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UNIFORM CONVERGENCE OF DOUBLE VILENKIN-FOURIER SERIES

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Abstract. In this paper we study the problem of uniform convergence for the rectangular partial sums of double Fourier series on a bounded Vilenkin group of functions of partial bounded oscillation.

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1. Introduction

Let N_+ denote the set of positive integers, and $N := N_+ \cup \{0\}$. Let $m_0, m_1, \ldots$ be a sequence of positive integers not less then 2. Denote by $Z_{m_k} = \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo m_k. Define the group G as the complete direct product of the groups Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s. If the sequence $m_0, m_1, \ldots$ is bounded, then G is called a bounded Vilenkin group. In this paper we consider only the bounded Vilenkin group. The elements of G can be represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$, $(x_j \in Z_{m_j})$. The group operation "+" in G is given by

$$x + y = ((x_0 + y_0) \, mod \, m_0, \ldots, (x_k + y_k) \, mod \, m_k, \ldots),$$

where $x := (x_0, \ldots, x_k, \ldots)$ and $y := (y_0, \ldots, y_k, \ldots) \in G$.

The inverse of operation "+" will be denoted by "−". It is easy to give a base for the neighborhoods of G :

$$I_0(x) := G, \qquad I_n(x) := \{y \in G : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\},$$

for some choice of $(x_j \in Z_{m_j})$, $j = 0, 1, \ldots, n-1$. Let $I_n(0) = I_n$.

We denote $e_n = (0, \ldots, 0, 1, 0, \ldots) \in G$ the element of G in which the nth coordinate is 1 and the rest are zeros $(n \in N)$.

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If we define the so-called generalized number system based on m in the following way: $M_0 := 1,\ M_{k+1} := m_k M_k\ (k \in N)$, then every $n \in N$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}\ (j \in N_+)$ and only a finite number of n_j's differ from zero, and $G^2 = G \times G$ is the product of the group G.

Define

$$z_{\alpha}^{(n)} := (x_0, x_1, ..., x_{n-1}, 0, 0, ...) \in G,$$

where

$$\alpha := \sum_{j=0}^{n-1} \left(\frac{x_j}{M_{j+1}} \right) M_n, \quad (x_j \in Z_{m_j}),\ j = 0, 1, ..., n-1.$$

Then it is easy to show that

$$G := \bigcup_{\alpha=0}^{M_n - 1} \left(I_n + z_{\alpha}^{(n)} \right). \tag{1.1}$$

Next, on the group G we introduce an orthonormal system, which is called Vilenkin system. We first define the complex valued functions $r_k(x) : G \to \mathbb{C}$, called the generalized Rademacher functions, in the following way:

$$r_k(x) := \exp\left(\frac{2\pi i x_k}{m_k} \right), \qquad (i^2 = -1,\ x \in G,\ k \in N).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G as follows (see [1]):

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \qquad (n \in N).$$

In the special case where $m_j \equiv 2\ (j \in N)$, the system ψ is called Walsh-Paley system.

For the system ψ the Dirichlet kernel is defined as follows:

$$D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in \mathbb{N}_+), \qquad D_0 = 0.$$

The following properties of the kernel D_n are well known (see, e.g., [20]).

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in G \backslash I_n. \end{cases} \tag{1.2}$$

and

$$\int_G D_n(t)\, d\mu(t) = 1, \quad n \in N_+. \tag{1.3}$$

Let $n = a_k M_k + n'$, with $0 < a_k < m_k$ and $0 \leq n' < M_k$, then

$$D_n(x) = \frac{1 - \psi_{M_k}^{a_k}(x)}{1 - \psi_{M_k}(x)} D_{M_k}(x) + \psi_{M_k}^{a_k}(x) D_{n'}(x). \tag{1.4}$$

The next property of the kernel D_n can be found in [19].

$$\left|D_k\left(z_\alpha^{(n)}\right)\right| < (p+1)\, M_n/\alpha \tag{1.5}$$

for all k, n, and α $(0 < \alpha < M_n)$, where $p = \sup_j m_j$.

The rectangular partial sums of the double Vilenkin-Fourier series are defined as follows:

$$S_{n,m}(f;x,y) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \widehat{f}(i,j)\, \psi_i(x)\, \psi_j(y),$$

where the number

$$\widehat{f}(i,j) = \int_{G^2} f(x,y)\, \overline{\psi}_i(x)\, \overline{\psi}_j(y)\, d\mu(x,y)$$

is called the (i,j)-th Vilenkin-Fourier coefficient of function f.

By $C(G^2)$ we denote the space of continuous functions on G^2 with the supremum norm:

$$\|f\|_C := \sup_{x,y\in G} |f(x,y)| \qquad (f \in C(G^2)).$$

The partial moduli of continuity of a function $f \in C(G^2)$ are defined by

$$\omega_1\left(f;\frac{1}{M_k}\right) := \sup_{x,y\in G,t\in I_k} |f(x-t,y) - f(x,y)|$$

and

$$\omega_2\left(f;\frac{1}{M_l}\right) := \sup_{x,y\in G,t\in I_l} |f(x,y-t) - f(x,y)|.$$

We also will use the notion of mixed modulus of continuity of a function $f \in C(G^2)$, defined as follows:

$$\omega_{1,2}\left(f;\frac{1}{M_k}\times\frac{1}{M_l}\right)$$

$$:= \sup_{(x,y)\in G^2} \sup_{(s,t)\in I_k\times I_l} |f(x-s,y-t) - f(x-s,y) - f(x,y-t) + f(x,y)|.$$

It is well known that there is a wide analogy between harmonic analysis on the bounded Vilenkin groups and the classical Fourier analysis. However, in the trigonometric case there is a class of functions such that their Fourier series are always convergent, and the convergence is uniform if the additional assumption of continuity of a function is made. An example of such class is the class of functions of bounded variation (BV) (see Jordan [15]).

The contributions of Wiener [21], Mercinkiewicz [17], Waterman [22], Chanturia [4], Kita and Yoneda [16], Akhobadze [2], Goginava [6] and their collaborators have shown that many of the results concerning the class of functions of bounded variation (BV) can be extended to more general classes. For Vilenkin system in one-dimensional case, the class of bounded fluctuation (BF) and the class of generalized bounded fluctuation (GBF) were introduced by Onneweer and Waterman [19].

In two-dimensional case, the class BV of functions of bounded variation was introduced and studied by Hardy [14]. An analogous result for double Walsh-Fourier series was obtained by Moricz [18]. Goginava [5] has proved that in Hardy's theorem there is no need to require the boundedness of mixed variation. In particular, in [5] it was proved that if f is a continuous function and has bounded partial variation, then its double trigonometric Fourier series converges uniformly on $[0,2\pi]^2$ in the Pringsheim sense. An analogous result for double Walsh-Fourier series was established in [7]. Different classes of generalized bounded variation for functions of two-variables were studied by Golubov [12], [13], Akhobadze [3], and Goginava and Sahakian [8]-[11]. In the present paper, we partially develop the above mentioned analogy for two-dimensional bounded Vilenkin groups, concerned with uniform convergence of Fourier series.

To state the main results of this paper, we first need to introduce the classes of functions of two variables of bounded variation and of partial bounded variation. Define

$$O_1(f;M_k,y):=\sum_{\alpha=0}^{M_k-1}\omega_1\left(f;I_k+z_\alpha^{(k)},y\right),$$

$$O_2(f;M_l,x):=\sum_{\beta=0}^{M_l-1}\omega_2\left(f;x,I_l+z_\beta^{(l)}\right),$$

and

$$O_{1,2}(f;M_k,M_l):=\sum_{\alpha=0}^{M_k-1}\sum_{\beta=0}^{M_l-1}\omega_{1,2}\left(f;\left(I_k+z_\alpha^{(k)}\right)\times\left(I_l+z_\beta^{(l)}\right)\right),$$

where

$$\omega_1\left(f,I_k+z_\alpha^{(k)},y\right):=\sup_{x,x'\in I_k+z_\alpha^{(k)}}\left|f(x,y)-f(x',y)\right|,$$

$$\omega_2\left(f,x,I_l+z_\beta^{(l)}\right):=\sup_{y,y'\in I_l+z_\beta^{(l)}}\left|f(x,y)-f(x,y')\right|,$$

and

$$\omega_{1,2}\left(f,\left(I_k+z_\alpha^{(k)}\right)\times\left(I_l+z_\beta^{(l)}\right)\right)$$
$$:=\sup_{x,x'\in I_k+z_\alpha^{(k)},\,y,y'\in I_l+z_\beta^{(l)}}\left|f(x,y)-f(x',y)-f(x,y')+f(x',y')\right|.$$

Definition 1.1. *We say that a function f is of Bounded Oscillation, and write $f \in BO(G^2)$, if it satisfies the following conditions:*

$$\sup_k O_1(f; M_k, 0) < \infty, \quad \sup_l O_2(f; M_l, 0) < \infty, \quad \sup_{k,l} O_{1,2}(f; M_k, M_l) < \infty. \tag{1.6}$$

We note that if $f \in BO(G^2)$, then $\sup_{y \in G} \sup_k O_1(f; M_k, y) < \infty$.

Indeed, in view of (1.6) and (??), we can write

$$\sup_{y\in G}\sup_k \sum_{\alpha=0}^{M_k-1} \omega_1\left(f, I_k + z_\alpha^{(k)}, y\right)$$

$$\leq \sup_{y\in G}\sup_k \sum_{\alpha=0}^{M_k-1} \sup_{x,x' \in I_k + z_\alpha^{(k)}} |f(x,y) - f(x,0) - f(x',y) + f(x',0)|$$

$$+\sup_k \sum_{\alpha=0}^{M_k-1} \sup_{x,x' \in I_k + z_\alpha^{(k)}} |f(x,0) - f(x',0)| < \infty.$$

Analogously, we can show that $\sup_{x\in G}\sup_l O_2(f; M_l, x) < \infty$.

Definition 1.2. *We say a bounded, measurable function f is of Partial Bounded Oscillation, and write $f \in PBO(G^2)$, if the following conditions hold:*

$$\sup_{y\in G}\sup_k O_1(f; M_k, y) < \infty, \quad \sup_{x\in G}\sup_l O_2(f; M_l, x) < \infty. \tag{1.7}$$

Define

$$\Delta_k^{(1)} f(x,y) := f(x - e_k, y) - f(x,y), \quad \Delta_l^{(2)} f(x,y) := f(x, y - e_l) - f(x,y), \tag{1.8}$$

and

$$\Delta_{k,l}^{(1,2)} f(x,y) := f(x - e_k, y - e_l) - f(x - e_k, y) - f(x, y - e_l) + f(x,y).$$

It is easy to see that

$$\left|\Delta_{k,l}^{(1,2)} f(x,y)\right| \leq \left|\Delta_k^{(1)} f(x,y)\right| + \left|\Delta_k^{(1)} f(x, y - e_l)\right|$$

and

$$\left|\Delta_{k,l}^{(1,2)} f(x,y)\right| \leq \left|\Delta_l^{(2)} f(x,y)\right| + \left|\Delta_l^{(2)} f(x - e_k, y)\right|.$$

2. Main results

In this section we state the main results of this paper.

Theorem 2.1. *Let $f \in C(G^2)$, and let the following conditions hold:*

(2.1)
$$\lim_{k\to\infty}\sum_{\alpha=1}^{M_k-1}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|=0,$$

(2.2)
$$\lim_{l\to\infty}\sum_{\beta=1}^{M_l-1}\frac{1}{\beta}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|=0,$$

(2.3)
$$\lim_{l,k\to\infty}\sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1}\frac{1}{\alpha}\frac{1}{\beta}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|=0$$

uniformly with respect to $(x,y)\in G^2$. Then the double Vilenkin-Fourier series of function f converges uniformly on G^2.

Theorem 2.2. *Let f be a continuous function on G^2 and $f \in PBO(G^2)$. Then the Fourier series of f converges uniformly on G^2.*

Corollary 2.1. *Let f be a continuous function on G^2 and $f \in BO(G^2)$. Then the Fourier series of f converges uniformly on G^2.*

3. Proof of Main Results

In this section we prove the main results of this paper, stated in Section 2.

Proof of Theorem 2.1. Let

$$n=\sum_{i=0}^{k}a_iM_i,\ \text{with}\ a_k\neq 0\ \text{and}\ 0\le a_i<m_i\ \text{for}\ 0\le i\le k,\ \text{and}\ n'=n-a_kM_k$$

and

$$m=\sum_{j=0}^{l}b_jM_j,\ \text{with}\ b_l\neq 0\ \text{and}\ 0\le b_j<m_j\ \text{for}\ 0\le j\le l,\ \text{and}\ m'=m-b_lM_l.$$

Then, in view of (1.3) and (1.4), we can write

(3.1)
$$S_{n,m}(f;x,y)-f(x,y)$$
$$=\int_{G^2}(f(x-s,y-t)-f(x,y))\,D_n(s)\,D_m(t)\,d\mu(s)\,d\mu(t)$$
$$=\int_{G^2}(f(x-s,y-t)-f(x,y))\left(1+\psi_{M_k}(s)+\ldots+\psi_{M_k}^{a_k-1}(s)\right)D_{M_k}(s)$$
$$\times\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)\,d\mu(s)\,d\mu(t)$$

$$+\int_{G^2}(f(x-s,y-t)-f(x,y))\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)$$
$$\times D_{M_l}(t)\psi_{M_k}^{a_k}(s)D_{n'}(s)\,d\mu(s)\,d\mu(t)$$
$$+\int_{G^2}(f(x-s,y-t)-f(x,y))\left(1+\psi_{M_k}(s)+\ldots+\psi_{M_k}^{a_k-1}(s)\right)$$
$$\times D_{M_k}(s)\psi_{M_l}^{b_l}(t)D_{m'}(t)\,d\mu(s)\,d\mu(t)$$
$$+\int_{G^2}(f(x-s,y-t)-f(x,y))\psi_{M_k}^{a_k}(s)D_{n'}(s)\psi_{M_l}^{b_l}(t)D_{m'}(t)\,d\mu(s)\,d\mu(t)$$
$$=: A_1+A_2+A_3+A_4.$$

From (1.2) we obtain

$$|A_1|\le M_kM_l\int_{I_k}\int_{I_l}|f(x-s,y-t)-f(x,y)| \tag{3.2}$$
$$\times\left|1+\psi_{M_k}(s)+\ldots+\psi_{M_k}^{a_k-1}(s)\right|\left|1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right|d\mu(s)\,d\mu(t)$$
$$\le M_kM_l\frac{1}{M_k}\frac{1}{M_l}\left(\omega_1\left(f;\frac{1}{M_k}\right)+\omega_2\left(f;\frac{1}{M_l}\right)\right)a_k\,b_l$$
$$\le p^2\left(\omega_1\left(f;\frac{1}{M_k}\right)+\omega_2\left(f;\frac{1}{M_l}\right)\right)=o(1),$$

as $l,k\to\infty$ uniformly with respect to $(x,y)\in G^2$.

We observe that if $t\in I_k$, $0\le\alpha<M_k$, then

$$D_{n'}\left(z_\alpha^{(k)}+t\right)=D_{n'}\left(z_\alpha^{(k)}\right).$$

Hence, in view of (1.1), we can write

$$A_2=\sum_{\alpha=0}^{M_k-1}\int_{I_k+z_\alpha^{(k)}}\int_G f(x-s,y-t)\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right) \tag{3.3}$$
$$\times D_{M_l}(t)\psi_{M_k}^{a_k}(s)D_{n'}(s)\,d\mu(s)\,d\mu(t)$$
$$=\int_{I_k}\int_G\sum_{\alpha=0}^{M_k-1}f\left(x-z_\alpha^{(k)}-s,y-t\right)\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)$$
$$\times D_{M_l}(t)D_{n'}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}(s)\,d\mu(s,t)$$
$$=\int_{I_k}\int_G f(x-s,y-t)\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)$$
$$\times D_{M_l}(t)D_{n'}(0)\psi_{M_k}^{a_k}(0)\psi_{M_k}^{a_k}(s)\,d\mu(s,t)$$
$$+\int_{I_k}\int_G\sum_{\alpha=1}^{M_k-1}f\left(x-z_\alpha^{(k)}-s,y-t\right)\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}(s)\,d\mu(s,t)=A_{21}+A_{22}.$$

It is clear that

$$\psi_{M_k}^{-a_k}(e_k)\,\psi_{M_k}^{a_k}(s)=e^{-\frac{2\pi i}{m_k}a_k}e^{\frac{2\pi i s_k}{m_k}a_k}=e^{\frac{2\pi i(s_k-1)}{m_k}a_k}=\psi_{M_k}^{a_k}(s-e_k)$$

and

$$0<c_1\le\left|1-\psi_{M_k}^{-a_k}(e_k)\right|\le 2.$$

We have

$$\psi_{M_k}^{-a_k}(e_k)\,A_{21}=\psi_{M_k}^{-a_k}(e_k)\int\limits_{I_k}\int\limits_{G}f(x-s,y-t)$$

$$\times\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(s)\,d\mu(s,t)$$

$$=\int\limits_{I_k}\int\limits_{G}f(x-s,y-t)\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)$$

$$D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(s-e_k)\,d\mu(s)\,d\mu(t)$$

$$=\int\limits_{I_k}\int\limits_{G}f(x-s-e_k,y-t)\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)$$

$$\times D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(s)\,d\mu(s,t).$$

Hence, we have

$$\left|A_{21}-\psi_{M_k}^{-a_k}(e_k)\,A_{21}\right|\le\int\limits_{I_k}\int\limits_{G}\left|\Delta_k^{(1)}f(x-s,y-t)\right.$$

$$\times\left.\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(s)\right|d\mu(s,t)$$

$$\le M_kM_l\frac{c}{M_k}\frac{1}{M_l}\omega_1\left(f,\frac{1}{M_k}\right)b_l\le cp\omega_1\left(f,\frac{1}{M_k}\right). \tag{3.4}$$

Analogously, in view of (1.5) and (2.1), we can write

$$\left|A_{22}-\psi_{M_k}^{-a_k}(e_k)\,A_{22}\right|$$

$$\le(p+1)\,M_k\int\limits_{I_k}\int\limits_{G}\left|\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)\right|$$

$$\times\sum_{\alpha=1}^{M_k-1}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)}-s,y\right)\psi_{M_k}^{a_k}(s)\right|d\mu(s,t)$$

$$\le c\,(p+1)\,M_kM_l\frac{c}{M_k}\frac{1}{M_l}o(1)\,b_l=o(1). \tag{3.5}$$

Combining (3.4) and (3.5) we get

$$A_2=o(1) \tag{3.6}$$

as $k \to \infty$ uniformly with respect to $(x, y) \in G^2$. Analogously, by (2.2) we have

$$A_3 = o(1) \tag{3.7}$$

as $l \to \infty$ uniformly with respect to $(x, y) \in G^2$.

For A_4 we can write

$$A_4 = \sum_{\alpha=0}^{M_k-1} \sum_{\beta=0}^{M_l-1} \int_{I_l+z_\beta^{(l)}} \int_{I_k+z_\alpha^{(k)}} \left(f(x-s, y-t) - f(x, y)\right) \tag{3.8}$$

$$\times \psi_{M_k}^{a_k}(s) D_{n'}(s) \psi_{M_l}^{b_l}(t) D_{m'}(t) d\mu(s) d\mu(t)$$

$$= \sum_{\alpha=0}^{M_k-1} \sum_{\beta=0}^{M_l-1} \int_{I_l} \int_{I_k} f\left(x - z_\alpha^{(k)} - s, y - z_\beta^{(l)} - t\right)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right) \psi_{M_k}^{a_k}\left(z_\alpha^{(k)}\right) \psi_{M_k}^{a_k}(s) D_{m'}\left(z_\beta^{(l)}\right) \psi_{M_l}^{b_l}\left(z_\beta^{(l)}\right) \psi_{M_l}^{b_l}(t) d\mu(s, t)$$

$$= \int_{I_l} \int_{I_k} f(x-s, y-t) D_{n'}(0) \psi_{M_k}^{a_k}(s) D_{m'}(0) \psi_{M_l}^{b_l}(t) d\mu(s, t)$$

$$+ \int_{I_l} \int_{I_k} \sum_{\alpha=1}^{M_k-1} f\left(x - z_\alpha^{(k)} - s, y - t\right) D_{m'}(0) \psi_{M_l}^{b_l}(t) D_{n'}\left(z_\alpha^{(k)}\right) \psi_{M_k}^{a_k}(s) d\mu(s, t)$$

$$+ \int_{I_l} \int_{I_k} \sum_{\beta=1}^{M_l-1} f\left(x - s, y - z_\beta^{(l)} - t\right) D_{n'}(0) \psi_{M_k}^{a_k}(s) D_{m'}\left(z_\beta^{(l)}\right) \psi_{M_l}^{b_l}(t) d\mu(s, t)$$

$$+ \int_{I_l} \int_{I_k} \sum_{\alpha=1}^{M_k-1} \sum_{\beta=1}^{M_l-1} f\left(x - z_\alpha^{(k)} - s, y - z_\beta^{(l)} - t\right) D_{n'}\left(z_\alpha^{(k)}\right) \psi_{M_k}^{a_k}(s)$$

$$\times D_{m'}\left(z_\beta^{(l)}\right) \psi_{M_l}^{b_l}(t) d\mu(s) d\mu(t) = A_{41} + A_{42} + A_{43} + A_{44}.$$

We have

$$\left|A_{41} - \psi_{M_k}^{-a_k}(e_k) A_{41}\right|$$

$$\leq \int_{I_l} \int_{I_k} \left|\Delta_k^{(1)} f(x-s, y-t) D_{m'}(0) \psi_{M_l}^{b_l}(t) D_{n'}(0) \psi_{M_k}^{a_k}(s)\right| d\mu(s, t)$$

$$\leq M_k M_l \frac{c}{M_k} \frac{1}{M_l} \omega_1\left(f, \frac{1}{M_k}\right) = o(1) \tag{3.9}$$

as $k, l \to \infty$ uniformly with respect to $(x, y) \in G^2$.

Analogously, we get

$$\left|A_{42} - \psi_{M_k}^{-a_k}(e_k) A_{42}\right| \leq (p+1) M_k$$

$$\times \int_{I_l}\int_{I_k} \left|D_{m'}(0)\,\psi^{b_l}_{M_l}(t)\right| \sum_{\alpha=1}^{M_k-1} \frac{1}{\alpha} \left|\Delta_k^{(1)} f\left(x - z_\alpha^{(k)} - s, y - t\right) \psi^{a_k}_{M_k}(s)\right| d\mu(s,t)$$

(3.10) $$\leq (p+1)\, M_k M_l \frac{1}{M_k M_l} o(1) = o(1) \text{ as } k,l \to \infty.$$

(3.11) $$A_{43} = o(1)$$

as $k,l \to \infty$ uniformly with respect to $(x,y) \in G^2$.

For A_{44} we can write

$$\psi^{-a_k}_{M_k}(e_k)\, A_{44} = \int_{I_l}\int_{I_k} \sum_{\alpha=1}^{M_k-1} \sum_{\beta=1}^{M_l-1} f\left(x - z_\alpha^{(k)} - s - e_k, y - z_\beta^{(l)} - t\right)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right) \psi^{a_k}_{M_k}(s)\, D_{m'}\left(z_\beta^{(l)}\right) \psi^{b_l}_{M_l}(t)\, d\mu(s,t),$$

$$\psi^{-b_l}_{M_l}(e_l)\, A_{44} = \int_{I_l}\int_{I_k} \sum_{\alpha=1}^{M_k-1} \sum_{\beta=1}^{M_l-1} f\left(x - z_\alpha^{(k)} - s, y - z_\beta^{(l)} - t - e_l\right)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right) \psi^{a_k}_{M_k}(s)\, D_{m'}\left(z_\beta^{(l)}\right) \psi^{b_l}_{M_l}(t)\, d\mu(s)\, d\mu(t),$$

$$\psi^{-a_k}_{M_k}(e_k)\, \psi^{-b_l}_{M_l}(e_l)\, A_{44} = \int_{I_l}\int_{I_k} \sum_{\alpha=1}^{M_k-1} \sum_{\beta=1}^{M_l-1} f\left(x - z_\alpha^{(k)} - s - e_k, y - z_\beta^{(l)} - t - e_l\right)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right) \psi^{a_k}_{M_k}(s)\, D_{m'}\left(z_\beta^{(l)}\right) \psi^{b_l}_{M_l}(t)\, d\mu(s,t).$$

So, using (2.3), we obtain

(3.12) $$\left|A_{44} - \psi^{-a_k}_{M_k}(e_k)\, A_{44} - \psi^{-b_l}_{M_l}(e_l)\, A_{44} + \psi^{-a_k}_{M_k}(e_k)\, \psi^{-b_l}_{M_l}(e_l)\, A_{44}\right|$$

$$= \left|1 - \psi^{-a_k}_{M_k}(e_k)\right| \left|1 - \psi^{-b_l}_{M_l}(e_l)\right| |A_{44}|$$

$$\leq (p+1)^2 M_k M_l \int_{I_l}\int_{I_k} \sum_{\alpha=1}^{M_k-1} \sum_{\beta=1}^{M_l-1} \frac{1}{\alpha}\frac{1}{\beta}$$

$$\times \left|\Delta_{k,l}^{(1,2)} f\left(x - z_\alpha^{(k)} - s, y - z_\beta^{(l)} - t\right) \psi^{a_k}_{M_k}(s)\, \psi^{b_l}_{M_l}(t)\right| d\mu(s)\, d\mu(t)$$

$$\leq (p+1)^2 M_k M_l \frac{1}{M_k M_l} o(1) = o(1)$$

as $k,l \to \infty$ uniformly with respect to $(x,y) \in G^2$. From (3.8)-(3.12) we get

(3.13) $$A_4 = o(1)$$

as $k,l \to \infty$ uniformly with respect to $(x,y) \in G^2$.

Combining (3.1), (3.2), (3.6) and (3.13) we complete the proof of the theorem. □

Proof of Theorem 2.2. In view of Theorem 2.1, it is enough to prove that the conditions (2.1)-(2.3) are fulfilled. Let $\theta(M_k)$ and $\eta(M_l)$ be sequences of natural numbers tending to infinity and depending on M_k and M_l, respectively. Using (1.7) we can write

$$\sum_{\alpha=1}^{M_k-1}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|$$

$$=\sum_{\alpha=1}^{\theta(M_k)}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|+\sum_{\alpha=\theta(M_k)+1}^{M_k-1}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|$$

$$\leq\omega_1\left(f,\frac{1}{M_k}\right)\log\theta\left(M_k\right)+\frac{c}{\theta\left(M_k\right)+1}. \tag{3.14}$$

Next, we can choose $\theta(M_k)$ so that both terms on the last relation tend to 0 as $k\to\infty$ uniformly with respect to $(x,y)\in G^2$, and (2.1) follows.

Analogously, by using (??), we get

$$\lim_{l\to\infty}\sum_{\beta=1}^{M_l-1}\frac{1}{\beta}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|=0 \tag{3.15}$$

as $l\to\infty$ uniformly with respect to $(x,y)\in G^2$, and (2.2) follows. To verify (2.3), we write

$$B:=\sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1}\frac{1}{\alpha}\frac{1}{\beta}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|$$

$$=\sum_{s=0}^{k-1}\sum_{r=0}^{l-1}\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\frac{1}{\alpha}\frac{1}{\beta}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|$$

$$\leq\sum_{s=0}^{k-1}\sum_{r=0}^{l-1}\frac{1}{M_s}\frac{1}{M_r}\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|\right)^{\frac{1}{2}}$$

From (1.8) and (??) we get

$$\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|$$

$$\leq 2pM_r\sup_y\sum_{\alpha=M_s}^{M_{s+1}-1}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|$$

and

$$\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|$$
$$\leq 2pM_s\sup_x\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|.$$

Hence, we can write

$$B\leq 2p\sum_{s=0}^{k-1}\sum_{r=0}^{l-1}\frac{1}{(M_s)^{\frac{1}{2}}}\frac{1}{(M_r)^{\frac{1}{2}}}$$
$$\times\sup_x\left(\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|\right)^{\frac{1}{2}}\sup_y\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|\right)^{\frac{1}{2}}$$
$$=2p\sum_{s=0}^{k-1}\frac{1}{(M_s)^{\frac{1}{2}}}\sup_y\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|\right)^{\frac{1}{2}}$$
$$\times\sum_{r=0}^{l-1}\frac{1}{(M_r)^{\frac{1}{2}}}\sup_x\left(\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|\right)^{\frac{1}{2}}$$
$$=2p\left(\sum_{s=0}^{\theta(k)-1}+\sum_{s=\theta(k)}^{k-1}\right)\left(\frac{1}{(M_s)^{\frac{1}{2}}}\sup_y\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|\right)^{\frac{1}{2}}\right)$$
$$\times\left(\sum_{r=0}^{\eta(l)-1}+\sum_{r=\eta(l)}^{l-1}\right)\left(\frac{1}{(M_r)^{\frac{1}{2}}}\sup_x\left(\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|\right)^{\frac{1}{2}}\right)$$
$$\leq 2p^2\left(\sqrt[2]{\omega_1\left(f;\frac{1}{M_k}\right)}\theta(k)+\frac{c}{(M_{\theta(k)})^{\frac{1}{2}}}\right)$$
$$\times\left(\sqrt[2]{\omega_2\left(f;\frac{1}{M_l}\right)}\eta(l)+\frac{c}{(M_{\eta(l)})^{\frac{1}{2}}}\right)$$

since we can choose $\theta(k)$ and $\eta(l)$ such that $\theta(k),\eta(l)\to\infty$ as $k,l\to\infty$,

$$\sqrt[2]{\omega_1\left(f;\frac{1}{M_k}\right)}\theta(k)\to 0 \text{ as } k,l\to\infty \quad\text{and}\quad \sqrt[2]{\omega_2\left(f;\frac{1}{M_l}\right)}\eta(l)\to 0 \text{ as } k,l\to\infty.$$

Therefore, we have

$$\lim_{l,k\to\infty}\sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1}\frac{1}{\alpha}\frac{1}{\beta}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|=0,\tag{3.16}$$

as $l,k\to\infty$ uniformly with respect to $(x,y)\in G^2$, and (2.3) follows.

Combining (3.14)-(3.16) we complete the proof of the theorem. □

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ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS FROM ANALYTIC BESOV SPACES INTO ZYGMUND TYPE SPACES

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Abstract. In this paper, we give some estimates for the essential norm of weighted composition operators from analytic Besov spaces into Zygmund type spaces. In particular, a new characterization for the boundedness and compactness of the weighted composition operators uC_φ is obtained.

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1. Introduction

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Recall that the essential norm of $T : X \to Y$ is its distance to the set of compact operators $K : X \to Y$, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \text{ is compact } \}.$$

Here $\|T\|_{X\to Y}$ denotes the operator norm of $T : X \to Y$.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$, $S(\mathbb{D})$ denote the set of all analytic self-maps of $\mathbb{D}$, and let $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined as follows:

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

When $u = 1$, we get the composition operator, denoted by C_φ. When $\varphi(z) = z$, we get the multiplication operator, denoted by M_u. A basic and interesting problem concerning concrete operators (such as composition operator, weighted composition

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operator, Toeplitz operator and Hankel operator) is to relate operator theoretic properties to their function theoretic properties of their symbols (for more information, we refer the reader to [2] and [28]).

For $0 < \alpha < \infty$, the Bloch type space $\mathcal{B}^\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha |f'(z)| < \infty.$$

If $\alpha = 1$, then $\mathcal{B}^\alpha$ is the Bloch space $\mathcal{B}$ (see [28] for more details of the Bloch spaces).

For $0 < \alpha < \infty$, the Zygmund type space, denoted by $\mathcal{Z}^\alpha$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha |f''(z)| < \infty.$$

It is easy to see that the space $\mathcal{Z}^\alpha$ is a Banach space with the above norm. If $\alpha = 1$, then $\mathcal{Z}^\alpha$ is the classical Zygmund space, denoted by $\mathcal{Z}$. When $1 < \alpha < \infty$, the space $\mathcal{Z}^\alpha$ coincides with the space $\mathcal{B}^{\alpha-1}$. In particular, we have $\mathcal{Z}^2 = \mathcal{B}$.

For $p \in (1, \infty)$, the analytic Besov space B_p is the space consisting of all $f \in H(\mathbb{D})$ such that

$$b_p(f)^p := \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. The quantity b_p is a seminorm and the norm is defined by $\|f\|_{B_p} = |f(0)| + b_p(f)$. In particular, B_2 is the classical Dirichlet space.

The compactness and essential norm of the operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ were studied in [19, 20, 24, 25, 27]. The boundedness, compactness and essential norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ were studied in [3, 11, 17, 18, 22, 29, 30]. See [1, 5–9, 12–16, 23, 26] for some results of composition operators, weighted composition operators and related operators mapping into the Zygmund space. In [5], the authors characterized the boundedness and compactness of the operator $uC_\varphi : B_p \to \mathcal{Z}$. In fact, under the assumption that $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded, $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{(1 - |z|^2)^\alpha |u(z)||\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} = 0,$$

$$\lim_{|\varphi(z)|\to 1} (1 - |z|^2)^\alpha |u''(z)| \Big(\log \frac{e}{1 - |\varphi(z)|^2}\Big)^{1-\frac{1}{p}} = 0$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1 - |z|^2)^\alpha |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{1 - |\varphi(z)|^2} = 0.$$

The purpose of this paper is to give some estimates for the essential norm of the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$, in particular, by using $u\varphi''$. Moreover, we give a new characterization of the boundedness and compactness for the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$. Throughout the paper, we will write $A \lesssim B$ if there exists a constant C such that $A \le CB$. The notation $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Essential norm of $uC_\varphi : B_p \to \mathcal{Z}^\alpha$

In this section, we give some estimates for the essential norm of the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$. For this purpose, we state some lemmas which will be used in the proofs of the main results.

Lemma 2.1 ([24]). *Let X and Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that the following conditions are satisfied.*

(1) The point evaluation functionals on Y are continuous.

(2) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(3) The operator $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Lemma 2.2 ([4]). *Let $1 < p < \infty$ and $f \in B_p$. Then the following statements hold.*

(i) $|f(z)| \lesssim \|f\|_{B_p}(\log \frac{2}{1-|z|^2})^{1-\frac{1}{p}}$ *for every $z \in \mathbb{D}$.*

(ii) $|f'(z)| \lesssim \frac{1}{1-|z|^2}\|f\|_{B_p}$ *for every $z \in \mathbb{D}$.*

Let $a \in \mathbb{D}$. Define the functions:

$$f_a(z) = \frac{\log \frac{e}{1-\bar{a}z}}{(\log \frac{e}{1-|a|^2})^{\frac{1}{p}}}, \qquad g_a(z) = \frac{(\log \frac{e}{1-\bar{a}z})^2}{(\log \frac{e}{1-|a|^2})^{1+\frac{1}{p}}}, \qquad h_a(z) = \frac{(\log \frac{e}{1-\bar{a}z})^3}{(\log \frac{e}{1-|a|^2})^{2+\frac{1}{p}}},$$

$$x_a(z) = \frac{(1-|a|^2)(a-z)}{(1-\bar{a}z)^2}, \qquad y_a(z) = \bar{a}\frac{(1-|a|^2)(a-z)^2}{(1-\bar{a}z)^3}, \quad z \in \mathbb{D}.$$

Now we are in position to state and prove our main results in this section.

Theorem 2.1. *Let $1 < p < \infty, 0 < \alpha < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ with $\|\varphi\|_\infty = 1$ be such that $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded. Then*

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \approx \max\{A, B, C, P, Q\} \approx \max\{E, F, G\}.$$

Here

$$A = \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha}, \qquad B = \limsup_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{Z}^\alpha}, C = \limsup_{|a|\to 1} \|uC_\varphi h_a\|_{\mathcal{Z}^\alpha},$$

$$P = \limsup_{|a|\to 1} \|uC_\varphi x_a\|_{\mathcal{Z}^\alpha}, Q = \limsup_{|a|\to 1} \|uC_\varphi y_a\|_{\mathcal{Z}^\alpha},$$

$$G = \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2},$$

$$E = \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |u''(z)| \Big(\log \frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}}$$

and

$$F = \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{1-|\varphi(z)|^2}.$$

Proof. First we prove that

$$\max\Big\{A, B, C, P, Q\Big\} \lesssim \|uC_\varphi\|_{e, \mathcal{B}_p \to \mathcal{Z}^\alpha}.$$

In [4] it was shown that $f_a, g_a, h_a, x_a, y_a \in \mathcal{B}_p$, the norms $\|f_a\|_{\mathcal{B}_p}$, $\|g_a\|_{\mathcal{B}_p}$, $\|h_a\|_{\mathcal{B}_p}$, $\|x_a\|_{\mathcal{B}_p}$, $\|y_a\|_{\mathcal{B}_p}$ are bounded by a constant independent of a, and f_a, g_a, h_a, x_a, y_a converge to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Thus, by Lemma 2.1, for any compact operator $K : \mathcal{B}_p \to \mathcal{Z}^\alpha$, we have

$$\lim_{|a|\to 1} \|Kf_a\|_{\mathcal{Z}^\alpha} = 0, \lim_{|a|\to 1} \|Kg_a\|_{\mathcal{Z}^\alpha} = 0, \lim_{|a|\to 1} \|Kh_a\|_{\mathcal{Z}^\alpha} = 0,$$

$$\lim_{|a|\to 1} \|Kx_a\|_{\mathcal{Z}^\alpha} = 0 \quad \text{and} \quad \lim_{|a|\to 1} \|Ky_a\|_{\mathcal{Z}^\alpha} = 0.$$

Since

$$\|uC_\varphi - K\|_{\mathcal{B}_p\to\mathcal{Z}^\alpha} \gtrsim \|(uC_\varphi - K)f_a\|_{\mathcal{Z}^\alpha} \geq \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} - \|Kf_a\|_{\mathcal{Z}^\alpha},$$

$$\|uC_\varphi - K\|_{\mathcal{B}_p\to\mathcal{Z}^\alpha} \gtrsim \|(uC_\varphi - K)g_a\|_{\mathcal{Z}^\alpha} \geq \|uC_\varphi g_a\|_{\mathcal{Z}^\alpha} - \|Kg_a\|_{\mathcal{Z}^\alpha},$$

$$\|uC_\varphi - K\|_{\mathcal{B}_p\to\mathcal{Z}^\alpha} \gtrsim \|(uC_\varphi - K)h_a\|_{\mathcal{Z}^\alpha} \geq \|uC_\varphi h_a\|_{\mathcal{Z}^\alpha} - \|Kh_a\|_{\mathcal{Z}^\alpha},$$

$$\|uC_\varphi - K\|_{\mathcal{B}_p\to\mathcal{Z}^\alpha} \gtrsim \|(uC_\varphi - K)x_a\|_{\mathcal{Z}^\alpha} \geq \|uC_\varphi x_a\|_{\mathcal{Z}^\alpha} - \|Kx_a\|_{\mathcal{Z}^\alpha}$$

and

$$\|uC_\varphi - K\|_{\mathcal{B}_p\to\mathcal{Z}^\alpha} \gtrsim \|(uC_\varphi - K)y_a\|_{\mathcal{Z}^\alpha} \geq \|uC_\varphi y_a\|_{\mathcal{Z}^\alpha} - \|Ky_a\|_{\mathcal{Z}^\alpha},$$

by taking $\limsup_{|a|\to 1}$ on both sides of these inequalities, we obtain

$$\|uC_\varphi - K\|_{\mathcal{B}_p\to\mathcal{Z}^\alpha} \gtrsim \max\Big\{A, B, C, P, Q\Big\}.$$

Therefore,

$$\|uC_\varphi\|_{e, \mathcal{B}_p\to\mathcal{Z}^\alpha} = \inf_K \|uC_\varphi - K\|_{\mathcal{B}_p\to\mathcal{Z}^\alpha} \gtrsim \max\Big\{A, B, C, P, Q\Big\}.$$

Next, we prove that

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim \max\Big\{E,F,G\Big\}.$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Define

$$k_j(z)=\frac{\log\frac{e}{1-\overline{\varphi(z_j)}z}}{(\log\frac{e}{1-|\varphi(z_j)|^2})^{\frac{1}{p}}}-\frac{(\log\frac{e}{1-\overline{\varphi(z_j)}z})^2}{(\log\frac{e}{1-|\varphi(z_j)|^2})^{1+\frac{1}{p}}}+\frac{1}{3}\frac{(\log\frac{e}{1-\overline{\varphi(z_j)}z})^3}{(\log\frac{e}{1-|\varphi(z_j)|^2})^{2+\frac{1}{p}}},$$

$$l_j(z)=\frac{(1-|\varphi(z_j)|^2)(\varphi(z_j)-z)}{(1-\overline{\varphi(z_j)}z)^2}-2\overline{\varphi(z_j)}\frac{(1-|\varphi(z_j)|^2)(\varphi(z_j)-z)^2}{(1-\overline{\varphi(z_j)}z)^3}$$

and

$$m_j(z)=\overline{\varphi(z_j)}\frac{(1-|\varphi(z_j)|^2)(\varphi(z_j)-z)^2}{(1-\overline{\varphi(z_j)}z)^3},$$

and observe that k_j,l_j and m_j belong to B_p and converge to zero uniformly on compact subsets of $\mathbb{D}$. Moreover, we have

$$|k_j(\varphi(z_j))|=\frac{1}{3}\Big(\log\frac{e}{1-|\varphi(z_j)|^2}\Big)^{1-\frac{1}{p}},\qquad k_j'(\varphi(z_j))=0,\qquad k_j''(\varphi(z_j))=0,$$

$$l_j(\varphi(z_j))=0,\qquad |l_j'(\varphi(z_j))|=\frac{1}{1-|\varphi(z_j)|^2},\qquad l_j''(\varphi(z_j))=0,$$

$$m_j(\varphi(z_j))=0,\qquad m_j'(\varphi(z_j))=0,\qquad |m_j''(\varphi(z_j))|=\frac{2|\varphi(z_j)|}{(1-|\varphi(z_j)|^2)^2}.$$

Then for any compact operator $K:B_p\to\mathcal{Z}^\alpha$, we get

$$\begin{aligned}\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi k_j\|_{\mathcal{Z}^\alpha}-\limsup_{j\to\infty}\|Kk_j\|_{\mathcal{Z}^\alpha}\\ &\gtrsim \limsup_{j\to\infty}(1-|z_j|^2)^\alpha|u''(z_j)|\Big(\log\frac{e}{1-|\varphi(z_j)|^2}\Big)^{1-\frac{1}{p}}\\ &= \limsup_{|\varphi(z)|\to1}(1-|z|^2)^\alpha|u''(z)|\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}}=E,\end{aligned}$$

$$\begin{aligned}\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi l_j\|_{\mathcal{Z}^\alpha}-\limsup_{j\to\infty}\|Kl_j\|_{\mathcal{Z}^\alpha}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)|}{1-|\varphi(z_j)|^2}\\ &= \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}=F\end{aligned}$$

and

$$\begin{aligned}\|uC_\varphi - K\|_{B_p\to\mathcal{Z}^\alpha} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi m_j\|_{\mathcal{Z}^\alpha} - \limsup_{j\to\infty}\|Km_j\|_{\mathcal{Z}^\alpha}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|}{(1-|\varphi(z_j)|^2)^2}\\ &= \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} = G.\end{aligned}$$

Hence, we have

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} = \inf_K\|uC_\varphi - K\|_{B_p\to\mathcal{Z}^\alpha} \gtrsim \max\{E,F,G\}.$$

Now we prove that

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \lesssim \max\{E,F,G\}, \qquad \|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \lesssim \max\{A,B,C,P,Q\}.$$

For $r\in[0,1)$, define $K_r : H(\mathbb{D})\to H(\mathbb{D})$ by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f\in H(\mathbb{D}).$$

It is obvious that $f_r\to f$ uniformly on compact subsets of $\mathbb{D}$ as $r\to1$. Moreover, the operator K_r is compact on B_p and $\|K_r\|_{B_p\to B_p}\le 1$. Let $\{r_j\}\subset(0,1)$ be a sequence such that $r_j\to1$ as $j\to\infty$. Then for any positive integer j, the operator $uC_\varphi K_{r_j} : B_p\to\mathcal{Z}^\alpha$ is compact. Hence, we have

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \le \limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{Z}^\alpha}. \tag{2.1}$$

Thus, we have only to show that

$$\limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{Z}^\alpha} \lesssim \max\{A,B,C,P,Q\}$$

and

$$\limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{Z}^\alpha} \lesssim \max\{E,F,G\}.$$

For any $f\in B_p$ such that $\|f\|_{B_p}\le 1$, we can write

$$\begin{aligned}&\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{Z}^\alpha}\\ =\ &|u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| + \|u\cdot(f-f_{r_j})\circ\varphi\|_*\\ &+|u'(0)(f-f_{r_j})(\varphi(0)) + u(0)(f-f_{r_j})'(\varphi(0))\varphi'(0)|.\end{aligned}\tag{2.2}$$

Here $\|g\|_* = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g''(z)|$. It is obvious that

$$\lim_{j\to\infty}|u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0 \tag{2.3}$$

and

$$\lim_{j\to\infty} |u'(0)(f - f_{r_j})(\varphi(0)) + u(0)(f - f_{r_j})'(\varphi(0))\varphi'(0)| = 0. \tag{2.4}$$

Next, we can write

$$\begin{aligned}
&\limsup_{j\to\infty} \|u\cdot(f - f_{r_j})\circ\varphi\|_*\\
= \;&\limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f - f_{r_j})(\varphi(z))||u''(z)|\\
&+\limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f - f_{r_j})(\varphi(z))||u''(z)|\\
&+\limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f - f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)|\\
&+\limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f - f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)|\\
&+\limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f - f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)|\\
&+\limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f - f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)|\\
= \;& Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6,
\end{aligned} \tag{2.5}$$

where $N \in \mathbb{N}$ is large enough such that $r_j \ge \frac{1}{2}$ for all $j \ge N$,

$$Q_1 = \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f - f_{r_j})(\varphi(z))||u''(z)|,$$

$$Q_2 = \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f - f_{r_j})(\varphi(z))||u''(z)|,$$

$$Q_3 = \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f - f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)|,$$

$$Q_4 = \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f - f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)|,$$

$$Q_5 = \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f - f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)|$$

and

$$Q_6 = \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f - f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)|.$$

Since $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded, applying the operator uC_φ to $1, z$ and z^2, we see that $u \in \mathcal{Z}^\alpha, u\varphi \in \mathcal{Z}^\alpha$ and $u\varphi^2 \in \mathcal{Z}^\alpha$. Hence, using the boundedness of φ and the triangle inequality, we get

$$\widetilde{K_1} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty$$

and

$$\widetilde{K_2} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\varphi'(z)|^2|u(z)| < \infty.$$

Next, since $f_{r_j} \to f$, $r_j f'_{r_j} \to f'$, as well as $r_j^2 f''_{r_j} \to f''$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, we have

$$Q_1 \leq \|u\|_{\mathcal{Z}^\alpha} \limsup_{j\to\infty} \sup_{|w|\leq r_N} |f(w) - f(r_j w)| = 0, \tag{2.6}$$

$$Q_3 \leq \widetilde{K_1} \limsup_{j\to\infty} \sup_{|w|\leq r_N} |f'(w) - r_j f'(r_j w)| = 0 \tag{2.7}$$

and

$$Q_5 \leq \widetilde{K_2} \limsup_{j\to\infty} \sup_{|w|\leq r_N} |f''(w) - r_j^2 f''(r_j w)| = 0. \tag{2.8}$$

We know that $Q_2 \leq \limsup_{j\to\infty}(S_1^j + S_2^j)$, where

$$S_1^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f(\varphi(z))||u''(z)|, S_2^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f(r_j\varphi(z))||u''(z)|.$$

Using the fact that $\|f\|_{B_p} \leq 1$ and Lemma 2.2, we obtain

$$\begin{aligned}
S_1^j &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f(\varphi(z))||u''(z)| \\
&\lesssim \frac{1}{3} \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |u''(z)| \Big(\log \frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi(f_a - g_a + \frac{1}{3}h_a)\|_{\mathcal{Z}^\alpha} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} + \sup_{|a|>r_N} \|uC_\varphi g_a\|_{\mathcal{Z}^\alpha} + \sup_{|a|>r_N} \|uC_\varphi h_a\|_{\mathcal{Z}^\alpha}.
\end{aligned}$$

Taking limit as $N \to \infty$, we obtain

$$\begin{aligned}
&\limsup_{j\to\infty} S_1^j \lesssim \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |u''(z)| \Big(\log \frac{c}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}} = E \\
\lesssim\ & \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi h_a\|_{\mathcal{Z}^\alpha} \\
=\ & A + B + C.
\end{aligned}$$

Similarly, $\limsup_{j\to\infty} S_2^j \lesssim E \lesssim A + B + C$, and hence, we get

$$Q_2 \lesssim E \lesssim A + B + C \lesssim \max\{A, B, C\}. \tag{2.9}$$

We have $Q_4 \leq \limsup_{j\to\infty}(S_3^j + S_4^j)$, where

$$S_3^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f'(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)|$$

and

$$S_4^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha r_j |f'(r_j\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)|.$$

Using the fact that $\|f\|_{B_p} \le 1$ and $B_p \subset \mathcal{B}$, we can write

$$
\begin{aligned}
S_3^j &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)| \\
&\lesssim \|f\|_{B_p} \sup_{|\varphi(z)|>r_N} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi(x_a-2y_a)\|_{\mathcal{Z}^\alpha} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi x_a\|_{\mathcal{Z}^\alpha} + \sup_{|a|>r_N} \|uC_\varphi y_a\|_{\mathcal{Z}^\alpha}.
\end{aligned}
$$

Taking limit as $N \to \infty$, we obtain

$$
\begin{aligned}
\limsup_{j\to\infty} S_3^j \lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} = F \\
\lesssim \limsup_{|a|\to 1} \|uC_\varphi x_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi y_a\|_{\mathcal{Z}^\alpha} = P+Q.
\end{aligned}
$$

Similarly, $\limsup_{j\to\infty} S_4^j \lesssim F \lesssim P+Q$, and hence, we get

$$Q_4 \lesssim F \lesssim P+Q. \tag{2.10}$$

We have $Q_6 \le \limsup_{j\to\infty}(S_5^j + S_6^j)$, where

$$S_5^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f''(\varphi(z))||\varphi'(z)|^2|u(z)|$$

and

$$S_6^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha r_j^2 |f''(r_j\varphi(z))||\varphi'(z)|^2|u(z)|.$$

Using the fact that $\|f\|_{B_p} \le 1$ and $B_p \subset \mathcal{B}$, we can write

$$
\begin{aligned}
S_5^j &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f''(\varphi(z))||\varphi'(z)|^2|u(z)| \\
&\lesssim \frac{1}{r_N}\|f\|_{B_p} \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |\varphi'(z)|^2|u(z)| \frac{|\varphi(z)|}{(1-|\varphi(z)|^2)^2} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \frac{2(1-|z|^2)^\alpha |\varphi'(z)|^2|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^2} \lesssim \sup_{|a|>r_N} \|uC_\varphi y_a\|_{\mathcal{Z}^\alpha}.
\end{aligned}
$$

Taking limit as $N \to \infty$, we obtain

$$\limsup_{j\to\infty} S_5^j \lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)|^2|u(z)|}{(1-|\varphi(z)|^2)^2} = G \lesssim Q.$$

Similarly, $\limsup_{j\to\infty} S_6^j \lesssim G \lesssim Q$, and hence, we have

$$Q_6 \lesssim G \lesssim Q. \tag{2.11}$$

Hence, by (2.2)-(2.11), we get

$$\limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{Z}^\alpha} = \limsup_{j\to\infty}\sup_{\|f\|_{B_p}\le 1}\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{Z}^\alpha}$$

$$(2.12)\quad = \limsup_{j\to\infty}\sup_{\|f\|_{B_p}\le 1}\|u\cdot(f-f_{r_j})\circ\varphi\|_* \lesssim E+F+G \lesssim A+B+C+P+Q.$$

Therefore, by (2.1) and (2.12), we obtain

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \lesssim E+F+G \lesssim \max\{E,F,G\}$$

and

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \lesssim A+B+C+P+Q \lesssim \max\{A,B,C,P,Q\}.$$

This completes the proof. Theorem 2.1 is proved. □

Theorem 2.2. *Let $1 < p < \infty$, $0 < \alpha < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ with $\|\varphi\|_\infty = 1$ be such that $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded. Then*

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \approx \max\left\{\limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{Z}^\alpha},\ \limsup_{n\to\infty}\|u\varphi^n\|_{\mathcal{Z}^\alpha}\right\}.$$

Proof. **The lower estimate.** For each nonnegative integer n, let $p_n(z) = z^n$. Then $p_n \in B_p$ and the sequence $\{p_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Thus, by Lemma 2.1, for any compact operator $K : B_p \to \mathcal{Z}^\alpha$, we have $\lim_{n\to\infty}\|Kp_n\|_{\mathcal{Z}^\alpha} = 0$. Hence

$$\|uC_\varphi - K\|_{B_p\to\mathcal{Z}^\alpha} \gtrsim \limsup_{n\to\infty}\|(uC_\varphi - K)p_n\|_{\mathcal{Z}^\alpha} \ge \limsup_{n\to\infty}\|uC_\varphi p_n\|_{\mathcal{Z}^\alpha}.$$

From the definition of essential norm, we get

$$(2.13)\qquad \limsup_{n\to\infty}\|u\varphi^n\|_{\mathcal{Z}^\alpha} \le \|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha}.$$

By Theorem 2.1 and (2.13), we get the desired lower estimate.

The upper estimate. For $a \in \mathbb{D}$, we define

$$\lambda_a(z) = \frac{1-|a|^2}{1-\bar{a}z}, \qquad \mu_a(z) = \frac{(1-|a|^2)^2}{(1-\bar{a}z)^2}, \qquad z\in\mathbb{D}.$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j\to\infty$. As shown in [4], $f_{\varphi(z_j)}, \lambda_{\varphi(z_j)}$ and $\mu_{\varphi(z_j)}$ belong to B_p and converge to zero uniformly on compact subsets of $\mathbb{D}$. Moreover, we have

$$f_{\varphi(z_j)}(\varphi(z_j)) = (M_{\varphi(z_j)})^{1-\frac{1}{p}}, \quad f'_{\varphi(z_j)}(\varphi(z_j)) = (M_{\varphi(z_j)})^{-\frac{1}{p}}\frac{\overline{\varphi(z_j)}}{1-|\varphi(z_j)|^2},$$

$$f''_{\varphi(z_j)}(\varphi(z_j)) = (M_{\varphi(z_j)})^{-\frac{1}{p}}\frac{(\overline{\varphi(z_j)})^2}{(1-|\varphi(z_j)|^2)^2},$$

$$\lambda_{\varphi(z_j)}(\varphi(z_j)) = 1, \quad \lambda'_{\varphi(z_j)}(\varphi(z_j)) = \frac{\overline{\varphi(z_j)}}{1-|\varphi(z_j)|^2}, \quad \lambda''_{\varphi(z_j)}(\varphi(z_j)) = \frac{2(\overline{\varphi(z_j)})^2}{(1-|\varphi(z_j)|^2)^2}.$$

$$\mu_{\varphi(z_j)}(\varphi(z_j)) = 1, \quad \mu'_{\varphi(z_j)}(\varphi(z_j)) = \frac{2\overline{\varphi(z_j)}}{1-|\varphi(z_j)|^2}, \quad \mu''_{\varphi(z_j)}(\varphi(z_j)) = \frac{6(\overline{\varphi(z_j)})^2}{(1-|\varphi(z_j)|^2)^2}.$$

Here $M_{\varphi(z_j)} = \log \frac{e}{1-|\varphi(z_j)|^2}$. Next, we can write

$$\begin{aligned}
\|uC_\varphi f_{\varphi(z_j)}\|_{\mathcal{Z}^\alpha} &\geq \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(uC_\varphi f_{\varphi(z_j)})''(z)| \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\Big|u''(z)f_{\varphi(z_j)}(\varphi(z)) + u(z)(\varphi'(z))^2 f''_{\varphi(z_j)}(\varphi(z)) \\
&\quad +(2u'(z)\varphi'(z) + u(z)\varphi''(z))f'_{\varphi(z_j)}(\varphi(z))\Big| \\
&\geq (1-|z_j|^2)^\alpha|u''(z_j)|(M_{\varphi(z_j)})^{1-\frac{1}{p}} \\
&\quad -\frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2}(M_{\varphi(z_j)})^{-\frac{1}{p}} \\
&\quad -\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2}(M_{\varphi(z_j)})^{-\frac{1}{p}}. \qquad (2.14)
\end{aligned}$$

$$\begin{aligned}
\|uC_\varphi \lambda_{\varphi(z_j)}\|_{\mathcal{Z}^\alpha} &\geq \frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2} \\
&\quad -2\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2} - (1-|z_j|^2)^\alpha|u''(z_j)| \qquad (2.15)
\end{aligned}$$

$$\begin{aligned}
\|uC_\varphi \mu_{\varphi(z_j)}\|_{\mathcal{Z}^\alpha} &\geq 6\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2} - (1-|z_j|^2)^\alpha|u''(z_j)| \\
&\quad -2\frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2}. \qquad (2.16)
\end{aligned}$$

Taking limit as $j \to \infty$ on both sides of (2.14), we get

$$\begin{aligned}
&\limsup_{j\to\infty}\|uC_\varphi f_{\varphi(z_j)}\|_{\mathcal{Z}^\alpha} \\
&+ \limsup_{j\to\infty}\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2}(M_{\varphi(z_j)})^{-\frac{1}{p}} \\
&+ \limsup_{j\to\infty}\frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2}(M_{\varphi(z_j)})^{-\frac{1}{p}} \\
\geq\ & \limsup_{j\to\infty}(1-|z_j|^2)^\alpha|u''(z_j)|(M_{\varphi(z_j)})^{1-\frac{1}{p}} \\
\geq\ & \limsup_{j\to\infty}(1-|z_j|^2)^\alpha|u''(z_j)|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\limsup_{|\varphi(z)|\to 1}\|uC_\varphi f_{\varphi(z)}\|_{\mathcal{Z}^\alpha}+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{-\frac{1}{p}}\\
&+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{-\frac{1}{p}}\\
\ge\ &\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}}\\
\ge\ &\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|.
\end{aligned}$$

Similarly, by (2.15) and (2.16), we get

$$\begin{aligned}
&\limsup_{|\varphi(z)|\to 1}\|uC_\varphi\lambda_{\varphi(z)}\|_{\mathcal{Z}^\alpha}+\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\\
\ge\ &\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}\\
&-\limsup_{|\varphi(z)|\to 1}2\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2},
\end{aligned}$$

$$\begin{aligned}
&\limsup_{|\varphi(z)|\to 1}\|uC_\varphi\mu_{\varphi(z)}\|_{\mathcal{Z}^\alpha}+\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\\
\ge\ &\limsup_{|\varphi(z)|\to 1}6\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\\
&-\limsup_{|\varphi(z)|\to 1}2\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}.
\end{aligned}$$

By the boundedness of $uC_\varphi : B_p\to\mathcal{Z}^\alpha$, we see that

$$\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{-\frac{1}{p}}=0$$

and

$$\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{-\frac{1}{p}}=0.$$

Thus, we can write

$$\begin{aligned}
E=\ &\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}}\\
\le\ &\limsup_{|\varphi(z)|\to 1}\|uC_\varphi f_{\varphi(z)}\|_{\mathcal{Z}^\alpha}\le\limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{Z}^\alpha},
\end{aligned}$$

$$
\begin{aligned}
G &= \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \\
&\le \frac{3}{2}\limsup_{|\varphi(z)|\to 1} \|uC_\varphi f_{\varphi(z)}\|_{\mathcal{Z}^\alpha} + \limsup_{|\varphi(z)|\to 1} \|uC_\varphi \lambda_{\varphi(z)}\|_{\mathcal{Z}^\alpha} + \frac{1}{2}\limsup_{|\varphi(z)|\to 1} \|uC_\varphi \mu_{\varphi(z)}\|_{\mathcal{Z}^\alpha} \\
&\le \frac{3}{2}\limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha} + \frac{1}{2}\limsup_{|a|\to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha},
\end{aligned}
$$

$$
\begin{aligned}
F &= \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{1-|\varphi(z)|^2} \\
&\le 4\limsup_{|\varphi(z)|\to 1} \|uC_\varphi f_{\varphi(z)}\|_{\mathcal{Z}^\alpha} + 3\limsup_{|\varphi(z)|\to 1} \|uC_\varphi \lambda_{\varphi(z)}\|_{\mathcal{Z}^\alpha} + \limsup_{|\varphi(z)|\to 1} \|uC_\varphi \mu_{\varphi(z)}\|_{\mathcal{Z}^\alpha} \\
&\le 4\limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} + 3\limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha}.
\end{aligned}
$$

By Theorem 2.1 and the last three inequalities we obtain

$$
\|uC_\varphi\|_{e,\mathcal{B}_\mu\to\mathcal{Z}^\alpha} \approx \max\Big\{E, F, G\Big\} \tag{2.17}
$$

$$
\le \max\Big\{ \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha}, \limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha}, \limsup_{|a|\to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha} \Big\}.
$$

Finally, we prove that

$$
\max\{\limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha},\ \limsup_{|a|\to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha}\} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}^\alpha}.
$$

Let $a \in \mathbb{D}$. For any fixed positive integer $n \ge 1$, it follows from triangle inequality, the fact that $\sup_{0\le k<\infty} \|u\varphi^k\|_{\mathcal{Z}^\alpha} < \infty$ and

$$
\lambda_a(z) = (1-|a|^2)\sum_{k=0}^{\infty} \bar{a}^k z^k, \quad z \in \mathbb{D},
$$

that we have

$$
\begin{aligned}
\|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha} &\le (1-|a|^2)\sum_{k=0}^{\infty} |a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} \\
&= (1-|a|^2)\sum_{k=0}^{n-1} |a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} + (1-|a|^2)\sum_{k=n}^{\infty} |a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} \\
&\le n(1-|a|^2)\sup_{0\le k\le n-1} \|u\varphi^k\|_{\mathcal{Z}^\alpha} + (1-|a|^2)\sum_{k=n}^{\infty} |a|^k \sup_{j\ge n} \|u\varphi^j\|_{\mathcal{Z}^\alpha} \\
&\lesssim n(1-|a|^2) + 2\sup_{k\ge n} \|u\varphi^k\|_{\mathcal{Z}^\alpha}.
\end{aligned}
$$

Letting $|a| \to 1$ in the above inequality, we get

$$
\limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}^\alpha}. \tag{2.18}
$$

Let $a \in \mathbb{D}$. Note that (see [28])

$$\frac{1}{(1-|a|)^{\beta}} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\beta)}{k!\Gamma(\beta)}|a|^k \quad \text{and} \quad \frac{\Gamma(k+\beta)}{k!} \approx k^{\beta-1}, \qquad k \to \infty,$$

for any fixed positive integer $n \geq 1$. Hence, using the triangle inequality, the fact that $u \in \mathcal{Z}^{\alpha}$, $\sup_{0\leq k<\infty} \|u\varphi^k\|_{\mathcal{Z}^{\alpha}} < \infty$, and

$$\mu_a(z) = (1-|a|^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(k+2)}{\Gamma(2)k!}\bar{a}^k z^k, \quad z \in \mathbb{D},$$

we obtain

$$\begin{aligned}
\|uC_{\varphi}\mu_a\|_{\mathcal{Z}^{\alpha}} &\leq (1-|a|^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(k+2)}{\Gamma(2)k!}|a|^k \|u\varphi^k\|_{\mathcal{Z}^{\alpha}} \\
&\lesssim (1-|a|^2)^2\|u\|_{\mathcal{Z}^{\alpha}} + (1-|a|^2)^2 \sum_{k=1}^{\infty} k|a|^k \|u\varphi^k\|_{\mathcal{Z}^{\alpha}} \\
&= (1-|a|^2)^2\|u\|_{\mathcal{Z}^{\alpha}} + (1-|a|^2)^2 \sum_{k=0}^{n-1} k|a|^k \|u\varphi^k\|_{\mathcal{Z}^{\alpha}} + (1-|a|^2)^2 \sum_{k=n}^{\infty} k|a|^k \|u\varphi^k\|_{\mathcal{Z}^{\alpha}} \\
&\leq (1-|a|^2)^2\|u\|_{\mathcal{Z}^{\alpha}} + n(n-1)(1-|a|^2)^2 \sup_{0\leq k\leq n-1} \|u\varphi^k\|_{\mathcal{Z}^{\alpha}} \\
&\quad + (1-|a|^2)^2 \sum_{k=n}^{\infty} k|a|^k \sup_{j\geq n} \|u\varphi^j\|_{\mathcal{Z}^{\alpha}} \\
&\lesssim (1-|a|^2)^2\|u\|_{\mathcal{Z}^{\alpha}} + n(n-1)(1-|a|^2)^2 \sup_{0\leq k\leq n-1} \|u\varphi^k\|_{\mathcal{Z}^{\alpha}} + 4\sup_{k\geq n} \|u\varphi^k\|_{\mathcal{Z}^{\alpha}}.
\end{aligned}$$

Letting $|a| \to 1$ in the above inequality, we get

$$\limsup_{|a|\to 1} \|uC_{\varphi}\mu_a\|_{\mathcal{Z}^{\alpha}} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}^{\alpha}}. \tag{2.19}$$

Therefore, by (2.17) - (2.19) we obtain the desired upper estimate:

$$\|uC_{\varphi}\|_{e,B_p\to\mathcal{Z}^{\alpha}} \lesssim \max\left\{ \limsup_{|a|\to 1} \|uC_{\varphi}f_a\|_{\mathcal{Z}^{\alpha}}, \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}^{\alpha}} \right\}.$$

The proof is complete. □

3. A new characterization of operator uC_{φ}:

In this section, we give a new characterization for the boundedness, compactness and essential norm of the operator $uC_{\varphi} : B_p \to \mathcal{Z}^{\alpha}$. For this purpose, we first state some definitions and lemmas.

Let $v : \mathbb{D} \to R_+$ be a continuous, strictly positive and bounded function. Here we call v a weight function. The weighted space, denoted by H_v^{∞}, is a space which consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

Observe that H_v^∞ is a Banach space with the norm $\|\cdot\|_v$. A weight v is called radial if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. The associated weight $\widetilde{v}$ of v is defined by

$$\widetilde{v} = \frac{1}{\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \le 1\}}, z \in \mathbb{D}.$$

When $v = v_\alpha(z) = (1-|z|^2)^\alpha (0 < \alpha < \infty)$, then it is easy to check that $\widetilde{v}_\alpha(z) = v_\alpha(z)$. In this case, instead of H_v^∞ we use the notation $H_{v_\alpha}^\infty$, that is,

$$H_{v_\alpha}^\infty = \{f \in H(\mathbb{D}) : \|f\|_{v_\alpha} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^\alpha < \infty\}.$$

When $v = v_{\log,p}(z) = \left((\log \frac{e}{1-|z|^2})^{1-\frac{1}{p}}\right)^{-1}$, then it is also easy to see that $\widetilde{v}_{\log,p} = v_{\log,p}$. Indeed, if

$$v(z) = \left(\max\{|g(w)|; |w| = |z|\}\right)^{-1}$$

is a weight for some $g \in H(\mathbb{D})$, then $\widetilde{v}(z) = v(z)$. Hence the statement follows with $g(z) = (\log \frac{e}{1-|z|^2})^{1-\frac{1}{p}}$.

Lemma 3.1 ([11]). *For $\alpha > 0$, we have $\lim_{k\to\infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.*

Also, we have the following result.

Lemma 3.2. *For $1 < p < \infty$, we have $\lim_{k\to\infty} (\log k)^{1-\frac{1}{p}} \|z^k\|_{v_{\log,p}} \approx 1$.*

Lemma 3.3 ([21]). *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.*

(a) The weighted composition operator $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)| < \infty.$$

(b) Suppose $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded. Then

$$\|uC_\varphi\|_{e,H_v^\infty\to H_w^\infty} = \lim_{s\to 1^-} \sup_{|\varphi(z)|>s} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)|.$$

Lemma 3.4 ([10]). *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.*

(a) The weighted composition operator $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if

$$\sup_{k\ge 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty,$$

with the norm comparable to the above supremum.

(b) Suppose $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded. Then $\|uC_\varphi\|_{e,H_v^\infty \to H_w^\infty} = \limsup_{k\to\infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}$.

Now we are in position to state and prove our main results in this section.

Theorem 3.1. *Let $1 < p < \infty$, $0 < \alpha < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the weighted composition operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded if and only if $u \in \mathcal{Z}^\alpha$,*

$$\sup_{j\ge1} j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha} < \infty, \quad \sup_{j\ge1} j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha} < \infty,$$

$$\sup_{j\ge1}(\log j)^{1-\frac{1}{p}}\|u''\varphi^j\|_{v_\alpha} < \infty.$$

Proof. Observe first that by Theorem 3 of [5], the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded if and only if

$$M_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|u''(z)|\left(\log\frac{e}{1-|a|^2}\right)^{1-\frac{1}{p}} < \infty,$$

$$M_2 = \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} < \infty$$

and

$$M_3 = \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} < \infty.$$

By Lemma 3.3, the condition $M_2 < \infty$ and the boundedness of the weighted composition operator $(2u'\varphi' + u\varphi'')C_\varphi : H_{v_1}^\infty \to H_{v_\alpha}^\infty$ are equivalent. By Lemma 3.4, this is equivalent to the following:

$$\sup_{j\ge1}\frac{\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_1}} < \infty.$$

By Lemma 3.3, the condition $M_3 < \infty$ and the boundedness of the operator $u\varphi'^2C_\varphi : H_{v_2}^\infty \to H_{v_\alpha}^\infty$ are equivalent. By Lemma 3.4, this is equivalent to the following:

$$\sup_{j\ge1}\frac{\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_2}} < \infty.$$

Next, by Lemma 3.3, the condition $M_1 < \infty$ and the boundedness of the operator $u''C_\varphi : H_{v_{\log,p}}^\infty \to H_{v_\alpha}^\infty$ are equivalent. By Lemma 3.4, this is equivalent to the following:

$$\sup_{j\ge1}\frac{\|u''\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_{\log,p}}} < \infty.$$

Finally, in view of Lemmas 3.1 and 3.2, we conclude that that the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded if and only if

$$\sup_{j\ge1} j\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_{v_\alpha} \approx \sup_{j\ge1}\frac{j\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_{v_\alpha}}{j\|z^{j-1}\|_{v_1}} < \infty,$$

$$\sup_{j\geq 1} j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha} \approx \sup_{j\geq 1}\frac{j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha}}{j^2\|z^{j-1}\|_{v_2}} < \infty$$

and

$$\max\left\{\|u\|_{\mathcal{Z}^\alpha}, \sup_{j\geq 1}(\log j)^{1-\frac{1}{p}}\|u''\varphi^j\|_{v_\alpha}\right\}$$

$$= \max\left\{\|u\|_{\mathcal{Z}^\alpha}, \sup_{j\geq 2}(\log(j-1))^{1-\frac{1}{p}}\|u''\varphi^{j-1}\|_{v_\alpha}\right\} \approx \sup_{j\geq 1}\frac{\|u''\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_{\log,p}}} < \infty,$$

and the result follows. Theorem 3.1 is proved. □

Theorem 3.2. *Let $1 < p < \infty$, $0 < \alpha < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ with $\|\varphi\|_\infty = 1$ be such that $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded. Then*

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \approx \max\{N_1, N_2, N_3\},$$

where

$$N_1 = \limsup_{j\to\infty} j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha},$$

and

$$N_2 = \limsup_{j\to\infty} j^2\|u(\varphi')^2\varphi^{j-1}\|_{v_\alpha} \quad \text{and} \quad N_3 = \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|u''\varphi^j\|_{v_\alpha}.$$

Proof. From the proof of Theorem 3.1 we see that the boundedness of $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is equivalent to the boundedness of the operators $(2u'\varphi' + u\varphi'')C_\varphi : H^\infty_{v_1} \to H^\infty_{v_\alpha}$, $u''C_\varphi : H^\infty_{v_{\log,p}} \to H^\infty_{v_\alpha}$ and $u\varphi'^2C_\varphi : H^\infty_{v_2} \to H^\infty_{v_\alpha}$.

The upper estimate. In view of Lemmas 3.1 - 3.4 we can write

$$\|(2u'\varphi' + u\varphi'')C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_\alpha}} = \limsup_{j\to\infty}\frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_1}}$$

$$= \limsup_{j\to\infty}\frac{j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha}}{j\|z^{j-1}\|_{v_1}} \approx \limsup_{j\to\infty} j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha},$$

$$\begin{aligned}\|u\varphi'^2C_\varphi\|_{e,H^\infty_{v_2}\to H^\infty_{v_\alpha}} &= \limsup_{j\to\infty}\frac{\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_2}} = \limsup_{j\to\infty}\frac{j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha}}{j^2\|z^{j-1}\|_{v_2}}\\ &\approx \limsup_{j\to\infty} j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha},\end{aligned}$$

$$\|u''C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_\alpha}} = \limsup_{j\to\infty}\frac{\|u''\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_{\log,p}}} = \limsup_{j\to\infty}\frac{(\log(j-1))^{1-\frac{1}{p}}\|u''\varphi^{j-1}\|_{v_\alpha}}{(\log(j-1))^{1-\frac{1}{p}}\|z^{j-1}\|_{v_{\log,p}}}$$

$$\approx \limsup_{j\to\infty}(\log(j-1))^{1-\frac{1}{p}}\|u''\varphi^{j-1}\|_{v_\alpha} = \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|u''\varphi^j\|_{v_\alpha}.$$

Thus, we have

$$\begin{aligned}\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} &\lesssim \|(2u'\varphi'+u\varphi'')C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_\alpha}} + \|u''C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_\alpha}}\\ &+\|u\varphi'^2C_\varphi\|_{e,H^\infty_{v_2}\to H^\infty_{v_\alpha}} \lesssim \max\{N_1,N_2,N_3\}.\end{aligned}$$

The lower estimate. From Theorem 2.1, Lemmas 3.1 – 3.4, and the above proof, we have

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim F = \|(2u'\varphi'+u\varphi'')C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_\alpha}} \approx \limsup_{j\to\infty} j\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_{v_\alpha},$$

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim G = \|u\varphi'^2C_\varphi\|_{e,H^\infty_{v_2}\to H^\infty_{v_\alpha}} \approx \limsup_{j\to\infty} j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha},$$

$$\begin{aligned}&\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim E = \|u''C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_\alpha}}\\ \approx\ &\limsup_{j\to\infty}(\log(j-1))^{1-\frac{1}{p}}\|u''\varphi^{j-1}\|_{v_\alpha} = \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|u''\varphi^{j}\|_{v_\alpha}.\end{aligned}$$

Therefore, $\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim \max\{N_1,N_2,N_3\}$, as desired. Theorem 3.2 is proved. □

From Theorem 3.2, we immediately get the following result.

Theorem 3.3. *Let $1<p<\infty$, $0<\alpha<\infty$, $u\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$ be such that $uC_\varphi : B_p\to\mathcal{Z}^\alpha$ is bounded. Then the operator $uC_\varphi : B_p\to\mathcal{Z}^\alpha$ is compact if and only if*

$$\limsup_{j\to\infty} j\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_{v_\alpha}=0, \qquad \limsup_{j\to\infty} j^2\|u(\varphi')^2\varphi^{j-1}\|_{v_\alpha}=0$$

and

$$\limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|u''\varphi^{j}\|_{v_\alpha}=0.$$

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MEROMORPHIC FUNCTIONS SHARING THREE VALUES WITH THEIR LINEAR DIFFERENTIAL POLYNOMIALS IN AN ANGULAR DOMAIN

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Abstract. Let f be a nonconstant meromorphic function of lower order $\mu(f) > 1/2$ in $\mathbb{C}$, and let a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers. We show that there exists an angular domain $D = \{z : \alpha \leq \arg z \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, such that if f share a_j $(j = 1, 2, 3)$ CM with its k-th linear differential polynomial $L[f]$ in D, then $f = L[f]$. This generalizes the corresponding results from Frank and Schwick [Results. Math. 22 (1992) 679-684], Zheng [Canad. Math. Bull. 47 (2004) 152-160] and Li-Liu-Yi [Results. Math. 68 (2015) 441-453].

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1. Introduction

We use $\mathbb{C}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to denote the whole complex plane and the extended complex plane, respectively. In what follows, we shall suppose that the reader is familiar with standard notations and fundamental results of the Nevanlinna theory (see [7, 14, 15]). For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic function of f and by $\delta(a, f)$ the Nevanlinna deficiency of f. Also, by $\lambda(f)$ and $\mu(f)$ we denote the order and the lower order of a meromorphic function f, respectively.

Let f and g be nonconstant meromorphic functions in the domain $D \subset \mathbb{C}$, and let $c \in \overline{\mathbb{C}}$. If $f - c$ and $g - c$ have the same zeros with the same multiplicities in D, then we say that f and g share c CM in D. If $f - c$ and $g - c$ only have the same zeros, then we say that f and g share c IM in D. The zeros of $f - c$ imply the poles of f when $c = \infty$.

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In 1979, Gundersen [6] and Mues-Steinmetz [10] have considered the uniqueness of a meromorphic function f and its derivative f' and obtained the following result.

Theorem A (see [6, 10]). *Let f be a nonconstant meromorphic function in $\mathbb{C}$, and let a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers. If f and f' share a_j $(j = 1, 2, 3)$ IM in $\mathbb{C}$, then $f = f'$.*

In 1992, Frank and Schwick [3] generalized Theorem A and proved the following result.

Theorem B (see [3]). *Let f be a nonconstant meromorphic function in $\mathbb{C}$ and a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers, and let k be a positive integer. If f and $f^{(k)}$ share a_j $(j = 1, 2, 3)$ IM in $\mathbb{C}$, then $f = f^{(k)}$.*

Remark 1.1. *Three IM shared values in Theorem B can be replaced by two CM shared values (see Frank and Weissenborn [4]).*

In 2004, Zheng [16] has extended Theorem B from complex plane to an angular domain, and proved the following theorem.

Theorem C (see [16]). *Let f be a transcendental meromorphic function of finite lower order $\mu(f)$ in $\mathbb{C}$ such that $\delta(a, f^{(p)}) > 0$ for some $a \in \overline{\mathbb{C}}$ and an integer $p \geq 0$. Let the pairs of real numbers $\{\alpha_j, \beta_j\}$ $(j = 1, \ldots, q)$ be such that*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 < \cdots \leq \alpha_q < \beta_q \leq \pi$$

with $\omega = \max\{\pi/(\beta_j - \alpha_j) : 1 \leq j \leq q\}$, and

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta\left(a, f^{(p)}\right)}{2}},$$

where $\sigma = \max\{\omega, \mu\}$. For a positive integer k, assume that f and $f^{(k)}$ share a_j $(j = 1, 2, 3)$ IM in $X := \bigcup_{j=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\}$, where a_j $(j = 1, 2, 3)$ are three distinct finite complex numbers such that $a \neq a_j$ $(j = 1, 2, 3)$. If $\lambda(f) > \omega$, then $f = f^{(k)}$.

In 2015, Li, Liu, and Yi [9] observed that Theorem C is invalid for $q \geq 2$, and proved the following more general result, which extends Theorem C (see [9, p. 443]).

Theorem D (see [9]). *Let f be a transcendental meromorphic function of finite lower order $\mu(f)$ in $\mathbb{C}$ and such that $\delta(a, f) > 0$ for some $a \in \mathbb{C}$. Assume that $q \geq 2$ pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfy the conditions:*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 < \cdots \leq \alpha_q < \beta_q \leq \pi$$

with $\omega = \max\{\pi/(\beta_j - \alpha_j) : 1 \le j \le q\}$, *and*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta(a,f)}{2}},$$

where $\sigma = \max\{\omega, \mu\}$. *For a k-th order linear differential polynomial* $L[f]$ *in* f *with constant coefficients given by*

$$L[f] = b_k f^{(k)} + b_{k-1} f^{(k-1)} + \cdots + b_1 f', \tag{1.1}$$

where k *is a positive integer,* $b_k, b_{k-1}, \cdots, b_1$ *are constants and* $b_k \neq 0$, *assume that* f *and* $L[f]$ *share* a_j $(j = 1,2,3)$ *IM in* $X = \bigcup_{j=1}^{q}\{z : \alpha_j \le \arg z \le \beta_j\}$, *where* a_j $(j = 1,2,3)$ *are three distinct finite complex numbers such that* $a \neq a_j$ $(j = 1,2,3)$. *If* $\lambda(f) \neq \omega$, *then* $f = L[f]$.

Based on Theorem D, we naturally arise the following question.

Question 1.1. Does there exist an angular domain $D = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$, such that if f and $L[f]$ share a_j $(j = 1,2,3)$ CM or IM in D, then $f = L[f]$ in Theorem D?

In this paper, we investigate the above question and prove the following result, which generalizes Theorems C and D.

Theorem 1.1. *Let* f *be a nonconstant meromorphic function of lower order* $\mu(f) > 1/2$ *in* $\mathbb{C}$, a_j $(j = 1,2,3)$ *be three distinct finite complex numbers, and let* $L[f]$ *be given by (1.1). Then there exists an angular domain* $D = \{z : \alpha \le \arg z \le \beta\}$, *where* $0 < \beta - \alpha \le 2\pi$, *such that if* f *and* $L[f]$ *share* a_j $(j = 1,2,3)$ *CM in* D, *then* $f = L[f]$.

As an immediate consequence of Theorem 1.1, we have the following result.

Corollary 1.1. *Let* f *be a nonconstant meromorphic function of lower order* $\mu(f) > 1/2$ *in* $\mathbb{C}$, a_j $(j = 1,2,3)$ *be three distinct finite complex numbers, and let* k *be a positive integer. Then there exists an angular domain* $D = \{z : \alpha \le \arg z \le \beta\}$, *where* $0 < \beta - \alpha \le 2\pi$, *such that if* f *and* $f^{(k)}$ *share* a_j $(j = 1,2,3)$ *CM in* D, *then* $f = f^{(k)}$.

In order to prove our results, we recall the Nevanlinna theory on an angular domain. Let f be a meromorphic function in $D = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. Nevanlinna [5, 11] defined the following symbols.

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right| + \log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t}, \tag{1.2}$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^{\omega}}\int_{\alpha}^{\beta}\log^{+}\left|f\left(re^{i\theta}\right)\right|\sin\omega\left(\theta-\alpha\right)d\theta, \tag{1.3}$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_m|<r}\left(\frac{1}{|b_m|^{\omega}}-\frac{|b_m|^{\omega}}{r^{2\omega}}\right)\sin\omega\left(\theta_m-\alpha\right), \tag{1.4}$$

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f), \tag{1.5}$$

where $\omega=\pi/(\beta-\alpha)$, and $b_m=|b_m|e^{i\theta_m}$ are the poles of f in D counting multiplicities. If we ignore their multiplicities, then we replace $C_{\alpha,\beta}(r,f)$ by $\overline{C}_{\alpha,\beta}(r,f)$. Also, $S_{\alpha,\beta}(r,f)$ will stand for the Nevanlinna's angular characteristic function in D.

Throughout the paper, we denote by $R(r,*)$ a quantity satisfying the following relation:

$$R(r,*)=O\left\{\log\left(rT(r,*)\right)\right\},\qquad \forall r\notin E, \tag{1.6}$$

where E denotes a set of positive real numbers with finite linear measure, which will not necessarily be the same in each occurrence.

Also, we will need the following definitions.

Definition 1.1. *(see [8, cf.1]). Assume that f is a meromorphic function of infinite order in $\mathbb{C}$. Then there exists a proximate order $\rho(r)$ of f such that:*

(i) $\rho(r)$ is continuous and nondecreasing for $r\geq r_0$, and $\rho(r)\to+\infty$ as $r\to+\infty$;

(ii) $U(r)=r^{\rho(r)}$ $(r\geq r_0)$ satisfies the condition:

$$\overline{\lim}_{r\to+\infty}\frac{\log U(R)}{\log U(r)}=1,\qquad R=r+\frac{r}{\log U(r)};$$

(iii) the following relation holds:

$$\overline{\lim}_{r\to+\infty}\frac{\log T(r,f)}{\rho(r)\log r}=1.$$

Definition 1.2. *(see [13, cf.1, 8]). Let f be a meromorphic function of infinite order in $\mathbb{C}$, and let $\rho(r)$ be the proximate order of f. A direction $\arg z=\theta_0$ is called a Borel direction of proximate order $\rho(r)$ of f if for arbitrarily small $\varepsilon>0$ the following relation holds:*

$$\overline{\lim}_{r\to+\infty}\frac{\log n(r,\theta_0,\varepsilon,f=a)}{\rho(r)\log r}=1$$

for all $a\in\overline{\mathbb{C}}$ except at most two exceptional values, where $n(r,\theta_0,\varepsilon,f=a)$ denotes the number of the zeros of $f-a$ counting multiplicities in the sector $|\arg z-\theta_0|<\varepsilon$, $|z|\leq r$.

Definition 1.3. *(see [12]). Let f be a meromorphic function of finite order $\lambda(f) > 0$ in $\mathbb{C}$. A direction $\arg z = \theta_0$ is called a Borel direction of order $\lambda(f)$ if for arbitrarily small $\varepsilon > 0$ the following relation holds:*

$$\overline{\lim}_{r\to+\infty}\frac{\log n(r,\theta_0,\varepsilon,f=a)}{\log r} = \lambda(f)$$

for all $a \in \overline{\mathbb{C}}$ except at most two exceptional values, where $n(r,\theta_0,\varepsilon,f=a)$ is as in Definition 1.2.

2. Some lemmas

Lemma 2.1. *(see [5, 11]). Let f be a meromorphic function in $\mathbb{C}$. Then for any $a \in \mathbb{C}$ the following relation holds:*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) = S_{\alpha,\beta}(r,f) + O(1).$$

Lemma 2.2. *(see [5, 11, cf.2]). Let f be a meromorphic function in $\mathbb{C}$. Then the following assertions hold:*

(i) for q (≥ 3) distinct complex numbers $a_j \in \overline{\mathbb{C}}$ ($j = 1,2,\cdots,q$) we have

$$(q-2)S_{\alpha,\beta}(r,f) \leq \sum_{j=1}^{q}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right) + R(r,f);$$

(ii) for a positive integer k we have

$$A_{\alpha,\beta}\left(r,\frac{f^{(k)}}{f}\right) + B_{\alpha,\beta}\left(r,\frac{f^{(k)}}{f}\right) = R(r,f);$$

(iii) if f is of finite order, then $R(r,f) = O(1)$;

(iv) if f is of infinite order and of proximate order $\rho(r)$, then $R(r,f) = O(\log U(r))$, where $U(r) = r^{\rho(r)}$ is as in Definition 1.

Lemma 2.3. *(see [7]). Let f be a meromorphic function in $\mathbb{C}$, and let $L[f]$ be given by (1.1). Then $T(r,L[f]) \leq (k+1)T(r,f) + O(\log rT(r,f))$.*

Lemma 2.4. *(see [12]). Let f be a meromorphic function of finite order $\lambda(f) > 0$ in $\mathbb{C}$. Then f has at least one Borel direction $\arg z = \theta_0$ ($0 \leq \theta_0 < 2\pi$) of order $\lambda(f)$.*

Using the same arguments applied in Lemma 1.3 of [15, p.14], we can easily obtain the following result.

Lemma 2.5. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, and let $a_j \in \mathbb{C}$ $(j = 1, 2, \cdots, q)$ be q distinct complex numbers. Then we have*

$$\sum_{j=1}^{q} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{f - a_j}\right) = (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) + O(1).$$

Lemma 2.6. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers, and let $L[f]$ be given by (1.1). Suppose that f and $L[f]$ share a_j $(j = 1, 2, 3)$ CM in $D = \{z : \alpha < \arg z < \beta\}$, where $0 < \beta - \alpha \leq 2\pi$. If $f \not\equiv L[f]$, then $S_{\alpha,\beta}(r, f) = R(r, f)$.*

Proof. By Lemma 2.5, the Nevanlinna basic reasoning (see [7], p. 5), the definition (1.5) of $S_{\alpha,\beta}(r, *)$, Lemma 2.1, and Lemma 2.2(ii), we can write

$$\begin{aligned}
\textstyle\sum_{j=1}^{3} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{f-a_j}\right) &= (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \textstyle\sum_{j=1}^{3} \frac{1}{f-a_j}\right) + O(1) \\
&\leq (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \textstyle\sum_{j=1}^{3} \frac{L[f]}{f-a_j}\right) + (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{L[f]}\right) + O(1) \\
&\leq \textstyle\sum_{j=1}^{3} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{L[f]}{f-a_j}\right) + S_{\alpha,\beta}\left(r, \frac{1}{L[f]}\right) + O(1) \leq S_{\alpha,\beta}(r, L[f]) + R(r, f).
\end{aligned}$$

that is,

$$\sum_{j=1}^{3} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{f - a_j}\right) \leq S_{\alpha,\beta}(r, L[f]) + R(r, f).$$

Therefore, we have

$$\sum_{j=1}^{3} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{f - a_j}\right) + \sum_{j=1}^{3} C_{\alpha,\beta} \left(r, \frac{1}{f - a_j}\right) \leq$$

$$\leq S_{\alpha,\beta}(r, L[f]) + \sum_{j=1}^{3} C_{\alpha,\beta} \left(r, \frac{1}{f - a_j}\right) + R(r, f).$$

which together with definition (1.5) of $S_{\alpha,\beta}(r, *)$ and Lemma 2.1 implies that

$$3S_{\alpha,\beta}(r, f) \leq S_{\alpha,\beta}(r, L[f]) + \sum_{j=1}^{3} C_{\alpha,\beta} \left(r, \frac{1}{f - a_j}\right) + R(r, f). \tag{2.1}$$

Next, since f and $L[f]$ share a_j $(j = 1, 2, 3)$ CM in D, by the Nevanlinna basic reasoning [7, p. 5], Lemma 2.1, the definition (1.5) of $S_{\alpha,\beta}(r, *)$, and Lemma 2.2(ii), we can write

$$\begin{aligned}
\sum_{j=1}^{3} C_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) &\le C_{\alpha,\beta}\left(r, \frac{1}{f-L[f]}\right) \le S_{\alpha,\beta}(r, f-L[f]) + O(1) \\
&\le (A_{\alpha,\beta}+B_{\alpha,\beta})(r, f-L[f]) + C_{\alpha,\beta}(r, f-L[f]) + O(1) \\
\le (A_{\alpha,\beta}+B_{\alpha,\beta})\left(r, \frac{f-L[f]}{f}\right) &+ (A_{\alpha,\beta}+B_{\alpha,\beta})(r, f) + C_{\alpha,\beta}(r, L[f]) + O(1) \\
&\le A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f) + kC_{\alpha,\beta}(r,f) + R(r,f) \\
\le S_{\alpha,\beta}(r,f) + \frac{k}{k+1}C_{\alpha,\beta}\left(r, f^{(k)}\right) + R(r,f) &< S_{\alpha,\beta}(r,f) + \frac{k}{k+1}C_{\alpha,\beta}(r, L[f]) + R(r,f) \\
&\le S_{\alpha,\beta}(r,f) + \frac{k}{k+1}S_{\alpha,\beta}(r, L[f]) + R(r,f).
\end{aligned}$$

Combining this with (2.1), we get

$$2S_{\alpha,\beta}(r,f) \le \frac{2k+1}{k+1}S_{\alpha,\beta}(r, L[f]) + R(r,f). \tag{2.2}$$

Set $F = 1/(f-c)$ and $L_1[f] = 1/(L[f]-c)$, where $c \in \mathbb{C}$ ($c \notin \{a_1, a_2, a_3\}$), and observe that f and $L[f]$ share a_j ($j = 1,2,3$) CM in D. Since f and $L[f]$ always share ∞ IM in D, F and $L_1[f]$ share 0 IM, and $1/(a_j - c)$ ($j = 1,2,3$) CM in D, then by Lemma 2.1, Lemma 2.2(i), the definition (1.6) of $R_{\alpha,\beta}(r,*)$, and Lemma 2.3 we get

$$\begin{aligned}
2S_{\alpha,\beta}(r, L_1[f]) &\le \sum_{j=1}^{3}\overline{C}_{\alpha,\beta}\left(r, \frac{1}{L_1[f]-1/(a_j-c)}\right) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{L_1[f]}\right) + R(r, L_1[f]) \\
&\le C_{\alpha,\beta}\left(r, \frac{1}{F-L_1[f]}\right) + R(r,f) \le S_{\alpha,\beta}(r, F-L_1[f]) + R(r,f) \\
&\le S_{\alpha,\beta}(r,F) + S_{\alpha,\beta}(r, L_1[f]) + R(r,f),
\end{aligned}$$

implying that

$$S_{\alpha,\beta}(r, L_1[f]) \le S_{\alpha,\beta}(r,F) + R(r,f).$$

Hence, by Lemma 2.1 we have

$$S_{\alpha,\beta}(r, L[f]) \le S_{\alpha,\beta}(r,f) + R(r,f). \tag{2.3}$$

In view of (2.2) and (2.3) we obtain the conclusion of the lemma. Lemma 2.6 is proved. ☐

Lemma 2.7. *Let f be a meromorphic function in $D = \{z : \alpha \le \arg z \le \beta\}$ ($0 < \beta - \alpha \le 2\pi$), and $\omega = \pi/(\beta-\alpha)$. Then for any $c \in \overline{\mathbb{C}}$ and arbitrarily small $\nu > 0$, we have*

$$n(r, D_\nu, f = c) \le Kr^{\omega}C_{\alpha,\beta}\left(2r, \frac{1}{f-c}\right),$$

where K is a positive constant, $D_\nu = \{z : \alpha+\nu \le \arg z \le \beta-\nu\}$, and $n(r, D_\nu, f = c)$ denotes the number of zeros of $f - c$ counting multiplicities in $D_\nu \cap \{z : |z| \le r\}$.

Proof. Let η_m be the zeros of $f - c$ counting multiplicities in D. Put $n(*) := n(*, D_\nu, f = c)$ for the sake of simplicity. Then for arbitrarily small $\nu > 0$ we can write

$$\begin{aligned}
C_{\alpha,\beta}\left(2r, \frac{1}{f-c}\right) &= 2 \sum_{1<|\eta_m|<2r, \alpha<\theta_m<\beta} \left(\frac{1}{|\eta_m|^\omega} - \frac{|\eta_m|^\omega}{(2r)^{2\omega}}\right) \sin\omega(\theta_m - \alpha) \\
&\ge 2 \sum_{1<|\eta_m|<r, \alpha+\nu<\theta_m<\beta-\nu} \left(\frac{1}{|\eta_m|^\omega} - \frac{|\eta_m|^\omega}{(2r)^{2\omega}}\right) \sin\omega(\theta_m - \alpha) \\
&\ge 2\sin(\omega\nu) \sum_{1<|\eta_m|<r, \alpha+\nu<\theta_m<\beta-\nu} \left(\frac{1}{|\eta_m|^\omega} - \frac{|\eta_m|^\omega}{(2r)^{2\omega}}\right) \\
&= 2\sin(\omega\nu)\left(\int_1^r \frac{dn(t)}{t^\omega} - \int_1^r \frac{t^\omega}{(2r)^{2\omega}} dn(t)\right) \\
&= 2\sin(\omega\nu)\left(\frac{n(r)}{r^\omega} + \omega\int_1^r \frac{n(t)}{t^{\omega+1}}dt - \frac{n(r)}{4^\omega r^\omega} + \frac{\omega}{(2r)^{2\omega}}\int_1^r t^{\omega-1} n(t)\, dt\right) \\
&\ge 2\sin(\omega\nu)\left(\frac{n(r)}{r^\omega} - \frac{n(r)}{4^\omega r^\omega}\right) = 2\sin(\omega\nu)\frac{n(r)}{r^\omega}\frac{4^\omega - 1}{4^\omega} \ge K\frac{n(r)}{r^\omega}.
\end{aligned}$$

Therefore

$$n(r) \le K r^\omega C_{\alpha,\beta}\left(2r, \frac{1}{f-c}\right),$$

where K is a positive constant not necessarily the same for each occurrence. This completes the proof of the lemma. Lemma 2.7 is proved. □

Lemma 2.8. *(see [13]). Let f be a meromorphic function of infinite order in $\mathbb{C}$, and let $\rho(r)$ be a proximate order of f. Then f has at least one Borel direction $\arg z = \theta_0$ $(0 \le \theta_0 < 2\pi)$ of proximate order $\rho(r)$.*

Lemma 2.9. *(see [12]). Let f be a meromorphic function of infinite order in $\mathbb{C}$, and let $\rho(r)$ be a proximate order of f. Then a direction $\arg z = \theta_0$ is a Borel direction of proximate order $\rho(r)$ of f, if and only if for arbitrarily small $\varepsilon > 0$ we have*

$$\limsup_{r\to+\infty} \frac{\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, f)}{\rho(r)\log r} = 1.$$

3. Proof of Theorem 1.1

Suppose that $f \not\equiv L[f]$. Since $\lambda(f) \ge \mu(f)$ and $\mu(f) > 1/2$, it follows that $\lambda(f) > 1/2$. Now we consider the following two cases.

Case 1. Assume that $1/2 < \lambda(f) < +\infty$. Choose ω such that $1/2 < \omega < \lambda(f)$, where $\omega = \pi/(\beta - \alpha)$ and $0 < \beta - \alpha \leq 2\pi$. Then for one given angular domain $D = \{z : \alpha \leq \arg z \leq \beta\}$, we have $\lambda(f) > \omega$. Thus, by Lemma 2.4, we can assume that f has at least one Borel direction $\arg z = \theta_0$ in D of order $\lambda(f)$. Therefore, in view of Definition 2.3 there exists a finite complex number c such that for arbitrarily small $\varepsilon > 0$,

$$\limsup_{r\to+\infty} \frac{\log n(r,\theta_0,\varepsilon, f = c)}{\log r} = \lambda(f) > \omega. \tag{3.1}$$

Next, since f and $L[f]$ share a_j $(j = 1,2,3)$ CM in D, by Lemma 2.6 and Lemma 2.2(iii), we have

$$S_{\alpha,\beta}(r,f) = R(r,f) = O(1). \tag{3.2}$$

On the other hand, for arbitrarily small $\nu > 0$, by Lemma 2.7 we get

$$n(r, D_\nu, f = c) \leq K r^\omega C_{\alpha,\beta}\left(2r, \frac{1}{f-c}\right), \tag{3.3}$$

where K is a positive constant, $D_\nu = \{z : \alpha+\nu \leq \arg z \leq \beta-\nu\}$, and $n(r, D_\nu, f = c)$ denotes the number of zeros of $f - c$ counting multiplicities in $D_\nu \cap \{z : |z| \leq r\}$. Thus, by (3.2), (3.3), and Lemma 2.1 it follows that

$$n(r,\theta_0,\varepsilon, f = c) \leq n(r, D_\nu, f = c) \leq$$
$$\leq K r^\omega C_{\alpha,\beta}\left(2r, \frac{1}{f-c}\right) \leq K r^\omega \left(S_{\alpha,\beta}(2r,f) + O(1)\right) \leq O(r^\omega),$$

and hence, we have

$$n(r,\theta_0,\varepsilon, f = c) = O(r^\omega).$$

This contradicts (3.1) and so we obtain $f \equiv L[f]$.

Case 2. Assume that $\lambda(f) = +\infty$ and $\rho(r)$ is a proximate order of f. Then in view of Lemma 8 we can assume that f has at least one Borel direction $\arg z = \theta_0$ in D of proximate order $\rho(r)$. Moreover, by Lemma 2.2(iv) and Lemma 2.6 we have

$$S_{\alpha,\beta}(r,f) = R(r,f) = O(\log U(r)), \qquad U(r) = r^{\rho(r)},$$

implying that

$$S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r,f) = O(\log U(r)), \qquad U(r) = r^{\rho(r)}. \tag{3.4}$$

Now by Lemma 2.9, for arbitrarily small $\varepsilon > 0$, we have

$$\limsup_{r\to+\infty} \frac{\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r,f)}{\rho(r)\log r} = 1. \tag{3.5}$$

Combining (3.4) and (3.5) we arrive at a contradiction. This completes the proof of Theorem 1.1.

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ON AN OPEN PROBLEM OF ZHANG AND XU

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Abstract. Taking an open problem in [25] into background we employ the idea of normal family to investigate the uniqueness problem of meromorphic functions sharing a non-zero polynomial which improves a number of existing results. Specially we rectify some errors and gaps in a recent result of P. Sahoo [15].

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1. Introduction, definitions and results

In this paper by meromorphic functions we always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let $a \in \mathbb{C}$. We say that f and g share a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM if $f-a$ and $g-a$ have the same zeros ignoring multiplicities.

We adopt the standard notation of value distribution theory (see [8]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly except a set of finite linear measure. A meromorphic function a is said to be a small function of f if $T(r,a)=S(r,f)$.

Throughout the paper, we denote by $\mu(f)$ and $\rho(f)$ the lower order and the order of f, respectively (see [8, 19]). Let f be a transcendental meromorphic function such that $\rho(f)=\rho\leq\infty$. A complex number a is said to be a Borel exceptional value (see [19]) if

$$\limsup_{r\longrightarrow\infty}\frac{\log^{+}N(r,a;f)}{\log r}<\rho.$$

A finite value z_0 is said to be a fixed point of $f(z)$ if $f(z_0) = z_0$. We will use the following definition:

$$\Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, W. K. Hayman (see [7], Corollary of Theorem 9) proved the following assertion.

Theorem A. [7] *Let f be a transcendental meromorphic function and let $n \in \mathbb{N}$ with $n \geq 3$. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, C. C. Yang and X. H. Hua [20] obtained the following uniqueness result corresponding to Theorem A.

Theorem B. [20] *Let f and g be two non-constant meromorphic functions, and let $n \in \mathbb{N}$ with $n \geq 11$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c, c_1, c_2 \in \mathbb{C}\backslash\{0\}$ and $(c_1c_2)^{n+1}c^2 = -1$ or $f \equiv tg$ for $t \in \mathbb{C}\backslash\{0\}$ such that $t^{n+1} = 1$.*

In 2002, using the idea of sharing fixed points, M. L. Fang and H. L. Qiu [5] further generalized and improved Theorem B by proving the following theorem.

Theorem C. [5] *Let f and g be two non-constant meromorphic functions, and let $n \in \mathbb{N}$ with $n \geq 11$. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $4(c_1c_2)^{n+1}c^2 = -1$ or $f = tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.*

For the last couple of years a number of astonishing results have been obtained regarding the value sharing of nonlinear differential polynomials, which are mainly the k-th derivative of some linear expression of f and g.

In 2010, J. F. Xu, F. Lü and H. X. Yi [17] studied the analogous problem corresponding to Theorem C, where in addition to the fixed point sharing problem, sharing of poles are also taken under consideration. More precisely, they proved the following theorems.

Theorem D. [17] *Let f and g be two non-constant meromorphic functions, and let $n, k \in \mathbb{N}$ such that $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, and f and g share*

∞ IM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ satisfying $4n^2(c_1c_2)^nc^2 = -1$, or $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^n = 1$.

Theorem E. [17] *Let f and g be two non-constant meromorphic functions such that $\Theta(\infty; f) > \frac{2}{n}$, and let $n, k \in \mathbb{N}$ such that $n \geq 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, and f and g share ∞ IM, then $f \equiv g$.*

Recently X. B. Zhang and J. F. Xu [25] further generalized and improved the results of [17] as follows (see [25], Theorem 1.3).

Theorem F. [25] *Let f and g be two transcendental meromorphic functions, p be a non-zero polynomial with $\deg(p) = l \leq 5$, $k, n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ such that $n > 3k + m + 7$, and let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \ldots + a_1w + a_0$ be a non-zero polynomial. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share p CM, and f and g share ∞ IM, then one of the following three cases hold:*

(1) $f(z) \equiv tg(z)$ for $t \in \mathbb{C}\setminus\{0\}$ such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,

(2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$;

(3) $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_iz^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$;
if $p(z)$ is not a constant, then $f(z) = c_1e^{cQ(z)}$, $g(z) = c_2e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$,
if $p(z)$ is a non-zero constant b, then $f(z) = c_3e^{cz}$, $g(z) = c_4e^{-cz}$, where $c, c_3, c_4 \in \mathbb{C} \setminus \{0\}$ such that $(-1)^ka_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

Zhang and Xu made the following observation in Remark 1.2 of [25]:
"From the proof of Theorem 1.3, we can see that the computation will be very complicated when $\deg(p)$ becomes large, so we are not sure whether Theorem 1.3 holds for the general polynomial p."

Also, at the end of the paper [25], the authors posed the following problem.

Open problem. What happens to Theorem 1.3 [25] if the condition "$l \leq 5$" is removed ?

Let us define $m^* = m$ if $P(z) \neq c_0$, and $m^* = 0$ if $P(z) \equiv c_0$.
Regarding the above problem, P. Sahoo [15] proved the following result.

Theorem G. [15] *Let f and g be two transcendental meromorphic functions, p be a non-constant polynomial of degree l, and let $k, n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ such that $n > \max\{3k + m^* + 6, k + 2l\}$. In addition, we suppose that either k, l are co-prime or $k > l$, when $l \geq 2$. Let $P(w)$ be as in Theorem F. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share p CM, and f and g share ∞ IM, then the following conclusions hold.*

(i) If $P(z) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$ is not a monomial, then either $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, 2, \ldots, m\}$, or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by $R(w_1, w_2) = w_1^n(a_m w_1^m + \ldots + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + \ldots + a_1 w_2 + a_0)$. In particular, when $m = 1$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$, then $f \equiv g$.

(ii) When $P(z) = c_0$ or $P(w) = a_m w^m$, then either $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+m^*} = 1$, or $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = p(z)$, $b, b_1, b_2 \in \mathbb{C} \setminus \{0\}$, and $c_0^2(nb)^2(b_1 b_2)^n = -1$ or $a_m^2((n+m)b)^2(b_1 b_2)^{n+m} = -1$.

Remark 1.1. Observing Theorem 1.1 of [15], it seems that the condition "$l \leq 5$" was removed. But unfortunately it is not the case. Actually the condition "$l \leq 5$" is replaced by the condition "$n > k + 2l$", with n depending on l. In the same paper the author claims that "Theorem 1.1 of [15] improves Theorem F by reducing the lower bound of n", but this is not true. For example, if we assume that $k = 1$, $m = 1$ and $l = 5$, then from Theorem F we get $n > 11$, while in Theorem G we have $n > 11$. On the other hand, we see that Theorem F holds for $k = l \leq 5$ but Theorem G does not hold.

Therefore, by the best knowledge of the authors, the above open problem is still open. Consequently one of the goals of this paper is to solve the above open problem without imposing any other conditions.

Remark 1.2. In the proof of Lemma 2.7 of [15], one can easily point out a gap. Indeed, from the relation

$$a_m^2(n+m)^2\alpha'\beta' e^{(n+m)(\alpha+\beta)} \equiv p^2$$

the authors conclude that α and β are polynomials. A question arises when $\alpha' = pe^{\gamma}$ and $\beta' = pe^{\delta}$. Actually the authors did not consider this case.

The above discussion is enough to make oneself inquisitive to investigate the accurate form of Theorem G. To state our main result we need the following definition, which also will be used throughout the paper.

Definition 1.1. [9, 10] *Let $k \in \mathbb{N} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f and g share the value a with weight k. We write f, g share (a, k) to mean that f and g share the value a with weight k. Also, we say that f, g share a value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞), respectively.*

Also, it is quite natural to ask the following questions.

Question 1. Can one remove the condition "Suppose that either k, l are co-prime or $k > l$, when $l \geq 2$" in Theorem G ?

Question 2. Can "CM" sharing in Theorems F and G be reduced to a finite weight sharing ?

In this paper, taking the possible answers of the above questions into background, we obtain the following result.

Theorem 1.1. *Let f, g be two transcendental meromorphic functions, and let $k, n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ be such that $n > 3k + m + 6$. Let p be a non-zero polynomial and $P(w)$ be defined as in Theorem F. If $[f^n P(f)]^{(k)} - p$, $[g^n P(g)]^{(k)} - p$ share $(0, k_1)$, where $k_1 = \left[\frac{3+k}{n+m-k-1}\right] + 3$ and f, g share $(\infty, 0)$, then one of the following three cases hold:*

(1) $f(z) \equiv tg(z)$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,

(2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$. In particular, when $m = 1$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$, then $f \equiv g$;

(3) $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$;
if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, and c_1, c, c_1, $c_2 \in \mathbb{C} \setminus \{0\}$ are such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$,

if $p(z)$ is a non-zero constant b, then $f(z) = c_3e^{cz}$, $g(z) = c_4e^{-cz}$, where c, c_3, $c_4 \in \mathbb{C} \setminus \{0\}$ are such that $(-1)^k a_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

Remark 1.3. Clearly Theorem 1.1 improves Theorems F and G. Also, in this paper we can remove the condition "$l \leq 5$" in Theorem F without imposing any other conditions and keeping all the conclusions intact.

The following definitions and notations will be used in the paper.

Definition 1.2. [11] *Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function. In an analogous manner we can define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.*

Definition 1.3. [10] *Let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then $N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq k)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.*

Definition 1.4. [2] *Let f and g be two non-constant meromorphic functions such that f and g share the value a IM for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p and also an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ $(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of f and g, where $p > q \geq 1$ $(q > p \geq 1)$. Also, we denote by $\overline{N}_E^{(1}(r, a; f)$ the reduced counting function of those a-points of f and g, where $p = q \geq 1$.*

Definition 1.5. [9, 10] *Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly, $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.*

Definition 1.6. [13] *Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.*

2. Lemmas

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M \ \forall \ z \in \mathbb{C}$, where $h^{\#}(z) = \frac{|h'(z)|}{1+|h(z)|^2}$ denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is a normal family in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [16]).

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H and V the functions defined as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \ V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right). \tag{2.1}$$

Lemma 2.1 ([18]). *Let f be a non-constant meromorphic function, and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([24]). *Let f be a non-constant meromorphic function and $k, p \in \mathbb{N}$. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.2}$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.3}$$

Lemma 2.3 ([12]). *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\overline{N}(r, 0; f \mid \geq k) + S(r, f).$$

Lemma 2.4 ([25]). *Let f and g be two non-constant meromorphic functions, $P(w)$ be defined as in Theorem F, and let $k, n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ be such that $n > 2k+m+1$. If $[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}$, then $f^n P(f) \equiv g^n P(g)$.*

Lemma 2.5 ([21], Lemma 6). *If $H \equiv 0$, then F, G share 1 CM. If further F, G share ∞ IM then F, G share ∞ CM.*

Lemma 2.6 ([25]). *Let f, g be non-constant meromorphic functions, $k, n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ be such that $n > k + 2$, and let $P(w)$ be defined as in Theorem F. Let $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to f with finitely many zeros and poles. If $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv \alpha^2$, f and g share ∞ IM, then $P(w)$ is reduced to a non-zero monomial, namely $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$.*

Lemma 2.7 ([6]). *Let $f(z)$ be a non-constant entire function and let $k \in \mathbb{N} \setminus \{1\}$. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a(\neq 0), b \in \mathbb{C}$.*

Lemma 2.8 ([8], Theorem 3.10). *Suppose that f is a non-constant meromorphic function and $k \in \mathbb{N} \setminus \{1\}$. If*

$$N(r, \infty, f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, \frac{f'}{f}),$$

then $f(z) = e^{az+b}$, where $a(\neq 0), b \in \mathbb{C}$.

Lemma 2.9 ([8], Lemma 3.5). *Suppose that F is meromorphic in a domain D, and set $f = \frac{F'}{F}$. Then for $n \in \mathbb{N}$, we have*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4}(f')^2 + P_{n-3}(f),$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

Lemma 2.10 ([4]). *Let f be a meromorphic function on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most 1.*

Lemma 2.11 ([19], Theorem 2.11). *Let f be a transcendental meromorphic function in the complex plane such that $\rho(f) > 0$. If f has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \rho(f)$ and $\rho(f)$ is a positive integer or ∞.*

Lemma 2.12 ([23]). *Let F be a family of meromorphic functions in the unit disc Δ such that all zeros of functions in F have multiplicity greater than or equal to l and all poles of functions in F have multiplicity greater than or equal to j, and let α be a real number satisfying $-l < \alpha < j$. Then F is not normal in any neighborhood of $z_0 \in \Delta$ if and only if there exist*

(i) points $z_n \in \Delta$, $z_n \to z_0$,

(ii) positive numbers ρ_n, $\rho_n \to 0^+$, and

(iii) functions $f_n \in F$,

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ *spherically locally uniformly in* $\mathbb{C}$, *where* g *is a non-constant meromorphic function. The function* g *may be taken to satisfy the normalization condition:* $g^{\#}(\zeta) \leq g^{\#}(0) = 1 (\zeta \in \mathbb{C})$.

Remark 2.1. Suppose that in Lemma 2.12, F is a family of holomorphic functions in the domain D and there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$, whenever $f = 0$. Then the real number α in Lemma 2.12 can be chosen to satisfy $0 \leq \alpha \leq k$. In that case, we also have $f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where g is a non-constant holomorphic function. The function g may be taken to satisfy the normalization condition: $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1 (\zeta \in \mathbb{C})$.

Lemma 2.13 ([19]). *Let* f_j $(j = 1, 2, 3)$ *be a meromorphic and* f_1 *be a non-constant functions. Suppose that* $\sum_{j=1}^{3} f_j = 1$ *and*

$$\sum_{j=1}^{3} N(r, 0; f_j) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_j) < (\lambda + o(1))T(r),$$

as $r \to +\infty$, $r \in I$, *where* I *is a set of* $r \in (0, \infty)$ *with infinite linear measure,* $\lambda < 1$ *and* $T(r) = \max_{1 \leq j \leq 3} T(r, f_j)$. *Then* $f_2 = 1$ *or* $f_3 = 1$.

Lemma 2.14 ([19], Theorem 1.24). *Let* f *be a non-constant meromorphic function, and let* $k \in \mathbb{N}$. *Suppose that* $f^{(k)} \not\equiv 0$, *then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.15. *Let* f, g *be two transcendental entire functions such that* f *and* g *have no zeros, and let* p *be a non-zero polynomial. Suppose that* $(f^n)'(g^n)' \equiv p^2$, *where* $n \in \mathbb{N}$. *Then the following assertions hold:*
(i) if $p(z)$ *is not a constant, then* $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(t)dt$, *and* $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *are such that* $(nc)^2(c_1c_2)^n = -1$,
(ii) if $p(z) = b \in \mathbb{C} \setminus \{0\}$, *then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* $c_3, c_4, d \in \mathbb{C} \setminus \{0\}$ *are such that* $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.

The proof follows from that of Theorem 1.3 of [25].

Lemma 2.16. *Let* f, g *be two transcendental meromorphic functions,* p *be a non-zero polynomial, and let* $k, n \in \mathbb{N}$ *be such that* $n > k$. *Suppose that* $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$,

where $(f^n)^{(k)} - p$ and $(g^n)^{(k)} - p$ share 0 CM and f, g share ∞ IM. Then the following assertions hold:
(i) if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, and $c, c_1, c_2 \in \mathbb{C}\setminus\{0\}$ are such that $(nc)^2(c_1c_2)^n = -1$,
(ii) if $p(z) = b \in \mathbb{C}\setminus\{0\}$, then $f(z) = c_3e^{dz}$, $g(z) = c_4e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}\setminus\{0\}$ are such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.

Proof. Suppose

$$(f^n)^{(k)}(g^n)^{(k)} \equiv p^2. \tag{2.4}$$

Since f and g share ∞ IM, from (2.4) one can easily infer that f and g are transcendental entire functions. Let $F_1 = \frac{(f^n)^{(k)}}{p}$ and $G_1 = \frac{(g^n)^{(k)}}{p}$. From (2.4) we get

$$F_1G_1 \equiv 1. \tag{2.5}$$

If $F_1 \equiv c_1^*G_1$, where $c_1^* \in \mathbb{C}\setminus\{0\}$, then by (2.5), F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv c_1^*G_1$.
Let

$$\Phi = \frac{(f^n)^{(k)} - p}{(g^n)^{(k)} - p}. \tag{2.6}$$

Then from (2.6) we have

$$\Phi = e^{\gamma_1}, \tag{2.7}$$

where γ_1 is an entire function. Let $f_1 = F_1$, $f_2 = \;\; e^{\gamma_1}G_1$ and $f_3 = e^{\gamma_1}$. Here f_1 is transcendental. Now from (2.7), we have $f_1 + f_2 + f_3 = 1$. Hence by Lemma 2.14 we get

$$\begin{aligned}\sum_{j=1}^{3} N(r,0;f_j) + 2\sum_{j=1}^{3}\overline{N}(r,\infty;f_j) &\le N(r,0;F_1) + N(r,0;e^{\gamma_1}G_1) + O(\log r)\\ &\le (\lambda + o(1))T(r),\end{aligned}$$

as $r \to +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1\le j\le 3} T(r, f_j)$.

So, by Lemma 2.13, we infer that either $e^{\gamma_1}G_1 = -1$ or $e^{\gamma_1} = 1$. But here the only possibility is that $e^{\gamma_1}G_1 = -1$, that is, $(g^n)^{(k)} = -e^{-\gamma_1}p$, and so from (2.4) we get

$$(f^n)^{(k)} = c_2^* e^{\gamma_1} p \text{ and } (g^n)^{(k)} = c_2^* e^{-\gamma_1} p, \tag{2.8}$$

where $c_2^* = \pm 1$. This shows that $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM. Let z_p be a zero of $f(z)$ of multiplicity p and z_q be a zero of $g(z)$ of multiplicity q. Since $n > k$, it follows that z_p will be a zero of $(f^n(z))^{(k)}$ of multiplicity $np - k$ and z_q will be a zero of $(g^n(z))^{(k)}$ of multiplicity $nq-k$. Since $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 0 CM, it follows that $z_p = z_q$ and $p = q$. Consequently f and g share 0 CM. Since $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$, we can take

$$f(z) = h_1(z)e^{\alpha(z)} \text{ and } g(z) = h_1(z)e^{\beta(z)}, \tag{2.9}$$

where $h_1(z)$ is a non-zero polynomial and α, β are two non-constant entire functions. We consider the following cases.

Case 1. Suppose 0 is a Picard exceptional value of both f and g. We consider the following sub-cases.

Sub-case 1.1. Let $\deg(p) = l \in \mathbf{N}$.

Since $N(r,0;f) = 0$ and $N(r,0;g) = 0$, we can take

$$f(z) = e^{\alpha(z)} \text{ and } g(z) = e^{\beta(z)}, \tag{2.10}$$

where α and β are two non-constant entire functions.

We deduce from (2.4) and (2.10) that either both α and β are transcendental entire functions, or both are polynomials. We consider the following sub-cases.

Sub-case 1.1.1. Let $k \in \mathbb{N}\setminus\{1\}$. We first suppose that both α and β are transcendental entire functions. Note that

$$S(r, n\alpha') = S(r, \frac{(f^n)'}{f^n}) \text{ and } S(r, n\beta') = S(r, \frac{(g^n)'}{g^n}).$$

Moreover we see that

$$N(r,0;(f^n)^{(k)}) \leq N(r,0;p^2) = O(\log r), \quad N(r,0;(g^n)^{(k)}) \leq N(r,0;p^2) = O(\log r).$$

From these and using (2.10) we have

$$N(r,\infty;f^n) + N(r,0;f^n) + N(r,0;(f^n)^{(k)}) = S(r,n\alpha') = S(r, \frac{(f^n)'}{f^n})$$

$$N(r,\infty;g^n) + N(r,0;g^n) + N(r,0;(g^n)^{(k)}) = S(r,n\beta') = S(r, \frac{(g^n)'}{g^n}). \tag{2.11}$$

Then from (2.11) and Lemma 2.8 we must have $f(z) = e^{a_3^* z + b_3^*}$ and $g(z) = e^{c_3^* z + d_3^*}$, where $a_3^*(\neq 0), b_3^*, c_3^*(\neq 0), d_3^* \in \mathbb{C}$. But these types of f and g do not agree with the relation (2.4).

Next, we suppose that α and β both are non-constant polynomials, since otherwise f and g reduce to polynomials contradicting that they are transcendental. Also, from (2.4) we get $\alpha + \beta = C_1 \in \mathbb{C}$, that is, $\alpha' = -\beta'$. Therefore $\deg(\alpha) = \deg(\beta)$. If

$\deg(\alpha) = \deg(\beta) = 1$, then we again get a contradiction from (2.4). Next, we suppose that $\deg(\alpha) = \deg(\beta) \geq 2$. Now from (2.10) and Lemma 2.9 we see that

$$(f^n)^{(k)} = \Big(n^k(\alpha')^k + \frac{k(k-1)}{2}n^{k-1}(\alpha')^{k-2}\alpha'' + P_{k-2}(\alpha')\Big)e^{n\alpha}.$$

Similarly we have

$$\begin{aligned}(g^n)^{(k)} &= \Big(n^k(\beta')^k + \frac{k(k-1)}{2}n^{k-1}(\beta')^{k-2}\beta'' + P_{k-2}(\beta')\Big)e^{n\beta}\\ &= \Big((-1)^k n^k(\alpha')^k - \frac{k(k-1)}{2}n^{k-1}(-1)^{k-2}(\alpha')^{k-2}\alpha'' + P_{k-2}(-\alpha')\Big)e^{n\beta}.\end{aligned}$$

Since $\deg(\alpha) \geq 2$, we observe that $\deg((\alpha')^k) \geq k\ \deg(\alpha')$, and so $(\alpha')^{k-2}\alpha''$ is either a non-zero constant or $\deg((\alpha')^{k-2}\alpha'') \geq (k-1)\ \deg(\alpha') - 1$. Also, we see that

$$\deg\left((\alpha')^k\right) > \deg\left((\alpha')^{k-2}\alpha''\right) > \deg\left(P_{k-2}(\alpha')\right) \left(\text{or } \deg\left(P_{k-2}(-\alpha')\right)\right).$$

Let

$$(\alpha(z))' = e_t z^t + e_{t-1}z^{t-1} + \ldots + e_0,$$

where $e_0, e_1, \ldots, e_t (\neq 0) \in \mathbb{C}$. Then we have

$$((\alpha)')^i = e_t^i z^{it} + i e_t^{i-1} e_{t-1} z^{it-1} + \ldots\ldots,$$

where $i \in \mathbb{N}$. Therefore we have

$$(f^n)^{(k)} = \Big(n^k e_t^k z^{kt} + kn^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots + (D_1 + D_2) z^{kt-t-1} + \ldots\Big)e^{n\alpha}$$

and

$$\begin{aligned}(g^n)^{(k)} &= \Big((-1)^k n^k e_t^k z^{kt} + (-1)^k kn^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots\\ &\quad + ((-1)^k D_1 + (-1)^{k-1} D_2) z^{kt-t-1} + \ldots\Big)e^{n\beta},\end{aligned}$$

where $D_1, D_2 \in \mathbb{C}$ are such that $D_2 = \frac{k(k-1)}{2} tn^{k-1} e_t^{k-1}$. Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM, we have

$$\begin{aligned}&n^k e_t^k z^{kt} + kn^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots + (D_1 + D_2) z^{kt-t-1} + \ldots\\ =\ & d_1^*\Big((-1)^k n^k e_t^k z^{kt} + (-1)^k kn^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots\\ &+ ((-1)^k D_1 + (-1)^{k-1} D_2) z^{kt-t-1} + \ldots\Big)\end{aligned}$$

where $d_1^* \in \mathbb{C} \setminus \{0\}$. From (2.12) we get $D_2 = 0$, that is, $\frac{k(k-1)}{2} tn^{k-1} e_t^{k-1} = 0$, which is impossible for $k \geq 2$.

Sub-case 1.1.2. Let $k = 1$. The result follows from Lemma 2.15.

Sub-case 1.2. Let $p(z) = b \in \mathbb{C} \setminus \{0\}$. Since $n > k$, we have $f \neq 0$ and $g \neq 0$.

Now using Sub-case 1.1 we can show that $f = e^{\alpha}$ and $g = e^{\beta}$, where α and β are non-constant entire functions. We now consider the following two sub-cases.

Sub-case 1.2.1. Let $k \geq 2$. We see that $N(r, 0; (f^n)^{(k)}) = 0$. It is clear that

$$f^n(z)(f^n(z))^{(k)} \neq 0 \text{ and } g^n(z)(g^n(z))^{(k)} \neq 0. \tag{2.12}$$

Then from (2.12) and Lemma 2.7 we must have $f(z) = e^{a_4^* z + b_4^*}$, $g(z) = e^{c_4^* z + d_4^*}$, where $a_4^*(\neq 0), b_4^*, c_4^*(\neq 0), d_4^* \in \mathbb{C}$. In view of (2.4) it is clear that $a_4^* + c_4^* = 0$. Finally, by (2.4) we take $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where c_3, c_4 and $d \in \mathbb{C} \setminus \{0\}$ are such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$.

Sub-case 1.2.2. Let $k = 1$. The result follows from Lemma 2.15.

Case 2. Suppose 0 is not a Picard exceptional value of f and g.

Let $H = f^n$, $\hat{H} = g^n$, $F = \frac{H}{p}$ and $G = \frac{\hat{H}}{p}$, and let $\mathcal{F} = \{F_\omega\}$ and $\mathcal{G} = \{G_\omega\}$, where $F_\omega(z) = F(z+\omega) = \frac{H(z+\omega)}{p(z+\omega)}$ and $G_\omega(z) = G(z+\omega) = \frac{\hat{H}(z+\omega)}{p(z+\omega)}$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are two families of meromorphic functions defined on $\mathbb{C}$. We now consider following two sub-cases.

Sub-case 2.1. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $F^{\#}(\omega) = F_\omega^{\#}(0) \leq M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 2.10 we have that $F(= \frac{f^n}{p})$ is of order at most 1. Now from (2.4) we have

$$\rho(f) = \rho(\frac{f^n}{p}) = \rho(f^n) = \rho((f^n)^{(k)}) = \rho((g^n)^{(k)}) = \rho(g^n) = \rho(\frac{g^n}{p}) = \rho(g) \leq 1. \tag{2.13}$$

Since f and g are transcendental entire functions, from (2.9) we have $\rho(f) > 0$ and $\rho(g) > 0$. We observe from (2.13) and Lemma 2.11 that $\mu(f) = \rho(f) = 1$ and $\mu(g) = \rho(g) = 1$. Now from (2.9) we get

$$f = h_1 e^{\alpha}, \quad g = h_1 e^{\beta}, \tag{2.14}$$

where α and β are non-constant polynomials of degree 1. From (2.4) we see that $\alpha + \beta = C_2 \in \mathbb{C}$, and so $\alpha' + \beta' = 0$. Again, from (2.14) we have

$$(f^n)^{(k)} = e^{n\alpha} \sum_{i=0}^{k} {}^kC_i (n\alpha')^{k-i} (h_1^n)^{(i)},$$

where we define $(h_1^n)^{(0)} = h_1^n$. Similarly we have

$$(g^n)^{(k)} = e^{n\beta} \sum_{i=0}^{k} {}^kC_i (-1)^{k-i} (n\alpha')^{k-i} (h_1^n)^{(i)}.$$

Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM, it follows that

$$\sum_{i=0}^{k} {}^kC_i(n\alpha')^{k-i}(h_1^n)^{(i)} \equiv d_2^* \sum_{i=0}^{k} {}^kC_i(-1)^{k-i}(n\alpha')^{k-i}(h_1^n)^{(i)}, \tag{2.15}$$

where $d_2^* \in \mathbb{C} \setminus \{0\}$. But from (2.15) we arrive at a contradiction.

Sub-case 2.2. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$ is not normal on $\mathbb{C}$. Then there exists at least one $z_0 \in \Delta$ such that $\mathcal{F}$ is not normal z_0, we assume that $z_0 = 0$. Now by Marty's theorem there exists a sequence of meromorphic functions $\{F(z+\omega_j)\} \subset \mathcal{F}$, where $z \in \{z : |z| < 1\}$ and $\{\omega_j\} \subset C$ is some sequence of complex numbers, such that $F^{\#}(\omega_j) \to \infty$ as $|\omega_j| \to \infty$.

Note that p has only finitely many zeros. So there exists a number $r > 0$ such that $p(z) \neq 0$ in $D = \{z : |z| \geq r\}$. Since p is a polynomial, for all $z \in \mathbb{C}$ satisfying $|z| \geq r$, we have

$$0 \leftarrow \left|\frac{p'(z)}{p(z)}\right| \leq \frac{M_1}{|z|} < 1, \quad p(z) \neq 0. \tag{2.16}$$

Also, since $w_j \to \infty$ as $j \to \infty$, without loss of generality we may assume that $|w_j| \geq r + 1$ for all j. Let $D_1 = \{z : |z| < 1\}$ and

$$F(w_j + z) = \frac{H(w_j + z)}{p(w_j + z)}.$$

Since $|w_j + z| \geq |w_j| - |z|$, it follows that $w_j + z \in D$ for all $z \in D_1$. Also, since $p(z) \neq 0$ in D, it follows that $p(\omega_j + z) \neq 0$ in D_1 for all j. Observing that $F(z)$ is analytic in D, we conclude that $F(\omega_j + z)$ is analytic in D_1. Therefore, all $F(\omega_j + z)$ are analytic in D_1. Also, from (2.8) we see that every zero of h_1 must be a zero of p. Thus, we have structured a family $\{F(\omega_j + z)\}$ of holomorphic functions such that $F(\omega_j + z) \neq 0$ in D_1 for all j.

Then by Lemma 2.12 there exist:

(i) points z_j, $|z_j| < 1$,

(ii) positive numbers ρ_j, $\rho_j \to 0^+$,

(iii) a subsequence $\{F(\omega_j+z_j+\rho_j\zeta)\}$ of $\{F(\omega_j+z)\}$, such that $h_j(\zeta) = \rho_j^{-k}F(\omega_j+z_j + \rho_j\zeta) \to h(\zeta)$, that is,

$$h_j(\zeta) = \rho_j^{-k}\frac{H(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h(\zeta) \tag{2.17}$$

spherically locally uniformly in $\mathbb{C}$, where $h(\zeta)$ is some non-constant holomorphic function such that $h^{\#}(\zeta) \leq h^{\#}(0) = 1$.

Now from Lemma 2.10 we see that $\rho(h) \leq 1$. In view of the proof of Zalcman's lemma (see [14, 22]), we see that $\rho_j = \frac{1}{F^{\#}(b_j)}$ and $F^{\#}(b_j) \geq F^{\#}(\omega_j)$, where $b_j = \omega_j + z_j$. By Hurwitz's theorem we see that $h(\zeta) \neq 0$. Note that

$$\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to 0 \quad \text{as} \quad j \to \infty. \tag{2.18}$$

Now we prove that

$$(h_j(\zeta))^{(k)} = \frac{H^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(k)}(\zeta). \tag{2.19}$$

To this end, note first that by (2.17) we have

$$\begin{aligned} \rho_j^{-k+1}\frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} &= h_j'(\zeta) + \rho_j^{-k+1}\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)}H(\omega_j + z_j + \rho_j\zeta) \\ &= h_j'(\zeta) + \rho_j\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}h_j(\zeta). \end{aligned} \tag{2.20}$$

Now from (2.22), (2.18) and (2.20) we observe that

$$\rho_j^{-k+1}\frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h'(\zeta).$$

Suppose

$$\rho_j^{-k+l}\frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(l)}(\zeta) \quad \text{and let} \quad G_j(\zeta) = \rho_j^{-k+l}\frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}.$$

Then $G_j(\zeta) \to h^{(l)}(\zeta)$. Note that

$$\begin{aligned} &\rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \\ = \;& G_j'(\zeta) + \rho_j^{-k+l+1}\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)}H^{(l)}(\omega_j + z_j + \rho_j\zeta) \\ = \;& G_j'(\zeta) + \rho_j\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}G_j(\zeta). \end{aligned} \tag{2.21}$$

So, from (2.18) and (2.21), we see that

$$\rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j + z_n + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h_j^{(l+1)}(\zeta).$$

Then by mathematical induction we get desired result (2.19). Let

$$(\hat{h}_j(\zeta))^{(k)} = \frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}. \tag{2.22}$$

From (2.4) we have

$$\frac{H^{(k)}(\omega_j + z_j + p_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}\frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} = 1,$$

and so, from (2.19) and (2.22), we get

$$(h_j(\zeta))^{(k)}(\hat{h}_j(\zeta))^{(k)} = 1. \tag{2.23}$$

Now from (2.19), (2.23) and the formula of higher derivatives we can deduce that $\hat{h}_j(\zeta) \to \hat{h}(\zeta)$, that is,

$$\frac{\hat{H}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to \hat{h}(\zeta), \tag{2.24}$$

spherically locally uniformly in $\mathbb{C}$, where $\hat{h}(\zeta)$ is some non-constant holomorphic function in the complex plane. By Hurwitz's theorem we see that $\hat{h}(\zeta) \neq 0$. Therefore, by (2.24) we have

$$(\hat{h}_j(\zeta))^{(k)} \to (\hat{h}(\zeta))^{(k)} \tag{2.25}$$

spherically locally uniformly in $\mathbb{C}$. From (2.19), (2.23) and (2.25) we get

$$(h(\zeta))^{(k)}(\hat{h}(\zeta))^{(k)} \equiv 1. \tag{2.26}$$

Since $\rho(h) \leq 1$, from (2.26) we see that

$$\rho(h) = \rho(h^{(k)}) = \rho(\hat{h}^{(k)}) = \rho(\hat{h}) \leq 1. \tag{2.27}$$

Since h and $\hat{h}$ are non-constant entire functions such that $h \neq 0$ and $\hat{h} \neq 0$, we can take $h = e^{\alpha_1}$ and $\hat{h} = e^{\beta_1}$, where α_1 and β_1 are non-constants entire functions. Consequently, $\rho(h) > 0$ and $\rho(\hat{h}) > 0$. Now we observe from (2.27) and Lemma 2.11 that $\mu(h) = \rho(h) = 1$ and $\mu(\hat{h}) = \rho(\hat{h}) = 1$. Therefore, we have

$$h(z) = \hat{c}_1 e^{\hat{c}z}, \quad \hat{h}(z) = \hat{c}_2 e^{-\hat{c}z}, \tag{2.28}$$

where $\hat{c}, \hat{c}_1, \hat{c}_2 \in \mathbb{C}\setminus\{0\}$ are such that $(-1)^k(\hat{c}_1\hat{c}_2)(\hat{c})^{2k} = 1$. Also, from (2.28) we have

$$\frac{h_j'(\zeta)}{h_j(\zeta)} = \rho_j \frac{F'(\omega_j + z_j + \rho_j\zeta)}{F(\omega_j + z_j + \rho_j\zeta)} \to \frac{h'(\zeta)}{h(\zeta)} = \hat{c}, \tag{2.29}$$

spherically locally uniformly in $\mathbb{C}$. From (2.28) and (2.29) we get

$$\rho_j\left|\frac{F'(\omega_j + z_j)}{F(\omega_j + z_j)}\right| = \frac{1 + |F(\omega_j + z_j)|^2}{|F'(\omega_j + z_j)|}\frac{|F'(\omega_j + z_j)|}{|F(\omega_j + z_j)|} = \frac{1 + |F(\omega_j + z_j)|^2}{|F(\omega_j + z_j)|} \to \left|\frac{h'(0)}{h(0)}\right| = |\hat{c}|,$$

which implies that

$$\lim_{j\to\infty} F(\omega_j + z_j) \neq 0, \infty. \tag{2.30}$$

From (2.29) and (2.30) we see that

$$h_j(0) = \rho_j^{-k} F(\omega_j + z_j) \to \infty. \tag{2.31}$$

Again, from (2.29) and (2.28) we have

$$h_j(0) \to h(0) = \hat{c}_1. \tag{2.32}$$

Now from (2.31) and (2.32) we arrive at a contradiction. Lemma 2.16 is proved. □

Lemma 2.17. *Let f, g be two transcendental meromorphic functions, and let $P(w)$ be defined as in Theorem F. Let $F = \frac{[f^nP(f)]^{(k)}}{p}$, $G = \frac{[g^nP(g)]^{(k)}}{p}$, where p is a non-zero polynomial, and $k, n \in \mathbb{N}$ and $m \in \mathbb{N}\setminus\{0\}$ are such that $n > 3k+m+3$. If f, g share ∞ IM and $H \equiv 0$, then either $[f^nP(f)]^{(k)}[g^nP(f)]^{(k)} \equiv p^2$, where $[f^nP(f)]^{(k)} - p$ and $[g^nP(g)]^{(k)} - p$ share 0 CM, or $f^nP(f) \equiv g^nP(g)$.*

Proof. Since $H \equiv 0$, by Lemma 2.5 we conclude that F and G share 1 CM. By integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{2.33}$$

where $a(\neq 0), b \in \mathbb{C}$. Now we consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$.

If $b = -1$, then from (2.33) we obtain

$$F \equiv \frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

So, in view of Lemmas 2.1 and 2.2 with $p = 1$ and the second fundamental theorem, we can write

$$\begin{aligned}
&(n+m)\,T(r,g) \le T(r,G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G)\\
\le\;& \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G) + S(r,g)\\
\le\;& \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^nP(g)) + \overline{N}(r,\infty;f) + S(r,g)\\
\le\;& \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^n) + N_{k+1}(r,0;P(g)) + S(r,g)\\
\le\;& 2\,\overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + S(r,g)\\
\le\;& (k+3+m)\,T(r,g) + S(r,g),
\end{aligned}$$

which is a contradiction since $n > k+3$. If $b \neq -1$, then from (2.33) we obtain

$$F - (1+\frac{1}{b}) \equiv \frac{-a}{b^2[G+\frac{a-b}{b}]},$$

and hence

$$\overline{N}(r,\frac{b-a}{b};G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f).$$

Using Lemmas 2.1 and 2.2 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b\neq 0$ and $a=b$. If $b=-1$, then from (2.33) we have $FG\equiv 1$, that is, $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv p^2$, where $[f^nP(f)]^{(k)}-p$ and $[g^nP(g)]^{(k)}-p$ share 0 CM. If $b\neq -1$, then from (2.33) we obtain

$$\frac{1}{F}\equiv\frac{bG}{(1+b)G-1}.$$

Therefore

$$\overline{N}(r,\frac{1}{1+b};G)=\overline{N}(r,0;F).$$

So, in view of Lemmas 2.1 and 2.2 with $p=1$ and the second fundamental theorem, we can write

$$\begin{aligned}
&(n+m)T(r,g)\\
\leq\ & \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{1}{1+b};G)+N_{k+1}(r,0;g^nP(g))-\overline{N}(r,0;G)+S(r,g)\\
\leq\ & \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+\overline{N}(r,0;F)+S(r,g)\\
\leq\ & \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+(k+1)\overline{N}(r,0;f)+T(r,P(f))\\
&+k\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
\leq\ & (k+2+m)\,T(r,g)+(2k+1+m)\,T(r,f)+S(r,f)+S(r,g).
\end{aligned}$$

Without loss of generality, we can assume that there exists a set I with infinite measure such that $T(r,f)\leq T(r,g)$ for $r\in I$. So, for $r\in I$, we have

$$(n-3k-3-m)\,T(r,g)\leq S(r,g),$$

which is a contradiction since $n>3k+3+m$.

Case 3. Let $b=0$. From (2.33) we obtain

$$F\equiv\frac{G+a-1}{a}. \tag{2.34}$$

If $a\neq 1$, then from (2.34) we obtain $\overline{N}(r,1-a;G)=\overline{N}(r,0;F)$. Similarly we can get a contradiction as in Case 2. Therefore $a=1$ and from (2.34) we obtain $F\equiv G$, that is, $[f^nP(f)]^{(k)}\equiv[g^nP(g)]^{(k)}$. Then by Lemma 2.4 we have $f^nP(f)\equiv g^nP(g)$. This completes the proof. Lemma 2.17 is proved. □

Lemma 2.18. *Let f and g be two transcendental meromorphic functions, $n, k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ such that $n > k+2$, and let p be a non-zero polynomial. Suppose that $[f^nP(f)]^{(k)} - p$, $[g^nP(g)]^{(k)} - p$ share 0 CM, and f, g share ∞ IM, where $P(w)$ is defined as in Theorem F. If $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv p^2$, then $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_iz^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$; if $p(z)$ is not a constant, then $f(z) = c_1e^{cQ(z)}$, $g(z) = c_2e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, and $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ are such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$, if $p(z)$ is a non-zero constant b, then $f(z) = c_3e^{cz}$, $g(z) = c_4e^{-cz}$, where $c, c_3, c_4 \in \mathbb{C} \setminus \{0\}$ are such that $(-1)^ka_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.*

The proof follows from Lemmas 2.6 and 2.16.

Lemma 2.19 ([1]). *Let f and g be two non-constant meromorphic functions sharing $(1, k_1)$, where $2 \leq k_1 \leq \infty$. Then*

$$\overline{N}(r,1;f|=2) + 2\,\overline{N}(r,1;f|=3) + \ldots + (k_1-1)\,\overline{N}(r,1;f|=k_1) + k_1\,\overline{N}_L(r,1;f)$$
$$+(k_1+1)\,\overline{N}_L(r,1;g) + k_1\,\overline{N}_E^{(k_1+1}(r,1;g) \leq N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 2.20. *Let f and g be two transcendental meromorphic functions, p be a non-zero polynomial, and let $F = [f^nP(f)]^{(k)}/p$, $G = [g^nP(g)]^{(k)}/p$, where $n, k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $P(w)$ is defined as in Theorem F. Suppose $H \not\equiv 0$. If f, g share $(\infty, 0)$ and F, G share $(1, k_1)$, where $0 \leq k_1 \leq \infty$, then $(n+m-k-1)\overline{N}(r,\infty;f) \leq (k+m+1)\big(T(r,f) + T(r,g)\big) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)$.*

Proof. Suppose that ∞ is an e.v.P of f and g, then the result follows immediately. Next, suppose that ∞ is not an e.v.P of f and g. Since $H \not\equiv 0$, we have $F \not\equiv G$. We claim that $V \not\equiv 0$. Suppose the opposite $V \equiv 0$. Then by integration we obtain
$1 - \frac{1}{F} = A(1 - \frac{1}{G})$,
where A is a constant such that $A \neq 0, 1$. Note that if z_0 ($p(z_0) \neq 0$) is a pole of f, then it is a pole of g as well. Hence, from the definition of F and G we have $\frac{1}{F(z_0)} = 0$ and $\frac{1}{G(z_0)} = 0$. So $A = 1$, which is a contradiction.

Next, suppose that z_0 is a pole of f with multiplicity q and a pole of g with multiplicity r such that $p(z_0) \neq 0$. Clearly z_0 is a pole of F with multiplicity $(n+m)q+k$ and a pole of G with multiplicity $(n+m)r+k$. Noting that f, g share $(\infty, 0)$ from the definition of V it follows that z_0 is a zero of V with multiplicity at least $n+m+k-1$. Now using the Milloux theorem (see [8], p. 55), and Lemma 2.1, we obtain from the definition of V that $m(r,V) = S(r,f) + S(r,g)$. Thus, using Lemma 2.1 and (2.3), we can write

$$
\begin{aligned}
& (n+m+k-1)\overline{N}(r,\infty;f) \le N(r,0;V) + O(\log r) \le T(r,V) + O(\log r) \\
\le\ & N(r,\infty;V) + m(r,V) + O(\log r) \\
\le\ & \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
\le\ & N_{k+1}(r,0;f^nP(f)) + N_{k+1}(r,0;g^nP(g)) + k\overline{N}(r,\infty;f) \\
& +k\overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
\le\ & N_{k+1}(r,0;f^n) + N_{k+1}(r,0;P(f)) + N_{k+1}(r,0;g^n) \\
& +N_{k+1}(r,0;P(g)) + 2k\overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
\le\ & (k+1)\overline{N}(r,0;f) + N(r,0;P(f)) + (k+1)\overline{N}(r,0;g) \\
& +N(r,0;P(g)) + 2k\overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g),
\end{aligned}
$$

implying that

$$
\begin{aligned}
(n+m-k-1)\overline{N}(r,\infty;f) \le\ & (k+m+1)(T(r,f) + T(r,g)) + \overline{N}_*(r,1;F,G) \\
& +S(r,f) + S(r,g).
\end{aligned}
$$

Lemma 2.20 is proved. □

3. Proof of the theorem

Let $F = \frac{[f^nP(f)]^{(k)}}{p}$ and $G = \frac{[g^nP(g)]^{(k)}}{p}$. Note that since f and g are transcendental meromorphic functions, p is a small function with respect to both $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$. Also, F, G share $(1,k_1)$ except the zeros of p, and f, g share $(\infty,0)$. Now we consider two cases.

Case 1. Let $H \not\equiv 0$.

From (2.1) it can easily be deduced that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$
\begin{aligned}
& N(r,\infty;H) \le \overline{N}_*(r,\infty;F,G) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2) \\
(3.1)\qquad & +\overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g).
\end{aligned}
$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is defined similarly.

Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So, we have

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \tag{3.2}$$

Using (3.2) and (3.3) we get

$$\begin{aligned}\overline{N}(r,1;F) &\leq N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2)\\ &\leq \overline{N}_*(r,\infty;f,g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\ &\quad+\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\ &\quad+\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).\end{aligned} \tag{3.3}$$

Now in view of Lemmas 2.3 and 2.19 we get

$$\begin{aligned}&\overline{N}_0(r,0;G')+\overline{N}(r,1;F|\geq 2)+\overline{N}_*(r,1;F,G)\\ \leq\ &\overline{N}_0(r,0;G')+\overline{N}(r,1;F|=2)+\overline{N}(r,1;F|=3)+\ldots+\overline{N}(r,1;F|=k_1)\\ &+\overline{N}_E^{(k_1+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G)\\ \leq\ &\overline{N}_0(r,0;G')-\overline{N}(r,1;F|=3)-\ldots-(k_1-2)\overline{N}(r,1;F|=k_1)\\ &-(k_1-1)\overline{N}_L(r,1;F)-k_1\overline{N}_L(r,1;G)-(k_1-1)\overline{N}_E^{(k_1+1}(r,1;F)\\ &+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G)\\ \leq\ &\overline{N}_0(r,0;G')+N(r,1;G)-\overline{N}(r,1;G)-(k_1-2)\overline{N}_L(r,1;F)\\ &-(k_1-1)\overline{N}_L(r,1;G)\\ \leq\ &N(r,0;G'\mid G\neq 0)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\\ \leq\ &\overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G).\end{aligned} \tag{3.4}$$

Hence, using (3.3), (3.4), Lemmas 2.2 and 2.20, and the second fundamental theorem, we can write

$$(n+m)T(r,f)\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)$$

$$\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)-N_0(r,0;F')+S(r,f)$$

$$\begin{aligned}
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))+\overline{N}(r,0;F\mid\geq 2)\\
&\quad+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G')\\
&\quad-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leq 3\,\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)-(k_1-2)\,\overline{N}_*(r,1;F,G)\\
&\quad-\overline{N}_L(r,1;G)+S(r,f)+S(r,g)\\
&\leq 3\,\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))\\
&\quad-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leq (3+k)\,\overline{N}(r,\infty;f)+(k+2)\,\overline{N}(r,0;f)+T(r,P(f))+(k+2)\,\overline{N}(r,0;g)\\
&\quad+T(r,P(g))-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leq (k+m+2)\,(T(r,f)+T(r,g))+(3+k)\overline{N}(r,\infty;f)\\
&\quad-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
&\leq (k+m+2)\,\big(T(r,f)+T(r,g)\big)+\frac{(3+k)(k+m+1)}{n+m-k-1}\,(T(r,f)+T(r,g))\\
&\quad+\frac{3+k}{n+m-k-1}\,\overline{N}_*(r,1;F,G)-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leq \left[k+m+2+\frac{(3+k)(k+m+1)}{n+m-k-1}\right]\,(T(r,f)+T(r,g))+S(r,f)+S(r,g).
\end{aligned}$$

In a similar way we can obtain

$$\begin{aligned}
(3.5)\qquad &(n+m)T(r,g)\\
\leq\ &\left[k+m+2+\frac{(3+k)(k+m+1)}{n+m-k-1}\right]\,(T(r,f)+T(r,g))+S(r,f)+S(r,g).
\end{aligned}$$

Adding (3.4) and (3.5) we get

$$\left[n-m-2k-4-\frac{(6+2k)(k+m+1)}{n+m-k-1}\right]\,\Big(T(r,f)+T(r,g)\Big)\leq S(r,f)+S(r,g).$$

Since the quantity in the third bracket can be written as

$$(3.6)\left[\frac{(n+m-k-1)^2-(2m+k+3)(n+m-k-1)-2(k+3)(k+m+1)}{n+m-k-1}\right],$$

by a simple computation one can easily verify that when

$$\begin{aligned}
n+m-k-1 &> 2m+2k+5\\
&> \frac{2m+k+3+\sqrt{(2m+k+3)^2+8(k+3)(k+m+1)}}{2},
\end{aligned}$$

that is, when $n > 3k + m + 6$, we obtain a contradiction from (3.6).

Case 2. Let $H \equiv 0$. Then by Lemma 2.17 we have either

$$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv p^2, \tag{3.7}$$

or

$$f^n P(f) \equiv g^n P(g). \tag{3.8}$$

From (3.8) we get

$$f^n(a_m f^m + a_{m-1} f^{m-1} + \ldots + a_0) = g^n(a_m g^m + a_{m-1} g^{m-1} + \ldots + a_0). \tag{3.9}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ into (3.9) we deduce that

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \ldots + a_0 g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = GCD(n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus, $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where $d = GCD(n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is non-constant, then by (3.9) f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$. In particular, when $P(w) = a_1 w + a_2$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$, then by Lemma 2.12 of [3], we have $f \equiv g$. Note that when $P(w) \equiv a_0$, then we must have $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^n = 1$. The remaining part of the proof follows from (3.7) and Lemma 2.18. □

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О СТРУКТУРЕ ФУНКЦИЙ, УНИВЕРСАЛЬНЫХ ДЛЯ ВЕСОВЫХ ПРОСТРАНСТВ $L^p_\mu[0,1]$, $p>1$

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Аннотация. В работе рассматриваются вопросы связанные со структурой универсальных функций для весовых пространств $L^p_\mu[0,1]$, $p>1$. Доказано существование измеримого множества $E \subset [0,1]$ со сколь угодно близкой к единице мерой и весовой функции $0 < \mu(x) \le 1$, равняющейся единице на E, таких, что надлежащим продолжением любой функции $f \in L^1(E)$ на $[0,1] \setminus E$ можно получить функцию $\tilde{f} \in L^1[0,1]$, универсальную для каждого класса $L^p_\mu[0,1]$, $p>1$ относительно подрядов – знаков ряда Фурье – Уолша.

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1. Введение

Вопросам существования функций или рядов, универсальных тем или иным смыслом в различных функциональных классах посвящено много работ. В частности, Дж. Биркхоф в 1929 г. доказал существование целой функции, которая универсальна относительно сдвигов [1]. В 1952г. Дж. Маклейн доказал аналогичный результат для другого типа универсальности, а именно, он показал, что существует целая функция универсальная относительно производных [2]. Далее, в 1975г. С. Воронин доказал теорему универсальности дзета – функции Римана [3], а в 1987г. К. Гроссе – Эрдман показал существование функции с универсальным рядом Тейлора [4]: существует функция $g(x) \in C^\infty(\mathcal{R})$ с $g(0) = 0$, ряд Тейлора которой в точке $x = 0$ локально — равномерно универсален в $C(\mathcal{R})$, т.е. для любой функции $f(x) \in C(\mathcal{R})$ с $f(0) = 0$ и числа $r > 0$, существует подпоследовательность

$$S_{n_k}(g,0) = \sum_{m=1}^{n_k} \frac{g^{(m)}(0)}{m!} x^m$$

частичных сумм ряда Тейлора функции $g(x)$, которая равномерно сходится к $f(x)$ на отрезке $|x| \le r$.

Было сделано, также, огромное количество исследований, относительно существования универсальных рядов (относительно подпоследовательностей частичных сумм, перестановок, подрядов, знаков коэффициентов и т.д.) по разным классическим ортогональным системам в различных функциональных пространствах. Наиболее общие результаты были получены Д. Е. Меньшовым [5], А. А. Талаляном [6], П. Л. Ульяновым [7] и их учениками (см. [8] – [23]).

Настоящая работа представляет очередной, на наш взгляд, интересный результат из серии исследований относительно существования и структуры функций с универсальными рядами Фурье – Уолша.

Прежде чем перейти к формулировке, дадим соответствующие определения.

Пусть $\{W_k\}$ – полная ортонормированная на $[0,1]$ система Уолша. $L^p(E)$ ($p \geq 1$) – класс всех тех измеримых на $E \subseteq [0,1]$ функций $f(x)$, для которых $\|f\|_{L^p(E)} = \left(\int_E |f(x)|^p dx\right)^{\frac{1}{p}} < +\infty$ и $L^p_\mu[0,1]$ (весовое пространство) – класс всех тех измеримых на $[0,1]$ функций $f(x)$, для которых $\|f\|_{L^p_\mu[0,1]} = \left(\int_0^1 |f(x)|^p \mu(x) dx\right)^{\frac{1}{p}} < +\infty$, где $0 < \mu(x) \leq 1$ – весовая функция.

Определение 1.1. *Будем говорить, что функция $U \in L^1[0,1]$ универсальна для класса $L^p(E)$ относительно системы $\{W_k\}$ в смысле знаков своих коэффициентов Фурье $c_k(U) = \int_0^1 U(x)W_k(x)dx$, если для каждой функции $f \in L^p(E)$ можно найти такие числа $\delta_k = \pm 1$, что ряд $\sum_{k=0}^{\infty} \delta_k c_k(U) W_k(x)$ сходится к $f(x)$ в метрике $L^p(E)$, т.е.*

$$\lim_{m \to +\infty} \int_E \left| \sum_{k=0}^{m} \delta_k c_k(U) W_k(x) - f(x) \right|^p dx = 0.$$

Определение 1.2. *Будем говорить, что функция $U \in L^1[0,1]$ универсальна для класса $L^p(E)$ относительно системы $\{W_k\}$ в смысле подпоследовательностей знаков коэффициентов Фурье, если для каждой функции $g \in L^p(E)$ можно найти такие числа $\sigma_k = \pm 1, 0$, что ряд $\sum_{k=0}^{\infty} \sigma_k c_k(U) W_k(x)$ сходится к $g(x)$ в метрике $L^p(E)$.*

Таким образом, определенные здесь универсальные функции, это функции с универсальными рядами Фурье – Уолша.

Замечание 1.1. *Легко видеть, что для классов $L^p[0,1]$, $p \geq 1$ не существуют определенные нами универсальные функции (ни в смысле знаков коэффициентов Фурье – Уолша, ни в смысле подпоследовательностей знаков коэффициентов Фурье – Уолша).*

Действительно, если бы для некоторого класса $L^p[0,1]$, $p \geq 1$ существовала такая функция $U \in L^1[0,1]$, то для функции $k_0 c_{k_0}(U) W_{k_0}(x)$, где $k_0 > 1$ любое натуральное число с условием $c_{k_0}(U) \neq 0$, нашлись бы такие числа $\delta_k = \pm 1$ или $\pm 1, 0$, что

$$\lim_{m\to\infty} \int_0^1 \left| \sum_{k=0}^{m} \delta_k c_k(U) W_k(x) - k_0 c_{k_0}(U) W_{k_0}(x) \right|^p dx = 0,$$

откуда сразу получаем противоречие: $\delta_{k_0} = k_0 > 1$.

Однако, полученные в работах [16] – [20] результаты показывают, что для пространств $L^p[0,1]$, $p \in (0,1)$; $L^p(E)$, $p \geq 1$, $E \subset [0,1]$ и $L^p_\mu[0,1]$, $p \geq 1$ (весовые пространства) картина иная.

В работе [16] доказано, что для любого числа $p \in (0,1)$ существует функция $U_p \in L^1[0,1]$ (ряд Фурье – Уолша которой имеет строго убывающие коэффициенты и сходится к ней по $L^1[0,1]$ норме), которая является универсальной для класса $L^p[0,1]$ в смысле знаков своих коэффициентов Фурье – Уолша. В [18] авторам удалось еще и описать структуру таких функций с точки зрения классических теорем Лузина – Меньшова [25], [26].

Далее, в [19] и [20] построены интегрируемые функции $g(x)$ (ряды Фурье – Уолша которых имеют строго убывающие коэффициенты и сходятся к ним по $L^1[0,1]$ норме) и весовые функции $0 < \mu(x) \leq 1$ так, чтобы, в первом случае, $g(x)$ была универсальной для весового пространства $L^1_\mu[0,1]$ в смысле знаков своих коэффициентов Фурье – Уолша, а во втором – универсальной для каждого класса $L^p_\mu[0,1]$, $p \geq 1$ в смысле подпоследовательностей знаков своих коэффициентов Фурье – Уолша. Более того, показано, что меру множества, на котором $\mu(x) = 1$, можно сделать сколь угодно близкой к единице.

Следующей теоремой описывается структура универсальных для классов $L^p(E)$, $p > 1$ функций:

Теорема 1.1. *Для любого числа $0 < \varepsilon < 1$ существует измеримое множество $E_\varepsilon \subset [0,1]$ с мерой $|E_\varepsilon| > 1 - \varepsilon$ такое, что для каждой функции $f \in L^1[0,1]$ можно найти функцию $\tilde{f} \in L^1[0,1]$, совпадающую с f на E_ε, которая универсальна для каждого класса $L^p(E)$, $p > 1$ в смысле подпоследовательностей знаков своего ряда Фурье – Уолша.*

Нам также удалось усилить этот результат и описать структуру универсальных функций для весовых пространств $L^p_\mu[0,1]$, $p > 1$:

Теорема 1.2. *Для любого числа $0 < \varepsilon < 1$ существует измеримое множество $E_\varepsilon \subset [0,1]$ с мерой $|E_\varepsilon| > 1-\varepsilon$ и весовая функция $0 < \mu(x) \le 1$, с $\mu(x) = 1$ на E_ε, такие, что для каждой функции $f \in L^1[0,1]$ можно найти функцию $\tilde{f} \in L^1[0,1]$, совпадающую с f на E_ε, которая универсальна для каждого класса $L^p_\mu[0,1]$, $p > 1$, в смысле подпоследовательностей знаков своего ряда Фурье – Уолша.*

В связи с результатами настоящей работы возникают следующие вопросы, ответы на которые нам не известны:

Вопрос 1. Справедливы ли теоремы 1.1–1.2 для универсальности в смысле знаков коэффициентов Фурье – Уолша?

Вопрос 2. Справедливы ли теоремы 1.1–1.2 для тригонометрической системы и/или для других классических ортонормированных систем?

Интересно было бы выяснить также существует ли абсолютно интегрируемая функция с универсальным в рассматриваемых здесь пространствах рядом Фурье – Уолша относительно подпоследовательностей частичных сумм. В конце приведем еще два интересных результата, непосредственно связанных с данной тематикой:

В [21] доказано, что если последовательность $\{a_k\}$ удовлетворяет условиям

$$a_0 \ge a_1 \ge \ldots \ge a_k \ge \ldots, \quad \lim_{k\to\infty} a_k = 0 \quad \text{и} \quad \sum_{k=0}^{\infty} a_k^2 = \infty, \tag{1.1}$$

то для любой п. в. конечной измеримой функции f, определенной на $[0,1]$, существует последовательность чисел $\{\delta_k\}$, $\delta_k = \pm 1, 0$, такая, что ряд $\sum_{k=0}^{\infty} \delta_k a_k W_k$ по системе Уолша сходится к f п.в. на $[0,1]$. В этой работе приведен еще и пример ряда Фурье – Уолша, коэффициенты которого удовлетворяют условиям (1.1) $\left(\sum_{n=1}^{+\infty} \frac{W_n}{\sqrt{n}}\right)$.

В [22] показано, что если для последовательности $\{a_k\}$ выполняются условия (1.1) то для любого числа $\varepsilon > 0$ существует измеримое множество $E \subset [0,1]$ с мерой $|E| > 1-\varepsilon$ такое, что для любой функции $f \in L^1[0,1]$ существуют функция $g \in L^1[0,1]$ и числа $\delta_k = \pm 1, 0$, такие, что $g(x) = f(x)$ для $x \in E$, а ряд $\sum_{k=0}^{\infty} \delta_k a_k W_k$ сходится к функции g в метрике $L^1[0,1]$.

2. Вспомогательные леммы

Функции системы Уолша $\{W_k(x)\}_{k=0}^{\infty}$, определяются функциями системы Радемахера

$$R_k(x) = \operatorname{sign}(\sin 2^k\pi x), \quad x \in [0,1], \quad k = 1,2,\ldots$$

следующим образом (см. [24]): $W_0(x) \equiv 1$, а для $k \geq 1$

$$W_k(x) = \prod_{i=1}^{l} R_{n_i+1}(x),$$

где $k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_l}$ $(n_1 > n_2 > \cdots > n_l)$.

Пусть $|E|$ – мера Лебега измеримого множества $E \subseteq [0,1]$, а $\chi_E(x)$ – ее характеристическая функция. Для системы Уолша при любом натуральном числе m верна (см. [24])

$$\sum_{k=0}^{2^m-1} W_k(x) = \begin{cases} 2^m, & \text{когда} \quad x \in [0,2^{-m}), \\ 0, & \text{когда} \quad x \in (2^{-m},1], \end{cases} \tag{2.1}$$

откуда для любого числа $p > 0$ имеем

$$\int_0^1 \left|\sum_{k=2^m}^{2^{m+1}-1} W_k(x)\right|^p dx = 2^{m(p-1)}. \tag{2.2}$$

Очевидно, что для любого натурального числа $M \in [2^m, 2^{m+1})$ и чисел $\{a_k\}_{k=2^m}^{2^{m+1}-1}$ верно

$$\left\|\sum_{k=2^m}^{M} a_k W_k\right\|_{L^1[0,1]} \leq \left\|\sum_{k=2^m}^{2^{m+1}-1} a_k W_k\right\|_{L^2[0,1]}. \tag{2.3}$$

Отметим, что из базисности системы Уолша в пространствах $L^p[0,1]$, $p > 1$ следует, что для любого числа $p > 1$ существует такая постоянная $C_p > 0$, что для каждой функции $f \in L^p[0,1]$ имеет место следующее неравенство

$$||S_n(f)||_{L^p[0,1]} \leq C_p||f||_{L^p[0,1]}, \quad n \in \mathbb{N}, \tag{2.4}$$

где $\{S_n(f)\}$ – частичные суммы ее разложения по системе Уолша [24].

Используя схемы доказательств Леммы 2 работы [19] и Леммы 2.2 работы [20], на основе соотношений (2.1) – (2.3) не трудно убедиться в справедливости следующей основной леммы:

Лемма 2.1. *Пусть $p > 1$, $n_0 \in \mathbb{N}$ и $\Delta = \left[\frac{l}{2^K}, \frac{l+1}{2^K}\right]$, $l \in [0, 2^K)$ есть двоичный интервал. Тогда для любых чисел $\varepsilon \in (0,1)$, $\gamma \neq 0$ и натурального числа q*

существуют измеримое множество $E_q \subset \Delta$ *с мерой* $|E_q| = (1 - 2^{-q})|\Delta|$ *и полиномы*

$$P_q(x) = \sum_{k=2^{n_0}}^{2^{n_q}-1} a_k W_k(x), \quad H_q(x) = \sum_{k=2^{n_0}}^{2^{n_q}-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1,$$

$$G_q(x) = \sum_{k=2^{n_0}}^{2^{n_q}-1} \sigma_k a_k W_k(x), \quad \sigma_k = \pm 1, 0,$$

по системе Уолша такие, что

1) $$0 < a_{k+1} \le a_k < \varepsilon, \quad \text{когда} \quad k \in [2^{n_0}, 2^{n_q} - 1),$$

2) $$P_q(x) \cdot \chi_{[2^{-n_0},1]}(x) = 0,$$

3) $$H_q(x) = \begin{cases} \begin{cases} \gamma, & \text{когда } x \in E_q \\ 0, & \text{когда } x \in [2^{-n_0}, 1] \setminus \Delta \end{cases}, & \text{если} \quad \Delta \subset [2^{-n_0}, 1], \\ 0, \quad \text{когда } x \in [2^{-n_0}, 1], & \text{если} \quad \Delta \subset [0, 2^{-n_0}], \end{cases}$$

4) $$G_q(x) = \begin{cases} \gamma, & \text{когда } x \in E_q, \\ 0, & \text{когда } x \in [0,1] \setminus \Delta, \end{cases}$$

5) $$\max_{2^{n_0} \le M < 2^{n_q}} \left\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k \right\|_{L^1[0,1]} < 3|\gamma||\Delta| + \varepsilon,$$

6) $$\max_{2^{n_0} \le M < 2^{n_q}} \left\| \sum_{k=2^{n_0}}^{M} \sigma_k a_k W_k \right\|_{L^p[0,1]} < 2^q C |\gamma||\Delta|^{1/p},$$

где C *есть постоянная определяемая пространством* $L^p[0,1]$.

7) $$\max_{2^{n_0} \le M < 2^{n_q}} \left\| \sum_{k=2^{n_0}}^{M} a_k W_k \right\|_{L^1[0,1]} < \varepsilon.$$

Лемма 2.1 позволяет установить вспомогательную Лемму 2.2.

Лемма 2.2. *Пусть* $p_0 > 1$, $n_0 \in \mathbb{N}$, $\varepsilon \in (0,1)$ *и* $f(x) = \sum_{m=1}^{\bar{\nu}_0} \bar{\gamma}_m \chi_{\tilde{\Delta}_m}(x)$ *есть такая ступенчатая функция, что* $\bar{\gamma}_m \neq 0$ *и* $\{\tilde{\Delta}_m\}_{m=1}^{\bar{\nu}_0}$ *– непересекающийся двоичные интервалы с* $\sum_{m=1}^{\bar{\nu}_0} |\tilde{\Delta}_m| = 1$. *Тогда можно найти измеримые множества* $E^{(1)} \subset [2^{-n_0}, 1]$, $E^{(2)} \subset [0,1]$ *и полиномы*

$$P(x) = \sum_{k=2^{n_0}}^{2^{n}-1} a_k W_k(x), \quad H(x) = \sum_{k=2^{n_0}}^{2^{n}-1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1,$$

$$G(x) = \sum_{k=2^{n_0}}^{2^n-1} \sigma_k a_k W_k(x), \quad \sigma_k = \pm 1, 0,$$

по системе Уолша, удовлетворяющие следующим условиям:

1) $$|E^{(1)}| > 1 - 2^{-n_0} - \varepsilon, \quad |E^{(2)}| > 1 - \varepsilon,$$

2) $$0 < a_{k+1} \leq a_k < \varepsilon, \quad k \in [2^{n_0}, 2^n - 1),$$

3) $$P(x) \cdot \chi_{[2^{-n_0},1]}(x) = 0,$$

4) $$f(x) = \begin{cases} H(x), & \text{когда } x \in E^{(1)}, \\ G(x), & \text{когда } x \in E^{(2)}, \end{cases}$$

5) $$\max_{2^{n_0} \leq M < 2^n} \left\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k \right\|_{L^1[0,1]} < 4\|f\|_{L^1[0,1]},$$

6) $$\max_{2^{n_0} \leq M < 2^n} \left\| \sum_{k=2^{n_0}}^{M} \sigma_k a_k W_k \right\|_{L^p(e)} \leq \|f\|_{L^p(e)} + \varepsilon$$

для любого измеримого множества $e \subseteq E^{(2)}$ *и числа* $1 < p \leq p_0$,

7) $$\max_{2^{n_0} \leq M < 2^n} \left\| \sum_{k=2^{n_0}}^{M} a_k W_k \right\|_{L^1[0,1]} < \varepsilon.$$

Доказательство. Выберем натуральное число

(2.5) $$q > \log_2 \frac{1}{\varepsilon},$$

и разделим отрезок $[0, 1]$ на короткие, непересекающиеся двоичные интервалы одинаковой длины $\{\Delta_j\}_{j=1}^{\nu_0}$ таким образом, что $|\Delta_j| \leq \min\{|\tilde{\Delta}_m|\}$, число 2^{-n_0} является точкой деления и выполняется неравенство

(2.6) $$\max_{j \in [1,\nu_0]} \left\{ 2^q C |\gamma_j| |\Delta_j|^{1/p_0} \right\} < \varepsilon,$$

где $\gamma_j = \tilde{\gamma}_m$, если $\Delta_j \subset \tilde{\Delta}_m$. Представим функцию $f(x)$ в виде $f(x) = \sum_{j=1}^{\nu_0} \gamma_j \chi_{\Delta_j}(x)$. Последовательно применяя лемму 2.1 для каждого из интервалов Δ_j, $j \in [1, \nu_0]$ и учитывая (2.5) и (2.6), найдем такие множества $E_q^{(j)} \subset \Delta_j$ с мерой

(2.7) $$|E_q^{(j)}| = (1 - 2^{-q})|\Delta_j| > (1 - \varepsilon)|\Delta_j|$$

и полиномы

$$P_q^{(j)}(x) = \sum_{k=2^{n_{j-1}}}^{2^{n_j}-1} a_k^{(j)} W_k(x), \quad H_q^{(j)}(x) = \sum_{k=2^{n_{j-1}}}^{2^{n_j}-1} \delta_k^{(j)} a_k^{(j)} W_k(x), \ \delta_k^{(j)} = \pm 1.$$

$$G_q^{(j)}(x) = \sum_{k=2^{n_{j-1}}}^{2^{n_j}-1} \sigma_k^{(j)} a_k^{(j)} W_k(x), \ \sigma_k^{(j)} = \pm 1, 0,$$

по системе Уолша, что

(2.8) $$\begin{cases} 0 < a_{k+1}^{(1)} \le a_k^{(1)} < \varepsilon, & \text{для} \quad k \in [2^{n_0}, 2^{n_1} - 1), \\ 0 < a_{k+1}^{(j)} \le a_k^{(j)} < a_{2^{n_{j-1}}-1}^{(j-1)}, & \text{для} \quad k \in [2^{n_{j-1}}, 2^{n_j} - 1), \ j \in [2, \nu_0], \end{cases}$$

(2.9) $$P_q^{(j)}(x) \cdot \chi_{[2^{-n_0}, 1]}(x) = 0,$$

(2.10) $$H_q^{(j)}(x) = \begin{cases} \begin{cases} \gamma_j, & \text{когда } x \in E_q^{(j)} \\ 0, & \text{когда } x \in [2^{-n_0}, 1] \setminus \Delta_j \end{cases}, & \text{если} \quad \Delta_j \subset [2^{-n_0}, 1], \\ 0, \quad \text{когда } x \in [2^{-n_0}, 1], & \text{если} \quad \Delta_j \subset [0, 2^{-n_0}], \end{cases}$$

(2.11) $$G_q^{(j)}(x) = \begin{cases} \gamma_j, & \text{когда } x \in E_q^{(j)}, \\ 0, & \text{когда } x \in [0, 1] \setminus \Delta_j. \end{cases}$$

(2.12) $$\max_{2^{n_{j-1}} \le M < 2^{n_j}} \left\| \sum_{k=2^{n_{j-1}}}^{M} \delta_k^{(j)} a_k^{(j)} W_k \right\|_{L^1[0,1]} < 3|\gamma_j||\Delta_j| + 2^{-j}\|f\|_{L^1[0,1]},$$

(2.13) $$\max_{2^{n_{j-1}} \le M < 2^{n_j}} \left\| \sum_{k=2^{n_{j-1}}}^{M} \sigma_k^{(j)} a_k^{(j)} W_k \right\|_{L^{p_0}[0,1]} < 2^q C |\gamma_j| |\Delta_j|^{1/p_0} < \varepsilon,$$

и

(2.14) $$\max_{2^{n_{j-1}} \le M < 2^{n_j}} \left\| \sum_{k=2^{n_{j-1}}}^{M} a_k^{(j)} W_k \right\|_{L^1[0,1]} < \frac{\varepsilon}{2^j}.$$

Определим множества

(2.15) $$E^{(1)} = \bigcup_{j:\ \Delta_j \subset [2^{-n_0}, 1]} E_q^{(j)}, \quad E^{(2)} = \bigcup_{j=1}^{\nu_0} E_q^{(j)}$$

и полиномы

$$P(x) = \sum_{j=1}^{\nu_0} P_q^{(j)}(x) = \sum_{k=2^{n_0}}^{2^{n_{\nu_0}}-1} a_k W_k(x), \quad H(x) = \sum_{j=1}^{\nu_0} H_q^{(j)}(x) = \sum_{k=2^{n_0}}^{2^{n_{\nu_0}}-1} \delta_k a_k W_k(x),$$

$$G(x) = \sum_{j=1}^{\nu_0} G_q^{(j)}(x) = \sum_{k=2^{n_0}}^{2^{n_{\nu_0}}-1} \sigma_k a_k W_k(x),$$

где $a_k = a_k^{(j)}$, $\sigma_k = \sigma_k^{(j)}$ и $\delta_k = \delta_k^{(j)}$, когда $k \in [2^{n_{j-1}}, 2^{n_j})$.

Утверждения 1) – 4) леммы 2.2 сразу получаются из соотношений (2.7) – (2.11) и (2.15). Далее, пусть M есть натуральное число из $[2^{n_0}, 2^{n_{\nu_0}})$. Тогда $M \in [2^{n_{m-1}}, 2^{n_m})$ для некотрого $m \in [1, \nu_0]$. Используя (2.11) – (2.14) имеем

$$\left\|\sum_{k=2^{n_0}}^{M} \delta_k a_k W_k\right\|_{L^1[0,1]} \le \sum_{j=1}^{\nu_0} \max_{2^{n_{j-1}} \le N < 2^{n_j}} \left\|\sum_{k=2^{n_{j-1}}}^{N} \delta_k a_k^{(j)} W_k\right\|_{L^1[0,1]} <$$

$$< 3\sum_{j=1}^{\nu_0} |\gamma_j||\Delta_j| + ||f||_{L^1[0,1]} < 4||f||_{L^1[0,1]},$$

$$\left\|\sum_{k=2^{n_0}}^{M} \sigma_k a_k W_k\right\|_{L^p(e)} \le \left\|\sum_{j=1}^{m-1} G_q^{(j)}\right\|_{L^p(e)} + \left\|\sum_{k=2^{n_{m-1}}}^{M} \sigma_k^{(m)} a_k^{(m)} W_k\right\|_{L^p(e)} \le$$

$$\le \left\|\sum_{j=1}^{m-1} \gamma_j \chi_{\Delta_j}\right\|_{L^p(e)} + \left\|\sum_{k=2^{n_{m-1}}}^{M} \sigma_k^{(m)} a_k^{(m)} W_k\right\|_{L^{p_0}[0,1]} < ||f||_{L^p(e)} + \varepsilon$$

для любого измеримого множества $e \subseteq E^{(2)}$ и числа $1 < p \le p_0$ и

$$\left\|\sum_{k=2^{n_0}}^{M} a_k W_k\right\|_{L^1[0,1]} \le \sum_{j=1}^{\nu_0} \max_{2^{n_{j-1}} \le N < 2^{n_j}} \left\|\sum_{k=2^{n_{j-1}}}^{N} a_k^{(j)} W_k\right\|_{L^1[0,1]} < \varepsilon.$$

Лемма 2.2 доказана. □

Теперь с помощью леммы 2.2 докажем основную лемму этого параграфа.

Лемма 2.3. *Для каждого числа $\eta \in (0,1)$ существует весовая функция $0 < \mu(x) \le 1$ с $|\{x \in [0,1] : \mu(x) = 1\}| > 1 - \eta$ такая, что для любых чисел $p_0 > 1$, $n_0 \in \mathbb{N}$, $\varepsilon \in (0,1)$ и ступенчатой функции $f(x) = \sum_{j=1}^{\nu_0} \gamma_j \chi_{\Delta_j}(x)$ с рациональными $\gamma_j \ne 0$ и условием $\sum_{j=1}^{\nu_0} |\Delta_j| = 1$, где $\{\Delta_j\}_{j=1}^{\nu_0}$ – непересекающиеся двоичные интервалы, можно найти измеримое множество $E \subset [2^{-n_0}, 1]$ и полиномы*

$$P(x) = \sum_{k=2^{n_0}}^{2^n - 1} a_k W_k(x), \quad H(x) = \sum_{k=2^{n_0}}^{2^n - 1} \delta_k a_k W_k(x), \quad \delta_k = \pm 1,$$

$$G(x) = \sum_{k=2^{n_0}}^{2^n - 1} \sigma_k a_k W_k(x), \quad \sigma_k = \pm 1, 0,$$

по системе Уолша, удовлетворяющие следующим условиям:

1) $$|E| > 1 - \varepsilon - 2^{-n_0},$$

2) $$0 < a_{k+1} \le a_k < \varepsilon, \quad k \in [2^{n_0}, 2^n - 1),$$

3) $$P(x) \cdot \chi_{[2^{-n_0},1]}(x) = 0,$$

4) $$H(x)=f(x),\quad \textit{когда}\quad x\in E.$$

5) $$\max_{2^{n_0}\le M<2^n}\left\|\sum_{k=2^{n_0}}^{M}\delta_k a_k W_k\right\|_{L^1[0,1]}<5\|f\|_{L^1[0,1]}.$$

6) $$\|f-G\|_{L^{p_0}_\mu[0,1]}<\varepsilon,$$

7) $$\max_{2^{n_0}\le M<2^n}\left\|\sum_{k=2^{n_0}}^{M}\sigma_k a_k W_k\right\|_{L^p_\mu[0,1]}<2\|f\|_{L^p_\mu[0,1]}+\varepsilon,\quad p\in(1,p_0].$$

8) $$\max_{2^{n_0}\le M<2^n}\left\|\sum_{k=2^{n_0}}^{M}a_k W_k\right\|_{L^1[0,1]}<\varepsilon.$$

Доказательство. Пусть $\eta\in(0,1)$, $N_0=1$ и $f_m(x)=\sum_{j=1}^{\nu_m}\gamma_j^{(m)}\chi_{\Delta_j^{(m)}}(x)$ есть последовательность всех ступенчатых функций с рациональными $\gamma_j^{(m)}\neq 0$ и условием $\sum_{j=1}^{\nu_m}|\Delta_j^{(m)}|=1$, где $\{\Delta_j^{(m)}\}_{j=1}^{\nu_m}$ непересекающиеся двоичные интервалы. Последовательным применением леммы 2 можно найти множества $E_m^{(1)}\subset[2^{-N_{m-1}},1]$, $E_m^{(2)}\subset[0,1]$ и полиномы

(2.16) $$P_m(x)=\sum_{k=2^{N_{m-1}}}^{2^{N_m}-1}a_k^{(m)}W_k(x),$$

(2.17) $$H_m(x)=\sum_{k=2^{N_{m-1}}}^{2^{N_m}-1}\delta_k^{(m)}a_k^{(m)}W_k(x),\quad \delta_k^{(m)}=\pm 1,$$

(2.18) $$G_m(x)=\sum_{k=2^{N_{m-1}}}^{2^{N_m}-1}\sigma_k^{(m)}a_k^{(m)}W_k(x),\quad \sigma_k^{(m)}=\pm 1,0,$$

по системе Уолша, удовлетворяющие для любого натурального числа m следующим условиям

(2.19) $$|E_m^{(1)}|>1-2^{-N_{m-1}}-2^{-m-1}\quad\text{и}\quad |E_m^{(2)}|>1-2^{-m-1},$$

(2.20) $$0<a_{k+1}^{(m)}\le a_k^{(m)}<\frac{\min\{1,\|f\|_{L^1[0,1]}\}}{4^{N_{m-1}}},\quad k\in[2^{N_{m-1}},2^{N_m}-1),$$

(2.21) $$P_m(x)=0,\quad \text{когда}\quad x\in\left[2^{-N_{m-1}},1\right],$$

(2.22) $$f_m(x)=\begin{cases}H_m(x), & \text{когда } x\in E_m^{(1)},\\ G_m(x), & \text{когда } x\in E_m^{(2)},\end{cases}$$

$$(2.23)\qquad \max_{2^{N_{m-1}}\le M<2^{N_m}}\left\|\sum_{k=2^{N_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k\right\|_{L^1[0,1]}<4\|f_m\|_{L^1[0,1]},$$

$$(2.24)\qquad \max_{2^{N_{m-1}}\le M<2^{N_m}}\left\|\sum_{k=2^{N_{m-1}}}^{M}\sigma_k^{(m)}a_k^{(m)}W_k\right\|_{L^p(e)}<\|f_m\|_{L^p(e)}+\frac{1}{2^{m+1}}$$

для любого измеримого подмножества $e\subseteq E_m^{(2)}$ и числа $1<p\le m$,

$$(2.25)\qquad \max_{2^{N_{m-1}}\le M<2^{N_m}}\left\|\sum_{k=2^{N_{m-1}}}^{M}a_k^{(m)}W_k\right\|_{L^1[0,1]}<\frac{1}{2^{m+1}}.$$

Используя соотношения (2.19), (2.22), (2.24) и схему примененную в лемме 2.4 работы [20] можно построить весовую функцию $0<\mu(x)\le 1$ с $|\{x\in[0,1]:\mu(x)=1\}|>1-\eta$ такую, что для любого натурального числа

$$(2.26)\qquad m>\bar{n}=[log_{1/2}\eta]+1$$

имеют место

$$(2.27)\qquad \|f_m-G_m\|_{L_\mu^m[0,1]}<\frac{1}{2^{m-1}},$$

и

$$(2.28)\qquad \left\|\sum_{k=2^{N_{m-1}}}^{M}\delta_k^{(m)}a_k^{(m)}W_k\right\|_{L_\mu^p[0,1]}<2\|f_m\|_{L_\mu^p[0,1]}+\frac{1}{2^{m-1}},$$

каковы бы не были $M\in[2^{N_{m-1}},2^{N_m})$ и $1<p\le m$.

Теперь, пусть числа $n_0\in\mathbb{N}$ и $\varepsilon\in(0,1)$ заданы. Из последовательности $\{f_m\}$ выберем такую функцию $f_{m_0}(x)=f(x)$, что

$$(2.29)\qquad m_0>\max\left\{\bar{n},\ p_0,\ \log_2\frac{8}{\varepsilon}\right\},\quad 2^{N_{m_0-1}}>\max\{2^{n_0},2/\varepsilon\}.$$

и для $k\in[2^{n_0},2^{N_{m_0}})$ положим (в соответствии с (2.16) – (2.18))

$$(2.30)\qquad a_k=\begin{cases}a^{(m_0)}_{2^{N_{m_0-1}}}, & \text{когда}\quad k\in[2^{n_0},2^{N_{m_0-1}}),\\ a_k^{(m_0)}, & \text{когда}\quad k\in[2^{N_{m_0-1}},2^{N_{m_0}}),\end{cases}$$

$$(2.31)\qquad \delta_k=\begin{cases}1, & \text{когда}\quad k\in[2^{n_0},2^{N_{m_0-1}}),\\ \delta_k^{(m_0)}=\pm1, & \text{когда}\quad k\in[2^{N_{m_0-1}},2^{N_{m_0}}),\end{cases}$$

$$\sigma_k = \begin{cases} 0, & \text{когда} \quad k \in \left[2^{n_0}, 2^{N_{m_0-1}}\right), \\ \sigma_k^{(m_0)} = \pm 1, 0 & \text{когда} \quad k \in \left[2^{N_{m_0-1}}, 2^{N_{m_0}}\right). \end{cases} \tag{2.32}$$

и

$$E = E_{m_0}^{(1)} \cap [2^{-n_0}, 1], \tag{2.33}$$

$$P(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0}}-1} a_k W_k(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0-1}}-1} a_k W_k(x) + P_{m_0}(x),$$

$$H(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0}}-1} \delta_k a_k W_k(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0-1}}-1} a_k W_k(x) + H_{m_0}(x),$$

$$G(x) = \sum_{k=2^{n_0}}^{2^{N_{m_0}}-1} \sigma_k a_k W_k(x) = G_{m_0}(x).$$

Убедимся, что функция $\mu(x)$, множество E и полиномы $P(x)$, $H(x)$ и $G(x)$ удовлетворяют всем условиям леммы 3. Утверждения 2), 6) и 7) непосредственно следуют из (2.20), (2.27) – (2.30) и (2.32). Используя (2.1), (2.19), (2.21), (2.22), (2.29) и (2.33) находим

$$|E| > 1 - 2^{-m_0-1} - 2^{-N_{m_0-1}} - 2^{-n_0} > 1 - \varepsilon - 2^{-n_0}, \quad \text{(утв. 1))}$$

$$P(x) = P_{m_0}(x) = 0, \quad \text{когда} \quad x \in [2^{-n_0}, 1], \quad \text{(утв. 3))}$$

$$H(x) = H_{m_0}(x) = f_{m_0}(x) = f(x), \quad \text{когда} \quad x \in E, \quad \text{(утв. 4))}.$$

Далее, обозначая

$$\max_{2^{n_0} \le M < 2^{N_{m_0-1}}} \left\| \sum_{k=2^{n_0}}^{M} a_k W_k \right\|_{L^1[0,1]} \equiv J,$$

на основе (2.23), (2.25), (2.29) и (2.31) получаем

$$\max_{2^{n_0} \le M < 2^{N_{m_0}}} \left\| \sum_{k=2^{n_0}}^{M} \delta_k a_k W_k \right\|_{L^1[0,1]} \le$$

$$\le J + \max_{2^{N_{m_0-1}} \le M < 2^{N_{m_0}}} \left\| \sum_{k=2^{N_{m_0-1}}}^{M} \delta_k^{(m_0)} a_k^{(m_0)} W_k \right\|_{L^1[0,1]} < J + 4\|f\|_{L^1[0,1]},$$

и

$$\max_{2^{n_0} \le M < 2^{N_{m_0}}} \left\| \sum_{k=2^{n_0}}^{M} a_k W_k \right\|_{L^1[0,1]} \le J + \max_{2^{N_{m_0-1}} \le M < 2^{N_{m_0}}} \left\| \sum_{k=2^{N_{m_0-1}}}^{M} a_k^{(m_0)} W_k \right\|_{L^1[0,1]} < J + \frac{\varepsilon}{2}.$$

Пусть, теперь, M произвольное натуральное число из $[2^{n_0}, 2^{N_{m_0-1}})$. Тогда $M \in [2^{n_1}, 2^{n_1+1})$ для некоторого $n_1 \in [n_0, N_{m_0-1})$ и, следовательно, с учетом (2.1), (2.20) и (2.29), приходим к следующему заключению:

$$\left\|\sum_{k=2^{n_0}}^{M} a_k W_k\right\|_{L^1[0,1]} < a^{(m_0)}_{2^{N_{m_0-1}}} \cdot \left\|\sum_{k=2^{n_0}}^{2^{n_1}-1} W_k\right\|_{L^1[0,1]} + a^{(m_0)}_{2^{N_{m_0-1}}} \cdot 2^{n_1} <$$

$$< \min\left\{\frac{\varepsilon}{2}, \|f\|_{L^1[0,1]}\right\},$$

чем окончательно доказываются утверждения 5) и 8). Лемма 2.3 доказано. □

3. Доказательство теоремы 1.2

Пусть $\varepsilon \in (0,1)$, n_0 – произвольное натуральное число больше $\log_2 \frac{8}{\varepsilon}$ и $f_m(x) = \sum_{j=1}^{\nu_m} \gamma_j^{(m)} \chi_{\Delta_j^{(m)}}(x)$ есть последовательность всех ступенчатых функций с рациональными $\gamma_j^{(m)} \neq 0$ и условием $\sum_{j=1}^{\nu_m} |\Delta_j^{(m)}| = 1$, где $\{\Delta_j^{(m)}\}_{j=1}^{\nu_m}$ – непересекающийся двоичные интервалы.

Применяя лемму 2.3 можно найти весовую функцию $0 < \mu(x) \leq 1$ с условием $|E_\varepsilon^{(1)}| > 1 - \frac{\varepsilon}{2}$, где $E_\varepsilon^{(1)} = \{x \in [0,1]\colon \mu(x) = 1\}$, множества $E_m \subset [2^{-n_{m-1}}, 1]$ и полиномы

$$P_m(x) = \sum_{k=2^{n_{m-1}}}^{2^{n_m}-1} a_k^{(m)} W_k(x), \tag{3.1}$$

$$H_m(x) = \sum_{k=2^{n_{m-1}}}^{2^{n_m}-1} \delta_k^{(m)} a_k^{(m)} W_k(x), \quad \delta_k^{(m)} = \pm 1, \tag{3.2}$$

$$G_m(x) = \sum_{k=2^{n_{m-1}}}^{2^{n_m}-1} \sigma_k^{(m)} a_k^{(m)} W_k(x), \quad \sigma_k^{(m)} = \pm 1, 0, \tag{3.3}$$

по системе Уолша, которые удовлетворяют следующим условиям для каждого натурального числа m:

$$|E_m| > 1 - \frac{\varepsilon}{2^{m+2}} - 2^{-n_{m-1}}, \tag{3.4}$$

$$\begin{cases} 0 < a_{k+1}^{(1)} \leq a_k^{(1)} < 1, \quad k \in [2^{n_0}, 2^{n_1} - 1), \\ 0 < a_{k+1}^{(m)} \leq a_k^{(m)} < \min\{2^{-m}, a^{(m-1)}_{2^{n_{m-1}}-1}\}, \ k \in [2^{n_{m-1}}, 2^{n_m} - 1), \end{cases} \tag{3.5}$$

$$P_m(x) \cdot \chi_{[2^{-n_{m-1}},1]}(x) = 0, \tag{3.6}$$

$$H_m(x) = f_m(x), \quad \text{когда} \quad x \in E_m, \tag{3.7}$$

$$\max_{2^{\nu_{m-1}} \le M < 2^{\nu_m}} \left\| \sum_{k=2^{\nu_{m-1}}}^{M} \delta_k^{(m)} a_k^{(m)} W_k \right\|_{L^1[0,1]} < 5\|f_m\|_{L^1[0,1]}, \tag{3.8}$$

$$\|f_m - G_m\|_{L^m_\mu[0,1]} < 2^{-m-2}, \tag{3.9}$$

$$\max_{2^{\nu_{m-1}} \le M < 2^{\nu_m}} \left\| \sum_{k=2^{\nu_{m-1}}}^{M} \sigma_k^{(m)} a_k^{(m)} W_k \right\|_{L^p_\mu[0,1]} < 2\|f_m\|_{L^p_\mu[0,1]} + 2^{-m-1} \tag{3.10}$$

для любого числа $1 < p \le m$,

$$\max_{2^{\nu_{m-1}} \le M < 2^{\nu_m}} \left\| \sum_{k=2^{\nu_{m-1}}}^{M} a_k^{(m)} W_k \right\|_{L^1[0,1]} < 2^{-m-1}. \tag{3.11}$$

Положим

$$P_0(x) = \sum_{k=0}^{2^{\nu_0}-1} a_k^{(0)} W_k(x) = \sum_{k=0}^{2^{\nu_0}-1} W_k(x) \tag{3.12}$$

и

(3.13)

$$a_k = a_k^{(m)} \quad \text{и} \quad c_k = a_k^{(m)} + 2^{-k(k+4)} \quad \text{для } k \in \begin{cases} [0, 2^{\nu_0}), & \text{если} \quad m = 0 \\ [2^{\nu_{m-1}}, 2^{\nu_m}), & \text{если} \quad m \ge 1. \end{cases}$$

Из (3.1), (3.5), (3.11) – (3.13) следует, что $c_k \searrow 0$ и ряд

$$\sum_{k=0}^{\infty} c_k W_k(x)$$

в метрике $L^1[0,1]$ сходится к некоторой функции $U \in L^1[0,1]$ (следовательно, $c_k = c_k(U)$ являются коэффициентами Фурье – Уолша функции U). Определим множество

$$E_\varepsilon^{(2)} = \left(\bigcap_{m=1}^{\infty} E_m \right) \subset E_m, \ m \in \mathbb{N}, \tag{3.14}$$

имеющего (на основе (3.4)) меру $|E_\varepsilon^{(2)}| > 1 - \frac{\varepsilon}{2}$.

Используя соотношения (2.1), (3.6) – (3.8), (3.11), (3.13), (3.14) и рассуждения сделанные в работе [15] заключаем, что для каждой функции $f \in L^1[0,1]$ можно найти функцию $\tilde{f} \in L^1[0,1]$, совпадающую с f на $E_\varepsilon^{(2)}$ и числа $\delta_k = \pm 1$ так, чтобы ряд $\sum_{k=0}^{\infty} \delta_k c_k W_k$ сходился бы к $\tilde{f}$ в метрике $L^1[0,1]$.

Теперь покажем, что для любого числа $p > 1$ и функции $g \in L^p_\mu[0,1]$ можно найти такие числа $\sigma_k = \pm 1, 0$, что ряд $\sum_{k=0}^{\infty} \sigma_k c_k W_k$ сходится к g в метрике

$L^p_\mu[0,1]$. В соответствии с (3.3) и (3.13) положим

$$\widetilde{G}_m(x) = \sum_{k=2^{n_{m-1}}}^{2^{n_m}-1} \sigma_k^{(m)} c_k W_k(x), \quad \sigma_k^{(m)} = \pm 1, 0, \quad m \in \mathbb{N}. \tag{3.15}$$

Согласно отношениям (3.9), (3.10), (3.13) и (3.15) для любых чисел $m \in \mathbb{N}$, $M \in [2^{n_{m-1}}, 2^{n_m}) \cap \mathbb{N}$ и $p \in (1, m]$ имеют место следующие неравенства:

$$\|f_m - \widetilde{G}_m\|_{L^m_\mu[0,1]} < \|f_m - G_m\|_{L^m_\mu[0,1]} + \sum_{k=2^{n_{m-1}}}^{2^{n_m}-1} 2^{-k-1} < 2^{-m-1}, \tag{3.16}$$

$$\left\| \sum_{k=2^{n_{m-1}}}^{M} \sigma_k^{(m)} c_k W_k \right\|_{L^p_\mu[0,1]} < \left\| \sum_{k=2^{n_{m-1}}}^{M} \sigma_k^{(m)} u_k^{(m)} W_k \right\|_{L^p_\mu[0,1]} + \tag{3.17}$$

$$+ \sum_{k=2^{n_{m-1}}}^{2^{n_m}-1} 2^{-k-1} < 2\|f_m\|_{L^p_\mu[0,1]} + 2^{-m}.$$

Пусть $p > 1$ и $g \in L^p_\mu[0,1]$. Из последовательности $\{f_m\}$ выберем такую функцию $f_{m_1}(x)$, что $m_1 > p$ и

$$\|g - f_{m_1}\|_{L^p_\mu[0,1]} < 2^{-2}. \tag{3.18}$$

Полагая

$$\sigma_k = \begin{cases} \sigma_k^{(m_1)} = \pm 1, 0, & \text{когда} \quad k \in [2^{n_{m_1-1}}, 2^{n_{m_1}}) \\ 0, & \text{когда} \quad k \in [0, 2^{n_{m_1-1}}) \end{cases}$$

и используя (3.15) – (3.18) находим

$$\left\| g - \sum_{k=0}^{2^{n_{m_1}}-1} \sigma_k c_k W_k \right\|_{L^p_\mu[0,1]} \le \|g - f_{m_1}\|_{L^p_\mu[0,1]} +$$

$$+ \left\| f_{m_1} - \widetilde{G}_{m_1} \right\|_{L^{m_1}_\mu[0,1]} < 2^{-2} + 2^{-m_1-1} < 2^{-1}$$

и

$$\max_{2^{n_{m_1-1}} \le M < 2^{n_{m_1}}} \left\| \sum_{k=2^{n_{m_1-1}}}^{M} \sigma_k c_k W_k \right\|_{L^p_\mu[0,1]} < 2\|f_{m_1}\|_{L^p_\mu[0,1]} + 2^{-m_1}.$$

Предположим, что для натурального числа $q > 1$ уже определены числа $m_1 < m_2 < \cdots < m_{q-1}$ и $\sigma_k = \pm 1, 0$, $k \in [0, 2^{n_{m_{q-1}}})$ таким образом, что для каждого натурального числа $j \in [1, q-1]$ выполняются следующие условия:

$$\sigma_k = \begin{cases} \sigma_k^{(m_j)} = \pm 1, 0, & \text{когда} \quad k \in [2^{n_{m_j-1}}, 2^{n_{m_j}}), \\ 0, & \text{когда} \quad k \notin \bigcup_{j=1}^{q-1} [2^{n_{m_j-1}}, 2^{n_{m_j}}), \end{cases}$$

$$\left\| g - \sum_{k=0}^{2^{n m_j}-1} \sigma_k c_k W_k \right\|_{L^p_\mu[0,1]} < 2^{-j}, \tag{3.19}$$

$$\max_{2^{n m_j - 1} \le M < 2^{n m_j}} \left\| \sum_{k=2^{n m_j - 1}}^{M} \sigma_k c_k W_k \right\|_{L^p_\mu[0,1]} < 2\|f_{m_j}\|_{L^p_\mu[0,1]} + 2^{-m_j}.$$

Из $\{f_m\}$ выберем такую функцию $f_{m_q}(x)$, что $m_q > m_{q-1}$ и

$$\left\| g - \sum_{k=0}^{2^{n m_{q-1}}-1} \sigma_k c_k W_k - f_{m_q} \right\|_{L^p_\mu[0,1]} < 2^{-q-1}, \tag{3.20}$$

и определим

$$\sigma_k = \begin{cases} \sigma_k^{(m_q)} = \pm 1, 0, & \text{когда} \quad k \in [2^{n m_q - 1}, 2^{n m_q}), \\ 0, & \text{когда} \quad k \notin \bigcup_{j=1}^{q} [2^{n m_j - 1}, 2^{n m_j}). \end{cases} \tag{3.21}$$

Основываясь на (3.15), (3.16), (3.20) и (3.21) получим

$$\left\| g - \sum_{k=0}^{2^{n m_q}-1} \sigma_k c_k W_k \right\|_{L^p_\mu[0,1]} \le \tag{3.22}$$

$$\le \left\| g - \sum_{k=0}^{2^{n m_{q-1}}-1} \sigma_k c_k W_k - f_{m_q} \right\|_{L^p_\mu[0,1]} + \left\| f_{m_q} - \widetilde{G}_{m_q} \right\|_{L^{m_q}_\mu[0,1]} < 2^{-q-1} + 2^{-m_q-1} < 2^{-q}.$$

Далее, из (3.19) и (3.20) следует, что

$$\|f_{m_q}\|_{L^p_\mu[0,1]} \le \left\| g - \sum_{k=0}^{2^{n m_{q-1}}-1} \sigma_k c_k W_k - f_{m_q} \right\|_{L^p_\mu[0,1]} +$$

$$+ \left\| g - \sum_{k=0}^{2^{n m_{q-1}}-1} \sigma_k c_k W_k \right\|_{L^p_\mu[0,1]} < 2^{-q-1} + 2^{-q+1} < 2^{-q+2}$$

и, следовательно, имея ввиду (3.17) и (3.21) для любого натурального числа $M \in [2^{n m_q - 1}, 2^{n m_q})$ имеем

$$\left\| \sum_{k=2^{n m_q - 1}}^{M} \sigma_k c_k W_k \right\|_{L^p_\mu[0,1]} < 2\|f_{m_q}\|_{L^p_\mu[0,1]} + 2^{-m_q} < 2^{-q+4}. \tag{3.23}$$

Таким образом, можно определить возрастающую последовательность индексов $\{m_q\}_{q=1}^{+\infty}$ и числа $\sigma_k = \pm 1, 0$ так, чтобы условия (3.21) – (3.23) имели место для каждого натурального числа q. Следовательно, мы получаем ряд

$$\sum_{k=0}^{+\infty} \sigma_k c_k W_k, \quad \sigma_k = \pm 1, 0,$$

который сходится к g в метрике $L^p_\mu[0,1]$.

Для завершения доказательства теоремы 1.2 остается положить $E_\varepsilon = E_\varepsilon^{(1)} \cap E_\varepsilon^{(2)}$.

Abstract. The paper is devoted to the questions relating the structure of universal functions for weighted spaces $L^p_\mu[0,1]$, $p > 1$. We prove existence of a measurable set $E \subset [0,1]$ with measure arbitrarily close to 1, and a weight function $0 < \mu(x) \leq 1$, equal to 1 on E, such that by suitable continuation of values of an arbitrary function $f \in L^1(E)$ on $[0,1] \setminus E$, a function $\tilde{f} \in L^1[0,1]$ can be obtained, which is universal for each class $L^p_\mu[0,1]$, $p > 1$, in the sense of subsequences of signs of its Fourier-Walsh coefficients.

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ON EXCESS OF RETRO BANACH FRAMES

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Abstract. The excess of a frame is the greatest number of elements that can be removed from a given frame, yet leave a set which is a frame for the underlying space. We present a characterization of retro Banach frames in Banach spaces with finite excess. A sufficient condition for the existence of a retro Banach frame with infinite excess is obtained.

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1. Introduction and preliminaries

Duffin and Schaeffer [6], while addressing some difficult problems from the theory of nonharmonic Fourier series introduced ***frames*** (or ***Hilbert frames***) for Hilbert spaces. Daubechies, Grossmann and Meyer [5] found a fundamental new application to wavelets and Gabor transforms in which frames continue to play an important role. For utility of frames in applied mathematics, see [1, 4].

A sequence (finite or countable) $\{f_k\} \subset \mathcal{H}$ is called a ***frame*** (or a ***Hilbert frame***) for a separable Hilbert space $\mathcal{H}$, if there exist constants $0 < \mathrm{A} \leq \mathrm{B} < \infty$ such that

$$\mathrm{A}\|f\|^2 \leq \sum_k |\langle f, f_k\rangle|^2 \leq \mathrm{B}\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

Gröchenig [9] generalized Hilbert frames to Banach spaces. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [7, 8] related to ***atomic decompositions***. An atomic decomposition allow a representation of every vector of the space via a series expansion in terms of a fixed sequence of vectors which we call ***atoms***. On the other hand, a Banach frame for a Banach space ensure reconstruction via a bounded linear operator or the ***synthesis operator***. Casazza, Han and Larson studied atomic decompositions and Banach frames

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in [3]. Han and Larson [10] defined a Schauder frame for a Banach space $\mathcal{X}$ to be an inner direct summand (that is, a compression) of a Schauder basis of $\mathcal{X}$. Retro Banach frames were introduced in [11] and were further studied in [14, 15].

In this paper, motivated by the recent work of Balan et al. [2] in the direction of excess of frames in Hilbert spaces, we present necessary and sufficient conditions for excess of retro Banach frames in Banach spaces.

In the remaining part of this section we recall some basic definitions and results which will be used throughout this paper. Let $\mathcal{X}$ be an infinite dimensional separable real (or complex) Banach space, and let $\mathcal{X}^*$ be the dual space (topological) of $\mathcal{X}$. For a sequence $\{f_k\} \subset \mathcal{X}$, by $[f_k]$ we denote the closure of $\text{span}\{f_k\}_k$ in the norm topology of $\mathcal{X}$. For a set L, let $|L|$ denote the number of elements in L. The set of positive integers is denoted by $\mathbb{N}$.

Definition 1.1 ([11]). A system $\mathcal{F} \equiv (\{x_n\}, \Theta)$ $(\{x_n\} \subset \mathcal{X}, \Theta : \mathcal{Z}_d \to \mathcal{X}^*)$ is called a *retro Banach frame* for $\mathcal{X}^*$ with respect to an associated Banach space of scalar valued sequences $\mathcal{Z}_d$, if the following conditions are fulfilled:

(*i*) $\{f(x_n)\} \in \mathcal{Z}_d$ for each $f \in \mathcal{X}^*$,

(*ii*) there exist positive constants $0 < \mathrm{A}_0 \leq \mathrm{B}_0 < \infty$ such that

$$\mathrm{A}_0\|f\| \leq \|\{f(x_n)\}\|_{\mathcal{Z}_d} \leq \mathrm{B}_0\|f\| \text{ for all } f \in \mathcal{X}^*,$$

(*iii*) Θ is a bounded linear operator such that $\Theta(\{f(x_n)\}) = f,\ f \in \mathcal{X}^*$.

The positive constants A_0 and B_0 are called the *lower* and *upper retro frame bounds* of the frame $\mathcal{F}$, respectively. The operator $\Theta : \mathcal{Z}_d \to \mathcal{X}^*$ is called the *retro pre-frame operator* (or simply *a reconstruction operator*) associated with $\mathcal{F}$. If there exists no reconstruction operator Θ_m such that $(\{x_n\}_{k\neq m}, \Theta_m)$ ($m \in \mathbb{N}$ is arbitrary) is a retro Banach frame for $\mathcal{X}^*$, then $\mathcal{F}$ is called an *exact* retro Banach frame for $\mathcal{X}^*$.

Lemma 1.1 ([12]). *Let $\mathcal{X}$ be a Banach space and $\{f_n\} \subset \mathcal{X}^*$ be a sequence such that*

$$\{x \in \mathcal{X} : f_n(x) = 0 \text{ for all } n \in \mathbb{N}\} = \{0\}.$$

Then $\mathcal{X}$ is linearly isometric to the Banach space $\mathcal{Z}_d = \{\{f_n(x)\} : x \in \mathcal{X}\}$, where the norm of $\mathcal{Z}_d$ is given by

$$\|\{f_n(x)\}\|_{\mathcal{Z}_d} = \|x\|_{\mathcal{X}}, x \in \mathcal{X}.$$

2. Main results

We start by the definition of excess of a retro Banach frame.

Definition 2.1. Let $\mathcal{T} \equiv (\{x_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_d$. The *excess* of $\mathcal{F}$, denoted by $e(\mathcal{F})$, is defined as follows:

$$e(\mathcal{F}) = \sup\Big\{|J| : (\{x_k\}_{k\notin J}, \Theta_0) \text{ is a retro Banach frame for } \mathcal{X}^*, J \subset \mathbb{N}\Big\}.$$

Remark 2.1. One may observe that $e(\mathcal{F})$ is an integer-valued (extended) function, and if $\mathcal{T}$ is an exact retro Banach frame, then $e(\mathcal{T}) = 0$.

Example 2.1. Let $\mathcal{X} = \ell^p$ $(1 \le p < \infty)$ and let $\{\chi_k\}$ be the sequence of canonical unit vectors in $\mathcal{X}$.

Define $\{x_k\} \subset \mathcal{X}$ as follows:

$$x_k = \chi_1 \ (1 \le k \le n), \quad \text{and} \quad x_k = \chi_{k-n}, \ k > n \quad (n \in \mathbb{N}). \tag{2.1}$$

Then, in view of Lemma 1.1, $\mathcal{Z}_d = \{\{f(x_k)\} : f \in \mathcal{X}^*\}$ is a Banach space with the norm

$$\|\{f(x_k)\}\|_{\mathcal{Z}_d} = \|f\|_{\mathcal{X}^*}, \ f \in \mathcal{X}^*.$$

Define $\Theta : \mathcal{Z}_d \to \mathcal{X}^*$ by $\Theta(\{f(x_k)\}) = f$, $f \in \mathcal{X}^*$. Then, Θ is a bounded linear operator such that $\mathcal{T} \equiv (\{x_k\}, \Theta)$ is a retro Banach frame for $\mathcal{X}^*$ with bounds $A = B = 1$.

Choose $J = \{1, 2, ..., k\} \subset \mathbb{N}$, where k is given in (2.1). Then, there exists a reconstruction operator Θ_o such that $\mathcal{F}_o \equiv (\{x_k\}_{k\notin J}, \Theta_o)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to the sequence space $\mathcal{Z}_{d_o} = \{\{f(x_k)\}_{k\notin J} : f \in \mathcal{X}^*\}$. Furthermore, $\mathcal{F}_o$ is exact. Therefore, $e(\mathcal{F}) = k$, which is finite.

Example 2.2. Let $\{z_k\} \subset \mathcal{X}$ be a sequence given by

$$z_{2k} = z_{2k-1} = \chi_k, \ k \in \mathbb{N}.$$

Then, there exists a retro pre-frame operator U such that $\Omega \equiv (\{z_k\}, U)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\Omega_d = \{\{f(z_k)\} : f \in \mathcal{X}^*\}$.

Choose $J = \{1, 3, 5, ..., 2k-1, ...\} \subset \mathbb{N}$. Then, there exists a reconstruction operator U_o such that $\Omega_o \equiv (\{z_k\}_{k\notin J}, U_o)$ is a retro Banach frame for $\mathcal{X}^*$. Hence $e(\Omega)$ is infinite.

Next, we characterize the finite excess of a retro Banach frame.

Theorem 2.1. *Let $\mathcal{F} \equiv (\{x_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_d$. Then $\mathcal{F}$ has finite excess if and only if for every infinite subset $J \subset \mathbb{N}$, we have*

$$[x_k]_{k\notin J} \neq \mathcal{X}.$$

Proof. Suppose first that $e(\mathcal{F})$ is finite. Assume, on the contrary, that there exists an infinite subset $J_o \subset \mathbb{N}$ such that

$$[x_k]_{k \notin J_o} = \mathcal{X}.$$

Then, there exists a reconstruction operator Θ_o such that $(\{x_k\}_{k \notin J_o}, \Theta_o)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to the Banach space of scalar-valued sequences $\mathcal{Z}_{d_o} = \{\{f(x_k)\}_{k \notin J_o} : f \in \mathcal{X}^*\}$. But $|J_o|$ is infinite. Hence $e(\mathcal{F})$ is infinite, which is impossible. Thus, the forward part is proved.

To prove the converse part, assume, on the contrary, that $e(\mathcal{F})$ is not finite. Then, for some suitable choice of J, there exists a number $k_1 \in J$ such that $(\{x_k\}_{k \neq k_1}, \Theta_{k_1})$ is a retro Banach frame for $\mathcal{X}^*$ with respect to some associated Banach space $\mathcal{Z}_{d_1}$. Therefore, there exist positive constants A_1 and B_1 such that

$$A_1\|f\| \leq \|\{f(x_k)\}_{k \neq k_1}\|_{\mathcal{Z}_{d_1}} \leq B_1\|f\| \text{ for each } f \in \mathcal{X}^*. \tag{2.2}$$

By using lower frame inequality in (2.2), we have $[x_k]_{k \neq k_1} = \mathcal{X}$. Hence

$$x_{k_1} \in [x_k]_{k \neq k_1} = \mathcal{X}.$$

Therefore, we can find a positive integer $n_1 \geq k_1$ such that

$$\operatorname{dist}\left(x_{k_1}, [x_k]^{n_1}_{\substack{k=1\\k \neq k_1}}\right) < \frac{1}{2}.$$

Since $e(\mathcal{F})$ is not finite, there exist a number $k_2 \in J$ and a reconstruction operator Θ_{k_2} such that $(\{x_k\}_{k \notin \{1,\dots,n_1\} \cup \{k_2\}}, \Theta_{k_2})$ is a retro Banach frame for $\mathcal{X}^*$ with respect to some $\mathcal{Z}_{d_2}$. Thus, we can find positive constants A_2 and B_2 such that

$$A_2\|f\| \leq \|\{f(x_k)\}_{k \notin \{1,\dots,n_1\} \cup \{k_2\}}\|_{\mathcal{Z}_{d_2}} \leq B_2\|f\| \text{ for each } f \in \mathcal{X}^*. \tag{2.3}$$

By using lower frame inequality in (2.3), we obtain

$$x_{k_2} \in [x_k]_{\substack{k \geq n_1+1\\k \neq k_2}} = \mathcal{X}.$$

Also, there exists a positive integer $n_2 \geq k_2$ such that

$$\operatorname{dist}\left(x_{k_2}, [x_k]^{n_2}_{\substack{k=1\\k \neq k_1, k_2}}\right) < \frac{1}{2^2}, \quad \operatorname{dist}\left(x_{k_1}, [x_k]^{n_2}_{\substack{k=1\\k \neq k_1, k_2}}\right) < \frac{1}{2^2}.$$

Continuing this process, we obtain a sequence $\{k_j\} \subset J$ and $\{n_j\} \subset \mathbb{N}$ such that

$$\operatorname{dist}\left(x_{k_j}, [x_k]^{n_l}_{\substack{k=1\\k \notin \{k_1, k_2, \dots, k_l\}}}\right) < \frac{1}{2^l}, \ 1 \leq j \leq l, \ l \in \mathbb{N}. \tag{2.4}$$

By using (2.4), we get

$$x_{k_j} \in [x_k]_{k \notin J} = \mathcal{X},$$

which is a contradiction. Hence $e(\mathcal{F})$ must be finite. Theorem 2.1 is proved. □

To conclude the paper, we show that if a given retro Banach frame for $\mathfrak{X}^*$ has finite excess associated with a certain nested sequence, then we can construct a retro Banach frame with infinite excess.

Theorem 2.2. *Let $\mathcal{J} \equiv (\{x_k\}, \Theta)$ be a retro Banach frame for $\mathfrak{X}^*$ with respect to $\mathcal{Z}_d$, and let $\mathbb{J}_1 \subset \mathbb{J}_2 \subset \ldots \subset \mathbb{J}_t \subset \ldots$ be a nested sequence of subsets of $\mathbb{N}$, where each $\mathbb{J}_n$ is finite. Assume that for each n, $\mathcal{F}_n \equiv (\{x_k\}_{k \notin \mathbb{J}_n}, \Theta_n)$ is a retro Banach frame for $\mathfrak{X}^*$ with respect to $\mathcal{Z}_{d_n}$, that is, for each n, $\mathrm{e}(\mathcal{F}_n)$ is finite. Then, there exist an infinite subset $J \subset \mathbb{N}$ and a reconstruction operator Θ such that $\mathcal{F} = (\{x_k\}_{k \notin J}, \Theta_n)$ is a retro Banach frame for $\mathfrak{X}^*$, and hence $\mathrm{e}(\mathcal{J})$ is infinite.*

Proof. Without loss of generality, let us write $\mathbb{J}_t = \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$. By hypothesis, $\mathcal{F}_1 \equiv (\{x_k\}_{k \notin \mathbb{J}_1}, \Theta_1)$ is a retro Banach frame for $\mathfrak{X}^*$ with respect to $\mathcal{Z}_{d_n}$. Hence, there exist finite positive constants a_1 and b_1 such that

$$a_1\|f\| \leq \|\{f(x_k\}_{k \neq 1}\|_{\mathcal{Z}_{d_1}} \leq b_1\|f\| \text{ for each } f \in \mathfrak{X}^*. \tag{2.5}$$

By using lower inequality in (2.5), we can find a positive integer $k_2 > k_1 = 1$ such that

$$\text{dist}\left(x_{k_1}, \left[\{x_k\}_{k \notin \bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=2}^{k_2-1}\right]\right) < \frac{1}{2}.$$

Again, by hypothesis, there exists a reconstruction operator Θ_{k_2} such that $\mathcal{F}_{k_2} \equiv (\{x_k\}_{k \notin \mathbb{J}_{k_2}}, \Theta_{k_2})$ is a retro Banach frame for $\mathfrak{X}^*$. By using lower frame inequality for $\mathcal{F}_{k_2}$, we can find a positive integer k_3 such that

$$\text{dist}\left(x_{k_1}, \left[\{x_k\}_{k \notin \bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_2+1}^{k_3-1}\right]\right) < \frac{1}{3}$$

and

$$\text{dist}\left(x_{k_2}, \left[\{x_k\}_{k \notin \bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_2+1}^{k_3-1}\right]\right) < \frac{1}{3}.$$

By induction, we can find a monotone increasing sequence of positive integers $\{k_j\}$ such that

$$\text{dist}\left(x_{k_j}, \left[\{x_k\}_{k \notin \bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_\ell+1}^{k_{\ell+1}-1}\right]\right) < \frac{1}{\ell+1},\ j = 1, 2.., \ell,\ (\ell \in \mathbb{N}). \tag{2.6}$$

Choose $\mathbb{I} = \{k_1, k_2, k_3, \ldots\}$. Then, since $\{x_k\}_{k \notin \bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_\ell+1}^{k_{\ell+1}-1} \subset \{x_k\}_{k \notin \mathbb{I}}$, by using (2.6), we obtain

$$\text{dist}(x_{k_j}, \{x_k\}_{k \notin \mathbb{I}}) \leq \text{dist}\left(x_{k_j}, \left[\{x_k\}_{k \notin \bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_\ell+1}^{k_{\ell+1}-1}\right]\right) < \frac{1}{\ell+1} \text{ for all } \ell \geq j.$$

Therefore, $\mathcal{Z}_\infty = \{\{f(x_k)\}_{k \notin \mathbb{I}} : f \in \mathfrak{X}^*\}$ is a Banach space of scalar valued sequences with norm given by $\|\{f(x_k)\}_{k \notin \mathbb{I}}\|_{\mathcal{Z}_\infty} = \|f\|_{\mathfrak{X}^*}$, $f \in \mathfrak{X}^*$. Define Θ_∞ :

$\mathcal{Z}_\infty \to \mathcal{X}^*$ by $\Theta_\infty(\{f(x_k)\}_{k\in\mathbb{I}}) = f$, $f \in \mathcal{X}^*$, and observe that Θ_∞ is a bounded linear operator such that $\mathcal{F} = (\{x_k\}_{k\in\mathbb{I}}, \Theta_\infty)$ is a retro Banach frame for $\mathcal{X}^*$ with bounds $A = B = 1$. Since $|\mathbb{I}|$ is infinite, $e(\mathcal{F})$ is also infinite. □

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