$$

We consider a question concerning asymptotic distribution (as $T \to \infty$) of the following tapered Toeplitz type quadratic functional of the process $X(t)$:

$$Q_T^h = \int_0^T \int_0^T \widehat{g}(t-s) h_T(t) h_T(s) X(t) X(s)\, dt\, ds, \tag{1.2}$$

where

$$\widehat{g}(t) = \int_{\mathbb{R}} e^{i\lambda t} g(\lambda) d\lambda, \quad t \in \mathbb{R}. \tag{1.3}$$

is the Fourier transform of some integrable even function $g(\lambda)$, $\lambda \in \mathbb{R}$, and $h_T(t) = h(t/T)$ with a taper function $h(\cdot)$ to be specified below.

We refer to $g(\lambda)$ and to its Fourier transform $\widehat{g}(t)$ as a *generating function* and *generating kernel* for the functional Q_T^h, respectively.

Throughout the paper we assume that the taper function $h(\cdot)$ satisfies the following assumption.

Assumption (T). The taper $h : \mathbb{R} \to \mathbb{R}$ is a continuous nonnegative function of bounded variation and of bounded support $[0,1]$, such that $H_2 \neq 0$, where

$$H_k := \int_0^1 h^k(t)dt, \quad k \in \mathbb{N} := \{1, 2, \ldots\}. \tag{1.4}$$

Remark 1.1. *The case where $h(t) = \mathbb{I}_{[0,1]}(t)$, where $\mathbb{I}_{[0,1]}(\cdot)$ denotes the indicator of the segment $[0,1]$, will be referred to as the non-tapered case, and the corresponding non-tapered quadratic functional will be denoted by Q_T.*

The limit distribution of the functional (1.2) is completely determined by the functions f, g and h, and depending on their properties it can be either Gaussian (that is, Q_T^h with an appropriate normalization obey central limit theorem), or non-Gaussian. We naturally arise the following two questions:

a) Under what conditions on f, g and h will the limits be Gaussian?

b) Describe the limit distributions, if they are non-Gaussian.

In this paper we discuss the question a), and obtain sufficient conditions in terms of functions f, g and h ensuring central limit theorems for a standard normalized tapered quadratic functional Q_T^h, extending the results of Ginovyan and Sahakyan [17] to the tapered case and sharpening the results of Ginovyan and Sahakyan [18] for the Gaussian case.

1.2. **Statistical motivation.** Quadratic functionals of the form (1.2) appear both in nonparametric and parametric estimation of the spectrum of the process $X(t)$ based on the tapered data:

$$\{h_T(t)X(t),\ 0 \le t \le T\}. \tag{1.5}$$

For instance, when we are interested in nonparametric estimation of a linear integral functional in $L^p(\mathbb{R})$, $p > 1$ of the form:

$$J = J(f) = \int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda, \tag{1.6}$$

where $g(\lambda) \in L^q(\mathbb{R}),\ 1/p + 1/q = 1$, then a natural statistical estimator for $J(f)$ is the linear integral functional of the empirical periodogram of the process $X(t)$. To define this estimator, we first introduce some notation.

Denote by $H_{k,T}(\lambda)$ the continuous-time tapered Dirichlet type kernel, defined by

$$H_{k,T}(\lambda) = \int_{\mathbb{R}} h_T^k(t)e^{-i\lambda t}dt = \int_0^T h_T^k(t)e^{-i\lambda t}dt. \tag{1.7}$$

Define the finite Fourier transform of the tapered data (1.5):

$$d_T^h(\lambda) = \int_0^T h_T(t)X(t)e^{-i\lambda t}dt, \tag{1.8}$$

and the tapered continuous periodogram $I_T^h(\lambda)$ of the process $X(t)$:

$$\begin{aligned} I_T^h(\lambda) : &= \frac{1}{C_T} d_T^h(\lambda)d_T^h(-\lambda) = \frac{1}{C_T}\left|\int_0^T h_T(t)X(t)e^{-i\lambda t}dt\right|^2 \\ &= \frac{1}{C_T}\int_0^T\int_0^T h_T(t)h_T(s)e^{-i\lambda(t-s)}X(t)X(s)dtds, \end{aligned} \tag{1.9}$$

where

$$C_T := 2\pi H_{2,T}(0) = 2\pi\int_0^T h_T^2(t)dt = 2\pi H_2\,T \neq 0. \tag{1.10}$$

Notice that for non-tapered case ($h(t) = \mathbb{I}_{[0,1]}(t)$), we have $C_T = 2\pi T$.

As an estimator J_T^h for functional $J(f)$, given by (1.6), based on the tapered data (1.5), we consider the averaged tapered periodogram (or a simple "plug-in"statistic), defined by

$$\begin{aligned} J_T^h &= J(I_T^h) := \int_{\mathbb{R}} I_T^h(\lambda)g(\lambda)d\lambda \\ &= \frac{1}{C_T}\int_0^T\int_0^T \widehat{g}(t-s)h_T(t)h_T(s)X(t)X(s)\,dt\,ds, \end{aligned} \tag{1.11}$$

where C_T is as in (1.10), and $\widehat{g}(t)$ is the Fourier transform of function $g(\lambda)$ given by (1.3).

In view of (1.2) and (1.11) we have

$$J_T^h = C_T^{-1}Q_T^h, \tag{1.12}$$

and thus, to study the asymptotic properties of the estimator J_T^h, we have to study the asymptotic distribution (as $T \to \infty$) of the tapered Toeplitz type quadratic functional Q_T^h given by (1.2).

Some brief history. The question of describing the asymptotic distribution of non-tapered Toeplitz type quadratic forms and functionals of stationary processes has a long history, and goes back to the classical monograph by Grenander and Szegö [23], where the problem was considered as an application of authors' theory of the asymptotic behavior of the trace of products of truncated Toeplitz matrices and operators.

Later the problem have been studied by a number of authors. Here we mention only some significant contributions. For discrete-time short memory processes, the problem was studied by Ibragimov [26] and M. Rosenblatt [29], in connection with statistical estimation of the spectral and covariance functions, respectively. Since 1986, there has been a renewed interest in this problem, related to the statistical inferences for long memory (long-range dependent) and intermediate memory (anti-persistent) processes (see, e.g., Avram [1], Fox and Taqqu [12], Giraitis and Surgailis [20], Giraitis and Taqqu [21], Hhas'minskii and Ibragimov [25], Ginovian and Sahakian [16], Terrin and Taqqu [30], and references therein). In particular, Avram [1], Fox and Taqqu [12], Ginovian and Sahakian [16], Giraitis and Surgailis [20], Giraitis and Taqqu [21] have obtained sufficient conditions for non-tapered quadratic form Q_T to obey the central limit theorem (CLT).

For continuous-time stationary Gaussian processes the problem of describing the asymptotic distribution of non-tapered Toeplitz type quadratic functionals was studied in a number of papers. We cite merely the papers Avram et al. [2, 3], Bai et al. [4, 5], Bryc and Dembo [7], Ginovyan [13, 14, 15], Ginovyan and Sahakyan [17], Ibragimov [26], where can be found additional references.

In spectral analysis of stationary processes, however, the data are frequently tapered before calculating the statistics of interest. Instead of the original data $\{X(t),\, 0 \le t \le T\}$ the tapered data $\{h(t)X(t),\, 0 \le t \le T\}$ with the data taper $h(t)$ are used for all further calculations. Benefits of tapering the data have been widely reported in the literature. For example, data-tapers are introduced to reduce leakage effects, especially in the case when the spectrum of the model contains high peeks. Other application of data-tapers is in situations in which some of the data values are missing. Also, the use of tapers leads to the bias reduction, which is especially important when dealing with spatial data. In this case, the tapers can be used to fight the so-called "edge effects" (see Brillinger [6], R. Dahlhaus [8, 9], R. Dahlhaus and H. Künsch [10], Guyon [24], and references therein).

Central and non-central limit theorems for tapered quadratic forms of a discrete-time long memory Gaussian stationary fields have been proved in Doukhan et al. [11]. A central limit theorem for tapered quadratic functionals Q_T^h, in the case where the underlying model $X(t)$ is a Lévy-driven continuous-time stationary linear process has been proved in Ginovyan and Sahakyan [18] with time-domain conditions.

Remark 1.2. *Recall that a stationary process $X(t)$ with spectral density $f(\lambda)$ is said to have (a) short memory, (b) long memory or (c) intermediate memory if $f(\lambda)$ (a) is bounded away from zero and infinity at $\lambda = 0$, (b) has a pole at $\lambda = 0$, or (c) vanishes at $\lambda = 0$, respectively.*

1.3. **The approach.** To study the asymptotic distribution (as $T \to \infty$) of the functional $\widetilde{Q}_T^h$, given by (1.2), we use the method of cumulants, the frequency-domain approach, and the technique of truncated tapered Toeplitz operators.

By $W_T^h(\psi)$ we denote the truncated tapered Toeplitz operator generated by a function $\psi \in L^1(\mathbb{R})$ defined as follows (see [19], [23], [26] for non-tapered case):

$$[W_T^h(\psi)u](t) = \int_0^T \hat{\psi}(t-s)h_T(t)h_T(s)u(s)ds, \quad u(t) \in L^2[0,T], \tag{1.13}$$

where $\hat{\psi}(\cdot)$ is the Fourier transform of $\psi(\cdot)$.

Let $W_T^h(f)$ and $W_T^h(g)$ be the truncated tapered Toeplitz operators generated by the spectral density f, and the generating function g, respectively. Similar to the non-tapered case, we have the following results (cf. [19], [23], [26], see also the proof of Lemma 4.8 below).

1. The quadratic functional Q_T^h in (1.2) has the same distribution as the sum $\sum_{j=1}^{\infty} \lambda_{j,T}^2 \xi_j^2$, where $\{\xi_j, j \geq 1\}$ are independent $N(0,1)$ Gaussian random variables and $\{\lambda_{j,T}, j \geq 1\}$ are the eigenvalues of the operator $W_T^h(f)\, W_T^h(g)$.
2. The characteristic function $\varphi(t)$ of Q_T^h is given by formula:

$$\varphi(t) = \prod_{j=1}^{\infty} |1 - 2it\lambda_{j,T}|^{-1/2}. \tag{1.14}$$

3. The k–th order cumulant $\chi_k(Q_T^h)$ of Q_T^h is given by formula:

$$\chi_k(Q_T) = 2^{k-1}(k-1)! \sum_{j=1}^{\infty} \lambda_{j,T}^k = 2^{k-1}(k-1)!\, \mathrm{tr}\, [W_T^h(f)\, W_T^h(g)]^k, \tag{1.15}$$

where $\mathrm{tr}[A]$ stands for the trace of an operator A.

Thus, to describe the asymptotic distributions of the quadratic functional Q_T^h, we have to control the traces and eigenvalues of the products of truncated tapered Toeplitz operators.

Throughout the paper the letters C, c and M with or without indices are used to denote positive constants, the values of which can vary from line to line. Also, by $\mathbb{I}_A(\cdot)$ we denote the indicator of a set $A \subset \mathbb{R}$.

The remainder of the paper is structured as follows. In Section 2 we state the main results of the paper – Theorems 2.1 – 2.5. In Section 3 we apply the results of Section 2 to show that the avaraged tapered periodogram is an asymptotically normal estimator for the linear spectral functional. In Section 4 we prove preliminary results that are used in the proofs of main results, and also represent independent interest. Section 5 is devoted to the proofs of results stated in Section 2.

2. Central limit theorems for tapered quadratic functional Q_T^h

Below we assume that $f, g \in L^1(\mathbb{R})$, and with no loss of generality, that $g \geq 0$. We use the following notation: By $\widetilde{Q}_T^h$ we denote the standard normalized quadratic functional:

$$\widetilde{Q}_T^h = T^{-1/2}\left(Q_T^h - \mathbb{E}[Q_T^h]\right). \tag{2.1}$$

Then by (1.15) we have

$$\chi_k(\widetilde{Q}_T^h) = \begin{cases} 0, & \text{for } k=1 \\ T^{-k/2}2^{k-1}(k-1)!\,\mathrm{tr}\,[W_T^h(f)W_T^h(g)]^k, & \text{for } k\geq 2. \end{cases} \tag{2.2}$$

We set

$$\sigma_h^2 := 16\pi^3 H_4 \int_{\mathbb{R}} f^2(\lambda)g^2(\lambda)\,d\lambda, \tag{2.3}$$

where H_4 is as in (1.4). The notation

$$\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2) \quad \text{as} \quad T\to\infty \tag{2.4}$$

means that the distribution of the random variable $\widetilde{Q}_T^h$ tends (as $T\to\infty$) to the centered normal distribution with variance σ_h^2.

The main results of the paper are the following theorems.

Theorem 2.1. *Assume that $f\cdot g \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, the taper function h satisfies assumption (T), and for $T\to\infty$*

$$\chi_2(\widetilde{Q}_T^h) = \frac{2}{T} tr\big[W_T^h(f)W_T^h(g)\big]^2 \longrightarrow \sigma_h^2, \tag{2.5}$$

where σ_h^2 is as in (2.3). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2)$ as $T\to\infty$.

Theorem 2.2. *Assume that the function*

$$\varphi(x_1,x_2,x_3) = \int_{\mathbb{R}} f(u)g(u-x_1)f(u-x_2)g(u-x_3)\,du \tag{2.6}$$

belongs to $L^2(\mathbb{R}^3)$ and is continuous at $(0,0,0)$, and the taper function h satisfies assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2)$ as $T\to\infty$.

Theorem 2.3. *Assume that $f(\lambda)\in L^p(\mathbb{R})$ $(p\geq 1)$ and $g(\lambda)\in L^q(\mathbb{R})$ $(q\geq 1)$ with $1/p+1/q\leq 1/2$, and the taper function h satisfies assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2)$ as $T\to\infty$.*

Theorem 2.4. *Let $f\in L^2(\mathbb{R}),\ g\in L^2(\mathbb{R}),\ fg\in L^2(\mathbb{R})$,*

$$\int_{\mathbb{R}} f^2(\lambda)g^2(\lambda-\mu)\,d\lambda \longrightarrow \int_{\mathbb{R}} f^2(\lambda)g^2(\lambda)\,d\lambda \quad \text{as} \quad \mu\to 0, \tag{2.7}$$

and let the taper function h satisfy assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2)$ as $T\to\infty$.

To state the next theorem, we need to introduce a class of slowly varying at zero functions. Recall that a function $u(\lambda)$, $\lambda \in \mathbb{R}$, is called slowly varying at zero if it is nonnegative and for any $t > 0$

$$\lim_{\lambda \to 0} \frac{u(t\lambda)}{u(\lambda)} \to 1.$$

Denote by $SV_0(\mathbb{R})$ the class of slowly varying at zero functions $u(\lambda)$, $\lambda \in \mathbb{R}$, satisfying the following conditions: for some $a > 0$, $u(\lambda)$ is bounded on $[-a, a]$, $\lim_{\lambda \to 0} u(\lambda) = 0$, $u(\lambda) = u(-\lambda)$ and $0 < u(\lambda) < u(\mu)$ for $0 < \lambda < \mu < a$. An example of a function belonging to $SV_0(\mathbb{R})$ is $u(\lambda) = |\ln|\lambda||^{-\gamma}$ with $\gamma > 0$ and $a = 1$.

Theorem 2.5. *Assume that the functions f and g are integrable on $\mathbb{R}$ and bounded outside any neighborhood of the origin, and satisfy for some $a > 0$*

$$f(\lambda) \le |\lambda|^{-\alpha} L_1(\lambda), \quad |g(\lambda)| \le |\lambda|^{-\beta} L_2(\lambda), \quad \lambda \in [-a, a], \tag{2.8}$$

for some $\alpha < 1$, $\beta < 1$ with $\alpha + \beta \le 1/2$, where $L_1(x)$ and $L_2(x)$ are slowly varying at zero functions satisfying

$$L_i \in SV_0(\mathbb{R}), \quad \lambda^{-(\alpha+\beta)} L_i(\lambda) \in L^2[-a, a], \; i = 1, 2. \tag{2.9}$$

Also, let the taper function h satisfy assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0, \sigma_h^2)$ as $T \to \infty$.

Remark 2.1. The conditions $\alpha < 1$ and $\beta < 1$ in Theorem 2.5 ensure that the Fourier transforms of f and g are well defined. Observe that when $\alpha > 0$ the process $X(t)$ may exhibit long-range dependence. We also allow here $\alpha + \beta$ to assume the critical value $1/2$. The assumptions $f \cdot g \in L^1(\mathbb{R})$, $f, g \in L^\infty(\mathbb{R} \setminus [-a, a])$ and (2.9) imply that $f \cdot g \in L^2(\mathbb{R})$, so that the variance σ_h^2 in (2.3) is finite.

Remark 2.2. In Theorem 2.5, the assumption that $L_1(x)$ and $L_2(x)$ belong to $SV_0(\mathbb{R})$ instead of merely being slowly varying at zero is done in order to deal with the critical case $\alpha + \beta = 1/2$. Suppose that we are away from this critical case, namely, $f(x) = |x|^{-\alpha} l_1(x)$ and $g(x) = |x|^{-\beta} l_2(x)$, where $\alpha + \beta < 1/2$, and $l_1(x)$ and $l_2(x)$ are slowly varying at zero functions. Assume also that $f(x)$ and $g(x)$ are integrable and bounded on $(-\infty, -a) \cup (a, +\infty)$ for any $a > 0$. We claim that Theorem 2.5 applies. Indeed, choose $\alpha' > \alpha$, $\beta' > \beta$ with $\alpha' + \beta' < 1/2$. Write $f(x) = |x|^{-\alpha'} |x|^{\delta} l_1(x)$, where $\delta = \alpha' - \alpha > 0$. Since $l_1(x)$ is slowly varying, when $|x|$ is small enough, for some $\epsilon \in (0, \delta)$ we have $|x|^{\delta} l_1(x) \le |x|^{\delta - \epsilon}$. Then one can bound $|x|^{\delta-\epsilon}$ by $c\,|\ln|x||^{-1} \in SV_0(\mathbb{R})$ for small $|x| < 1$. Hence one has when $|x| < 1$ is small enough, $f(x) \le |x|^{-\alpha'} \left(c\,|\ln|x||^{-1}\right)$. Similarly, when $|x| < 1$ is small enough, one has $g(x) \le |x|^{-\beta'}\left(c\,|\ln|x||^{-1}\right)$. All the assumptions in Theorem 2.5 are now readily checked with α, β replaced by α' and β', respectively.

Remark 2.3. The analogs of Theorems 2.1 - 2.5 for non-tapered case ($h(t) = \mathbb{I}_{[0,1]}(t)$) were proved in Ginovyan and Sahakyan [17].

Remark 2.4. In Ginovyan and Sahakyan [18] was proved a central limit theorem for tapered functional Q_T^h for more general case where $X(t)$ is a Lévy-driven stationary linear process. Specifically, in [18] was proved the following result (see [18], Theorem 5.1). Let $\{X(t),\ t \in \mathbb{R}\}$ be a stationary linear process defined by

$$X(t) = \int_{\mathbb{R}} a(t-s)\xi(ds),$$

where $a(\cdot)$ is a filter from $L^2(\mathbb{R})$, and $\xi(t)$ is a Lévy process satisfying the conditions: $\mathbb{E}\xi(t) = 0$, $\mathbb{E}\xi^2(1) = 1$ and $\mathbb{E}\xi^4(1) < \infty$. Assume that the filter $a(\cdot)$ and the generating kernel $\hat{g}(\cdot)$ are such that

$$a(\cdot) \in L^p(\mathbb{R}) \cap L^2(\mathbb{R}), \quad \hat{g}(\cdot) \in L^q(\mathbb{R}), \quad 1 \le p, q \le 2, \quad \frac{2}{p} + \frac{1}{q} \ge \frac{5}{2}, \tag{2.10}$$

and the taper h satisfies assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0, \sigma^2_{L,h})$ as $T \to \infty$, where

$$\sigma^2_{L,h} = 16\pi^3 H_4 \int_{\mathbb{R}} f^2(\lambda) g^2(\lambda) d\lambda + \kappa_4 4\pi^2 H_4 \left[\int_{\mathbb{R}} f(\lambda) g(\lambda) d\lambda\right]^2. \tag{2.11}$$

Notice that if the underlying process $X(t)$ is Gaussian, then in formula (2.11) we have only the first term and so $\sigma^2_{L,h} = \sigma^2_h$, because in this case $\kappa_4 = 0$. On the other hand, the condition (2.10) is more restrictive than the conditions in Theorems 2.1 - 2.5. Thus, for Gaussian processes Theorems 2.1 - 2.5 improve the above stated result.

3. An application

In this section we apply the results of Section 2 to prove that the statistic J_T^h given by (1.11) is an asymptotically normal estimator for the linear functional $J(f)$ given by (1.6). To state the corresponding result we introduce the L^p-Hölder class and set up an assumption.

Given numbers $p \ge 1$, $0 < \alpha < 1$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we set $\beta = \alpha + r$ and denote by $H_p(\beta)$ the L^p-Hölder class, that is, the class of those functions $\psi(\lambda) \in L^p(\mathbb{R})$, which have r-th derivatives in $L^p(\mathbb{R})$ and with some positive constant C satisfy

$$||\psi^{(r)}(\cdot + h) - \psi^{(r)}(\cdot)||_p \le C|h|^\alpha.$$

Assumption (A). Let the spectral density $f(\lambda) \in H_p(\beta_1)$, $\beta_1 > 0$, $p \geq 1$ and let the generating function $g(\lambda) \in H_q(\beta_2)$, $\beta_2 > 0$, $q \geq 1$ with $1/p + 1/q = 1$. Assume that one of the conditions a)–d) is fulfilled:

a) $\beta_1 > 1/p$, $\beta_2 > 1/q$

b) $\beta_1 \leq 1/p$, $\beta_2 \leq 1/q$ and $\beta_1 + \beta_2 > 1/2$

c) $\beta_1 > 1/p$, $1/q - 1/2 < \beta_2 \leq 1/q$

d) $\beta_2 > 1/q$, $1/p - 1/2 < \beta_1 \leq 1/p$.

Theorem 3.1. *Let the functionals $J = J(f)$ and $J_T^h = J(I_T^h)$ be defined by (1.6) and (1.11), respectively. Then under the conditions (A) and (T) the statistic J_T^h is an asymptotically normal estimator for functional J. More precisely, we have*

$$T^{1/2}\left[J_T^h - J\right] \xrightarrow{d} \eta \quad \text{as} \quad T \to \infty, \tag{3.1}$$

where η is a normally distributed random variable with mean zero and variance $\sigma_h^2(J)$ given by

$$\sigma_h^2(J) = 4\pi e(h) \int_{\mathbb{R}} f^2(\lambda) g^2(\lambda) d\lambda, \quad e(h) := H_4 H_2^{-2}, \tag{3.2}$$

and H_k is as in (1.4).

Remark 3.1. In Theorem 2.3 of Ginovyan and Sahakyan [18] was proved the asymptotic normality of the estimator J_T^h for more general case where $X(t)$ is a Lévy-driven stationary linear process, but under the additional restrictive condition (2.10). Thus, for Gaussian processes Theorem 3.1 improve Theorem 2.3 of Ginovyan and Sahakyan [18].

4. Preliminaries

For a number k $(k = 2, 3, \ldots)$ and a taper function h satisfying assumption (T) consider the following "tapered" Fejér type kernel function:

$$\Phi_{k,T}^h(\mathbf{u}) = \frac{G_{k,T}(\mathbf{u})}{(2\pi)^{k-1} H_{k,T}(0)}, \quad \mathbf{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{R}^{k-1}, \tag{4.1}$$

where

$$G_{k,T}(\mathbf{u}) = H_{1,T}(u_1) \cdots H_{1,T}(u_{k-1}) H_{1,T}\left(-\sum_{j=1}^{k-1} u_j\right), \tag{4.2}$$

and the function $H_{k,T}$ is defined by (1.7) with $H_{k,T}(0) = T \cdot H_k \neq 0$ (see (1.4)).

Remark 4.1. Observe that by a change of variables $u_1 = x_1 - x_2$, $u_2 = x_2 - x_3$, …, $u_{k-1} = x_{k-1} - x_k$, the function $G_{k,T}(\mathbf{u})$ in (4.2) can be written in the following "symmetric" form:

$$G_{k,T}(\mathbf{x}) = H_{1,T}(x_1 - x_2) H_{1,T}(x_2 - x_3) \cdots H_{1,T}(x_{k-1} - x_k) H_{1,T}(x_k - x_1), \tag{4.3}$$

where $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$.

In Lemma 3.4 of Ginovyan and Sahakyan [18], it was proved that, similar to the classical Fejér kernel, the "tapered"kernel $\Phi^h_{k,T}(\mathbf{u})$ is an approximation identity. In particular, it was shown that the kernel $\Phi^h_{k,T}$ possesses the following property.

Lemma 4.1. *If a function $\psi(\mathbf{u}) \in L^1(\mathbb{R}^{k-1}) \bigcap L^{k-2}(\mathbb{R}^{k-1})$ is continuous at $\mathbf{v} = (v_1, \ldots, v_{k-1})$ $(k = 2, 3, \ldots)$, then*

$$\lim_{T\to\infty} \int_{\mathbb{R}^{k-1}} \psi(\mathbf{u} + \mathbf{v}) \Phi^h_{k,T}(\mathbf{u}) d\mathbf{u} = \Psi(\mathbf{v}), \tag{4.4}$$

where $\mathbf{u} = (u_1, \ldots, u_{k-1})$ and $\Phi^h_{k,T}(\mathbf{u})$ is defined by (4.1) and (4.2).

The next lemma contains a formula for trace of product of truncated tapered Toeplitz operators.

Lemma 4.2. *Let $W^h_T(f)$ and $W^h_T(g)$ be the truncated tapered Toeplitz operators generated by functions $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, respectively. Then*

$$tr \left[W^h_T(f) W^h_T(g)\right]^2 = \int_{\mathbb{R}^4} G_T(\mathbf{x}) f(x_1) g(x_2) f(x_3) g(x_4)\, d\mathbf{x}, \tag{4.5}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $G_T(\mathbf{x}) := G_{4,T}(\mathbf{x})$, that is,

$$G_T(\mathbf{x}) := H_{1,T}(x_1 - x_2) H_{1,T}(x_2 - x_3) H_{1,T}(x_3 - x_4) H_{1,T}(x_4 - x_1), \tag{4.6}$$

and $H_{1,T}(\cdot)$ is as in (1.7) with $k = 1$.

Proof. It is easy to check that the result follows from (1.1), (1.3), (1.7), (1.13), and the formula for traces of integral operators (see [22], §3.10). Lemma 4.2 is proved.

Denote

$$\mu_T(A) = \frac{1}{T} \int_A G_T(\mathbf{x})\, d\mathbf{x}, \tag{4.7}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $G_T(\mathbf{x}))$ is as in (4.6), and let $C_{loc}(\mathbb{R}^n)$ be the space of continuous functions on $\mathbb{R}^n$ with bounded support.

Lemma 4.3. *If $\phi \in C_{loc}(\mathbb{R}^4)$, then*

$$\lim_{T\to\infty} \int_{\mathbb{R}^4} \phi(\mathbf{x})\, d\mu_T = 8\pi^3 H_4 \int_{\mathbb{R}} \phi(u, u, u, u) du, \tag{4.8}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mu_T(A)$ is as in (4.7) and H_4 is as in (1.4).

Proof. Making a change of variables

$$x_1 = u, \quad x_1 - x_2 = u_1, \quad x_2 - x_3 = u_2, \quad x_3 - x_4 = u_3,$$

in view of (4.1), (4.2) and (4.7), we can write

$$\begin{aligned}\int_{\mathbb{R}^4} \phi(\mathbf{x})\, d\mu_T &= \frac{1}{T}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \phi(u, u-u_1, u-u_1-u_2, u-u_1-u_2-u_3)du \\ &\times\ H_{1,T}(u_1)H_{1,T}(u_2)H_{1,T}(u_3)H_{1,T}(-u_1-u_2-u_3)\, du_1\, du_2\, du_3 \\ &= 8\pi^3 H_4 \int_{\mathbb{R}^3} \Psi(\mathbf{u})\, \Phi_T^h(\mathbf{u})\, d\mathbf{u}, \qquad (4.9)\end{aligned}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\Phi_T^h(\mathbf{u}) := \Phi_{4,T}^h(\mathbf{u})$ and

$$\Psi(\mathbf{u}) = \int_{\mathbb{R}} \phi(u, u-u_1, u-u_1-u_2, u-u_1-u_2-u_3)\, du.$$

It is not difficult to check that the function Ψ satisfies conditions of Lemma 4.1 and

$$\lim_{\mathbf{u}\to(0,0,0)} \Psi(\mathbf{u}) = \int_{\mathbb{R}} \phi(u, u, u, u)\, du. \qquad (4.10)$$

Hence applying Lemma 4.1 from (4.9) and (4.10) we get (4.8). Lemma 4.3 is proved.

Lemma 4.4. *Let* $\phi(u_1, u_2, u_3, u_4) = \prod_{i=1}^{4} \phi_i(u_i)$, *where* $\phi_i \in L^1(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$, $i = 1, 2, 3, 4$. *Then the asymptotic relation (4.8) holds.*

Proof. Suppose $\|\phi_i\|_\infty \le M < \infty$, $i = 1, 2, 3, 4$. Using Lusin's theorem for any $\varepsilon > 0$ we can find functions φ_i, ψ_i, $i = 1, 2, 3, 4$, satisfying

$$\phi_i = \varphi_i + \psi_i, \quad \varphi_i \in C_{loc}(\mathbb{R}), \quad \|\psi_i\|_{L^1(\mathbb{R})} \le \varepsilon, \quad \|\varphi_i\|_C \le M. \qquad (4.11)$$

Therefore

$$\int_{\mathbb{R}^4} \phi d\mu_T = \int_{\mathbb{R}^4} \prod_{i=1}^{4} (\varphi_i + \psi_i) d\mu_T = \int_{\mathbb{R}^4} \prod_{i=1}^{4} \varphi_i d\mu_T + I_T, \qquad (4.12)$$

where by (4.11) and Lemma 4.5 below

$$\begin{aligned}|I_T| &\le \sum_{j=1}^{4} \int_{\mathbb{R}^4} |\psi_j| \prod_{i=1, i\ne j}^{4} (|\varphi_i| + |\psi_i|) d|\mu_T| \\ &\le C_M \sum_{j=1}^{4} \int_{\mathbb{R}^4} |\psi_j| d|\mu_T| \le C_M \sum_{j=1}^{4} \|\psi_j\|_{L^1(\mathbb{R})} \le C_M \varepsilon. \qquad (4.13)\end{aligned}$$

By Lemma 4.3 we have

$$\begin{aligned}\lim_{T\to\infty} \int_{\mathbb{R}^4} \prod_{i=1}^{4} \varphi_i(u_i) d\mu_T &= \int_{\mathbb{R}} \prod_{i=1}^{4} \varphi_i(u) du \\ &= \int_{\mathbb{R}} \prod_{i=1}^{4} [\phi_i(u) - \psi_i(u)]\, du = \int_{\mathbb{R}} \phi(u, u, u, u) du + J, \qquad (4.14)\end{aligned}$$

where

$$|J| \le \sum_{j=1}^{4} \int_{\mathbb{R}} |\psi_j(u)| \prod_{i=1, i\ne j}^{4} (|\phi_i(u)| + |\varphi_i(u)|) du \le C_M \varepsilon. \qquad (4.15)$$

From (4.12) – (4.15) we get (4.8). Lemma 4.4 is proved.

Lemma 4.5. *If $f \in L^1(\mathbb{R})$, then the following inequalities hold:*

$$1) \int_{\mathbb{R}^4} |f(x_i)| d|\mu_T| \le C_1 \|f\|_{L^1(\mathbb{R})}, \quad i = 1, 2, 3, 4, \tag{4.16}$$

$$2) \int_{\mathbb{R}^4} |f(x_i) f(x_j)| d|\mu_T| \le C_2 \|f\|^2_{L^2(\mathbb{R})}, \quad i, j = 1, 2, 3, 4, \; i \ne j. \tag{4.17}$$

where C_1 and C_2 are absolute constants, and μ_T is as in (4.7).

Proof. Since h is a function of bounded variation with support on $[0, 1]$, in view of (1.7), for $T > 0$ we have

$$|H_{1,T}(x)| \le C_h T \psi_T(x), \quad \text{where} \quad \psi_T(x) = \frac{1}{1 + T|x|}, \quad x \in \mathbb{R}. \tag{4.18}$$

We use the following inequality for function $\psi_T(x)$, which was proved in Ginovyan and Sahakyan [17] (see proof of Lemma 5):

$$T \int_{\mathbb{R}} \psi_T(x+u) \psi_T(x+v) dx \le C_\delta \psi_T^{1-\delta}(u - v), \quad \delta > 0, \quad u, v \in \mathbb{R}. \tag{4.19}$$

To prove (4.16) for $i = 1$ (say), we use (4.6), (4.7) and the inequality (4.19) with $\delta = 1/4$ to obtain

$$\begin{aligned} \int_{\mathbb{R}^4} |f(x_1)| d|\mu_T| \qquad & \le C T^3 \int_{\mathbb{R}^4} |f(x_1)| \psi_T(x_1 - x_3) \psi_T(x_3 - x_2) \\ & \times \psi_T(x_4 - x_1) \psi_T(x_2 - x_4) dx_1 dx_2 dx_3 dx_4 \\ & \le C T \int_{\mathbb{R}} |f(x_1)| \int_{\mathbb{R}} \psi_T^{3/2}(x_1 - x_2) dx_2 dx_1 \le C_1 \|f\|_{L^1(\mathbb{R})}. \end{aligned}$$

This proves (4.16). To prove (4.17) for $i = 1, \; j = 2$ (say), we use (4.6), (4.7), the inequality (4.19) with $\delta = 1/4$, and Cauchy inequality to obtain

$$\begin{aligned} & \int_{\mathbb{R}^4} |f(x_1) f(x_2)| d|\mu_T| \le C T^3 \int_{\mathbb{R}^4} |f(x_1) f(x_2)| \psi_T(x_1 - x_3) \psi_T(x_3 - x_2) \psi_T(x_4 - x_1) \\ & \le C T \int_{\mathbb{R}^2} |f(x_1) f(x_2)| \psi_T^{3/2}(x_1 - x_2) dx_1 dx_2 \\ & \le C \left\{ T \int_{\mathbb{R}^2} f^2(x_1) \psi_T^{3/2}(x_1 - x_2) dx_1 dx_2 \right\}^{1/2} \\ & + \left\{ T \int_{\mathbb{R}^2} f^2(x_2) \psi_T^{3/2}(x_1 - x_2) dx_1 dx_2 \right\}^{1/2} \le C_2 \int_{\mathbb{R}} f^2(x) dx. \end{aligned}$$

Lemma 4.5 is proved.

Lemma 4.6. *Let $\psi(u) \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, with $1 < p < \infty$, and let the taper function h satisfy assumption (T). Then*

$$\lambda_T := ||W_T^h(\psi)||_\infty = o\big(T^{1/p}\big) \quad as \quad T \to \infty. \tag{4.20}$$

Proof. Let N_T be a positive function of T, which we will specify later. We set

$$M_T = \{\lambda \in \mathbb{R}; \; |\psi(\lambda)| > N_T\}. \tag{4.21}$$

We have

$$\lambda_T = ||W_T^h(\psi)||_\infty = \sup_{||u||_2=1} |(W_T^h(\psi)u, u)| =$$

$$\sup_{||u||_2=1} \left| \int_0^T \int_0^T \widehat{\psi}(t-s)u(t)u(s)h(t)h(s)\,dt\,ds \right| =$$

$$\sup_{||u||_2=1} \left| \int_0^T \int_0^T \left[\int_{\mathbb{R}} e^{i\lambda(t-s)}\psi(\lambda)\,d\lambda \right] u(t)u(s)h(t)h(s)\,dt\,ds \right|. \tag{4.22}$$

A square integrable function $u(t)$ is also integrable on $[0, T]$. Hence, switching the order of integration in (4.22), we get

$$\lambda_T = \sup_{||u||_2=1} \left| \int_{\mathbb{R}} \psi(\lambda) \left[\int_0^T u(t)h(t)e^{it\lambda}\,dt \int_0^T u(s)h(s)e^{-is\lambda}\,ds \right] d\lambda \right|$$

$$\leq \sup_{||u||_2=1} \int_{\mathbb{R}} |\psi(\lambda)| \left| \int_0^T u(t)h(t)\,e^{i\lambda t} dt \right|^2 d\lambda. \tag{4.23}$$

Since for $u(t) \in L^2[0, T]$ with $||u||_2 = 1$ and h is bounded, we have $\left|\int_0^T u(t)h(t)\,e^{i\lambda t}dt\right|^2 \leq C_h T$, and by Plancherel's theorem from (4.23) we obtain

$$\lambda_T \leq C_h \left(N_T + T \int_{M_T} |\psi(\lambda)|\,d\lambda \right), \tag{4.24}$$

where M_T is as in (4.21). We show that for every p $(1 < p < \infty)$

$$\int_{M_T} |\psi(\lambda)|\,d\lambda \leq \gamma_T^p\, N_T^{(1-p)}, \tag{4.25}$$

where

$$\gamma_T = \left(\int_{M_T} |\psi(\lambda)|^p\,d\lambda \right)^{1/p}. \tag{4.26}$$

Indeed, by Hölder inequality

$$\int_{M_T} |\psi(\lambda)|\,d\lambda \leq \gamma_T \left[m(M_T) \right]^{(p-1)/p}, \tag{4.27}$$

where $m(M_T)$ is the Lebesgue measure of the set M_T. Next, by Chebyshev inequality

$$m(M_T) \leq \gamma_T^p\, N_T^{-p}. \tag{4.28}$$

A combination of (4.27) and (4.28) yields (4.25). Now from (4.24) and (4.25) we have

$$\lambda_T \leq C_h \left(N_T + T\gamma_T^p\, N_T^{(1-p)} \right). \tag{4.29}$$

If $\psi \in L^\infty(\mathbb{R})$, then putting $N_T = \|\psi\|_\infty$ for all $T > 0$, we will have $\gamma_T = 0$ and (4.29) implies $\lambda_T = O(1)$.

Now suppose $\psi \notin L^\infty(\mathbb{R})$ and for fixed $T > 0$ consider the function

$$F(N) = N - T^{1/p} \left(\int_{\{\lambda : |\psi(\lambda)| > N\}} |\psi(\lambda)|^p\,d\lambda \right)^{1/p}, \qquad N \in [0, \infty).$$

Since $F(0) < 0$ and $\lim_{N\to\infty} F(N) = +\infty$ there exists a positive number $N = N_T$ with $F(N_T) = 0$, that is,

$$N_T = T^{1/p}\left(\int_{\{\lambda:|\psi(\lambda)|>N_T\}} |\psi(\lambda)|^p\right) = T^{1/p}\gamma_T. \tag{4.30}$$

It is easy to see that for $\psi \notin L^\infty(\mathbb{R})$ the equality (4.30) implies $\lim_{T\to\infty} N_T = \infty$. Hence $\gamma_T = o(1)$ and from (4.29) and (4.30) we obtain $\lambda_T < C_h T^{1/p}\gamma_T = o(T^{1/p})$ as $T \to \infty$. Lemma 4.6 is proved.

Lemma 4.7. *Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and let $W_T^h(\psi)$ be the tapered truncated Toeplitz operator defined by (1.13) with taper function h satisfying assumption (T). Then*

$$\frac{1}{T}||W_T^h(\psi)||_2^2 \longrightarrow 2\pi H_4||\psi||_2^2 \quad as \quad T\to\infty, \tag{4.31}$$

where H_4 is as in (1.4)*.*

Proof. Using the formula for Hilbert–Schmidt norm of integral operators (see [22]), by (1.13) we have

$$||W_T^h(\psi)||_2^2 = \int_0^T\int_0^T |\widehat{\psi}(t-s)|^2|h_T(t)h_T(s)|^2\,dt\,ds. \tag{4.32}$$

Using the change of variables $t-s=u$ and taking into account that by assumption (T) the taper function h is supported on $[0,1]$, from (4.32) we get

$$||W_T^h(\psi)||_2^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\psi}(u)|^2|h_T(s+u)h_T(s)|^2\,du\,ds. \tag{4.33}$$

Next, taking into account that $h_T(t) = h(t/T)$ and using the change of variables $s/T = v$, from (4.33) we can write

$$\begin{aligned}\frac{1}{T}||W_T^h(\psi)||_2^2 &= \frac{1}{T}\int_{\mathbb{R}} |\widehat{\psi}(u)|^2| \left[\int_0^T |h(s/T+u/T)h(s/T)|^2\,ds\right] du\\ &= \int_{\mathbb{R}} |\widehat{\psi}(u)|^2| \left[\int_0^1 |h(v+u/T)h(v)|^2\,dv\right] du.\end{aligned} \tag{4.34}$$

For the inside integral on the right-hand side of (4.34), in view of (1.4), we have

$$\lim_{T\to\infty}\int_0^1 |h(v+u/T)h(v)|^2\,dv = \int_0^1 h^4(v)\,dv = H_4. \tag{4.35}$$

Finally, using Parseval-Plancherel's equality, from (4.34) and (4.35), we obtain (4.31). Lemma 4.7 is proved.

Lemma 4.8. *Let $Y(t)$, $t \in \mathbb{R}$, be a real-valued, centered, separable stationary Gaussian process with the spectral density $f_Y(\lambda) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and let $h_T(t) = h(t/T)$ with a taper function h satisfying assumption (T). Define*

$$L_T^h := \int_0^T [h_T(t)Y(t)]^2\,dt. \tag{4.36}$$

Then the distribution of the normalized quadratic functional

$$\widetilde{L}_T^h := T^{-1/2}\left(L_T^h - \mathbb{E}L_T^h\right) \tag{4.37}$$

tends (as $T\to\infty$) to the normal distribution $N(0,\sigma_Y^2)$ with variance

$$\sigma_Y^2 = 4\pi H_4 \int_{\mathbb{R}} f_Y^2(\lambda)\,d\lambda, \tag{4.38}$$

where H_4 is as in (1.4)*.*

Proof. Let $R(t)$ be the covariance function of $Y(t)$. For $T>0$ denote by $\lambda_j=\lambda_j(T)$, $j\in\mathbb{N}$, the eigenvalues of the operator $W_T^h(f_Y)$ (see (1.13)), and let $e_j(t)=e_j(t,T)\in L_2[0.T]$, $j\in\mathbb{N}$, be the corresponding orthonormal eigenfunctions, that is,

$$\int_0^T K(t-s)e_j(s)\,ds = \lambda_j e_j(t), \quad j\in\mathbb{N}, \tag{4.39}$$

where $K(t-s):=R(t-s)h_T(t)h_T(s)$. Since by Mercer's theorem (see, e.g., [22], §3.10)

$$K(t-s) = \sum_{j=1}^{\infty} \lambda_j e_j(t)e_j(s) \tag{4.40}$$

with positive and summable eigenvalues $\{\lambda_j\}$, we have the Karhunen–Loéve expansion:

$$h_T(t)Y(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\xi_j e_j(t), \tag{4.41}$$

where $\{\xi_j\}$ are independent $N(0,1)$ random variables. Therefore by (4.37) and (4.41)

$$\widetilde{L}_T^h = T^{-1/2}\sum_{j=1}^{\infty} \lambda_j(\xi_j^2-1). \tag{4.42}$$

Denote by $\chi_k(\widetilde{L}_T^h)$ the k-th order cumulant of $\widetilde{L}_T^h$. By (4.42) (cf. (2.2)) we have

$$\chi_k(\widetilde{L}_T^h) = \begin{cases} 0, & \text{for } k=1,\\ (k-1)!\,2^{k-1}T^{-k/2}\mathrm{tr}[W_T^h(f_Y)]^k, & \text{for } k\ge 2.\end{cases} \tag{4.43}$$

By (4.43) and Lemma 4.7 we have

$$\chi_2(\widetilde{L}_T^h) = \frac{2}{T}||W_T^h(f_Y)||_2^2 \longrightarrow 4\pi H_4\int_{\mathbb{R}} f_Y^2(\lambda)\,d\lambda \quad \text{as} \quad T\to\infty. \tag{4.44}$$

Next, by (4.43) for $k\ge 3$, we have

$$|\chi_k(\widetilde{L}_T^h)| \le C\,\frac{1}{T}||W_T^h(f_Y)||_2^2\,T^{1-k/2}\lambda_T^{k-2}, \tag{4.45}$$

where $\lambda_T = ||W_T^h(f_Y)||_\infty$. By Lemmas 4.6 and 4.7 the right hand side of (4.45) tends to zero as $T\to\infty$. Lemma 4.8 is proved.

The next lemma, which is the well-known Hardy-Littlewood type embedding theorem for the Hölder classes $H_p(\beta)$ (see Nikol'skii [28]), will be used in the proof of Theorem 3.1.

Lemma 4.9. *Let $\psi(\lambda) \in H_p(\beta)$ with $\beta > 0$ and $p \geq 1$. The following assertions hold:*

a) if $\beta \leq 1/p$ and $p < p_1 < p/(1-\beta p)$, then

$$\psi(\lambda) \in H_{p_1}(\beta - \frac{1}{p} + \frac{1}{p_1})$$

b) if $\beta > 1/p$, then $\psi(\lambda)$ is continuous and $||\psi||_\infty < \infty$.

5. Proofs

Since the proofs of Theorems 2.3 and 2.4 are almost the same (with some minor changes) as in the non-tapered case given in Ginovyan and Sahakyan [17], here we prove only Theorems 2.1, 2.2 and 2.5.

Proof of Theorem 2.1. By Theorem 16.7.2 from [27] the underlying process $X(t)$ admits the moving average representation

$$X(t) = \int_{\mathbb{R}} \widehat{a}(t-s)\, d\xi(s), \tag{5.1}$$

where $\widehat{a}(\cdot)$ is a function from $L^2(\mathbb{R})$, and $\xi(s)$ is a process with orthogonal increments such that $\mathbb{E}[d\xi(s)] = 0$ and $\mathbb{E}|d\xi(s)|^2 = ds$. Moreover the spectral density $f(\lambda)$ can be represented as

$$f(\lambda) = 2\pi\, |a(\lambda)|^2, \tag{5.2}$$

where $a(\lambda)$ is the inverse Fourier transform of $\widehat{a}(t)$. We set

$$a_1(\lambda) = (2\pi)^{1/2}\, a(\lambda) \cdot [g(\lambda)]^{1/2}, \tag{5.3}$$

where $g(\lambda)$ is the generating function, and consider a process $Y(t)$ $(t \in \mathbb{R})$ defined by

$$Y(t) = \int_{\mathbb{R}} \widehat{a}_1(t-s)\, d\xi(s), \tag{5.4}$$

where $\widehat{a}_1(t)$ is the Fourier transform of $a_1(\lambda)$ and $\xi(s)$ is as in (5.1). Since $fg \in L^1(\mathbb{R})$ by Parseval-Plancherel's identity we have

$$\int_{\mathbb{R}} |\widehat{a}_1(t)|^2\, dt = 2\pi \int_{\mathbb{R}} |a_1(\lambda)|^2\, d\lambda = 4\pi^2 \int_{\mathbb{R}} f(\lambda)\, g(\lambda)\, d\lambda < \infty. \tag{5.5}$$

So, $Y(t)$ is well-defined stationary process with spectral density

$$f_Y(\lambda) := |a_1(\lambda)|^2 = 2\pi\, f(\lambda)\, g(\lambda). \tag{5.6}$$

Since by assumption $f(\lambda)g(\lambda) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the process $Y(t)$ defined by (5.4) satisfies the conditions of Lemma 4.8. Hence Lemma 4.8 and Lemma 5.1 that follows imply Theorem 2.1.

Lemma 5.1. *Under assumptions of Theorem 2.1*

$$\mathrm{Var}(Q_T^h - L_T^h) = o(T) \quad \text{as} \quad T \to \infty, \tag{5.7}$$

where Q_T^h and L_T^h are as in (1.2) and (4.36) respectively.

Proof. By (1.2) and (5.1) we have

$$Q_T^h = \int_{\mathbb{R}^2} \left[\int_0^T \int_0^T \widehat{g}(t-s)\widehat{a}(t-u_1)\widehat{a}(s-u_2)h_T(t)h_T(s)\,dt\,ds \right] d\xi(u_1)\,d\xi(u_2). \tag{5.8}$$

Similarly, by (4.36) and (5.4)

$$L_T^h = \int_{\mathbb{R}^2} \left[\int_0^T \widehat{a}_1(t-u_1)\widehat{a}_1(t-u_2)h_T^2(t)\,dt \right] d\xi(u_1)\,d\xi(u_2). \tag{5.9}$$

Setting

$$d_{1T}(u_1,u_2) := \int_0^T \int_0^T \widehat{g}(t-s)\widehat{a}(t-u_1)\widehat{a}(s-u_2)h_T(t)h_T(s)\,dt\,ds \tag{5.10}$$

and

$$\begin{aligned} d_{2T}(u_1,u_2) &:= \int_0^T \int_0^T \widehat{a}_1(t-u_1)\widehat{a}_1(s-u_2)h_T^2(t)\,dt\,ds \\ &= \int_0^T \widehat{a}_1(t-u_1)\widehat{a}_1(t-u_2)h_T^2(t)\,dt, \end{aligned} \tag{5.11}$$

from (5.8)–(5.11) we get

$$Q_T^h - L_T^h = \int_{\mathbb{R}^2} \left[d_{1T}(u_1,u_2) - d_{2T}(u_1,u_2) \right] d\xi(u_1)\,d\xi(u_2). \tag{5.12}$$

Since the underlying process $X(t)$ is Gaussian, we obtain

$$\mathrm{Var}(Q_T^h - L_T^h) = 2\int_{\mathbb{R}^2} \left[d_{1T}(u_1,u_2) - d_{2T}(u_1,u_2) \right]^2 du_1\,du_2. \tag{5.13}$$

We set

$$p_1(\lambda_1,\lambda_2,\mu) = a(\lambda_1)a(\lambda_2)g(\mu), \tag{5.14}$$

$$p_2(\lambda_1,\lambda_2,\mu) = a_1(\lambda_1)a_1(\lambda_2)\delta(\mu) = a(\lambda_1)a(\lambda_2)[g(\lambda_1)]^{1/2}[g(\lambda_2)]^{1/2}. \tag{5.15}$$

By Parseval-Plancherel's identity we have

$$\begin{aligned} &\int_{\mathbb{R}^2} d_{iT}^2(u_1,u_2)\,du_1\,du_2 \\ &= (2\pi)^2 \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} H_{1,T}(\lambda_1-\mu)H_{1,T}(\mu-\lambda_2)p_i(\lambda_1,\lambda_2,\mu)\,d\mu \right|^2 d\lambda_1\,d\lambda_2 \\ &= (2\pi)^2 T||p_i||_T^2, \qquad i=1,2, \end{aligned} \tag{5.16}$$

where $H_{1,T}(u)$ is given by (1.7), $||p||_T^2 = (p,p)_T$,

$$(p,p')_T = \int_{\mathbb{R}^4} p(\lambda_1,\lambda_2,\lambda_3)\overline{p'(\lambda_1,\lambda_2,\lambda_4)}\,d\mu_T, \tag{5.17}$$

and the measure μ_T is defined by (4.7).

As in (5.16) (see also (5.13)), we have

$$\mathrm{Var}(Q_T - L_T) = 8\pi^2 T \|p_1 - p_2\|_T^2. \tag{5.18}$$

For any $K > 0$ we consider the sets

$$E_1^K = \{u \in \mathbb{R} : |a(u)| < K\}, \qquad E_2^K = \{u \in \mathbb{R} : g(u) < K\}, \tag{5.19}$$

and denote

$$\begin{aligned} p_1^K(\mathbf{u}) &= p_1(\mathbf{u})\chi_1^K(u_1)\chi_1^K(u_2)\chi_2^K(u_3), \\ p_2^K(\mathbf{u}) &= p_2(\mathbf{u})\chi_1^K(u_1)\chi_1^K(u_2)\chi_2^K(u_1)\chi_2^K(u_2), \end{aligned} \tag{5.20}$$

where $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $\chi_j^K(u)$ is the characteristic function of the set E_j^K, $j = 1, 2$. Then

$$\|p_1 - p_2\|_T^2 \le 3\left(\|p_1^K - p_2^K\|_T^2 + \|p_1 - p_1^K\|_T^2 + \|p_2 - p_2^K\|_T^2\right). \tag{5.21}$$

Now, (5.14), (5.15) and (5.20) imply that $\|p_1^K - p_2^K\|_T^2 = \int_{\mathbb{R}^4} \Gamma d\mu_T$, where $\Gamma = \Gamma(u_1, u_2, u_3, u_4)$ is a sum of four functions satisfying the conditions of Lemma 4.4. Since $\Gamma(u, u, u, u) = 0$ for $u \in \mathbb{R}$, applying Lemma 4.4 we get

$$\lim_{T\to\infty} \|p_1^K - p_2^K\|_T = \int_{\mathbb{R}} \Gamma(u, u, u, u)du = 0. \tag{5.22}$$

Next, according to (5.17) we have

$$\|p_1\|_T^2 = \|p_1^K + (p_1 - p_1^K)\|_T^2 = \|p_1\|_T^2 + 2(p_1^K, p_1 - p_1^K)_T + \|p_1 - p_1^K\|_T^2.$$

Therefore

$$\|p_1 - p_1^K\|_T^2 \le \big|\|p_1\|_T^2 - \|p_1^K\|_T^2\big| + 2\big|(p_1^K, p_1 - p_1^K)_T\big|. \tag{5.23}$$

By (2.5), (5.16) and Lemma 4.2 we have

$$\|p_1\|_T^2 = (2\pi)^{-2}\frac{1}{T}\mathrm{tr}\left[W_T^h(f)W_T^h(g)\right]^2 \to 2\pi H_4 \int_{\mathbb{R}} f^2(u)g^2(u)du, \tag{5.24}$$

while according to Lemma 4.4 and (5.16)

$$\|p_1^K\|_T^2 \to 2\pi H_4 \int_{F_K} f^2(u)g^2(u)du, \tag{5.25}$$

where $F_K := \{u \in \mathbb{R} : f(u) < K,\ g(u) < K\}$. From (5.24) and (5.25) we get

$$\lim_{T\to\infty}\left(\|p_1\|_T^2 - \|p_1^K\|_T^2\right) = 2\pi H_4 \int_{\mathbb{R}\setminus F_K} f^2(u)g^2(u)du = o(1) \quad \text{as} \quad K \to \infty. \tag{5.26}$$

To estimate the last term on the right hind side of (5.23) we denote

$$\Gamma_K(u_1, u_2, u_3, u_4) = p_1^K(u_1, u_2, u_3)\left[p_1(u_1, u_2, u_4) - p_1^K(u_1, u_2, u_4)\right].$$

From (5.19) and (5.20) for $\Gamma_K(u_1, u_2, u_3, u_4) \neq 0$ we have

$$|a(u_1)| < K,\ |a(u_2)| < K,\ g(u_3) < K,\ g(u_4) > K, \tag{5.27}$$

Next, for any $L > K$ and $\mathbf{u} = (u_1, u_2, u_3, u_4)$ we have

$$(p_1^K, p_1 - p_1^K)_T = \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) d\mu_T = \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) \chi_2^L(u_4) d\mu_T + I, \tag{5.28}$$

where with some constant C_K depending on K

$$|I| \le C_K \int_{\mathbb{R}^4} g(u_4) \left(1 - \chi_2^L(u_4)\right) d|\mu_T|. \tag{5.29}$$

It follows from (5.14), (5.15) and (5.20) that $\Gamma_K(\mathbf{u})\chi_2^L(u_4)$ is a linear combination of functions satisfying the conditions of Lemma 4.4. Applying Lemma 4.4 and taking into account that $\Gamma_K(u, u, u, u) = 0$ for $u \in \mathbb{R}$ (see (5.27)), we obtain

$$\lim_{T\to\infty} \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) \chi_2^L(u_4) d\mu_T = \int_{\mathbb{R}} \Gamma_K(u, u, u, u) \chi_2^L(u) du = 0. \tag{5.30}$$

For given $\varepsilon > 0$ and sufficiently large L by (4.16) we get

$$C_K \int_{\mathbb{R}^4} g(u) \left(1 - \chi_2^L(u)\right) d|\mu_T| \le C_K \int_{\{u:\ g(u)>L\}} g(u) du \le \varepsilon. \tag{5.31}$$

From (5.28) – (5.31) we obtain

$$\lim_{T\to\infty} (p_1^K, p_1 - p_1^K)_T = 0.$$

This combined with (5.23) and (5.26) yields

$$\lim_{T\to\infty} \|p_1 - p_1^K\|_T = 0. \tag{5.32}$$

Finally, we prove that

$$\lim_{T\to\infty} \|p_2 - p_2^K\|_T = 0. \tag{5.33}$$

Indeed, according to (5.15), (5.20) and (4.17), we have

$$\begin{aligned} \|p_2 - p_2^K\|_T &\le \int_{\mathbb{R}^4} [1 - \chi_1^K(u_1)] f(u_1) g(u_1) f(u_2) g(u_2) d|\mu_T| \\ &\quad + \int_{\mathbb{R}^4} [1 - \chi_1^K(u_2)] f(u_1) g(u_1) f(u_2) g(u_2) d|\mu_T| \\ &\le \int_{\{u:|f(u)|>\sqrt{K}\}} f^2(u) g^2(u) du + \int_{\{u:|g(u)|>K\}} f^2(u) g^2(u) du = o(1), \end{aligned}$$

when $K \to \infty$ (uniformly on T). A combination of (5.18), (5.22), (5.32) and (5.33) yields (5.7). This completes the proof of Lemma 5.1. Theorem 2.1 is proved.

Proof of Theorem 2.2. By a change of variables $x_1 = u$, $x_1 - x_2 = u_1$, $x_2 - x_3 = u_2$, $x_3 - x_4 = u_3$, in view of (4.1), (4.2), (4.5) and (4.6), we can write

$$\begin{aligned} \operatorname{tr} \left[W_T^h(f) W_T^h(g)\right]^2 &= \int_{\mathbb{R}^3} \int_{\mathbb{R}} f(u) g(u - u_1) f(u - u_1 - u_2) g(u - u_1 - u_2 - u_3) du \\ &\times\ H_{1,T}(u_1) H_{1,T}(u_2) H_{1,T}(u_3) H_{1,T}(-u_1 - u_2 - u_3) du_1 du_2 du_3 \\ &=:\ 8\pi^3 H_4 \int_{\mathbb{R}^3} \Psi(\mathbf{u}) \Phi_T^h(\mathbf{u}) d\mathbf{u}, \end{aligned} \tag{5.34}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\Phi_T^h(\mathbf{u}) := \Phi_{4,T}^h(\mathbf{u})$ is defined by (4.1), $\Psi(\mathbf{u}) := \varphi(u_1, u_1 + u_2, u_1 + u_2 + u_3)$ and $\varphi(u_1, u_2, u_3)$ is defined by (2.6). By Theorem 2.1 and (5.34) we need to prove that

$$\lim_{T\to\infty} \int_{\mathbb{R}^3} \Psi(\mathbf{u})\Phi_T^h(\mathbf{u})d\mathbf{u} = \int_{\mathbb{R}} f^2(x)g^2(x)dx. \tag{5.35}$$

Now, since both functions $\varphi(u_1, u_2, u_3)$ and $\Psi(u_1, u_2, u_3) = \varphi(u_1, u_1+u_2, u_1+u_2+u_3)$ are square integrable and continuous at $(0,0,0)$, and

$$\Psi(0,0,0) = \int_{\mathbb{R}} f^2(x)g^2(x)dx,$$

from Lemma 4.1 we obtain (5.35). Theorem 2.2 is proved.

Proof of Theorem 2.5. In view of (4.5) and (4.7), we need to prove that (2.8) and (2.9) imply

$$\lim_{T\to\infty} \frac{1}{T}\int_{\mathbb{R}^4} f(x_1)f(x_2)g(x_3)g(x_4)d\mu_T = 8\pi^3 H_4 \int_{\mathbb{R}} f^2(x)g^2(x)dx. \tag{5.36}$$

If $\alpha, \beta \geq 0$, then (2.8), (2.9) imply $f \in L^{1/\alpha}(\mathbb{R})$, $g \in L^{1/\beta}(\mathbb{R})$, and the result follows from Theorem 2.3. Assuming $\beta < 0$, from (2.8) we have $g \in L^\infty(\mathbb{R})$.

Denote

$$\overline{f}(x) = \begin{cases} 0, & \text{if} \quad x \in \left[-\frac{a}{2}, \frac{a}{2}\right] \\ f(x), & \text{otherwise} \end{cases}, \qquad \overline{g}(x) = \begin{cases} 0, & \text{if} \quad x \in [-a, a] \\ g(x), & \text{otherwise,} \end{cases}$$

where the number $a > 0$ is as in the statement of the theorem, and let $\underline{f} = f - \overline{f}$, $\underline{g} = g - \overline{g}$. Then we have

$$\begin{aligned} \frac{1}{T}\int_{\mathbb{R}^4} f(x_1)f(x_2)g(x_3)g(x_4)d\mu_T &= \frac{1}{T}\int_{\mathbb{R}^4} \overline{f}(x_1)f(x_2)g(x_3)g(x_4)d\mu_T \\ &+ \frac{1}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\overline{f}(x_2)g(x_3)g(x_4)d\mu_T + \frac{1}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)g(x_3)g(x_4)d\mu_T \\ &= I_T^1 + I_T^2 + I_T^3. \end{aligned} \tag{5.37}$$

Since $\overline{f},\ g \in L^\infty(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ we obtain

$$\begin{aligned} \lim_{T\to\infty} I_T^1 &= 8\pi^3 H_4 \int_{\mathbb{R}} \overline{f}(x)f(x)g^2(x)dx = 8\pi^3 H_4 \int_{|x|>\frac{a}{2}} f^2(x)g^2(x)dx, \\ \lim_{T\to\infty} I_T^2 &= 8\pi^3 H_4 \int_{\mathbb{R}} \underline{f}(x)\overline{f}(x)g^2(x)dx = 0. \end{aligned} \tag{5.38}$$

Next, we can write

$$\begin{aligned} I_T^3 &= \frac{1}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\underline{g}(x_4)d\mu_T + \frac{1}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\overline{g}(x_4)d\mu_T \\ &+ \frac{1}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\underline{g}(x_4)d\mu_T + \frac{1}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\overline{g}(x_4)d\mu_T = \sum_{i=1}^4 J_T^i. \end{aligned} \tag{5.39}$$

We have

$$J_T^1 = \frac{1}{T}\int_{[-a,a]^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\underline{g}(x_4)d\mu_T.$$

Arguments similar to those leading to equality (4.3) from [16] may be used to prove that

$$\lim_{T\to\infty} J_T^1 = 8\pi^3 H_4 \int_{-a}^{a} \underline{f}^2(x)g^2(x)dx = 8\pi^3 H_4 \int_{-a/2}^{a/2} f^2(x)g^2(x)dx. \tag{5.40}$$

Since $f(x_1)f(x_2) \in L^1(\mathbb{R}^2)$ for any $\varepsilon > 0$ we can find $\delta > 0$ satisfying

$$\int_{|x_1-x_2|<\delta} |f(x_1)f(x_2)|dx_1dx_2 < \varepsilon.$$

Because $g \in L^\infty(\mathbb{R})$, in view of (4.18) and (4.19) for sufficiently large T we obtain

$$\begin{aligned}
|J_T^2| &\le C\cdot T \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} |f(x_1)f(x_2)| \int_{-a/2}^{a/2} \psi_T(x_1-x_3)\psi_T(x_2-x_3) \\
&\quad \times \int_{|x_4|>a} x_4^{-2}dx_4dx_3dx_1dx_2 \\
&\le C \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} |f(x_1)f(x_2)|(1+T|x_1-x_2|)^{-1/2}dx_1dx_2 \\
&\le C \int_{|x_1-x_2|<\delta} |f(x_1)f(x_2)|dx_1dx_2 \\
&\quad + (1+T\delta)^{-1/2} \int_{|x_1-x_2|\ge\delta} |f(x_1)f(x_2)|dx_1dx_2 \le 2\varepsilon.
\end{aligned}$$

This means that

$$\lim_{T\to\infty} J_T^2 = 0. \tag{5.41}$$

Likewise, we get

$$\lim_{T\to\infty} J_T^3 = 0. \tag{5.42}$$

To estimate the integral J_T^4 in (5.39) note that in this case $|x_i - x_j| > \frac{a}{2}$, $i = 1, 2$, $j = 3, 4$. Therefore

$$\begin{aligned}
|J_T^4| &\le \frac{C}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\overline{g}(x_4)dx_1dx_2dx_3dx_4 \\
&\le \frac{C}{T}\|f\|^2_{L^1(\mathbb{R})}\|g\|^2_{L^1(\mathbb{R})} \to 0 \quad \text{as} \quad T\to\infty.
\end{aligned} \tag{5.43}$$

From (5.39) – (5.43) we obtain

$$\lim_{T\to\infty} I_T^3 = 8\pi^3 H_4 \int_{-a/2}^{a/2} f^2(x)g^2(x)dx,$$

which combined with (5.37) and (5.38) yields (5.36). Theorem 2.5 is proved.

Proof of Theorem 3.1. Taking into account the equality

$$T^{1/2}\left[J_T^h - J\right] = T^{1/2}\left[\mathbb{E}(J_T^h) - J\right] + T^{1/2}\left[J_T^h - \mathbb{E}(J_T^h)\right], \tag{5.44}$$

to prove the theorem we have to establish the following two asymptotic relations:

$$T^{1/2}\left[\mathbb{E}(J_T^h) - J\right] \to 0 \quad \text{as} \quad T \to \infty, \tag{5.45}$$

$$T^{1/2}\left[J_T^h - E(J_T^h)\right] \xrightarrow{d} \eta \sim N\left(0, \sigma_h^2(J)\right) \quad \text{as} \quad T \to \infty, \tag{5.46}$$

where $\sigma_h^2(J)$ is given by (3.2).

Observe first that the relation (5.45) is an immediate consequence of Theorem 2.1 of Ginovyan and Sahakyan [18], since under each of the conditions a)-d) in assumption (A), we have $\beta_1 + \beta_2 > 1/2$.

Now we proceed to show that the relation (5.46) follows from Theorem 2.3. To do this we need to show that, under the assumption (A), there exist numbers p_1 ($p_1 > p$) and q_1 ($q_1 > q$), such that $H_p(\beta_1) \subset L_{p_1}$, $H_q(\beta_2) \subset L_{q_1}$ and $1/p_1 + 1/q_1 \leq 1/2$.

The case $\beta_1 > 1/p$, $\beta_2 > 1/q$ is obvious, since in view of Lemma 4.9 b) we have $H_p(\beta_1) \subset L_\infty$ and $H_q(\beta_2) \subset L_\infty$.

Let $\beta_1 \leq 1/p$, $\beta_2 \leq 1/q$ and $\beta_1 + \beta_2 > 1/2$. For an arbitrary number $\varepsilon > 0$ satisfying $\beta_1 > \varepsilon$ and $\beta_2 > \varepsilon$, we set

$$\frac{1}{p_1} = \frac{1}{p} - \beta_1 + \varepsilon \quad \text{and} \quad \frac{1}{q_1} = \frac{1}{q} - \beta_2 + \varepsilon.$$

It is easy to see that $p < p_1 < p/(1-\beta_1 p)$ and $q < q_1 < q/(1-\beta_2 q)$. Hence by Lemma 4.9 a) we obtain $H_p(\beta_1) \subset L_{p_1}$ and $H_q(\beta_2) \subset L_{q_1}$. On the other hand, we have

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p} + \frac{1}{q} - (\beta_1 + \beta_2) + 2\varepsilon = 1 - (\beta_1 + \beta_2) + 2\varepsilon.$$

Since $\beta_1 + \beta_2 > 1/2$, choosing ε sufficiently small, we obtain $1/p_1 + 1/q_1 \leq 1/2$.

Now let $\beta_1 > 1/p$ and $1/q - 1/2 < \beta_2 \leq 1/q$. By Lemma 4.9 b) we have $H_p(\beta_1) \subset L_\infty$. For an arbitrary number $\varepsilon > 0$ satisfying $\beta_2 > \varepsilon$, we set

$$\frac{1}{q_1} = \frac{1}{q} - \beta_2 + \varepsilon.$$

Obviously $q < q_1 < q/(1-\beta_2 q)$, and hence $H_q(\beta_2) \subset L_{q_1}$ by Lemma 4.9 a).
Further, we have

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{q} - \beta_2 + \varepsilon.$$

Since $1/q - \beta_2 < 1/2$, choosing ε sufficiently small we obtain $1/p_1 + 1/q_1 \leq 1/2$.

The case $\beta_2 > 1/q$ and $1/p - 1/2 < \beta_1 \leq 1/p$ can be treated similarly.

Thus, we can apply Theorem 2.3, to obtain that

$$Q_T^h = T^{-1/2}\left(Q_T^h - \mathbb{E}[Q_T^h]\right) \xrightarrow{d} \eta \sim N(0, \sigma_h^2) \quad \text{as} \quad T \to \infty, \tag{5.47}$$

where Q_T^h and $\sigma_h^2 = \sigma_h^2(Q)$ are given by (1.2) and (2.3), respectively.

Also, in view of (1.10), (1.12) and (2.3), we have

$$(5.48) \quad \sigma_h^2(J) = \frac{1}{4\pi^2 H_2^2}\sigma_h^2(Q) = 4\pi e(h)\int_{\mathbb{R}} f^2(\lambda)g^2(\lambda)d\lambda, \quad e(h) = H_4 H_2^{-2}.$$

Putting together (5.47) and (5.48), we obtain the relation (5.46). Theorem 3.1 is proved.

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ON A WEAK TYPE ESTIMATE FOR SPARSE OPERATORS OF STRONG TYPE

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Abstract. We define sparse operators of strong type on abstract measure spaces with ball-bases. Weak and strong type inequalities for such operators are proved.

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1. Introduction

The sparse operators are very simple positive operators recently appeared in the study of weighted estimates of Calderón-Zygmund and other related operators. It was proved that some well-known operators (Calderón-Zygmund operators, martingale transforms, maximal function, Carleson operators, etc.) can be dominated by sparse operators, and this kind of dominations imply a series of deep results for the mentioned operators (see [1, 2, 4 – 7]). In particular, Lerner's [6] norm domination of the Calderón-Zygmund operators by sparse operators gave a simple alternative proof to the A_2-conjecture solved by Hytönen [3]. Lacey [5] established a pointwise sparse domination for the Calderón-Zygmund operators with an optimal condition (Dini condition) on the modulus of continuity, getting a logarithmic gain to the result previously proved by Conde-Alonso and Rey [1]. The paper [5] also proves a pointwise sparse domination for the martingale transforms, providing a short approach to the A_2-theorem proved by Treil-Thiele-Volberg [8]. For the Carleson operators norms sparse domination was proved by Di Plinio and Lerner [2], while the pointwise domination follows from a general result proved later in [4].

In this paper we consider sparse operators based on ball-bases in abstract measure spaces. The concept of ball-basis was introduced by the first author in [4]. Based on ball-basis the paper [4] defines a wide class of operators (including, in particular, the

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above mentioned operators) that can be pointwisely dominated by sparse operators. Some estimates of sparse operators in abstract spaces were obtained in [4]. In this paper we define a stronger version of sparse operators, and prove weak and strong type estimates for such operators.

We first recall the definition of the ball-basis from [4].

Definition 1.1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. A family of sets $\mathfrak{B} \subset \mathfrak{M}$ is said to be a ball-basis if it satisfies the following conditions.*

B1) $0 < \mu(B) < \infty$ *for any ball* $B \in \mathfrak{B}$.

B2) For any two points $x, y \in X$ *there exists a ball* $B \ni x, y$.

B3) If $E \in \mathfrak{M}$, *then for any* $\varepsilon > 0$ *there exists a finite or infinite sequence of balls* B_k, $k = 1, 2, \ldots$, *such that*

$$\mu\left(E \bigtriangleup \bigcup_k B_k\right) < \varepsilon.$$

B4) For any $B \in \mathfrak{B}$ *there is a ball* $B^* \in \mathfrak{B}$ *(called a* hull *of* B*) satisfying the conditions:*

$$\bigcup_{A \in \mathfrak{B}:\, \mu(A) \le 2\mu(B),\, A \cap B \neq \varnothing} A \subset B^*, \quad \mu(B^*) \le \mathcal{K}\mu(B),$$

where $\mathcal{K}$ *is a positive constant.*

A ball-basis $\mathfrak{B}$ is said to be doubling if there is a constant $\eta > 1$ such that for any $A \in \mathfrak{B}$, $A^* \neq X$, one can find a ball $B \in \mathfrak{B}$ to satisfy

$$A \subsetneq B, \quad \mu(B) \le \eta \cdot \mu(A). \tag{1.1}$$

In [4], it was shown that the condition (1.1) in the definition can equivalently be replaced by a stronger condition $\eta_1 \le \mu(B)/\mu(A) \le \eta_2$, where $\eta_2 > \eta_1 > 1$. It is well-known the non-standard features of non-doubling bases in many problems of analysis.

One can easily check that the family of Euclidean balls in $\mathbb{R}^n$ forms a ball-basis and it is doubling. An example of non-doubling ball-basis can serve us the martingale-basis defined as follows. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $\{\mathfrak{B}_n : n \in \mathbb{Z}\}$ be a collection of measurable sets such that 1) each $\mathfrak{B}_n$ is a finite or countable partition of X, 2) for each n and $A \in \mathfrak{B}_n$ the set A is a union of sets $A' \in \mathfrak{B}_{n+1}$, 3) the collection $\mathfrak{B} = \cup_{n \in \mathbb{Z}} \mathfrak{B}_n$ generates the σ-algebra $\mathfrak{M}$, 4) for any points $x, y \in X$ there is a set $A \in \mathfrak{B}$ such that $x, y \in A$. One can easily check that $\mathfrak{B}$ satisfies all the ball-basis conditions B1)-B4). On the other hand, it is not always doubling. Obviously, it is doubling if and only if $\mu(\mathrm{pr}(B)) \le c\mu(B)$, $B \in \mathfrak{B}$, where $\mathrm{pr}(B)$ (parent of B) denotes the minimal ball satisfying $B \subsetneq \mathrm{pr}(B)$.

Let $\mathfrak{B}$ be a ball-basis in a measure space $(X, \mathfrak{M}, \mu)$. For $f \in L^r(X)$, $1 \le r < \infty$, and a ball $B \in \mathfrak{B}$ we set

$$\langle f \rangle_{B,r} = \left(\frac{1}{\mu(B)} \int_B |f|^r \right)^{1/r}, \quad \langle f \rangle^*_{B,r} = \sup_{A \in \mathfrak{B} : A \supset B} \langle f \rangle_{A,r}.$$

A collection of balls $\mathcal{S} \subset \mathfrak{B}$ is said to be sparse or γ-sparse if for any $B \in \mathcal{S}$ there is a set $E_B \subset B$ such that $\mu(E_B) \ge \gamma \mu(B)$ and the sets $\{E_B : B \in \mathcal{S}\}$ are pairwise disjoint, where $0 < \gamma < 1$ is a constant. We associate with $\mathcal{S}$ the operators:

$$\mathcal{A}_{\mathcal{S},r} f(x) = \sum_{A \in \mathcal{S}} \langle f \rangle_{A,r} \cdot \mathbb{I}_A(x), \quad \mathcal{A}^*_{\mathcal{S},r} f(x) = \sum_{A \in \mathcal{S}} \langle f \rangle^*_{A,r} \cdot \mathbb{I}_A(x),$$

called sparse and strong type sparse operators, respectively. The weak-L^1 estimate of $\mathcal{A}_{\mathcal{S},1}$ in $\mathbb{R}^n$ (case $r = 1$) as well as its boundedness on L^p $(1 < p < \infty)$ were proved by Lerner [6]. The L^p-boundedness of $\mathcal{A}_{\mathcal{S},r}$ for general ball-bases was shown by the first author in [4].

We will say that a constant is admissible if it depends only on p and on the constants $\mathcal{K}$ and γ from the above definitions, and the notation $a \lesssim b$ will stand for the inequality $a \le c \cdot b$, where $c > 0$ is an admissible constant. The main result of this paper is the weak-L^r estimate of $\mathcal{A}^*_{\mathcal{S},r}$ generated by general ball-bases. More precisely, we have the following result.

Theorem 1.1. *A sparse operator of strong type $\mathcal{A}^*_{\mathcal{S},r}$, $1 \le r < \infty$, corresponding to a general ball-basis, is a bounded operator on L^p for $r < p < \infty$, and satisfies the weak-L^r estimate, that is,*

$$\left\| \mathcal{A}^*_{\mathcal{S},r}(f) \right\|_p \lesssim \|f\|_p, \quad r < p < \infty, \tag{1.2}$$

$$\mu\left\{ \mathcal{A}^*_{\mathcal{S},r}(f) > \lambda \right\} \lesssim \frac{\|f\|_r^r}{\lambda^r}, \quad \lambda > 0. \tag{1.3}$$

The proof of L^p-boundedness of $\mathcal{A}^*_{\mathcal{S},r}$ is simple and uses the duality argument as in [6]. Lerner's [6] proof of weak-L^1 estimate in $\mathbb{R}^n$ applies the standard Calderón-Zygmund decomposition argument. The Calderón-Zygmund decomposition may fail if the ball-basis is not doubling, so for the weak-L^r estimate in the case of general ball-basis we apply the function flattening technique displayed in Lemma 2.7. That is, we reconstruct the function $f \in L^r$ around the big values to get a λ-bounded function $g \in L^{2r}$, having ball averages of f dominated by those of g. As a result we will have $\|\mathcal{A}^*_{\mathcal{S},r} f\|_{r,\infty} \lesssim \|\mathcal{A}^*_{\mathcal{S},r} g\|_{2r,\infty}$, reducing the weak-$L^r$ estimate of $\mathcal{A}^*_{\mathcal{S},r}$ to weak-L^{2r}.

2. Auxiliary lemmas

Recall some definitions and propositions from [4]. We say that a set $E \subset X$ is bounded if $E \subset B$ for a ball $B \in \mathfrak{B}$.

Lemma 2.1 ([4]). *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$. If $E \subset X$ is bounded and $\mathcal{G}$ is a family of balls with $E \subset \bigcup_{G \in \mathcal{G}} G$, then there exists a finite or infinite sequence of pairwise disjoint balls $G_k \in \mathcal{G}$ such that $E \subset \bigcup_k G_k^*$.*

Definition 2.1. *For a set $E \in \mathfrak{M}$ a point $x \in E$ is said to be a density point if for any $\varepsilon > 0$ there exists a ball $B \ni x$ such that $\mu(B \cap E) > (1-\varepsilon)\mu(B)$. We say that a measure space $(X, \mathfrak{M}, \mu)$ satisfies the density property if almost all points of any measurable set are density points.*

Lemma 2.2 ([4]). *Any ball-basis satisfies the density property.*

The L^r maximal function associated to the ball-basis $\mathfrak{B}$ we denote by

$$M_r f(x) = \sup_{B \in \mathfrak{B}:\, x \in B} \langle f \rangle_{B,r}$$

Lemma 2.3 ([4]). *If $1 \le r < p \le \infty$, then the maximal function M_r satisfies the strong L^p and weak-L^r inequalities.*

Definition 2.2. *We say that $B \in \mathfrak{B}$ is a λ-ball for a function $f \in L^r(X)$ if*

$$\langle f \rangle_{B,r} > \lambda.$$

If, in addition, there is no λ-ball $A \supset B$ satisfying $\mu(A) \ge 2\mu(B)$, then B is said to be a maximal λ-ball for f.

Lemma 2.4. *Let the function $f \in L^r(X)$ have bounded support, and let $\lambda > 0$. There exist pairwise disjoint maximal λ-balls $\{B_k\}$ such that*

$$G_\lambda = \{x \in X : M_r f(x) > \lambda\} \subset \bigcup_k B_k^*. \tag{2.1}$$

Proof. Since f has bounded support, one can easily check that the set G_λ is also bounded. Besides, any λ-ball is in some maximal λ-ball. Thus we conclude that $G_\lambda = \bigcup_\alpha B_\alpha$, where each B_α is a maximal λ-ball. Applying the above covering lemma, we find a sequence of pairwise disjoint balls B_k such that

$$G_\lambda \subset \bigcup_k B_k^*$$

and so we have (2.1). □

Let $B \subset (a, b)$ be a Lebesgue measurable set. For a given positive real $\kappa \le |B|$ denote

$$a(\kappa, B) = \inf\{a' : |(a, a') \cap B)| \ge \kappa\}, \quad L(\kappa, B) = (a, a(\kappa, G)) \cap B.$$

Observe that $L(\kappa, B)$ determines the "leftmost"set of measure κ in B and $a(\kappa, B)$ does not depend on the choice of a.

Lemma 2.5. *Let $A \subset B \subset (a, b)$ be Lebesgue measurable sets on the real line, and let $0 < \kappa \le |A|$. Then we have*

$$|L(\kappa, B) \triangle L(\kappa, A)| \le 2|B \setminus A|.$$

Proof. Obviously, we have $a \le a(\kappa, B) \le a(\kappa, A) \le b$. Since $|L(\kappa, B)| = |L(\kappa, B)|$, the sets

$$L(\kappa, B) \setminus L(\kappa, A) = \big((a, a(\kappa, B)) \cap (B \setminus A)\big),$$
$$L(\kappa, A) \setminus L(\kappa, B) = \big((a(\kappa, B), a(\kappa, A)) \cap A\big).$$

have the same measure. So, we get

$$|L(\kappa, B) \triangle L(\kappa, A)| = 2 \left|\big((a, a(\kappa, B)) \cap (B \setminus A)\big)\right| \le 2|B \setminus A|.$$

Lemma 2.6. *Let $(X, \mathfrak{M}, \mu)$ be a non-atomic measure space and G_k be a finite or infinite sequence of measurable sets in X. If a sequence of numbers $\xi_k \ge 0$ satisfies $\sum_k \xi_k < \infty$ and the condition*

$$\sum_{j:\, \mu(G_j) \le \mu(G_k),\, G_j \cap G_k \ne \varnothing} \xi_j \le \mu(G_k), \quad k = 1, 2, \ldots, \tag{2.2}$$

then there exist pairwise disjoint measurable sets $\tilde{G}_k \subset G_k$ such that

$$\mu(\tilde{G}_k) = \xi_k, k = 1, 2, \ldots. \tag{2.3}$$

Proof. Without loss of generality we can suppose that $\mu(G_k)$ is decreasing. Since the measure space is non-atomic, we can also suppose that G_k are Lebesgue measurable sets in $\mathbb{R}$. We first assume that the sequence G_k, $k = 1, 2, \ldots, n$, is finite. We apply backward induction. The existence of $\tilde{G}_n \subset G_n$ satisfying $\mu(\tilde{G}_n) = \xi_n$ follows from (2.2), since the latter implies $\xi_n \le \mu(G_n)$ and we have that the measure is non-atomic. We define $\tilde{G}_n$ to be the leftmost set in G_n, that is, $\tilde{G}_n = L(\xi_n, G_n)$. Suppose by induction we have defined pairwise disjoint sets $\tilde{G}_k \subset G_k$ satisfying (2.3) for $l \le k \le n$. From (2.2) it follows that

$$\mu\left(G_{l-1} \setminus \bigcup_{k=l}^{n} \tilde{G}_k\right) \ge \mu(G_{l-1}) - \sum_{l \le j \le n:\, G_j \cap G_{l-1} \ne \varnothing} \mu(\tilde{G}_j) \ge \xi_{l-1}.$$

Hence we can define $\tilde{G}_{l-1} = L(\xi_{l-1}, G_{l-1} \setminus \bigcup_{k=l}^{n} \tilde{G}_k)$. To proceed the general case we apply the finite case that we have proved. Then for each n we find a family of pairwise disjoint sets $G_k^{(n)}$, $k = 1, 2, \ldots, n$ such that $\mu(G_k^{(n)}) = \xi_k$ for $1 \le k \le n$. Applying Lemma 2.5 and analyzing once again the leftmost selection argument of the tilde sets, one can observe that

$$\mu(G_k^{(n+1)} \triangle G_k^{(n)}) \le \sum_{j=k}^{n} \mu(G_{n+1}^{(n+1)} \cap G_j^{(n)}) \le \xi_{n+1}.$$

So, we conclude that

$$\mu(G_k^{(m)} \triangle G_k^{(n)}) \le \sum_{k=n+1}^{m} \xi_k, \quad m > n \ge k.$$

The last inequality implies that for a fixed k the sequence $\mathbb{I}_{G_k^{(m)}}$ converges in L^1-norm as $m \to \infty$. Moreover, one can see that the limiting function is again an indicator function of a set $\tilde{G}_k$, and the sequence $\tilde{G}_k$ satisfies the conditions of the lemma. □

Lemma 2.7. *Let $(X, \mathfrak{M}, \mu)$ be a non-atomic measure space, and let $f \in L^r(X)$, $1 \le r < \infty$, be a boundedly supported positive function. Then for any $\lambda > 0$ there exists a measurable set $E_\lambda \subset X$ such that*

$$\mu(E_\lambda) \lesssim \|f\|_r^r / \lambda^r, \quad \{x \in X : M_r f(x) > \lambda\} \subset E_\lambda, \tag{2.4}$$

and the function

$$g(x) = f(x) \cdot \mathbb{I}_{X \setminus E_\lambda}(x) + \lambda \cdot \mathbb{I}_{E_\lambda}(x) \tag{2.5}$$

satisfies the conditions:

$$g(x) \le \lambda \text{ a.e. on } X, \quad \langle f \rangle_{B,r} \lesssim \langle g \rangle_{B^*,r} \text{ whenever } B \in \mathfrak{B},\ B \not\subset E_\lambda. \tag{2.6}$$

Proof. Applying Lemma 2.4 we find a sequence of pairwise disjoint maximal λ-balls B_k satisfying (2.1). Thus, applying the density property (Lemma 2.2), one can conclude that

$$f(x) \le \lambda \text{ for a.a. } x \in X \setminus \bigcup_k B_k^*. \tag{2.7}$$

Given B_k, we associate the family of balls

$$\mathfrak{B}_k = \{B \in \mathfrak{B} : B \cap B_k^* \ne \varnothing,\ \mu(B) > 2\mu(B_k^*)\}. \tag{2.8}$$

Observe that if one of these families, say $\mathfrak{B}_{k_0}$, is empty, then in view of conditions B2) and B4), one can easily check that $X \subset B_{k_0}^{**}$. Then defining $E_\lambda = X$, the claim of the lemma will be satisfied. Hence we can assume that each $\mathfrak{B}_k$ is nonempty, and so, there is a ball $G_k \in \mathfrak{B}_k$ such that

$$\mu(G_k) \le 2 \inf_{B \in \mathfrak{B}_k} \mu(B). \tag{2.9}$$

From λ-maximality of B_k and the inequality $\mu(G_k) > 2\mu(B_k^*)$, we get $B_k^* \subset G_k^*$, $\langle f \rangle_{G_k^*, r} \le \lambda$. This implies

$$\frac{1}{\lambda^r} \int_{G_k^*} f^r \le \mu(G_k^*) \le c \cdot \mu(G_k), \tag{2.10}$$

where $c > 0$ is an admissible constant. Denote

$$D_1 = B_1^*, \quad D_k = B_k^* \setminus \cup_{1 \le j \le k-1} B_j^*, \ k \ge 2,$$

and consider the numerical sequence $\xi_k = \frac{\delta}{\lambda^r} \int_{D_k} f^r$, $k = 1, 2, \ldots$, for some constant $\delta > 0$. Taking into account (2.10), for a small admissible constant $\delta > 0$ we obtain

$$\begin{aligned} \bigcup_{j:\, \mu(G_j) \le \mu(G_k),\, G_j \cap G_k \ne \varnothing} \xi_j &= \frac{\delta}{\lambda^r} \bigcup_{j:\, \mu(G_j) \le \mu(G_k),\, G_j \cap G_k \ne \varnothing} \int_{D_j} f^r \\ &\le \frac{\delta}{\lambda^r} \int_{G_k^*} f^r \le c\delta\mu(G_k) \le \mu(G_k), \end{aligned}$$

which gives condition (2.2). Since our measure space in non-atomic, applying Lemma 2.6, we find pairwise disjoint subsets $\tilde{G}_k \subset G_k$ such that

$$\mu(\tilde{G}_k) = \frac{\delta}{\lambda^r} \int_{D_k} f^r, \quad k = 1, 2, \ldots. \tag{2.11}$$

The disjointness of the sets D_k implies

$$\sum_k \mu(\tilde{G}_k) = \frac{\delta}{\lambda^r} \sum_k \int_{D_k} f^r \lesssim \frac{\|f\|_r^r}{\lambda^r}. \tag{2.12}$$

From the λ-maximality and disjointness property of B_k, we get

$$\mu\left(\bigcup_k B_k^{**}\right) \lesssim \sum_k \mu\left(B_k\right) \le \frac{1}{\lambda^r} \sum_k \int_{B_k} f^r \le \frac{\|f\|_r^r}{\lambda^r}. \tag{2.13}$$

Denote $E_\lambda = \left(\bigcup_k \tilde{G}_k\right) \bigcup (\bigcup_k B_k^{**})$. From (2.12) and (2.13) we get $\mu(E_\lambda) \lesssim \|f\|_r^r / \lambda^r$, and (2.7) implies (2.6). Hence it remains to prove that the function g satisfies (2.6). Take a ball $B \in \mathfrak{B}$ with $B \not\subset E_\lambda$. First of all observe that for each B_k satisfying $B \cap B_k^* \ne \varnothing$ we have $\mu(B) > 2\mu(B_k^*)$, since otherwise we would have $B \subset B_k^{**} \subset E_\lambda$, which is not true. Thus, whenever $B \cap B_k^* \ne \varnothing$ we have $B \in \mathfrak{B}_k$, then we get $\mu(G_k) \le 2\mu(B)$, and so $\tilde{G}_k \subset G_k \subset B^*$. Besides, from (2.7) and the definition of g it follows that $f(x) \le g(x)$ a.e. on $X \setminus \cup_k B_k^*$. Hence, using (2.11) and the disjointness of $\tilde{G}_k$, we can write

$$\langle f\rangle_{B,r}^r = \frac{1}{\mu(B)}\left(\int\limits_{B\cap(\cup_k B_k^*)} f^r + \int\limits_{B\setminus\cup_k B_k^*} f^r\right) \le \frac{1}{\mu(B)}\left(\sum_{k:\, B_k^*\cap B\neq\varnothing}\int\limits_{B\cap D_k} f^r + \int\limits_{B\setminus\cup_k B_k^*} g^r\right)$$

$$\le \frac{1}{\mu(B)}\left(\sum_{k:\, B_k^*\cap B\neq\varnothing}\int_{D_k} f^r + \int_B g^r\right) = \frac{1}{\mu(B)}\left(\sum_{k:\, B_k^*\cap B\neq\varnothing}\frac{\lambda^r\mu(\tilde{G}_k)}{\delta} + \int_B g^r\right)$$

$$= \frac{1}{\delta\mu(B^*)}\left(\sum_{k:\, B_k^*\cap B\neq\varnothing}\int_{\tilde{G}_k} g^r + \int_{B^*} g^r\right) \lesssim \langle g\rangle_{B^*,r}^r.$$

This implies (2.6). □

3. Proof of Theorem 1.1

Proof of L^p-boundedness. For any $B \in \mathcal{S}$ we have $\langle f\rangle_{B,r}^* \le M_r f(x)$ for all $x \in B$, and therefore $\langle f\rangle_{B,r}^* \le \langle M_r f\rangle_{B,r}$, $B \in \mathfrak{B}$. Let E_B be the disjoint portions of the sparse collection of balls satisfying $\mu(E_B) \ge \gamma \cdot \mu(B)$. Also, suppose that $r < p < \infty$ and $q = p/(p-1)$. Thus, for positive functions $f \in L^p$ and $g \in L^q(X)$, we can write

$$\int_X \mathcal{A}_{\mathcal{S},r}^* f \cdot g d\mu \le \sum_{B\in\mathcal{S}} \langle M_r f\rangle_{B,r}\int_B g d\mu = \sum_{B\in\mathcal{S}}\langle M_r f\rangle_{B,r}\cdot\langle g\rangle_{B,1}\cdot\mu(B)$$
$$\le \gamma^{-1}\sum_{B\in\mathcal{S}}\langle M_r f\rangle_{B,r}\cdot(\mu(E_B))^{1/p}\cdot\langle g\rangle_{B,1}\cdot(\mu(E_B))^{1/q}$$
$$\le \gamma^{-1}\left(\sum_{B\in\mathcal{S}}\langle M_r f\rangle_{B,r}^p\cdot\mu(E_B)\right)^{1/p}\cdot\left(\sum_{B\in\mathcal{S}}\langle g\rangle_{B,1}^q\cdot\mu(E_B)\right)^{1/q}$$
$$\le \gamma^{-1}\|M_r(M_r f)\|_p\|M_1(g)\|_q \lesssim \|M_r f\|_p\cdot\|g\|_q \lesssim \|f\|_p\cdot\|g\|_q,$$

which completes the proof of L^p-boundedness. □

Proof of weak-L^r estimate. Without loss of generality, we can assume that our measure space $(X, \mathfrak{M}, \mu)$ is non-atomic, since any measure space can be extended to a non-atomic measure space by splitting the atoms as follows. Suppose $A \subset \mathfrak{M}$ is the family of atomic elements of the measure space $(X, \mathfrak{M}, \mu)$, that is, for any $a \in A$ we have $\mu(a) > 0$ and there is no proper $\mathfrak{M}$-measurable set in a. We can suppose that each atom is continuum and let $(a, \mathfrak{M}_a, \mu_a)$ be a a non-atomic measure space on $a \in A$ such that $\mu_a(a) = \mu(a)$. Denote by $\mathfrak{M}'$ the σ-algebra on X generated by $\mathfrak{M}$ and by all $\mathfrak{M}_a$, $a \in A$. Let μ' be an extension of μ such that $\mu'(E) = \mu_a(E)$ for any $\mathfrak{M}_a$-measurable set $E \subset a$. Hence, $(X, \mathfrak{M}', \mu')$ provides a non-atomic extension of the measure space $(X, \mathfrak{M}, \mu)$.

Now let f be a $\mathfrak{M}$-measurable function. The balls are $\mathfrak{M}$-measurable, and so they can not contain an atom a partially. Thus, the left and right sides of inequality (1.3) are not changed if we consider $(X, \mathfrak{M}', \mu')$ instead of the initial measure space. Hence, we can suppose that $(X, \mathfrak{M}, \mu)$ is itself non-atomic. Applying Lemma 2.7, we find a function g satisfying the conditions of the lemma. From (2.6) we get $\langle f\rangle^*_{B,r} \le \langle g\rangle^*_{B,r}$ for any $B \in \mathcal{S}$ with $B \not\subset E_\lambda$ and hence, $\mathcal{A}^*_{\mathcal{S},r} f(x) \le \mathcal{A}^*_{\mathcal{S},r} g(x)$, $x \in X \setminus E_\lambda$. Therefore, using the L^{2r} bound of $\mathcal{A}^*_{\mathcal{S},r}$, we obtain

$$\mu\{x \in X : \mathcal{A}^*_{\mathcal{S},r} f(x) > \lambda\} \le \mu(E_\lambda) + \mu\{x \in X \setminus E_\lambda : \mathcal{A}^*_{\mathcal{S},r} g(x) > \lambda\}$$
$$\lesssim \frac{\|f\|_r^r}{\lambda^r} + \frac{1}{\lambda^{2r}} \int_{X\setminus E_\lambda} |g|^{2r} \le \frac{\|f\|_r^r}{\lambda^r} + \frac{\lambda^r}{\lambda^{2r}} \int_{X\setminus E_\lambda} f^r \le \frac{2\|f\|_r^r}{\lambda^r}.$$

This completes the proof of theorem 1.1. □

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КОРРЕКТНАЯ РАЗРЕШИМОСТЬ ЗАДАЧИ ДИРИХЛЕ В ПОЛУПРОСТРАНСТВЕ ДЛЯ РЕГУЛЯРНЫХ УРАВНЕНИЙ

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Аннотация. В работе изучается задача Дирихле в полупространстве для регулярных гипоэллиптических уравнений. Применяя специальное интегральное представление, строются приближенные решения для данной задачи, и тем самым доказывается корректная разрешимость.

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1. Введение

В работе рассматривается задача Дирихле в полупространстве для специальных (мультиоднородных) регулярных гипоэллиптических уравнений с нулевыми граничными условиями. Задачи такого типа появляются при изучении мультианизотропных процессов и трудность их изучения заключается в том, что соответствующий полный символ не обобщенно однородный, как для эллиптических или полуэллиптических уравнений (см. [1]-[10]), а мультиоднородный, и построение приближенного решения для таких уравнений представляет от себя трудность. Но, применяя специальное интегральное представление функций, которое охватывает все вершины вполне правильного многогранника Ньютона (см. [11]-[14]), удается построить приближенные решения через интегральные операторы. Аналогичные вопросы во всем пространстве $\mathbb{R}^n$ были изучены в работе [15]. В данной работе изучается вопрос о разрешимости задачи Дирихле в соболевских пространствах $W_p^{\mathfrak{N}}(\mathbb{R}_+^n)$ $(1 < p < \infty)$.

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Пусть $\mathbb{R}^n$ есть n-мерное евклидовое пространство, а $\mathbb{Z}_+^n$ – множество мультииндексов из $\mathbb{R}^n$. Элементы $\mathbb{R}^n$ и $\mathbb{Z}_+^n$ будем представлять в виде (ξ,τ), (α,α_n), где, соответственно, $\xi \in \mathbb{R}^{n-1}$, $\tau \in \mathbb{R}^1$, $\alpha \in \mathbb{Z}_+^{n-1}$, $\alpha_n \in \mathbb{Z}_+^1$.

Для мультииндекса $\alpha \in \mathbb{Z}_+^{n-1}$ обозначим $|\alpha| = \alpha_1 + \cdots + \alpha_{n-1}$, $D_k = \frac{1}{i}\frac{\partial}{\partial x_k}$ $(k = 1,\dots,n)$. $D^\alpha = D_1^{\alpha_1}\dots D_{n-1}^{\alpha_{n-1}}$ есть обобщенная производная по С.Л. Соболеву порядка $|\alpha|$

Для данного набора мультииндексов из $\mathbb{Z}_+^{n-1}$ обозначим через $\mathfrak{N}$ наименьший выпуклый многогранник, содержащий все точки этого набора. Многогранник $\mathfrak{N}$ называется вполне правильным, если имеет вершину в начале координат и на всех координатных осях, а внешние нормали всех $(n-2)$-мерных некоординатных граней имеют положительные координаты. Пусть $\mathfrak{N}_i^{n-2}$ $(i = 1,\dots,I_{n-2})$ есть $(n-2)$-мерные некоординатные грани многогранника $\mathfrak{N}$, $\partial'\mathfrak{N}$ – множество всех тех мультииндексов, которые принадлежат хотя бы одной $(n-2)$-мерной некоординатной грани многогранника $\mathfrak{N}$, $\mathfrak{N}^{(0)} = \mathfrak{N} \setminus \partial'\mathfrak{N}$, $\{\alpha^1,\alpha^2,\dots,\alpha^M\}$ – множество всех вершин многогранника $\mathfrak{N}$ отличных от нуля. И пусть μ^i $(i = 1,\dots,I_{n-2})$ есть такая внешняя нормаль грани $\mathfrak{N}_i^{n-2}$, при которой уравнение гиперплоскости, содержащей данную грань, задается формулой $(\alpha,\mu^i) = 1$. Предположим, что многогранник $\mathfrak{N}$ имеет $(n-2)$-мерные грани, содержащие точки $\{\alpha^1,\dots,\alpha^{n-1}\} \setminus \{\alpha^i\}$ $(i = 1,\dots,n-1)$, где $\alpha^i = (0,\dots,0,l_i,0,\dots,0)$. Внешнюю нормаль данной грани обозначим через μ^i $(i = 1,\dots,n-1)$. Обозначим также через $\mu^0 = \left(\mu_1^0,\dots,\mu_{n-1}^0\right) = (1/l_1,\dots,1/l_{n-1})$. Пусть $\gamma = (\gamma_1,\dots,\gamma_{n-1})$ есть точка пересечения гиперплоскостей, содержащих $(n-2)$-мерные грани с внешними нормалями $\mu^1,\dots,\mu^{n-1}$, и для определенности предположим, что $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-r} \le \gamma_{n-r+1} \le \cdots \le \gamma_{n-1}$, где $r = 0,1,\dots,n-2$. Предположим также, что $\max\limits_{i=1,\dots,I_{n-2}} |\mu^i| - \min\limits_{i=1,\dots,r+1} |\mu^i| < 1$. Обозначим через $\mathfrak{M} \subset \mathbb{R}^n$ многогранник в $\mathbb{Z}_+^n$, имеющий вершины $\beta^1,\dots,\beta^{M+1}$, где $\beta^i = (\alpha^i,0)$ $(i = 1,\dots,M)$, $\beta^{M+1} = (0,\dots,0,2m)$ и в $\mathbb{R}_+^n$ рассмотрим дифференциальный оператор $P(D_x,D_{x_n})$ с постоянными действительными коэффициентами a_i $(i = 1,\dots,M)$

$$P(D_x,D_{x_n}) = D_{x_n}^{2m} + \sum_{i=1}^{M} a_i D^{\alpha^i} \tag{1.1}$$

с полным символом $P(\xi,\xi_n) = \xi_n^{2m} + \sum\limits_{i=1}^{M} a_i \xi^{\alpha^i}$.

Предположим, что оператор (1.1) есть регулярный оператор, то есть существует постоянная $C>0$ такая, что для любого $(\xi,\xi_n)\in\mathbb{R}^n$ имеет место неравенство

$$|P(\xi,\xi_n)|\geq C\left(\sum_{i=1}^{M}\left|\xi^{\alpha^i}\right|+\xi_n^{2m}\right). \tag{1.2}$$

Примерами регулярных операторов являются эллиптические, квазиэллиптические операторы, а также операторы типа $P(D)=\sum\limits_{\alpha\in\partial'\mathfrak{N}}D^\alpha$, где $\mathfrak{N}$ – произвольный вполне правильный многогранник с вершинами, имеющие четные координаты. Для вполне правильного многогранника $\mathfrak{M}$ обозначим через $W_p^{\mathfrak{M}}\left(\mathbb{R}_+^n\right)=\{f:$ $f\in L_p\left(\mathbb{R}_+^n\right), D^{\beta^i}f\in L_p\left(\mathbb{R}_+^n\right), i=1,\dots,M+1\}$ и называем мультианизотропным пространством Соболева с нормой $\|f\|_{W_p^{\mathfrak{M}}(\mathbb{R}_+^n)}=\sum\limits_{i=1}^{M+1}\left\|D^{\beta^i}f\right\|_{L_p\left(\mathbb{R}_+^n\right)}+\|f\|_{L_p\left(\mathbb{R}_+^n\right)}$.

Из регулярности следует, что вершины многогранника $\mathfrak{N}$ имеют четные координаты, и при действительных коэффициентах a_i $(i=1,\dots,M)$ многочлен $P(\xi,\tau)$ по τ имеет ровно m корней с положительными и отрицательными мнимыми частями. Для любого фиксированного ξ обозначим через $\tau_i^{\pm}(\xi)$ $(i=1,\dots,m)$ эти корни. Обозначим также

$$M^+(\xi,\tau)=\prod_{j=1}^{m}(\tau-\tau_j^+(\xi))=\sum_{i=0}^{m}b_i(\xi)\tau^{m-i},$$

$$M_k^+(\xi,\tau)=\sum_{i=0}^{k}b_i(\xi)\tau^{m-i},\quad \chi=\left(|\mu^0|+\frac{1}{2m}\right)\left(1-\frac{1}{p}\right),$$

где $p>1$ некоторое число. В $\mathbb{R}_+^n$ рассмотрим следующую задачу Дирихле:

$$P(D_x,D_{x_n})U = f(x,x_n),\quad x_n>0,\ x\in\mathbb{R}^{n-1}, \tag{1.3}$$

$$\left.\frac{\partial^i U}{\partial x_n^i}\right|_{x_n=0} = 0,\quad i=0,1,\dots,m-1. \tag{1.4}$$

В настоящей работе изучается разрешимость задачи (1.3)-(1.4), а именно будут доказаны следующие теоремы, которые суть основные результаты статьи.

Теорема 1.1. *Если $f\in L_p(\mathbb{R}_+^n)$ $(1<p<\infty)$ и имеет компактный носитель, то при $\chi>1$ задача (1.3)-(1.4) имеет единственное решение U из класса $W_p^{\mathfrak{M}}(\mathbb{R}_+^n)$, и для некоторой постоянной $C>0$ (не зависящей от f) имеет место оценка*

$$\|U\|_{W_p^{\mathfrak{M}}(\mathbb{R}_+^n)}\leq C\|f\|_{L_p(\mathbb{R}_+^n)}. \tag{1.5}$$

А при $\chi\leq 1$ имеет место следующая теорема.

Теорема 1.2. *Пусть $\chi \le 1$ и $f \in L_p(\mathbb{R}^n_+)$ $(1 < p < \infty)$ с компактным носителем удовлетворяет следующим условиям ортогональности:*

$$\int\limits_{\mathbb{R}^{n-1}} x^s f(x, x_n) dx = 0 \tag{1.6}$$

при $|s| = 0, 1, \ldots, L-1$, где L – натуральное число, определяемое из неравенств

$$\chi + L\mu^0_{min} > 1 \ge \chi + (L-1)\mu^0_{min}, \tag{1.7}$$

где $\mu^0_{min} = \min\limits_{i=1,\ldots,n-1} \mu^0_i$. Тогда для любой такой функции f задача (1.3)-(1.4) имеет единственное решение из класса $W^{\mathfrak{M}}_p(\mathbb{R}^n_+)$, для которой имеет место неравенство (1.5).

2. Построение приближенного решения задачи Дирихле в $\mathbb{R}^n_+$

Как и в работах [12]-[15], для параметра $\nu > 0$ и натурального числа k обозначим

$$\rho_{\mathfrak{N}}(\xi) = \left(\xi^{2\alpha^1} + \cdots + \xi^{2\alpha^M}\right)^{\frac{1}{2}}, \quad G_0(\xi, \nu) = e^{-(\nu\rho_{\mathfrak{N}}(\xi))^{2k}},$$
$$G_1(\xi, \nu) = (-2k)(\nu\rho_{\mathfrak{N}}(\xi))^{2k-1} e^{-(\nu\rho_{\mathfrak{N}}(\xi))^{2k}},$$
$$G_2(\xi, \nu) = (-2k)\nu^{2k-1}(\rho_{\mathfrak{N}}(\xi))^{2k} e^{-(\nu\rho_{\mathfrak{N}}(\xi))^{2k}},$$

а $\hat{G}_i(t, \nu)$ $(i = 0, 1, 2)$ есть преобразования Фурье для соответствующих функций.

В работах [12]-[14] изучено усреднение функции $f \in L_p(\mathbb{R}^{n-1})$ через ядро $G_0(\xi, \nu)$:

$$f_\nu(x) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int\limits_{\mathbb{R}^{n-1}} f(t)\hat{G}_0(t - x, \nu) dt$$

и почти для всех $x \in \mathbb{R}^{n-1}$ получено интегральное представление

$$f(x) = \lim_{h \to 0} \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int\limits_h^{h^{-1}} d\nu \int\limits_{\mathbb{R}^{n-1}} f(t)\hat{G}_2(t - x, \nu) dt. \tag{2.1}$$

Применяя (2.1), построим приближенное решение задачи (1.3)-(1.4).

Так как оператор $P(D_x, D_{x_n})$ регулярный (см. неравенство (1.2)), то корни многочлена $P(\xi, \tau)$ по τ имеют вид $\tau^{\pm}_k(\xi) = \sqrt[2m]{\sum\limits_{i=1}^{M} a_i \xi^{\alpha^i}} \cdot \omega^{\pm}_k$ $(k = 1, \ldots, m)$, где $\omega^{\pm}_k$ $(k = 1, \ldots, m)$ – корни $\sqrt[2m]{-1}$, следовательно, для некоторых положительных постоянных δ и δ_1 имеют места соотношения

$$\delta \sqrt[2m]{\rho_{\mathfrak{N}}(\xi)} \le |Im\tau_k(\xi)| \le \delta_1 \sqrt[2m]{\rho_{\mathfrak{N}}(\xi)} \ \ (k = 1, \ldots, m), \tag{2.2}$$

так как $\sum\limits_{i=1}^{M} a_i\xi^{\alpha^i}$ эквивалентно выражению $\rho_{\mathfrak{N}}(\xi)$.

Как и в работе [8], обозначим

$$G^+(\xi) = \{\lambda \in \mathbb{C}; |\lambda| < 2\rho_{\mathfrak{N}}^{\frac{1}{2m}}(\xi); Im(\lambda) > \delta\rho_{\mathfrak{N}}^{\frac{1}{2m}}(\xi)\},$$
$$G^-(\xi) = \{\lambda \in \mathbb{C}; |\lambda| < 2\rho_{\mathfrak{N}}^{\frac{1}{2m}}(\xi); Im(\lambda) < -\delta\rho_{\mathfrak{N}}^{\frac{1}{2m}}(\xi)\},$$

а $\Gamma^+(\xi)$ и $\Gamma^-(\xi)$ соответствующие границы этих областей. Рассмотрим следующие контурные интегралы:

$$J_+(\xi, x_n) = \frac{1}{2\pi}\int\limits_{\Gamma^+(\xi)} \frac{e^{ix_n\lambda}}{P(\xi,\lambda)}d\lambda, \quad J_-(\xi, x_n) = \frac{1}{2\pi}\int\limits_{\Gamma^-(\xi)} \frac{e^{ix_n\lambda}}{P(\xi,\lambda)}d\lambda,$$

$$J_j(\xi, x_n) = \frac{1}{2\pi i}\int\limits_{\Gamma^+(\xi)} \frac{e^{ix_n\lambda}M^+_{m-j}(\xi,\lambda)}{M^+(\xi,\lambda)}d\lambda,$$

$$I_j(\xi, x_n) = \left(\frac{\partial}{\partial y_n}\right)^{j-1} J_-(\xi, y_n - x_n)\bigg|_{y_n=0}, \quad j = 1, \dots, m.$$

Лемма 2.1. *При $x_n > 0$ и $\xi \neq 0$ имеют место оценки*

(2.3)
$$\left|\xi^\alpha D_\xi^\alpha D_{x_n}^k J_+(\xi, x_n)\right| + \left|\xi^\alpha D_\xi^\alpha D_{x_n}^k J_-(\xi, -x_n)\right| \leq C(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}(k+1)-1} e^{-\delta x_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}},$$

(2.4)
$$\left|\xi^\alpha D_\xi^\alpha D_{x_n}^k J_j(\xi, x_n)\right| \leq C(\rho_{\mathfrak{N}}(\xi))^{\frac{k}{2m}-\frac{j-1}{2m}} e^{-\delta x_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}, \ j = 1, \dots, m$$

с некоторыми постоянными $C > 0$ и $\delta > 0$, не зависящими от ξ и x_n.

Доказательство. Как и при доказательстве леммы 2 работы [8], достаточно дифференцировать только подынтегральные выражения. При $\alpha = 0$ для $D_{x_n}^k J_+(\xi, x_n)$ имеем

$$D_{x_n}^k J_+(\xi, x_n) = \frac{1}{2\pi}\int\limits_{\Gamma^+(\xi)} \frac{\lambda^k e^{ix_n\lambda}}{P(\xi,\lambda)}d\lambda.$$

Применив свойства контурного интеграла, из неравенства (2.2) имеем, что

$$\left|D_{x_n}^k J_+(\xi, x_n)\right| \leq C(\rho_{\mathfrak{N}}(\xi))^{\frac{k}{2m}-1+\frac{1}{2m}} e^{-\delta x_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}.$$

Пусть теперь $\alpha \neq 0$. Допустим, что $\alpha = (1, 0, \dots, 0)$, $k = 0$. Тогда имеем

$$|\xi_1 D_{\xi_1} J_+(\xi, x_n)| \leq C|\xi_1 D_{\xi_1}\rho_{\mathfrak{N}}(\xi)| \int\limits_{\Gamma^+(\xi)} \frac{e^{i\lambda x_n}}{\left(\lambda^{2m} + \rho_{\mathfrak{N}}(\xi)\right)^2}d\lambda \leq$$

$$C\rho_{\mathfrak{N}}(\xi)(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}-2} e^{-\delta x_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}},$$

так как $|\xi^\alpha D_\xi^\alpha \rho_{\mathfrak{N}}(\xi)| \leq C\rho_{\mathfrak{N}}(\xi)$.

Аналогично оцениваются $\xi^\alpha D_\xi^\alpha D_{x_n}^k J_-(\xi,-x_n)$. Для оценки

$$D_{x_n}^k J_j(\xi,x_n)=\frac{1}{2\pi i}\int\limits_{\Gamma^+(\xi)}\frac{\lambda^k e^{ix_n\lambda}M_{m-j}^+(\xi,\lambda)}{M^+(\xi,\lambda)}d\lambda$$

нужно учитывать (см. [7]), что

$$\frac{1}{2\pi i}\int\limits_{\Gamma^+(\xi)}\frac{\lambda^{k-1}M_{m-j}^+(\xi,\lambda)}{M^+(\xi,\lambda)}d\lambda=\delta_j^k,\quad j,k=1,\dots,m.$$

□

Лемма 2.2. *Имеют места следующие тождества*

$$D_{x_n}^{k-1}\left(J_+(\xi,x_n)+J_-(\xi,-x_n)\right)\Big|_{x_n=0}=\delta_{2m}^k,\quad k=1,\dots,2m \tag{2.5}$$

$$D_{x_n}^{k-1}J_j(\xi,x_n)\Big|_{x_n=0}=\delta_j^k,\quad k,j=1,\dots,m. \tag{2.6}$$

Доказательство. Как и при доказательстве леммы 2.1, имеем

$$\left(D_{x_n}^{k-1}\left(J_+(\xi,x_n)+J_-(\xi,-x_n)\right)\right)\Big|_{x_n=0}=$$
$$\frac{1}{2\pi}\left(\int\limits_{\Gamma^+(\xi)}\lambda^{k-1}\frac{e^{ix_n\lambda}}{P(\xi,\lambda)}d\lambda+\int\limits_{\Gamma^-(\xi)}\lambda^{k-1}\frac{e^{ix_n\lambda}}{P(\xi,\lambda)}d\lambda\right)\Bigg|_{x_n=0}=$$
$$\frac{1}{2\pi}\int\limits_{\Gamma(\xi)}\lambda^{k-1}\frac{e^{ix_n\lambda}}{P(\xi,\lambda)}d\lambda\Big|_{x_n=0}=\delta_{2m}^k,\quad k=1,\dots,2m,$$

где $\Gamma(\xi)$ – контур, охватывающий все корни $\tau_j^\pm(\xi)$, $j=1,\dots,m$.

Для $D_{x_n}^{k-1}J_j(\xi,x_n)$ имеем

$$D_{x_n}^{k-1}J_j(\xi,x_n)\Big|_{x_n=0}=\frac{1}{2\pi i}\int\limits_{\Gamma^+(\xi)}\frac{\lambda^{k-1}e^{ix_n\lambda}M_{m-j}^+(\xi,\lambda)}{M^+(\xi,\lambda)}d\lambda\Big|_{x_n=0}=\delta_j^k,$$

$k,j=1,\dots,m.$ □

Теперь переходим к построению приближенного решения задачи (1.3)-(1.4). Будем применять методы работ [8] и [12], то есть построим приближенное решение, как и в работе [8], с применением специальных ядер $G_j(\xi,\nu)$ $(j=0,1,2)$ из работы [12]. Для этого определим следующие функции:

$$U_h^+(x,x_n) =$$
$$\frac{1}{(2\pi)^{n-1}}\int\limits_h^{h^{-1}}\int\limits_0^{x_n}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}} e^{i(x-y)\xi}G_2(\xi,\nu)J_+(\xi,x_n-y_n)f(y,y_n)d\xi dy dy_n d\nu,$$
$$U_h^-(x,x_n) =$$
$$-\frac{1}{(2\pi)^{n-1}}\int\limits_h^{h^{-1}}\int\limits_{x_n}^{\infty}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}} e^{i(x-y)\xi}G_2(\xi,\nu)J_-(\xi,x_n-y_n)f(y,y_n)d\xi dy dy_n d\nu,$$
$$U_{jh}(x,x_n) =$$
$$\frac{1}{(2\pi)^{n-1}}\int\limits_h^{h^{-1}}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}} e^{i(x-y)\xi}G_2(\xi,\nu)J_j(\xi,x_n)\int\limits_0^{\infty}I_j(\xi,y_n)f(y,y_n)dy_n d\xi dy d\nu,$$
$$j=1,\ldots,m,$$

$$U_h(x,x_n) = U_h^+(x,x_n) + U_h^-(x,x_n) + \sum_{j=1}^{m} U_{jh}(x,x_n) \tag{2.7}$$

и докажем, что $U_h(x,x_n)$ – приближенные решения задачи (1.3)-(1.4).

3. Вспомогательные леммы

Для доказательства основных теорем нам понадобятся следующие вспомогательные леммы.

Лемма 3.1. *Если $\beta^i = (\alpha^i,0)$ или $\beta^{M+1} = (0,\ldots,0,2m)$, где $\alpha^i \in \partial'\mathfrak{N}$ ($i = 1,\ldots,M$), то $D^{\beta^i}U_h^{\pm} \in L_p(\mathbb{R}^n)$ ($i = 1,\ldots,M+1$) и для некоторой положительной постоянной C (не зависящей от h) имеет место неравенство*

$$\left\| D^{\beta^i}(U_h^+ + U_h^-) \right\|_{L_p(\mathbb{R}_+^n)} \le C\|f\|_{L_p(\mathbb{R}_+^n)}, \tag{3.1}$$

причем при $h_1,h_2 \to 0$

$$\left\| D^{\beta^i}(U_{h_1}^+ + U_{h_1}^-) - D^{\beta^i}(U_{h_2}^+ + U_{h_2}^-) \right\|_{L_p(\mathbb{R}_+^n)} \to 0. \tag{3.2}$$

Доказательство. В начале рассмотрим случай $\beta^{M+1}=(0,\dots,0,2m)$. Из представления $U_h^{\pm}(x,x_n)$, как и в работе [8], имеем, что

$$D_{x_n}^{2m}(U_h^+(x,x_n)+U_h^-(x,x_n))=$$

$$\frac{1}{(2\pi)^{n-1}}D_{x_n}^{2m}\int\limits_h^{h^{-1}}\int\limits_0^{x_n}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}e^{i(x-y)\xi}G_2(\xi,\nu)(J_+(\xi,x_n-y_n)+$$

$$J_-(\xi,x_n-y_n))f(y,y_n)d\xi dy dy_n d\nu-$$

$$\frac{1}{(2\pi)^{n-1}}\int\limits_h^{h^{-1}}\int\limits_0^{\infty}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}e^{i(x-y)\xi}G_2(\xi,\nu)D_{x_n}^{2m}J_-(\xi,x_n-y_n)f(y,y_n)d\xi dy dy_n d\nu.$$

Отсюда и из тождеств (2.5) имеем

$$D_{x_n}^{2m}(U_h^+(x,x_n)+U_h^-(x,x_n))=$$

$$\frac{1}{(2\pi)^{n-1}}\int\limits_h^{h^{-1}}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}e^{i(x-y)\xi}G_2(\xi,\nu)f(y,x_n)d\xi dy d\nu+$$

$$\frac{1}{(2\pi)^{n-1}}\int\limits_h^{h^{-1}}\int\limits_0^{x_n}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}e^{i(x-y)\xi}G_2(\xi,\nu)D_{x_n}^{2m}J_+(\xi,x_n-y_n)f(y,y_n)d\xi dy dy_n d\nu-$$

$$\frac{1}{(2\pi)^{n-1}}\int\limits_h^{h^{-1}}\int\limits_{x_n}^{\infty}\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}e^{i(x-y)\xi}G_2(\xi,\nu)D_{x_n}^{2m}J_-(\xi,x_n-y_n)f(y,y_n)d\xi dy dy_n d\nu=$$

$$f_h(x,x_n)+v_h^+(x,x_n)+v_h^-(x,x_n).$$

Из интегрального представления (2.1) следует, что для некоторой постоянной $C>0$ $\|f_h\|_{L_p(\mathbb{R}^n_+)}\le C\|f\|_{L_p(\mathbb{R}^n_+)}$. Оценим $v_h^{\pm}(x,x_n)$. Так как они оцениваются аналогичным образом, то оценим $v_h^+(x,x_n)$. Как и в работе [8], представим $v_h^+(x,x_n)$ в виде

$$v_h^+(x,x_n)=\int\limits_{\mathbb{R}^n}e^{ix\xi+ix_n\xi_n}\mu(\xi,\xi_n)K(\xi,h)\hat{f}_\theta(\xi,\xi_n)d\xi d\xi_n, \tag{3.3}$$

где

$$K(\xi,h)=\frac{1}{(2\pi)^{\frac{n-1}{2}}}\int\limits_h^{h^{-1}}G_2(\xi,\nu)d\nu,$$

$$\hat{f}_\theta(\xi,\xi_n)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}e^{-i\xi y-i\xi_n y_n}\theta(y_n)f(y,y_n)dy dy_n,$$

где $\theta(y_n)$ – функция Хевисайда, а

$$\mu(\xi,\xi_n)=\sqrt{2\pi}\int\limits_0^\infty e^{-i\xi_n y_n} D_{y_n}^{2m} J_+(\xi,y_n)dy_n.$$

После представления (3.3) остается показать, что функция $\mu(\xi,\xi_n)K(\xi,h)$ удовлетворяет условиям теоремы П.И. Лизоркина о мультипликаторах [16], то есть является (L_p,L_p)-мультипликатором, который равномерно ограничен по h. Так как произведение (L_p,L_p)-мультипликаторов тоже (L_p,L_p)-мультипликатор, то достаточно доказать, что каждый из множителей μ и K является (L_p,L_p)-мультипликатором. Рассмотрим $K(\xi,h)$. Имеем

$$|K(\xi,h)|=\frac{1}{(2\pi)^{\frac{n-1}{2}}}\left|\int\limits_h^{h^{-1}}(2k)\nu^{2k-1}(\rho_{\mathfrak{N}}(\xi))^{2k}e^{-(\nu\rho_{\mathfrak{N}}(\xi))^{2k}}d\nu\right|=\frac{2k\int\limits_{h\rho_{\mathfrak{N}}(\xi)}^{h^{-1}\rho_{\mathfrak{N}}(\xi)}t^{2k-1}e^{-t^{2k}}dt}{(2\pi)^{\frac{n-1}{2}}}$$

и, как показано в работе [15], $K(\xi,h)$ является мультипликатором, который равномерно ограничен по h.

Изучим $\mu(\xi,\xi_n)$, то есть докажем, что при $\xi_j\neq 0$ $(j=1,\dots,n)$ $|\xi^\beta D_\xi^\beta\mu(\xi)|\le C$, где $\beta_i=0$ или $\beta_i=1$. При $\beta_n=1$ имеем

$$\xi^\beta\xi_n D_\xi^\beta D_{\xi_n}\mu(\xi,\xi_n)=\xi^\beta D_\xi^\beta\int\limits_0^\infty y_n D_{y_n}(e^{-i\xi_n y_n})D_{y_n}^{2m}J_+(\xi,y_n)dy_n=$$

$$-\xi^\beta D_\xi^\beta\int\limits_0^\infty e^{-i\xi_n y_n}\left(D_{y_n}^{2m}J_+(\xi,y_n)+y_n D_{y_n}^{2m+1}J_+(\xi,y_n)\right)dy_n.$$

Отсюда, применяя неравенство (2.3), получаем

$$\left|\xi^\beta\xi_n D_\xi^\beta D_{\xi_n}\mu(\xi,\xi_n)\right|\le$$

$$C\int\limits_0^\infty\left((\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}e^{-\delta y_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}+y_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{m}}e^{-\delta y_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}\right)dy_n\le C=C(\delta).$$

При $\beta_n=0$, опять применяя неравенство (2.3), имеем

$$\left|\xi^\beta D_\xi^\beta\mu(\xi,\xi_n)\right|\le C\int\limits_0^\infty(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}e^{-\delta y_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}dy_n\le C(\delta),$$

то есть $\mu(\xi,\xi_n)K(\xi,h)$ является равномерно ограниченным по h (L_p,L_p)-мультипликатором и, следовательно, для некоторой постоянной $C>0$

$$\left\|v_h^+\right\|_{L_p(\mathbb{R}_+^n)}\le C\|f\|_{L_p(\mathbb{R}_+^n)}.$$

Пусть теперь $\beta = (\alpha, 0)$, где $\alpha \in \partial'\mathfrak{N}$. Аналогично, как и в предыдущем случае, достаточно оценить $D^\alpha U_h^+(x, x_n)$. Из представления $U_h^+(x, x_n)$ и из соображений предыдущего случая, имеем, что

$$D^\alpha U_h^+(x, x_n) = \int\limits_0^{x_n} \int\limits_{\mathbb{R}^{n-1}} e^{ix\xi} \xi^\alpha K(\xi, h) J_+(\xi, x_n - y_n) \hat{f}_\theta(\xi, y_n) d\xi dy_n.$$

После обозначения

$$\mu(\xi, \xi_n) = \xi^\alpha \int\limits_0^\infty e^{-i\xi_n y_n} J_+(\xi, y_n) dy_n$$

получим, что

$$D^\alpha U_h^+(x, x_n) = \int\limits_{\mathbb{R}^n} e^{ix\xi + ix_n\xi_n} \mu(\xi, \xi_n) K(\xi, h) \hat{f}_\theta(\xi, \xi_n) d\xi d\xi_n.$$

Как доказано выше, $K(\xi, h)$ есть равномерно ограниченный по h (L_p, L_p)-мультипликатор, следовательно, достаточно показать, что $\mu(\xi, \xi_n)$ есть (L_p, L_p)-мультипликатор. Применяя лемму 2.1, имеем, что

$$|\mu(\xi, \xi_n)| \le \left| \frac{\xi^\alpha}{\rho_\mathfrak{N}(\xi)} \right| \int\limits_0^\infty (\rho_\mathfrak{N}(\xi))^{\frac{1}{2m}} e^{-\delta y_n (\rho_\mathfrak{N}(\xi))^{\frac{1}{2m}}} dy_n.$$

Второй множитель ограничен, а для первого множителя имеем: так как $\alpha \in \partial'\mathfrak{N}$, то, как показано в работе [17], $\xi^\alpha / \rho_\mathfrak{N}(\xi)$ является (L_p, L_p)-мультипликатором.

Докажем ограниченность $\xi^\beta D_\xi^\beta \mu(\xi, \xi_n)$, где $\beta = (\beta_1, \ldots, \beta_n)$ – вектор с координатами 0 или 1. Сперва оценим $\xi_k D_{\xi_k} \mu(\xi, \xi_n)$ $(k = 1, \ldots, n-1)$. Имеем

$$\xi_k D_{\xi_k} \mu(\xi, \xi_n) = \alpha_k \xi^\alpha \int\limits_0^\infty e^{-i\xi_n y_n} J_+(\xi, y_n) dy_n + \xi^\alpha \int\limits_0^\infty e^{-i\xi_n y_n} \xi_k D_{\xi_k} J_+(\xi, y_n) dy_n$$

и, применяя лемму 2.1, имеем, что

$$|\xi_k D_{\xi_k} \mu(\xi, \xi_n)| \le C \frac{\xi^\alpha}{(\rho_\mathfrak{N}(\xi))} \int\limits_0^\infty (\rho_\mathfrak{N}(\xi))^{\frac{1}{2m}} e^{-\delta y_n (\rho_\mathfrak{N}(\xi))^{\frac{1}{2m}}} dy_n \le C.$$

Пусть $k = n$. Имеем

$$|\xi_n D_{\xi_n}\mu(\xi,\xi_n)| = \left|\xi^\alpha \int\limits_0^\infty y_n D_{y_n} e^{-i\xi_n y_n} J_+(\xi,y_n)dy_n\right| =$$

$$\left|\xi^\alpha \int\limits_0^\infty \left(e^{-i\xi_n y_n} J_+(\xi,y_n) + y_n D_{y_n} J_+(\xi,y_n)\right)dy_n\right| \le$$

$$C\frac{\xi^\alpha}{\rho_{\mathfrak{N}}(\xi)} \int\limits_0^\infty \left((\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}} e^{-\delta y_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}} + y_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{m}} e^{-\delta y_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}\right)dy_n \le$$

$$C = C(\delta).$$

Аналогично оцениваются $\xi^\beta D_\xi^\beta \mu(\xi,\xi_n)$, где $\beta_i = 0$ или $\beta_i = 1$ $(i = 1,\ldots,n)$, и тем самым неравенство (3.1) доказано. Доказательство неравенства (3.2) проводится аналогичным образом. □

Переходим к оценке членов $U_{j,h}$, $j = 1,\ldots,m$.

Лемма 3.2. *Пусть* $\beta^i = (\alpha^i, 0)$, *где* $\alpha^i \in \partial'\mathfrak{N}$ $(i = 1,\ldots,M)$ *или* $\beta^{M+1} = (0,\ldots,0,2m)$. *Тогда для некоторой постоянной* $C > 0$ *имеет место неравенство*

$$\left\|D^{\beta^i} U_{jh}\right\|_{L_p(\mathbb{R}^n_+)} \le C\|f\|_{L_p(\mathbb{R}^n_+)}, \quad (j = 1,\ldots,m), \tag{3.4}$$

причем при $h_1, h_2 \to 0$

$$\left\|D^{\beta^i} U_{jh_1} - D^{\beta^i} U_{jh_2}\right\|_{L_p(\mathbb{R}^n_+)} \to 0, \quad (i = 1,\ldots,M+1). \tag{3.5}$$

Доказательство. Пусть $\beta = (\alpha, 0)$, где $\alpha \in \partial'\mathfrak{N}$. Тогда, как и при доказательстве леммы 3.1, имеем, что

$$D_x^\alpha U_{jh}(x,x_n) = \int\limits_{\mathbb{R}^{n-1}} e^{ix\xi}\xi^\alpha K(\xi,h)J_j(\xi,x_n)\int\limits_0^\infty I_j(\xi,y_n)\hat{f}(\xi,y_n)dy_n d\xi, \quad (j = 1,\ldots,m).$$

В связи с тем, что функции $J_j(\xi,x_n)$ удовлетворяют условиям (2.4), получаем

$$J_j(\xi,x_n)\int\limits_0^\infty I_j(\xi,y_n)\hat{f}(\xi,y_n)dy_n =$$

$$-\int\limits_0^\infty D_{z_n}\left(J_j(\xi,x_n+z_n)\int\limits_0^\infty I_j(\xi,y_n+z_n)\hat{f}(\xi,y_n)dy_n\right)dz_n.$$

Применяя данное тождество, получаем

$$-D_x^\alpha U_{jh}(x,x_n)=\int\limits_{\mathbb{R}^{n-1}}\int\limits_{-\infty}^{+\infty} e^{ix\xi}\theta(x_n+z_n)D_{z_n}J_j(\xi,x_n+z_n)\xi^\alpha K(\xi,h)\theta(z_n)\cdot$$
$$\left(\int\limits_0^\infty I_j(\xi,y_n+z_n)\hat f(\xi,y_n)dy_n\right)dz_n d\xi+$$
$$\int\limits_{\mathbb{R}^{n-1}}\int\limits_{-\infty}^{+\infty} e^{ix\xi}\theta(x_n+z_n)J_j(\xi,x_n+z_n)\xi^\alpha K(\xi,h)\theta(z_n)\cdot$$
$$\left(\int\limits_0^\infty D_{z_n}I_j(\xi,y_n+z_n)\hat f(\xi,y_n)dy_n\right)dz_n d\xi=\Phi_1(x,x_n)+\Phi_2(x,x_n).$$

Каждое из этих слагаемых оценивается аналогично. Оценим $\Phi_1(x,x_n)$. Как и в работе [8], представим $\Phi_1(x,x_n)$ в виде

$$\Phi_1(x,x_n)=-\int\limits_{\mathbb{R}^n} e^{ix\xi-ix_n\xi_n}\mu(\xi,\xi_n)\int\limits_{-\infty}^{+\infty} e^{-i\xi_n t_n}\cdot$$

$$\left(\theta(t_n)\int\limits_{-\infty}^{+\infty}\xi^\alpha(\rho_{\mathfrak{N}}(\xi))^{-\frac{j-1}{2m}}I_j(\xi,y_n+t_n)K(\xi,h)\theta(y_n)\hat f(\xi,y_n)dy_n\right)dt_n d\xi d\xi_n, \tag{3.6}$$

где

$$\mu(\xi,\xi_n)=\int\limits_0^\infty e^{i\xi_n z_n}(\rho_{\mathfrak{N}}(\xi))^{\frac{j-1}{2m}}D_{z_n}J_j(\xi,z_n)dz_n$$

и покажем, что $\mu(\xi,\xi_n)$ – (L_p,L_p)-мультипликатор. Из леммы 2.1 следует, что

$$|D_{z_n}J_j(\xi,z_n)|=\left|\frac{1}{2\pi i}\int\limits_{\Gamma^+(\xi)}\frac{\lambda e^{iz_n\lambda}M^+_{m-j}(\xi,\lambda)}{M^+(\xi,\lambda)}d\lambda\right|\le C(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}-\frac{j-1}{2m}}e^{-\delta z_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}},$$

следовательно, для некоторой постоянной C имеем, что $|\mu(\xi,\xi_n)|\le C$.

Оценим $\xi_n D_{\xi_n}\mu(\xi,\xi_n)$. Учитывая лемму 2.1, имеем

$$\xi_n D_{\xi_n}\mu(\xi,\xi_n)=\int\limits_0^\infty z_n D_{z_n}e^{i\xi_n z_n}(\rho_{\mathfrak{N}}(\xi))^{\frac{j-1}{2m}}D_{z_n}J_j(\xi,z_n)dz_n=$$
$$-\int\limits_0^\infty e^{i\xi_n z_n}(\rho_{\mathfrak{N}}(\xi))^{\frac{j-1}{2m}}D_{z_n}J_j(\xi,z_n)dz_n-\int\limits_0^\infty z_n e^{i\xi_n z_n}(\rho_{\mathfrak{N}}(\xi))^{\frac{j-1}{2m}}D^2_{z_nz_n}J_j(\xi,z_n)dz_n.$$

Как уже оценили выше, первое слагаемое ограничено некоторой постоянной, а второе слагаемое по лемме 2.1 оценивается выражением

$$\int_0^\infty z_n(\rho_{\mathfrak{N}}(\xi))^{\frac{j-1}{2m}}(\rho_{\mathfrak{N}}(\xi))^{-\frac{j-1}{2m}+\frac{1}{m}}e^{-\delta z_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}dz_n \le C(\delta).$$

Аналогично оцениваются выражения $\xi^\beta \xi_n D_\xi^\beta D_{\xi_n}\mu(\xi,\xi_n)$, где $\beta_i = 0$ или 1.

Теперь обозначим через $F(\xi,t_n)$ то выражение в формуле (3.6), которое находится в скобках и покажем, что для некоторой постоянной $C>0$ имеет место неравенство

$$\left\| \int_{\mathbb{R}^{n-1}} e^{iy\xi}F(\xi,t_n)d\xi \right\|_{L_p(\mathbb{R}^n)} \le C\|f\|_{L_p(\mathbb{R}^n_+)}. \tag{3.7}$$

Отсюда по теореме о мультипликаторах (см. [16]) имеем, что для некоторой постоянной $C>0$

$$\|\Phi_1\|_{L_p(\mathbb{R}^n)} \le C\|f\|_{L_p(\mathbb{R}^n_+)}. \tag{3.8}$$

Применяя свойства преобразования Фурье для левой части неравенства (3.7), имеем, что

$$\left\| \int_{\mathbb{R}^{n-1}} e^{iy\xi}F(\xi,t_n)d\xi \right\|_{L_p(\mathbb{R}^n)} \le \left\| \int_{\mathbb{R}^n} e^{iy\xi - it_n\xi_n}\widetilde{\mu}(\xi,\xi_n)K(\xi,h)\hat{f}_\theta(\xi,\xi_n)d\xi d\xi_n \right\|_{L_p(\mathbb{R}^n)},$$

где

$$\widetilde{\mu}(\xi,\xi_n) = \int_0^\infty e^{i\xi_n z_n}\xi^\alpha(\rho_{\mathfrak{N}}(\xi))^{-\frac{j-1}{2m}}I_j(\xi,z_n)dz_n.$$

Теперь нужно показать, что $\widetilde{\mu}(\xi,\xi_n)$ есть (L_p,L_p)-мультипликатор. Из определения $I_j(\xi,x_n)$ $(j=1,\dots,m)$ и из леммы 2.1 следует, что

$$|I_j(\xi,x_n)| \le C(\rho_{\mathfrak{N}}(\xi))^{\frac{j}{2m}-1}e^{-\delta x_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}},$$

следовательно, для $\widetilde{\mu}(\xi,\xi_n)$ имеем, что

$$|\widetilde{\mu}(\xi,\xi_n)| \le C\int_0^\infty (\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}\frac{\xi^\alpha}{\rho_{\mathfrak{N}}(\xi)}e^{-\delta x_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}dx_n \le C = C(\delta),$$

так как $\xi^\alpha/\rho_{\mathfrak{N}}(\xi)$ при $\alpha \in \partial'\mathfrak{N}$ является (L_p, L_p)-мультипликатором. Теперь оценим $\xi_n D_{\xi_n}\widetilde{\mu}(\xi,\xi_n)$. Имеем

$$\xi_n D_{\xi_n}\widetilde{\mu}(\xi,\xi_n) = \int\limits_0^\infty \xi_n z_n e^{i\xi_n z_n}\xi^\alpha(\rho_{\mathfrak{N}}(\xi))^{-\frac{j-1}{2m}} I_j(\xi,z_n)dz_n =$$

$$\int\limits_0^\infty z_n D_{z_n}(e^{i\xi_n z_n})\xi^\alpha(\rho_{\mathfrak{N}}(\xi))^{-\frac{j-1}{2m}} I_j(\xi,z_n)dz_n =$$

$$-\frac{\xi^\alpha}{\rho_{\mathfrak{N}}(\xi)}(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}}\left(\int\limits_0^\infty e^{i\xi_n z_n} I_j(\xi,z_n)dz_n + \int\limits_0^\infty e^{i\xi_n z_n} z_n D_{z_n} I_j(\xi,z_n)dz_n\right).$$

Первый множитель $\xi^\alpha/\rho_{\mathfrak{N}}(\xi)$ ограничен, оценим остальные. Из леммы 2.1 получим

$$(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}}\left|\int\limits_0^\infty e^{i\xi_n z_n} I_j(\xi,z_n)dz_n\right| \le$$

$$C(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}}\int\limits_0^\infty (\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}+\frac{j-1}{2m}-1} e^{-\delta z_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}dz_n \le C = C(\delta).$$

Для второго слагаемого имеем

$$(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}}\left|\int\limits_0^\infty e^{i\xi_n z_n} z_n D_{z_n} I_j(\xi,z_n)dz_n\right| \le$$

$$C(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}}\int\limits_0^\infty (\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}+\frac{j}{2m}-1} z_n e^{-\delta z_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}dz_n \le C(\delta).$$

Если $\beta_n = 0$, то для $\xi_k D_{\xi_k}\widetilde{\mu}(\xi,\xi_n)$ $(k = 1,\dots,n-1)$ имеем

$$\xi_k D_{\xi_k}\widetilde{\mu}(\xi,\xi_n) = \xi_k \int\limits_0^\infty e^{i\xi_n z_n} D_{\xi_k}\left(\frac{\xi^\alpha}{\rho_{\mathfrak{N}}(\xi)}(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}} I_j(\xi,z_n)\right)dz_n.$$

Если производная берется по $\xi^\alpha/\rho_{\mathfrak{N}}(\xi)$, то выражения $\xi_k D_k(\xi^\alpha/\rho_{\mathfrak{N}}(\xi))$, $k = 1,\dots,n-1$ ограничены. Если берется по $(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}}$, то

$$\left|\xi_k D_k(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}}\right| \le \left|\xi_k(\rho_{\mathfrak{N}}(\xi))^{-\frac{j-1}{2m}}\right| \cdot |D_{\xi_k}\rho_{\mathfrak{N}}(\xi)| \le$$

$$\left|\frac{\xi_k D_{\xi_k}\rho_{\mathfrak{N}}(\xi)}{\rho_{\mathfrak{N}}(\xi)}\right|(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}} \le C(\rho_{\mathfrak{N}}(\xi))^{1-\frac{j-1}{2m}}.$$

Если же производная берется по $I_j(\xi, z_n)$, то из леммы 2.1 получим

$$|\xi_k D_{\xi_k} I_j(\xi, z_n)| = \left|\xi_k \frac{\partial}{\partial \xi_k}\left(\frac{\partial}{\partial y_n}\right)^{j-1} J_-(\xi, y_n - x_n)\right|\Big|_{y_n=0} \leq$$
$$C(\rho_{\mathfrak{N}}(\xi))^{\frac{j-1}{2m}-1+\frac{1}{2m}} e^{-\delta x_n (\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}.$$

То есть из всех соображений следует, что $\widetilde{\mu}(\xi, \xi_n)$ является (L_p, L_p)-мультипликатором, следовательно, выполняется неравенство (3.8). Остальные оценки доказываются аналогичным образом. □

Наконец, переходим к оценке $U_h(x, x_n)$. Для простоты записи будем изучать тот случай, когда вполне правильный многогранник $\mathfrak{N}$ имеет одну вершину анизотропности $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ и $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}$. Для таких многоугольников в работе [13] (общий случай см. [14]) доказаны следующие оценки (см. леммы 1.1 и 1.5).

Лемма 3.3. *Пусть $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}$. Тогда для любого мультииндекса $m = (m_1, \ldots, m_{n-1})$ и любого четного выпрямляющего числа N существуют постоянная C_0 и набор векторов $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$, $\beta = (\beta_1, \ldots, \beta_{n-2}, 0)$, $\ldots$, $\sigma = (\sigma_1, 0, \ldots, 0)$ таких, что для любого $\nu : 0 < \nu < 1$ имеют места неравенства*

(3.9)
$$\left|D^m \hat{G}_r(t, \nu)\right| \leq C_0 \nu^{-\max\limits_{i=1,\ldots,I_{n-2}}\left(|\mu^i| + (m, \mu^i)\right)} \cdot \frac{1}{1 + \nu^{-N}(t^{N\alpha} + t^{N\beta} + \cdots + t_1^{N\sigma_1})},$$

где $r = 0, 1$.

Лемма 3.4. *Для любого мультииндекса m и натурального числа N существует постоянная C_0, такая, что при $\nu > 1$ имеют места неравенства*

$$\left|D^m \hat{G}_r(t, \nu)\right| \leq C_0 \nu^{-\left(|\mu^0| + (m, \mu^0)\right)} \cdot \frac{1}{1 + \nu^{-N}(t_1^{Nl_1} + \cdots + t_{n-1}^{Nl_{n-1}})}, \tag{3.10}$$

где $r = 0, 1$.

Замечание 3.1. *В работах [13]-[14] в соответствующих леммах участвуют также многочлены по $|\ln \nu|$, но так как эти слагаемые не влияют на сходимость интеграла по ν, то здесь и в дальнейшем, для простоты записи, мы коэффициенты логарифмического многочлена считаем нулями, кроме C_0.*

Применяя леммы 3.3 и 3.4, докажем следующее утверждение.

Лемма 3.5. *Если $f \in L_p(\mathbb{R}^n_+)$ имеет компактный носитель $K = suppf$ и при $\chi \leq 1$ выполняются условия ортогональности (1.6), то существует постоянная $C = C(K) > 0$, что при любом $h > 0$ имеет место неравенство*

$$\left\|U_h^+ + U_h^-\right\|_{L_p(\mathbb{R}^n_+)} \leq C\|f\|_{L_p(\mathbb{R}^n_+)} \tag{3.11}$$

и при $h_1, h_2 \to 0$

$$\left\|(U_{h_1}^+ + U_{h_1}^-) - (U_{h_2}^+ + U_{h_2}^-)\right\|_{L_p(\mathbb{R}^n_+)} \to 0. \tag{3.12}$$

Доказательство. Пусть $\chi > 1$. Проведем оценку функции $U_h^+(x, x_n)$ ($U_h^-(x, x_n)$ оценивается аналогично). Как и в работе [8], вводя обозначения

$$K_+(\nu, x, x_n) = \int\limits_{\mathbb{R}^{n-1}} e^{ix\xi} G_2(\xi, \nu) J_+(\xi, x_n)\theta(x_n) d\xi$$

и, применяя неравенство Минковского, имеем

$$\left\|U_h^+\right\|_{L_p(\mathbb{R}^n_+)} \leq \int\limits_1^{h^{-1}} \left\|\int\limits_{\mathbb{R}^n} K_+(\nu, x-y, x_n-y_n)\theta(y_n) f(y, y_n) dy dy_n\right\|_{L_p(\mathbb{R}^n)} d\nu +$$

$$\int\limits_h^1 \left\|\int\limits_{\mathbb{R}^n_+} K_+(\nu, x-y, x_n-y_n)\theta(y_n) f(y, y_n) dy dy_n\right\|_{L_p(\mathbb{R}^n)} d\nu = A_{1,h} + A_{2,h}.$$

Оценим каждое слагаемое по отдельности. Сперва оценим A_{2h}. Применяя неравенство Юнга, имеем

$$A_{2,h} \leq \int\limits_h^1 \|K_+(\nu, x, x_n)\|_{L_1(\mathbb{R}^n)} d\nu \cdot \|f\|_{L_p(\mathbb{R}^n_+)}.$$

Представим $\|K_+(\nu, x, x_n)\|_{L_1(\mathbb{R}^n)}$ в виде

$$\|K_+(\nu, x, x_n)\|_{L_1(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} \frac{1}{1 + \nu^{-N}(x^{N\alpha} + x^{N\beta} + \cdots + x_1^{N\sigma_1})} \cdot$$

$$\left|\int\limits_{\mathbb{R}^{n-1}} \left(1 + \nu^{-N}\left(D_\xi^{N\alpha} + \cdots + D_{\xi_1}^{N\sigma_1}\right)\right) e^{ix\xi} G_2(\xi, \nu) J_+(\xi, x_n)\theta(x_n) d\xi\right| dx_n dx.$$

Далее имеем

$$A=\int\limits_{\mathbb{R}^1}\left|\int\limits_{\mathbb{R}^{n-1}} e^{ix\xi}G_2(\xi,\nu)J_+(\xi,x_n)\theta(x_n)d\xi\right|dx_n\le$$

$$C\int\limits_0^\infty\int\limits_{\mathbb{R}^{n-1}}\nu^{2k-1}(\rho_{\mathfrak{N}}(\xi))^{2k}e^{-(\nu\rho_{\mathfrak{N}}(\xi))^{2k}}\left|\int\limits_{\Gamma^+(\xi)}\frac{e^{ix_n\lambda}}{P(\lambda,\xi)}d\lambda\right|d\xi dx_n\le$$

$$C\int\limits_{\mathbb{R}^{n-1}}\int\limits_0^\infty\nu^{2k-1}(\rho_{\mathfrak{N}}(\xi))^{2k}e^{-(\nu\rho_{\mathfrak{N}}(\xi))^{2k}}(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}-1}e^{-\delta x_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}}dx_nd\xi.$$

В последнем интеграле обозначив $t_n=x_n(\rho_{\mathfrak{N}}(\xi))^{\frac{1}{2m}}$ и применив лемму 3.3 (см. неравенство (3.9)), после преобразования $\xi=\nu^{-\mu^1}\eta$, получаем

$$A\le C\nu^{-\max\limits_{i=1,\ldots,I_{n-2}}|\mu^i|}.$$

Теперь оценим

$$B=\nu^{-N}\int\limits_{\mathbb{R}^1}\left|\int\limits_{\mathbb{R}^{n-1}}D_\xi^{N\alpha}e^{ix\xi}G_2(\xi,\nu)J_+(\xi,x_n)\theta(x_n)d\xi\right|dx_n.$$

После интегрирования по частям имеем, что

$$B\le C\nu^{-N}\int\limits_0^\infty\int\limits_{\mathbb{R}^{n-1}}\sum_{\gamma+\beta=N\alpha}\left|D_\xi^\gamma(G_2(\xi,\nu))\right|\cdot\left|D_\xi^\beta\int\limits_{\Gamma^+(\xi)}\frac{e^{ix_n\lambda}}{\lambda^{2m}+\rho_{\mathfrak{N}}(\xi)}d\lambda\right|d\xi dx_n.$$

Если применить замену переменных $\xi=\nu^{-\mu^i}\eta$ (для некоторого $i=1,\ldots,I_{n-2}$), то последний интеграл примет вид

$$B\le C\nu^{-N-|\mu^i|+(N\alpha,\mu^i)}.$$

$$\int\limits_0^\infty\int\limits_{\mathbb{R}^{n-1}}\sum_{\gamma+\beta=N\alpha}\left|\eta^{-\beta}D_\eta^\gamma\left(\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\right)^{2k}e^{-\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\right)^{2k}}\right)\right|\cdot$$

$$\left|\eta^\beta D_\eta^\beta\int\limits_{\Gamma^+(\eta)}\frac{e^{i\frac{x_n}{\nu^{\frac{1}{2m}}}\nu^{\frac{1}{2m}}\lambda}}{(\lambda\nu^{\frac{1}{2m}})^{2m}+\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)}d\lambda\right|d\eta dx_n.$$

Наконец, если в контурном интеграле сделать замену переменных $\nu^{\frac{1}{2m}}\lambda = \tau$ и применить лемму 2.1, имеем

$$B \le C\nu^{-N-|\mu^i|-\frac{1}{2m}+(N\alpha,\mu^i)}\cdot$$

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} \sum_{\gamma+\beta=N\alpha} \left|\eta^{-\beta}\Big(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\Big)^{-1} D_\eta^\gamma \left(\Big(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\Big)^{2k} e^{-\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\right)^{2k}}\right)\right|\cdot$$

$$\Big(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\Big)^{\frac{1}{2m}} e^{-\frac{\delta x_n}{\nu^{1/(2m)}}\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\right)^{\frac{1}{2m}}} d\eta dx_n \le C\nu^{-N-|\mu^i|+(N\alpha,\mu^i)}\cdot$$

$$\int_{\mathbb{R}^{n-1}} \sum_{\gamma+\beta=N\alpha} \left|\eta^{-\beta}\Big(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\Big)^{-1} D_\eta^\gamma \left(\Big(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\Big)^{2k} e^{-\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^i}\eta)\right)^{2k}}\right)\right| d\eta.$$

Последнее неравенство получили после изменения места интегрирования и как выше оценки интеграла по x_n. Теперь, так как $(N\alpha,\mu^i)-N \ge 0$ $(i=1,\dots,I_{n-2})$ и для любого $\alpha \in \partial'\mathfrak{N}$ $(\alpha,\mu^i) \le 1$, а $\nu^{1-(\alpha,\mu^i)} \le 1$, то, выбирая k настолько большим, чтобы все степени η были положительными, имеем, что

$$B \le \nu^{-\max\limits_{i=1,\dots,I_{n-2}}|\mu^i|}.$$

В итоге получим, что

$$A_{2,h} \le C\int_h^1 \nu^{-\max\limits_{i=1,\dots,I_{n-2}}|\mu^i|} \int_{\mathbb{R}^{n-1}} \frac{dx}{1+\nu^{-N}(x^{N\alpha}+x^{N\beta}+\dots+x^{N\sigma})} d\nu\cdot \|f\|_{L_p(\mathbb{R}^n_+)}.$$

В последнем интеграле сделав замену переменных $x=\nu^{\mu^1}t$ и применив лемму 1.2 работы [13] о том, что интеграл по x сходится, имеем, что

$$A_{2,h} \le C\int_h^1 \nu^{-\max\limits_{i=1,\dots,I_{n-2}}|\mu^i|+|\mu^1|} d\nu\cdot \|f\|_{L_p(\mathbb{R}^n_+)} \le C\|f\|_{L_p(\mathbb{R}^n_+)},$$

так как по условию на многогранник $\mathfrak{N}$ $\max\limits_{i=1,\dots,I_{n-2}}|\mu^i| - \min\limits_{i=1,\dots,r+1}|\mu^i| < 1$.

Теперь переходим к оценке $A_{1,h}$, где $\nu > 1$. Опять, применяя неравенство Юнга, получаем

$$A_{1,h} \le \int_1^{h^{-1}} \|K_+(\nu,x,x_n)\|_{L_p(\mathbb{R}^n)}\cdot\|f\|_{L_1(\mathbb{R}^n_+)}.$$

Оценим первый множитель. Имеем

$$\|K_+(\nu,x,x_n)\|_{L_p(\mathbb{R}^n_+)} = \Bigg(\int\limits_{\mathbb{R}^{n-1}}\int\limits_0^{\infty}\left(\frac{1}{1+\nu^{-N}(x_1^{l_1}+\cdots+x_{n-1}^{l_{n-1}})}\right)^p \cdot \tag{3.13}$$

$$\left|\int\limits_{\mathbb{R}^{n-1}}\left(1+\nu^{-N}\left(D_{\xi_1}^{l_1}+\cdots+D_{\xi_{n-1}}^{l_{n-1}}\right)\right)e^{ix\xi}G_2(\xi,\nu)J_+(\xi,x_n)d\xi\right|^p dxdx_n\Bigg)^{\frac{1}{p}}.$$

Достаточно оценить один из интегралов (оценка остальных слагаемых проводится аналогично).

$$I_1 = \left|\int\limits_{\mathbb{R}^{n-1}}\nu^{-N}D_{\xi_1}^{Nl_1}e^{ix\xi}G_2(\xi,\nu)J_+(\xi,x_n)d\xi\right| \le$$

$$C\nu^{-N}\sum_{\gamma+\beta=Nl_1}\int\limits_{\mathbb{R}^{n-1}}\left|D_{\xi_1}^{\gamma}\left(\nu^{2k-1}(\rho_{\mathfrak{N}}(\xi))^{2k}e^{-(\nu\rho_{\mathfrak{N}}(\xi))^{2k}}\right)\right|\left|D_{\xi_1}^{\beta}\int\limits_{\Gamma^+(\xi)}\frac{e^{ix_n\lambda}d\lambda}{\lambda^{2m}+\rho_{\mathfrak{N}}(\xi)}\right|d\xi =$$

$$C\nu^{-N}\sum_{\gamma+\beta=Nl_1}\int\limits_{\mathbb{R}^{n-1}}\left|D_{\xi_1}^{\gamma}(\nu\rho_{\mathfrak{N}}(\xi))^{2k}e^{-(\nu\rho_{\mathfrak{N}}(\xi))^{2k}}\right|\left|D_{\xi_1}^{\beta}\int\limits_{\Gamma^+(\xi)}\frac{e^{i\frac{x_n}{\nu^{\frac{1}{2m}}}\nu^{\frac{1}{2m}}\lambda}d\lambda}{(\lambda\nu^{\frac{1}{2m}})^{2m}+\nu\rho_{\mathfrak{N}}(\xi)}\right|d\xi.$$

Если после обозначения в контурном интеграле $\nu^{\frac{1}{2m}}\lambda=\tau$, сделать замену переменных $\xi=\nu^{-\mu^0}\eta$ и применить формулу дифференцирования (1.8) работы [13], получим, что

$$I_1 \le C\nu^{-\left(|\mu^0|+\frac{1}{2m}\right)}.$$

$$\sum_{\gamma+\beta=Nl_1}\int\limits_{\mathbb{R}^{n-1}}\eta^{-\beta}\sum_{r+\sigma=\gamma}C_{|\gamma|}^{|r|}D^r\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{2k}e^{-\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{2k}}\cdot \tag{3.14}$$

$$\sum_{r^1+\cdots+r^{|\sigma|}=\sigma}\prod_{j=1}^{|\sigma|}D_{\eta}^{r_j}\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{2k}\cdot\left|\eta^{\beta}D_{\eta}^{\beta}\int\limits_{\Gamma^+(\eta)}\frac{e^{i\frac{x_n}{\nu^{\frac{1}{2m}}}\tau}}{\tau^{2m}+\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)}d\tau\right|d\eta.$$

Последний множитель (по β) по лемме 2.1 оценивается выражением

$$C\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{\frac{1}{2m}-1}e^{-\delta\frac{x_n}{\nu^{\frac{1}{2m}}}\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{\frac{1}{2m}}}.$$

Учитывая, что при $\alpha^i=(0,\ldots,0,l_i,0,\ldots,0)$ $(\alpha^i,\mu^0)=1$ $(i=1,\ldots,n)$, а для мультианизотропной вершины $\alpha=(\alpha_1,\ldots,\alpha_{n-1})$: $(\alpha,\mu^0)>1$, следовательно, при $\nu>1$ $\nu^{1-(\alpha,\mu^0)}<1$, то, подбирая k настолько большим, чтобы в первом множителе формулы (3.14) степени η были положительными, получим, что при некоторой постоянной $C > 0$

$$I_1 \le C\nu^{-\left(|\mu^0|+\frac{1}{2m}\right)}\cdot$$

$$\int\limits_{\mathbb{R}^{n-1}} \left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{\frac{1}{2m}} \eta^{\rho} e^{-\left(\eta_1^{l_1}+\dots+\eta_{n-1}^{l_{n-1}}\right)^{2k}} e^{-\delta\frac{x_n}{\nu^{\frac{1}{2m}}}\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{\frac{1}{2m}}} d\eta_1\dots d\eta_{n-1},$$

где $\rho = (\rho_1, \dots, \rho_{n-1})$ некоторый мультииндекс, который получается при группировки степеней ξ_i $(i = 1, \dots, n-1)$.

Поставляя оценки для I_k $(k = 1, \dots, n-1)$ в формуле (3.13) и применяя обобщенное неравенство Минковского, имеем, что

$$\|K_+(\nu, x, x_n)\|_{L_p(\mathbb{R}^n_+)} \le C\nu^{-\left(|\mu^0|+\frac{1}{2m}\right)} \left(\int\limits_{\mathbb{R}^{n-1}} \left(\frac{dx}{1+\nu^{-N}\left(x_1^{Nl_1}+\dots+\eta_{n-1}^{Nl_{n-1}}\right)}\right)^p\right)^{\frac{1}{p}}\cdot$$

$$\int\limits_{\mathbb{R}^{n-1}} \left(\int\limits_0^\infty \left(\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{\frac{1}{2m}} \eta^{\rho} e^{-\left(\eta_1^{l_1}+\dots+\eta_{n-1}^{l_{n-1}}\right)^{2k}} e^{-\delta\frac{x_n}{\nu^{\frac{1}{2m}}}\left(\nu\rho_{\mathfrak{N}}(\nu^{-\mu^0}\eta)\right)^{\frac{1}{2m}}}\right)^p dx_n\right)^{\frac{1}{p}} d\xi.$$

В последнем интеграле, если сделать замену переменных $x = \nu^{\mu^0}\tau$, $x_n = \nu^{\frac{1}{2m}}\tau_n$ и учитывать, что интеграл по ξ сходится, то после выбора натурального числа N таким, чтобы интеграл по x тоже был сходящимся, имеем, что

$$\|K_+(\nu, x, x_n)\|_{L_p(\mathbb{R}^n_+)} \le C\nu^{-\left(|\mu^0|+\frac{1}{2m}\right)+\left(|\mu^0|+\frac{1}{2m}\right)\frac{1}{p}},$$

то есть

$$A_{1,h} \le C\int\limits_1^{h^{-1}} \nu^{-\left(|\mu^0|+\frac{1}{2m}\right)+\left(|\mu^0|+\frac{1}{2m}\right)\frac{1}{p}} d\nu \cdot \|f\|_{L_1(\mathbb{R}^n_+)}.$$

Так как по предположению $\chi = |\mu^0| + \frac{1}{2m} - \left(|\mu^0| + \frac{1}{2m}\right)\frac{1}{p} > 1$, то интеграл по ν сходится. В итоге имеем, что $A_{1,h} \le C$.

Пусть теперь $\chi \le 1$ и L такое число, что выполняются условия (1.7). Так как функция f удовлетворяет условиям ортогональности (1.6), то $\hat{f}(\xi, y_n)$ можно представить в виде

$$\hat{f}(\xi, y_n) =$$

$$\int\limits_0^1 \cdots \int\limits_0^1 \left(\int\limits_{\mathbb{R}^{n-1}} e^{-i\lambda_L\dots\lambda_1 y\xi}(-iy\xi)^L f(y, y_n)dy\right) \lambda_{n-1}\lambda_{n-2}^2\dots\lambda_1^{L-1} d\lambda_L\dots d\lambda_1.$$

Тогда, применяя неравенство Минковского для $A_{1,h}$, имеем

$$A_{1,h} \leq C \sum_{|\beta|=L} \int_1^{h^{-1}} \int_0^1 \cdots \int_0^1 \left\| \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} e^{i(x-\lambda_L\ldots\lambda_1 y)\xi} G_2(\xi,\nu) y^\beta (i\xi)^\beta \theta(x_n - y_n) \cdot \right.$$

$$\left. \lambda_{L-1}\lambda_{L-2}^2 \ldots \lambda_1^{L-1} \theta(y_n) f(y,y_n) d\xi dy dy_n \right\|_{L_p(\mathbb{R}^n)} d\lambda_L \ldots d\lambda_1 d\nu.$$

Отсюда, применяя неравенство Юнга, получим

$$A_{1,h} \leq C \sum_{|\beta|=L} \int_1^{h^{-1}} \left\| \int_{\mathbb{R}^{n-1}} e^{ix\xi} G_2(\xi,\nu)(i\xi)^\beta \theta(x_n) J_+(\xi,x_n) d\xi \right\|_{L_p(\mathbb{R}^n)} d\nu \cdot$$

$$\left\| y^\beta f(y,y_n) \right\|_{L_1(\mathbb{R}^n)}. \tag{3.15}$$

Норма по $L_p(\mathbb{R}^n_+)$ в (3.15) оценивается как выражение $K_+(\nu,x,x_n)$ в лемме 3.5, откуда для данной нормы имеем оценку (при $\nu > 1$)

$$C\nu^{-\left(\left(|\mu^0| + \frac{1}{2m}\right) + (\beta,\mu^0)\right) + \left(|\mu^0| + \frac{1}{2m}\right)\frac{1}{p}}.$$

Следовательно, интеграл по ν будет сходится, если $\chi + (\beta,\mu^0) > 1$. Но так как по условию (1.7) $\chi + (\beta,\mu^0) > \chi + |\beta|\mu^0_{min} = \chi + L\mu^0_{min} > 1$, то по выбору числа L имеем, что интеграл по ν сходится, и если функция f удовлетворяет условиям ортогональности (1.6), то

$$A_{1,h} \leq C \sum_{|\beta|=L} \left\| y^\beta f(y,y_n) \right\|_{L_1(\mathbb{R}^n_+)}.$$

Доказательство неравенства (3.11) для нормы $\left\| U_h^- \right\|_{L_p(\mathbb{R}^n)}$ и неравенства (3.12) проводится аналогично. □

Наконец для оценки $U_{jh}(x,x_n)$ $(j = 1,\ldots,m)$ имеем

Лемма 3.6. *Пусть выполняются условия леммы 3.3. Тогда существует постоянная $C = C(K) > 0$, что для любого $h > 0$*

$$\|U_{jh}\|_{L_p(\mathbb{R}^n_+)} \leq C\|f\|_{L_p(\mathbb{R}^n_+)}, \quad (j = 1,\ldots,m) \tag{3.16}$$

и при $h_1, h_2 \to 0$

$$\|U_{jh_1} - U_{jh_2}\|_{L_p(\mathbb{R}^n_+)} \to 0. \tag{3.17}$$

Доказательство не отличается от доказательств предыдущих лемм с применением леммы работы [8], поэтому ее мы опускаем.

Теперь мы готовы доказать, что функции $U_h(x, x_n)$ являются приближенными решениями нашей задачи, то есть имеет место

Лемма 3.7. *Если* $h \to 0$, *то*

$$\|P(D_x, D_{x_n})U_h - f\|_{L_p(\mathbb{R}^n_+)} \to 0 \tag{3.18}$$

и для любого $h > 0$

$$\left(\frac{\partial}{\partial x_n}\right)^{j-1} U_h(x, x_n)\bigg|_{x_n=0} = 0, \quad (j = 1, \ldots, m). \tag{3.19}$$

Доказательство. По определению функции $U_h(x, x_n)$ (см. формулу (2.7)) имеем

$$P(D_x, D_{x_n})U_h = P(D_x, D_{x_n})(U_h^+ + U_h^-) + \sum_{k=1}^{m} P(D_x, D_{x_n})U_{kh}.$$

Из определения функций $J_j(\xi, x_n)$ $(j = 1, \ldots, m)$ следует, что $P(\xi, D_{x_n})J_j(\xi, x_n) \equiv 0$ $(j = 1, \ldots, m)$. Остается оценить первое слагаемое. Применяя на функции $U_h^+ + U_h^-$ оператор $P(D_x, D_{x_n})$ и учитывая лемму 2.2 (см. формулы (2.5), (2.6)), получим, что

(3.20)

$$P(D_x, D_{x_n})(U_h^+ + U_h^-) = \frac{1}{2\pi^{n-1}} \int\limits_h^{h^{-1}} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}^{n-1}} e^{i(x-y)\xi} G_2(\xi, \nu) f(y, x_n) d\xi dy d\nu.$$

По интегральному предствлению (2.1) правая часть формулы (3.20) почти всюду стремится к $f(x, x_n)$ при $h \to 0$, откуда следует соотношение (3.18). Докажем соотношение (3.19). Как это делали при доказательстве леммы 3.1, прибавляя и отнимая к $U_h(x, x_n) - y$ выражение

$$\frac{1}{2\pi^{n-1}} \int\limits_h^{h^{-1}} \int\limits_0^{x_n} e^{i(x-y)\xi} G_2(\xi, \nu) I_-(\xi, x_n - y_n) f(y, y_n) d\xi dy dy_n d\nu$$

и, применяя лемму 2.2 (формулу (2.5)), имеем

$$\left(\frac{\partial}{\partial x_n}\right)^{j-1} (U_h^+ + U_h^-)\bigg|_{x_n=0} =$$

$$-\frac{1}{2\pi^{n-1}} \int\limits_h^{h^{-1}} \int\limits_0^{\infty} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}^{n-1}} e^{i(x-y)\xi} G_2(\xi, \nu) I_j(\xi, y_n) f(y, y_n) d\xi dy dy_n d\nu. \tag{3.21}$$

Теперь вычислим

$$\left(\frac{\partial}{\partial x_n}\right)^{j-1}\sum_{k=1}^{m}U_{hk}(x,x_n)\bigg|_{x_n=0},\quad (j=1,\dots,m). \tag{3.22}$$

Из определения функций $U_{hk}(x,x_n)$ $(k=1,\dots,m)$ и из леммы 2.2 (см. формулу (2.6)) после вычисления (3.22) получим формулу (3.21), но со знаком минус. То есть действительно выполняются соотношения (3.19). □

4. Доказательства основных теорем

Доказательство теоремы 1.1. Из лемм 3.1, 3.2, 3.5 и 3.6 следует, что существует функция $U(x,x_n)\in W_p^{\mathfrak{M}}(\mathbb{R}_+^n)$ такая, что $\|U_h-U\|_{W_p^{\mathfrak{M}}(\mathbb{R}_+^n)}$ при $h\to 0$, и для некоторой постоянной $C>0$ имеет место неравенство (1.5). В силу леммы 3.7 эта функция является решением задачи (1.3)-(1.4), приближенными решениями которого являются функции $U_h(x,x_n)$, задаваемые формулой (2.7). Докажем единственность этого решения. То есть докажем, что однородная задача (1.3)-(1.4) имеет нулевое решение. Применим методы работ [8] и [18]. Пусть сначала решение $U(x,x_n)\in W_p^{\mathfrak{M}}(\mathbb{R}_+^n)$ краевой задачи (1.3)-(1.4) имеет компактный носитель. Тогда, применяя преобразование Фурье по x, имеем, что

$$P(\xi,D_{x_n})\hat{U}(\xi,x_n)=0,\ \ x_n>0$$

$$\left(\frac{\partial}{\partial x_n}\right)^{j-1}\hat{U}(\xi,x_n)\bigg|_{x_n=0}=0,\ \ (j=1,\dots,m)$$

и $|\hat{U}(\xi,x_n)|\to 0$ при $x_n\to+\infty$. Так как для любого $\xi\in\mathbb{R}^{n-1}\setminus\{0\}$ выполняются граничные условия, то $\hat{U}(\xi,x_n)=0$ при $\xi\neq 0$. В силу непрерывности следует, что $U(x,x_n)\equiv 0$. То есть решение задачи (1.3)-(1.4), имеющий компактный носитель по x, единственное. По неравенству Фридрихса

$$\|U\|_{L_p(\mathbb{R}_+^n)}\le C\sum_{\alpha\in\partial'\mathfrak{N}}\|D^\alpha U\|_{L_p(\mathbb{R}_+^n)}\le C_1\|P(D_x,D_{x_n})U\|_{L_p(\mathbb{R}^n)}. \tag{4.1}$$

Рассмотрим общий случай. Пусть $f(x,x_n)\equiv 0$. Покажем, что для любого решения задачи (1.3)-(1.4) в любом компакте $K\subset\mathbb{R}_+^{n-1}$ $\|U\|_{L_p(\mathbb{R}_+^1\times K)}=0$. Так как $U\in W_p^{\mathfrak{M}}(\mathbb{R}_+^n)$, то для любого $\varepsilon>0$ существует $U_\varepsilon\in W_p^{\mathfrak{M}}(\mathbb{R}_+^n)$ такая, что ее носитель по x принадлежит K и $\|U-U_\varepsilon\|_{W_p^{\mathfrak{M}}(\mathbb{R}_+^1\times K)}<\varepsilon$. Следовательно, из (4.1), учитывая, что $P(D_x, D_{x_n})U = 0$, имеем, что

$$\|U\|_{L_p(\mathbb{R}^1_+\times K)} \leq \|U-U_\varepsilon\|_{L_p(\mathbb{R}^1_+\times K)} + \|U_\varepsilon\|_{L_p(\mathbb{R}^1_+\times K)} \leq \varepsilon + C\|P(D_x, D_{x_n})U_\varepsilon\|_{L_p(\mathbb{R}^1_+\times K)}$$
$$= C\|P(D_x, D_{x_n})(U-U_\varepsilon)\|_{L_p(\mathbb{R}^1_+\times K)} + \varepsilon \leq C\|U-U_\varepsilon\|_{W_p^{\mathfrak{M}}(\mathbb{R}^1_+\times K)} + \varepsilon \leq (C+1)\varepsilon.$$

Для любого $K \subset \mathbb{R}^{n-1}$, то есть $U \equiv 0$ в $\mathbb{R}^n_+$, и теорема 1.1 доказана.

Теорема 1.2 доказывается аналогичным образом.

Abstract. In this paper we study the Dirichlet problem in the half-space for regular hypoelliptic equations. Applying a special integral representation, we construct approximate solutions for this problem and thereby prove correct solvability of the problem.

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ИНТЕРАССОЦИАТИВНОСТЬ С ПОМОЩЬЮ СВЕРХТОЖДЕСТВ

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Аннотация. Мы расширяем понятие интерассоциативности с помощью сверхтождеств ассоциативности и описываем множество полугрупп $\{i,j\}$-интерассоциативных к свободной и коммутативной свободной полугруппе, где $i,j = 1,2,3$.

MSC2010 number: 03C05, 03C85, 08A05, 08B20.

Ключевые слова: интерассоциативность; сверхтождество; полугруппа; свободная полугруппа.

1. Введение

Известно [1, 2], что если в q-алгебре или e-алгебре выполняется нетривиальное сверхтождество ассоциативности, то оно может быть одного из следующих видов:

$$X(Y(x,y),z) = Y(x,X(y,z)), \qquad (ass)_1$$

$$X(Y(x,y),z) = X(x,Y(y,z)), \qquad (ass)_2$$

$$X(X(x,y),z) = Y(x,Y(y,z)) \qquad (ass)_3,$$

(здесь X,Y – функциональные переменные, а x,y,z – предметные переменные).

Понятие интерассоциативности впервые ввел Зупник в работе [3]. В дальнейшем оно расширялось в работах [4 – 10]. В итоге получилось следующее определение для полугрупп.

Определение 1.1. *Полугруппа $Q(\circ)$ называется интерассоциативной к полугруппе $Q(\cdot)$, если выполняются следующие тождества:*

(1.1) $$x\cdot(y\circ z) = (x\cdot y)\circ z,$$

(1.2) $$x\circ(y\cdot z) = (x\circ y)\cdot z.$$

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Если еще выполняется тождество

$$x \circ (y \cdot z) = (x \cdot y) \circ z,$$

то полугруппа $Q(\circ)$ называется сильно интерассоциативной к полугруппе $Q(\cdot)$. Здесь выполняется также следующее тождество:

$$x \cdot (y \circ z) = (x \cdot y) \circ z = x \circ (y \cdot z) = (x \circ y) \cdot z.$$

Дадим более общее определение.

Определение 1.2. *Для заданных $i, j = 1, 2, 3$ полугруппу $Q(\circ)$ мы назовем $\{i, j\}$-интерассоциативной к полугруппе $Q(\cdot)$, если в алгебре $Q(\circ, \cdot)$ с двумя бинарными операциямн выполняются сверхтождества ассоциативности $(ass)_i$ и $(ass)_j$. Если $i = j$, то будем говорить просто о $\{i\}$-интерассоциативности. Множество всех полугрупп $\{i, j\}$-интерассоциативных к полугруппе $Q(\cdot)$ обозначим через $Int_{\{i,j\}}Q(\cdot)$; если $i = j$, то пишем просто: $Int_{\{i\}}Q(\cdot)$.*

В данном выше определении, поставив $i = j = 1$ и $i = 1$, $j = 2$, получим понятия интерассоциативности и сильной интерассоциативности, соответственно.

Пусть X произвольное непустое множество. Свободную полугруппу и свободную коммутативную полугруппу над алфавитом X обозначим, соответственно, через $\mathcal{F}(X)(\cdot)$ и $\mathcal{F}C(X)(\cdot)$. Свободную полугруппу с единицей обозначим через $\mathcal{F}^1(X)(\cdot)$. Для полугруппы $Q(\cdot)$ и ее фиксированного элемента $x \in Q$ можно определить бинарную операцию

$$a *_x b = a \cdot x \cdot b, \qquad \text{для любых } a, b \in Q.$$

В итоге получим полугруппу $Q(*_x)$, которая называется вариантом полугруппы $Q(\cdot)$ [1, 11].

Понятие варианта имеет тесную связь с понятием $\{i, j\}$-интерассоциативности.

В работах Горбаткова [12, 13] доказаны следующие утверждения.

Теорема 1.1. *Для $|X| \geqslant 4$ имеют место следующие равенства:*

$$Int_{\{1\}}\mathcal{F}C(X)(\cdot) = Int_{\{1,2\}}\mathcal{F}C(X)(\cdot) = \{\mathcal{F}C(X)(*_x) \,|\, x \in \mathcal{F}C(X)\} \cup \{\mathcal{F}C(X)(\cdot)\}.$$

Теорема 1.2. *Полугруппа $\mathcal{F}(X)(\circ)$ $\{1\}$-интерассоциативна к $\mathcal{F}(X)(\cdot)$ тогда и только тогда, когда*

$$u \circ w = u_\ell(u^{(1)} \circ w^{(0)})w_r, \qquad \text{для всех } u, w \in \mathcal{F}(X),$$

где $u^{(1)}$ *– последняя буква слова* u, $w^{(0)}$ *– первая буква слова* w, *а* u_ℓ, w_r *– слова, получающиеся из слов* u, w *сокращением букв* $u^{(1)}$, $w^{(0)}$ *соответственно.*

В работе [14] рассмотрена $\{3\}$-интерассоциативность, где получен следующий результат.

Теорема 1.3. *Если* $Q(\circ) \in Int_{\{3\}}Q(\cdot)$ *и выполняется следующее квазитождество:*

$$x \cdot x = y \cdot y \Rightarrow x = y,$$

то полугруппы $Q(\circ)$ *и* $Q(\cdot)$ *совпадают.*

2. Основные результаты

В этом разделе для произвольного множества X мы получаем описание множеств $Int_{\{2\}}\mathcal{F}(X)(\cdot)$, $Int_{\{1,2\}}\mathcal{F}(X)(\cdot)$, $Int_{\{3\}}\mathcal{F}(X)(\cdot)$, $Int_{\{3\}}\mathcal{F}C(X)(\cdot)$ и описание $Int_{\{2\}}\mathcal{F}C(X)(\cdot)$, когда $|X| \geqslant 4$.

Сперва докажем следующие леммы:

Лемма 2.1. *Пусть* $Q(\circ) \in Int_{\{2\}}Q(\cdot)$. *Допустим, что для некоторого* $a \in Q$ *отображение* $\chi_a : Q \to Q$, $\chi_a(x) = ax$ *инъективно. Тогда* $Q(\circ) \in Int_{\{1\}}Q(\cdot)$.

Доказательство. Имеем следующие тождества

$$(x \circ y)z = x(y \circ z), \tag{2.1}$$

$$(xy) \circ z = x \circ (yz). \tag{2.2}$$

Тогда из цепочки равенств

$$x((yz) \circ t) \overset{(2.1)}{=} (x \circ (yz))t \overset{(2.2)}{=} ((xy) \circ z)t \overset{(2.1)}{=} xy(z \circ t) = x(y(z \circ t))$$

и из условий леммы, получаем

$$(yz) \circ t = y(z \circ t),$$

т.е. тождество (1.1), из которого вместе с тождествами (2.1), (2.2) вытекает тождество (1.2), а тождества (1.1), (1.2) вместе с полугрупповыми тождествами ассоциативности дают сверхтождество $(ass)_1$. □

Если в определении отображения χ_a каждый элемент умножить справа на a и требовать соответствующее условие, тогда доказательство соответствующей леммы можно провести аналогично.

Лемма 2.2. *Если $Q(\circ) \in Int_{\{2\}}Q(\cdot)$, то в алгебре $Q(\circ, \cdot)$ имеет место тождество*

$$(a \circ b)cd = ab(c \circ d).$$

Доказательство. Имеем

$$(a \circ b)cd \overset{(2.1)}{=} a(b \circ c)d \overset{(2.1)}{=} ab(c \circ d). \qquad \square$$

Теорема 2.1. *При $|X| \geqslant 3$ имеют место следующие равенства*

$$Int_{\{2\}}\mathcal{F}(X)(\cdot) = Int_{\{1,2\}}\mathcal{F}(X)(\cdot) = \{\mathcal{F}(X)(\cdot)\}.$$

Доказательство. Первое равенство следует из Леммы 2.1. Возьмем $a, b \in X$, $a \neq b$ и используем Лемму 2.2. Из тождества

$$(a \circ b)ab = ab(a \circ b) \tag{2.3}$$

следует:

$$a \circ b = ab\,\varphi(a, b), \qquad \varphi : X \times X \to \mathcal{F}^1(X),$$
$$a \circ b = \psi(a, b)\,ab, \qquad \psi : X \times X \to \mathcal{F}^1(X).$$

Подставляя последние равенства в (2.3), получаем

$$\varphi(a, b) = \psi(a, b), \qquad a \neq b,\ \forall a, b \in X.$$

Выберем еще $c, d \in X$, $c \neq d$ и по лемме 2.2 имеем

$$(a \circ b)cd = ab(c \circ d),$$
$$ab\,\varphi(a, b)\,cd = ab\,\varphi(c, d)\,cd \Rightarrow \varphi(a, b) = \varphi(c, d),$$

и φ постоянное отображение, если ее аргументы не совпадают. Пусть

$$\varphi(a, b) = x \in \mathcal{F}^1(X) \qquad a, b \in X,\ a \neq b,$$
$$a \circ b = abx = xab.$$

Пусть $a, b, c \in X$, $a \neq b$, $b \neq c$. Используя (2.1), получаем

$$(a \circ b)c = a(b \circ c) \Rightarrow abxc = abcx \Rightarrow xc = cx.$$

Индукцией по длине слова x, легко выводим $x = c^n$, $n \in \mathbb{N}$. Если $c = a$, тогда $x = a^m$, $m \in \mathbb{N}$, а если $c = d$, где $d \neq a$ и $d \neq b$, то получим $x = d^k$, $k \in \mathbb{N}$. Следовательно, $x = \emptyset$. Далее $x = \emptyset \Rightarrow a \circ b = ab$, где $a \neq b$.

Если $a \neq c$, то из (2.1) имеем:

$$(a \circ a)c = a(a \circ c) = aac \Rightarrow a \circ a = aa$$

и операции $\circ$ и $\cdot$ совпадают на множестве $X \times X$, но по теореме 1.2 они совпадают и на множестве $\mathcal{F}(X) \times \mathcal{F}(X)$. $\square$

Нетрудно заметить, что $\mathcal{F}C(X)(\cdot)$ также удовлетворяет условию Леммы 2.1 и с учетом теоремы 1.1 заключаем что справедливо следующее утверждение.

Теорема 2.2. *При $|X| \geqslant 4$ имеют место следующие равенства:*

$$Int_{\{2\}}\mathcal{F}C(X)(\cdot) = \{\mathcal{F}C(X)(*_x) \,|\, x \in \mathcal{F}C(X)\} \cup \{\mathcal{F}C(X)(\cdot)\}.$$

В работе Горбаткова [13] описано множество $Int_{\{1\}}\mathcal{F}(X)$ в случае, когда $|X| = 2$. Используя этот метод, прямой проверкой убеждаемся, что теорема 2.1 верна и в случае, когда $|X| = 2$.

Теорема 2.3. *При $|X| = 2$ имеют место следующие равенства:*

$$Int_{\{2\}}\mathcal{F}(X)(\cdot) = Int_{\{1,2\}}\mathcal{F}(X)(\cdot) = \{\mathcal{F}(X)(\cdot)\}.$$

Теорема 2.4. *Пусть $|X| = 1$ и $X = \{a\}$. Тогда имеем $Int_{\{1\}}\mathcal{F}(X)(\cdot) = Int_{\{2\}}\mathcal{F}(X)(\cdot) = Int_{\{1,2\}}\mathcal{F}(X)(\cdot) = \{\mathcal{F}(X)(*_x) \,|\, x \in \mathcal{F}C(X)\} \cup \{\mathcal{F}(X)(\cdot)\} \cup \{\mathcal{F}(X)(\Delta)\}$, где $a^m \Delta a^n = a^{m+n-1}$, $m, n \in \mathbb{N}$.*

Доказательство. Первое и второе равенства вытекают из Леммы 2.1 и из коммутативности операции. Предположим, что $\mathcal{F}(X)(\circ) \in Int_{\{1\}}\mathcal{F}(X)(\cdot)$. При $m, n > 1$ имеем $a^m \circ a^n = (a^{m-1}a) \circ a^n \overset{(1.1)}{=} a^{m-1}(a \circ a^n) = a^{m-1}(a \circ (aa^{n-1})) \overset{(1.2)}{=} a^{m-1}(a \circ a)a^{n-1}$. Если $m = 1$ или $n = 1$, то это равенство очевидно. Рассмотрим следующие случаи:

(i) $a \circ a = a \Rightarrow a^m \circ a^n = a^{m+n-1} = a^m \Delta a^n$, т.е. получаем полугруппу $\mathcal{F}(X)(\Delta)$.

(ii) $a \circ a = aa$. В этом случае операции $\circ$ и $\cdot$ совпадают.

(iii) $a \circ a = a^k$, $k > 2$, тогда $a^m \circ a^n = a^m a^{k-2} a^n$.

Обозначим $a^{k-2} = x$ и заключаем, что полугруппы $\mathcal{F}(X)(\circ)$ и $\mathcal{F}(X)(*_x)$ совпадают. $\square$

В заключении заметим, что полугруппы $\mathcal{F}(X)(\cdot)$ и $\mathcal{F}C(X)(\cdot)$ удовлетворяют требованию теоремы 1.3 и, следовательно, верны следующие равенства:

$Int_{\{3\}}\mathcal{F}(X)(\cdot) = \{\mathcal{F}(X)(\cdot)\}$, $Int_{\{3\}}\mathcal{F}C(X)(\cdot) = \{\mathcal{F}C(X)(\cdot)\}$.

Abstract. In this paper the concept of interassociativity via hyperidentities of associativity is extended and characterized the semigroups which are $\{i, j\}$-interassociative to free semigroups and free commutative semigroups, where $i, j = 1, 2, 3$.

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A DIFFERENCE ANALOGUE OF CARTAN'S SECOND MAIN THEOREM FOR MEROMORPHIC MAPPINGS

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Abstract. In this paper, we prove a difference analogue of Cartan's second main theorem for a meromorphic mapping on $\mathbb{C}^m$ intersecting a finite set of fixed hyperplanes in general position on $\mathbb{P}^n(\mathbb{C})$. As an application, we prove a uniqueness theorem for a class of holomorphic curves by inverse images of $n+4$ hyperplanes. This result is so far the best result about the uniqueness problem for holomorphic curves by inverse images of hyperplanes.

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1. Introduction and main results

Recently, Nevanlinna theory have been studied for difference operators. In 2006, R. Halburd and R. Korhonen [6, 7] have built the second main theorem for a difference operator of meromorphic functions. Since then, many authors have studied applications of Nevanlinna theory for difference operators. In 2014, R. Halburd, R. Korhonen and K. Tohge [8] proved a difference analogue of Cartan's second main theorem for holomorphic curves. In 2016, T. B. Cao and R. Korhonen [1] gave a new version of the difference second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables.

However, to the best of our knowledge, a little is known concerning uniqueness problem of holomorphic curves by applying difference second main theorems. When one applies inequalities of type second main theorem, it is often crucial to have an inequality with truncated counting functions. For instance, all the existing constructions of unique range sets depend on the second main theorem with truncated counting functions. The above quoted results motivate us to consider the difference second main theorem for holomorphic curves intersecting hyperplanes with the level of truncation. In order to reduce the number of hyperplanes in the uniqueness problem, we first establish a difference analogue of Cartan's second main theorem

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with truncated level 1. As an application of this result, we prove a uniqueness theorem for holomorphic curves by inverse images of $n+4$ hyperplanes.

To state our results, we first recall some notation and notions from Nevanlinna theory. We set

$$|z|^2=\sum_{j=1}^{m}|z_j|^2 \quad \text{for all} \quad z=(z_1,\dots,z_m)\in\mathbb{C}^m,$$
$$S_m(r)=\{z\in\mathbb{C}^m:|z|=r\},\quad \overline{B}_m(r)=\{z\in\mathbb{C}^m:|z|\le r\},$$
$$d=\partial+\overline{\partial},\qquad d^c=\frac{1}{4\pi i}(\partial-\overline{\partial}),$$
$$\omega_m=dd^c\log|z|^2,\quad \sigma_m=d^c\log|z|^2\wedge{\omega_m}^{m-1}(z),\nu_m(z)=dd^c|z|^2.$$

Let ν be a divisor in $\mathbb{C}^m$. We set supp$\nu=\overline{\{z:\nu(z)\ne 0\}}$, and define the counting function of ν by

$$N_\nu(r)=\int_1^r\frac{n(t)}{t^{2m-1}}dt,\quad 1<r<+\infty,$$

where $n(t)=\int_{\mathrm{supp}\nu\cap\overline{B}_m(t)}\nu_m^{m-1}$ for $m\ge 2$, and $n(t)=\sum_{|z|\le t}\nu(z)$ for $m=1$. Let M be a positive integer, we define ν^M by $\nu^M(z)=\min\{M,\nu(z)\}$ and the counting function of ν^M by

$$N_\nu^M(r)=\int_1^r\frac{n^M(t)}{t^{2m-1}}dt,\quad 1<r<+\infty,$$

where $n^M(t)=\int_{\mathrm{supp}\,\min\{M,\nu\}\cap\overline{B}_m(t)}\nu_m^{m-1}$ for $m\ge 2$, and $n^M(t)=\sum_{|z|\le t}\min\{M,\nu(z)\}$ for $m=1$. When $M=1$, we get the reduced counting function $\overline{N}_\nu(r)$.

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For a set $\alpha=(\alpha_1,\dots,\alpha_m)$ of nonnegative integers, we set $|\alpha|=\alpha_1+\dots+\alpha_m$ and $D^{|\alpha|}F=\dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1}z_1\dots\partial^{\alpha_m}z_m}$. We define the zero divisor ν_F of F by

$$\nu_F=\max\{p:D^{|\alpha|}F(z)=0\quad\text{for all}\quad \alpha:|\alpha|<p\}.$$

Let ϕ be a nonzero meromorphic function on $\mathbb{C}^m$. For each $z_0\in\mathbb{C}^m$, the zero divisor ν_ϕ of ϕ is defined as follows. We choose nonzero holomorphic functions F and G defined on a neighborhood U of z_0 such that $\phi=\dfrac{F}{G}$ on U and $\dim(F^{-1}(0)\cap G^{-1}(0))\le m-2$, then we put $\nu_\phi=\nu_{\phi=0}=\nu_F$, and $\nu_{\phi=\infty}=\nu_G$ is called the polar divisor of ϕ. For each $a\in\mathbb{P}^1(\mathbb{C})$ with $\phi^{-1}(a)\ne\mathbb{C}^m$, the counting function of an a-point of ϕ is defined as follows. We denote by $\nu_\phi(a)$ the a-divisor of ϕ. This means that if $\phi=(\phi_0:\phi_1)$ is an expression reducing ϕ, then the a-divisor $\nu_\phi(a)$ is the divisor associated with the holomorphic function $\phi_1-a\phi_0$. Thus, we have $\nu_\phi(a)=\sum_{z\in\mathbb{C}^m}\nu_{\phi_1-a\phi_0}(z)$. We define

$$n_\phi(r,a)=\int_{\mathrm{supp}\nu_\phi(a)\cap\overline{B}_m(r)}\nu_\phi(a)\nu_m^{m-1}$$

outside a set analysis of codimension 2, that is, $\dim((\phi_1 - a\phi_0)^{-1}(0) \cap \phi_0^{-1}(0)) \leq m-2$ for all $m \geq 1$ and $r > 0$, where $\mathrm{supp}\nu_\phi(a)$ denotes the closure of the set $\{z \in \mathbb{C}^m : \nu_\phi(a)(z) \neq 0\}$. The *counting function* of an a-point of ϕ is defined by

$$N_\phi(r,a) = \int_1^r \frac{n_\phi(t,a)}{t^{2m-1}} dt.$$

The *proximity function* of ϕ is defined by

$$m_\phi(r,a) = \begin{cases} \int_{S_m(r)} \log^+ \dfrac{1}{|\phi(z)-a|}\sigma_m(z), & a \neq \infty \\ \int_{S_m(r)} \log^+ |\phi(z)|\sigma_m(z), & a = \infty \end{cases}.$$

The *characteristic function* of ϕ is defined by $T_\phi(r) = m_\phi(r,\infty) + N_\phi(r,\infty)$. We also define $T_\phi(r,a) := m_\phi(r,a) + N_\phi(r,a), \quad a \neq \infty$. In some cases, we also use the notation: $T_\phi(r,a) = T(r, \dfrac{1}{\phi - a})$ and $m_\phi(r,a) = m(r, \dfrac{1}{\phi - a})$. The first main theorem states that $T_\phi(r,a) = T_\phi(r) + O(1)$. The difference operator of a meromorphic function ϕ is defined by

$$\Delta_{\mathbf{c}}(\phi) = \phi(z_1 + c_1, \ldots, z_m + c_m) - \phi(z_1, \ldots, z_m),$$

where $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{C}^m$. The hyper-order of ϕ is defined by

$$\varsigma(\phi) = \limsup_{r\to\infty} \frac{\log\log T_\phi(r)}{\log r}.$$

Let f be a meromorphic map of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. For arbitrary fixed homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$, we can choose holomorphic functions $f_0, f_1, \ldots, f_n$ defined on $\mathbb{C}^m$ such that $I_f = \{z \in \mathbb{C}^m : f_0(z) = \cdots = f_n(z) = 0\}$ is of dimension at most $m-2$ and $f = (f_0 : \cdots : f_n)$. Usually, the function $\tilde{f} = (f_0, \ldots, f_n) : \mathbb{C}^m \longrightarrow \mathbb{C}^{n+1}$ is called a reduced representation of f. Set $||\tilde{f}(z)|| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. The characteristic function of f is defined by

$$T_f(r) = \int_{S_m(r)} \log ||\tilde{f}(z)|| \sigma_m(z),$$

where the above definition is independent (up to an additive constant) of the choice of the reduced representation of f. The order of f and the hyper-order f of are defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \quad \text{and} \quad \varsigma(f) = \limsup_{r\to\infty} \frac{\log\log T_f(r)}{\log r},$$

respectively.

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$, and let

$$L(z_0, \ldots, z_n) = \sum_{j=0}^n a_j z_j$$

be a linear form defined on H, where $a_j \in \mathbb{C},\ j = 0, \dots, n$, are constants. Denote by $\mathbf{a} = (a_0, \dots, a_n)$ the non-zero vector associated with H, and define

$$L(\tilde{f}) = (H, f) = (\mathbf{a}, \tilde{f}) = \sum_{j=0}^{n} a_j f_j.$$

Under the assumption that $(\mathbf{a}, \tilde{f}) \not\equiv\ 0$ for $1 < r < +\infty$, the proximity function of f with respect to H is defined as follows:

$$m_f(r, H) = \int_{S_m(r)} \log \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}, \tilde{f})(z)|} \sigma_m(z),$$

where the above definition is independent (up to an additive constant) of the choice of the reduced representation of f. The counting function of f is defined to be $N_{\nu_{(H,f)}}(r)$, meaning that $N_{\nu(H,f)}(r) = N_{(H,f)}(r, 0)$. In some cases, we use the notation $N_f(r, H)$ instead of $N_{\nu(H,f)}(r)$.

The Casorati determinant of f is defined by

$$W_{\mathbf{c}}(f) = W_{\mathbf{c}}(f_0, \dots, f_n) = \begin{vmatrix} f_0(z) & f_1(z) & \cdot & f_n(z) \\ f_0(z + \mathbf{c}) & f_1(z + \mathbf{c}) & \cdot & f_n(z + \mathbf{c}) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(z + n\mathbf{c}) & f_1(z + n\mathbf{c}) & \cdot & f_n(z + n\mathbf{c}) \end{vmatrix},$$

where $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{C}^m \setminus \{0\}$.

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve, the Casorati type determinant of f is defined by

$$D_c(f) = D_c(f_0, \dots, f_n) = \begin{vmatrix} f_0'(z) & f_1'(z) & \cdot & f_n'(z) \\ f_0(z + c) & f_1(z + c) & \cdot & f_n(z + c) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(z + nc) & f_1(z + nc) & \cdot & f_n(z + nc) \end{vmatrix},$$

where $c \in \mathbb{C} \setminus \{0\}$.

Let F be a nonzero holomorphic function on $\mathbb{C}^m$ and $z_0 = (z_{0,1}, \dots, z_{0,m}) \in \mathbb{C}^m$ be such that $F(z_0) = 0$ with multiple $p \in \mathbb{N}^*$, then

$$F(z) = \sum_{|k|=p}^{\infty} b_k (z - z_0)^k$$

on a neighborhood of z_0, where $b_k \in \mathbb{C}$ and

$$(z - z_0)^k = (z_1 - z_{0,1})^{k_1} \dots (z_m - z_{0,m})^{k_m}, k_1 + \dots + k_m = |k|, k = (k_1, \dots, k_m) \in \mathbb{N}^m.$$

Observe that on a neighborhood of z_0, we also have

$$F(z + \mathbf{c}) = \sum_{|k|=q}^{\infty} c_k (z - z_0)^k, \ \ q \geq 0,$$

where c_k are complex constants.

We denote by $\overset{\sim\mathbf{c}}{N}_F(r,0)$ the counting function at all zeros z_0 of $F(z)$, and observe that z_0 is also a zero of $F(z+\mathbf{c})$ in the following sense. If z_0 is a zero of $F(z)$ with multiplicity $p \geq 1$ and also is a zero of $F(z+\mathbf{c})$ with multiplicity $q \geq 1$, then z_0 is counted $p-q$ times in $\overset{\sim\mathbf{c}}{N}_F(r,0)$. If $q=0$, the point z_0 is counted p times in $\overset{\sim\mathbf{c}}{N}_F(r,0)$. If $F(z)=0$ implies $F(z+\mathbf{c})=0$, then we denote by $\widetilde{N}_{F(z+\mathbf{c})}(r,0)$ the counting function at the points $F(z+\mathbf{c})=0$ when $F(z)=0$ with counting multiplicity. This means that if z_0 is a zero of $F(z)$ with multiple $p \geq 1$ and z_0 also is a zero of $F(z+\mathbf{c})$ with multiple $q \geq 1$, then z_0 is counted q times in $\widetilde{N}_{F(z+\mathbf{c})}(r,0)$. We have $N_F(r,0) = \overset{\sim\mathbf{c}}{N}_F(r,0) + \widetilde{N}_{F(z+\mathbf{c})}(r,0)$. Note that $\overset{\sim\mathbf{c}}{N}_F(r,0)$ may be negative, positive or zero if $F(z) \equiv F(z+\mathbf{c})$.

The following definition was given in Korhonen et. al [9].

Definition 1.1. *Let $n \in \mathbb{N}^*$, $c \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{P}^1(\mathbb{C})$. An a-point z_0 of a meromorphic function $h(z)$ is said to be n-successive and c-separated if the n meromorphic functions $h(z+jc)$ $(j=1,\dots,n)$ take the value a at $z=z_0$ with multiplicity not less than that of $h(z)$ at $z=z_0$. All the other a-points of $h(z)$ are called n-aperiodic of pace c. By $\overset{\sim[n,c]}{N_h}(r,a)$ we denote the counting function of n-aperiodic zeros of the function $h-a$ of pace c.*

Therefore, we denote by $\overset{\sim[n,c]}{N}_{(H,g)}(r,0)$ the counting function of the n-aperiodic zeros of function (H,g) for holomorphic curve $g: \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$. Also, we denote by $N_h^{[n,c]}(r,a)$ (resp. $\overline{N}_h^{[n,c]}(r,a)$) the counting with multiplicity (resp. without counting multiplicity) function of n-successive and c-separated a-points of a function h.

Recall that the hyperplanes $H_1,\dots,H_q,\ q>n$, in $\mathbb{P}^n(\mathbb{C})$ are said to be in general position if for any distinct $i_1,\dots,i_{n+1} \in \{1,\dots,q\}$, we have $\bigcap_{k=1}^{n+1} \operatorname{supp}(H_{i_k}) = \emptyset$, which is equivalent to the $H_{i_1},\dots,H_{i_{n+1}}$ being linearly independent.

In this paper, we consider the following family of meromorphic maps:

$$\mathcal{F} = \Big\{ f: \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}) \text{ such that } T_{f_i}(r) \leq O(T_f(r)) \text{ for all } i=0,\dots,n \Big\}.$$

Observe that $\mathcal{F} \neq \emptyset$, since $f = (f_0 : f_1 : \cdots : f_n) \in \mathcal{F}$, where $f_i = 1$ for some $i \in \{0,\dots,n\}$. Indeed, we have for all $j \neq i$,

$$T_{f_j}(r) = \int_{S_m(r)} \log^+ |f_j(z)| \sigma_m(z) \leq \int_{S_m(r)} \log(1 + \max_{t \in \{0,\dots,n\} \setminus i} \{|f_t(z)|\}) \sigma_m(z)$$
$$\leq \int_{S_m(r)} \log ||\tilde{f}(z)|| \sigma_m(z) + O(1) = T_f(r) + O(1).$$

Now we are in position to state the main results of this paper. The next theorem is a difference analogue of Cartan's second main theorem.

Theorem 1.2. *Let $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve in $\mathcal{F}$ with $\varsigma(f) = \varsigma < 1$, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that the image of f is not contained in $H_j, j = 1, \ldots, q$. Suppose that $D_c(f) \not\equiv 0$. Then for any $1 < r < +\infty$, we have*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q}(\tilde{N}{}^{[n,c]}_{(H_j,f)}(r,0) + \overline{N}^{[n,c]}_{(H_j,f)}(r,0)) + S(r,f),$$

r lies outside of a possible exceptional set $E \subset [1,\infty)$ of finite logarithmic measure.

As an immediate consequence of Theorem 1.2, we have the following result.

Corollary 1.1. *Let $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve in $\mathcal{F}$ with $\varsigma(f) = \varsigma < 1$, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that the image of f is not contained in $H_j, j = 1, \ldots, q$. Suppose that $D_c(f) \not\equiv 0$ and for any $1 < r < +\infty$,*

$$\sum_{j=1}^{q}\tilde{N}{}^{[n,c]}_{(H_j,f)}(r,0) = S(r,f).$$

Then we have

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q}\overline{N}^{[n,c]}_{(H_j,f)}(r,0) + S(r,f) \leq \sum_{j=1}^{q}\overline{N}_{(H_j,f)}(r,0) + S(r,f)$$

for all r lying outside a of possible exceptional set $E \subset [1,\infty)$ of finite logarithmic measure.

Next, we consider the family $\mathcal{G} \subset \mathcal{F}$ of holomorphic curves with the following properties:

(i) $D_c(f) \not\equiv 0$ for all $f \in \mathcal{G}$;

(ii) Let $H_1, \ldots, H_q, q \geq n+4$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that the image of f is not contained in $H_j, j = 1, \ldots, q$, and $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for all $i \neq j$, and $f \in \mathcal{G}$. We also assume that $\sum_{j=1}^{q}\tilde{N}{}^{[n,c]}_{(H_j,f)}(r,0) = S(r,f)$ for all $f \in \mathcal{G}$.

(iii) $\varsigma(f) = \varsigma < 1$ for all $f \in \mathcal{G}$.

As an application of Corollary 1.1, we have the following uniqueness theorem for holomorphic curves from $\mathcal{G}$.

Theorem 1.3. *Let f and g be two holomorphic curves in $\mathcal{G}$, and let $H_1, \ldots, H_q, q \geq n+4$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Suppose that $f(z) = g(z)$ on $\cup_{j=1}^{q}(f^{-1}(H_j) \cup g^{-1}(H_j))$. Then we have $f \equiv g$.*

Remark 1.1. In 2010, Z. Chen and Q. Yan [3] have proved a uniqueness theorem for holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ by inverse images of $2n+3$ hyperplanes.

Our Theorem 1.3 gives a uniqueness theorem for holomorphic curves by inverse images of $n+4$ hyperplanes.

Theorem 1.4. *Let $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic non-degenerate linear map in $\mathcal{F}$ with $\varsigma(f) = \varsigma < 1$, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that $H_j(f(0)) \neq 0, j = 1, \ldots, q$. Then for any $1 < r < +\infty$, we have*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q}(\tilde{N}^{\mathbf{c}}_{(H_j,f)}(r,0) + \tilde{N}^{n}_{(H_j,f(z+\mathbf{c}))}(r,0)) + S(r,f),$$

r lies outside of a possible exceptional set $E \subset [1,\infty)$ of finite logarithmic measure.

2. Some results from Nevanlinna theory

In this section we state some known results from Nevanlinna theory that will be used in the proofs of the theorems.

Lemma 2.1 ([1]). *Let f be a non-constant meromorphic function in $\mathbb{C}^m$ such that $f(0) \neq 0, \infty$, and let $\mathbf{c} \in \mathbb{C}^m$. If $\varsigma(f) = \varsigma < 1$, then*

$$m(r, \frac{f(z+\mathbf{c})}{f(z)}) = S(r,f),$$

for all $r > 0$ outside of a possible exceptional set $E \subset [1,+\infty)$ of finite logarithmic measure $\int_E dt/t < +\infty$.

Lemma 2.2 ([8, 10]). *Let $f : \mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function, and let $\mathbf{c} \in \mathbb{C}^m$. If $\varsigma(f) = \varsigma < 1$, then $T_{f(z+\mathbf{c})}(r) \leq T_f(r) + o(T_f(r))$, where $r \to \infty$ outside of an exceptional set of finite logarithmic measure.*

Lemma 2.3 ([8]). *Let f be a non-constant meromorphic function, $\varepsilon > 0$ and $c \in \mathbb{C}$. If $\varsigma(f) < 1$ and $\varepsilon > 0$, then*

$$m(r, \frac{f(z+c)}{f(z)}) = o(\frac{T_f(r)}{r^{1-\varsigma-\varepsilon}})$$

for all r outside of a set of finite logarithmic measure.

3. Proof of Theorems

We first prove a number of lemmas.

Lemma 3.1. *Let $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve in $\mathcal{F}$ with hyper-order $\varsigma(f) < 1$, and let $H_1, \ldots, H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$ such that the image of f is not contained in $H_j, j = 1, \ldots, q$. Let $\mathbf{a}_j$ by the non-zero* *vector associated with $H_j, j=1,\dots,q$. Suppose that $D_c(f)\not\equiv 0$. Then the following inequality*

$$\int_0^{2\pi}\max_K\sum_{l\in K}\log\frac{\|\tilde f(re^{i\theta})\|}{|(\mathbf{a}_l,\tilde f)(re^{i\theta})|}\frac{d\theta}{2\pi}\leqslant (n+1)T_f(r)-N_{D_c(f)}(r,0)+S(r,f)$$

holds for all r outside of an exceptional set of finite logarithmic measure. Here the maximum is taken over all subsets K of $\{1,\dots,q\}$ such that $\mathbf{a}_l$, $l\in K$ are linearly independent.

Proof. Let $K\subset\{1,\dots,q\}$ be a set such that $\mathbf{a}_l$ $(l\in K)$ are linearly independent. Without loss of generality, we may assume that $q\geqslant n+1$ and $\#K=n+1$. Let $\mathcal{T}$ be the set of all injective maps $\mu:\{0,1,\dots,n\}\to\{1,\dots,q\}$. Then we can write

$$\begin{aligned}
\int_0^{2\pi}\max_K\sum_{l\in K}\log\frac{\|\tilde f(re^{i\theta})\|}{|(\mathbf{a}_j,\tilde f)(re^{i\theta})|}\frac{d\theta}{2\pi}&=\int_0^{2\pi}\max_{\mu\in\mathcal{T}}\sum_{l=0}^{n}\log\frac{\|\tilde f(re^{i\theta})\|}{|(\mathbf{a}_{\mu(l)},\tilde f)(re^{i\theta})|}\frac{d\theta}{2\pi}\\
&=\int_0^{2\pi}\max_{\mu\in\mathcal{T}}\log\left\{\frac{\|\tilde f(re^{i\theta})\|^{n+1}}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(re^{i\theta})|}\right\}\frac{d\theta}{2\pi}\\
&\le\int_0^{2\pi}\max_{\mu\in\mathcal{T}}\log\left\{\frac{\|\tilde f(re^{i\theta})\|^{n+1}}{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}\right\}\frac{d\theta}{2\pi}\\
&+\int_0^{2\pi}\max_{\mu\in\mathcal{T}}\log\left\{\frac{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(re^{i\theta})|}\right\}\frac{d\theta}{2\pi}+O(1)\\
&=\int_0^{2\pi}\log\max_{\mu\in\mathcal{T}}\left\{\frac{\|\tilde f(re^{i\theta})\|^{n+1}}{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}\right\}\frac{d\theta}{2\pi}\\
&+\int_0^{2\pi}\log\max_{\mu\in\mathcal{T}}\left\{\frac{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(re^{i\theta})|}\right\}\frac{d\theta}{2\pi}+O(1)\\
&\le\int_0^{2\pi}\log\sum_{\mu\in\mathcal{T}}\frac{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(re^{i\theta})|}\frac{d\theta}{2\pi}\\
&+\int_0^{2\pi}\log\sum_{\mu\in\mathcal{T}}\frac{\|\tilde f(re^{i\theta})\|^{n+1}}{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}\frac{d\theta}{2\pi}+O(1).
\end{aligned}$$

By the property of Casorati-type determinant, we see that $|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\ldots,(\mathbf{a}_{\mu(n)},\tilde{f}))| = C_{1,\mu}|D_c(f_0,\ldots,f_n)|$, where $C_{1,\mu}>0$ is a constant. So, we obtain

$$\begin{aligned}\int_0^{2\pi}\max_K\sum_{l\in K}\log\frac{\|\tilde{f}(re^{i\theta})\|}{|(\mathbf{a}_l,\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi} &\leqslant \int_0^{2\pi}\log\sum_{\mu\in T}\frac{|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\ldots,(\mathbf{a}_{\mu(n)},\tilde{f}))(re^{i\theta})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi}\\ &\quad+\int_0^{2\pi}\log\frac{\|\tilde{f}(re^{i\theta})\|^{n+1}}{|D_c(f_0,\ldots,f_n)(re^{i\theta})|}\frac{d\theta}{2\pi}+O(1).\end{aligned}\tag{3.1}$$

We have

$$\begin{aligned}&\frac{D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\ldots,(\mathbf{a}_{\mu(n)},\tilde{f}))(z)}{\prod\limits_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde{f})(z)}\\ &=\begin{vmatrix}\frac{(a_{\mu(0)},\tilde{f})'(z)}{(\mathbf{a}_{\mu(0)},\tilde{f})(z)} & \frac{(\mathbf{a}_{\mu(1)},\tilde{f})'(z)}{(\mathbf{a}_{\mu(1)},\tilde{f})(z)} & . & \frac{(\mathbf{a}_{\mu(n)},\tilde{f})'(z)}{(\mathbf{a}_{\mu(n)},\tilde{f})(z)}\\ \frac{(\mathbf{a}_{\mu(0)},\tilde{f})(z+c)}{(\mathbf{a}_{\mu(0)},\tilde{f})(z)} & \frac{(\mathbf{a}_{\mu(1)},\tilde{f})(z+c)}{(\mathbf{a}_{\mu(1)},\tilde{f})(z)} & . & \frac{(\mathbf{a}_{\mu(n)},\tilde{f})(z+c)}{(\mathbf{a}_{\mu(n)},\tilde{f})(z)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{(\mathbf{a}_{\mu(0)},\tilde{f})(z+nc)}{(\mathbf{a}_{\mu(0)},\tilde{f})(z)} & \frac{(\mathbf{a}_{\mu(1)},\tilde{f})(z+nc)}{(\mathbf{a}_{\mu(1)},\tilde{f})(z)} & . & \frac{(\mathbf{a}_{\mu(n)},\tilde{f})(z+nc)}{(\mathbf{a}_{\mu(n)},\tilde{f})(z)}\end{vmatrix}.\end{aligned}$$

By Lemma 2.3, we obtain

$$m(r,\frac{(\mathbf{a}_{\mu(l)},\tilde{f})(z+jc)}{(\mathbf{a}_{\mu(l)},\tilde{f})(z)})=o(T_{(\mathbf{a}_{\mu(l)},\tilde{f})(z)}(r)),\tag{3.2}$$

for all $r>0$ outside of a possible exceptional set $E\subset[1,+\infty)$ of finite logarithmic measure $\int_E dt/t<+\infty$, for all $l=0,\ldots,n$ and for all $j=1,\ldots,n$. We have

$$T_{(\mathbf{a}_{\mu(l)},\tilde{f})(z)}(r)\le\sum_{j=0}^{n}T_{f_j}(r)+O(1)\le O(T_f(r))$$

for all $l=0,\ldots,n$. Thus, (3.2) implies

$$\|\quad m(r,\frac{(\mathbf{a}_{\mu(l)},\tilde{f})(z+jc)}{(\mathbf{a}_{\mu(l)},\tilde{f})(z)})=o(T_f(r)),\tag{3.3}$$

for all $l=0,\ldots,n$ and for all $j=1,\ldots,n$.

From (3.3) and the lemma on the logarithmic derivative, for any $\mu\in\mathcal{T}$, we have

$$\int_0^{2\pi}\log^+\frac{|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\ldots,(\mathbf{a}_{\mu(n)},\tilde{f}))(re^{i\theta})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi}\leqslant S(r,f).$$

This implies that

$$\int_0^{2\pi} \log \sum_{\mu\in\mathcal{T}} \frac{|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\ldots,(\mathbf{a}_{\mu(n)},\tilde{f}))(re^{i\theta})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi}$$

$$\leqslant \int_0^{2\pi} \log^+ \sum_{\mu\in\mathcal{T}} \frac{|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\ldots,(\mathbf{a}_{\mu(n)},\tilde{f}))(re^{i\theta})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi}$$

$$\leqslant \sum_{\mu\in\mathcal{T}} \int_0^{2\pi} \log^+ \frac{|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\ldots,(\mathbf{a}_{\mu(n)},\tilde{f}))(re^{i\theta})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi} + O(1) \leqslant S(r,f). \tag{3.4}$$

Now the statement of the lemma follows from (3.1), (3.4) and Jensen's formula. Lemma 3.1 is proved. □

Lemma 3.2. *(see [5]) Let $f_0, f_1, \ldots, f_n$ be linearly independent meromorphic functions in $\mathbb{C}^m$, and let $f = (f_0, f_1, \ldots, f_n)$. Then there are multi-indices $\nu_i \in \mathbb{Z}_+^m$, $i = 1, \ldots, n$ such that $0 < |\nu_i| \le i$ and $f, \partial^{\nu_1} f, \ldots, \partial^{\nu_n} f$ are linearly independent over $\mathbb{C}^m$.*

Fix multi-indices $\nu_i \in \mathbb{Z}_+^m$ with $\nu_0 = 0$ and $|\nu_i| > 0$ $(i = 1, \ldots, n)$, and set $l = |\nu_1| + \cdots + |\nu_n|$. For meromorphic functions $f_0, \ldots, f_n$ in $\mathbb{C}^m$, the Wronskian determinant is defined by

$$W(f_0,\ldots,f_n) = W_{\nu_1\ldots\nu_n}(f_0,\ldots,f_n) = \begin{vmatrix} f_0 & f_1 & \cdot & f_n \\ \partial^{\nu_1} f_0 & \partial^{\nu_1} f_1 & \cdot & \partial^{\nu_1} f_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{\nu_n} f_0 & \partial^{\nu_n} f_1 & \cdot & \partial^{\nu_n} f_n \end{vmatrix}.$$

Observe that if $f_0, f_1, \ldots, f_n$ are linearly independent meromorphic functions in $\mathbb{C}^m$, then $W(f_0, \ldots, f_n) \not\equiv 0$.

Lemma 3.3. *Let $f = (f_0 : \cdots : f_n) : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a non-degenerate meromorphic map in $\mathcal{F}$ with $\varsigma(f) < 1$, and let $H_1, \ldots, H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$ such that $H_j(f(0)) \neq 0, j = 1, \ldots, q$. Let $\mathbf{a}_j$ be the non-zero vector associated with $H_j, j = 1, \ldots, q$. Then the following inequality*

$$\int_{S_m(r)} \max_K \sum_{l\in K} \log \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_l,\tilde{f})(z)|}\sigma_m(z) \leqslant (n+1)T_f(r) - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f)$$

holds for all r outside of an exceptional set of finite logarithmic measure. Here the maximum is taken over all subsets K of $\{1, \ldots, q\}$ such that $\mathbf{a}_l$ $(l \in K)$ are linearly independent.

Proof. By Lemma 3.2, there are multi-indices $\nu_i \in \mathbb{Z}_+^m (i = 1, \ldots, n)$ such that $0 < |\nu_i| \le i$ and $\tilde{f}(z+\mathbf{c}), \partial^{\nu_1}\tilde{f}(z+\mathbf{c}), \ldots, \partial^{\nu_n}\tilde{f}(z+\mathbf{c})$ are linearly independent over

$\mathbb{C}^m$. Therefore, we have $W(f_0(z+\mathbf{c}),\dots,f_n(z+\mathbf{c}))\not\equiv 0$. Let $K\subset\{1,\dots,q\}$ be a set such that $\mathbf{a}_l$ $(l\in K)$ are linearly independent. Without loss of generality, we may assume that $q\geqslant n+1$ and $\#K=n+1$. Let $\mathcal{T}$ be the set of all injective maps $\mu:\{0,1,\dots,n\}\to\{1,\dots,q\}$. Then, we can write

$$\begin{aligned}
\int_{S_m(r)}\max_K\sum_{l\in K}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_j,\tilde f)(z)|}\sigma_m(z) &= \int_{S_m(r)}\max_{\mu\in\mathcal{T}}\sum_{l=0}^{n}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\sigma_m(z)\\
&= \int_{S_m(r)}\max_{\mu\in\mathcal{T}}\log\left\{\frac{\|\tilde f(z)\|^{n+1}}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\right\}\sigma_m(z)\\
&\le \int_{S_m(r)}\max_{\mu\in\mathcal{T}}\log\left\{\frac{\|\tilde f(z)\|^{n+1}}{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})|}\right\}\sigma_m(z)\\
&+ \int_{S_m(r)}\max_{\mu\in\mathcal{T}}\log\left\{\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\right\}\sigma_m(z)+O(1)\\
&= \int_{S_m(r)}\log\max_{\mu\in\mathcal{T}}\left\{\frac{\|\tilde f(z)\|^{n+1}}{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}\right\}\sigma_m(z)\\
&+ \int_{S_m(r)}\log\max_{\mu\in\mathcal{T}}\left\{\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\right\}\sigma_m(z)+O(1)\\
&\le \int_{S_m(r)}\log\sum_{\mu\in\mathcal{T}}\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\sigma_m(z)\\
&\quad+ \int_{S_m(r)}\log\sum_{\mu\in\mathcal{T}}\frac{\|\tilde f(z)\|^{n+1}}{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}\sigma_m(z)+O(1).
\end{aligned}$$

By the property of Wronskian determinant, we get $|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|=C_{2,\mu}|W(f_0,\dots,f_n)(z+\mathbf{c})|$, where $C_{2,\mu}>0$ is a constant. So, we obtain

$$\begin{aligned}
(3.5)\qquad &\int_{S_m(r)}\max_K\sum_{l\in K}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_l,\tilde f)(z+\mathbf{c})|}\sigma_m(z)\\
&\leqslant \int_{S_m(r)}\log\sum_{\mu\in T}\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\sigma_m(z)\\
&\quad+\int_{S_m(r)}\log\frac{\|\tilde f(z)\|^{n+1}}{|W(f_0,\dots,f_n)(z+\mathbf{c})|}\sigma_m(z)+O(1).
\end{aligned}$$

Next, we have

$$\frac{W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})}{\prod\limits_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z)}$$

$$=\frac{W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})}{\prod\limits_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}\cdot\frac{\prod\limits_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{\prod\limits_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z)}$$

$$=\begin{vmatrix} 1 & 1 & \cdot & 1\\ \frac{\partial^{\nu_1}(\mathbf{a}_{\mu(0)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(0)},\tilde f)(z+\mathbf{c})} & \frac{\partial^{\nu_1}(\mathbf{a}_{\mu(1)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(1)},\tilde f)(z+\mathbf{c})} & \cdot & \frac{\partial^{\nu_1}(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^{\nu_n}(\mathbf{a}_{\mu(0)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(0)},\tilde f)(z+\mathbf{c})} & \frac{\partial^{\nu_n}(\mathbf{a}_{\mu(1)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(1)},\tilde f)(z+\mathbf{c})} & \cdot & \frac{\partial^{\nu_n}(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})}\end{vmatrix}$$

$$\cdot\frac{\prod\limits_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{\prod\limits_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z)}.$$

By Lemma 2.1, we obtain

$$m\Big(r,\frac{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(l)},\tilde f)(z)}\Big)=o(T_{(\mathbf{a}_{\mu(l)},\tilde f)(z)}(r)),\tag{3.6}$$

for all $r>0$ outside of a possible exceptional set $E\subset[1,+\infty)$ of finite logarithmic measure $\int_E dt/t<+\infty$, for all $l=0,\dots,n$.

By Lemma 2.2, we have

$$T_{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}(r)\le T_{(\mathbf{a}_{\mu(l)},\tilde f)(z)}(r)+S(r,f)\le\sum_{j=0}^{n}T_{f_j}(r)+S(r,f)\le O(T_f(r))$$

for all $l=0,\dots,n$. Thus, (3.6) implies that

$$\|\quad m\Big(r,\frac{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(l)},\tilde f)(z)}\Big)=o(T_f(r)),\tag{3.7}$$

for all $l=0,\dots,n$.

Hence, by the lemma on the logarithmic derivative of several variables, for any $\mu\in\mathcal{T}$, we have

$$\int_{S_m(r)}\log^+\frac{\partial^{\nu_i}(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}\sigma_m(z)=o(T_{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}(r))=S(r,f),$$

for all $l=0,\dots,n$ and $i=1,\dots,n$. Therefore

$$\int_{S_m(r)}\log^+\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})|}\sigma_m(z)\leqslant S(r,f).\tag{3.8}$$

Next, in view of (3.7) and (3.8), we have

$$\begin{aligned}\int_{S_m(r)} \log \sum_{\mu\in\mathcal{T}} \frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\sigma_m(z)\\
\leqslant \int_{S_m(r)} \log^+ \sum_{\mu\in\mathcal{T}} \frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})|}\sigma_m(z)\\
+\sum_{l=0}^{n}\int_{S_m(r)}\log^+\Big|\frac{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(l)},\tilde f)(z)}\Big|\sigma_m(z)\leqslant S(r,f).\end{aligned}\tag{3.9}$$

The statement of the lemma follows from (3.5), (3.9) and Jensen's formula. □

Lemma 3.4. *Let $f=(f_0:\dots:f_n):\mathbb{C}^m\longrightarrow\mathbb{P}^n(\mathbb{C})$ be a meromorphic map, and let $H_1,\dots,H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that the image of f is not contained in $H_j, j=1,\dots,q$. Let $\mathbf{a}_j$ be the vector associated with H_j for $j=1,\dots,q$. Then*

$$\sum_{j=1}^{q} m_f(r,H_j)\leqslant \int_{S_m(r)}\max_K\sum_{l\in K}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_l,\tilde f)(z)|}\sigma_m(z)+O(1),$$

where the maximum is taken over all subsets K of $\{1,\dots,q\}$ such that $\#K=n+1$.

Proof. Let $\mathbf{a}_j=(a_{j,0},\dots,a_{j,n})$ be the associated vector of H_j, $1\leqslant j\leqslant q$, and let $\mathcal{T}$ be the set of all injective maps $\mu:\{0,1,\dots,n\}\longrightarrow\{1,\dots,q\}$. Since by hypothesis $H_1,\dots,H_q$ are in general position, for any $\mu\in\mathcal{T}$, the vectors $\mathbf{a}_{\mu(0)},\dots,\mathbf{a}_{\mu(n)}$ are linearly independent.

Let $\mu\in\mathcal{T}$, we have

$$(\tilde f,\mathbf{a}_{\mu(t)})=a_{\mu(t),0}f_0+\cdots+a_{\mu(t),n}f_n,\quad t=0,1,\dots,n.\tag{3.10}$$

Solve the system of linear equations (3.10), to get

$$f_t=b_{\mu(t),0}(\mathbf{a}_{\mu(0)},\tilde f)+\cdots+b_{\mu(t),n}(\mathbf{a}_{\mu(n)},\tilde f),\quad t=0,1,\dots,n,$$

where $\Big(b_{\mu(t),j}\Big)_{t,j=0}^{n}$ is the inverse of the matrix $\Big(a_{\mu(t),j}\Big)_{t,j=0}^{n}$. So, there is a constant C_μ to satisfy

$$\|\tilde f(z)\|\leqslant C_\mu\max_{0\leqslant t\leqslant n}|(\mathbf{a}_{\mu(t)},\tilde f)(z)|.$$

Set $C=\max_{\mu\in\mathcal{T}}C_\mu$. Then for any $\mu\in\mathcal{T}$, we have

$$\|\tilde f(z)\|\leqslant C\max_{0\leqslant t\leqslant n}|(\mathbf{a}_{\mu(t)},\tilde f)(z)|.$$

For any $z\in\mathbb{C}^m\setminus\{\cup_{j=1}^{q}(H_j(\tilde f))^{-1}(0)\cup I_f\}$, there exists a mapping $\mu\in\mathcal{T}$ such that for $j\notin\{\mu(0),\dots,\mu(n)\}$,

$$0<|(\mathbf{a}_{\mu(0)},\tilde f)(z)|\leqslant|(\mathbf{a}_{\mu(1)},\tilde f)(z)|\leqslant\dots\leqslant|(\mathbf{a}_{\mu(n)},\tilde f)(z)|\leqslant|(\mathbf{a}_j,\tilde f)(z)|.$$

Therefore, we have

$$\prod_{j=1}^{q}\frac{\|\tilde f(z)\|}{|(\mathbf{a}_j,\tilde f)(z)|}\leqslant C^{q-n-1}\max_{\mu\in\mathcal{T}}\prod_{t=0}^{n}\frac{\|\tilde f(z)\|}{|(\mathbf{a}_{\mu(t)},\tilde f)(z)|}.$$

Next, we have

$$\sum_{j=1}^{q}m_f(r,H_j)=\sum_{j=1}^{q}\int_{S_m(r)}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_j,\tilde f)(z)|}\,\sigma_m(z)=\int_{S_m(r)}\log\prod_{j=1}^{q}\frac{\|\tilde f(z)\|}{|(\mathbf{a}_j,\tilde f)(z)|}\,\sigma_m(z)$$

$$\leqslant\int_{S_m(r)}\log\max_{\mu\in\mathcal{T}}\prod_{t=0}^{n}\frac{\|\tilde f(z)\|}{|(\mathbf{a}_{\mu(t)},\tilde f)(z)|}\,\sigma_m(z)+O(1)=\int\limits_{S_m(r)}\max_{\mu\in T}\log\prod_{t=0}^{n}\frac{\|\tilde f(z)\|}{|(\mathbf{a}_{\mu(t)},\tilde f)(z)|}\,\sigma_m(z)$$

$$+O(1)=\int_{S_m(r)}\max_{\mu\in T}\sum_{t=0}^{n}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_{\mu(t)},\tilde f)(z)|}\,\sigma_m(z)+O(1).$$

Finally, we obtain

$$\sum_{j=1}^{q}m_f(r,H_j)\leqslant\int_{S_m(r)}\max_{K}\sum_{j\in K}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_j,\tilde f)(z)|}\,\sigma_m(z)+O(1).$$

This completes the proof of lemma 3.4. □

Now we are in position to prove the main results of this paper.

Proof of Theorem 1.2. By Lemmas 3.1 and 3.4, we obtain

$$\begin{aligned}\sum_{j=1}^{q}m_f(r,H_j)&\leqslant\int_0^{2\pi}\max_{K}\sum_{l\in K}\log\frac{\|\tilde f(re^{i\theta})\|}{|(\mathbf{a}_l,\tilde f)(re^{i\theta})|}\frac{d\theta}{2\pi}\\&\leqslant(n+1)T_f(r)-N_{D_c(f)}(r,0)+S(r,f).\end{aligned}\tag{3.11}$$

By the first main theorem, we get $T_f(r)=N_{(H_j,f)}(r,0)+m_f(r,H_j)+O(1)$ for any $j\in\{1,\dots,q\}$. So, from (3.11), we have

$$(q-n-1)T_f(r)\leqslant\sum_{j=1}^{q}N_{(H_j,f)}(r,0)-N_{D_c(f)}(r,0)+S(r,f).$$

For $z_0\in\mathbb{C}$, we may assume that z_0 is a zero of $(\mathbf{a}_j,\tilde f)$ for $1\leqslant j\le q_1\le n$, and $(\mathbf{a}_j,\tilde f)$ does not vanish at z_0 for $j>q_1$. Without loss of generality, we may assume that $z_0\in\mathbb{C}$ is an n-successive and c-separated zero of $(\mathbf{a}_j,\tilde f)$ for $1\leqslant j\leqslant p_1\le q_1\le n$. Hence, there exist integers k_j $(j=1,\dots,q)$ and nowhere vanishing holomorphic functions g_j $(j=1,\dots,q)$, defined on a neighborhood U of z_0, such that

$$(\mathbf{a}_j,\tilde f)(z)=(z-z_0)^{k_j}g_j(z),\ \text{for } j=1,\dots,q,$$

where $k_j=0$ for $q_1<j\leqslant q$. Also, we can assume that $k_j\geqslant 2$ for $1\leqslant j\le p_0$, and $k_j=1$ for $p_0<j\leqslant p_1$. From the definition of n-successive and c-separated 0-point, we have

$$(\mathbf{a}_j,\tilde f)(z+kc)=(z-z_0)^{l_j}h_j^k(z),\ \text{for } j=1,\dots,p_1,$$

for all $k = 1, \dots, n$, where h_j^k $(j = 1, \dots, p_1)$ are nowhere vanishing holomorphic functions, defined on a neighborhood U of z_0, and $l_j \geq k_j, 1 \leq j \leq p_1$. Let $\mathcal{T}$ be the set of all injective maps $\mu : \{0, 1, \dots, n\} \to \{1, \dots, q\}$. By a property of Wronskian, there exists a constant $\mathcal{C}_\mu \neq 0$ such that

$$D_c(f) = \mathcal{C}_\mu . \begin{vmatrix} (\mathbf{a}_{\mu(0)}, \tilde{f})' & (\mathbf{a}_{\mu(1)}, \tilde{f})' & \cdot & (\mathbf{a}_{\mu(n)}, \tilde{f})' \\ (\mathbf{a}_{\mu(0)}, \tilde{f})(z+c) & (\mathbf{a}_{\mu(1)}, \tilde{f})(z+c) & \cdot & (\mathbf{a}_{\mu(n)}, \tilde{f})(z+c) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{a}_{\mu(0)}, \tilde{f})(z+nc) & (\mathbf{a}_{\mu(1)}, \tilde{f})(z+nc) & \cdot & (\mathbf{a}_{\mu(n)}, \tilde{f})(z+nc) \end{vmatrix}$$
$$= \prod_{j=1}^{p_0} (z - z_0)^{k_j - 1} h(z),$$

where $h(z)$ is a holomorphic function on U. Then $D_c(f)$ vanishes at z_0 with order at least $\sum_{j=1}^{p_0} (k_j - 1)$. By the definitions of $N^{[n,c]}_{(H_j,f)}(r, 0)$ and $N_{D_c(f)}(r, 0)$, we have

$$\sum_{j=1}^{q} N^{[n,c]}_{(H_j,f)}(r, 0) - N_{D_c(f)}(r, 0) \leqslant \sum_{j=1}^{q} \overline{N}^{[n,c]}_{(H_j,f)}(r, 0).$$

Therefore, we get

$$(q - n - 1)T_f(r) \leqslant \sum_{j=1}^{q} (\widetilde{N}^{[n,c]}_{(H_j,f)}(r, 0) + \overline{N}^{[n,c]}_{(H_j,f)}(r, 0)) + S(r, f),$$

r lies outside of a exceptional set $E \subset [1, \infty)$ of finite logarithmic measure. □

Proof of Theorem 1.3. We denote $f = (f_0 : \cdots : f_n)$ and $g = (g_0 : \cdots : g_n)$, and assume that $f \not\equiv g$. Then there are two numbers $\alpha, \beta \in \{0, \dots, n\}$, $\alpha \neq \beta$ such that $f_\alpha g_\beta \not\equiv f_\beta g_\alpha$. Assume that $z_0 \in \mathbb{C}$ is a zero of (H_j, f) for some $j = 1, \dots, q$, then z_0 is a zero of at most n entire functions $(H_t, f), t \in \{1, \dots, q\}$. Since $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for all $i \neq j$, then z_0 is a zero of one entire function (H_j, f) for some $j \in \{1, \dots, q\}$. From condition $f(z) = g(z)$, when $z \in \cup_{j=1}^q (f^{-1}(H_j) \cup g^{-1}(H_j))$, we get $f(z_0) = g(z_0)$. This implies that z_0 is a zero of $\frac{f_\alpha}{f_\beta} - \frac{g_\alpha}{g_\beta}$. Therefore, we have

$$\sum_{j=1}^{q} \overline{N}_{(H_j,f)}(r, 0) \leq N_{\frac{f_\alpha}{f_\beta} - \frac{g_\alpha}{g_\beta}}(r, 0) \leq T_f(r) + T_g(r) + O(1).$$

Applying Corollary 1.1, we obtain

$$\|(q - n - 1)T_f(r) \leq T_f(r) + T_g(r) + o(T_f(r)). \tag{3.12}$$

Similarly, we get

$$\|(q - n - 1)T_g(r) \leq T_f(r) + T_g(r) + o(T_g(r)). \tag{3.13}$$

Finally, combining (3.12) and (3.13), we obtain $\|(q - n - 3)(T_f(r) + T_g(r)) \leq o(T_f(r)) + o(T_g(r))$, which contradicts the condition $q \geq n + 4$. Hence $f \equiv g$. □

Proof of Theorem 1.4. By Lemmas 3.3 and 3.4, we have

$$\sum_{j=1}^{q} m_f(r,H_j) \leqslant \int_{S_m(r)} \max_K \sum_{l\in K} \log \frac{\|\tilde f(z)\|}{|(\mathbf{a}_l,\tilde f)(z)|}\sigma_m(z)$$

$$\leqslant (n+1)T_f(r) - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f). \tag{3.14}$$

Next, by the first main theorem, we get

$$T_f(r) = N_{(H_j,f)}(r,0) + m_f(r,H_j) + O(1)$$

for any $j \in \{1,\dots,q\}$. So, in view of (3.14), we can write

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} N_{(H_j,f)}(r,0) - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f)$$

$$= \sum_{j=1}^{q} [\widetilde{N}^{\mathbf{c}}_{(H_j,f)}(r,0) + \widetilde{N}_{(H_j,f(z+\mathbf{c}))}(r,0)] \quad - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f)$$

$$= \sum_{j=1}^{q} \widetilde{N}^{\mathbf{c}}_{(H_j,f)}(r,0) + \sum_{j=1}^{q} \widetilde{N}_{(H_j,f(z+\mathbf{c}))}(r,0) - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f).$$

We assume that z_0 is a zero of (H_j,f) with multiple $k_j > 0, 1 \le j \le q_1 \le n$, and $k_j > n$ when $1 \le j \le q_0$, $k_j < n$ when $q_0 < j \le q_1$ and $k_j = 0$ when $q_1 < j \le q$. Hence, we may assume that z_0 is also a zero of $(H_j, f(z+\mathbf{c}))$ with multiple l_j, $l_j \ge 0, 1 \le j \le q_1$, and $l_j > n$ when $1 \le j \le p_0$, $1 \le l_j \le n$ when $p_0 < j \le p_1$ and $l_j = 0$ when $p_1 < j \le q_1$.

Therefore, it is easy to see that z_0 is counted in $N_{W(f(z+\mathbf{c}))}(r,0)$ with order at least $\sum_{j=1}^{p_0}(l_j - n)$. Then, we have

$$\sum_{j=1}^{q} \widetilde{N}_{(H_j,f(z+\mathbf{c})}(r,0) - N_{W(f(z+\mathbf{c}))}(r,0) \le \sum_{j=1}^{q} \widetilde{N}^{n}_{(H_j,f(z+\mathbf{c}))}(r,0).$$

Finally, we get

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} (\widetilde{N}^{\mathbf{c}}_{(H_j,f)}(r,0) + \widetilde{N}^{n}_{(H_j,f(z+\mathbf{c}))}(r,0)) + S(r,f).$$

This completes the proof of theorem 1.3. □

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