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SWW SEQUENCES AND THE INFINITE ERGODIC RANDOM WALK

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Abstract. This article is concerned with demonstrating the power and simplicity of *sww* (special weakly wandering) sequences. We calculate an *sww* growth sequence for the infinite measure preserving random walk transformation. From this we obtain the first explicit *eww* (exhaustive weakly wandering) sequence for the transformation. The exhaustive property of the *eww* sequence is a "gift" from the *sww* sequence and requires no additional work. Indeed we know of no other method for finding explicit *eww* sequences for the random walk map or any other infinite ergodic transformation. The result follows from a detailed analysis of the proof of Theorem 3.3.12 in [1] as applied to the random walk transformation from which an *sww* growth sequence is obtained. We explain the significance of *sww* sequences in the construction of *eww* sequences.

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1. Introduction

Every nonsingular invertible transformation T of a Lebesgue space $(X, \mathcal{B}, m)$ with no finite invariant measure equivalent with m has exhaustive weakly wandering sequences (defined below). However, for most transformations no explicit exhaustive weakly wandering sequences are known, and in particular given a specific T, it is not clear how to find an explicit exhaustive weakly wandering sequence for it.

In this article, we examine the infinite measure preserving random walk transformation (see next section) and derive an *sww* growth sequence for it (defined below). Using this we prove that the integer sequence $\{16^{i+4} : i = 1, 2, 3, \ldots\}$, and every infinite subsequence of it, is an exhaustive weakly wandering sequence for the random walk transformation. The method employed here is general enough that it applies to a wide range of other maps.

For the sake of completeness and clarity of exposition, we repeat some results that were presented in [1].

1.1. **History.** The definition of mixing in ergodic theory, for a transformation T preserving a probability measure μ is

$$\lim_{n\to\infty} \mu(T^n A \cap B) = \mu(A)\mu(B) \tag{1.1}$$

for all measurable sets A and B. There have been several attempts to extend this definition of mixing to ergodic transformations that preserve a σ-finite infinite measure. The first attempt in this direction was by Eberhard Hopf when in 1937 in his famous book Ergodentheorie [2] he devoted to it, section 17 titled "Ein Beispiel für Mischung bei unendlichem $m(\Omega)$". His goal was to extend the notion of mixing for finite measure preserving transformations to infinite measure preserving transformations. He presented a slight variation of the random walk transformation on the integers. His example started with the classic random walk on the nonnegative integers: for $n > 0$, $n \to \{n-1, n+1\}$ with probability $\{\frac{1}{2}, \frac{1}{2}\}$ and $0 \to \{0, 1\}$ with probability $\{\frac{1}{2}, \frac{1}{2}\}$. This he considered as a map of the infinite strip $[0,\infty) \times [0,1)$ which preserved the infinite Lebesgue measure. Being in an infinite measure space he replaced equation (1.1) with a ratio version. However he was only able to prove (equation 17.1 in [2])

$$\frac{m(A \cap T^n B)}{m(C \cap T^n D)} \to \frac{m(A)m(B)}{m(C)m(D)}, \quad n \to \infty \tag{1.2}$$

for Jordan measurable sets of finite measure with $m(C)m(D) \neq 0$. Then he concluded that if the above were shown to be true for all measurable sets of finite measure then "metric transitivity"(that is, ergodicity) of T would follow. He then ended the section with "Dieser Beweis verlagt jedoch tiefere Hilfsmittel."

Now we know this cannot be done. In 1964, Hajian and Kakutani [3] defined weakly wandering sets and showed that all infinite measure preserving ergodic transformations possess weakly wandering sets. These are sets with an infinite number of mutually disjoint images under the transformation T. Replacing the sets C and D with the same weakly wandering set in equation (1.2) shows the convergence fails. Further historical details and attempts at defining mixing in infinite measure spaces can be found in Lenci [4].

In what follows we do more. We discuss the random walk transformation T on the integers and exhibit some properties of it that show how far T is from possessing any type of "mixing"feature. We do this by showing the existence and construction of specific *eww* sequences (defined below) that T possesses. We also exhibit some number theoretic properties of these sequences.

1.2. **Definitions and Preliminaries.** We consider transformations T that are invertible onto maps defined on a σ-finite Lebesgue measure space $(X, \mathcal{B}, m)$. As usual, all statements are to be understood as "modulo sets of measure zero and all sets will be measurable by assumption or construction. We assume all the transformations T we consider are measurable $[A \in \mathcal{B} \iff TA \in \mathcal{B}]$, and non-singular $[m(A) = 0 \iff m(TA) = 0]$. We say T is a measure preserving transformation if $m(TA) = m(A)$ for all $A \in \mathcal{B}$. Two measures m and μ defined on the same measurable space $(X, \mathcal{B})$ are equivalent ($m \sim \mu$) if m and μ have the same sets of measure zero. There are many equivalent definitions of ergodicity. We use the following.

- T is *ergodic* if $TA = A$ implies $m(A) = 0$ or $m(X \setminus A) = 0$.

An ergodic transformation T is an *infinite ergodic* transformation if it is a measure preserving transformation defined on the infinite measure space $(X, \mathcal{B}, m)$.

Following [1], we consider the following infinite sequences of integers $\{n_i\}$ associated to an infinite ergodic transformation T.

Definition 1.1.

- $\{n_i\}$ is a *weakly wandering* (*ww*) sequence for T if for some set A of positive measure $T^{n_i}A \cap T^{n_j}A = \emptyset$ for $i \neq j$.
- $\{n_i\}$ is an *exhaustive weakly wandering (eww)* sequence for T if for some set A of positive measure $X = \bigcup_{i=0}^{\infty} T^{n_i}A(disj)$.
- $\{n_i\}$ is a *special* (or at times called *strongly) weakly wandering (sww)* sequence for T if there exists a set A of positive measure such that for $i, j, k, l \geq 0$ and $i > j$ we have $T^{n_i - n_k + k'}A \cap T^{n_j - n_l + l'}A = \emptyset$ whenever one of the indices $\{i, j, k, l\}$ is larger than all the others or $i = l > k$.
- We call the set A above, a *ww, eww*, or *sww set* respectively (for T, with the sequence $\{n_i\}$), and at times we say $\{n_i\}$ is a *ww, eww*, or *sww sequence* (for T with the set A).

The definition of *ww* sequences first appeared in [3] where it was shown that they exist for every infinite ergodic transformation. There are many examples of infinite ergodic transformations in the literature; however, for almost any example, it has not been possible to exhibit a specific *ww* sequence for the transformation. There is one notable exception: the infinite ergodic example T in [5] which was constructed for the purpose of exhibiting an explicit *ww* sequence for it. In that example it was noticed that the constructed *ww* sequence happened to be an *eww* sequence. Except for the transformation T in [5] and some similar ones, it is not that easy to construct specific *ww* sequences for any of the known infinite ergodic transformations — though we know they must exist. However, to our knowledge, it is practically impossible to construct *eww* sequences for any of those transformations. The construction of *ww* sequences entails showing for some sequence and some set W the mutually disjoint images of the set W. For *eww* sequences on the other hand one needs to show further that the mutually disjoint images of the set W fill up the whole space X.

In [6], Jones and Krengel present a proof that *eww* sequences exist for all infinite ergodic transformations. In outline, their proof is a complicated back-and-forth induction existence proof. They build their sequence one integer at a time while simultaneously adjusting their set. The set is built up in a two step process. At each step they must *take a bit away* from the set so that it will be disjoint for the next integer and then they have to *add a bit back* in order to build up the set to be exhaustive. As a practical matter, no one to date has been able to use this method to construct an actual *eww* sequence for any transformation.

To overcome this difficulty *sww* sequences were introduced in [1]. The definition of an *sww* sequence appears to be more complicated than that of an *eww* sequence. However it is designed in such a way that it can be easily applied. The construction of *sww* sequences is similar to the construction of *ww* sequences. By this we mean that both sequences are concerned only with the construction of a set A whose images under the sequence are mutually disjoint, and this is relatively easy. Once the set A is constructed in the case of an *sww* sequence, a second easily performed automatic construction produces the *derived* set W. For ergodic transformations the fact that the mutually disjoint images of the derived set W are exhaustive follows from the definitions.

In addition, *sww* sequences give a lot more. When the transformation is ergodic, not only is the *sww* sequence an *eww* sequence for the associated derived set, but every infinite subsequence of it is again an *eww* sequence with a similarly defined derived set W. This hereditary property follows from the definitions of both *ww* and *sww* sequences but not for *eww* sequences. That is, if the images of a set A are mutually disjoint under a sequence, they are still mutually disjoint for any infinite subsequence of it; but the set may not be exhaustive for the same sequence. For example, the *eww* sequence given in [5] has many infinite subsequences which are not *sww* (just remove any single non-zero integer from the sequence).

1.3. The Derived Set. To clarify the comments made above, and to make this article self contained, we make some general observations and discuss a few results that are covered in [1] and will be used in the sequel.

For a sequence of integers $\{n_i : i > 0\}$ and any set A with $m(A) > 0$,

let $n_0 = 0$, $A_0 = A$, and $W_0 = T^{-n_0}A_0$,

$A_1 = TA \setminus \bigcup_{r=0}^{\infty} T^{n_r}W_0$, and $W_1 = \bigcup_{i=0}^{1} T^{-n_i}A_i$,

and in general for $p \geq 2$

$$A_p = T^pA \setminus \bigcup_{r=0}^{\infty} T^{n_r}W_{p-1}, \text{ and } W_p = \bigcup_{i=0}^{p} T^{-n_i}A_i. \tag{1.3}$$

Let us call the set $W = \bigcup_{p=0}^{\infty} W_p$ the *derived* set from the set A and the sequence $\{n_i\}$. Then for any $p > 0$ we have

$\bigcup_{r=0}^{\infty} T^{n_r}W \supset T^pA \cup \bigcup_{r=0}^{\infty} T^{n_r}W_{p-1}$ which implies $\bigcup_{r=0}^{\infty} T^{n_r}W \supset \bigcup_{p=0}^{\infty} T^pA$.

From the above we conclude with the following remark:

Remark 1.1. Let W be the derived set from the set A and the sequence $\{n_i\}$. If $\{n_i\}$ is an *sww* sequence then

$$\bigcup_{i=0}^{\infty} T^{n_i}W(\text{disj}) \supset \bigcup_{p=0}^{\infty} T^pA. \tag{1.4}$$

To show (1.4) it is enough to show $T^{n_i}W \cap T^{n_j}W = \emptyset$ for $i, j \geq 0$ and $i > j$. For this it is sufficient to show that

$$T^{n_i-n_k}A_k \cap T^{n_j-n_l}A_l = \emptyset \quad \text{for} \quad i,j,k,l \geq 0, \quad \textit{and} \quad i > j. \tag{1.5}$$

It is clear from (1.3) that for any integer $r > 0$

$$A_p \cap T^{n_r-n_s}A_s = \emptyset \quad \text{if} \quad p > s. \tag{1.6}$$

If $i = k > \max\{j, l\}$ then (1.5) follows from (1.6). In all the other cases we note that $A_p \subset T^pA$ for all $p \geq 0$, and (1.5) then follows from the properties defining the *sww* sequence $\{n_i\}$.

For the next theorem we define the following:

Definition 1.2. Let $\{0 < N_1 < N_2 < \cdots\}$ be an increasing sequence of positive integers. Then, for any increasing sequence of positive integers $\{0 = n_0 < n_1 < n_2 < \cdots\}$

(I) If $n_i - n_{i-1} \geq N_i$ for all $i \geq 1 \implies \{n_i\}$ is a *ww* sequence for T then $\{N_i\}$ is a *ww growth sequence* for T,
and

(II) If $n_i - 2n_{i-1} \geq N_i$ for all $i \geq 1 \implies \{n_i\}$ is an *sww* sequence for T then $\{N_i\}$ is an *sww growth sequence* for T.

Theorem 1.1. *Let T be a measurable and nonsingular transformation defined on $(X, \mathcal{B}, m)$, and suppose there is a set A of positive measure satisfying $\lim_{n\to\infty} m(T^n A \cap A) = 0$. Then there exists an increasing sequence of positive integers $\{N_i\}$ which is both a ww and an sww growth sequence for T.*

Proof. The proof of the Theorem is contained in detail in [1]. Here we sketch a proof and show the similarity of the role of *ww* and *sww* sequences in constructing the *ww* and *sww* set A_0 for each. Later, we apply this construction to find an explicit *sww* growth sequence for the random walk transformation on the integers.

Let A be a set of positive measure with $m(A) < \infty$, and suppose

$$\lim_{n\to\infty} m(T^n A \cap A) = 0. \tag{1.7}$$

For positive $\epsilon < m(A)$, and for $i \geq 1$ let $\epsilon_i = \dfrac{\epsilon}{2(2i+1)i^3 2^i}$.

Using (1.7) we choose an increasing sequence of positive integers $\{0 < N_1' < N_2' < \cdots\}$ such that for each $i \geq 1$, $m(T^n A \cap A) \leq \epsilon_i$ for all $n \geq N_i'$. We let $N_i = N_i' + i$ for $i \geq 1$.

To show $\{N_i\}$ is a *ww* growth sequence we let $\{0 = n_0 < n_1 < n_2 < \cdots\}$ be any increasing sequence of integers satisfying $n_i - n_{i-1} \geq N_i$ for $i \geq 1$.

(1.8) For $i > 0$ and $0 \leq j < i$ we have $n_i - n_j \geq n_i - n_{i-1} \geq N_i$.

Next we let,

$$A' = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} T^{n_i - n_j} A \cap A.$$

It is not too difficult to show that $m(A') \leq \epsilon$ and the set $A_0 = A \setminus A'$ is a *ww* set with the sequence $\{n_i\}$.

To show $\{N_i\}$ is an *sww* growth sequence we let $\{0 = n_0 < n_1 < n_2 < \cdots\}$ be any increasing sequence of integers satisfying $n_i - 2n_{i-1} \geq N_i$ for $i \geq 1$. For each

$i \geq 1$ we consider the set of integers $S_i = \{s : s = an_i + bn_j + cn_k + dn_l + e\}$ where $a \in \{1,2\}$, $b,c,d \in \{0,\pm 1\}$, $e \in \{0,\pm 1,\pm 2,\ldots,\pm i\}$, $0 \leq j,k,l < i$.

In the sets S_i we also require that at most two of the numbers b,c,d be negative. Then the cardinality of S_i, $|S_i| \leq 2i^3 3^3(2i+1)$. Since $\{n_i : i \geq 0\}$ is an increasing sequence of positive integers we have for $s \in S_i$

$s = an_i + bn_j + cn_k + dn_l + e \geq n_i - 2n_{i-1} - i \geq N_i - i = N_i'$.

Similarly as before we let

$$A' = \bigcup_{i=1}^{\infty} \bigcup_{s \in S_i} T^s A \cap A.$$

Again, it is not difficult to show that $m(A') \leq \epsilon$ and the set $A_0 = A \setminus A'$ is an *sww* set with the sequence $\{n_i\}$. □

Remark 1.2. For an ergodic transformation T the following is an immediate consequence of the definition:

$$m(A) > 0 \Rightarrow \bigcup_{p=0}^{\infty} T^p A = X. \tag{1.9}$$

Then (1.4) in Remark 1.1 together with (1.9) above imply that all *sww* sequences for an ergodic transformation are *eww* sequences for T.

Next for an ergodic transformation T we extend the definition of *ww* and *sww* growth sequences to *eww growth sequences* for T.

(III) An increasing sequence $\{0 < N_1 < N_2 < \cdots\}$ of positive integers is an *eww growth sequence* for an ergodic transformation T if any increasing sequence $\{0 = n_0 < n_1 < n_2 < \cdots\}$ of positive integers that satisfies $n_i - 2n_{i-1} \geq N_i$ for $i \geq 1 \implies \{n_i\}$ is an *eww* sequence for T.

Then for ergodic transformations every *sww* growth sequence is an *eww* growth sequence. Finally we conclude with the following Corollary to Theorem 1.1.

Corollary 1.1. *Every infinite ergodic transformation T that posesses a set A of positive measure with $\lim_{n\to\infty} m(T^n A \cap A) = 0$ has eww growth sequences.*

2. Infinite Measure Preserving Random Walk on the Integers

2.1. Random Walk on the Integers. We begin, as did Hopf (page 61 of [2]), with the Baker's transformation S defined on the unit square $Z = \{(x,y) : 0 \leq x < 1,\ 0 \leq y < 1\}$:

$$S(x,y) = \begin{cases} (2x, y/2) & \text{if } 0 \leq x < \frac{1}{2}, \\ (2x-1, (y+1)/2) & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

The map S is obviously a finite (probability) measure preserving invertible transformation. Hopf [2] proved it was mixing in the sense that (1.1) holds for all measurable sets. The proof begins by analyzing how S operates on dyadic rectangles. It is then a standard approximation argument to extend the mixing result from dyadic rectangles to all measurable sets. From this mixing property it follows that the map is ergodic (i.e. metrically transitive). There are now multiple proofs of the ergodicity of the Baker's map and in fact a lot more is known. For example it is well-known to be Bernoulli. We extend the transformation S to the two-sided infinite strip $\{(x,y) : -\infty < x < \infty,\ 0 \leq y < 1\}$ by a skew product construction as follows. Identify each square $\{(x,y) : n \leq x < n+1,\ 0 \leq y < 1\}$ as (Z,n). The infinite strip $(-\infty,\infty)\times[0,1)$ with area measure is the space $Z\times\mathbf{Z} = \bigcup_{n\in\mathbf{Z}}(Z,n)$ with the measure which is the product of the Lebesgue area measure on Z and the counting measure on the integers. Consider the skewing function $\phi : Z \to \{-1,1\}$ defined by $\phi(x,y) = -1$ if $0 \leq x < 1/2$ and $\phi(x,y) = 1$ if $1/2 \leq x < 1$. The random walk transformation on $Z \times \mathbf{Z}$ is (see p. 62–63, [1])

$$T((x,y),n) = (S(x,y), n + \phi(x,y))$$

We refer to this map T as infinite measure preserving random walk on $\mathbf{Z}$. This example is a variation of Hopf's example on the one sided infinite strip $[0,\infty) \times [0,1)$.

Theorem 2.1. *The infinite measure preserving random walk transformation is ergodic.*

Although Hopf never completed the proof of the ergodicity of the random walk transformation on the non-negative integers its ergodicity and that of the random walk on the integers T are now well known (see [7] and [4]). An elementary proof of the ergodicity of T can be given by examining the induced transformation on $(Z,0)$ and recognizing it as a finite measure preserving Bernoulli map (similar to recognizing the Baker's map as Bernoulli). More precisely, the induced map on every square $Z \times \{n\}$ is a Bernoulli map, and each square $Z \times \{n\}$ can be mapped to any portion of every other square.

2.2. An sww sequence for Random Walk. We are now in a position to derive an explicit *sww* growth sequence for T and we emphasize how simple and short it is once one has the *sww* definition. Specifically we duplicate the steps of the proof given in Theorem 1.1 to the random walk transformation T.

The necessary inequalities used in calculating the *sww growth* sequence come from the next lemma.

Lemma 2.1. *For the infinite measure preserving random walk transformation T described above, the set $(Z,0)$ satisfies the inequality*

$$\frac{1}{\sqrt{5k}} < m\left(T^{2k}(Z,0)\cap(Z,0)\right) < \frac{1}{\sqrt{2k}} \quad \textit{for all } k>0.$$

Proof. For an odd integer $n>0$ we have $m(T^n(Z,0)\cap(Z,0)) = 0$, and for an even integer $n=2k$, $k\geq 1$ we have:

$$m\left(T^{2k}(Z,0)\cap(Z,0)\right) = \binom{2k}{k}\frac{1}{2^{2k}} = \frac{(2k)!}{k!k!2^{2k}} = \frac{k!2^k(2k-1)!!}{k!2^k(2k)!!} = \frac{(2k-1)!!}{(2k)!!}.$$

Using induction it is easy to show:

$$\frac{(2k-1)!!}{(2k)!!} < \frac{1}{\sqrt{2k+1}} \quad \text{for } k\geq 1.$$

It is also easy to show:

$$\frac{1}{\sqrt{4k+1}} < \frac{(2k-1)!!}{(2k)!!} \quad \text{for } k\geq 1.$$

Combining the above we get:

$$\frac{1}{\sqrt{5k}} < \frac{1}{\sqrt{4k+1}} < m\left(T^{2k}(Z,0)\cap(Z,0)\right) < \frac{1}{\sqrt{2k+1}} < \frac{1}{\sqrt{2k}} \quad \text{for } k\geq 1.$$

□

We use the lemma above to get an *sww* growth sequence for the random walk transformation. This will also be an *sww* and an *eww* sequence.

Theorem 2.2. *The sequence $\{N_i = 16^{i+4} : i\geq 1\}$ is both an sww growth sequence and an eww sequence for T the infinite measure preserving random walk transformation.*

Proof. For the random walk transformation T we showed in Lemma 2.1

$$m(T^{2k}(Z,0)\cap(Z,0)) < \frac{1}{\sqrt{2k}}.$$

Therefore specializing the part of the proof given after the statement of Theorem 1.1 to the random walk T, the set $A=(Z,0)$ and $\epsilon = 1/2$, we can choose N_i' so that for all $n\geq N_i'$

$$m(T^{2n}(Z,0)\cap(Z,0)) < \frac{1}{\sqrt{2N_i'}} \leq \frac{1}{4(2i+1)i^3 2^i}.$$

From this we conclude that $N_i' \geq 8(2i+1)i^6 4^i$ and we have the growth sequence $8(2i+1)i^6 4^i + i$. This can be "neatened" to the growth sequence $16(2i+1)i^6 4^i$ which can be bounded by

$$N_i = 4^{2i+8} = 16^{i+3},\ i\geq 1.$$

This is also an *sww* growth sequence.

Clearly this implies that the sequence 16^{i+4} is also a growth sequence.

Since T is ergodic, we use Condition II of Definition 1.2 comparing $\{16^{i+4}\}$ to the previous growth sequence $\{16^{i+3}\}$ obtaining

$$16^{i+4} - 2 \cdot 16^{(i-1)+4} = (16-2) \cdot 16^{i+3}$$

which shows that $\{16^{i+4}\}$ is also an *eww* sequence for the random walk transformation. □

3. Application to Tilings of the Integers

As a special case consider the integers $\mathbf{Z}$ with the counting measure μ and denote the translation transformation $T : (\mathbf{Z}, \mu) \to (\mathbf{Z}, \mu)$, $T(n) = n + 1$. This is an ergodic, infinite measure preserving, invertible transformation, albeit with an atomic measure, and we can consider the analog of Theorem 1.1 for this map.

First we note that an infinite subset of integers $\{n_i : i \geq 1\}$ (denoted simply by $\{n_i\}$) is weakly wandering for T in this context means there exists another subset $\{m_j\}$ of the integers such that

$$\{n_i\} + \{m_j\} = \{n_i\} \oplus \{m_j\}$$

By this it is meant that the sum is *direct*, $n_i + m_i = n_j + m_j$ if and only if $n_i = n_j$ and $m_i = m_j$.

Further, to say that $\{n_i\}$ is *eww* means there exists $\{m_j\}$ which is direct with $\{n_i\}$ and the sum contains all integers, i.e., $\{n_i\} \oplus \{m_j\} = \mathbf{Z}$. This says that $\{n_i\}$ *tiles* the integers $\mathbf{Z}$ and we call $\{n_i\}$ a *tile*.

The case when $\{n_i\}$ (or $\{m_j\}$) is finite is a very active area of research with many open questions. This finite case has been studied using a wide range of techniques including cyclotomic polynomials, fourier analysis and the theory of finite cyclic groups. None of these methods however apply in the case when both $\{n_i\}$ and $\{m_j\}$ are infinite. This is the situation in which we are interested in obtaining an analog of Theorem 1.1.

In [1] it is shown that the following provides an analog of part II of Theorem 1.1 and replaces the eww growth condition by a limit.

Theorem 3.1. *Any infinite sequence $\{n_i\} = \{n_0 = 0 < n_1 < n_2 \cdots\}$ of nonnegative integers satisfying $\lim_{i\to\infty} n_i - 2n_{i-1} = \infty$ tiles the integers.*

A surprising consequence of this, which emphasizes the difference between finite and infinite tiles, is that such an infinite tile has the hereditary property that any finite set of non-zero integers can be removed and the resulting sequence still tiles the integers. This is not true for finite tiles and is not true for all infinite tiles.

This theorem was first proved using ergodic theory techniques for the translation transformation T [8], but J. Schmerl (private communication) gave a strictly combinatorial proof which appears in [1].

Note that, the analogous *ww* growth condition for part I of Theorem 1.1 is not true: There exist sequences of integers $\{n_i\}$ which satisfy $\lim(n_i - n_{i-1}) = \infty$, yet there is no infinite subset $\{m_i\}$ with which $\{n_i\}$ is direct let alone tiles the integers.

4. Questions

In this section we gather a few questions about the random walk transformation.

Question 1. The *eww* sequence obtained in Theorem 2.2 has the derived set W as an *eww* set associated with it. Is the measure of W infinite or finite?

Question 2. Transformations can have many different *eww* sequences and sets. Does the random walk transformation T have another *eww* sequence whose *eww* set has finite measure?

Question 3. If S is a nonsingular transformation which commutes with the random walk T is it measure preserving?

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JOSEPH MECKE'S LAST FRAGMENTARY MANUSCRIPTS - A COMPILATION

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Abstract. Summarising results from Joseph Mecke's last fragmentary manuscripts, the generating function and the Laplace transform for nonnegative random variables are considered. The concept of thickening of a random variable, as an inverse operation to thinning (which is usually applied to point processes) is introduced, based on generating functions, and a characterisation of thickable random variables is given. Further, some new relations between exponential distributions and their interpretation in terms of Poisson point processes are derived with the help of the Laplace transform.

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1. Introduction

Joseph Mecke passed away in February 2014, a few days after his 76th birthday. Until his last days, he was dealing with mathematical problems, and he wrote fragments of manuscripts, saved on his computer. His brother, Norbert Mecke, was able to identify the corresponding files; he handed them over to the authors of the present paper, in order to see whether some of the material can be published. The present paper is the result of this compilation.

As emphasized in the introduction of [8], Joseph Mecke preferred to work deep into problems in order to reach a clear insight and a maximum of mathematical elegance. After his last paper published in a journal [6], he formulated several new ideas and a wider working agenda. The fragments compiled here date from July 2011 to December 2011 and then from February 2013 to June 2013.

Joseph Mecke made outstanding contributions to the theory of point processes, mainly in the 1960s and early 1970s. Nowadays the Campbell-Mecke formula (Mecke himself referred to it as the 'refined Campbell formula') and the Slivnyak-Mecke formula (see [12], referred to as the Mecke formula in [4]) are cited oftentimes. Since the late 1970s, Joseph Mecke worked in stochastic geometry, a field in which he applied the point process theory strikingly. Thus he contributed to a sound mathematical foundation of this field, proving rigorously quite a few new results.

At a first glance, the content of the present paper – involving generating functions and the Laplace transform for nonnegative random variables – seems to be far away from the main subjects of Joseph Mecke's work, described above. It is not so surprising, however, because in his earlier work he applied and appreciated these powerful tools. Although they appear only occasionally in his published proofs during a long career, this use often gave deeper insight into a problem.

We (the authors of the present paper) remember a situation in a seminar (in 2006) when we dealt with the length distribution of I-segments in planar STIT tessellations. We had found an expression for the density of this distribution which looked rather strange and we had no clue how to interpret it. Joseph Mecke immediately started his calculation (using the Laplace transform) and soon he revealed this 'mysterious' distribution as a mixture of exponential distributions. Meanwhile, much more is known about STIT tessellations, and there are other methods to prove the mentioned result. But Joseph Mecke opened a door – as he did it in many other cases.

Probably, the present paper will inspire other mathematicians to study and to generalize some of the problems which Joseph Mecke considered.

2. Nonnegative integer-valued random variables and generating functions

2.1. Generating function. We denote $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbf{N} = \mathbf{N}_0 \setminus \{0\}$, and $\mathbf{1}\{\cdot\}$ the indicator function which has the value 1, if the condition in braces is fulfilled and with value 0 otherwise.

Generating functions are widely used in mathematics and they play also an important role in probability theory. In this paper they are considered for nonnegative integer-valued random variables to introduce later the concept of thinning and thickening of a random variable. Let ζ be a discrete random variable taking values in $\mathbf{N}_0$ with distribution

$$\mathfrak{L}(\zeta) = \sum_{k=0}^{\infty} a_k \delta_k, \tag{2.1}$$

where $a_k \geq 0$, $\sum_{k=0}^{\infty} a_k = 1$ and δ_k the Dirac measure assigning mass 1 to k. The corresponding generating function $G : [0,1] \to [0,1]$ is defined by

$$G(x) = \mathbf{E}(x^\zeta) = \sum_{k=0}^{\infty} a_k x^k, \quad 0 \leq x \leq 1. \tag{2.2}$$

Note that in the following we will consider the series on the right-hand side also for general $x \in \mathbb{R}$ if it is defined.

Recall that a function $G : [0,1] \to [0,1]$ is a generating function of a nonnegative integer-valued random variable if and only if $G(1) = 1$, $\lim_{x \nearrow 1} G(x) = 1$, $G(0) \geq 0$ and all derivatives of G are finite and nonnegative on $[0,1)$ (see [3]). Furthermore, the uniqueness theorem conveys that two random variables have the same generating function if and only if they have identical distributions.

Examples:

(a) If ζ is almost surely (a.s.) constant, $P(\zeta = m) = 1$ for some $m \in \mathbb{N}_0$, then $G(x) = x^m$.

(b) If ζ has a two-point distribution, $\mathfrak{L}(\zeta) = (1-r)\delta_m + r\delta_n$, $m, n \in \mathbb{N}_0$, $r \in (0,1)$, then the generating function is $G(x) = (1-r)x^m + rx^n$.

(c) For a random variable ζ which is Poisson-distributed with parameter $\lambda > 0$ we have $G(x) = e^{\lambda(x-1)}$.

(d) If ζ has a binomial distribution with parameters $n \in \mathbb{N}$ and $r \in (0,1)$ the generating function is $G(x) = (1 - r + rx)^n$, which is the n-th power of a generating function of a Bernoulli random variable with parameter r.

(e) The generating function of a geometric random variable ζ with parameter r and distribution $\mathfrak{L}(\zeta) = \sum_{k=0}^{\infty}(1-r)^k r \delta_k$ is $G(x) = \frac{r}{1-x(1-r)}$. A negative binomial random variable with distribution $\mathfrak{L}(\zeta) = \sum_{k=0}^{\infty} \binom{-n}{k}(r-1)^k r^n \delta_k$ (parameters $r \in (0,1)$ and $n \in (0,\infty)$) has the generating function $G(x) = \left(\frac{r}{1-x(1-r)}\right)^n$ which is the n-th power of the generating function of a geometric random variable with parameter r.

2.2. Thinning and thickening. Thinning is an operation applied to point processes, see [2] and the references therein. Given a realization, for each single point it is decided (independently of the other points) whether it survives or not. If the survival probability is p for all points, and if ζ is the (finite) random number of points before thinning, then the distribution of the number of the thinned point process is described in Definition 2.1. In this definition, thinning is introduced for arbitrary nonnegative integer-valued random variables. And one can ask whether there is an inverse operation to thinning. So, given a nonnegative integer-valued random variable, can this be the result of thinning of a 'thicker' one, and if so, what is its distribution? This will be formalized in Definition 2.2 and studied in this section.

Definition 2.1. *Let $\alpha_1, \alpha_2, \ldots$ be independent and identically distributed random variables with the two-point distribution $\mathfrak{L}(\alpha_k) = (1-p)\delta_0 + p\delta_1$, $k \in \mathbb{N}$. For a nonnegative integer-valued random variable ζ the* thinning with parameter $p \in (0,1)$ *is defined as the random variable*

$$\mathfrak{D}_p\zeta = \sum_{k=1}^{\zeta} \alpha_k. \tag{2.3}$$

If ζ has the distribution given in (2.1), then the distribution of $\mathfrak{D}_p\zeta$ can be written as

$$\mathfrak{L}(\mathfrak{D}_p\zeta) = \sum_{k=0}^{\infty} a_k \left((1-p)\delta_0 + p\delta_1\right)^{*k}, \tag{2.4}$$

where $*$ denotes the convolution of measures. This means that $\mathfrak{L}(\mathfrak{D}_p\zeta)$ is the mixture of binomial distributions with parameters k and p, weighted with a_k, respectively.

Because $P(\mathfrak{D}_p\zeta = m) = \sum_{k=m}^{\infty} a_k \binom{k}{m} p^m (1-p)^{k-m}$, a straightforward calculation yields the generating function G_p of $\mathfrak{D}_p\zeta$,

$$G_p(x) = G(1 - p + px). \tag{2.5}$$

Examples.

(a) For an a.s. constant ζ, $P(\zeta = m) = 1$ for some $m \in \mathbb{N}_0$, the generating function of the thinning is $G_p(x) = (1 - p + px)^m$. The uniqueness theorem yields that $\mathfrak{D}_p\zeta$ has a binomial distribution with parameters m and p, which is obvious in this case.

(b) For ζ with a two-point distribution, $\mathfrak{L}(\zeta) = (1-r)\delta_m + r\delta_n$, the thinning $\mathfrak{D}_p\zeta$ is a mixture of two binomial distributions with weights $1-r$ and r and parameters m, n respectively, and p.

(c) For a Poisson-distributed ζ with parameter $\lambda > 0$ the generating function of its thinning is $G_p(x) = e^{\lambda p(x-1)}$, which confirms the well-known fact, that $\mathfrak{D}_p\zeta$ has again a Poisson-distribution with parameter $p\lambda$.

(d) If ζ has a binomial distribution with parameters n and r, then the thinning $\mathfrak{D}_p\zeta$ is again binomially distributed but with parameters n and pr.

(e) Also for geometric and negative binomial distributions thinning retaines the type of the distribution. In both cases the parameter r of ζ changes over to $q = \frac{r}{r+p(1-r)}$ of $\mathcal{D}_p\zeta$.

Now consider $G(x)$, represented by a series as in (2.2), for arbitrary $x \in \mathbb{R}$ if the value of this series is defined. For $0 < p < 1$ let us formally modify the function (2.5) to

(2.6) $$G_{\frac{1}{p}}(x) = G\left(1 - \frac{1}{p} + \frac{1}{p}x\right), \qquad \text{if this is defined for all } 0 \le x \le 1,$$

As a function of x, this is not necessarily a generating function of a random variable.

Definition 2.2. *Let ζ be a nonnegative integer-valued random variable with generating function G. We say that ζ is* p-thickable *for $0 < p < 1$, if the function $G_{\frac{1}{p}}(x) = G(1 - \frac{1}{p} + \frac{1}{p}x)$ is defined for all $x \in [0,1]$, and if it is the generating function of a nonnegative integer-valued random variable. Such a variable will be denoted by $\mathcal{D}_{\frac{1}{p}}\zeta$.*

A combination of formulas (2.5) and (2.6) yields that

$$\left(G_{\frac{1}{p}}\right)_p = (G_p)_{\frac{1}{p}} = G.$$

This means that thinning and thickening are somehow mutually inverse operations. But note, that $\left(G_{\frac{1}{p}}\right)_p = G$ is meaningful only for those p for which the distribution is p-thickable. The other equation, $(G_p)_{\frac{1}{p}} = G$, holds for all $0 < p < 1$. This confirms the meaning of thickening as the inverse operation of thinning.

First we investigate the nonnegative integer-valued random variables given in the examples above whether they are p-thickable or not. From the characterization of a generating function it follows that all the (right-hand side) derivatives at $x = 0$ are nonnegative. Hence if, for a fixed $p \in (0,1)$, the function $G_{\frac{1}{p}}$ given in (2.6) is the generating function of a nonnegative integer-valued random variable then

(2.7) $$G^{(\ell)}\left(1 - \frac{1}{p}\right) \ge 0 \quad \text{for all } \ell = 0,1,2,\ldots,$$

where $G^{(\ell)}$ denotes the ℓ-th derivative of G.

Examples:

(a) An a.s. constant ζ with $P(\zeta = m) = 1$ is p-thickable for all $0 < p < 1$ if $m = 0$, and it is not thickable for any $0 < p < 1$ if m is a positive integer. For $m = 0$ we have $G = G_p = G_{\frac{1}{p}} = 1$. In contrast, if $m > 0$, then $G(x) = x^m$ and hence $G_{\frac{1}{p}}(x) = G(1 - \frac{1}{p} + \frac{1}{p}x) = (1 - \frac{1}{p} + \frac{1}{p}x)^m$. If m is odd, then $G_{\frac{1}{p}}(x) < 0$

for $0 \leq x < 1 - p$. And, if m is even, then the first derivative at $x = 0$ is negative. This contradicts the necessary condition given in (2.7).

(b) Analogous considerations show that ζ with a two-point distribution is not thickable for any $0 < p < 1$.

(c) If ζ is Poisson-distributed with parameter $\lambda > 0$ then $G_{\frac{1}{p}}(x) = e^{\lambda \frac{1}{p}(x-1)}$ which yields that $\mathcal{D}_{\frac{1}{p}}\zeta$ has a Poisson-distribution with parameter $\frac{1}{p}\lambda$. Therefore the Poisson-distributions are p-thickable for all $0 < p < 1$.

(d) If ζ has a binomial distribution with parameters n and r, i.e. the generating function is $G(x) = (1-r+rx)^n$, then ζ is p-thickable if and only if $r \leq p < 1$. For $r > p$ the function $G_{\frac{1}{p}} = (1-r\frac{1}{p}+r\frac{1}{p}x)^n$ is no longer a generating function. This follows with the same argument for the derivative which was given for ζ a.s. constant. For $r \leq p < 1$ the p-thickening of ζ is again binomially distributed with parameters n and $\frac{r}{p}$. In particular, the r-thickening of ζ is the constant n.

(e) As in the case of thinning also thickening retains the type of geometric and negative binomial distributions. Thickening is possible for all $p \in (0,1)$ and the new parameter for $\mathcal{D}_{\frac{1}{p}}\zeta$ is $\frac{r}{r+\frac{1-r}{p}}$.

2.3. Characterization of unbounded thickability. In the examples above we have seen that some of the nonnegative integer-valued random variables are p-thickable for all $0 < p < 1$ and others only for some p.

Definition 2.3. *A nonnegative integer-valued random variable is called* unbounded thickable *if it is p-thickable for all* $p \in (0,1)$.

Random variables with a Poisson, a geometrical or a negative binomial distribution are unbounded thickable. A random variable with a binomial distribution is not unbounded thickable. In the following theorem the class of unbounded thickable random variables is described.

Theorem 2.1. *(Characterization of unbounded thickability)*

A nonnegative integer-valued random variable ζ with generating function G as in (2.2) is p-thickable for all $p \in (0,1)$ if and only if it has a Cox distribution (a mixture of Poisson distributions and the constant 0), i.e. if and only if there exists a probability measure Q on $[0,\infty)$ such that

$$G(x) = \int_{[0,\infty)} e^{t(x-1)} Q(dt), \quad 0 \leq x \leq 1. \tag{2.8}$$

Proof. If a function G satisfies (2.8), then obviously $G(1) = 1$, $G(0) \geq 0$ and $\lim_{x \nearrow 1} G(x) = 1$, because for all $0 \leq x \leq 1$, $t \geq 0$, the function $e^{t(x-1)}$ is monotone in x and $0 < e^{t(x-1)} \leq 1$. Furthermore, because for all $\ell = 0, 1, \dots$ the function $t^\ell e^{t(x-1)}$, $t \geq 0$ can be dominated on $(0, \infty)$ by a constant, we obtain that the derivatives $G^{(\ell)}(x) = \int t^\ell e^{t(x-1)} Q(dt) \geq 0$ and they are finite. Hence, G is indeed the generating function of a nonnegative integer-valued random variable. With analogous arguments it can be shown, that also $G_{\frac{1}{p}}(x) = G(1 - \frac{1}{p} + \frac{1}{p}x) = \int e^{\frac{t}{p}(x-1)} Q(dt)$ is the generating function of a nonnegative integer-valued random variable.

Now we show that (2.8) is necessary for unbounded thickability. If ζ is p-thickable for all $p \in (0, 1)$, then according to (2.7)

$$G^{(\ell)}(x) \geq 0 \quad \text{for all } x < 0, \ \ell = 0, 1, 2, \dots$$

For $s \geq 0$ we define $L(s) = G(1 - s)$, which implies

$$(-1)^\ell L(s)^{(\ell)}(s) \geq 0 \text{ for all } s > 0, \ \ell = 0, 1, 2, \dots$$

i.e. L is completely monotone on $(0, \infty)$. Furthermore, L is right-continuous at 0 (because the generating function G is left-continuous at 1) and $L(0) = G(1) = 1$. Hence the characterization theorem for Laplace transforms (also referred to as moment generating functions: see [3]) yields that L is the Laplace transform of a probability measure Q on the half-axis $[0, \infty)$, and hence

$$G(1 - s) = \int e^{-ts} Q(dt), \quad s \geq 0,$$

or, equivalently,

$$G(x) = \int e^{t(x-1)} Q(dt), \quad x \leq 1.$$

Examples: Referring to the examples above, special Cox distributions are:

(c) The Poisson distribution with parameter λ, and according to (2.8), $Q = \delta_\lambda$.

(e) The negative binomial distribution with parameters n and r, where Q is the gamma distribution with parameters n and $\frac{r}{1-r}$. In the particular case of a geometric distribution, we have $n = 1$ and hence Q is the exponential distribution with parameter $\frac{r}{1-r}$.

2.4. **Relations to point processes.** In [1], R.V. Ambartzumian introduced the concept of $1/p$-*condensation* of point processes ($p \in (0, 1]$) as the inverse operation to thinning. Hence condensation is also related to splitting of point processes. Moreover, he provided a sufficient condition for 2-condensability (which is related to 1/2-thickability considered in the present paper) of point processes in $\mathbb{R}^d$, $d \geq 1$. It remains an open problem to characterize the class of all point processes which are $1/p$-condensable if a value $p \in (0,1)$ is fixed. Recently, thinning, splitting and condensation were studied in [9, 10], and in [10] a generalized concept of thinning is introduced, both for nonnegative integer-valued random variables and for point processes.

In an early paper [5] (where thinning is named 'Auswürfelverfahren'), J. Mecke already proved that a point process Φ on the real axis $\mathbf{R}$ is a Cox process if and only if *for any* $p \in (0,1]$ exists a point process Φ_p, such that the p-thinning of Φ_p has the same distribution as Φ (Satz 4.2 ibidem). This means that a point process on the real axis is unbounded (i.e. for all $p \in (0,1]$) condensable if and only if it is a Cox process. This result immediately implies Theorem 2.1 of the present paper. But the proof given here is much shorter and more elegant than that one in [5]. And vice versa, with the help of the generating functional for point processes (see [7] or [2]), one can easily deduce Satz 4.2 in [5] from Theorem 2.1.

2.5. M-transform. The generating function G of a nonnegative integer-valued random variable ζ in (2.2) can also be interpreted as the cumulative distribution function (c.d.f.) of a probability measure concentrated on the interval $[0,1]$. Consequently, in this section we consider the problem how to find for a given ζ a random variable ξ whose c.d.f. F_ξ coincides with G on $[0,1]$. To avoid complications due to $P(\zeta = 0) > 0$, i.e. $a_0 > 0$, in this section only positive integer-valued random variables with values in $\mathbf{N}$ are considered.

Proposition 2.1. *Let $\eta_1, \eta_2, \ldots$ be i.i.d. random variables with uniform distribution on the interval* $(0,1)$ *and* ζ, *independent of this sequence, a positive integer-valued random variable with generating function G as in (2.2) with $a_0 = 0$. Then the random variable*

$$\xi := \max\{\eta_1, \ldots, \eta_\zeta\}$$

has the c.d.f. F_ξ with $F_\xi(x) = G(x)$ for all $x \in [0,1]$. .

Proof. Straightforward calculations yield for $0 \le x \le 1$

$$\begin{aligned} F_\xi(x) &= P(\xi \le x) = \sum_{k=1}^{\infty} P\big(\max\{\eta_1, \ldots, \eta_\zeta\} \le x | \zeta = k\big) \cdot P(\zeta = k) \\ &= \sum_{k=1}^{\infty} P\big(\max\{\eta_1, \ldots, \eta_k\} \le x\big) \cdot P(\zeta = k) = \sum_{k=1}^{\infty} x^k a_k. \end{aligned}$$

Under the assumptions of Proposition 2.1 the following transform of a positive integer-valued random variable can be specified.

As the Laplace transform depends only on the distribution of a random variable, we can also speak of the Laplace transform of a distribution. Since $1 - L_\zeta$ is continuous, nondecreasing with $1 - L_\zeta(0) = 0$ and $\lim_{s\to\infty} 1 - L_\zeta(s) = 1$, it can also be interpreted as the c.d.f. F_ξ of some nonnegative random variable ξ, i.e. $F_\xi = 1 - L_\zeta$. Equivalently, $L_\zeta = 1 - F_\xi$ is the survival function of ξ.

An open problem: Let be given a nonnegative random variable ζ and a sequence $\eta_1, \eta_2, \ldots$ of i.i.d. random variables, uniformly distributed on $(0, 1)$. Find a random variable (if it exists) $\xi = \xi(\zeta, \eta_1, \eta_2, \ldots)$ which transforms $\zeta, \eta_1, \eta_2, \ldots$ such that $F_\xi = 1 - L_\zeta$. And as for the M-transform in Section 2.5 we could ask for an inverse transform: For a given ξ find a random variable ζ with Laplace transform L_ζ equal to the survival function of ξ.

3.2. Laplace transform and generating function. Recall that for a nonnegative integer-valued random variable ζ with generating function G, the Laplace transform is $L_\zeta(s) = G(e^{-s})$ for all $s \geqslant 0$.

Now we consider an arbitrary nonnegative random variable.

Proposition 3.1. *Let ζ be a nonnegative random variable with Laplace transform L_ζ and define for all $t > 0$ the function $G_t : [0, 1] \to [0, 1]$ by*

$$G_t(x) = L_\zeta(t(1 - x)) \qquad \text{for all } x \in [0, 1].$$

(1) Then for all $t > 0$ the function G_t is the generating function of a nonnegative integer-valued random variable.

(2) If, for all $t > 0$, κ_t is a nonnegative integer-valued random variable with generating function G_t, then

$$\lim_{t\to\infty} L_{(\kappa_t/t)}(s) = L_\zeta(s),$$

which implies that for $t \to \infty$ the random variables κ_t/t converge in distribution to ζ.

Proof. As it can be seen in the proof of Theorem 2.1, G_t is the probability generating function of a nonnegative integer random variable, κ_t say. Now define the nonnegative random variable $\beta_t = \kappa_t/t$ which has the Laplace transform L_{β_t} with values

$$L_{\beta_t}(s) = L_{\kappa_t}\left(\frac{s}{t}\right) = G_t\left(e^{-\frac{s}{t}}\right) = L_\zeta\left(t\left(1 - e^{-\frac{s}{t}}\right)\right).$$

This yields

$$\lim_{t\to\infty} L_{\beta_t}(s) = \lim_{a\to 0} L_\zeta\left(\frac{1 - e^{-sa}}{a}\right) = L_\zeta(s).$$

An open problem is again the construction of the random variables κ_i as a transform of a given ζ.

3.3. Roots of survival functions. Let $\eta_1, \eta_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with c.d.f. F. As it is well-known, for $n \in \mathbb{N}$ the survival function of the random variable $\zeta := \min\{\eta_1, \ldots \eta_n\}$ is $1 - F_\zeta(x) = (1 - F(x))^n$ for all $x \geq 0$. This immediately yields for the survival function of η_1 that

$$1 - F = \sqrt[n]{1 - F_\zeta}. \tag{3.1}$$

How can a random variable η with c.d.f. F according to (3.1) be generated from a sequence $\zeta_1, \zeta_2, \ldots$ of i.i.d. copies of ζ?

Proposition 3.2. *Let $\zeta_1, \zeta_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with c.d.f. F_ζ and α a random variable, geometrically distributed with parameter $1/n$, $n \in \mathbb{N}$, and independent from the sequence. Further, define the sequence $\xi_1, \xi_2, \ldots$ of record times by*

$$\xi_1 = 1, \quad \xi_2 = \min\{k > \xi_1 : \zeta_k \geq \zeta_{\xi_1}\}, \quad \ldots \quad \xi_{m+1} = \min\{k > \xi_m : \zeta_k \geq \zeta_{\xi_m}\}, \quad \ldots$$

Then the random variable ζ_{ξ_α} has a c.d.f. F satisfying (3.1).

Proof. As it is well-known (see e.g. [11]), the process $\zeta_{\xi_1}, \zeta_{\xi_2}, \ldots$ of records can be represented as a Poisson point process on $[0, \infty)$.

Given F_ζ, define the measure μ on $[0, \infty)$ (with the Borel σ-algebra) by

$$\exp(-\mu([0, x))) = 1 - F_\zeta(x) \quad \text{for all } x > 0. \tag{3.2}$$

This measure can be interpreted as a failure measure for ζ. If F_ζ has the density f_ζ, then for $x > 0$ with $F_\zeta(x) < 1$, the failure rate of ζ is $\frac{\partial \mu([0,x))}{\partial x} = \frac{f_\zeta(x)}{1 - F_\zeta(x)}$. Note that μ is not necessarily a Radon measure.

Now let Ψ be a Poisson point process on the positive half-axis with intensity measure μ, and denote the ordered sequence of its points by $\beta_1 \leq \beta_2 \leq \ldots$ This implies $\mathfrak{L}(\beta_m) = \mathfrak{L}(\zeta_{\xi_m})$ for $m = 1, 2, \ldots$ Now consider the Poisson point process Ψ' generated from Ψ by independent thinning with the probability $1 - (1/n)$ for deleting a point from Ψ. Then Ψ' has the intensity measure $\mu' = (1/n)\mu$. Therefore, according to (3.2) its first point β_1' (in the ordered point set) has the c.d.f. satisfying (3.1). Furthermore, if α is geometrically distributed and independent from all the other random variables, we obtain that $\mathfrak{L}(\beta_1') = \mathfrak{L}(\zeta_{\xi_\alpha})$, which completes the proof. □

3.4. Relations between exponential distributions. It is well-known, that the minimum of finitely many independent and exponentially distributed random variables is exponentially distributed as well. Furthermore, the sum of n i.i.d. exponentially distributed random variables with parameter $\lambda > 0$ has an Erlang distribution, which is a special gamma distribution with parameters n and λ. In order to study the sum of not necessarily identically distributed random variables, we consider now particular convolutions of exponential distributions.

Some of the results have an interpretation concerning Poisson point processes. The intervals between the points of a homogeneous Poisson point process on the real axis are i.i.d. exponentially distributed.

Denote the exponential distribution with parameter $\lambda > 0$ by $\mathrm{E}[\lambda]$ and by $\mathrm{E}^{*k}[\lambda]$ its k-fold convolution, $k \in \mathbf{N}$.

Theorem 3.1. *For all* $0 < \lambda < \infty$ *and* $0 < p < 1$,

$$\mathrm{E}[p\lambda] = p\sum_{k=0}^{\infty}(1-p)^k \mathrm{E}^{*(k+1)}[\lambda]. \tag{3.3}$$

Proof. The proof is easy, using the Laplace transform $L(s) = \lambda/(\lambda + s)$, $s \geq 0$, for the exponential distribution with parameter $\lambda > 0$, and the fact that the Laplace transform of a k-fold convolution of a distribution is just the k-th power of the Laplace transform of the respective distribution.

This result has also an interesting interpretation in terms of Poisson point processes on the positive real axis. Let Φ be a homogeneous Poisson point process on $(0, \infty)$ with intensity λ. Then the coordinate of the first point of Φ has the exponential distribution $\mathrm{E}[\lambda]$, and the coordinate of the $(k+1)$-st point has the distribution $\mathrm{E}^{*(k+1)}[\lambda]$. Now consider the independent thinning of Φ where the points are deleted with probability $1-p$. This yields an homogeneous Poisson point process with intensity $p\,\lambda$. Thus the coordinate of the first point of the thinned point process has the distribution $\mathrm{E}[p\lambda]$. The probability that this first point of the thinned process (i.e. the first point which survived the independent thinning procedure) is the $(k+1)$-st point of Φ is $p(1-p)^k$. This is expressed by (3.3).

Decomposing the summands in (3.3) for $k \geq 1$ as

$$(1-p)^k \mathrm{E}^{*(k+1)}[\lambda] = (1-p)(1-p)^{k-1}\mathrm{E}[\lambda] * \mathrm{E}^{*(k)}[\lambda]$$

straightforwardly yields:

Corollary 3.1. *For all* $0 < \lambda < \infty$ *and* $0 < p < 1$

$$p\mathrm{E}[\lambda] + (1-p)(\mathrm{E}[\lambda] * \mathrm{E}[p\lambda]) = \mathrm{E}[p\lambda].$$

Substituting λ by λ_2 and p by λ_1/λ_2 for $0 < \lambda_1 < \lambda_2 < \infty$, this immediately supplies:

Corollary 3.2. *For all* $0 < \lambda_1 < \lambda_2 < \infty$

$$\mathrm{E}[\lambda_1] = \frac{\lambda_1}{\lambda_2}\mathrm{E}[\lambda_2] + \frac{\lambda_2 - \lambda_1}{\lambda_2}(\mathrm{E}[\lambda_1] * \mathrm{E}[\lambda_2])$$

or equivalently,

$$\mathrm{E}[\lambda_1] * \mathrm{E}[\lambda_2] = \frac{\lambda_2}{\lambda_2 - \lambda_1}\mathrm{E}[\lambda_1] - \frac{\lambda_1}{\lambda_2 - \lambda_1}\mathrm{E}[\lambda_2].$$

Now, we formulate the main result of this section for the convolution of two exponentially distributed random variables. Similarly as in Theorem 3.1 it is given as a mixture of Erlang distributions. Note that the two exponential distributions have different parameters.

Theorem 3.2. *For all* $0 < \lambda_1 < \lambda_2 < \infty$ *and* $p = \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}\right)^2$

$$\mathrm{E}[\lambda_1] * \mathrm{E}[\lambda_2] = (1-p)\sum_{k=0}^{\infty} p^k\, \mathrm{E}^{*2(k+1)}[\tfrac{1}{2}(\lambda_2 + \lambda_1)]. \tag{3.4}$$

Proof. Let L denote the Laplace transform of the distribution on the right-hand side of (3.4). Then, for $s \geq 0$ and $p = \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}\right)^2$,

$$\begin{aligned} L(s) &= \left(1 - \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}\right)^2\right)\sum_{k=0}^{\infty}\left(\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}\right)^{2k}\left(\frac{\frac{1}{2}(\lambda_2 + \lambda_1)}{\frac{1}{2}(\lambda_2 + \lambda_1) + s}\right)^{2k+2} \\ &= \left(1 - \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}\right)^2\right)\left(\frac{\lambda_2 + \lambda_1}{\lambda_2 + \lambda_1 + 2s}\right)^2 \frac{1}{1 - \frac{(\lambda_2 - \lambda_1)^2}{(\lambda_2 + \lambda_1 + 2s)^2}} \\ &= \frac{\lambda_1}{\lambda_1 + s}\cdot\frac{\lambda_2}{\lambda_2 + s}, \end{aligned}$$

and the term in the last line is just the product of the Laplace transforms of $\mathrm{E}[\lambda_1]$ and $\mathrm{E}[\lambda_2]$.

Alternatively, the result in Theorem 3.2 also follows from an iterated application of the equation given in the next corollary.

Corollary 3.3. *For all* $0 < \lambda_1 < \lambda_2 < \infty$ *and* $p = \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}\right)^2$

$$\mathrm{E}[\lambda_1] * \mathrm{E}[\lambda_2] = \mathrm{E}^{*2}[\tfrac{1}{2}(\lambda_1 + \lambda_2)] * \Big((1-p)\,\delta_0 + p\,(\mathrm{E}[\lambda_1] * \mathrm{E}[\lambda_2])\Big). \tag{3.5}$$

Again the proof is straightforward using the Laplace transforms.

Concluding remarks and acknowledgment. In Joseph Mecke's fragments almost no references are given. Therefore we cannot reconstruct and cite the sources which he probably used. Consequently, we do not claim priority concerning all details. We are indebted to Hans Zessin for his valuable comments and hints.

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A MOMENT CONDITION AND NON-SYNTHETIC DIAGONALIZABLE OPERATORS ON THE SPACE OF FUNCTIONS ANALYTIC ON THE UNIT DISK

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Abstract. Examples are given of (continuous, linear) operators on the space of functions analytic on the open unit disk in the complex plane having the monomials as eigenvectors, but which fail spectral synthesis (that is, which have closed invariant subspaces which are not the closed linear span of any collection of eigenvectors).

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1. Introduction

The main topic of this paper concerns invariant subspaces of a particular class of complete operators $T : \mathfrak{X} \to \mathfrak{X}$ acting on a complete metrizable vector space $\mathfrak{X}$ (recall that a subspace $\mathcal{M}$ of a complete metrizable vector space $\mathfrak{X}$ is *invariant* for an operator $T : \mathfrak{X} \to \mathfrak{X}$ if $Tx \in \mathcal{M}$ whenever $x \in \mathcal{M}$). Any complete operator has an abundance of invariant subspaces, namely the closed linear span of arbitrary collections of its eigenvectors. In fact, it may be tempting to believe that these are all of the invariant subspaces of a complete operator. However, this is not always the case, even when $\mathcal{H}$ is a Hilbert space having an orthonormal basis of eigenvectors for the operator (see Wolff's Example below). Any complete operator, all of whose invariant subspaces are the closed linear span of some collection of its eigenvectors, is said to **admit spectral synthesis.** The operators of study in this paper are the so-called *diagonal operators*, which by definition act on the space $\mathcal{H}(\mathbb{D})$ of functions analytic on the open unit disk in the complex plane and have as eigenvectors the monomials. The purpose of this paper is to produce a rich class of examples of diagonal operators on $\mathcal{H}(\mathbb{D})$ which fail spectral synthesis.

The problem of determining which complete operators admit spectral synthesis remains open, even when $\mathfrak{X}$ is a Hilbert space having an orthonormal basis of eigenvectors for T. In fact, it wasn't until 1921, with the advent of an example due to Wolff, that it was known that there existed examples of non-synthetic operators of this type. In particular, if $T : \mathcal{H} \to \mathcal{H}$ is an operator acting on a Hilbert space $\mathcal{H}$ having an orthonormal basis of eigenvectors for T with associated eigenvalues $\{\lambda_n\}$, then

$$T\left(\sum_{n=0}^{\infty} a_n e_n\right) \equiv \sum_{n=0}^{\infty} a_n \lambda_n e_n \tag{1.1}$$

for all $\sum_{n=0}^{\infty} a_n e_n$ in $\mathcal{H}$. Moreover, it is not difficult to see (p. 270 of [1]) that T fails spectral synthesis if and only if there exists a non-trivial sequence $\{w_n\} \in \ell^1$ for which the Moment Condition

$$0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$$

holds for all $k \geq 0$.

Wolff's elegant construction [2] of such an example (which uses only Laurent series) may also be found in [3].

There are numerous conditions known to be equivalent to the Moment Condition (1.1) holding for all $k \geq 0$ whenever $\{\lambda_n\}$ is a bounded sequence of distinct complex numbers. For instance, it follows from the Fubini-Tonelli Theorem that

$$0 \equiv \sum_{n=0}^{\infty} \frac{w_n}{z - \lambda_n} = \sum_{k=0}^{\infty} \left[\frac{1}{z^{k+1}} \sum_{n=0}^{\infty} w_n \lambda_n^k\right]$$

whenever $|z| > \sup|\lambda_n|$. Moreover, condition (1.1) holds for all $k \geq 0$ if and only if the Dirichlet series $g(z) \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ vanishes identically on the complex plane since $g \equiv 0$ if and only if

$$g^{(k)}(0) = \sum_{n=0}^{\infty} w_n \lambda_n^k$$

for all $k \geq 0$. This, in turn, is equivalent to the measure $\mu \equiv \sum_{n=0}^{\infty} w_n \delta_{\{\lambda_n\}}$ (the sum of weighted point masses) annihilating the monomials since

$$\int z^k d\mu = \sum_{n=0}^{\infty} w_n \lambda_n^k.$$

If the points $\{\lambda_n\}$ lie in a Jordan region Ω and accumulate only on its boundary, then $\sum_{n=0}^{\infty} w_n/(z - \lambda_n) \equiv 0$ whenever $|z| > \sup|\lambda_n|$ where $\{w_n\}$ is a non-trivial sequence in ℓ^1 if and only if $\{\lambda_n\}$ is a dominating sequence for Ω; that is, if and only if

$$\sup\{|f(z)| : z \in \Omega\} = \sup\{|f(\lambda_n)| : n \geq 0\}$$

for all functions f bounded and analytic on Ω (see Theorem 3 on p. 107 of Brown, Shields, and Zeller [4]). If Ω is the open unit disk, then this condition is equivalent to almost every point of the unit circle (with respect to Lebesgue arc length measure) being the non-tangential limit point of $\{\lambda_n\}$. Deep connections to operator theory are provided by work of Sarason [5] and [6] who shows that that the Borel series $\sum_{n=0}^{\infty} w_n/(z-\lambda_n) \equiv 0$ whenever $|z| > \sup|\lambda_n|$ for some non-trivial sequence $\{w_n\}$ in ℓ^1 if and only if there exists a closed invariant subspace for the diagonal operator D having eigenvalues $\{\lambda_n\}$ which is not invariant for the adjoint D^* of D. This condition, in turn, is equivalent to the weakly closed algebra generated by D and the identity operator not containing D^*. For more on the connections between Borel series and complete normal operators, please see Wermer [1], Scroggs [7] and Nikolskii [3], [8].

The study of Borel series has a rich and fabled history. Of particular interest has been conditions for a function analytic on a region to be representable as a Borel series, and conditions for such a representation, if one exists, to be unique. In particular, the seminal work of Leontev [9], Korobeinik [10], Leont' eva [11], and Brown, Shields, and Zeller [4], amongst others, has examined the extent to which the existence of non-trivial expansions of zero by Dirichlet series $\sum_{n=0}^{\infty} w_n e^{\lambda_n z} \equiv 0$ on regions Ω in the complex plane imply (and, under additional conditions, is equivalent to) the ability to represent an arbitary function $f(z)$ analytic on Ω as a Dirichlet series $f(z) = \sum_{n=0}^{\infty} a_n e^{\lambda_n z}$ on Ω. It follows from the preceding comments that the non-uniqueness of any such representation is equivalent to the existence of Borel series which vanish identically on Ω. In 1959, Makarov [12] showed that for every sequence of complex numbers $\{\lambda_n\}$ for which $|\lambda_n| \to \infty$, there exists a sequence of complex numbers $\{w_n\}$ for which the moment condition (1.1) holds for all $k \geq 0$ where the coefficients $\{w_n\}$ satisfy the decay rate $0 < \sum_{n=0}^{\infty} |w_n| \cdot |\lambda_n^k| < \infty$. In addition to Wolff's example [2], in which the coefficients $\{w_n\}$ are in ℓ^1, Denjoy [13] in 1924 and Leont' eva [11] in the late 1960's gave examples of Borel series which vanish identically where the coefficients satisfy various decay rates just shy of exponential decay (see p. 26 of [14]).

There has also been particular interest regarding the converse, namely the so-called unicity problem, which is to determine the rate at which $\{|w_n|\}$ must decrease so that $\sum_{n=0}^{\infty} w_n/(z-\lambda_n)$ does not extend analytically to a region containing $\{\lambda_n\}$. Borel [15], Carleman [16], Gonchar [17], and Poincare all determined decay rates in the unicity problem in their investigations on Borel series, which were focused mainly on issues regarding quasianalyticity and analytic continuation. In 1968 Makarov gave such a decay rate depending on a given arbitrary sequence $\{\lambda_n\}$ (see 5.7.8(c)(vii) on p. 128 of [3]). A rather definitive result was obtained by Sibilev in 1995 when the eigenvalues $\{\lambda_n\}$ are bounded (see the theorem on p. 146 of [18]). For more on the history of Borel series and a discussion of generalized analytic continuation, please see the recent monograph of Ross and Shapiro [14].

The purpose of this paper is to provide a rich class of examples of diagonal operators acting on $\mathcal{H}(\mathbb{D})$ which fail spectral synthesis. The main result of this paper, Theorem 1, appears in Section 2 below and improves upon previous results in the literature. When endowed with the topology of uniform convergence on compacta, $\mathcal{H}(\mathbb{D})$ is an example of a complete locally convex topological vector space. Using the Radius of Convergence Formula, it follows that a function $\sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{H}(\mathbb{D})$ if and only if $\limsup |a_n|^{1/n} < 1$. Moreover, if $\{\lambda_n\}$ is any sequence of distinct complex numbers, then the map for which $D(z^n) \equiv \lambda_n z^n$ extends by linearity to an operator on all of $\mathcal{H}(\mathbb{D})$ if and only if $\limsup |\lambda_n|^{1/n} \leq 1$ (see [19]). In particular, the set of eigenvalues of a diagonal operator on $\mathcal{H}(\mathbb{D})$ need not be bounded. It's known that the diagonal operator $D(\sum_{n=0}^{\infty} a_n z^n) \equiv \sum_{n=0}^{\infty} a_n \lambda_n z^n$ fails spectral synthesis if and only if the moment condition (1.1) holds for all $k \geq 0$ for some non-trivial sequence $\{w_n\}$ of complex numbers for which $\limsup |w_n|^{1/n} < 1$ (please see Theorem 3 on p. 1214 of [19] for this and other conditions equivalent to non-synthesis).

In [20], Anderson, Khavinson, and Shapiro, give a detailed analysis of the moment condition (1.1) for all $k \geq 0$ where the eigenvalues $\lambda_n = n^p$ are powers of n with $p > 0$. Their study focuses on questions concerning the analytic continuation of Dirichlet series and Fredholm's method for examining gap series and its connections to partial differential equations. They show, amongst other results, that the moment condition $0 = \sum_{n=0}^{\infty} w_n (n^p)^k$ holds for all $k \geq 0$ where $0 < \limsup |w_n|^{1/n} < 1$ if and only if $p > 2$, and moreover, that no solution exists for integral $p > 2$ for which

$$0 < \limsup |w_n|^{1/n} < e^{-\pi \tan^{-1}(\pi/p)}$$

(see Theorem 3.1 on p. 464 of [20]). In view of which, the moment condition holding and hence a diagonal operator admitting spectral synthesis is intimately related to the growth rate of the eigenvalues $\{\lambda_n\}$ of the diagonal operator. In some cases, the growth rate of the eigenvalues alone determines the spectral synthesis; for instance, Leontev [9] has shown that the moment condition (1.1) does not hold for all $k \geq 0$ whenever $\limsup |w_n|^{1/n} < 1$ if $\{\lambda_n\}$ exhibits linear growth (that is, whenever $0 < \liminf |\lambda_n|/n \leq \limsup |\lambda_n|/n < \infty$) whether or not the λ_n are positive). However, it is known that the distribution of the points λ_n throughout the complex plane, as well as their growth, typically plays a role in determining spectral synthesis. For example, the diagonal operator on $\mathcal{H}(D)$ having eigenvalues $\{\sqrt{n}\}$ admits spectral synthesis by Theorem 3.1 of [20], while the diagonal operator on $\mathcal{H}(D)$ with eigenvalues $\{\lambda_n\}$ comprising the integer lattice $\mathbb{Z} \times i\mathbb{Z} \equiv \{m + in : m, n \in \mathbb{Z}\}$ fails spectral synthesis (see [21]), although $|\lambda_n| \approx n^{1/2}$.

The result of Anderson, Khavinson, and Shapiro mentioned above suggests that the slower the growth rate of $\{|\lambda_n|\}$, the harder it is for the moment condition to hold, and hence for the associated diagonal operator on $\mathcal{H}(D)$ with eigenvalues $\{\lambda_n\}$ to fail spectral synthesis. Nonetheless, in this paper, we demonstrate that there exist diagonal operators acting on $\mathcal{H}(D)$ whose eigenvalues have growth rate $|\lambda_n| \approx n^{\beta}$ for any $\beta < 1$ which fail spectral synthesis. The examples produced do not require that the eigenvalues $\{\lambda_n\}$ assume any particular form, only that they satisfy a particular growth rate and are regularly distributed (in a sense made precise in the next section).

2. Examples of non-synthetic diagonal operators on $\mathcal{H}(D)$

In this section, we show that a diagonal operator on $\mathcal{H}(D)$ fails spectral synthesis whenever its eigenvalues have order of growth less than one, are regularly distributed with respect to a proximate order $\rho(r)$, and satisfy a separation criterion, definitions of which we now provide for the convenience of the reader.

The relationship between the growth of an entire function and the distribution of its zeros is well-known. It is often convenient to measure the growth of an entire function using a so-called *proximate order*, or function $\rho(r)$ for which $\lim_{r\to\infty} \rho(r) \equiv \rho \geq 0$ and $\lim_{r\to\infty} r\rho'(r) \ln r = 0$ (see p. 32 of [22]). A set of points in the complex plane is said to have an *angular density* $\Delta(\psi)$ *of index* $\rho(r)$ if for all but a countable set of values η and θ for which $0 < \eta < \theta \leq 2\pi$, the limit

$$\Delta(\eta, \theta) \equiv \lim_{r\to\infty} \frac{n(r, \eta, \theta)}{r^{\rho(r)}}$$

exists where here $n(r, \eta, \theta)$ denotes the number of points of the set lying within the sector $\{z : |z| \leq r; \eta < \arg z < \theta\}$ (see p. 89 of [22]).

A sequence $\{a_n\}$ of distinct complex numbers satisfies **Condition (C)** if there exists a positive number $d > 0$ such that the set of closed balls

$$\overline{\{B(a_n; d|a_n|^{1-\rho(|a_n|)/2})\}}$$

are pairwise disjoint, while the sequence satisfies **Condition (C′)** if the points all lie inside sectors with a common vertex at the origin but with no other points in common, and which are such that if one arranges the points of the set $\{a_n\}$ within any one of these sectors in order of increasing moduli, then for all points which lie inside the same sector it is true that $|a_{k+1}| - |a_k| > d|a_k|^{1-\rho(|a_k|)}$ (see p. 95 of [22]). In the following theorem, the conditions that the points are regularly distributed with respect to $\rho(r)$ and $\rho < 1/2$ ensure that there exist coefficients $\{w_n\}$ for which $0 = \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ while the separation condition ensures that $\limsup |w_n|^{1/n} < 1$.

Theorem 2.1. *Let $\rho(r)$ be any proximate order for which $\rho \equiv \lim_{r\to\infty} \rho(r) \in (0, 1/2)$ and let $\{a_n\}$ be any sequence of distinct complex numbers whose angular density $\Delta(\psi)$ has index $\rho(r)$, satisfies either Condition (C) or Condition (C′), and is such that $\liminf |a_n|^{\rho(|a_n|)}/n > 0$. Then the diagonal operator having eigenvalues $\{|a_n|^{1/q} e^{i(\arg a_n + 2\pi j)/q} : 0 \le j < q; 0 \le n\}$ fails to admit spectral synthesis on $\mathcal{H}(\mathrm{D})$ whenever q is any integer for which $q > 1/\rho$.*

An outline of the proof is as follows: Let $\{a_n\}$ be any sequence of complex numbers satisfying the hypotheses of Theorem 2.1. Then $S(z) \equiv f(z^q)$ is an entire function having only simple zeros at the points

$$\lambda_n = |a_n|^{1/q} e^{i(\arg a_n + 2\pi j)/q}$$

for $0 \le j < q$ and $0 \le n$ and $f(z) \equiv \Pi_{n=0}^{\infty}(1 - z/\lambda_n)$ is a canonical product having only simple zeros at the points λ_n. Since the points $\{a_n\}$ are separated, it follows that $|S(\lambda)| \ge e^{\alpha|\lambda|^{\beta}}$ for all λ on a sequence of circles C_r whose radii increase to infinity, where here $\beta > 1$. Using this estimate and the Residue Theorem, we see that

$$0 \leftarrow \int_{C_r} \frac{e^{\lambda z}}{S(\lambda)} d\lambda \to \sum_{n=0}^{\infty} \frac{e^{\lambda_n}}{S'(\lambda_n)}.$$

It follows from estimates for S near the points λ_n obtained using the Inverse Function Theorem and Schwarz's Lemma, that $\limsup (1/|S'(\lambda_n)|^{1/n}) < 1$. Hence, the moment condition holds and the result follows (see Theorem 3 on p. 1214 of [19]).

Proof. Let $\{a_n\}$ be any enumeration of the set of points

$$\{|a_n|^{1/q} e^{i(\arg a_n + 2\pi j)/q} : 0 \le j < q; 0 \le n\}$$

where here $n(r)$ denotes the number of points a_n for which $|a_n| \le r$. Hence $n(r) \ge .5\Delta r^{\rho(r)}$ for all r sufficiently large. Since $\{|a_i|\}$ is increasing, it follows that $|a_t| \le r$ where $t \equiv .5\Delta r^{\rho(r)}$ for all r sufficiently large. Since $\rho(r) \to \rho$, we have that $t = .5\Delta r^{\rho(r)} \ge .5\Delta r^{\rho/2}$ for all r sufficiently large. Thus $|a_{.5\Delta r^{\rho/2}}| \le |a_t| \le r$ or $|a_n| \le r = (2n/\Delta)^{2/\rho}$ for all n sufficiently large. Hence

$$\hat{r} \le \{(2+d)|a_{n+1}|\}^{1/q} \le (2+d)^{1/q}(2/\Delta)^{2/(qp)}(n+1)^{2/(qp)}$$

for all n sufficiently large. Moreover,

$$|S(\hat{r}_n e^{i\theta})| \ge e^{(\epsilon/2)\hat{r}^{qp(\hat{r}_n^q)}}$$

for all n sufficiently large, and so it follows that

$$\sum_{k=0}^{\infty} \frac{e^{\lambda_k z}}{S'(\lambda_k)} = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{C_n} \frac{e^{\lambda z}}{S(\lambda)} d\lambda = 0$$

for all $z \in \mathbf{C}$ since $qp(r_n^q) \to qp > 1$. In order to deduce that the diagonal operator having eigenvalues $\{\lambda_k\}$ fails to admit spectral synthesis, by Theorem 3 on p. 1214 of [19], it suffices to show that $\limsup (1/|S'(\lambda_k)|)^{1/k} < 1$. To this end, let k be any positive integer. Then

$$\lambda_n = |a_n|^{1/q} e^{i(\arg a_n + 2\pi j)/q}$$

for some integer n and $j \in \{0, 1, ..., q-1\}$. Since $S(z) = f(z^q)$, we have that $S'(z) = qz^{q-1}f'(z^q)$, and so

$$|S'(\lambda_k)| = |q\lambda_k^{q-1} f'(a_n)| = q|a_n|^{q-1}|f'(a_n)|.$$

We now estimate $|f'(a_n)|$ using the Inverse Function Theorem and Schwarz's Lemma. Since the closed balls $\{\overline{B(a_n, r_n)}\}$ are pairwise disjoint, there exist radii $\bar{r}_n \in (r_n, 1 + r_n)$ for which the open balls $\{B(a_n, \bar{r}_n)\}$ are pairwise disjoint and stay inside E^c. Hence

$$|f(re^{i\theta})| \ge e^{(\epsilon/2)r^{\rho(r)}} \ge e^{(\epsilon/2)(|a_n| - \bar{r}_n)^{\rho((|a_n| - \bar{r}_n))}} \equiv \alpha_n$$

whenever $re^{i\theta} \in \partial B(a_n, \bar{r}_n)$ with r sufficiently large. It follows from the Inverse Function Theorem (p. 234 of [23]) that the restriction

$$f|_{f^{-1}(B(0,\alpha_n))} : f^{-1}(B(0, \alpha_n)) \to B(0, \alpha_n)$$

of f to $f^{-1}(B(0, \alpha_n))$ has analytic inverse f^{-1}. Hence

$$g(z) \equiv (1/\bar{r}_n)(f^{-1}(\alpha_n z) - a_n) : B(0,1) \to B(0,1)$$

is analytic with $g(0) = 0$. By Schwarz's Lemma, we have that

$$1 \geq |g'(0)| = |\frac{a_n}{\tilde{r}_n f'(f^{-1}(0))}| = \frac{a_n}{\tilde{r}_n |f'(a_n)|}$$

whence

$$\frac{1}{|f'(a_n)|} \leq \frac{\tilde{r}_n}{e^{(\epsilon/2)\{|a_n|-\tilde{r}_n\}^{\rho(\{|a_n|-\tilde{r}_n\})}}}$$

for all r sufficiently large. Since $(n-1)q \leq k < nq$, we have that

$$\limsup\,(1/|S'(\lambda_k)|)^{1/k} = \limsup \frac{1}{\{q|a_n|^{(q-1)/q} \cdot |f'(a_n)|\}^{1/k}}$$

$$\leq \limsup \frac{1}{\{q|a_n|^{(q-1)/q} \cdot |f'(a_n)|\}^{1/(nq)}}.$$

Since $\Delta \equiv \lim_{r \to \infty} n(r)/r^{\rho(r)}$, we have that $n(r) \leq 1.5\Delta r^{\rho(r)}$ for all r sufficiently large. Since $\{|a_n|\}$ is increasing, it follows that $|a_t| \geq r$ where $t \equiv 1.5\Delta r^{\rho(r)}$ for all r sufficiently large. Since $\rho(r) \to \rho$, we have that $t = 1.5\Delta r^{\rho(r)} \leq 1.5\Delta r^{2\rho}$ for all r sufficiently large, and so $|a_s| = |a_{1.5\Delta r^{2\rho}}| \geq r$ where $s = 1.5\Delta r^{2\rho}$. Hence $|a_n| \geq (2n/(3\Delta))^{1/(2\rho)}$ for all n sufficiently large. Hence

$$\limsup\,(1/|S'(\lambda_k)|) \leq \limsup \frac{1}{|f'(a_n)|^{1/(nq)}}$$

$$\leq (\limsup \frac{\tilde{r}_n^{1/n}}{e^{(1/n)(\epsilon/2)\{|a_n|-\tilde{r}_n\}^{\rho(\{|a_n|-\tilde{r}_n\})}}})^{1/q}.$$

However, for all n sufficiently large,

$$\tilde{r}_n \leq 1 + r_n \leq 1 + d|a_n|^{1-\rho} \leq 2d|a_n| \leq 2d(\Delta n/2)^{2/\rho}$$

and so

$$\limsup\,(1/|S'(\lambda_k)|)^{1/k} \leq (\limsup \frac{1}{e^{(1/n)(\epsilon/2)\{|a_n|-\tilde{r}_n\}^{\rho(\{|a_n|-\tilde{r}_n\})}}})^{1/q}.$$

Since $\rho(r) \to \rho$, we have that $\tilde{r}_n \leq d|a_n|^{1-\rho/4}$ for all n sufficiently large and so $|a_n| - \tilde{r}_n \geq .5|a_n|$ for all n sufficiently large. Since $r^{\rho(r)}$ is increasing for all r sufficiently large (see p. 33 of [22]) and $L(r) \equiv r^{\rho(r)-\rho}$ is slowly increasing (see p. 33 of [22], it follows that

$$\{|a_n| - \tilde{r}_n\}^{\rho(|a_n|-\tilde{r}_n)} \geq (.5|a_n|)^{\rho(.5|a_n|)} \geq (.5|a_n|)^{\rho(.5|a_n|-\rho)}\{.5|a_n|\}^{\rho}$$

$$= L(.5|a_n|)\{.5|a_n|\}^{\rho} \geq \frac{.9}{2^{\rho}} L(|a_n|) \cdot |a_n|^{\rho} \geq \frac{1}{2^{1+\rho}} |a_n|^{\rho(|a_n|)}$$

for all n sufficiently large. Since

$$\liminf |a_n|^{\rho(|a_n|)}/n \equiv \delta > 0$$

by hypothesis, it follows that

$$\limsup \left(1/|S'(\lambda_k)|\right)^{1/k} \leq \left(\limsup \frac{1}{e^{(1/n)(\epsilon/2)(|a_n|-r_n)^{\rho(|a_n|-r_n)}}}\right)^{1/q} \leq e^{-\epsilon\delta/(2q)} < 1.$$

The result follows.

Examples. If $\rho > 0$, then $a_n \equiv n^{1/\rho}$ is a sequence of complex numbers having proximate order $\rho(r) \equiv \rho$ with $\liminf |a_n|^{\rho(|a_n|)}/n > 0$. If $\rho > 0$, then $a_n \equiv n^{1/\rho} \ln n$ is a sequence of complex numbers having proximate order $\rho(r) \equiv \rho + (\ln \ln r)/\ln r$ with $\liminf |a_n|^{\rho(|a_n|)}/n > 0$.

It follows from Theorem 3.1 of [20] that the diagonal operator D on $\mathcal{H}(\mathbf{D})$ having eigenvalues $\{n^{1/3}\}$ admits spectral synthesis. However, it follows from the preceding theorem that the diagonal operator on $\mathcal{H}(\mathbf{D})$ having eigenvalues $\{n^{1/3}e^{2\pi ij/6} : 0 \leq j < 6\}$ consisting of six copies of $\{n^{1/3}\}$ placed on the six rays $\{re^{2\pi ij/6} : r > 0\}$ where $0 \leq j < 6$ fails spectral synthesis. In fact, a similar conclusion holds for any sequence of eigenvalues $\{n^{\beta}\}$ whenever $\beta < 1$. In particular, if $\beta < 1$, then for any integer $q > 2/\beta$, we have that $\rho \equiv 1/(q\beta) < 1/2$. Hence the diagonal operator on $\mathcal{H}(\mathbf{D})$ having eigenvalues $\{|a_n|^{1/q}e^{2\pi ij/q} : 0 \leq j < q\}$ fails spectral synthesis by the preceding theorem, where here $a_n \equiv n^{1/\rho}$. In this case, $a_n^{1/q} = n^{\beta}$. In fact, we need only choose points $\{a_n\}$ having proximate order $\rho(r) \equiv r$, which places only mild conditions on how they are distributed throughout the complex plane. This example is in contrast to examples mentioned earlier where the points $\{n^{1/3}e^{2\pi ij/6} : 0 \leq j < 6\}$ lie on six rays, or the eigenvalues $\mathbf{Z} \times i\mathbf{Z} = \{m + in : m, n \in \mathbf{Z}\}$ form a lattice.

It is possible to obtain examples of diagonal operators on $\mathcal{H}(\mathbf{D})$ which fail spectral synthesis by perturbing the eigenvalues of a diagonal operator on $\mathcal{H}(\mathbf{D})$ which is known to fail spectral synthesis; however, some care must be taken. Recall that a linear map D for which $D(z^n) = \lambda_n z^n$ extends to an operator on all of $\mathcal{H}(\mathbf{D})$ if and only if $\limsup |\lambda_n|^{1/n} \leq 1$. In this case, D fails spectral synthesis if and only if there exists a non-trivial sequence of complex numbers $\{w_n\}$ for which the moment condition $0 = \sum_{n=0}^{\infty} w_n \lambda_n^k$ holds for all $k \geq 0$ where here $\limsup |w_n|^{1/n} < 1$.

It may be tempting to believe in this case that adding points to this list of eigenvalues produces another diagonal operator which fails spectral synthesis (simply by making their coefficients zero in the moment condition). However, this requires moving the position of the existing eigenvalues $\{\lambda_n\}$ and the coefficients $\{w_n\}$ which in turn typically changes the values of both $\limsup |\lambda_n|^{1/n}$ and $\limsup |w_n|^{1/n}$. This poses difficulties even when simply rearranging the eigenvalues. For instance, suppose that D is a diagonal operator on $\mathcal{H}(\mathbf{D})$ which fail spectral synthesis. It follows from the result due to Sibilev mentioned above that the eigenvalues $\{\lambda_n\}$ are unbounded (see p. 146 of [18] or [19]). In view of which, there is some rearrangement $\{\lambda_{i(n)}\}$ of $\{\lambda_n\}$ for which $\limsup|\lambda_{i(n)}|^{1/n} = \infty$. That is, there does exist a continuous linear map $\tilde{D}$ for which $\tilde{D}(z^n) = \lambda_{i(n)}z^n$ for all $i \geq 0$. Even if such a rearrangement yields a new diagonal operator $\tilde{D}$ on $\mathcal{H}(\mathbf{D})$ having eigenvalues $\{\lambda_{i(n)}\}$, it need not be the case that $\tilde{D}$ fails spectral synthesis (see, for example, Example 4.3 on p. 58 of [21]). It is known, however, that adding or deleting any finite list of eigenvalues of a non-synthetic diagonal operator on $\mathcal{H}(\mathbf{D})$ produces a new diagonal operator on $\mathcal{H}(\mathbf{D})$ failing synthesis (see, for example, [19]), but that adding a countable list of eigenvalues to an operator admitting synthesis may produce an operator failing synthesis. For example, the diagonal operator on $\mathcal{H}(\mathbf{D})$ having eigenvalues $\{n^{1/3}\}$ fails spectral synthesis while the diagonal operator on $\mathcal{H}(\mathbf{D})$ having eigenvalues $\{n\}$ admits spectral synthesis (see Theorem 3.1 of [19]). The extent to which rearranging, adding, or deleting eigenvalues effects the synthesis or non-synthesis of a diagonal operator is explored in [21].

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WEIGHTED NORM INEQUALITIES FOR AREA FUNCTIONS RELATED TO SCHRÖDINGER OPERATORS

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Abstract. Let $L=-\Delta+V$ be a Schrödinger operator, where Δ is the Laplacian operator on $\mathbb{R}^n$, and V is a nonnegative potential belonging to certain reverse Hölder class. In this paper, we establish some weighted norm inequalities for area functions related to Schrödinger operators and their commutators.

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1. Introduction

In this paper, we consider the Schrodinger differential operator on $\mathbb{R}^n$ $(n\geq 3)$:

$$L=-\Delta+V(x),$$

where Δ is the Laplacian operator on $\mathbb{R}^n$, and V is a nonnegative potential belonging to certain reverse Holder class.

A nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to the class B_q $(1<q\leq\infty)$ if there exists a constant $C>0$ such that the reverse Holder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}V^q(y)dy\right)^{1/q}\leq C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}V(y)dy\right) \tag{1.1}$$

holds for every $x\in\mathbb{R}^n$ and $0<r<\infty$, where $B(x,r)$ denotes the ball centered at x and radius r. In particular, if V is a nonnegative polynomial, then $V\in B_\infty$. It is worth to point out that if $V\in B_q$ for some $q>1$, then there exist $\epsilon>0$, depending only n, and a constant C (as in (1.1)) such that $V\in B_{q+\epsilon}$. Throughout this paper, we always assume that $0\not\equiv V\in B_{n/2}$.

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The study of the Schrodinger operator $L = -\Delta + V$ has recently attracted much attention (see [1, 2, 5, 6, 12, 15], and references therein). In particular, in Shen [12] it was proved that the Schrodinger type operators: $\nabla(-\Delta+V)^{-1}\nabla$, $\nabla(-\Delta+V)^{-1/2}$, $(-\Delta+V)^{-1/2}\nabla$ with $V \in B_n$, and $(-\Delta+V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, are standard Calderón-Zygmund operators.

Recently, Bongioanni et al. (see [1]) proved the $L^p(\mathbb{R}^n)(1 < p < \infty)$ boundedness for commutators of Riesz transforms associated with Schrodinger operators with $BMO(\rho)$ functions (which include the class BMO functions), and then, in [2], they established the weighted boundedness of Riesz transforms, fractional integrals and Littlewood-Paley functions associated with Schrödinger operator with weights from the class A_p^ρ, which includes the class of Muckenhoupt weights. Very recently, in [13, 14], one of the authors of this paper has established weighted norm inequalities for some Schrodinger type operators, which include commutators of Riesz transforms, fractional integrals, and Littlewood-Paley functions related to Schrödinger operators (see also [3, 4]).

In this paper, we continue our research to study weighted norm inequalities for area functions related to Schrödinger operators and their commutators. To state the main result of this paper, we first introduce some definitions. The area function S_Q related to Schrödinger operators is defined by

$$S_Q(f)(x) = \left(\int_0^\infty \int_{|x-y|<t} |Q_t(f)(y)|^2 \, dy \frac{dt}{t^{n+1}}\right)^{1/2},$$

where

$$(Q_t f)(x) = t^2 \left(\frac{dT_s}{ds}\Big|_{s=t^2} f \right)(x), \quad T_s = e^{-sL}, \quad (x,t) \in \mathbb{R}^{n+1}_+ = (0,\infty) \times \mathbb{R}^n$$

The commutator of S_Q with $b \in BMO(\rho)$ is defined by

$$S_{Q,b}(f)(x) = \left(\int_0^\infty \int_{|x-y|<t} |Q_t((b(x)-b(\cdot))f)(y)|^2 \, dy \frac{dt}{t^{n+1}}\right)^{1/2}.$$

The following two theorems are the main results of this paper.

Theorem 1.1. *Let $1 < p < \infty$. If $\omega \in A_p^\rho$ (to be defined in Section 2), then there exists a constant C such that*

$$\|S_Q(f)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

If $\omega \in A_1^\rho$, *then there exists a constant* $C > 0$ *such that for any* $\lambda > 0$

$$\omega(\{x \in \mathbb{R}^n : \ |S_Q(f)(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\omega(x)dx.$$

The next theorem contains weighted norm inequalities for the commutator $S_{Q,b}$.

Theorem 1.2. *Let* $b \in BMO(\rho)$ *(to be defined in Section 2) and* $1 < p < \infty$. *If* $\omega \in A_p^\rho$, *then there exists a constant* C *such that*

$$\|S_{Q,b}(f)\|_{L^p(\omega)} \leq C\|b\|_{BMO(\rho)}\|f\|_{L^p(\omega)}.$$

If $\omega \in A_1^\rho$, *then there exists a constant* $C > 0$ *such that for any* $\lambda > 0$

$$\omega(\{x \in \mathbb{R}^n : \ |S_{Q,b}(f)(x)| > \lambda\}) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) \omega(x)dx.$$

The rest of the paper is organized as follows. In Section 2, we introduce some notation and state some basic results. In Section 3, we establish a number of lemmas, which play a crucial role in this paper. Finally, in Section 4 , we prove our main results - Theorems 1.1 and 1.2.

Throughout the paper, we let C to denote constants that are independent of the main parameters involved, but whose value may vary from line to line. The notation $A \sim B$ means that there exists a constant $C > 1$ such that $1/C \lesssim A/B \lesssim C$.

2. Preliminaries

We first recall some notation. Given a ball $B = B(x, r)$ and a number $\lambda > 0$, by λB we will denote the λ-dilated ball, which is the ball with the same center x and with radius λr. Similarly, by $Q(x, r)$ we will denote the cube centered at x with the side length r, and $\lambda Q(x, r) := Q(x, \lambda r)$ (here and below only cubes with sides parallel to the coordinate axes are considered). Given a Lebesgue measurable set E and a weight ω, by $|E|$ we denote the Lebesgue measure of E and $\omega(E) := \int_E \omega dx$. For $0 < p < \infty$, by $L^p(\omega)$ we denote the L^p-weighted space with norm $\|f\|_{L^p(\omega)} := (\int_{\mathbb{R}^n} |f(y)|^p \omega(y)dy)^{1/p}$.

The function $m_V(x)$ is defined by

$$\rho(x) = \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \ \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1 \right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, we have $m_V(x) = 1$ for $V = 1$ and $m_V(x) \sim (1 + |x|)$ for $V = |x|^2$.

Lemma 2.1. *(see [12]). There exist constants $l_0 > 0$ and $C_0 > 1$ such that*

$$\frac{1}{C_0}\left(1+|x-y|m_V(x)\right)^{-l_0} \le \frac{m_V(x)}{m_V(y)} \le C_0\left(1+|x-y|m_V(x)\right)^{l_0/(l_0+1)}.$$

In particular, $m_V(x) \sim m_V(y)$ if $|x-y| < C/m_V(x)$.

For a ball $B = B(x_0, r)$ with center at x_0 and radius r and a number $\theta > 0$, we denote $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$.

A weight will always mean a nonnegative locally integrable function. As in [2], we say that a weight ω belongs to the class $A_p^{\rho,\theta}$ $(1 < p < \infty)$, if there is a constant C such that for all balls $B = B(x, r)$,

$$\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega(y)\,dy\right)\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \le C.$$

Also, we say that a nonnegative function ω satisfies the $A_1^{\rho,\theta}$ condition if there exists a constant C such that for all balls B,

$$M_V^\theta(\omega)(x) \le C\omega(x), \text{ a.e. } x \in \mathbb{R}^n,$$

where

$$M_V^\theta f(x) = \sup_{x\in B}\frac{1}{\Psi_\theta(B)|B|}\int_B |f(y)|\,dy.$$

Since $\Psi_\theta(B) \ge 1$, we obviously have $A_p \subset A_p^{\rho,\theta}$ for $1 \le p < \infty$, where A_p denotes the class of classical Muckenhoupt weights (see [7] and [9]). Note that in some cases we have the embedding $A_p \subset\subset A_p^{\rho,\theta}$ for $1 \le p < \infty$. Indeed, let $\theta > 0$ and $0 \le \gamma \le \theta$, then it is easy to check that $\omega(x) = (1+|x|)^{-(n+\gamma)} \notin A_\infty := \bigcup_{p\ge1} A_p$ and $\omega(x)dx$ is not a doubling measure, but $\omega(x) = (1+|x|)^{-(n+\gamma)} \in A_1^{\rho,\theta}$ provided that $V = 1$ and $\Psi_\theta(B(x_0, r)) = (1+r)^\theta$.

Also, we remark that in the above definitions of $A_p^{\rho,\theta}$ $(p \ge 1)$ and $M_{V,\theta}$, the balls can be replaced by cubes because $\Psi_\theta(B) \le \Psi_\theta(2B) \le 2^\theta\Psi_\theta(B)$. For $V = 0$ and $\theta = 0$, instead of $M_{0,0}f(x)$ we use the notation $Mf(x)$, which is the classical Hardy-Littlewood maximal function. It is easy to see that $|f(x)| \le M_V^\theta f(x) \le Mf(x)$ for a.e. $x \in \mathbb{R}^n$ and $\theta \ge 0$. For convenience, in the rest of this paper, for a fixed $\theta \ge 0$, instead of $\Psi_\theta(B)$ and $A_p^{\rho,\theta}$ we use the notation $\Psi(B)$ and A_p^ρ, respectively.

The next lemma follows from the definition of the class A_p^ρ $(1 \le p < \infty)$.

Lemma 2.2. *Let $1 \le p < \infty$. Then the following assertions hold.*

(i) If $1 \le p_1 < p_2 < \infty$, then $A_{p_1}^\rho \subset A_{p_2}^\rho$.

(ii) $\omega \in A_p^\rho$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}^\rho$, where $1/p + 1/p' = 1$.

In [1], Bongioanni et al. have introduced a new space $BMO(\rho)$ defined by

$$\|f\|_{BMO(\rho)} = \sup_{B\subset\mathbb{R}^n} \frac{1}{\Psi(B)|B|}\int_B |f(x) - f_B|dx < \infty,$$

where $f_B = \frac{1}{|B|}\int_B f(y)dy$, $\Psi(B) = (1 + r/\rho(x_0))^{\theta}$, $B = B(x_0, r)$, and $\theta > 0$.

In particular, in [1] it was proved the following result for the space $BMO(\rho)$.

Lemma 2.3. *Let $\theta > 0$ and $1 \leq s < \infty$. If $b \in BMO(\rho)$, then*

$$\left(\frac{1}{|B|}\int_B |b - b_B|^s\right)^{1/s} \leq C_{\theta,s}\|b\|_{BMO(\rho)}\left(1 + \frac{r}{\rho(x)}\right)^{\theta'},$$

for all $B = B(x, r)$ with $x \in \mathbf{R}^n$ and $r > 0$, where $\theta' = (l_0 + 1)\theta$.

Obviously, the classical BMO is properly embedded into $BMO(\rho)$. More examples can be found in [1].

Applying Lemma 2.3, one of the the authors of this paper proved the following John-Nirenberg type inequality for space $BMO(\rho)$ (see [13]).

Proposition 2.1. *Let $f \in BMO(\rho)$. There exist positive constants γ and C such that*

$$\sup_B \frac{1}{|B|}\int_B \exp\left\{\frac{\gamma}{\|f\|_{BMO(\rho)}\Psi_{\theta'}(B)}|f(x) - f_B|\right\} dx \leq C,$$

where $f_B = \frac{1}{|B|}\int_B f(y)dy$, $\Psi_{\theta'}(B) = (1+r/\rho(x_0))^{\theta'}$, $B = B(x_0, r)$, and $\theta' = (l_0+1)\theta$.

We remark that in the above definitions of A_p^{ρ}, $BMO(\rho)$ and M_V, the balls can be replaced by cubes.

We also will need the dyadic maximal operator $M_{V,\eta}^{\triangle}f(x)$ and the dyadic sharp maximal operator $M_{V,\eta}^{\sharp}f(x)$, which for $0 < \eta < \infty$ are defined by the following formulas:

$$M_{V,\eta}^{\triangle}f(x) = \sup_{x\in Q(\text{dyadic cube})} \frac{1}{\Psi(Q)^{\eta}|Q|}\int_Q |f(x)|\, dx$$

and

$$\begin{aligned} M_{V,\eta}^{\sharp}f(x) &= \sup_{x\in Q, r<\rho(x_0)} \frac{1}{|Q|}\int_{Q(x_0,r)} |f(y) - f_Q|\, dy + \sup_{x\in Q, r\geq\rho(x_0)} \frac{1}{\Psi(Q)^{\eta}|Q|}\int_{Q(x_0,r)} |f|\, dy \\ &\simeq \sup_{x\in Q, r<\rho(x_0)} \inf_C \frac{1}{|Q|}\int_{Q(x_0,r)} |f(y) - C|\, dy + \sup_{x\in Q, r\geq\rho(x_0)} \frac{1}{\Psi(Q)^{\eta}|Q|}\int_{Q(x_0,r)} |f|\, dx, \end{aligned}$$

where $Q'_{x_0}s$ denotes the dyadic cube $Q(x_0, r)$ and $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$.

The following versions of dyadic maximal and dyadic sharp maximal operators:

$$M_{\delta,\eta}^{\triangle}f(x) = M_{V,\eta}^{\triangle}(|f|^{\delta})^{1/\delta}(x)$$

and

$$M^{\sharp}_{\delta,\eta}f(x) = M^{\sharp}_{V,\eta}(|f|^{\delta})^{1/\delta}(x)$$

will be the main tools in our scheme.

In [13], one of the the authors of this paper proved the following results.

Theorem 2.1. *Let $0 < p,\ \eta,\ \delta < \infty$ and $\omega \in A_\infty$. There exists a positive constant C such that*

$$\int_{\mathbf{R}^n} M^{\triangle}_{\delta,\eta}f(x)^p\omega(x)dx \le C\int_{\mathbf{R}^n} M^{\sharp}_{\delta,\eta}f(x)^p\omega(x)dx.$$

Further, let $\varphi : (0,\infty) \to (0,\infty)$ be a doubling function, then there exists a positive constant C such that

$$\sup_{\lambda>0}\varphi(\lambda)\omega(\{x \in \mathbf{R}^n :\ M^{\triangle}_{\delta,\eta}f(x) > \lambda\}) \le C\sup_{\lambda>0}\varphi(\lambda)\omega(\{x \in \mathbf{R}^n :\ M^{\sharp}_{\delta,\eta}f(x) > \lambda\})$$

for any smooth function f for which the left hand-hand side is finite.

Proposition 2.2. *Let $1 < p < \infty$ and $\omega \in A^\rho_p$. If $p < p_1 < \infty$, then*

$$\int_{\mathbf{R}^n} |M_V f(x)|^{p_1}\omega(x)dx \le C_p\int_{\mathbf{R}^n} |f(x)|^{p_1}\omega(x)dx.$$

Further, let $1 \le p < \infty$, then $\omega \in A^\rho_p$ if and only if

$$\omega(\{x \in \mathbf{R}^n :\ M_V f(x) > \lambda\}) \le \frac{C_p}{\lambda^p}\int_{\mathbf{R}^n} |f(x)|^p\omega(x)dx.$$

From Proposition 2.2 it follows that M_V may be unbounded on $L^p(\omega)$ for all $\omega \in A^\rho_p$ and $1 < p < \infty$. We will need a variant of maximal operator $M_{V,\eta}$ $(0 < \eta < \infty)$ defined as follows:

$$M_{V,\eta}f(x) = \sup_{x\in B}\frac{1}{(\Psi(B))^\eta|B|}\int_B |f(y)|\,dy.$$

Theorem 2.2. *Let $1 < p < \infty$ and $p' = p/(p-1)$, and let $\omega \in A^\rho_p$. Then there exists a constant $C > 0$ such that*

$$\|M_{V,p'}f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Finally, we recall some basic definitions and facts about Orlicz spaces, referring to [11] for a complete account.

A function $B(t) : [0,\infty) \to [0,\infty)$ is called a Young's function if it is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $B \to \infty$ as $t \to \infty$. For a Young's function B, we define the B-average of a function f over a cube Q by means of the following Luxemberg norm:

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 :\ \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}.$$

If A, B and C are Young's functions such that

$$A^{-1}(t)B^{-1}(t) \le C^{-1}(t),$$

where A^{-1} is the Young's complementary function associated with A, then we have

$$\|fg\|_{C,R} \le 2\|f\|_{A,R}\|g\|_{B,R}.$$

The examples to be considered in our study will be $A^{-1}(t) = \log(1+t)$, $B^{-1}(t) = t/\log(e+t)$ and $C^{-1}(t) = t$. Then $A(t) \sim e^t$ and $B(t) \sim t\log(e+t)$, which give the generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |fg|\,dy \le \|f\|_{A,Q}\|g\|_{B,Q}.$$

For these examples, if $b \in BMO(\rho)$ and b_Q denotes its B-average over the cube Q, in view of Proposition 2.1, we get

$$\|(b - b_Q)/\Psi_{\theta'}(Q)\|_{expL,Q} \le C\|b\|_{BMO(\rho)},$$

where $\theta' = (1 + l_0)\theta$.

Also, we define the corresponding maximal functions:

$$M_B f(x) = \sup_{Q:x\in Q} \|f\|_{B,Q}$$

and

$$M_{V,B} f(x) = \sup_{Q:x\in Q} \Psi(Q)^{-1}\|f\|_{B,Q}.$$

3. Some lemmas

In this section, we establish some estimates, which will play a crucial role in the proofs of the main results of this paper. We first introduce some notation and definitions. We define the space $\mathrm{B} = L^2(\mathbb{R}_+^{n+1}, dydt/t^n)$ to be the set of measurable functions $a : \mathbb{R}_+^{n+1} \to \mathbb{C}$ endowed the norm $|a|_{\mathrm{B}} = (\int_{\mathbb{R}_+^{n+1}} |a(y,t)|^2\,dydt/t^n)^{1/2} < \infty$. By $\mathcal{M}(\mathbb{R}^n)$ we denote the set of measurable functions $a : \mathbb{R}^n \to \mathbb{C}$, and by $\mathcal{M}(\mathbb{R}^n, \mathrm{B})$ we denote the set of Bochner-measurable functions $h : \mathbb{R}^n \to \mathrm{B}$. The space $L^p(\mathbb{R}^n, \mathrm{B})$ is defined to be the set of functions $h \in \mathcal{M}(\mathbb{R}^n, \mathrm{B})$ endowed the finite norm:

$$\|h\|_{L^p(\mathbb{R}^n,\mathrm{B})} = \left(\int_{\mathbb{R}^n} |h(x)|_{\mathrm{B}}^p\,dx\right)^{1/p}.$$

We define $s_Q(f)(x) = \left(\int_0^\infty |Q_t f(x)|^2 \frac{dt}{t}\right)^{1/2}$. It is known that $\|s_Q(f)\|_2 = \frac{1}{\sqrt{8}}\|f\|_2$ (see Lemma 4.1 of [6]).

Let $\varphi \leq 1$ be a nonnegative infinitely differentiable function on $\mathbb{R}_+$ such that $\varphi(s) = 1$ for $0 < s < 1$ and $\varphi(s) = 0$ for $s \geq 2$. Then the function $\varphi_t(x,y) := \frac{1}{t}\varphi(\frac{|x-y|}{t})$ satisfies

$$(3.1) \qquad |\varphi_t(x,y) - \varphi_t(x',y)| \leq C\frac{|x-x'|}{t^2}\chi_{[0,2]}\left(\frac{\min\{|x-y|,|x'-y|\}}{t}\right),$$

for $|x-y| > 2|x-x'|$.

Now we consider an operator $\tilde{S} : \mathcal{M}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{R}^n, \mathbb{B})$ defined as follows:

$$(3.2) \qquad \tilde{S}f(x) = \left\{\tilde{S}_{(y,t)}f(x) := t^{1/2}\varphi_t(x,y)Q_tf(y)\right\}_{(y,t)\in\mathbb{R}^{n+1}_+},$$

which has an associated kernel given by

$$(3.3) \qquad \tilde{K}(x,z) = \left\{t^{1/2}\varphi_t(x,y)Q_t(y,z)\right\}_{(y,t)\in\mathbb{R}^{n+1}_+}.$$

We first recall some properties of the function Q_t.

Lemma 3.1. *(see [6]) There exist positive constants c and $\delta_0 \leq 1$ such that for every $l \geq 0$ there is a constant C_l so that the following inequalities hold:*

(a) $|Q_t(x,y)| \leq C_l t^{-n}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-l}\exp\left(-\frac{c|x-y|^2}{t^2}\right)$,

(b) $|Q_t(x+h,y)-Q_t(x,y)| \leq C_l\left(\frac{|h|}{t}\right)^{\delta_0} t^{-n}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-l}\exp\left(-\frac{c|x-y|^2}{t^2}\right)$
for all $|h| \leq t$;

Lemma 3.2. *Let $\tilde{K}(x,z)$ and δ_0 be as above, then for any $l \geq 0$ we have*

$$(3.4) \qquad |\tilde{K}(x,z)|_{\mathbb{B}} \leq \frac{C_l}{(1+|x-z|(\rho(x)^{-1}+\rho(z)^{-1}))^l}\frac{1}{|x-y|^n},$$

(3.5)

$$|\tilde{K}(x,z)-\tilde{K}(x',z)|_{\mathbb{B}} \leq \frac{C_l}{(1+|x-z|(\rho(x)^{-1}+\rho(z)^{-1}))^l}\frac{|x-x'|^{\delta_0}}{|x-z|^{n+\delta_0}}, \text{ if } \quad |x-z| > 2|x-x'|.$$

(3.6)

$$|\tilde{K}(z,x)-\tilde{K}(z,x')|_{\mathbb{B}} \leq \frac{C_l}{(1+|x-z|(\rho(x)^{-1}+\rho(z)^{-1}))^l}\frac{|x-x'|^{\delta_0}}{|x-z|^{n+\delta_0}}, \text{ if } \quad |x-z| > 2|x-x'|.$$

Proof. We adapt the arguments applied in the proof of Theorem 4.1 of [8]. Without loss of generality, we can assume that $\rho(z) < |x-z|$. We first prove the inequality

(3.4). By Lemma 3.1(a), for any $N, l > 0$ we can write

$$\begin{aligned}
|\tilde{K}(x,z)|_{\mathrm{B}}^2 &= \int_0^\infty \int_{\mathbf{R}^n} \frac{|\varphi_t(x,y)|^2 |Q_t(y,z)|^2}{t^{n-1}}\, dy dt \\
&\le C \int_{\mathbf{R}^n} \int_{|x-y|/2}^{\infty} \frac{1}{t^{n+1}} \left(\frac{(1+|y-z|/t)^{-N}}{t^n} \right)^2 \left(\frac{t}{\rho(z)} \right)^{-2l} dt dy \\
&\le C \int_{A_1 \cup A_2 \cup A_3} \int_{|x-y|/2}^{\infty} \frac{1}{t^{3d+1}} (1+|y-z|/t)^{-2N} \left(\frac{t}{\rho(z)} \right)^{-2l} dt dy \\
&:= J_1 + J_2 + J_3,
\end{aligned}$$

where the sets A_1, A_2 and A_3, constituting a partition of $\mathbf{R}^n$, are given by

$A_1 = \{y \in \mathbf{R}^n : |y-z| > 2|x-z|\}$,

$A_2 = \{y \in \mathbf{R}^n : \frac{1}{2}|x-z| < |y-z| \le 2|x-z|\}$,

$A_3 = \{y \in \mathbf{R}^n : |y-z| \le \frac{1}{2}|x-z|\}$.

For $y \in A_1$, we have $|x-z| \le \frac{1}{2}|y-z| \le |x-y| \le 2|y-z|$, and hence

$$\begin{aligned}
J_1 &\le C \int_{|x-y| \ge |x-z|} \int_{|x-y|/2}^{\infty} \frac{1}{t^{3n+1}} \left(\frac{t}{\rho(z)} \right)^{-2l} dt dy \\
&\le C\rho(z)^{2l} \int_{|x-y| \ge |x-z|} \frac{1}{|x-y|^{3n+2l}}\, dy \\
&\le \frac{C}{|x-z|^{3n}} \left(\frac{|x-z|}{\rho(z)} \right)^{-2l}.
\end{aligned}$$

For $y \in A_2$, we have $|x-y| \le 3|x-z|$ and $|x-z| \sim |y-z|$, and hence

$$\begin{aligned}
J_2 &\le C \int_{A_2} \left(\int_{|x-y|/2}^{3|x-z|} \frac{1}{t^{3n+1}} \frac{t^{2N}}{|y-z|^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dt + \int_{3|x-z|}^{\infty} \frac{1}{t^{3n+1}} \left(\frac{t}{\rho(z)} \right)^{-2l} dt \right) dy \\
&:= J_{2a} + J_{2b}.
\end{aligned}$$

For J_{2a}, we have

$$\begin{aligned}
J_{2a} &\le \frac{C\rho(z)^{2l}}{|x-z|^{2N}} \int_{|x-y|<3|x-z|} \int_{|x-y|/2}^{\infty} \frac{1}{t^{3n-2N+1+2l}}\, dt dy \\
&\le \frac{C}{|x-z|^{2N}} \int_{|x-y|<3|x-z|} \frac{1}{|x-y|^{3n-2N+2l}}\, dy \\
&\le \frac{C}{|x-z|^{2n}} \left(\frac{|x-z|}{\rho(z)} \right)^{-2l},
\end{aligned}$$

in which we take $n < N - l < \frac{3}{2}n$, and for J_{2b}, we get

$$J_{2b} \le \frac{C\rho(z)^{2l}}{|x-z|^{3n+2l}} \int_{|x-y|<3|x-z|} dy \le \frac{C}{|x-z|^{2n}} \left(\frac{|x-z|}{\rho(z)} \right)^{-2l}.$$

Thus, from the above inequalities it follows that

$$J_2 \le \frac{C}{|x-z|^{2n}} \left(\frac{|x-z|}{\rho(z)} \right)^{-2l}.$$

Finally, for $y \in A_3$, we have $|y-z| < \frac{1}{2}|x-z|$ and $|x-z| \sim |x-y|$, and hence

$$\begin{aligned} J_3 &\le C \int_{|y-z| \le \frac{1}{2}|x-z|} \int_{|x-y|/2}^{\infty} \frac{\rho(z)^{2l} dt}{t^{3n+1+2l}} dy \\ &\le \frac{C\rho(z)^{2l}}{|x-z|^{3n+2l}} \int_{|x-y|<3|x-z|} dy \\ &\le \frac{C}{|x-z|^{2n}} \left(\frac{|x-z|}{\rho(z)} \right)^{-2l}. \end{aligned}$$

From the above inequalities and Lemma 2.3, we obtain the inequality (3.4).

To prove the inequality (3.5), let us consider $|x-z| > 2|x-x'|$, denote $a = \min\{|x-y|, |x'-y|\}$, and define $B = \{y : |x-y| > 2|x-x'|\}$. Then, by Lemma 3.1(b) and the inequality (3.1), we can write

$$\begin{aligned} |\bar{K}(x,z) - \bar{K}(x',z)|_{\mathbb{B}}^2 &= \int_{\mathbb{R}^n} \int_0^{\infty} |\varphi_t(x,y) - \varphi_t(x',y)|^2 |Q_t(y,z)|^2 \frac{dtdy}{t^{n-1}} \\ &\le C \int_B \int_{a/2}^{\infty} \frac{|x-x'|^2}{t^{3n+1+2}} \frac{t^{2N}}{(t+|y-z|)^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy \\ &\quad + C \int_{B^c} \int_{a/2}^{\infty} \frac{1}{t^{3n+1}} \frac{t^{2N}}{(t+|y-z|)^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy := I + II. \end{aligned}$$

For $y \in B$, we have $|x'-y| > |x-y|/2$, and hence $a > |x-y|/2$. Denoting $B_1 = B \cap \{y : |x-y| \ge |x-z|/2\}$, we get

$$\begin{aligned} I &\le C|x-x'|^2 \int_{B_1} \int_{|x-y|/4}^{\infty} \frac{1}{t^{3n+1+2}} \frac{t^{2N}}{(t+|y-z|)^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy \\ &\quad + C|x-x'|^2 \int_{B \setminus B_1} \int_{|x-y|/4}^{\infty} \frac{1}{t^{3n+1+2}} \frac{t^{2N}}{(t+|y-z|)^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy \\ &:= I_1 + I_2. \end{aligned}$$

For $y \in B_1$, we have

$$\begin{aligned} I_1 &\le C|x-x'|^2 \int_{|x-y| \ge |x-z|/2} \int_{|x-y|/4}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2l+2}} dtdy \\ &\le C|x-x'|^2 \int_{|x-y| \ge |x-z|/2} \frac{\rho(z)^{2l}}{|x-y|^{3n+2l+2}} dy \\ &\le C \left(\frac{|x-z|}{\rho(z)} \right)^{-2l} \frac{|x-x'|^2}{|x-z|^{2n+2}}. \end{aligned}$$

For $y \in B \setminus B_1$, we have $|y-z| \sim |x-z|$, and hence

$$\begin{aligned} I_2 &\le C|x-x'|^2 \int_{B \setminus B_1} \int_{|x-y|/4}^{|x-z|} \frac{1}{t^{3n+1+2}} \frac{t^{2N}}{|y-z|^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy \\ &\quad + C|x-x'|^2 \int_{B \setminus B_1} \int_{|x-z|}^{\infty} \frac{1}{t^{3n+1+2}} \frac{t^{2N}}{|y-z|^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy := I_{2a} + I_{2b}. \end{aligned}$$

Then

$$\begin{aligned}I_{2a} &\le C\frac{|x-x'|^2}{|x-z|^{2N}}\int_{B\setminus B_1}\left(\int_{|x-y|/4}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2-2N+2l}}\,dt\right)dy\\ &\le C\frac{|x-x'|^2}{|x-z|^{2N}}\int_{|x-y|\le|x-z|/2}\frac{\rho(z)^{2l}}{|x-y|^{3n+2-2N+2l}}\,dy\\ &\le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^2}{|x-z|^{2n+2}},\end{aligned}$$

where $n+1<N-l<\frac{3n+2}{2}$, and

$$\begin{aligned}I_{2b} &\le C|x-x'|^2\int_{B\setminus B_1}\int_{|x-z|}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2+2l}}\,dtdy\\ &\le C\frac{\rho(z)^{2l}|x-x'|^2}{|x-z|^{3n+2+2l}}\int_{|x-y|\le|x-z|/2}dy\\ &\le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^2}{|x-z|^{2n+2}}.\end{aligned}$$

In this way, we get

$$I\le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^2}{|x-z|^{2n+2}}.$$

To estimate II, we notice that if $y\in B^c$, then $|x'-y|\le 3|x-x'|$, and hence

$$II\le\left(\int_{B_c}\int_{|x-y|/2}^{\infty}+\int_{|x'-y|\le 3|x-x'|}\int_{|x'-y|/2}^{\infty}\right)\frac{\rho(z)^{2l}}{t^{3n+2l+1}}\frac{t^{2N}}{(t+|y-z|)^{2N}}\,dtdy:=II_1+II_2.$$

Since the above two integrals are similar, we estimate only II_1, the estimate for II_2 can be obtained similarly.

We consider the set $(B^c)_1=B^c\cap\{y:|x-y|\ge|x-z|/2\}$, and notice that for $y\in(B^c)_1$, we have $|x-y|\sim|x-z|$ and $|y-z|\le 2|x-z|$, and for $y\in B^c\setminus(B^c)_1$, we have $|x-z|\sim|y-z|$ and $|x-y|<|x-z|/2$. Thus, we can write

$$\begin{aligned}II_1 &\le C\int_{(B^c)_1}\int_{|x-y|/2}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2l}}\,dtdy C\int_{B^c\setminus(B^c)_1}\int_{|x-y|/2}^{|x-z|}\frac{\rho(z)^{2l}}{t^{3n+1+2l}}\frac{t^{2N}}{|y-z|^{2N}}\,dtdy\\ &\quad+\int_{B^c\setminus(B^c)_1}\int_{|x-z|}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2l}}\,dtdy:=II_{1a}+II_{1b}+II_{1c}.\end{aligned}$$

Then, for II_{1a} we have

$$\begin{aligned}II_{1a} &\le C\int_{(B^c)_1}\frac{\rho(z)^{2l}}{|x-y|^{3n+2l}}dy\\ &\le\frac{C\rho(z)^{2l}}{|x-z|^{2n+2+2l}}\int_{|x-y|\le 3|x-x'|}\frac{dy}{|x-y|^{n-2}}\\ &\le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^2}{|x-z|^{2n+2}}.\end{aligned}$$

For II_{1b}, we take $N = n + 1$, to obtain

$$\begin{aligned} II_{1b} &\leq C \int_{B^*\setminus(B^*)_1} \int_{|x-y|/2}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2l}} \frac{t^{2n+2}}{|y-z|^{2n+2}} \, dt dy \\ &\leq \frac{C\rho(z)^{2l}}{|x-z|^{2n+2+2l}} \int_{|x-y|\leq 2|x-x'|} \frac{1}{|x-y|^{n-2}} \, dy \\ &\leq C \left(\frac{|x-z|}{\rho(z)} \right)^{-2l} \frac{|x-x'|^2}{|x-z|^{2n+2}}. \end{aligned}$$

For II_{1c}, since $|x-z| > 2|x-y|$, we have

$$\begin{aligned} II_{1c} &\leq C \int_{B^*\setminus(B^*)_1} \frac{\rho(z)^{2l}}{|x-z|^{3n+2l}} dy \\ &\leq \frac{C\rho(z)^{2l}}{|x-z|^{2n+2+2l}} \int_{|x-y|\leq 2|x-x'|} \frac{dy}{|x-y|^{n-2}} \\ &\leq C \left(\frac{|x-z|}{\rho(z)} \right)^{-2l} \frac{|x-x'|^2}{|x-z|^{2n+2}}. \end{aligned}$$

Combining the above inequalities and using Lemma 2.1, we get the inequality (3.5).

Now we proceed to prove the inequality (3.6). To this end, we consider $|x-z| > 2|x-x'|$ and define $E = \{y : |y-z| \geq |x-z|/2\}$. Note that $Q_t(x,y) = Q_t(y,x)$, and hence we can apply Lemma 3.1(b) to obtain

$$\begin{aligned} |\tilde{K}(z,x) - \tilde{K}(z,x')|_{\mathbb{B}}^2 &= \int_{\mathbb{R}^n} \int_{|y-z|/2}^{\infty} |\varphi_t(z,y)|^2 |Q_t(y,x) - Q_t(y,x')|^2 \frac{dt}{t^{n-1}} dy \\ &\leq \int_{\mathbb{R}^n} \int_{|y-z|/2}^{\infty} \frac{1}{t^{n+1}} |Q_t(x,y) - Q_t(x',y)|^2 \, dt dy \\ &\leq C|x-x'|^{2\delta_0} \int_{\mathbb{R}^n} \int_{|y-z|/2}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}} \, dt dy \\ &\leq C|x-x'|^{2\delta_0} \int_{E} \int_{|y-z|/2}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}} \, dt dy \\ &\quad + C|x-x'|^{2\delta_0} \int_{E^c} \int_{|y-z|/2}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}} \, dt dy \\ &:= III_1 + III_2. \end{aligned}$$

For III_1, we then have

$$III_1 \leq C|x-x'|^{2\delta_0} \int_E \frac{\rho(z)^{2l}}{|y-z|^{3n+2\delta_0+2l}} dy \leq C \left(\frac{|x-z|}{\rho(z)} \right)^{-2l} \frac{|x-x'|^{2\delta_0}}{|x-z|^{2n+2\delta_0}}.$$

If $y \in E^c$, then $|y-z| < |x-z|/2 < |x-y| < 2|x-z|$, and hence

$$I_2 \leq C|x-x'|^{2\delta_0} \int_{E^c} \left(\int_{|y-z|/2}^{|x-z|} + \int_{|x-z|}^{\infty} \right) \frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}} \, dt dy := III_{2a} + III_{2b}.$$

For III_{2a} and III_{2b}, we have the following estimates:

$$\begin{aligned}
III_{2a} &\le C\frac{|x-x'|^{2\delta_0}}{|x-z|^{2N}}\int_{E^c}\int_{|y-z|/2}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0-2N+2l}}\,dt\,dy\\
&\le C\frac{|x-x'|^{2\delta_0}}{|x-z|^{2N}}\int_{|y-z|\le|x-z|/2}\frac{\rho(z)^{2l}}{|y-z|^{3n+2\delta_0-2N+2l}}\,dy\\
&\le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^{2\delta_0}}{|x-z|^{2n+2\delta_0}},
\end{aligned}$$

(we take $d+\delta_0<N-l<(3n+2\delta_0)/2$), and

$$\begin{aligned}
III_{2b} &\le C|x-x'|^{2\delta_0}\int_{E^c}\int_{|x-z|}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}}\,dt\,dy\\
&\le C\frac{\rho(z)^{2l}|x-x'|^{2\delta_0}}{|x-z|^{3n+2\delta_0+2l}}\int_{|y-z|\le|x-z|/2}dy \le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^{2\delta_0}}{|x-z|^{2n+2\delta_0}}.
\end{aligned}$$

Combining the above inequalities, we get the inequality (3.6). □

Lemma 3.3. *Let $0<p,\eta<\infty$ and let $\omega\in A_\infty^\rho$, then the following inequalities hold:*

$$\int_{\mathbf{R}^n}|\widetilde{S}(f)(x)|_{\mathbf{B}}^p\omega(x)dx\le C\int_{\mathbf{R}^n}|M_{V,\eta}f(x)|^p\omega(x)dx$$

$$\sup_{\lambda>0}\lambda^p\omega(\{x\in\mathbf{R}^n:\ |\widetilde{S}(f)(x)|_{\mathbf{B}}>\lambda\})\le C\sup_{\lambda>0}\lambda^p\omega(\{x\in\mathbf{R}^n:\ M_{V,\eta}f(x)>\lambda\}),$$

Proof. By Fubini's theorem and the property of $s_Q f$, we have

$$\||\widetilde{S}(f)|_{\mathbf{B}}\|_2\le C\|s_Q(f)\|_2\le C\|f\|_2.$$

By Lemma 3.2 and the theory of vector valued singular integrals the result will be proved by showing that the kernel $\widetilde{K}$ of $\widetilde{S}$ is a standard vector valued Calderón-Zygmund kernel, and so $|\widetilde{S}|_{\mathbf{B}}$ is bounded on $L^p(\mathbf{R}^n)(1<p<\infty)$ and of weak type $(1,1)$. In view of the inequality $|f(x)|\le M_{\delta,\eta}^{\triangle}f(x)$ a.e $x\in\mathbf{R}^n$, and Theorem 2.1, to prove the lemma, we need only to show that for any $0<\eta<\infty$ and $0<\delta<\eta/(\eta+1)$ the following inequality holds:

$$M_{\delta,\eta}^{\sharp}(|\widetilde{S}(f)|_{\mathbf{B}})(x)\le CM_{V,\eta}(f)(x),\ \text{a.e}\ x\in\mathbf{R}^n, \tag{3.7}$$

We fix $x\in\mathbf{R}^n$ and assume that $x\in Q=Q(x_0,r)$ (dyadic cube). Decompose $f=f_1+f_2$, where $f_1=f\chi_{Q}$ with $Q=Q(x,8\sqrt{n}r)$. To prove the inequality (3.7), we consider the following two possible cases: $r<\rho(x_0)$ and $r\ge\rho(x_0)$.

Case 1: $r<\rho(x_0)$. Let $C_Q=|\widetilde{S}(f)(x_0)|_{\mathbf{B}}$. Since $0<\delta<1$, we can write

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_C\left||\widetilde{S}(f)(y)|_{\mathbf{B}}^\delta-C_Q^\delta\right|dy\right)^{1/\delta}\le\left(\frac{1}{|Q|}\int_C\left||\widetilde{S}(f)(y)|_{\mathbf{B}}-|\widetilde{S}(f_2)(x_0)|_{\mathbf{B}}\right|^\delta dy\right)^{1/\delta}\\
&\le C\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_1)(y)|_{\mathbf{B}}^\delta\,dy\right)^{1/\delta}+C\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_2)(y)-\widetilde{S}(f_2)(x_0)|_{\mathbf{B}}^\delta\,dy\right)^{1/\delta}=I+II.
\end{aligned}$$

To estimate the term I, we recall that $|\widetilde{S}(f)|_{\mathbb{B}}$ is of weak type $(1,1)$, and note that $\rho(x)\sim\rho(x_0)$ for any $x\in Q$ and $\Psi(Q)\sim 1$. Hence we can apply Kolmogorov's inequality (see [10]), to obtain

$$I \le \frac{C}{|Q|}\||\widetilde{S}(f)|_{\mathbb{B}}\|_{L^{1,\infty}} \le \frac{C}{|Q|}\int_Q |f(y)|\,dy \le CM_{V,\eta}f(x). \tag{3.8}$$

To estimate the term II, we let $Q_k=Q(x_0,2^{k+1}r)$ and $\alpha=\eta+1$. Then, taking $l\ge\theta\alpha$ and using (3.5), we obtain

$$\begin{aligned}
II &\le \frac{C}{|Q|}\int_Q |\widetilde{S}(f_2)(y)-\widetilde{S}(f_2)(x_0)|_{\mathbb{B}}\,dy\\
&\le \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus Q}|\widetilde{K}(y,\omega)-\widetilde{K}(x_0,\omega)|_{\mathbb{B}}|f(\omega)|d\omega dy\\
&\le \frac{C}{|Q|}\int_Q\int_{|x_0-\omega|>2r}|\widetilde{K}(y,\omega)-\widetilde{K}(x_0,\omega)|_{\mathbb{B}}|f(\omega)|d\omega dy\\
&\le \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^kr\le|x_0-\omega|<2^{k+1}r}|\widetilde{K}(y,\omega)-\widetilde{K}(x_0,\omega)|_{\mathbb{B}}|f(\omega)|d\omega dz dy
\end{aligned}$$

$$\begin{aligned}
&\le C_l\sum_{k=2}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r m_V(x_0))^l(2^{k+1}r)^n}\int_{Q_k}|f(\omega)|d\omega\\
&\le C_l\sum_{k=1}^{\infty}\frac{2^{-k\delta_0}}{(1+2^kr/\rho(x_0))^{l-\alpha\theta}}\frac{1}{(1+2^kr/\rho(x_0))^{\alpha\theta}|Q_k|}\int_{Q_k}|f(\omega)|d\omega\\
&\le C_l\sum_{k=1}^{\infty}2^{-k\delta_0}M_{V,\eta}(f)(x)\le C_lM_{V,\eta}(f)(x).
\end{aligned}\tag{3.9}$$

Case 2: $r\ge\rho(x_0)$. In this case, noting that $\alpha_1:=\eta/\delta\ge\eta+1$, we get

$$\begin{aligned}
\frac{C}{\Psi(Q)^{\alpha_1}}\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_1)(y)|_{\mathbb{B}}^{\delta}dy\right)^{1/\delta} &\le \frac{C}{\Psi(Q)^{\alpha_1}}\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_1)(y)|_{\mathbb{B}}^{\delta}dy\right)^{1/\delta}\\
&\quad+\frac{C}{\Psi(Q)^{\alpha_1}}\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_2)(y)|_{\mathbb{B}}^{\delta}dy\right)^{1/\delta}\\
&=: I_1+II_1.
\end{aligned}$$

For I_1, similar to I, we have the following estimate

$$\begin{aligned}
I_1 &\le \frac{C}{\Psi(Q)^{\alpha_1}}\frac{1}{|Q|}\||\widetilde{S}(f_1)|_{\mathbb{B}}\|_{L^{1,\infty}}\\
&\le \frac{C}{\Psi(Q)^{\eta}(\Psi(Q)|Q|)}\int_Q|f(y)|\,dy\le CM_{V,\eta}f(x).
\end{aligned}\tag{3.10}$$

As for II_1, taking $l = \alpha_1\theta + 1$, and using (3.5) and Lemma 2.1, we get

$$
\begin{aligned}
II_1 &\le \frac{C}{|Q|}\int_Q |\widetilde{S}(f_2)(y)|_{\mathbb{B}}\,dy \le \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus\widetilde{Q}} |\widetilde{K}(y,\omega)|_{\mathbb{B}}|f(\omega)|d\omega dy\\
&\le \frac{C}{|Q|}\int_Q\sum_{k=3}^{\infty}\int_{2^k r\le|x_0-\omega|<2^{k+1}r} |\widetilde{K}(y,\omega)|_{\mathbb{B}}|f(\omega)|d\omega dy\\
&\le C_l\sum_{k=3}^{\infty}\frac{1}{(1+2^k r/\rho(x_0))^l|Q_k|}\int_{Q_k}|f(\omega)|d\omega \qquad (3.11)\\
&\le C_l\sum_{k=1}^{\infty}\frac{1}{(1+2^k r/\rho(x_0))^{l-\theta\alpha_1}}\frac{1}{(1+2^k r/\rho(x_0))^{\theta\alpha_1}|Q_k|}\int_{Q_k}|f(\omega)|d\omega\\
&\le C_l\sum_{k=1}^{\infty}2^{-k}M_{V,\eta}(f)(x) \le C_l M_{V,\eta}(f)(x).
\end{aligned}
$$

From (3.8)–(3.11), we get (3.7). Lemma 3.3 is proved. □

Lemma 3.4. *Let $b\in BMO(\rho)$ and $(l_0+1)\le\eta<\infty$, and let $0<2\delta<\epsilon<1$. Then for any $f\in C_0^\infty(\mathbb{R}^n)$ and the following inequality holds:*

(3.12)
$$M^{\sharp}_{\delta,\eta}(|[b,\widetilde{S}]f|_{\mathbb{B}})(x) \le C\|b\|_{BMO(\rho)}(M^{\triangle}_{\epsilon,\eta}(|\widetilde{S}(f)|_{\mathbb{B}})(x)+M_{L\log L,V,\eta}(f)(x)),\ \text{a.e. } x\in\mathbb{R}^n,$$

Proof. Observe first that for any constant λ we have

$$[b,\widetilde{S}]f(x) = (b(x)-\lambda)\widetilde{S}(f)(x) - \widetilde{S}((b-\lambda)f)(x).$$

As above, we fix $x\in\mathbb{R}^n$ and assume that $x\in Q=Q(x_0,r)$ (dyadic cube). Decompose $f=f_1+f_2$, where $f_1=f\chi_{\bar{B}}$ with $\tilde{Q}=Q(x,8\sqrt{n}r)$.

To prove the inequality (3.12), again we consider the following two possible cases: $r<\rho(x_0)$ and $r\ge\rho(x_0)$.

Case 1: $r<\rho(x_0)$. We first fix $\lambda=b_{\bar{Q}}$, the average of b on Q. Since $0<\delta<1$, we then can write

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_C \big|\, |[b,\widetilde{S}]f(y)|^{\delta}_{\mathbb{B}} - |\widetilde{S}((b-b_{\bar{Q}})f)(x_0)|^{\delta}_{\mathbb{B}}\big|\,dy\right)^{1/\delta}\\
&\le\left(\frac{1}{|Q|}\int_Q \big||[b,\widetilde{S}]f(y)|_{\mathbb{B}} - \widetilde{S}((b-b_{\bar{Q}})f_2)(x_0)\big|^{\delta}_{\mathbb{B}}\,dy\right)^{1/\delta}\\
&\le C\left(\frac{1}{|Q|}\int_Q |(b(y)-b_{\bar{Q}})|\widetilde{S}f(y)|_{\mathbb{B}}|^{\delta}\,dy\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q |\widetilde{S}((b-b_{\bar{Q}})f_1)(y)|^{\delta}_{\mathbb{B}}\,dy\right)^{1/\delta}\\
&\quad + C\left(\frac{1}{|Q|}\int_Q |\widetilde{S}((b-b_{\bar{Q}})f_2)(y) - \widetilde{S}((b-b_{\bar{Q}})f_2)(x_0)|^{\delta}_{\mathbb{B}}\,dy\right)^{1/\delta} =: I+II+III.
\end{aligned}
$$

Then for any $1 < \gamma < \epsilon/\delta$, note that $\rho(x) \sim \rho(x_0)$ for any $x \in Q$ and $\Psi(Q) \sim 1$. Hence, by Lemma 2.3, for the term I we obtain the estimate

$$
\begin{aligned}
I &\leq C\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|b(y)-b_{\tilde{Q}}|^{\delta\gamma'}dy\right)^{\gamma'/\delta}\left(\frac{1}{|Q|}\int_{Q}|\widetilde{S}f(y)|_{\mathbb{B}}^{\delta\gamma}dy\right)^{\delta\gamma} \\
&\leq C\|b\|_{BMO(\rho)}M^{\Delta}_{\epsilon,\eta}(|\widetilde{S}f(y)|_{\mathbb{B}})(x),
\end{aligned}
\tag{3.13}
$$

where $1/\gamma' + 1/\gamma = 1$.

To estimate the term II, we recall that $|\widetilde{S}|_B$ is of weak type $(1,1)$, and note that $\rho(x) \sim \rho(x_0)$ for any $x \in Q$ and $\Psi(\tilde{Q}) \sim 1$. Hence, by Kolmogorov's inequality and Proposition 2.1, we get

$$
\begin{aligned}
II &\leq \frac{C}{|Q|}\||\widetilde{S}((b-b_{\tilde{Q}})f_1)|_{\mathbb{B}}\|_{L^{1,\infty}} \\
&\leq \frac{C}{|Q|}\int_{\tilde{Q}}|(b-b_{\tilde{Q}})f(y)|\,dy \leq CM_{L\log L,V,\eta}f(x).
\end{aligned}
\tag{3.14}
$$

For the term III, we let $b_{Q_k} = b_{Q(x_0,2^{k+1}r)}$ and $\theta' = (l_0+1)\theta$. Then, in view of Lemmas 2.1 and 3.2, we can write

(3.15)

$$
\begin{aligned}
III &\leq \frac{C}{|Q|}\int_{Q}|\widetilde{S}((b-b_{\tilde{Q}})f_2)(y)-\widetilde{S}((b-b_{\tilde{Q}})f_2)(x_0)|_{\mathbb{B}}\,dy \\
&\leq \frac{C}{|Q|}\int_{Q}\int_{|x_0-\omega|>2r}|\widetilde{K}(y,\omega)-\widetilde{K}(x,\omega)|_{\mathbb{B}}|(b(\omega)-b_{\tilde{Q}})f(\omega)|d\omega dy \\
&\leq \frac{C}{|Q|}\int_{Q}\sum_{k=2}^{\infty}\int_{2^k r\leq|x_0-\omega|<2^{k+1}r}|\widetilde{K}(y,\omega)-\widetilde{K}(x,\omega)|_{\mathbb{B}}|(b(\omega)-b_{\tilde{Q}})f(\omega)|d\omega dy \\
&\leq C_l\sum_{k=1}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r/\rho(x_0))^l|Q_k|}\int_{Q_k}|b(\omega)-b_{\tilde{Q}}|f(\omega)|d\omega \\
&\leq C_l\sum_{k=1}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r/\rho(x_0))^{l-(\eta+1)\theta'}} \\
&\qquad\times\frac{1}{(1+2^k r/\rho(x_0))^{(\eta+1)\theta'}|Q_k|}\int_{Q_k}|b(\omega)-b_{Q_k}||f(\omega)|d\omega \\
&+C_l\sum_{k=1}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r/\rho(x_0))^{l-\theta'(\eta+2)}} \\
&\qquad\times\frac{1}{(1+2^k r/\rho(x_0))^{\theta'(\eta+2)}|Q_k|}|b_{\tilde{Q}}-b_{Q_k}|\int_{Q_k}|f(\omega)|d\omega \\
&\leq C_l\sum_{k=1}^{\infty}2^{-k\delta_0}\|b\|_{BMO(\rho)}M_{L\log L,V,\eta}(f)(x)+C_l\|b\|_{BMO(\rho)}M_{V,\eta}(f)(x)\sum_{k=1}^{\infty}k2^{-k\delta_0} \\
&\leq C_l\|b\|_{BMO(\rho)}M_{L\log L,V,\eta}(f)(x),
\end{aligned}
$$

where $l = (\eta+2)\theta'$, and in the last inequality we have used the following inequalities:

$$
M_{V,\eta}(f)(x) \leq M_{L\log L,V,\eta}(f)(x) \quad\text{and}\quad |b_{\tilde{Q}}-b_{Q_k}| \leq C(1+2^k r/\rho(x_0))^{\theta}\|b\|_{BMO(\rho)}.
$$

Case 2: $r \geq \rho(x_0)$. Since $0 < 2\delta < \epsilon < 1$, we have $a = \eta/\delta$ and $\epsilon/\delta > 2$. Hence, we can write

$$\begin{aligned}
&\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q \|[b,\widetilde{S}]f(y)\|_{\mathbb{B}}^{\delta}\,dy\right)^{1/\delta}\\
&=\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |(b(y)-b_Q)\widetilde{S}(f)(y)-\widetilde{S}((b-b_Q)f)(y)|_{\mathbb{B}}^{\delta}\,dy\right)^{1/\delta}\\
&\leq C\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |(b(y)-b_Q)\widetilde{S}(f)(y)|_{\mathbb{B}}^{\delta}\,dy\right)^{1/\delta}\\
&\quad+C\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |\widetilde{S}((b-b_Q)f_1)(y)|_{\mathbb{B}}^{\delta}\,dy\right)^{1/\delta}\\
&\quad+C\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |\widetilde{S}((b-b_Q)f_2)(y)|_{\mathbb{B}}^{\delta}\,dy\right)^{1/\delta} := I_1+II_1+III_1.
\end{aligned}$$

Then noting that $l_0+1 \leq \eta$ for any $2 \leq \gamma < \epsilon/\delta$, by Lemma 2.3 we obtain the following estimate for I_1:

(3.16)
$$\begin{aligned}
I_1 &\leq C\frac{1}{\Psi_{\theta'}(Q)}\left(\frac{1}{|Q|}\int_Q |b(y)-b_Q|^{\delta\gamma'}\,dy\right)^{1/(\gamma'\delta)}\\
&\quad\times\frac{\Psi_{\theta'}(Q)}{\Psi(Q)^{a-\eta/(2\delta)}}\left(\frac{1}{\Psi(Q)^{\eta}|Q|}\int_Q |\widetilde{S}(f)(y)|_{\mathbb{B}}^{\delta\gamma}\,dy\right)^{1/(\delta\gamma)}\\
&\leq C\|b\|_{BMO(\rho)}M^{\Delta}_{\epsilon,\eta}(|\widetilde{S}(f)|_{\mathbb{B}})(x),
\end{aligned}$$

where $1/\gamma'+1/\gamma=1$.

To estimate II_1, we recall that $|\widetilde{S}|_{\mathbb{B}}$ is of weak type $(1,1)$, and use Kolmogorov's inequality and Proposition 2.1, to obtain

(3.17)
$$\begin{aligned}
II_1 &\leq \frac{C}{\Psi(Q)^a}\frac{1}{|Q|}\||\widetilde{S}((b-b_Q)f_1)|_{\mathbb{B}}\|_{L^{1,\infty}}\\
&\leq \frac{C}{\Psi(Q)^a}\frac{1}{|Q|}\int_Q |(b-b_Q)f(y)|\,dy\\
&\leq CM_{L\log L,V,\eta}f(x).
\end{aligned}$$

Finally, to estimate III_1, we let $b_{Q_k} = b_{Q(x_0,2^{k+1}r)}$ and $\theta' = (l_0+1)\theta$. Then, we use Lemmas 2.1 and 3.2 with $l = (\eta+2)\theta'+1$, to obtain

(3.18)

$$\begin{aligned}
III &\le \frac{C}{|Q|}\int_Q |\tilde{S}((b-b_{\tilde{Q}})f_2)(y)|_{\mathbb{B}}\,dy \\
&\le \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus\tilde{Q}} |\tilde{K}(y,\omega)|_{\mathbb{B}}|(b(\omega)-b_{\tilde{Q}})f(\omega)|d\omega dy \\
&\le \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^k r\le|x_0-\omega|<2^{k+1}r} |\tilde{K}(y,\omega)||(b(\omega)-b_{\tilde{Q}})f(\omega)|d\omega dy \\
&\le C_l\sum_{k=1}^{\infty}\frac{1}{(1+2^k r/\rho(x_0))^l|Q_k|}\int_{Q_k}|b(\omega)-b_{\tilde{Q}}|f(\omega)|d\omega \\
&\le C_l\sum_{k=1}^{\infty}\frac{1}{(1+2^k/\rho(x_0))^{l-(\eta+2)\theta'}} \\
&\qquad\times\frac{1}{(1+2^k r/\rho(x_0))^{(\eta+2)\theta'}|Q_k|}\int_{Q_k}|b(\omega)-b_{Q_k}||f(\omega)|d\omega \\
&+C_l\sum_{k=1}^{\infty}\frac{1}{(1+2^k r/\rho(x_0))^{l-(\eta+2)\theta'}} \\
&\qquad\times\frac{1}{(1+2^k r/\rho(x_0))^{(\eta+2)\theta'}|Q_k|}|b_{\tilde{Q}}-b_{Q_k}|\int_{Q_k}|f(\omega)|d\omega \\
&\le C_l\sum_{k=1}^{\infty}2^{-k}\|b\|_{BMO(\rho)}M_{L\log L,V,\eta}(f)(x)+C_l\|b\|_{BMO(\rho)}M_{V,\eta}(f)(x)\sum_{k=1}^{\infty}k2^{-k} \\
&\le C_l\|b\|_{BMO(\rho)}M_{L\log L,V,\eta}(f)(x).
\end{aligned}$$

From (3.13)–(3.18), we get (3.12). Lemma 3.4 is proved. □

Finally, we recall the following results proved in [13, 14].

Lemma 3.5. *Let $0<\eta<\infty$ and $M_{V,\eta/2}f$ be locally integrable. Then there exist positive constants C_1 and C_2 independent of f and x such that*

$$C_1M_{V,\eta}M_{V,\eta+1}f(x)\le M_{L\log L,V,\eta+1}f(x)\le C_2M_{V,\eta/2}M_{V,\eta/2}f(x).$$

Lemma 3.6. *Let $2\le\eta<\infty$, $\omega\in A_1^\rho$ and $B(t)=t\log(e+t)$. Then there exists a constant $C>0$ such that for all $t>0$*

$$\omega(\{x\in\mathbb{R}^n:\ M_{B,V,\eta}f(x)>t\})\le C\int_{\mathbb{R}^n}B\left(\frac{|f(x)|}{t}\right)\omega(x)dx. \tag{3.19}$$

Proof. Let K be any compact subset in $\{x\in\mathbb{R}^n: M_{L\log L,\varphi,\eta}(f)(x)>\lambda\}$. For any $x\in K$, by a standard covering lemma, it is possible to choose cubes $Q_1,\cdots,Q_m$ with pairwise disjoint interiors such that $K\subset\bigcup_{j=1}^m 3Q_j$ and $\|f\|_{L\log L,\varphi,Q_j}>\lambda$, $j=1,\cdots,m$. This implies

$$\Psi(Q_j)^2|Q_j|\le\int_{Q_j}\frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)dy.$$

From the last inequality we obtain

$$\begin{aligned}\omega(3Q_j) &\le C\Psi(Q_j)\omega(Q_j) = C\Psi(Q_j)^2|Q_j|\frac{\omega(Q_j)}{\Psi(Q_j))|Q_j|} \\ &\le C\frac{\omega(Q_j)}{\Psi(Q_j)|Q_j|}\int_{Q_j}\frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)dy \\ &\le C\inf_{Q_j}\omega(x)\int_{Q_j}\frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)dy \\ &\le C\int_{Q_j}\frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)\omega(y)\,dy,\end{aligned}$$

implying (3.19). □

4. Proof of main results

Proof of Theorem 1.1. We first notice that

$$S_Q(f)(x) \le C|\tilde{S}(f)(x)|_{\mathbb{B}} \text{ for every } x \in \mathbb{R}^n. \tag{4.1}$$

Thus, the desired results follow from (4.1), Lemmas 3.2, Theorem 2.2 and Proposition 2.2. □

Proof of Theorem 1.2. We first notice that

$$S_{Q,b}(f)(x) \le C|[b,\tilde{S}]f(x)|_{\mathbb{B}} \text{ for every } x \in \mathbb{R}^n. \tag{4.2}$$

Using arguments similar to those applied in [10], the inequality (4.2), Lemmas 3.3-3.6, Proposition 2.1, and Theorems 1.1, 2.1 and 2.2, we can obtain the desired results. □

Remark. It can be shown that the analogs of Theorems 1.1 and 1.2 hold for spaces $BMO_{\theta_1}(\rho)$ and $A_p^{\theta_2}$ if $\theta_1 \ne \theta_2$.

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О НЕПРЕРЫВНЫХ СЕЛЕКЦИЯХ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ С ПОЧТИ ВЫПУКЛЫМИ ЗНАЧЕНИЯМИ

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Аннотация. Доказано, что через каждую точку графика непрерывного многозначного отображения с почти выпуклыми и звездными значениями проходит непрерывная селекция этого отображения.

MSC2010 number: 26E25; 49J52; 46J05.

Ключевые слова: многозначное отображение; звездное множество; почти выпуклость; селектор.

1. Введение

Одним из важных проблем в теории многозначных отображений является вопрос существования однозначных аппроксимаций и селекций с определенными свойствами. Вопрос о существовании селекций, обладающих некоторыми топологическими свойствами весьма интересен и находит разнообразные приложения во многих областях математики. Задача о существовании непрерывных селекций мультиотображения, восходящая к классической теореме Э. Майкла (см. [15]) получила в дальнейшем широкое развитие и нашла многочисленные приложения в теории дифференциальных включениях, управляемых системах и в общей топологии (см. [1, 4]). Теорема Майкла утверждает, что всякое полунепрерывное снизу отображение с выпуклыми значениями допускает непрерывную селекцию.

В работах [1, 16] приведены примеры, иллюстрирующие важность условия выпуклости многозначного отображения. А в статье [9] (пример 1(A)) построен пример непрерывного отображения со звездными значениями, недопускающий ни одного непрерывного селектора. Тем не менее, существование непрерывных селекций может быть доказано и для некоторых классов отображений с

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невыпуклыми значениями. Так например в статье [9] рассмотрен некоторый подкласс (отображения с звездноподобными или **паравыпуклыми** значениями) непрерывных многозначных отображений со звездными значениями, допускающие непрерывные селекции (см. теорему 1 из [9]). В общем невыпуклом случае в статье [10] к каждому замкнутому множеству M ставят в соответсвие некоторую функцию $h : R_+ \to R_+$ невыпуклости множества M. Доказано(см. [10], теорема 5.1), что если a полунепрерывное снизу отображение, функции выпуклости $h_{a(x)}$ значений которого строго меньше некоторой монотонно неубывающей функции $\alpha : (0, \infty) \to [0, 1)$, то a имеет непрерывную однозначную селекцию. Однако, определение функции h имеет описательный характер и довольно сложно построить эту функцию для каждого замкнутого множества M.

Отметим также, что в статьях [11, 12] методом касательных конусов выделяются дифференцируемые или дифференцируемые по направлениям локальные селекции от многозначных отображений как с выпуклыми так и невыпуклыми значениями.

В настоящей статье рассматривается вопрос существования непрерывных селекций для нового класса многозначных отображений с невыпуклыми значениями, точнее отображениями с **почти выпуклыми** значениями. Понятие почти выпуклости введено в работах [7, 8]. Потребность изучения таких множеств возникла в теории дифференциальных игр [5].

2. Некоторые обозначения и определения

Пусть X – метрическое а Y – банахово пространства. В дальнейшем $B_r(a)$ – замкнутый шар с центром a радиуса r; $M \subseteq Y$ – замкнутое множество а $diam(M)$- диаметр множества M, $conv\{M\}$ – выпуклая оболочка множества M. Положим

$$Pr_M(x) \equiv \{y \in M / \|x - y\| = \inf_{z \in M} \|x - z\| \equiv d(x, M)\}.$$

Напомним определения многозначного отображения и селектора. Пусть 2^Y совокупность всех непустых подмножеств из Y, а E – подмножество пространства X.

Отображение $a : E \longrightarrow 2^Y$ называется многозначным отображением. Непрерывное однозначное отображение $y : E \longrightarrow Y$ называется непрерывной селекцией (непрерывным селектором) отображения a, если $y(x) \in a(x),\ x \in E$.

Отображение $a : E \longrightarrow 2^Y$ называется полунепрерывным снизу в $x_0 \in E$, если для любого $\varepsilon > 0$ существует такое $\delta > 0$, что

$$a(x_0) \subseteq a(x) + B_\varepsilon(0), \ \forall x \in E \cap B_\delta(x_0).$$

Отображение $a : E \longrightarrow 2^Y$ называется полунепрерывным сверху в $x_0 \in E$, если для любого $\varepsilon > 0$ существует такое $\delta > 0$, что

$$a(x) \subseteq a(x_0) + B_\varepsilon(0) \ \forall x \in E \cap B_\delta(x_0).$$

Если отображение полунепрерывно снизу и сверху в x_0, то оно называется непрерывным в этой точке (см. [1], определение 1.2.43 непрерывности в смысле Хаусдорфа). Множество

$$graph(a) = \{(x, y) \in E \times R^m, \ y \in a(x)\}.$$

называется графиком отображения a.

Определение 2.1. *(см. [3]). Пусть $M \subseteq Y$. Положим*

$$M^0 = \{x \in M : \ \lambda x + (1-\lambda)y \in M, \ \forall y \in M, \lambda \in [0,1]\}.$$

Подмножество $M^0 \subseteq M$ называется ядром звездности множества M. Если $M^0 \neq \emptyset$, то множество M называется звездным. Нетрудно показать, что M^0- выпуклое множество. Очевидно, что если $M-$ выпуклое множество, то $M = M^0$.

Определение 2.2. *(см. [7]). Множество $M \subset Y$ удовлетворяет условию почти выпуклости с константой $\theta \geq 0$, если для любых*

$$x_j \in M, \lambda_j \geq 0, j \in J,$$

где $J-$ конечное множество индексов, таких, что $\sum_{j\in J} \lambda_j = 1$, выполняется

$$\sum_{j \in J} \lambda_j x_j \in M + \theta r^2 B_1(0),$$

где $r = \max_{i,j \in J} \|x_i - x_j\|$.

Если нет необходимости уточнять константу θ, то будем просто говорить, что множество M почти выпукло. Заметим, что если $\theta = 0$, то $M-$ выпуклое множество. Класс почти выпуклых множеств достаточно широк.

3. Примеры

Пример 3.1. Множество $M = \{a, b\}$ состоящих из двух точек почти выпукло. Действительно, имеем

$$conv\{a,b\} \subseteq M + \frac{1}{2\|a-b\|}\|a-b\|^2 B_1(0),$$

т.е. в этом случае константу почти выпуклости θ можно выбрать $1/(2\|a-b\|)$.

Пример 3.2. Дуга на окружности является почти выпуклым множеством. Это непосредственно следует из достаточного условия почти выпуклости, доказанного в [8], Теорема 2. Найдем константу почти выпуклости. Предположим, что дуга M меньше полуокружности и множество $Q = \{x_1, x_2, ..., x_k\}$ находится на этой дуге. Пусть $A = x_1, B = x_k, d = diam(Q) = AB$. Тогда как видно из рисунка 1 множество $conv\{Q\}$ находится на α-окрестности множества M, где $\alpha = CD$. Имеем

$$DC = R - \sqrt{R^2 - \frac{d^2}{4}}.$$

Теперь число θ выберем из неравенства $DC \leq \theta d^2$, т.е.

$$\frac{1}{R + \sqrt{R^2 - \frac{d^2}{4}}} \leq \theta.$$

Очевидно, что этому неравенству удовлетворяют числа $\theta \geq 1/4R$. Если дуга больше полуокружности, то она почти выпукла по теореме 3 из [8] с некоторой константой ϑ. Тогда, как видно из рисунка 1, если $Q = \{a, b\}$, то множество $conv\{Q\}$ находится в β- окрестности дуги, где $\beta = \|a - b\|/2$. Таким образом

$$\theta \geq \frac{1}{2\|a-b\|}.$$

Значит, если $\|a - b\| \to 0$, то $\theta \to \infty$.

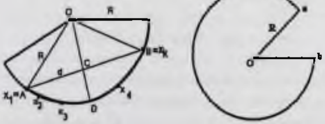

Рис. 1

Пример 3.3. Окружность M с радиусом R является почти выпуклым множеством с константой $\theta \geq 1/(\sqrt{3}R)$. Действительно, пусть множество $Q = \{x_1, x_2, ..., x_k\} \subset M$. Рассмотрим две случаи. Если $0 \notin conv\{Q\}$. Это означает, что множество находится в некоторой полуокружности. Тогда из примера 3.2 следует включение

$$conv\{Q\} \subseteq M + \frac{1}{4R}(diam(Q))^2 B_1(0). \tag{3.1}$$

Если $0 \in intQ$. Тогда в $convQ$ существует некоторый остроугольный треугольник, содержащий внутри себя центр окружности 0. Значит, окружность описана к этому треугольнику. Следовательно, длина некоторой стороны треугольника больше или равно $R\sqrt{3}$. Отсюда

$$diam(Q) \geq \sqrt{3}R.$$

Очевидно, что множество Q находится в R-окрестности множества M. Теперь выберем число θ из условия

$$R \leq \theta(diam(Q))^2. \tag{3.2}$$

Это неравество имеет место, если $\theta \geq 1/(\sqrt{3}R)$. Если точка O находится на границе множества $conv\{Q\}$, то $diam(Q) = 2R$. Тогда неравенство (2) выполняется, если $\theta \geq 1/4R$. В общем случае, имея ввиду и включение(1), имеем

$$conv\{Q\} \subset M + \frac{1}{\sqrt{3}R}(diam(Q))^2 B_1(0).$$

Отсюда $M-$ почти выпуклое множество с константой $1/(\sqrt{3}R)$.

Приведем пример множества, являющегося почти выпуклым и звездным, но не выпуклым.

Пример 3.4. Окрашенная область M на рис.2 с замкнутой границей $ACBDA$ является звездным множеством. Покажем, что оно почти выпукло. Выберем число $\theta > 0$ из условия

$$DE = R - \sqrt{R^2 - \frac{d^2}{4}} \leq \theta d^2, \text{ где } d = AB.$$

Это неравенство очевидным образом выполняется, если положим $\theta = \frac{1}{4R}$. Нетрудно заметить также, что область M является почти выпуклым множеством с константой θ. Заметим, что если $DO = R \to \infty$, то $\theta \to 0$, а область $ACDBA$ превращается в треугольник ACB.

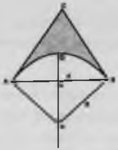

Рис. 2. Множество почти выпуклое и звездное

4. Свойства почти выпуклых множеств

Предложение 4.1. *([8], Теорема 3). Пусть замкнутое множество $M \subseteq R^n$ удовлетворяет условию почти выпуклости с константой $\theta > 0$. Если $\varepsilon \leq 1/(16\theta)$, то отображение $x \longrightarrow Pr_M(x)$ однозначно на множестве $M + B_\varepsilon(0)$ и*

$$\|Pr_M(x_1) - Pr_M(x_2)\| \leq 2\|x_1 - x_2\|.$$

Следует отметить, что если M – выпукло и замкнуто, то любая точка из R^n имеет единственную проекцию на M и оператор проектирования Pr_M удовлетворяет условию Липшица с постоянной 1.

Замечание 4.1. *Ф. Кларк и другие [14] определили понятие проксимально гладкого множества как множества, функция расстояния до которого от некоторой точки пространства непрерывно дифференцируема в некоторой окрестности этого множества, за исключением самого множества. В той же работе показано (см. теорема 4.11 [14]), что в гильбертовых пространствах условие проксимальной гладкости множества эквивалентно тому, что метрическая проекция на это множество любой точки из достаточно малой окрестности множества существует, единственна и непрерывно зависит от проецируемой точки. Затем в статье [13] доказан аналогичный результат в некоторых равномерно выпуклых и гладких банаховых пространствах. Из предложения 4.1 непосредственно следует, что если $M \subset R^n$ и почти выпукло, то оно и проксимально гладко.*

Таким образом, в пространстве R^n почти выпуклые множества составляют некоторый подкласс в семействе проксимально гладких множеств.

Предложение 4.2. *Если $M \subseteq R^n$ – замкнутое, звездное и почти выпуклое множество, то для достаточно малых $\varepsilon > 0$ множество $M + B_\varepsilon(0)$ также звездно и почти выпукло.*

Доказательство. Если замкнутое множество M – почти выпукло с константой θ, то известно (см. [8], Теорема 3, Следствие 3) что, если $\varepsilon < 1/(16\theta)$, то множество $M + B_\varepsilon(0)$ является почти выпуклым с константой 16θ. Нетрудно также показать, что $(M^0 + B_\varepsilon(0)) \subseteq (M + B_\varepsilon(0))^0$. Значит, M – звездное множество. □

Теорема 4.1. *Пусть непрерывное многозначное отображение $a : [a, b] \longrightarrow 2^{R^n}$ с почти выпуклыми значениями и с константой θ. Тогда через любую точку его графика проходит непрерывная селекция этого отображения.*

Доказательство. Поскольку отображение a непрерывно по Хаусдорфу на отрезке $[a, b]$ то оно равномерно непрерывно на этом отрезке. Это значит, что для любого $\varepsilon > 0$ найдется число $\delta > 0$ такое, что при разбиении отрезка на частичные сегменты $[x_{i-1}, x_i]$, длины которых меньше δ колебание отображения a на каждом таком частичном сегменте будет меньше ε. Выберем $\varepsilon < 1/(16\theta)$. Тогда

$$a(x_{i-1}) \in a(x) + B_\varepsilon(0)\ x \in [x_{i-1}, x_i].$$

Пусть $\overline{y_0} \in a(x_0)$. Положим $y_0(x) = Pr_{a(x)}\overline{y_0}$ $(x \in [x_0, x_1])$. Поскольку, согласно предложению 4.1, проекция точки $\overline{y_0}$ на множество $a(x)$ единственно и отображение a непрерывно, то непрерывным будет и отображение y_0 (см. [2], глава 3, п.5, лемма 3, стр. 344). Выберем точку $y_0(x_1)$ и спроектируем ее на множество $a(x)$ $(x \in [x_1, x_2].)$ Положим $y_1(x) = Pr_{a(x)}y_0(x_1)$. Оно также будет непрерывным отображение по вышеуказанным причинам.

Продолжая аналогично, мы построим непрерывное отображение $y(x)$, определенное на целом отрезке $[a, b]$ такое, что

$$y(x) = y_i(x),\ (x \in [x_{i-1}, x_i],\ i = 1, 2, ..., n.)$$

Теорема 4.1 доказана. □

Замечание 4.2. *Отображение y, построенное в теореме 4.1 зависит также от начальной точки $\overline{y_0}$. Используя предложение 4.1 легко заметить, что отображение y удовлетворяет условию Липшица относительно переменной $\overline{y_0}$*

равномерно по x. Следовательно, отображение y непрерывно по совокупности переменных (x, y_0).

Существует пример непрерывного многозначного отображения $a : R^2 \longrightarrow R^2$ с почти выпуклыми и компактными значениями, который не допускает ни одного непрерывного селектора.

Пример 4.1. Пусть

$$a(x) = S_1 \setminus B_{\|x\|}(\frac{x}{\|x\|}), \ x \neq 0, \ a(0) = S_1.$$

В [1](пример 1.4.6., стр. 58) доказано, что отображение a является непрерывным и не допускает непрерывного селектора. Заметим еще, что отображение a с почти выпуклыми значениями. Действительно, так как множество $a(x)$– дуга на единичной окружности S_1, то как отмечено выше в примере 3.3 оно почти выпукла. Причем если дуга $a(x)$ меньше полуокружности, то она почти выпукла с постоянной $\theta = 1/4$. Если дуга $a(x)$ больше полуокружности, то она почти выпукла с некоторой константой θ. А единичная окружность S_1 почти выпукла с константой $1/\sqrt{3}$.

5. Основные результаты

Пусть $a : E \to 2^{R^m}$– многозначное отображение. Определим отображение a_0 по правилу: $a_0(x) \equiv (a(x))^0 \ \forall x \in E$. Очевидно, что многозначное отображение $a_0 : E \to 2^{R^m}$ имеет выпуклые значения.

Теорема 5.1. *Пусть E– компактое подмножество метрического пространства X а $a : E \to 2^{R^m}$– непрерывное многозначное отображение с компактными, звездными и почти выпуклыми значениями. Предположим также, константы $\theta(x)$ почти выпуклости множеств $a(x)$, $x \in E$ удовлетворяют условию:*

$$\sup_{x \in E} \theta(x) = \eta < \infty.$$

Пусть $(x_0, y_0) \in graph(a)$. Тогда существует непрерывная селекция y отображения a, проходящая через точку (x_0, y_0), т.е.

$$y(x_0) = y_0, \ y(x) \in a(x) \ \forall x \in E.$$

Доказательство основано на утверждениях следующих лемм.

Лемма 5.1. *Пусть X– метрическое, а Y– банахово пространства. $a : X \to 2^Y$ – и $b : X \to 2^Y$ – многозначные отображения со звездными и компактными значениями. Пусть отображения a, a_0 и b, b_0 непрерывны в точке x_0 и*

$$0 \subseteq int\ (a_0(x_0) - b_0(x_0)). \tag{5.1}$$

Тогда отображение $c(x) \equiv a(x) \bigcap b(x)$ непрерывно в x_0.

Доказательство. Сначала докажем полунепрерывность снизу отображения c в точке x_0. Поскольку полунепрерывное снизу отображение $\Gamma \equiv a_0{}^* - b_0$ имеет выпуклые, замкнутые значения и справедливо включение (5.1), то существуют некоторое число $\tau > 0$ и окрестность U точки x_0, такие, что

$$B_\tau(0) \subseteq \Gamma(x) = (a_0(x) - b_0(x))\ \forall x \in U. \tag{5.2}$$

Дейтвительно, так как отображение Γ полунепрерывно в точке x_0, то существуют число $\tau > 0$ и окрестность U точки x_0 такие, что

$$B_{2\tau}(0) \subseteq \Gamma(x) + B_\tau(0).$$

Отсюда для произвольного непрерывного линейного функционала y^*, $\| y^* \| = 1$ имеем

$$\max_{u \in B_{2\tau}(0)} < y^*, u > \leq \max_{u \in \Gamma(x)} < y^*, u > + \max_{u \in B_\tau(0)} < y^*, u > .$$

Отсюда

$$2\tau \leq \max_{u \in \Gamma(x))} < y^*, u > + \tau,$$

т.е. $\tau \leq \max_{u \in \Gamma(x)} < y^*, u >$. Отсюда, так как $\Gamma(x)$– выпуклое замкнутое множество в банаховом пространстве Y, то

$$B_\tau(0) \subseteq \Gamma(x),\ \forall x \in U.$$

Далее, так как многозначное отображение b полунепрерывно сверху в окрестности U, то оно ограничено на этой окресности, т.е. существует ограниченое множество G, такое, что $b(x) \subseteq G$, $\forall x \in U$. Пусть $diam(G) = D$. Пусть $\varepsilon > 0$ и такое, что $\varepsilon < 2D$.Положим $\alpha = \tau\varepsilon/(2D - \varepsilon)$ и выберем $\tau > 0$ настолько малым, что $\alpha < \varepsilon/2$. Поскольку a и b являются полунепрерывными снизу отображениями в x_0, то можно найти такую окрестность $U \subseteq U$ точки x_0, что

$$a(x_0) \subseteq a(x) + B_{\alpha/2}(0),\ b(x_0) \subseteq b(x) + B_{\alpha/2}(0),\ x \in \overline{U}.$$

Пусть точка $x \in U$. Тогда для любого $y \in c(x_0)$ существует вектор $y_x \in b(x)$ такой, что

$$\overline{y}_x \in a(x) + B_\alpha(0) \text{ и } \|y - \overline{y}_x\| \leq \alpha. \tag{5.3}$$

Положим $\theta = \tau/(\alpha+\tau) < 1$. Умножим включение (5.3) на θ и замечая, что $\theta\alpha = (1-\theta)\tau$, получим

$$\theta \overline{y}_x \in \theta a(x) + \theta\alpha B_1(0) = \theta a(x) + (1-\theta)\tau B_1(0). \tag{5.4}$$

Умножим включение (5.2) на $(1-\theta)$, получим

$$(1-\theta)\tau B_1(0) \subseteq (1-\theta)a_0(x) - (1-\theta)b_0(x).$$

Отсюда и из (5.4) получим, что существует вектор $y' \in b_0(x)$ такой, что

$$\theta\overline{y}_x + (1-\theta)y' \in a(x) \tag{5.5}$$

С другой стороны, так как $y_x \in b(x)$ и $y' \in b_0(x)$, то

$$\overline{y} \equiv \theta\overline{y}_x + (1-\theta)y' \in b(x). \tag{5.6}$$

Из соотношений (5.5) и (5.6) следует, что $y \in c(x)$. Проверим, что $\|y-\overline{y}\| \le \varepsilon$. Действительно,

$$\|y-\overline{y}\| \le \|y - (\theta\overline{y}_x + (1-\theta)y')\| = \|\theta y + (1-\theta)y - \theta\overline{y}_x - (1-\theta)y'\| \le$$

$$\le \theta\|y - y_x\| + (1-\theta)\|y-y'\| \le \alpha + \frac{\alpha}{\alpha+\tau}D \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Таким образом $c(x_0) \subseteq c(x) + B_\varepsilon(0)\ \forall x \in U$. Это означает, что отображение c полунепрерывно снизу в x_0. Аналогично доказывается полунепрерывность сверху отображения c. □

Пример 5.1. Пусть для каждого $t \in [0,1/2]$ множество $a(t)$ есть область с замкнутой границей $OADBCO$ на рис.3. Тогда $a_0(t)$- треугольник ODC. Положим $b(t) \equiv \{(x_1,x_2) \in [0,1]\times[0,1]/x_2 = tx_1\}, t \in R$. Легко заметить, что отображения a и b со звездными значениями непрерывны, но их пересечение $a \cap b$ разрывно в точке $t = 1/2$. Это связано с тем, что здесь условие (5.1) леммы 5.1 не выполняется в точке $t = 1/2$.

Приведем пример непрерывного многозначного отображения a такое, что отображение a_0 не является непрерывным.

Пример 5.2. Пусть область на рис. 4 с границей $OAFDHO$- множество $a(t)$, $t \in [1/2,1]$. Оно является звездным множеством а его ядро $a_0(t)$- множество с границей $OFEHO$. При $t=1$ множеством значений отображения a_0 является квадрат $OAEH$. Очевидно, что многозначное отображение $a:[1/2,1] \longrightarrow 2^{R^2}$ непрерывно во всех точках отрезка $[1/2,1]$, но отображение a_0 терпит разрыв в

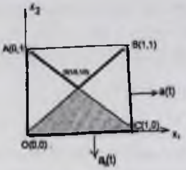

Рис. 3. Пересечение непрерывных отображений со звездными значениями

точке 1. Отметим также, что значения $a(t)$, $t \in [1/2, 1)$ отображения a не являются почти выпуклыми, поскольку любая точка на биссектрисе угла $\angle AFD$ имеет две проекции на множество $a(t)$, что противоречит предложению 4.1.

В общем случае имеет место следующий результат о непрерывности отображения a_0.

Предложение 5.1. *Пусть $E \subseteq X$ – компактное подмножество метрического пространства X, Y – банахово пространство; отображение $a : E \to 2^Y$ со звездными компактными значениями непрерывно. Тогда внутренность точек, где a_0 не непрерывно, пуста.*

Доказательство. Сначала покажем, что отображение $a_0 : E \to 2^{R^m}$ полунепрерывно сверху. Пусть $x_n \to x_0$, $y_n \in a_0(x_n), y_n \to y_0$. Покажем, что $y_0 \in a_0(x)$. Пусть $z_0 \in a(x_0)$. Так как a является полунепрерывным снизу отображением, то существует последовательность $z_n \in a(x_n)$ такая, что $z_n \to z_0$. С другой стороны, поскольку $y_n \in a_0(x_n)$, то для любого $\lambda \in [0, 1]$ имеем

$$\lambda z_n + (1 - \lambda) y_n \in a(x_n).$$

Отсюда следует, что $\lambda z_0 + (1 - \lambda) y_0 \in a(x_0)$. Это означает, чт о $y_0 \in a_0(x_0)$. Итак, отображение a_0 имеет замкнутый график. Теперь согласно теореме 10 [6] (гл. 1, п.1, стр.118) внутренность множества точек, где a_0 не непрерывно, пуста. □

Как иллюстрацию этого утверждения можно рассматривать пример 5.2, где отображение a_0 определено на отрезке $[1/2, 1]$ и оно разрывно только в точке $t = 1$. Однако, если значения непрерывного отображения a являются звездными и почти выпуклыми множествами, то отображение a_0 будет непрерывным. А именно, верна следующая лемма.

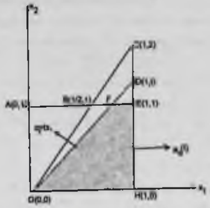

Рис. 4. Отображение a - непрерывно, а отображение a_0 не непрерывно

Лемма 5.2. *Пусть $E \subseteq X$ – компактное подмножество метрического пространства X; отображение $a : E \to 2^{R^m}$ непрерывно. Пусть далее множества $a(x)$ компактны, звездны и удовлетворяют условию выпуклости с некоторой константой $\theta(x)$. Предположим, что для каждого x $int\ a_0(x) \neq \emptyset$ и $\eta = \sup_{x \in E} \theta(x) < \infty$. Тогда отображение a_0 непрерывно.*

Доказательство. Полунепрерывность сверху отображения a_0 доказана в предложении 5.1. Покажем, что оно и полунепрерно снизу. Пусть $y_0 \in int\ a_0(x_0)$. Предположим, что существуют последовательность $x_k \to x_0$ и число $\delta > 0$ такие, что $d(y_0, a_0(x_k)) \geq \delta$ для достаточно больших k. Тогда можно считать, что $B_\delta(y_0) \subseteq a_0(x_0)$, но для больших k $B_\delta(y_0) \bigcap a_0(x_k) = \emptyset$. Так как отображение a точечно непрерывно (см. [1], теорема 1.38, стр. 45), то существует такая окрестность $B_{\delta_0}(y_0) \subseteq B_\delta(y_0)$, что $B_{\delta_0}(y_0) \subseteq a(x_k)$ для достаточно больших k. Отсюда, поскольку $y_0 \notin a_0(x_k)$, то существует такая точка $y_k \in a(x_k)$, которая не видна из точки y_0, т.е. на отрезке $[y_0, y_k]$ существует точка $\overline{y_k} \notin a(x_k)$. По замкнутости множества $a(x_k)$ существует шар V_k с центром $\overline{y_k}$ и такой, что $V_k \bigcap a(x_k) = \emptyset$. Будем сдвигать этот шар от точки $\overline{y_k}$ к y_k по отрезку $[y_0, y_k]$. В силу компактности $a(x_k)$ среди этих шаров существует такой шар V_k', который касается множества $a(x_k)$ только в одной точке $z_k \in a(x_k)$. Очевидно, что касательная к V_k' в точке z_k гиперплоскость L_k сильно отделяет точку y_0 от множества $a_0(x_k)$. Пусть H_{z_k} – полупространство, содержащее точку y_0. Заметим, что по построению в этом полупространстве находится шар V_k'. Не нарушая общности, можно считать, что $z_k \to z_0 \in a(x_0)$. Так как отображение a удовлетворяет условию выпуклости с определенной константой, то существует шар $\widetilde{V_k}$ фиксированного радиуса $r = 1/8\eta$, который также касается множества в z_k и который находится в полупространстве H_{z_k}. (см. лемму 2.7 из [7], теорему 1 из [8]). Можно считать, что последовательность шаров $\widetilde{V_k}$ в метрике Хаусдорфа сходится к некоторому шару V_0 радиуса r. Предельная гиперплоскость L_0 касается шара V_0 в точке z_0. Заметим, что если $u \in int\ V_0$, то существует число $\varepsilon_0 > 0$ такое, что $B_{\varepsilon_0}(u) \subseteq \widetilde{V_k}$ для достаточно больших k. Отсюда следует, что $int\ V_0 \bigcap a(x_0) = \emptyset$. Заметим также, что в предельном замкнутом полупространстве H_{z_0} находится шар V_0 и точка y_0. Значит, в шаре $B_\delta(y_0)$ существуют точки, которые не видны из z_0. Но это невозможно, поскольку шар $B_\delta(y_0)$ целиком входит в ядро множества $a(x_0)$. Полученное противоречие и доказывает лемму 5.2. □

Лемма 5.3. *Пусть $E \subseteq X$ – компактное подмножество метрического пространства X; $a : E \to 2^{R^m}$ – многозначное отображение с компактными звездными значениями и такое, что для любого x $int\ a_0(x) \neq \emptyset$. Предположим также, что отображения a и a_0 непрерывны. Тогда для любого $(x_0, y_0) \in graf(a)$ существует непрерывное отображение $y(x)$ такое, что $y(x) \in a(x)$ $\forall x \in E$ и $y(x_0) = y_0$.*

Доказательство. Поскольку отображение a_0 полунепрерывно снизу, то существует непрерывное отображение $\bar{y}(x)$ такое, что $\bar{y}(x) \in int a_0(x)$ $\forall x \in R^n$. Действительно, поскольку отображение a_0 точечно непрерывно, то для любого $y \in int\ a_0(x)$ существуют такие окрестности $V(y)$, $U_y(x)$, что $V(y) \subseteq a_0(x')$ $\forall x' \in U_y(x)$. Пусть $U_y = \bigcup_{x \in E} U_y(x)$. Система открытых множеств $\{U_y\}_{y \in Y}$, ($Y \equiv \bigcup_{x \in E} a_0(x)$) образует открытое покрытие компактного множества E. Пусть $\{U_{y_j}\}_{j \in J}$ конечное подпокрытие этого покрытия. Рассмотрим $\{p_{y_j}\}_{j \in J}$ – разбиение единицы, соотвествующее покрытию $\{U_{y_j}\}_{j \in J}$ и определим непрерывное отображение y следующим образом: $\bar{y}(x) = \sum_{j \in J} p_{y_j}(x) y_j$. Нетрудно проверить, что $\bar{y}(x) \in int a_0(x)$, $x \in E$. Рассмотрим отображение b следующим образом:

$$b(x) = \{y : y = \lambda y_0 + (1 - \lambda)\bar{y}(x),\ \lambda \in [0, 1]\}.$$

Очевидно, что оно полунепрерывно снизу и для любого x имеем $0 \in int(a_0(x) - b(x))$. Тогда согласно лемме 5.1 отображение $c(x) \equiv a(x) \bigcap b(x)$ полунепрерывно снизу. Ясно также, что оно имеет выпуклые замкнутые значения. Значит, согласно теореме Майкла через точку $(x_0, y_0) \in graf(c)$ проходит непрерывная селекция y этого отображения. □

Доказательство теоремы 5.1. Пусть $\epsilon < 1/(16\eta)$. Рассмотрим многозначное отображение $a(x) + B_\varepsilon(0)$. Оно по предложению 5.1 удовлетворяет всем предположениям леммы 5.3. Поэтому через точку $(x_0, y_0) \in graph(a)$ проходит непрерывное одночначное отображение $\bar{y}$ такое, что $\bar{y}(x) \in a(x) + B_\varepsilon(0)$, $x \in E$. Так как $\bar{y}(x) \in a(x) + B_{1/(16\theta(x))}(0), x \in E$, то согласно предложению 5.1 проекция $y(x)$ точки $\bar{y}(x)$ на множество $a(x)$ однозначно. Поскольку отображения $a, \bar{y}$ непрерывны, то как уже отмечено выше, непрерывным будет и отображение y. Очевидно, что отображение y является искомым. □

Приведем достаточное условие о существовании непрерывных селекций многозначных отображений с почти выпуклыми значениями (без условия звездности). Верна следующая теорема.

Теорема 5.2. *Пусть $E \subseteq X-$ компактное подмножество метрического пространства X; отображение $a : E \to 2^{R^m}$ непрерывно, причем для любого $x \in E$ множество $a(x)$ компактно и удовлетворяет условию выпуклости с константой $\theta(x)$. Допустим также, что*

$$\eta = \sup_{x \in E} \theta(x) < \infty, \quad diam(a(x)) \le \frac{1}{4\theta(x)}. \tag{5.7}$$

Тогда через любую точку графика отображения a проходит непрерывная селекция этого отображения.

Доказательство. Сначала предположим, что $int\ a(x) \neq \emptyset$. Покажем, что отображение a точечно непрерывно. Пусть $y_0 \in int\ a(x_0)$. Тогда, в силу полунепрерывности снизу отображения a для любого $\varepsilon > 0$ можно выбрать окрестность U точки x_0 таким образом, что $y_0 + B_{2\varepsilon} \subseteq a(x) + B_\varepsilon(0), x \in U$. Отсюда следует, что

$$y_0 + B_\varepsilon(0) \subseteq \bigcap_{s \in B_\varepsilon(0)} (a(x) + B_\varepsilon(0) - s). \tag{5.8}$$

Так как $a(x)$ удовлетворяет условию выпуклости с константой $\theta(x)$, то по лемме 2.11 [7](см. также [8], теорема 5) при $\varepsilon \le 1/16\eta$ правая часть включения (5.8) равна $a(x)$. Поэтому $y_0 + B_\varepsilon(0) \subseteq a(x)\ \forall x \in U$. Покажем теперь, что существует такое непрерывное отображение $\bar{y}(x)$, что $\bar{y}(x) \in a(x) + (diam(a(x)))^2\theta(x)B_1(0)$. Действительно, пусть $u_x \in int\ a(x)$. Тогда, в силу точечной непрерывности, существует такая окрестность $U(x)$, что $u_x \in int\ a(x)\ \forall x \in U_x \equiv U(x)$.

Семейство открытых окрстностей $\{U_x\}_{x\in E}$ образует покрытие компактного множества . Пусть $\{U_{x_j}\}_{j\in J}$ конечное покрытие этого покрытия. Рассмотрим $\{p_j\}_{j\in J}$– разбиение единицы, соответствующее этому покрытию и определим непрерывное отображение $\bar{y}$ следующим образом: $\bar{y}(x) = \sum_{j\in J} p_j(x)u_j$. Пусть $J(x) = \{j \in J : x \in U_{x_j}\}$. Тогда, если $x \in U(x_j)$, то $u_j \in a(x)$, поэтому имеем

$$\bar{y}(x) = \sum_{j\in J(x)} p_j(x)u_j \in a(x) + \theta(\max_{i,j\in J(x)} \|u_j - u_i\|)^2 B_1(0) \subseteq$$

$$\subseteq a(x) + \theta(x)(diam(a(x)))^2 B_1(0).$$

Теперь, если $\theta(x)(diam(a(x)))^2 \leq 1/16\theta(x)$, т.е. $diam(a(x)) \leq 1/4\theta(x)$, то согласно предложению 4.1 существует единственная проекция $y(x)$ точки $\bar{y}(x)$ на множестве $a(x)$. Так как отображение a с компактными значениями непрерывно, то отображение $y(x)$ также будет непрерывным. Теперь рассмотрим общий случай. Положим $b(x) = a(x) + B_\varepsilon(0)$, $\varepsilon < 1/(16\eta)$. По предложению 4.2 множество $b(x)$ почти выпукло с константой $4\theta(x)$. Очевидно также, что $diam(b(x)) = diama(x)+\varepsilon$. Теперь, согласно вышеуказанному, через любую точку его графика проходит непрерывная селекция этого отображения, если $diamb(x) \leq 1/(4(4\theta(x))) = 1/16\theta(x)$. Отсюда, если $diama(x) \leq 1/16\theta(x) - \varepsilon < 1/4\theta(x)$, то существует непрерывное отображние y такое, что $y(x_0) = y_0$, $\bar{y}(x) \in b(x) \forall x \in E$. Очевидно, что отображение $y(x) = Pr_{a(x)}\bar{y}(x)$ будет искомым. □

Замечание 5.1. *Для примера 4.1 неравенство (5.7) теоремы 5.2 не выполняется. Действительно, для единичной окружности S_1 имеем $diam(S_1) = 2$, $\theta = 1/\sqrt{3}$, поэтому неравенство $diam(S_1) \leq 1/4\theta$ не имеет место. Неравенство $sup_{x\in E}\theta(x) < \infty$ также не имеет место, поскольку пример 3.2 показывает, что $sup_{x\in E}\theta(x) = \infty$.*

Abstract. It is proved that through each point of the graph of a continuous set-valued mapping with almost convex and star-like values can be passed a continuous selection of that mapping.

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SOME FORMULAS FOR THE GENERALIZED ANALYTIC FEYNMAN INTEGRALS ON THE WEINER SPACE

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Abstract. In this paper, we analyze the analytic Feynman integrals on the Wiener space. We define a new concept of analytic Feynman integral on the Wiener space, which is called the generalized analytic Feynman integral, to explain various physical circumstances. Furthermore, we evaluate the generalized analytic Feynman integrals for several important classes of functionals. We also establish various properties of these generalized analytic Feynman integrals. We conclude the paper by giving several applications involving the Cameron-Storvick theorem and quantum mechanics.

MSC2010 numbers: 28C20, 60J65.

Keywords: Schrodinger equation; diffusion equation; (non)harmonic oscillator; Feynman-Kac formula; Cameron-Storvick theorem.

1. Introduction

Let $C_0[0,T]$ denote the one-parameter Wiener space, that is, the space of continuous real-valued functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote the Wiener measure. Observe that $(C_0[0,T],\mathcal{M},m)$ is a complete measure space, and denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm(x).$$

Feynman [5] has introduced an integral over a space of paths, and used his integral in a formal way in his approach to quantum mechanics. Since then the notion of Feynman integral was developed and was applied in various theories. For the procedure of analytic continuation, to define the analytic Feynman integral, we refer the reader

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to [5], [12]- [15], [18, 20]. Many mathematicians have studied the analytic Feynman integrals of functionals in several classes of functionals (see, [1] – [4], [6, 8, 10, 16, 18, 21, 22]). The differential equation

$$\frac{\partial}{\partial t}\psi(u,t) = \frac{1}{2\lambda}\Delta\psi(u,t) - V(u)\psi(u,t) \tag{1.1}$$

is called a diffusion equation with initial condition $\psi(u,0) = \varphi(u)$, where Δ is the Laplacian and V is an appropriate potential function. Many mathematicians have considered the Wiener integral of functionals of the form $F(\lambda^{-\frac{1}{2}}x + u)$, where u is a real number. It is a well-known fact that the Wiener integral of the functional

$$\exp\left\{-\int_0^T V(\lambda^{-\frac{1}{2}}x(t) + u)dt\right\}\varphi(\lambda^{-\frac{1}{2}}x(T) + u) \tag{1.2}$$

gives solutions of the diffusion equation (1.1) by the Feynman-Kac formula. In the case where time is replaced by imaginary time, this diffusion equation becomes the Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi(u,t) = -\frac{1}{2}\Delta\psi(u,t) + V(u)\psi(u,t) \tag{1.3}$$

with initial condition $\psi(u,0) = \varphi(u)$. Hence, a solution of Schrödinger equation (1.3) is obtained via an analytic Feynman integral. In particular, the authors found the solutions of the diffusion equation (1.1) and the Schrodinger equation (1.3) for the harmonic oscillator $V(u) = \frac{k}{2}u^2$ (for a more detailed study see [8, 23]). On the other hand, it is not easy to find the solutions of the diffusion equation (1.1) and the Schrodinger equation (1.3) with respect to nonharmonic oscillator.

In this paper we consider the following functional:

$$\exp\left\{-\int_0^T V(\lambda^{-\frac{1}{2}}x(t) + h(t))dt\right\}\varphi(\lambda^{-\frac{1}{2}}x(T) + h(t)), \tag{1.4}$$

where $h(t)$ is a continuous function on $[0,T]$. When $h(t)$ is a constant function, then the functional F in (1.4) reduces to that of in (1.2). That is, our functional (1.4) is more general than that of in (1.2). Therefore, the results and formulas for functional (1.2) will be special cases of the results and formulas obtained in this paper for functional (1.4).

2. Preliminaries and Definitions

A subset B of $C_0[0,T]$ is said to be scale-invariant measurable if ρB is $\mathcal{M}$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set if $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [11]. Throughout this paper we will assume that each functional $F : C_0[0,T] \to \mathbb{C}$ that we consider is scale-invariant measurable and that for each $\rho > 0$

$$\int_{C_0[0,T]} |F(\rho x)| dm(x) < \infty.$$

For $v \in L_2[0,T]$ and $x \in C_0[0,T]$, let $\langle v, x\rangle$ denote the Paley-Wiener-Zygmund (PWZ) stochastic integral. The following assertions hold:

(1) For each $v \in L_2[0,T]$, the PWZ stochastic integral $\langle v, x\rangle$ exists for a.e. $x \in C_0[0,T]$.

(2) If $v \in L_2[0,T]$ is a function of bounded variation on $[0,T]$, then $\langle v, x\rangle$ is equal to the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_0[0,T]$.

(3) The PWZ stochastic integral $\langle v, x\rangle$ has the expected linearity property.

(4) The PWZ stochastic integral $\langle v, x\rangle$ is a Gaussian process with mean 0 and variance $\|v\|_2^2$.

For a more detailed study of the PWZ stochastic integral see [7]– [10].

Now we define the analytic Feynman integral of functionals on Wiener space.

Definition 2.1. *Let* $\mathbb{C}$ *denote the set of complex numbers,* $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$, *and let* $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0$ *and* $Re(\lambda) \geq 0\}$. *Let* $F : C_0[0,T] \to \mathbb{C}$ *be a measurable functional such that for each* $\lambda > 0$ *the Wiener integral*

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x) dm(x)$$

exists. If there exists a function $J^*(\lambda)$ *analytic in* $\mathbb{C}_+$ *such that* $J^*(\lambda) = J(\lambda)$ *for all* $\lambda > 0$, *then* $J^*(\lambda)$ *is defined to be the analytic Wiener integral of* F *over* $C_0[0,T]$ *with parameter* λ, *and for* $\lambda \in \mathbb{C}_+$ *we write*

$$J^*(\lambda) = \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x) dm(x).$$

Let $q \neq 0$ *be a real number and let* F *be a functional such that* $J^*(\lambda)$ *exists for all* $\lambda \in \mathbb{C}_+$. *If the following limit exists, we call it the analytic Feynman integral of* F

with parameter q, and write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)dm(x) = \lim_{\lambda\to -iq} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)dm(x),$$

where $\lambda \to -iq$ through values in $\mathbf{C}_+$.

The following theorem provides a well-known integration formula which we will use several times in this paper.

Theorem 2.1. *Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal set of functions in L^2, and let $f : \mathbf{R}^n \to \mathbf{C}$ be Borel measurable. Let $|\vec{v}| = \sqrt{v_1^2 + \cdots + v_n^2}$, and let*

$$F(x) = f((\alpha_1, x), \cdots, (\alpha_n, x)) \equiv f((\vec{\alpha}, x)).$$

Then

$$\begin{aligned}\int_{C_0[0,T]} F(x)dm(x) &= \int_{C_0[0,T]} f((\vec{\alpha}, x))dm(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbf{R}^n} f(\vec{v}) \exp\left\{-\frac{|\vec{v}|^2}{2}\right\} d\vec{v}\end{aligned} \tag{2.1}$$

in the sense that if either side of (2.1) exists, then both sides exist and the equality holds.

3. An analogue of the analytic Feynman integral

Now we explain the importance of the functionals given by equation (1.4). For a constant k, when the potential function is given by $V(u) = \frac{k}{2}u^2$, then the equation (1.1) is called a diffusion equation for harmonic oscillator with potential V. For $\xi \in \mathbb{R}$, the function

$$V_1(u) \equiv V(u+\xi) = \frac{k}{2}(u+\xi)^2$$

is the translation of V, and so, the equation (1.1) is called a diffusion equation for harmonic oscillator with potential V_1. However, for an appropriate function $h(t)$ on $[0, T]$, the function

$$V_2(u) \equiv V(u + h(u)) = \frac{k}{2}(u + h(u))^2$$

might be a nonharmonic oscillator.

Example 3.1. Let $h(u) = u^2$ defined on $[0, T]$. Then

$$V_3(u) = \frac{k}{2}(u^2 + 2u^3 + u^4).$$

In this case, the equation (1.1) is called a diffusion equation for a nonharmonic oscillator with potential V_3 because it contains the "u^3"term. The above facts show that in certain physical circumstances the status of the harmonic oscillator can be exchanged by the status of the nonharmonic oscillator, which can be explained by studying the Wiener integral of the functional given by (1.4).

Example 3.2. For $\gamma \in \mathbb{R}$ let $h(u) = -u + \sqrt{u^2(u^2-\gamma^2)}$ defined on $[0,T]$. Then

$$V_4(u) = \frac{k}{2}u^2(u^2-\gamma^2).$$

In this case, the equation (1.1) is called a diffusion equation for double-well potential V_4. Thus, the functionals considered in this paper are more useful in applications than the functionals considered in the earlier papers [1] – [4], [6, 8, 10, 12, 23].

Now we are ready to state the definition of a generalized analytic Feynman integral.

Definition 3.1. *Let $h \in C_0[0,T]$ be given, and let $F : C_0[0,T] \longrightarrow \mathbb{C}$ be such that the function space integral*

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)dm(x)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the modified analytic function space integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$J^*(\lambda) = \int_{C_0[0,T]}^{\mathrm{an}_\lambda^h} F(x)dm(x).$$

Let $q \neq 0$ be a real number and let F be a functional such that the integral $\int_{C_0[0,T]}^{\mathrm{an}_\lambda^h} F(x)dm(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it a modified generalized analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q^h} F(x)dm(x) = \lim_{\lambda\to -iq} \int_{C_0[0,T]}^{\mathrm{an}_\lambda^h} F(x)dm(x),$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

Remark 3.1. If $h(t) \equiv 0$ on $[0,T]$, then we can write

$$\int_{C_0[0,T]}^{\mathrm{an}_\lambda^h} F(x)dm(x) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)dm(x)$$

and

$$\int_{C_0[0,T]}^{\mathrm{anf}_q^*} F(x)dm(x) = \int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)dm(x).$$

4. Examples involving generalized analytic Feynman integrals

In this section we establish the existence of the generalized analytic Feynman integrals for several classes of functionals. Let $M(L_2[0,T])$ be the class of all complex valued countably additive Borel measures f on $L_2[0,T]$.

4.1. The Banach algebra $\mathcal{S}$. Let $\mathcal{S}$ be the class of functionals of the form:

$$F(x) = \int_{L_2[0,T]} \exp\{i(v,x)\}df(v) \tag{4.1}$$

for s-a.e. $x \in C_0[0,T]$ for some $f \in M(L_2[0,T])$. One can show that $\mathcal{S}$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{L_2[0,T]} |df(v)|.$$

Example 4.1. Let $h(t) = \int_0^t z_h(s)ds$ for some $z_h \in L_2[0,T]$ and let $F \in \mathcal{S}$ be given by equation (4.1). Then for all $\lambda > 0$, we have

$$\begin{aligned} &\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)dm(x) \\ &= \int_{C_0[0,T]}\int_{L_2[0,T]} \exp\{i(v,\lambda^{-\frac{1}{2}}x + h)\}df(v)dm(x) \\ &= \int_{L_2[0,T]} \exp\Big\{-\frac{1}{2\lambda}\|v\|_2 + i(v,z_h)_2\Big\}df(v). \end{aligned} \tag{4.2}$$

But the expression above can be extended to the open right-hand plane $\lambda = p - iq$ with $p > 0$. Then letting $p \to 0$ we obtain that

$$\int_{C_0[0,T]}^{\mathrm{anf}_q^*} F(x)dm(x) = \int_{L_2[0,T]} \exp\Big\{-\frac{i}{2q}\|v\|_2 + i(v,z_h)_2\Big\}df(v) \tag{4.3}$$

and that

$$\left|\int_{C_0[0,T]}^{\mathrm{anf}_q^*} F(x)dm(x)\right| \le \|f\| < +\infty$$

for all $q \in \mathbb{R} - \{0\}$.

4.2. The class $\mathcal{A}_n^{(p)}$. Let $\mathcal{A}_n^{(p)}$ be the class of all functionals of the form:

$$F(x) = f(\langle\alpha_1, x\rangle, \cdots, \langle\alpha_n, x\rangle) \equiv f(\langle\vec{\alpha}, x\rangle),$$

where $f \in L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$ and $\{\alpha_1, \cdots, \alpha_n\}$ is an orthonormal set in $L_2[0,T]$.

Example 4.2. Let $h(t) = \int_0^t z_h(s)ds$ for some $z_h \in L_2[0,T]$ and let $F \in \mathcal{A}_n^{(p)}$. Then for all $q \in \mathbb{R} - \{0\}$, the generalized analytic Feynman integral of F exists and is given by formula

$$\int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x) = \left(\frac{-iq}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{\frac{iq}{2}\sum_{j=1}^{n}(u_j - (\alpha_j, z_h)_2)^2\right\} d\vec{u}. \tag{4.4}$$

Furthermore, we have

$$\left|\int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x)\right| \leq \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} |f(\vec{u})|d\vec{u} \leq \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \|f\|_1 < +\infty.$$

4.3. The class of Fourier-type functionals. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable functions $f(\vec{u})$ together with all their derivatives each of which decays at infinity faster than any polynomial of $|\vec{u}|^{-1}$. Let $\hat{f}$ be the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$, that is,

$$\hat{f}(\vec{\xi}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp(i\vec{u}\cdot\vec{\xi})d\vec{u}, \tag{4.5}$$

where $\vec{u}$ and $\vec{\xi}$ are in $\mathbb{R}^n$ and $\vec{u}\cdot\vec{\xi} = u_1\xi_1 + \cdots + u_n\xi_n$.

Note that the Fourier transform is an isomorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In addition, $\Delta^k f$ and $\widehat{\Delta^k f}$ are elements of $\mathcal{S}(\mathbb{R}^n)$ for all $k = 1, 2, \cdots$, where Δ denotes the Laplacian.

Next, following [9], we introduce the Fourier-type functionals. Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal set of functions in L^2. For $f \in \mathcal{S}(\mathbb{R}^n)$, we set

$$\Delta^k F(x) = (\Delta^k f)(\langle\vec{\alpha}, x\rangle), \quad k = 0, 1, \cdots \tag{4.6}$$

and

$$\widehat{\Delta^k F}(x) = \widehat{\Delta^k f}(\langle\vec{\alpha}, x\rangle), \quad k = 0, 1, \cdots. \tag{4.7}$$

The functionals in (4.6) and (4.7) are called Fourier-type functionals defined on the Wiener space $C_0[0,T]$.

Example 4.3. Let $\widehat{\Delta^k F}$ be as in (4.7), and let $h(t) = \int_0^t z_h(s)ds$ for some $z_h \in L_2[0,T]$. Then it is not hard to show that for all $q \neq 0$, the generalized analytic Feynman integral of $\widehat{\Delta^k F}$ exists and is given by the formula:

$$\text{(4.8)} \quad \int_{C_0[0,T]}^{\mathrm{anf}_q^h} \widehat{\Delta^k F}(x)dm(x)\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} (\Delta^k f)(\vec{v}) \exp\left\{-\frac{i\|\vec{v}\|^2}{2q} + i\vec{v}\cdot(\vec{\alpha}, z_h)_2\right\} d\vec{v}$$

for each $k = 0, 1, \cdots$, and hence

$$\left|\int_{C_0[0,T]}^{\mathrm{anf}_q^h} \widehat{\Delta^k F}(x)dm(x)\right| \leq \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} |(\Delta^k f)(\vec{v})| d\vec{v} < +\infty.$$

5. Properties of generalized analytic Feynman integrals

The following lemma is useful in establishing various relationships among generalized analytic Feynman integrals.

Lemma 5.1. *(1)* (Translation theorem). *Let F be a Wiener integrable functional, and let $x_0(t) = \int_0^t z_0(s)ds$ for some $z_0 \in L_2[0,T]$. Then*

$$\text{(5.1)} \quad \int_{C_0[0,T]} F(x+x_0)dm(x) = \exp\left\{-\frac{1}{2}\|z_0\|_2^2\right\} \int_{C_0[0,T]} F(x)\exp\{(z_0, x)\}dm(x).$$

(2) (Fubini theorem for Wiener integrals). *Let F be a Wiener integrable functional on $C_0[0,T]$. Then for all non-zero real numbers p_1 and p_2,*

$$\text{(5.2)} \quad \begin{aligned} &\int_{C_0[0,T]} \left(\int_{C_0[0,T]} F(p_1x_1 + p_2x_2)dm(x_1)\right)dm(x_2) \\ &= \int_{C_0[0,T]} F(\sqrt{p_1^2 + p_2^2}\,x)dm(x) \\ &= \int_{C_0[0,T]} \left(\int_{C_0[0,T]} F(p_1x_1 + p_2x_2)dm(x_2)\right)dm(x_1). \end{aligned}$$

In Theorem 5.1 below, we list several relationships in a table format.

Theorem 5.1. *Let F be as in Lemma 5.1. Let $h_j(t) = \int_0^t z_j(s)ds$ for some $z_j \in L_2[0,T]$, $j = 1,2,3$, and let $H_q(x) = F(x)\exp\{(-iq)(z_3, x)\}$ for $q \in \mathbb{R} - \{0\}$. Then for all non-zero real numbers q_1 and q_2 with $q_1 + q_2 \neq 0$, we have the following relationships:*

1. Commutative:

$$\int_{C_0[0,T]}^{\mathrm{anf}_{q_2}^{h_2}} \left(\int_{C_0[0,T]}^{\mathrm{anf}_{q_1}^{h_1}} F(x+y)dm(x)\right)dm(y) = \int_{C_0[0,T]}^{\mathrm{anf}_{q_1}^{h_1}} \left(\int_{C_0[0,T]}^{\mathrm{anf}_{q_2}^{h_2}} F(x+y)dm(y)\right)dm(x).$$

2. Fubini theorem: $\int_{C_0[0,T]}^{\mathrm{anf}_{q_2}^{h_2}}\left(\int_{C_0[0,T]}^{\mathrm{anf}_{q_1}^{h_1}} F(x+y)dm(x)\right)dm(y) \doteq \int_{C_0[0,T]}^{\mathrm{anf}_{\frac{q_1q_2}{q_1+q_2}}^{h_1+h_2}} F(z)dm(z)$.

3. Translation theorem: $\int_{C_0[0,T]}^{\mathrm{anf}_q^{h}} F(x)dm(x) \doteq \exp\left\{\frac{iq}{2}\|z_3\|_2^2\right\}\int_{C_0[0,T]}^{\mathrm{anf}_q^{h_2}} H_q(x)dm(x)$.

4. Integration formula: $\int_{C_0[0,T]}^{\mathrm{anf}_{-q}^{-h}}\int_{C_0[0,T]}^{\mathrm{anf}_q^{h}} F(x+y)dm(x)dm(y) = 0$.

Proof of Relationship 1:

First, using the symmetric property, for all $\lambda, \beta > 0$, we have

$$\begin{aligned}
&\int_{C_0^2[0,T]} F(\lambda^{-\frac{1}{2}}x + \beta^{-\frac{1}{2}}y + h_1 + h_2)d(m\times m)(x,y)\\
&\doteq \int_{C_0^2[0,T]} F(\beta^{-\frac{1}{2}}y + \lambda^{-\frac{1}{2}}x + h_2 + h_1)d(m\times m)(y,x).
\end{aligned}$$

It can be analytically continued in λ and β for (λ,β), and so we have for all $(\lambda,\beta) \in \mathbf{C}_+ \times \mathbf{C}_+$,

$$\begin{aligned}
&\int_{C_0[0,T]}^{\mathrm{anp}_\beta^{h_2}}\left(\int_{C_0[0,T]}^{\mathrm{anp}_\lambda^{h_1}} F(x+y)dm(x)\right)dm(y)\\
&\doteq \int_{C_0[0,T]}^{\mathrm{an}_\lambda^{h_1}}\left(\int_{C_0[0,T]}^{\mathrm{an}_\beta^{h_2}} F(x+y)d\mu(y)\right)d\mu(x).
\end{aligned} \tag{5.3}$$

Next, let E be a subset of $\mathbf{C}_+ \times \mathbf{C}_+$ containing the point $(-iq_1, -iq_2)$ and be such that $(\lambda,\beta) \in E$ implies that $\lambda + \beta \neq 0$. Then the function

$$\mathcal{H}(\lambda,\beta) \equiv \int_{C_0[0,T]}^{\mathrm{anp}_\beta^{h_2}}\left(\int_{C_0[0,T]}^{\mathrm{anp}_\lambda^{h_1}} F(y+z)dm(y)\right)dm(z)$$

is continuous on E and is uniformly continuous on E provided that E is compact. By the continuity of $\mathcal{H}$ and equation (5.3), the Relationship 1 follows.

Proof of Relationship 2:

Using equation (5.2), it follows that for $\lambda > 0$ and $\beta > 0$,

$$\begin{aligned}
&\int_{C_0^2[0,T]} F(\lambda^{-\frac{1}{2}}x + \beta^{-\frac{1}{2}}y + h_1 + h_2)d(m\times m)(x,y)\\
&\doteq \int_{C_0[0,T]} F(\sqrt{\lambda^{-1}+\beta^{-1}}z + h_1 + h_2)dm(z).
\end{aligned}$$

This last expression is defined for $\lambda > 0$ and $\beta > 0$. For $\beta > 0$ it can be analytically continued in $\lambda \in \mathbf{C}_+$. Also, for $\lambda > 0$ it can be analytically continued in $\beta \in \mathbf{C}_+$. Therefore $\lambda \in \mathbf{C}_+, \beta \in \mathbf{C}_+$ implies that $\frac{\lambda\beta}{\lambda+\beta} \in \mathbf{C}_+$, and hence it can be analytically continued into $\mathbf{C}_+$ to equal the generalized analytic Wiener integral:

$$\int_{C_0[0,T]}^{\mathrm{an}_\gamma^{h_1+h_2}} F(z)dm(z), \tag{5.4}$$

where $\gamma = \frac{\lambda\beta}{\lambda+\beta}$. Next, note that for all $q_1, q_2 \in \mathbb{R} - \{0\}$ with $q_1 + q_2 \neq 0$, if $\lambda \to -iq_1$ and $\beta \to -iq_2$, then $\frac{\lambda\beta}{\lambda+\beta} \to -i\frac{q_1 q_2}{q_1+q_2}$. Now, using this fact and equation (5.4), we can write

$$\begin{aligned}
&\int_{C_0[0,T]}^{\mathrm{anf}_{q_2}^{h_2}} \left(\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q_1}^{h_1}} F(x+y)dm(x) \right) dm(y) \\
&\doteq \lim_{\beta\to -iq_2} \lim_{\lambda \to -iq_1} \int_{C_0[0,T]}^{\mathrm{an}_\gamma^{h_1+h_2}} F(z)dm(z) \\
&\doteq \lim_{\frac{\lambda\beta}{\lambda+\beta} \to -i\frac{q_1q_2}{q_1+q_2}} \int_{C_0[0,T]}^{\mathrm{an}_\gamma^{h_1+h_2}} F(z)dm(z) \\
&\doteq \int_{C_0[0,T]}^{\mathrm{anf}_{\frac{q_1q_2}{q_1+q_2}}^{h_1+h_2}} F(z)dm(z),
\end{aligned}$$

which completes the proof of Relationship 2.

Proof of Relationship 3:

Using equation (5.1) with $G_\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$ (instead of F) and $x_0(t) = \lambda^{\frac{1}{2}}h_3(t)$, we can write

$$\begin{aligned}
&\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h_3)dm(x) \doteq \int_{C_0[0,T]} G_\lambda(x+x_0)dm(x) \\
&\doteq \exp\left\{-\frac{\lambda}{2}\|z_3\|_2^2\right\} \int_{C_0[0,T]} G_\lambda(x)\exp\{\lambda^{\frac{1}{2}}(z_3, x)\} \\
&\doteq \exp\left\{-\frac{\lambda}{2}\|z_3\|_2^2\right\} \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x)\exp\{\lambda^{\frac{1}{2}}(z_3,)\} \\
&\doteq \exp\left\{-\frac{\lambda}{2}\|z_3\|_2^2\right\} \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x)\exp\{\lambda(z_3, \lambda^{-\frac{1}{2}}x)\}.
\end{aligned}$$

It can be analytically continued in $\lambda \in \mathbf{C}_+$, and hence we have established Relationship 3 as $\lambda \to -iq$.

Proof of Relationship 4:

In view of equation (5.2), it follows that for all nonzero real numbers γ and β,

$$\begin{aligned}
&\int_{C_0[0,T]}\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + \beta^{-\frac{1}{2}}y + h_j - h_j)dm(x_1)dm(x_2) \\
&= \int_{C_0[0,T]} F(\sqrt{\lambda^{-1}+\beta^{-1}}x)dm(x).
\end{aligned}$$

Let $\lambda \to -iq$ and $\beta \to -i(-q) = iq$. Then $\lambda^{-1} + \beta^{-1} \to 0$, and hence Relationship 4 follows.

6. Application to the Cameron-Storvick theorem

In our first application, we establish the generalized Cameron-Storvick theorem for the generalized analytic Feynman integral. To do this, we need to define the concept of first variation of functionals on $C_0[0,T]$.

Definition 6.1. *Let F be a functional defined on $C_0[0,T]$. Then the first variation of F is defined by the formula:*

$$\delta F(x|u) = \frac{\partial}{\partial k} F(x+kw)\Big|_{k=0}, \quad x, w \in C_0[0,T], \tag{6.1}$$

if it exists.

Now we are ready to state the generalized Cameron-Storvick theorem for the generalized analytic Feynman integrals.

Theorem 6.1. (Generalized Cameron-Storvick theorem). *Let F be an m-integrable functional on $C_0[0,T]$ such that*

$$\sup_{|k| \le \eta} |\delta F(x + h|w)|$$

is an m-integrable functional on $C_0[0,T]$, and let $w(t) = \int_0^t z_w(s)ds$ for some $z_w \in L_2[0,T]$. Then

$$\begin{aligned} \int_{C_0[0,T]}^{\text{anf}_q^h} \delta F(x|w)dm(x) &\doteq \int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x) \\ &+ iq(z_w, h)_2 \int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x) - iq \int_{C_0[0,T]}^{\text{anf}_q^h} (z_w, x)F(x)dm(x). \end{aligned} \tag{6.2}$$

Proof of Theorem 6.1: First, let $F_h(x) = F(x+h)$ and $G_\lambda(x) = F_h(\lambda^{-\frac{1}{2}}x)$. Then for $\lambda > 0$, we obtain that

$$\begin{aligned}\int_{C_0[0,T]} \delta F(\lambda^{-\frac{1}{2}}x + h|w)dm(x) &= \frac{\partial}{\partial k}\left[\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h + kw)dm(x)\right]\Bigg|_{k=0} \\ &= \frac{\partial}{\partial k}\left[\int_{C_0[0,T]} F_h(\lambda^{-\frac{1}{2}}x + kw)dm(x)\right]\Bigg|_{k=0} \\ &= \frac{\partial}{\partial k}\left[\int_{C_0[0,T]} F_h(\lambda^{-\frac{1}{2}}(x + \lambda^{\frac{1}{2}}kw))dm(x)\right]\Bigg|_{k=0} \\ &= \frac{\partial}{\partial k}\left[\int_{C_0[0,T]} G_\lambda(x + x_0)dm(x)\right]\Bigg|_{k=0},\end{aligned}$$

where $x_0(t) = \lambda^{\frac{1}{2}}kw(t) = \int_{t_0}^{t} \lambda^{\frac{1}{2}}kz_w(s)ds$. Now applying the translation theorem for functional G_λ, we get

$$\begin{aligned}&\int_{C_0[0,T]} \delta F(\lambda^{-\frac{1}{2}}x + h|w)dm(x) \\ &= \frac{\partial}{\partial k}\left[\exp\left\{-\frac{\lambda k^2}{2}\|z_w\|_2^2\right\}\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)\exp(\lambda^{\frac{1}{2}}k(z_w, x))dm(x)\right]\Bigg|_{k=0} \\ &= \frac{\partial}{\partial k}\left[\exp\left\{-\frac{\lambda k^2}{2}\|z_w\|_2^2\right\}\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)\right. \\ &\qquad \left.\cdot\exp\{\lambda k(z_w, \lambda^{-\frac{1}{2}}x + h) - \lambda k(z_w, h))\}dm(x)\right]\Bigg|_{k=0}.\end{aligned} \tag{6.3}$$

The last expression in (6.3) can be decomposed into three terms

$$\begin{aligned}\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)dm(x) - \lambda(z_w, h)_2 \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)dm(x) \\ + \lambda \int_{C_0[0,T]} (z_w, \lambda^{-\frac{1}{2}}x + h)F(\lambda^{-\frac{1}{2}}x + h)dm(x).\end{aligned}$$

It can be analytically continued in $\lambda \in \mathbf{C}_+$, and hence, we have

$$\begin{aligned}\int_{C_0[0,T]}^{\mathrm{anf}_q} \delta F(x|w)dm(x) &\doteq \int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)dm(x) \\ -\lambda(z_w, h)_2 \int_{C_0[0,T]}^{\mathrm{anf}_q} &F(x)dm(x) + \lambda \int_{C_0[0,T]}^{\mathrm{anf}_q} (z_w, x)F(x)dm(x).\end{aligned}$$

Passing to the limit as $\lambda \to -iq$, we obtain the desired equation.

From Theorem 6.1 we have the following corollary, which is known as ordinary Cameron-Storvick theorem for the analytic Feynman integral.

Corollary 6.1. *Let $h(t) \equiv 0$ on $[0,T]$ and $(z_w, h)_2 = 0$. Then for the analytic Feynman integral we have*

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} \delta F(x|w)dm(x) \doteq \int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)dm(x) - iq\int_{C_0[0,T]}^{\mathrm{anf}_q} (z_w, x)F(x)dm(x).$$

We conclude this section by giving two relationships concerning generalized analytic Feynman integrals. From Theorems 5.1 and 6.1 we have the following relationships, which we state without any conditions.

Relationship R1: (Cameron-Storvick theorem, and Relationships 1 and 2 from Theorem 5.1).

$$\int_{C_0[0,T]}^{\mathrm{anf}_{q_2}^{h_2}} \left(\int_{C_0[0,T]}^{\mathrm{anf}_{q_1}^{h_1}} \delta F(x+y|w)dm(x) \right) dm(y) \doteq \int_{C_0[0,T]}^{\mathrm{anf}^{h_1+h_2}_{\frac{q_1 q_2}{q_1+q_2}}} \delta F(x|w)dm(x)$$
$$\doteq \int_{C_0[0,T]}^{\mathrm{anf}^{h_1+h_2}_{\frac{q_1 q_2}{q_1+q_2}}} F(x)dm(x) + i\frac{q_1 q_2}{q_1+q_2}(z_w, h_1+h_2)_2 \int_{C_0[0,T]}^{\mathrm{anf}^{h_1+h_2}_{\frac{q_1 q_2}{q_1+q_2}}} F(x)dm(x)$$
$$- i\frac{q_1 q_2}{q_1+q_2} \int_{C_0[0,T]}^{\mathrm{anf}^{h_1+h_2}_{\frac{q_1 q_2}{q_1+q_2}}} (z_w, x)F(x)dm(x).$$

To state the next relationship, we first give some observations. Let F and G be functionals on $C_0[0,T]$, and let H_q be as in Theorem 5.1. Then for all $x, w \in C_0[0,T]$ we have $\delta(FG)(x|w) = \delta F(x|w)G(x) + F(x)\delta G(x|w)$, provided that it exists. Also, note that $\delta H(x|w) = (-iq)(z_3, x)H(x)$, where $H(x) = \exp\{(-iq)(z_3, x)\}$. Hence we have

$$\delta(H_q)(x|w) = \delta F(x|w)\exp\{(-iq)(z_3,x)\} - iq(z_3,x)F(x)\exp\{(-iq)(z_3,x)\} \tag{6.4}$$

provided that they exist.

Relationship R2: (Relationship 3 from Theorem 5.1 and equation (6.4)).

$$\int_{C_0[0,T]}^{\mathrm{anf}_q^{h_3}} \delta F(x|w)dm(x) \doteq \exp\left\{\frac{iq}{2}\|z_3\|_2^2\right\} \int_{C_0[0,T]}^{\mathrm{anf}_q^{h_3}} \delta F(x|w)\exp\{(-iq)(z_3,x)\}dm(x)$$
$$\doteq \exp\left\{\frac{iq}{2}\|z_3\|_2^2\right\} \Bigg[\int_{C_0[0,T]}^{\mathrm{anf}_q^{h_3}} \delta H_q(x|w)dm(x)$$

$$+ iq \int_{C_0[0,T]}^{\mathrm{anf}_q^{h_3}} (z_3, x) F(x) \exp\{(-iq)(z_3, x)\} dm(x) \Big]$$

$$= \exp\left\{ \frac{iq}{2} \|z_3\|_2^2 \right\} \Big[\int_{C_0[0,T]}^{\mathrm{anf}_q^{h_3}} H_q(x) dm(x) + iq (z_w, h_3)_2 \int_{C_0[0,T]}^{\mathrm{anf}_1^{h_3}} H_q(x) dm(x)$$

$$- iq \int_{C_0[0,T]}^{\mathrm{anf}_q^{h_3}} (z_w, x) H_q(x) dm(x) + iq \int_{C_0[0,T]}^{\mathrm{anf}_q^{h_3}} (z_3, x) F(x) \exp\{(-iq)(z_3, x)\} dm(x) \Big].$$

7. Application to quantum mechanics

The equation (1.3) with $V(u) = a^2 u^2, a \in \mathbb{R} - \{0\}$ is called diffusion equation for harmonic oscillator:

$$\frac{\partial}{\partial t} \psi(u, t) = \frac{1}{2\lambda} \Delta \psi(u, t) - a^2 u^2 \psi(u, t) \tag{7.1}$$

with the initial condition $\psi(u, 0) = \varphi(u)$. Hence the solution of the diffusion equation for harmonic oscillator is given by

$$\int_{C_0[0,T]} \varphi(\lambda^{-1/2} x(T)) \exp\left\{ -\frac{a^2}{\lambda} \int_0^T x^2(s) ds \right\} dm(x).$$

Also, when time is replaced by imaginary time, the equation (7.1) becomes the Schrodinger equation for harmonic oscillator:

$$i \frac{\partial}{\partial t} \psi(u, t) = -\frac{1}{2} \Delta \psi(u, t) + a^2 u^2 \psi(u, t) \tag{7.2}$$

with the initial condition $\psi(u, 0) = \varphi(u)$. In [8, 16], the authors have described an approach for finding solutions for the diffusion equation for the harmonic oscillator (7.1) and the Schrodinger equation for harmonic oscillator (7.2) as follows.

(1) Note that there is a function f_m in $S(\mathbb{R}^m)$ so that $\widehat{f_m}(\vec{\xi}) = \exp\left\{ -a^2 \sum_{j=1}^m \beta_j \xi_j^2 \right\}$. In fact, f_m is given by the inverse Fourier transform of $\exp\left\{ -a^2 \sum_{j=1}^m \beta_j \xi_j^2 \right\}$.

Now, let $V_m(x) = f_m((\vec{\alpha}, x))$. Then V_m is a Fourier-type functional, and so, $\widehat{V_m}$ is also a Fourier-type functional. Furthermore, we have

$$\widehat{V_m}(x) = \exp\left\{ -a^2 \sum_{j=1}^m \beta_j (\alpha_j, x)^2 \right\} \tag{7.3}$$

and

$$\lim_{m \to +\infty} \widehat{V_m}(x) = \exp\left\{ -a^2 \int_0^T x^2(s) ds \right\},$$

for a.e. $x \in C_0[0,T]$, where $\beta_m = \left(\frac{T}{(m-\frac{1}{2})\pi}\right)^2$. Also, we have $|\widehat{V_m}(x)| \le 1$ for all $m = 1, 2, \cdots$, and

$$\lim_{m\to\infty} \varphi(x(T))\widehat{V_m}(x) = \varphi(x(T)) \exp\left\{ -a^2 \int_0^T x^2(s)ds \right\}$$

for a.e. $x \in C_0[0,T]$.

(2) The solution of the diffusion equation for harmonic oscillator (7.1) is the limit of Wiener integrals for Fourier-type functionals. Assume that φ is a bounded function. Then the limit of Wiener integrals for the Fourier-type functionals

$$\lim_{m\to\infty} \int_{C_0[0,T]} \varphi(\lambda^{-\frac{1}{2}}x(T))\widehat{V_m}(\lambda^{-\frac{1}{2}}x)dm(x)$$

is a solution of the diffusion equation for harmonic oscillator (7.1). Furthermore, the solution of the Schrodinger equation for harmonic oscillator (7.2) is the limit of analytic Feynman integrals for the Fourier-type functionals,

$$\lim_{m\to\infty} \int_{C_0[0,T]}^{anf_1} \varphi(x(T))\widehat{V_m}(x)dm(x)$$

is a solution of the Schrodinger equation for harmonic oscillator (7.2).

(3) In particular, we can choose the following initial condition:

$$\psi(u,0) = \varphi(u) = \begin{cases} A, & |u| \le L/2 \\ 0, & |u| > L/2, \end{cases}$$

where A is a real constant. In view of the Schrodinger equation this condition corresponds to a pulse wave packet with constant amplitude A in the given range of $|u| \le L/2$ (see [17, 19]). Then the solution of the diffusion equation for harmonic oscillator with the wave packet is:

$$A \lim_{m\to\infty} \prod_{j=1}^{m} \left(\frac{(j-\frac{1}{2})^2\pi^2\lambda}{2a^2T + (j-\frac{1}{2})^2\pi^2\lambda} \right) = A\,\mathrm{sech}\left(\sqrt{\frac{2a^2T}{\lambda}} \right).$$

Furthermore, the solution of the Schrödinger equation for harmonic oscillator with the wave packet is:

$$\lim_{m\to\infty} \int_{C_0[0,T]}^{anf_1} \varphi(x(T))\widehat{V_m}(x)dm(x) = A \lim_{m\to\infty} \prod_{j=1}^{m} \left(-\frac{(j-\frac{1}{2})^2\pi^2 i}{2a^2T - (j-\frac{1}{2})^2\pi^2 i} \right)$$

$$= A\mathrm{sech}\left(\sqrt{\frac{2a^2T}{-i}} \right) = A\sec\left(\sqrt{-i2a^2T} \right).$$

It was not easy to obtain the solutions for the diffusion equation and the Schrödinger equation for nonharmonic oscillators. However, we would like to obtain the solutions of these equations, by using the generalized analytic Feynman integral introduced in Section 3. Given the potential function $V(u) = a^2u^2, a \in \mathbb{R} - \{0\}$, if we take $h(u)$ so that $V(u + h(u))$ is the potential function for the nonharmonic oscillator, then we can conclude that the solution of the diffusion equation for nonharmonic oscillator is the limit of Wiener integrals for Fourier-type functionals. That is, the limit of the Wiener integrals for the Fourier-type functionals:

$$\lim_{m\to\infty} \int_{C_0[0,T]} \varphi(\lambda^{-\frac{1}{2}}x(T) + h(T))\widehat{V_m}(\lambda^{-\frac{1}{2}}x + h)dm(x)$$

is a solution of the diffusion equation for nonharmonic oscillator, and the solution of the Schrodinger equation for nonharmonic oscillator is the limit of analytic Feynman integrals for the Fourier-type functionals. Furthermore,

$$\lim_{m\to\infty} \int_{C_0[0,T]}^{\mathrm{anf}_1^h} \varphi(x(T))\widehat{V_m}(x)dm(x)$$

is a solution of the Schrodinger equation for the nonharmonic oscillator (7.2).

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