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ОБ M^*-МНОЖЕСТВАХ РЯДОВ ПО СИСТЕМЕ ФРАНКЛИНА

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Аннотация. Доказывается, что множество E является M^*-множеством или AM^*-множеством для системы Франклина, тогда и только тогда, когда E содержит в себе непустое совершенное множество.

MSC2010 number: 42C10; 42C20.

Ключевые слова: Система Франклина; M^*-множество.

1. Введение

Напомним, что множество $E \subset [a, b]$ называется U-множеством (множеством единственности) для системы $\{\varphi_n(x)\}_{n=0}^{\infty}$, $x \in [a, b]$, если из условия $\sum_{n=0}^{\infty} a_n\varphi_n(x) = 0$, $x \in [a, b]\setminus E$, следует, что все коэффициенты a_n равны нулю. В противном случае множество E называется M-множеством, т.е. $E \subset [a, b]$ является M-множеством, если существует нетривиальный ряд $\sum_{n=0}^{\infty} b_n\varphi_n(x)$, частичные суммы которого вне E всюду сходятся к нулю.

Классическая теорема Кантора гласит (см. [1] стр.191 или [2]), что пустое множество является U-множеством для тригонометрической системы. Далее, Юнг доказал (см. [1] стр.792, или [3]), что любое счетное множество является U-множеством для тригонометрической системы. Очевидно, что любое множество $E \subset [-\pi, \pi]$, с положительной мерой является M-множеством для тригонометрической системы. Действительно, для этого нужно рассмотреть ряд Фурье характеристической функции множества F, где $F \subset E$ некоторое замкнутое множество положительной меры.

Долгое время оставался открытым вопрос: является ли всякое множество меры нуль U-множеством для тригонометрической системы? В 1916 году Д. Е. Меньшовым [4] был приведен пример нетривиального тригонометрического ряда, сумма которого почти всюду (п.в.) равна нулю.

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[0]Исследования выполнены при финансовой поддержке ГКН МОН РА в рамках научного проекта 15T–1A006

Известны также примеры множеств меры нуль, которые являются U-множествами для тригонометрической системы (см. [1], [5] и [6]).

Независимо несколькими авторами [7]-[9] было доказано, что пустое множество является U-множеством для системы Хаара. Известно, что любое одноточечное множество является M-множеством для системы Хаара (см. [10]).

Ф. Г. Арутюняном и А. А. Талаляном [11], в частности, доказано что если ряд по системе Хаара $\sum_{n=0}^{\infty} a_n \chi_n(x)$, с коэффициентами

$$a_n = o(\sqrt{n}), \tag{1.1}$$

всюду, кроме быть может, некоторого счетного множества сходится к нулю, то все коэффициенты этого ряда равны нулю.

Следуя Г. М. Мушегяну [12], множество E назовем M^*-множеством для системы Хаара, если существует нетривиальный ряд $\sum_{n=0}^{\infty} a_n \chi_n(x)$, с коэффициентами (1.1), такое что $\sum_{n=0}^{\infty} a_n \chi_n(x) = 0$, для любого $x \in [0,1] \setminus E$. В той же работе Г.М. Мушегян доказал, что множество E является M^*-множеством для системы Хаара, тогда и только тогда, когда E содержит непустое совершенное подмножество.

В настоящей работе доказывается аналог вышеупомянутого результата Г.М. Мушегяна для системы Франклина.

2. Основной результат

Для формулировки основного результата, напомним определение системы Франклина.

Пусть $n = 2^{\mu} + \nu$, где $\mu = 0, 1, 2, \ldots$, и $1 \le \nu \le 2^{\mu}$. Обозначим

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}}, & \text{для } 0 \le i \le 2\nu, \\ \frac{i-\nu}{2^{\mu}}, & \text{для } 2\nu < i \le n. \end{cases} \tag{2.1}$$

Через S_n обозначим пространство функций, непрерывных и кусочно линейных на $[0;1]$ с узлами $\{s_{n,i}\}_{i=0}^{n}$, т.е. $f \in S_n$, если $f \in C[0;1]$ и линейная на каждом отрезке $[s_{n,i-1}; s_{n,i}]$, $i = 1, 2, \ldots, n$. Ясно, что $\dim S_n = n + 1$ и множество $\{s_{n,i}\}_{i=0}^{n}$ получается добавлением точки $s_{n,2\nu-1}$ к множеству $\{s_{n-1,i}\}_{i=0}^{n-1}$. Поэтому, существует единственная, с точностью до знака, функция $f_n \in S_n$, которая ортогональна S_{n-1} и $\|f_n\|_2 = 1$. Полагая $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x - 1)$, $x \in [0;1]$, получим ортонормированную систему $\{f_n(x)\}_{n=0}^{\infty}$, которая эквивалентным образом определена в работе [13] и называется системой Франклина.

Недавно, в работах [14], [15] была доказана теорема типа Кантора для системы Франклина, т.е. доказана, что если ряд $\sum_{n=0}^{\infty} a_n f_n(x) = 0$, $x \in [0,1]$, то все коэффициенты a_n равны нулю. Тем самым, доказано, что пустое множество является U-множеством для системы Франклина.

Для $n = 2^\mu + \nu$, где $\mu = 0,1,2\ldots,$ и $1 \leq \nu \leq 2^\mu$, обозначим $t_n := s_{\mu,2\nu-1}$ (см. (2.1)), и $[n] := \mu$.

Систематическое изучение системы Франклина началось с работ [17], [18], где получены многие важные свойства этой системы. В частности, получены знаменитые экспоненциальные оценки для функций Франклина и ядер Дирихле системы Франклина. З. Чисельским доказано существование постоянных $C_1 > 0$, $C_2 > 0$, $q_1 \in (0,1)$, $q_2 \in (0,1)$, таких что

$$|f_n(x)| \leq C_1 \cdot 2^{\frac{[n]}{2}} \cdot q_1^{2^{[n]}|x-t_n|}, \quad x \in [0,1], \tag{2.2}$$

$$|K_n(x,t)| \leq C_2 \cdot 2^{[n]} \cdot q_2^{2^{[n]}|x-t|}, \quad x,t \in [0,1], \tag{2.3}$$

где

$$K_n(x,t) = \sum_{k=0}^{n} f_k(x) f_k(t), \tag{2.4}$$

является n-ым ядром Дирихле системы Франклина.

Пусть x_0-некоторое число из $[0,1]$ и $a_k = f_k(x_0)$. Из (2.4) и (2.3) следует, что $\sum_{k=0}^{\infty} a_k f_k(x) = 0$, когда $x \neq x_0$. Очевидно также, что $a_0 = 1$. Следовательно, любое одноточечное множество является M-множеством для системы Франклина.

Определение 2.1. *Множество E назовем M^*-множеством для системы Франклина, если существует нетривиальный ряд $\sum_{n=0}^{\infty} a_n f_n(x)$, с коэффициентами (1.1) такой, что $\sum_{n=0}^{\infty} a_n f_n(x) = 0$, для любого $x \in [0,1] \setminus E$.*

В работе [14] анонсировано, а в [15] доказано, что любое счетное множество не является M^*-множеством для системы Франклина. В работе [16] доказана более общая теорема.

Теорема 2.1. *Пусть ряд $\sum_{k=0}^{\infty} a_k f_k(x)$, с коэффициентами (1.1), сходится по мере к интегрируемой функции и всюду, кроме быть может, некоторого счетного множества, выполняется $\sup_n \left|\sum_{k=0}^{n} a_k f_k(x)\right| < \infty$. Тогда ряд $\sum_{k=0}^{\infty} a_k f_k(x)$ является рядом Фурье-Франклина этой функции.*

В настоящей работе доказывается полный аналог теоремы Г.М. Мушегяна для системы Франклина.

Теорема 2.2. *Для того, чтобы множество E являлось M^*-множеством для системы Франклина, необходимо и достаточно, чтобы E содержало непустое совершенное подмножество.*

Доказательство. **Необходимость.** Допустим множество E является M^* - множеством для системы Франклина. Тогда существует нетривиальный ряд $\sum\limits_{k=0}^{\infty} a_k f_k(x)$, с коэффицентами (1.1), который всюду вне E сходится к нулю. В силу теоремы 2.1 множество E-несчетно. Обозначим $S_n(x) = \sum_{k=0}^{n} a_k f_k(x)$, $n = 0, 1, \dots$, $x \in [0,1]$. Очевидно, что множество

$$B := \{x \in [0,1] : S_n(x) \not\to 0\} = \bigcup_{m=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} \left\{x \in [0,1] : |S_n(x)| > \frac{1}{m}\right\}$$

является борелевским множеством и содержится в E. В силу теоремы 2.1 множество B-несчетно. Всякое несчетное борелевское множество содержит непустое совершенное подмножество. Следовательно, множество E содержит непустое совершенное подмножество. Необходимость доказана.

Достаточность. Нужно доказать, что любое непустое совершенное множество меры нуль является M^*-множеством. Пусть P-непустое совершенное множество меры нуль и $a := \min\{x : x \in P\}$, $b := \max\{x : x \in P\}$. Тогда $P = [a,b] \setminus (\bigcup_{k=1}^{\infty}(a_k, b_k))$, где интервалы (a_i, b_i) взаимно непересекаются и не имеют общих концов. Причем количество невырожденных интервалов (a_k, b_k) счетно. По индукции определим отрезки Δ_j, $j = 0, 1, 2, \dots$. Положим $\Delta_0 := [0, a]$, $\Delta_1 := [b, 1]$. Обозначим $\delta_1 = \sup_{k\geq 1}(b_k - a_k)$. Ясно, что $0 < \delta_1 < 1$. Поэтому существует j_1, такое что $(b_{j_1} - a_{j_1}) > \frac{\delta_1}{2}$. Положим $\Delta_2 = [a_{j_1}, b_{j_1}]$.

Допустим определены отрезки Δ_q, числа δ_q j_q, $q = 1, 2, \dots, m$, со свойствами:

(1) $\Delta_{q+1} = [a_{j_q}, b_{j_q}]$;

(2) $b_{j_q} - a_{j_q} > \frac{\delta_q}{2}$, $j_q \notin \{j_1, \dots, j_{q-1}\}$;

(3) $\delta_q = \sup\{b_k - a_k : k \neq j_p, p = 1, 2, \dots, q-1\}$.

Обозначим $\delta_{m+1} = \sup\{b_k - a_k : k \neq j_p, p = 1, 2, \dots, m\}$. Очевидно, что $0 < \delta_{m+1} < 1$. Поэтому найдется $j_{m+1} \notin \{j_1, j_2, \dots, j_m\}$, такое что $b_{j_{m+1}} - a_{j_{m+1}} > \frac{\delta_{m+1}}{2}$. Положив $\Delta_{m+2} = [a_{j_{m+1}}, b_{j_{m+1}}]$, получим отрезок Δ_{m+2} и число δ_{m+1}, обладающие свойствами: $|\Delta_{m+2}| = b_{j_{m+1}} - a_{j_{m+1}} > \frac{\delta_{m+1}}{2}$ и $\delta_{m+1} = \sup\{b_k - a_k : k \neq j_p, p = 1, 2, \dots, m\}$. Таким образом, по индукции определим отрезки Δ_q и числа δ_q j_q,

$q = 1, 2, \ldots$, которые удовлетворяют 1.-3., $\Delta_0 = [0, a]$, $\Delta_1 = [b, 1]$. Условие 2. обеспечивает, чтобы для каждого индекса k существовало единственное q, такое чтобы выполнялось $(a_{j_q}, b_{j_q}) = (a_k, b_k)$. Поэтому

$$\Delta_p \bigcap \Delta_q = \emptyset, \quad \text{когда} \quad p \neq q, \tag{2.5}$$

и

$$P = [0, 1] \setminus \left([0, a) \bigcup (b, 1] \bigcup \left(\bigcup_{k=2}^{\infty} (a_{j_k}, b_{j_k}) \right) \right).$$

Определим функции $\psi_k(x)$, $k = 1, 2, \ldots$, следующим образом. Областью определения функции ψ_k является множество $D_k := \bigcup_{j=0}^{k} \Delta_j$. Полагается

$$\psi_1(x) = \begin{cases} 0, & \text{когда} \quad x \in \Delta_0; \\ 1, & \text{когда} \quad x \in \Delta_1. \end{cases} \tag{2.6}$$

Далее, для $k \geq 2$, функция ψ_k определяется по формуле

$$\psi_k(x) = \begin{cases} \psi_{k-1}(x), & \text{когда} \quad x \in D_{k-1} \\ \frac{\max_{t<x, t \in D_{k-1}} \psi_{k-1}(t) + \min_{t>x, t \in D_{k-1}} \psi_{k-1}(t)}{2}, & \text{когда} \quad x \in \Delta_k. \end{cases} \tag{2.7}$$

Очевидно, что функции ψ_k принимают двоично-рациональные значения. Учитывая, что множество P имеет меру нуль, выполняется (2.5) и интервалы (a_k, b_k) не имеют общих концов, получим что любое двоично-рациональное значение $r \in [0, 1]$ принимается функциями ψ_k при всех k начиная с некоторого k_0, зависящего от r.

Положим

$$\psi(x) = \sup_{k \geq 1} \max_{t \leq x, t \in D_k} \psi_k(t), \quad x \in [0, 1].$$

Очевидно, что ψ-неубывающая функция, принимающая все двоично-рациональные значения из отрезка $[0, 1]$. Следовательно, ψ-непрерывна на отрезке $[0, 1]$. Из (2.6) и (2.7) следует, что

$$\psi(0) = 0 \quad \text{и} \quad \psi(1) = 1. \tag{2.8}$$

Кроме того, из (2.8) следует, что функция ψ на отрезках Δ_k принимает постоянные значения.

Положим

$$a_n = \int_0^1 f_n(t) d\psi(t), \quad n = 0, 1, 2, \ldots, \tag{2.9}$$

и

$$S_n(x) = \sum_{k=0}^{n} a_k f_k(x), \quad n = 0, 1, 2, \ldots. \tag{2.10}$$

Из (2.4), (2.9) и (2.10) имеем

$$S_n(x) = \int_0^1 K_n(x,t)d\psi(t), \quad n = 0, 1, 2, \ldots.$$

Пусть $x \notin P$. Поскольку P-замкнутое множество, то существует $\eta > 0$, такое что

$$(x - \eta, x + \eta) \bigcap P = \emptyset.$$

Следовательно, на $(x - \eta, x + \eta)$ функция ψ-постоянна и поэтому

$$(2.11) \quad |S_n(x)| = \left| \int_0^1 K_n(x,t)d\psi(t) \right| \le \max_{|x-t|\ge\eta} |K_n(x,t)| V(\psi) = \max_{|x-t|\ge\eta} |K_n(x,t)|.$$

Из (2.11) и (2.3) следует

$$|S_n(x)| \le C_2 \cdot 2^{[n]} \cdot q_2^{2^{[n]}\eta}.$$

Отсюда имеем

$$(2.12) \qquad \sum_{n=0}^{\infty} a_n f_n(x) = 0, \quad \text{когда } x \notin P.$$

Из (2.8) имеем

$$a_0 = \int_0^1 d\psi(t) = 1.$$

Для завершения доказательства теоремы нам остается доказать, что

$$(2.13) \qquad a_n = o\left(\sqrt{n}\right).$$

Пусть $n = 2^\mu + \nu$, где $\mu = 0, 1, 2, \ldots$, и $1 \le \nu \le 2^\mu$. Тогда из (2.1), (2.9) имеем

$$(2.14) \qquad |a_n| = \left| \int_0^1 f_n(t)d\psi(t) \right| \le \sum_{i=1}^{n} \left| \int_{s_{n,i-1}}^{s_{n,i}} f_n(t)d\psi(t) \right| \le$$

$$\sum_{i=1}^{n} (\psi(s_{n,i}) - \psi(s_{n,i-1})) \max_{t\in[s_{n,i-1}, s_{n,i}]} |f_n(t)| \le$$

$$\max_{1\le i\le n} (\psi(s_{n,i}) - \psi(s_{n,i-1})) \sum_{i=1}^{n} \max_{t\in[s_{n,i-1}, s_{n,i}]} |f_n(t)|$$

Из (2.2) следует, что

$$(2.15) \qquad \sum_{i=1}^{n} \max_{t\in[s_{n,i-1}, s_{n,i}]} |f_n(t)| \le C_3 \cdot \sqrt{n}, \quad \text{где } C_3 - \text{некоторая постоянная}$$

А из непрерывности функции ψ и (2.1) следует, что

$$(2.16) \qquad \lim_{n\to\infty} \max_{1\le i\le n} (\psi(s_{n,i}) - \psi(s_{n,i-1})) = 0.$$

Из (2.14)-(2.16) следует (2.13). □

3. Заключительные замечания

Замечание 3.1. *Для ряда, построенного при доказательстве теоремы 2.2, помимо условий (2.12) и (2.13) выполняется также следующее условие:*

$$\sum_{n=0}^{\infty} |a_n f_n(x)| < \infty, \quad \text{когда} \quad x \notin P. \tag{3.1}$$

Доказательство. Пусть $x \notin P$. Тогда $x \in (a_i, b_i)$, для некоторого i или $x \in [0, a)$ или $x \in (b, 1]$ (если эти интервалы непусты). Обсудим только случай $x \in (a_i, b_i)$. $\Delta_j = (a_i, b_i)$, для некоторого i. Пусть $\eta = \min\{x - a_i, b_i - x\}$. Фиксируем некоторое натуральное k, с условием $2^{-k} < \eta$ и оценим

$$\sum_{[n]=k} |a_n f_n(x)|.$$

Обозначим $H_p = [\frac{p-1}{2^{k+1}}, \frac{p}{2^{k+1}}]$, $p = 1, 2, ..., 2^{k+1}$. Через $\rho(t, A)$ обозначим расстояние точки t до множества A. Учитывая, что функция ψ на (a_i, b_i) постоянна, для a_n, с условием $[n] = k$, получим

$$|a_n| \le \sum_{p=1}^{2^{k+1}} \left| \int_{H_p} f_n(t) d\psi(t) \right| \le \sum_{p:\rho(x, H_p) > \eta} \max_{t \in H_p} |f_n(t)| V_p(\psi), \tag{3.2}$$

где $V_p(\psi) = \psi(\frac{p}{2^{k+1}}) - \psi(\frac{p-1}{2^{k+1}})$.

Из (2.2) имеем

$$\max_{t \in H_p} |f(t)| \le C_1 \cdot 2^{\frac{k}{2}} \cdot q_1^{2^k \rho(t_n, H_p)}. \tag{3.3}$$

Снова применяя (2.2), из (3.2) и (3.3) получаем

$$\sum_{[n]=k} |a_n f_n(x)| \le C_3 \cdot 2^k \cdot \sum_{[n]=k} q_1^{2^k |x - t_n|} \sum_{p:\rho(x, H_p) > \eta} q_1^{2^k \rho(t_n, H_p)} V_p(\psi). \tag{3.4}$$

Учитывая, что $\sum_{p=1}^{2^{k+1}} V_p(\psi) = \psi(1) - \psi(0) = 1$ и $|x - t_n| + \rho(t_n, H_p) \ge \rho(x, H_p)$, из (3.4) получим

$$\sum_{[n]=k} |a_n f_n(x)| \le C_3 \cdot 2^k \sum_{[n]=k} \sum_{p:\rho(x, H_p) > \eta} q_1^{2^k \rho(x, H_p)} V_p(\psi) \le C_2 \cdot 2^{2k} \cdot q_1^{2^k \eta}. \tag{3.5}$$

Выполнение (3.1) следует из (3.5). ☐

Определение 3.1. *Множество E назовем AM^*-множеством для системы Франклина, если существует нетривиальный ряд $\sum_{n=0}^{\infty} a_n f_n(x)$, с коэффициентами $a_n = o(\sqrt{n})$, который вне E всюду абсолютно сходится к нулю, т.е.*

$$\sum_{n=0}^{\infty} a_n f_n(x) = 0 \quad и \quad \sum_{n=0}^{\infty} |a_n f_n(x)| < \infty \quad когда \quad x \notin E.$$

Замечание 3.1 указывает на то, что для системы Франклина класс M^*-множеств совпадает с классом AM^*-множеств.

Abstract. In this paper, we prove that a set E is an M^*-set or an AM^*-set for the Franklin system if and only if E contains a nonempty perfect set.

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ALMOST EVERYWHERE CONVERGENCE OF STRONG NORLUND LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

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Abstract. In this paper we study the maximal operator for a class of subsequences of strong Nörlund logarithmic means of Walsh-Fourier series. For such a class we prove the almost everywhere strong summability for every integrable function f.

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1. Introduction

We denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \le p \le 2^n$. Denote $I_N := [0, 2^{-N})$ and $I_N(x) := I_N + x$.

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1, & \text{if} \quad x \in [0,1/2) \\ -1, & \text{if} \quad x \in [1/2,1) \end{cases}, \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \ge 1.$$

Let $w_0, w_1, \ldots$ denote the Walsh functions, that is, $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a nonnegative integer with $n_1 > n_2 > \cdots > n_s$, then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

Given $x \in \mathbb{I}$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{1.1}$$

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where each $x_k = 0$ or 1, will be called a dyadic expansion of x. If $x \in \mathbb{I}\backslash\mathbf{Q}$, then (1.1) is uniquely determined. For the dyadic expansion $x \in \mathbf{Q}$ we choose the one for which $\lim_{k\to\infty} x_k = 0$.

The dyadic addition of $x, y \in \mathbb{I}$ in terms of the dyadic expansion of x and y is defined by

$$\rho(x, y) := x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

If $f \in L^1(\mathbb{I})$, then by

$$\hat{f}(n) = \int_{\mathbb{I}} f(x) \, w_n(x) dx$$

we denote the n-th Fourier coefficient of f.

The partial sums of Fourier series with respect to the Walsh system are defined by

$$S_M(x; f) = \sum_{m=0}^{M-1} \hat{f}(m) \, w_m(x).$$

For $n \in \mathbb{N}$ let us introduce the projections

$$E_n(x; f) := S_{2^n}(x; f) = 2^n \int_{I_n(x)} f(s) \, ds \qquad (f \in L_1(\mathbb{I}), x \in \mathbb{I}),$$

and

$$E^*(x; f) := \sup_{n \in \mathbb{N}} E_n(x; |f|).$$

The question of almost everywhere convergence is one of the important questions in the theory of Fourier series. It is well known that for Walsh and trigonometric Fourier series the logarithmic means defined by

$$\frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k(f)}{k}, \quad l_n = \sum_{k=1}^{n} \frac{1}{k}$$

have a nice behavior, in the sense that, for each integrable on the unit interval function f, these means converge to f almost everywhere. Thus, to examine the logarithmic means is a good idea, because for the partial sums there are divergence results. For instance, for Walsh system it is known that for each measurable function ϕ satisfying $\phi(u) = o(u\sqrt{\log u})$ there exists an integrable function f such that

$$\int_{\mathbb{I}} \phi(|f(x)|) dx < \infty,$$

and the Walsh-Fourier series of f diverges everywhere (see [1]).

The notion of Norlund logarithmic means is similar to that of logarithmic means, the difference is that the denominators are taken in the reversed order. More precisely, the Norlund logarithmic means are defined by

$$L_n(f) := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}.$$

In [5, 6] it is proved that these means are much more closer to the partial sums than the logarithmic means. More precisely, we proved that in the function class above (see the result of Bochkarev [1]), there exist a function and a set with positive measure, such that the Walsh-Norlund logarithmic means of the function diverge on that set. This also says that, in this point of view, not all classical summation methods improve the convergence properties of the partial sums. On the other hand, in [9], the author studied the maximal operator for a class of Norlund logarithmic means of Walsh-Fourier series, where only the logarithmic means of order 2^n was considered. For such subsequence we have proved the almost everywhere convergence for every integrable function f. In [22], Memić enlarged the convergence class of subsequences given in [9].

The strong summability problem, that is, the convergence of the strong means

$$H_n^{T,p}(x;f) := \frac{1}{n+1}\sum_{k=0}^{n}\left|S_k^T(x,f)-f(x)\right|^p,\quad x\in\mathbb{T},\quad p>0, \tag{1.2}$$

was first considered by Hardy and Littlewood in [18], where by $S_k^T(x,f)$ we denote the partial sums of Fourier series with respect to trigonometric system. They showed that for any $f\in L_r(\mathbb{T})$ $(1<r<\infty)$ the strong means tend to 0 a.e., as $n\to\infty$. The Fourier series of $f\in L_1(\mathbb{T})$ is said to be (H,p)-summable at $x\in T$, if the strong means (1.2) converge to 0 as $n\to\infty$. The (H,p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [21] for $p=2$, and later by Zygmund [31] for the general case $1\leq p<\infty$.

In [25], Schipp investigated the strong (H,p)- and BMO-summability of Walsh-Fourier series. Among others, he gave a characterization of points at which the Walsh-Fourier series of an integrable function is (H,p)- and BMO-summable. This result is an analogue of Gabisonia's result, obtained in [4], that characterizes the points of strong summability with respect to the trigonometric system.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems, see Schipp [31, 34], Fridli and Schipp [2, 3], Leindler [20], Totik [29], Rodin [24], Weisz [40], Goginava, Gogoladze

[13, 12], Gogoladze [15, 16], Glukhov [17], Goginava [10, 11], Goginava, Gogoladze, Karagulyan [14] Gát, Goginava, Karagulyan [7, 8], Karagulyan [19], Oskolkov [23].

In this paper we study the maximal opearator for a class of subsequences of strong Norlund logarithmic means of Walsh-Fourier series. For such a class we prove the almost everywhere strong summability for every integrable function f.

2. Main results

The strong logarithmic means are defined by

$$L_n^{(p)}(x;f) := \left(\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{|S_k(x;f)|^p}{n-k}\right)^{1/p}.$$

Let

$$m_n := 2^{\alpha_1(n)} + 2^{\alpha_2(n)} + \cdots + 2^{\alpha_r(n)}, \tag{2.1}$$

where

$$\alpha_1(n) > \alpha_2(n) > \cdots > \alpha_r(n) \geq 0, \quad r = r(n).$$

and

$$m_n^{(i)} := 2^{\alpha_{i+1}(n)} + 2^{\alpha_{i+2}(n)} + \cdots + 2^{\alpha_r(n)}, \quad i = 0, 1, \dots, r-1. \tag{2.2}$$

The following are the main results of this paper.

Theorem 2.1. *Let $p > 0$ and*

$$\overline{\lim_{n\to\infty}} \frac{1}{l_{m_n}^{1/p}} \sum_{s=0}^{r-1} l_{m_n^{(s)}}^{1/p} < \infty. \tag{2.3}$$

Then

$$\lambda \left|\left\{x : \sup_n L_{m_n}^{(p)}(f) > \lambda\right\}\right| \leq c(p) \|f\|_1, \quad f \in L_1(\mathbb{I}).$$

By making use the well-known density argument due to Marcinkiewicz and Zygmund we can show that Corollary 2.1 follows from Theorem 2.1.

Corollary 2.1. *Let the condition (2.3) be satisfied and $f \in L_1(\mathbb{I})$. Then*

$$\frac{1}{l_{m_n}} \sum_{j=0}^{m_n-1} \frac{|S_j(x;f) - f(x)|^p}{m_n - j} \to 0, \quad n \to \infty$$

for a. e. $x \in \mathbb{I}$ and for any $p > 0$.

Corollary 2.2. *Let $f \in L_1(\mathbb{I})$, $m_n := 2^n + \gamma_n$, $\gamma_n \leq 2^{n^{1/(1+p)}}$ and $p > 0$. Then*

$$\frac{1}{l_{m_n}} \sum_{j=0}^{m_n-1} \frac{|S_j(x;f) - f(x)|^p}{m_n - j} \to 0, \quad n \to \infty$$

for a. e. $x \in \mathbb{I}$.

Corollary 2.3. *Let $f \in L_1(\mathbb{I})$ and $p > 0$. Then*

$$\frac{1}{l_{2^n}} \sum_{j=0}^{2^n-1} \frac{|S_j(x;f) - f(x)|^p}{2^n - j} \to 0, \quad n \to \infty$$

for a. e. $x \in \mathbb{I}$.

3. Auxiliary propositions

In [25], Schipp introduced the following operator ($p > 1$)

$$V_n^{(p)}(x;f) := \left(\sum_{l=0}^{2^n-1} \left(\int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) f(x+t+e_j)\, dt \right)^q \right)^{1/q},$$

$$\frac{1}{p} + \frac{1}{q} = 1, e_j := 2^{-j-1}.$$

Set

$$V_*^{(p)}(x;f) := \sup_n \left| V_n^{(p)}(x;f) \right|.$$

The proof of the next lemma can be found in [25] (for $p = 2$) and in [7] (for $p > 2$).

Lemma 3.1. *Let $p \geq 2$. Then*

$$\sup_\lambda \lambda \left| \left\{ x \in \mathbb{I} : V_*^{(p)}(x;|f|) > \lambda \right\} \right| \leq c(p) \|f\|_1 .$$

Set

$$H_n^{(p)}(x;f) := \left(\frac{1}{n} \sum_{m=0}^{n-1} |S_m(x;f)|^p \right)^{1/p}.$$

Lemma 3.2. *Let $p \geq 2$. The following inequality holds:*

$$H_{2^n}^{(p)}(x;f) \leq c V_n^{(p)}(x;|f|) .$$

Proof of Lemma 3.2. Observe first that for $p = 2$ the lemma was proved in [25]. Let

$$\varepsilon_{ij} := \begin{cases} -1, & \text{if } j = 0, 1, ..., i-1 \\ 1, & \text{if } j = i. \end{cases}$$

In [25], Schipp proved that

$$D_m(t) = \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus I_{k+1}}(t) \sum_{j=0}^{k} \varepsilon_{kj} 2^{j-1} w_m(t \dotplus e_j) - \frac{1}{2} w_m(t) + (m+1/2)\mathbb{I}_{I_n}(t), \quad m < 2^n. \tag{3.1}$$

We can write

$$2^{n/p} H_{2^n}^{(p)}(x;f) = \left\{ \sum_{m=0}^{2^n-1} |S_m(x;f)|^p \right\}^{1/p} = \sup_{\{\alpha_n\}} \left| \sum_{m=0}^{2^n-1} \alpha_m(x) S_m(x;f) \right|, \tag{3.2}$$

by taking the supremum over all $\{\alpha_n\}$ for which

$$\left(\sum_{m=0}^{2^n-1} |\alpha_m(x)|^q \right)^{1/q} \leq 1, \quad 1/p + 1/q = 1.$$

Let us assume that $p \geq 2$. From (3.1) we have

$$\left| \sum_{m=0}^{2^n-1} \alpha_m(x) S_m(x;f) \right|$$

$$\leq \left| \sum_{m=0}^{2^n-1} \alpha_m(x) \int_I f(x \dotplus t) \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus I_{k+1}}(t) \sum_{j=0}^{k} \varepsilon_{kj} 2^{j-1} w_m(t \dotplus e_j)\, dt \right|$$

$$+ \left| \sum_{m=0}^{2^n-1} \alpha_m(x) \int_I f(x \dotplus t) \frac{w_m(t)}{2} dt \right|$$

$$+ \left| \sum_{m=0}^{2^n-1} \alpha_m(x) \int_I f(x \dotplus t)(m+1/2)\mathbb{I}_{I_n}(t)\, dt \right|$$

$$:= J_1 + J_2 + J_3.$$

Since

$$\left| \sum_{k=j}^{n-1} \varepsilon_{kj} \mathbb{I}_{I_k \setminus I_{k+1}}(t) \right| \leq \mathbb{I}_{I_j}(t),$$

for J_1 we get

$$J_1 \leq \int_I |f(x \dotplus t)| \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) \times \left| \sum_{m=0}^{2^n-1} \alpha_m(x) w_m(t \dotplus e_j) \right| dt. \tag{3.3}$$

Set

$$P_n(x;t) := \sum_{m=0}^{2^n-1} \alpha_m(x) w_m(t).$$

It is easy to see that $\tilde{\mathbb{I}}_{I_j}(t) = \mathbb{I}_{I_j}(t \dotplus e_j)$. Then from (3.3) we have

$$J_1 \le \int_{\mathbb{I}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t)\, |f(x+t+e_j)|\, |P_n(x;t)|\, dt \tag{3.4}$$

$$= \sum_{l=0}^{2^n-1} \int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t)\, |f(x+t \dotplus e_j)|\, |P_n(x;t)|\, dt$$

$$= \sum_{l=0}^{2^n-1} \left| P_n\left(x; \frac{l}{2^n}\right)\right| \int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t)\, |f(x \dotplus t + e_j)|\, dt$$

$$\le \left(\sum_{l=0}^{2^n-1} \left| P_n\left(x; \frac{l}{2^n}\right)\right|^p \right)^{1/p}$$

$$\times \left(\sum_{l=0}^{2^n-1} \left(\int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t)\, |f(x \dotplus t + e_j)|\, dt \right)^q \right)^{1/q}.$$

First use Holder's inequality and Hausdorf-Young inequality to obtain $(p \ge 2, 1/p + 1/q = 1)$

$$\left(\sum_{l=0}^{2^n-1} \left| P_n\left(x; \frac{l}{2^n}\right)\right|^p \right)^{1/p} = 2^{n/p} \left(\sum_{l=0}^{2^n-1} \int_{l2^{-n}}^{(l+1)2^{-n}} |P_n(x;t)|^p\, dt \right)^{1/p}$$

$$= 2^{n/p} \left(\int_{\mathbb{I}} |P_n(x;t)|^p\, dt \right)^{1/p} = 2^{n/p} \sup_{\|h\|_q \le 1} \int_{\mathbb{I}} P_n(x;t)\, h(t)\, dt$$

$$= 2^{n/p} \sup_{\|h\|_q \le 1} \sum_{m=0}^{2^n-1} \alpha_m(x)\, \widehat{h}(m)$$

$$\le 2^{n/p} \sup_{\|h\|_q \le 1} \left(\sum_{m=0}^{2^n-1} |\alpha_m(x)|^q \right)^{1/q} \left(\sum_{m=0}^{2^n-1} \left|\widehat{h}(m)\right|^p \right)^{1/p}$$

$$\le c2^{n/p} \sup_{\|h\|_q \le 1} \|h\|_q = c2^{n/p}.$$

Consequently, from (3.4) we obtain the estimate

$$J_1 \le c2^{n/p} V_*^{(p)}(x; |f|), \quad p \ge 2. \tag{3.5}$$

For J_2 we can write

$$J_2 \le c \int_{\mathbb{I}} |f(x+t \dotplus e_0)| \left| \sum_{m=0}^{2^n-1} \alpha_m(x)\, w_m(e_0)\, w_m(t) \right| dt \tag{3.6}$$

$$\leq \int_{I} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{1}_{I_j}(t) |f(x+t+e_j)| |P_n'(x;t)| dt \leq c2^{n/p} V_n(x;|f|), \quad p \geq 2,$$

where

$$P_n'(x;t) := \sum_{m=0}^{2^n-1} a_m(x) w_m(e_0) w_m(t).$$

Analogously, we can write

$$J_3 \leq c2^{(1+1/p)n} \int_{I_n} |f(x+t)| dt \leq c2^{n/p} V_n(x;|f|), \quad p \geq 2. \tag{3.7}$$

Combining (3.2) and (3.5)-(3.7) we complete the proof of the lemma. □

4. Proof of Theorem 2.1

Observe first that in view of (2.1) and (2.2) we can write

$$L_{m_n}^{(p)}(x;f) \leq \left(\frac{1}{l_{m_n}} \sum_{j=0}^{2^{\alpha_1(n)}-1} \frac{|S_j(x;f)|^p}{2^{\alpha_1(n)}-j}\right)^{1/p} + \left(\frac{1}{l_{m_n}} \sum_{j=0}^{m_n^{(1)}-1} \frac{|S_{j+2^{\alpha_1(n)}}(x;f)|^p}{m_n^{(1)}-j}\right)^{1/p}.$$

Since for $j = 0, 1, ..., 2^{\alpha_1(n)} - 1$

$$S_{j+2^{\alpha_1(n)}}(x;f) = S_{2^{\alpha_1(n)}}(x;f) + w_{2^{\alpha_1(n)}}(x) S_j(x;fw_{2^{\alpha_1(n)}}),$$

we obtain

$$L_{m_n}^{(p)}(x;f) \leq \left(\frac{1}{l_{m_n}} \sum_{j=0}^{2^{\alpha_1(n)}-1} \frac{|S_j(x;f)|^p}{2^{\alpha_1(n)}-j}\right)^{1/p} + \left(\frac{l_{m_n^{(1)}}}{l_{m_n}}\right)^{1/p} |S_{2^{\alpha_1(n)}}(x;f)|$$
$$+ \left(\frac{l_{m_n^{(1)}}}{l_{m_n}}\right)^{1/p} \left(\frac{1}{l_{m_n}^{(1)}} \sum_{j=0}^{m_n^{(1)}-1} \frac{|S_j(x;fw_{2^{\alpha_1(n)}})|^p}{m_n^{(1)}-j}\right)^{1/p}.$$

Iterating the last inequality we obtain

$$L_{m_n}^{(p)}(x;f) \leq \sum_{s=0}^{r-1} \left(\frac{l_{m_n^{(s)}}}{l_{m_n}}\right)^{1/p} \left(\frac{1}{l_{m_n}^{(s)}} \sum_{j=0}^{2^{\alpha_{s+1}(n)}-1} \frac{|S_j(x;fw_{2^{\alpha_1(n)}} \cdots w_{2^{\alpha_s(n)}})|^p}{2^{\alpha_{s+1}(n)}-j}\right)^{1/p}$$
$$+ \sum_{s=0}^{r-2} \left(\frac{l_{m_n^{(s+1)}}}{l_{m_n}}\right)^{1/p} \left|S_{2^{\alpha_{s+1}(n)}}(x;fw_{2^{\alpha_1(n)}} \cdots w_{2^{\alpha_s(n)}})\right|.$$

Next, since

$$D_{2^k-j} = D_{2^k} - w_{2^k-1} D_j, \, j = 1, 2, ..., 2^k - 1,$$

we can write

$$L^{(p)}_{m_n}(x;f) \le \sum_{s=0}^{r-1}\left(\frac{l_{m_n^{(s)}}}{l_{m_n}}\right)^{1/p}\left(\frac{1}{l^{(s)}_{m_n}}\sum_{j=1}^{2^{\alpha_{s+1}(n)}}\frac{\left|S_{2^{\alpha_{s+1}(n)}-j}\left(x;f w_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}}\right)\right|^p}{j}\right)^{1/p}$$

$$+\sum_{s=0}^{r-2}\left(\frac{l_{m_n^{(s+1)}}}{l_{m_n}}\right)^{1/p}\left|S_{2^{\alpha_{s+1}(n)}}(x;|f|)\right| \le 2\sum_{s=0}^{r-1}\left(\frac{l_{m_n^{(s+1)}}}{l_{m_n}}\right)^{1/p}\left|S_{2^{\alpha_{s+1}(n)}}(x;|f|)\right|$$

$$+\sum_{s=0}^{r-1}\left(\frac{l_{m_n^{(s)}}}{l_{m_n}}\right)^{1/p}\left(\frac{1}{l^{(s)}_{m_n}}\sum_{j=1}^{2^{\alpha_{s+1}(n)}}\frac{\left|S_j\left(x;f w_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}} w_{2^{\alpha_{s+1}(n)}-1}\right)\right|^p}{j}\right)^{1/p}. \tag{4.1}$$

Let $p \ge 2$. Then using Lemma 3.2, we can write

$$\sum_{j=1}^{2^{\alpha_{s+1}(n)}-1}\frac{\left|S_j\left(x;f w_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}} w_{2^{\alpha_{s+1}(n)}-1}\right)\right|^p}{j} \tag{4.2}$$

$$= \sum_{l=0}^{\alpha_{s+1}(n)-1}\sum_{j=2^l}^{2^{l+1}-1}\frac{\left|S_j\left(x;f w_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}} w_{2^{\alpha_{s+1}(n)}-1}\right)\right|^p}{j}$$

$$\le \sum_{l=0}^{\alpha_{s+1}(n)-1} 2^{-l}\sum_{j=2^l}^{2^{l+1}-1}\left|S_j\left(x;f w_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}} w_{2^{\alpha_{s+1}(n)}-1}\right)\right|^p$$

$$\le 2\sum_{l=0}^{\alpha_{s+1}(n)-1}\left(H^{(p)}_{2^{l+1}}\left(x;f w_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}} w_{2^{\alpha_{s+1}(n)}-1}\right)\right)^p$$

$$\le 2\sum_{l=0}^{\alpha_{s+1}(n)-1}\left(V^{(p)}_{l+1}(x;|f|)\right)^p \le 2\alpha_{s+1}(n)\left(V^{(p)}_*(x;|f|)\right)^p.$$

Combining (4.1) and (4.2), and taking into account the condition (2.3) of the theorem, we obtain

$$L^{(p)}_{m_n}(x;f) \le c\left\{E^*(x;|f|) + V^{(p)}_*(x;|f|)\right\}, \quad p \ge 2. \tag{4.3}$$

Now let $0 < p < 2$. Since

$$H^{(p)}_{2^{l+1}}(x;f) \le H^{(2)}_{2^{l+1}}(x;f),$$

we can write

$$L^{(p)}_{m_n}(x;f) \le c\left\{E^*(x;|f|) + V^{(2)}_*(x;|f|)\right\}, \quad 0 < p < 2. \tag{4.4}$$

Finally, taking into account the inequality

$$\lambda\left|\{x \in \mathbb{I} : E^*(x;|f|) > \lambda\}\right| \le c\|f\|_1, \quad f \in L_1(\mathbb{I}),$$

from estimates (4.3), (4.4) and Lemma 3.1, we conclude the proof of the theorem. Theorem 2.1 is proved.

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CONJUGATE FUNCTIONS AND THE MODULUS OF SMOOTHNESS OF FRACTIONAL ORDER

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Abstract. In the present paper, estimates of the partial moduli of smoothness of fractional order of the conjugate functions of several variables are obtained in the space $C(\mathbf{T}^n)$. The accuracy of the obtained estimates is established by appropriate examples.

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1. Introduction

Let $\mathbb{R}^n$ $(n \geq 1;\ \mathbb{R}^1 = \mathbb{R})$ be the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$ with real coordinates. Let B be an arbitrary non-empty subset of the set $M = \{1, \ldots, n\}$. Denote by $|B|$ the cardinality of B. Let x_B be such a point in $\mathbb{R}^n$ whose coordinates with indices in $M \setminus B$ are zero.

As usual, let $\mathbf{T} = [-\pi, \pi]$ and let $\mathbf{C}(\mathbf{T}^n)$ $(\mathbf{C}(\mathbf{T}^1) = \mathbf{C}(\mathbf{T}))$ denote the space of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ that are 2π-periodic in each variable, endowed with the norm

$$\|f\| = \max_{x \in \mathbf{T}^n} |f(x)|.$$

If $f \in L(\mathbf{T}^n)$, then following Zhizhiashvili [14 , p. 182], the function

$$\overline{f}_B(x) = \Big(-\frac{1}{2\pi}\Big)^{|B|} \int_{\mathbf{T}^{|B|}} f(x + s_B) \prod_{i \in B} \cot\frac{s_i}{2}\, ds_B$$

we call the conjugate function of n variables with respect to those variables whose indices form the set B (with $\overline{f}_B = \overline{f}$ for $n = 1$).

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Suppose that $f \in \mathbb{C}(\mathbb{T}^n)$, $1 \le i \le n$, and $h \in \mathbb{R}$. For each $x \in \mathbb{R}^n$, we consider the difference of fractional order α $(\alpha > 0)$:

$$\Delta_i^\alpha(h)\, f(x) = \sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} f(x_1, \ldots, x_{i-1}, x_i + j\,h, x_{i+1}, \ldots, x_n),$$

where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$ for $j \ge 1$, and $\binom{\alpha}{j} = 1$ for $j = 0$.

Then we define the partial modulus of smoothness of order α of a function f with respect to the variable x_i by the equality (see ([2], [10]):

$$\omega_{\alpha,i}(f;\delta) = \sup_{|h|\le\delta} \|\Delta_i^\alpha(h)\, f\|.$$

For $n = 1$ we write $\Delta_i^\alpha(h)\, f(x) \equiv \Delta^\alpha(h)\, f(x)$ and $\omega_{\alpha,i}(f;\delta) \equiv \omega_\alpha(f;\delta)$.

Definition 1.1. We say that a function φ is almost decreasing in $[a,b]$ if there exists a positive constant A such that $\varphi(t_1) \ge A\,\varphi(t_2)$ for $a \le t_1 \le t_2 \le b$.

Definition 1.2. If for $f \in \mathbb{C}(\mathbb{T})$ there exists a function $g \in \mathbb{C}(\mathbb{T})$ such that $\lim_{h\to 0+} \|h^{-\alpha}\Delta^\alpha(h)f - g\| = 0$, then g is called the Liouville-Grunwald derivative of order $\alpha > 0$ of f in the $\mathbf{C}(\mathbf{T})$-norm, and is denoted by $D^\alpha f$.

Let Φ_α $(\alpha > 0)$ be the set of nonnegative, continuous functions $\varphi(\delta)$ defined on $[0,1)$ and satisfying the following conditions:

1. $\varphi(\delta) = 0$,
2. $\varphi(\delta)$ is nondecreasing,
3. $\int_0^\delta \frac{\varphi(t)}{t}dt = O(\varphi(\delta))$,
4. $\delta^\alpha \int_\delta^1 \frac{\varphi(t)}{t^{\alpha+1}}dt = O(\varphi(\delta))$.

Note that when $\alpha = k$ is an integer number, then the class Φ_α coincides with the well-known class of Bari-Stechkin of order k (see [1]).

Let φ be a nonnegative, nondecreasing continuous function defined on $[0,1)$ with $\varphi(\delta) = 0$. Then by $H_i^\alpha(\varphi; \mathbb{C}(\mathbb{T}^n))$ $(i = 1, \ldots, n)$ we denote the set of all functions $f \in \mathbb{C}(\mathbb{T}^n)$ such that

$$\omega_{\alpha,i}(f;\delta) = O\big(\varphi(\delta)\big), \quad \delta \to 0+, \quad i = 1, \ldots, n,$$

and define

$$H^\alpha(\varphi; \mathbb{C}(\mathbf{T}^n)) = \bigcap_{i=1}^{n} H_i^\alpha(\varphi; \mathbb{C}(\mathbf{T}^n)).$$

In the theory of real-valued functions there is a well-known theorem by Privalov on the invariance of Lipschitz classes under the conjugate operator $\tilde{f}$. An analogous

result, in terms of modulus of smoothness of fractional order, has been obtained by Samko and Yakubov in [8], where they proved that the generalized Hölder class $H^{\alpha}(\varphi;\mathbb{C}(\mathbb{T}))$ ($\varphi \in \Phi_\alpha, \alpha > 0$) is invariant under the operator $\tilde{f}$. In the paper [9] by Simonov and Tikhonov, embedding theorems for generalized Weyl-Nikol'skii classes and for generalized Lipschitz classes are obtained. In the paper [12] by Simonov, Besov-Nikol'skii classes are considered and embedding theorems for some classes of functions are established.
In the present paper, we obtain exact estimates of the partial moduli of smoothness of fractional order of the conjugate functions of several variables in the space $H(\varphi;\mathbb{C}(\mathbb{T}^n))$, provided that $\varphi \in \Phi_\alpha$, $\alpha > 0$. Notice that similar results for classical moduli of smoothness (that is, when the moduli of continuity of different orders satisfy Zygmund's condition) were obtained in the papers [3], [5] – [7], [13].

Now we state some auxiliary results that will be used in the proof of the main result of this paper.

Lemma 1.1 (see [4]). *Let $f \in \mathbb{C}(\mathbb{T})$, and let $\omega_k(f;t)$ and $\omega_{k+1}(f;t)$ be the moduli of continuity of f of k-th and $(k+1)$-th orders, respectively. Then for all $t \in [0,1)$ the following inequality holds:*

$$\omega_k(f;t^2) \leq A\omega_{k+1}(f;t),$$

where A is a constant, which is independent of f.

Lemma 1.2. *Let $f \in H^{\alpha}(\varphi;\mathbb{C}(\mathbb{T}^n))$ and $\varphi \in \Phi_\alpha$, $\alpha > 0$. Then the following asymptotic relations hold:*

$$(1)\quad \omega_{\alpha,i}(\tilde{f}_{\{i\}};\delta) = O(\varphi(\delta)),\ \ i = 1,\ldots,n,\ \delta \to 0+,$$

$$(2)\quad \omega_{\alpha,k}(\tilde{f}_{\{i\}};\delta) = O(\varphi(\delta)|\ln\delta|),\ \ i,k = 1,\ldots,n, i \neq k,\ \delta \to 0+.$$

Proof. The statement (1) of the lemma is a multivariate version of Theorem 2 from [8] and can be proved exactly in the same way with some minor changes. So, we have to prove only the statement (2) of the lemma.

Let $h_{\{k\}} = (\underbrace{0,\ldots,0}_{k-1},h,0,\ldots 0)$. For a given α, there exists a natural number p such that $p-1 < \alpha \leq p$.

By the definitions of the difference of fractional order and the conjugate function, we can write

$$(-2\pi)\Delta_k^{\alpha}(h)\,\tilde{f}_i(x) =$$

$$= \sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} \int_0^{h^{2^{-p+1}}} [f(x - jh_{\{k\}} + s_{\{i\}}) - f(x - jh_{\{k\}} - s_{\{i\}})] \cot \frac{s_i}{2} ds_i +$$

$$+ \int_{h^{2^{p-1}}}^{\pi} \Delta^{\alpha}_{s}(h) f(x + s_{\{i\}}) \cot \frac{s_i}{2} ds_i - \int_{h^{2^{p-1}}}^{\pi} \Delta^{\alpha}_{s}(h) f(x - s_{\{i\}}) \cot \frac{s_i}{2} ds_i =$$

$$= \sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} I_j(x, h_{\{k\}}) + J_1(x, h_{\{k\}}) + J_2(x, h_{\{k\}}).$$

For each j $(j = 1, \dots, \infty)$ we have

$$\|I_j(x, h_{\{k\}})\| \leq A \int_0^{h^{2^{-p+1}}} \frac{\omega_{1,i}(f; s_i)}{s_i} ds_i,$$

where A is a constant independent of f.

Taking into account Lemma 1 and substituting s_i by $s_i^{2^{-p+1}}$, we get

$$\|I_j(x, h_{\{k\}})\| \leq A_1 \int_0^{h^{2^{-p+1}}} \frac{\omega_{p,i}(f; s_i^{2^{-p+1}})}{s_i} ds_i \leq A_2 \int_0^h \frac{\omega_{p,i}(f; s_i)}{s_i} ds_i,$$

where A_1 and A_2 are constants independent of f.

Now using the inequality $\omega_{p,i}(f; s_i) \leq C\omega_{\alpha,i}(f; s_i)$ (see [11]), where C is a constant independent of f, we obtain

$$\|I_j(x, h_{\{k\}})\| \leq A_3 \int_0^h \frac{\omega_{\alpha,i}(f; s_i)}{s_i} ds_i, \quad j = 1, \dots, \infty,$$

where A_3 is a constant independent of f.

It is easy to see that

$$\|J_i(x, h_{\{k\}})\| \leq A_4 \omega_{\alpha,k}(f, h) |\ln h|, \quad i = 1, 2,$$

where A_4 is a constant independent of f.

In view of the above estimates for $I_j(x, h_{\{k\}})$ $(j = 1, \dots, \infty)$ and $J_i(x, h_{\{k\}})$ $(i = 1, 2)$, and the condition $\varphi \in \Phi_\alpha$, we complete the proof of the statement (2). Lemma 1.2 is proved. □

The next two lemmas can be proved in the same way as the statement (Lemma 3) given in [1, pp. 498-499].

Lemma 1.3. *If $\varphi \in \Phi_\alpha (\alpha > 0)$, then the function $\frac{\varphi(t)}{t^\alpha}$ is almost decreasing in* $[0, 1]$.

Lemma 1.4. *If $\varphi \in \Phi_\alpha (\alpha > 0)$, then there exists a real number β $(0 < \beta < \alpha)$ such that the function $\frac{\varphi(t)}{t^\beta}$ is almost decreasing in* $[0, 1]$.

Notice that Lemma 1.4 actually implies Lemma 1.3.

2. Estimates for the partial moduli of smoothness of fractional order of the conjugate functions

The following theorem is the main result of this paper.

Theorem 2.1. *The following assertions hold:*

(a) *Let* $f \in H^{\alpha}(\varphi; \mathbb{C}(\mathbb{T}^n))$ *and* $\varphi \in \Phi_{\alpha}$, $\alpha > 0$. *Then*

$$\omega_{\alpha,i}(\bar{f}_B; \delta) = O(\varphi(\delta)|\ln\delta|^{|B|-1}), \quad i \in B, \ \delta \to 0+, \tag{2.1}$$

$$\omega_{\alpha,i}(\bar{f}_B; \delta) = O(\varphi(\delta)|\ln\delta|^{|B|}), \quad i \in M\backslash B, \ \delta \to 0+. \tag{2.2}$$

(b) *For each* $B \subseteq M$ *there exists a function* G *such that* $G \in H(\varphi; \mathbb{C}(\mathbb{T}^n))$ *and*

$$\omega_{\alpha,i}(\widetilde{G}_B; \delta) \geq C\varphi(\delta)|\ln\delta|^{|B|-1} \quad i \in B, \ 0 \leq \delta \leq \delta_0, \tag{2.3}$$

$$\omega_{\alpha,i}(\bar{G}, \delta_B, \delta) \geq C\varphi(\delta)|\ln\delta|^{|B|}, \quad i \in M \setminus B, \ 0 \leq \delta \leq \delta_0, \tag{2.4}$$

where C *and* δ_0 *are positive constants.*

It should be noted that, for the case of modulus of continuity of first order, the theorem was proved in [7].

Proof. Part (a) of the theorem follows from Lemma 1.2. So, we have to prove only part (b). Without loss of generality, we carry out the proof of part (b) for the case $B = \{1, \dots, n-1\}$.

We consider a strictly decreasing sequence of positive numbers $(b_l)_{l\geq 1}$ satisfying the following conditions:

1. $\sum_{l=0}^{\infty} b_l \leq 1 \quad (b_0 = 0)$;
2. $\sum_{i=l+1}^{\infty} b_i < b_l$;
3. $\varphi^{-1}(b_{l+1}) < (\varphi^{-1}(b_l))^{[illegible]}$, where $\varphi^{-1}(b_l)$ $(l = 1, 2, \dots)$ is a certain element of the set $\{t : \varphi(t = b_l)\}$ and β $(0 < \beta < \alpha)$ satisfies the condition of Lemma 1.4.

We se

$$\tau_l = 2\sum_{j=0}^{l-1} \varphi^{-1}(b_j), \quad \tau_l^* = \tau_l + \varphi^{-1}(b_l).$$

For any $l = 1, 2, ..$, let us consider the functions g_l and h_l in $\mathbb{T}$, defined as follows:

$$g_l(u) = \begin{cases} 0, & -\pi \leq u \leq 0, \\ \dfrac{u^{\alpha}}{(\tau_l^* - \tau_l)^{\alpha}}, & 0 < u \leq \tau_l^* - \tau_l, \\ 1, & \tau_l^* - \tau_l < u \leq \pi - \tau_l^* + \tau_l, \\ \dfrac{(\pi - u)^{\alpha}}{(\tau_l^* - \tau_l)^{\alpha}}, & \pi - \tau_l^* + \tau_l < u \leq \pi. \end{cases}$$

and

$$h_l(u) = \begin{cases} \dfrac{(u-\tau_l)^{\alpha}(\tau_l^*-u)^{\alpha}}{(\tau_l^*-\tau_l)^{2\alpha}}, & \tau_l \le u \le \tau_l^*, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we define the functions G_l in $\mathbb{T}^n$ as follows:

$$G_l(x_1,\dots,x_n) = b_l \prod_{i=1}^{n-1} g_l(x_i) h_l(x_n), \quad l = 1,2,\dots$$

and consider the function G defined by the series

$$G(x_1,\dots,x_n) = \sum_{l=1}^{\infty} G_l(x_1,\dots,x_n).$$

We extend the function G 2π-periodically in each variable to the whole space $\mathbb{R}^n$.

We claim that

$$G \in H^{\alpha}(\varphi; \mathbb{C}(\mathbb{T}^n)).$$

Let $0 < h < \varphi^{-1}(b_1)$. Then we have

$$\|\Delta_n^{\alpha}(h)G\| \le \sum_{l=1}^{\infty} \|\Delta_n^{\alpha}(h)G_l\| = \sum_{l=1}^{\infty} I_l(h).$$

Let us estimate each $I_l(h)$ ($l = 1,2,\dots$) from above.

For given h, there exists a number N such that $\tau_{N+1}^* - \tau_{N+1} \le h < \tau_N^* - \tau_N$.

Let $l = 1,\dots,N$. It is known (see [2]) that if a function of one variable $f \in \mathbb{C}(\mathbb{T})$ has fractional derivative of order α ($\alpha > 0$), then

$$\omega_{\alpha}(f;\delta) \le C\delta^{\alpha}\|D^{\alpha}f\| (\delta > 0), \ C = const > 0.$$

In our case, using the definition of the function G_l and this fact for the variable x_n, we can conclude that

$$I_l(h) \le A_1 h^{\alpha} \frac{b_l}{(\tau_l^* - \tau_l)^{\alpha}}, \quad A_1 = const.$$

If $l = N+1, \dots$, then we have

$$I_l(h) \le A_2 b_l, \quad A_2 = const.$$

Therefore

$$\|\Delta_n^{\alpha}(h)G\| \le A_1 \sum_{l=1}^{N} h^{\alpha} \frac{b_l}{(\tau_l^* - \tau_l)^{\alpha}} + A_2 \sum_{l=N+1}^{\infty} b_l.$$

If $\tau_{N+1}^* - \tau_{N+1} \le h \le (\tau_N^* - \tau_N)^{\frac{\alpha}{\alpha-\beta}}$, then by Lemma 4 and by the construction of the sequence $(b_l)_{l\ge 1}$, with some constant A_3, we obtain

$$\|\Delta_n^{\alpha}(h)G\| \le A_1 \sum_{l=1}^{N} \frac{b_l}{(\tau_l^* - \tau_l)^{\alpha}} h^{\beta} h^{\alpha-\beta} + A_2 \sum_{l=N+1}^{\infty} b_l \le A_3 \varphi(h).$$

If $(\tau_N^* - \tau_N)^{\frac{\alpha}{\alpha-\beta}} \le h \le \tau_N^* - \tau_N$, then by Lemmas 3 and 4, and by the construction of the sequence $(b_l)_{l\ge 1}$, we get

$$\|\Delta_n^\alpha(h)G\| \le A_1 \sum_{l=1}^{N-1} \frac{b_l}{(\tau_l^* - \tau_l)^\beta} h^\beta h^{\alpha-\beta}$$

$$+A_1 \frac{b_N}{(\tau_N^* - \tau_N)^\alpha} h^\alpha + A_2 \sum_{l=N+1}^{\infty} b_l \le A_3 \varphi(h), \quad A_3 = const.$$

Hence, we have

$$\omega_{\alpha,n}(G;\delta)(h)G = O(\varphi(\delta)), \quad \delta \to 0+.$$

Analogously, we can show that

$$\omega_{\alpha,i}(G;\delta)(h)G = O(\varphi(\delta)), \delta \to 0+, \quad i = 1, \dots, n-1.$$

Hence

$$G \in H^\alpha(\varphi; C(\mathbb{T}^n)).$$

Now we proceed to prove the inequalities (2.3) and (2.4).

Let $h = \tau_l^* - \tau_l$. According to the definition of the conjugate function and the function G, we obtain

$$\Delta_n^\alpha(h)\tilde{G}_{\{1,\dots,n-1\}}(0,\dots,0,\frac{\tau_l^*+\tau_l}{2}) =$$

$$= \Big(-\frac{1}{2\pi}\Big)^{n-1} \sum_{j=1}^{\infty} \int_{\mathbb{T}^{n-1}} \Delta_n^\alpha(h) G_j(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}) \prod_{i=1}^{n-1} \cot\frac{s_i}{2}\, ds_i =$$

$$= \Big(-\frac{1}{2\pi}\Big)^{n-1} \int_{\mathbb{T}^{n-1}} \Big[\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}+kh)\Big] \prod_{i=1}^{n-1} \cot\frac{s_i}{2}\, ds_i.$$

Now using the inequality $\left|\binom{\alpha}{k}\right| \le C_1 k^{-\alpha-1}$ $(k = 1, 2, \dots)$ (see [9]), the construction of the sequence $(b_l)_{l\ge 1}$ and the definition of the function G_j, we can write

$$\Big|\sum_{j=l+1}^{\infty}\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}+kh)\Big| \le C_2 \sum_{j=l+1}^{\infty} b_j \prod_{i=1}^{n-1} g_j(s_i),$$

$$\Big|\sum_{j=1}^{l-1}\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}+kh)\Big| \le$$

$$\le \sum_{j=1}^{l-1} b_j \prod_{i=1}^{n-1} g_j(s_i) \sum_{k=[\frac{2\pi+\tau_1-\frac{\tau_l^*+\tau_l}{2}}{h}]+1}^{\infty} \left|\binom{\alpha}{k}\right| \le C_3 h^\alpha \sum_{j=1}^{l-1} b_j \prod_{i=1}^{n-1} g_j(s_i).$$

$$\Big|\sum_{k=1}^{\infty}(-1)^k \binom{\alpha}{k} G_j(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}+kh)\Big| \le b_l \prod_{i=1}^{n-1} g_i(s_i) \sum_{k=[\frac{2\pi}{h}-\frac{1}{2}]+1}^{\infty} \Big|\binom{\alpha}{k}\Big| \le$$

$$\le C_4 h^{\alpha} b_l \prod_{i=1}^{n-1} g_i(s_i),$$

where C_i $(i = 1,\dots,4)$ are positive constants and the symbol $[a]$ denotes the integer part of a real number a.

Hence, we can conclude that with some constants C_5 and C_6

$$|\Delta_n^{\alpha}(h)\,\tilde{G}_{\{1,\dots,n-1\}}(0,\dots,0,\frac{\tau_l^*+\tau_l}{2})| \ge C_5 \int_{[0,\pi]^{n-1}} G_l(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}) \prod_{i=1}^{n-1} s_i^{-1}\,ds_i$$

$$\ge C_6 b_l |\ln(\tau_l^*-\tau_l)|^{n-1}.$$

Thus, the inequality (4) is proved. Now prove the inequality (3). Without loss of generality, we can take $i = n-1$.

Let $h = \tau_l^* - \tau_l$. Then in view of the definition of conjugate function, we can write

$$\Delta_{n-1}^{\alpha}(-h)\,\tilde{G}_{\{1,\dots,n-1\}}(0,\dots,0,\frac{\tau_l^*+\tau_l}{2}) =$$

$$= \Big(-\frac{1}{2\pi}\Big)^{n-1} \int_{\mathbb{T}^{n-1}} \Delta_{n-1}^{\alpha}(-h)\, G(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}) \prod_{i=1}^{n-1} \cot\frac{s_i}{2}\,ds_i =$$

$$= \Big(-\frac{1}{2\pi}\Big)^{n-1} \int_{[0,\pi]^{n-2}} \int_{\mathbb{T}} \Delta_{n-1}^{\alpha}(-h)\, G(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}) \prod_{i=1}^{n-1} \cot\frac{s_i}{2}\,ds_i =$$

$$= \Big(-\frac{1}{2\pi}\Big)^{n-1} \int_{[0,\pi]^{n-2}} \int_0^{\pi} G(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2})(\Delta^{\alpha}(h)\cot\frac{s_{n-1}}{2})ds_{n-1} \prod_{i=1}^{n-2} \cot\frac{s_i}{2}\,ds_i.$$

Next, using the definition of the function G, we obtain

$$|\Delta_{n-1}^{\alpha}(-h)\,\tilde{G}_{\{1,\dots,n-1\}}(0,\dots,0,\frac{\tau_l^*+\tau_l}{2})| =$$

$$= \Big(\frac{1}{2\pi}\Big)^{n-1} \Big| \int_{[0,\pi]^{n-2}} \Big[\int_0^{\pi} G_l(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2})(\Delta^{\alpha}(h)\cot\frac{s_{n-1}}{2})ds_{n-1}\Big] \prod_{i=1}^{n-2} \cot\frac{s_i}{2}\,ds_i\Big|$$

$$\ge C_7 b_l \int_{\tau_l^*-\tau_l}^{1} \cdots \int_{\tau_l^*-\tau_l}^{1} \prod_{i=1}^{n-2} s_i^{-1} \frac{h^{\alpha}}{s_{n-1}^{\alpha+1}}\,ds_i$$

$$\ge C_8 b_l |\ln(\tau_l^*-\tau_l)|^{n-2},$$

where C_7 and C_8 are positive constants. Thus, the inequality (2.3) is proved. □

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ON THE SOLVABILITY OF A MIXED PROBLEM FOR AN ONE-DIMENSIONAL SEMILINEAR WAVE EQUATION WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper, for an one-dimensional semilinear wave equation we study a mixed problem with a nonlinear boundary condition. The questions of uniqueness and existence of global and blow-up solutions of this problem are investigated, depending on the nonlinearity nature appearing both in the equation and in the boundary condition.

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Keywords: Semilinear wave equation; nonlinear boundary condition; a priori estimate; local, global and blow-up solutions.

1. Introduction. The statement of the problem

In this paper, in the domain $D_T = \{(x,t) \in \mathbb{R}^2 : \ 0 < x < l, \ 0 < t < T\}$ of the plane of independent variables x and t, we consider a mixed problem of determination of a solution $u(x,t)$ of a semilinear wave equation of the form:

$$Lu = u_{tt} - u_{xx} + g(u) = f(x,t), \quad (x,t) \in D_T, \tag{1.1}$$

satisfying the initial conditions:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 \le x \le l, \tag{1.2}$$

and the boundary conditions:

$$u_x(0,t) = F[u(0,t)] + \alpha(t), \quad u_x(l,t) = \beta(t)u(l,t) + \gamma(t), \quad 0 \le t \le T, \tag{1.3}$$

where g, f, φ, ψ, α, β, γ and F are given functions, and u is the unknown real function.

Note that for $f \in C(\overline{D}_T)$, $g \in C(\mathbb{R})$, $F \in C^1(\mathbb{R})$, $\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$, $\alpha, \beta, \gamma \in C^1([0,T])$, necessary conditions of solvability of the problem (1.1)-(1.3) in the class

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$C^2(\bar{D}_T)$ are the following second order consistency conditions:

$$\varphi'(0) = F[\varphi(0)] + \alpha(0), \quad \psi'(0) = F'[\varphi(0)]\psi(0) + \alpha'(0),$$
(1.4)
$$\varphi'(l) = \beta(0)\varphi(l) + \gamma(0), \quad \psi'(l) = \beta'(0)\varphi(l) + \beta(0)\psi(l) + \gamma'(0).$$

We set $\Gamma = \Gamma_1 \cup \omega_0 \cup \Gamma_2$, where $\Gamma_1 : x = 0,\ 0 \leq t \leq T$; $\omega_0 : t = 0,\ 0 < x \leq l$; $\Gamma_2 : x = l,\ 0 < t \leq T$.

Definition 1.1. *Let the functions*

$$f \in C(\bar{D}_T), \quad g,\ F \in C(\mathbb{R}),$$
(1.5)
$$\varphi \in C^1([0,l]), \quad \psi \in C([0,l]), \quad \alpha,\ \beta,\ \gamma \in C([0,T])$$

satisfy the following first order consistency conditions:

(1.6) $$\varphi'(0) = F[\varphi(0)] + \alpha(0), \quad \varphi'(l) = \beta(0)\varphi(l) + \gamma(0).$$

A function u is said to be a strong generalized solution of the problem (1.1)-(1.3) of the class C in the domain D_T if $u \in C(\bar{D}_T)$, and there exists a sequence of functions $u_n \in C^2(\bar{D}_T)$ such that the following conditions are satisfied:

(1.7) $$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|Lu_n - f\|_{C(\bar{D}_T)} = 0,$$

(1.8) $$\lim_{n\to\infty} \|u_n(\cdot, 0) - \varphi\|_{C^1(\omega_0)} = 0, \quad \lim_{n\to\infty} \|u_{nt}(\cdot, 0) - \psi\|_{C(\omega_0)} = 0.$$

(1.9) $$\lim_{n\to\infty} \big\|u_{nx}(0, \cdot) - F[u_n(0, \cdot)] - \alpha(\cdot)\big\|_{C(\Gamma_1)} = 0,$$

(1.10) $$\lim_{n\to\infty} \big\|u_{nx}(l, \cdot) - \beta(\cdot)u_n(l, \cdot) - \gamma(\cdot)\big\|_{C(\Gamma_2)} = 0.$$

Remark 1.1. In the case $\alpha = 0$ and $\gamma = 0$, in Definition 1.1 we assume that the sequence u_n is such that $u_n \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma_1, \Gamma_2) := \{v \in C^2(\bar{D}_T) : (v_x - F(v))|_{\Gamma_1} = 0,\ (v_x - \beta v)|_{\Gamma_2} = 0\}$.

Remark 1.2. It is clear that the classical solution $u \in C^2(\bar{D}_T)$ of the problem (1.1)-(1.3) is a strong generalized solution of that problem of the class C in the domain D_T.

Note that nonlinear boundary conditions of the form (1.3) arise, for instance, in the description of the process of longitudinal vibrations of a spring in the case of elastic fixing one of its endpoints, when tension is not subjected to linear Hooke's law and is a nonlinear function of blending (see [1], p. 41], as well as, in the description of processes in the distributed self-vibrating systems (see [2], p. 405 and [3]).

The problem (1.1)-(1.3) in the case of one-dimensional spatial variable, as well as, its multivariate version has been studied in a number of papers (see, e.g., [4]-[8], and references therein). On the whole, in these papers the solution $u = u(x,t)$ of the problems of interest are considered in the energetic spaces, when the solution and its partial derivatives for a fixed t belong to Sobolev spaces with respect to the spatial variables. In the paper [9], for equation (1.1) was investigated the mixed problem, when at the endpoint $x = l$ is imposed Dirichlet homogeneous condition. When jumping from this case to the case of Robin type boundary condition (see condition (1.3) with $x = l$), additional difficulties arise not only of technical nature, but also in obtaining a priori estimate of the solution, as well as, in construction of a representation of a solution of the corresponding linear problem, which plays an essential role in obtaining of an existence theorem.

In this paper, we study the problem (1.1)-(1.3) in the class of continuous functions for sufficiently broad classes of nonlinear functions, appearing both in the problem (1.1) – (1.3).

The paper is organized as follows. In Section 2, under some conditions imposed on functions $g, F, \alpha, \beta, \gamma$ appearing in equation (1.1), we obtain a priori estimate for a strong generalized solution u of the problem (1.1)-(1.3) of the class C in the domain D_T in the sense of Definition 1.1. In Section 3, we reduce the problem (1.1)-(1.3) to an equivalent system of Volterra type nonlinear integral equations in the class of continuous functions. Section 4 is devoted to the proof of local solvability of the problem (1.1)-(1.3) in variable t. In Section 5, we prove a uniqueness theorem for a solution of the nonlinear mixed problem (1.1)-(1.3). In Section 6, we consider the question of solvability on the whole in the domain $D_T, T \leq l$ of the problem (1.1)-(1.3) in the class of continuous functions, as well as, the question of existence of a global classical solution of this problem in the domain D_∞. Finally, in Section 7, we consider the question of existence of a blow-up solution of the problem (1.1)-(1.3).

2. An a priori estimate of a solution of the problem (1.1)-(1.3)

Consider the following conditions:

(2.1) $$G(g;s) := \int_0^s g(s_1)ds_1 \geq -M_1 s^2 - M_2, \quad \int_0^s F(s_1)ds_1 \geq -M_3 \quad \forall s \in \mathbb{R},$$

(2.2) $$\alpha = \gamma = 0, \quad \beta \in C^1([0,T]), \quad \beta(t) \leq 0, \quad \beta'(t) \geq 0, \quad 0 \leq t \leq T,$$

where $M_i = const \geq 0$, $1 \leq i \leq 3$.

Lemma 2.1. *Let the conditions (2.1) and (2.2) be satisfied. Then for a strong generalized solution u of the problem (1.1)-(1.3) of the class C in the domain D_T in the sense of Definition 1.1 the following a priori estimate is fulfilled:*

$$\|u\|_{C(\overline{D}_T)} \leq c_1\|f\|_{C(\overline{D}_T)} + c_2\|\varphi\|_{C^1(\omega_0)} + c_3\|\psi\|_{C(\omega_0)} + c_4\|G(|g|;|\varphi|)\|^{\frac{1}{2}}_{C(\omega_0)}$$

$$+c_5\|F\|_{C([-|\varphi(0)|,|\varphi(0)|])} + c_6, \tag{2.3}$$

where $c_i = c_i(M_1, M_2, M_3, l, T, \beta(0))$, $1 \leq i \leq 6$ are positive constants, independent of functions u, f, φ and ψ.

Proof. Let u be a strong generalized solution u of the problem (1.1)-(1.3) of the class C in the domain D_T. Then by (2.2), Definition 1.1 and Remark 1.1, there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(D_T, \Gamma_1, \Gamma_2)$, such that the limiting relations (1.7) and (1.8) are satisfied.

Denote

$$f_n = Lu_n, \tag{2.4}$$

$$\varphi_n = u_n|_{\omega_0}, \quad \psi_n = u_{nt}|_{\omega_0}. \tag{2.5}$$

Multiplying both sides of equality (2.4) by u_{nt} and integrating over the domain D_τ, $0 < \tau \leq T$, we obtain

$$\frac{1}{2}\int\limits_{D_\tau}(u_{nt}^2)_t dxdt - \int\limits_{D_\tau} u_{nxx}u_{nt}dxdt + \int\limits_{D_\tau}\left[G(g;u_n)\right]_t dxdt = \int\limits_{D_\tau} f_n u_{nt} dxdt. \tag{2.6}$$

We set $\omega_\tau : t = \tau$, $0 \leq x \leq l$; $0 \leq \tau \leq T$. Let $\nu = (\nu_x, \nu_t)$ be the unit vector of the exterior normal to ∂D_τ. It is easy to see that

$$\begin{gathered} \nu_x|_{\omega_\tau} = 0, \quad 0 \leq \tau \leq T, \quad \nu_x|_{\Gamma_1} = -1, \quad \nu_x|_{\Gamma_2} = 1, \\ \nu_t|_{\Gamma_1 \cup \Gamma_2} = 0, \quad \nu_t|_{\omega_0} = -1, \quad \nu_t|_{\omega_\tau} = 1, \quad 0 < \tau \leq T. \end{gathered} \tag{2.7}$$

Applying integration by parts (Green's formula), and taking into account (2.5), (2.7), and that $u_n \in \overset{0}{C}{}^2(D_T, \Gamma_1, \Gamma_2)$, we can write

$$\frac{1}{2}\int_{D_\tau}(u_{nt}^2)_t dxdt + \int_{D_\tau}\left[G(g;u_n)\right]_t dxdt = \frac{1}{2}\int_{\partial D_\tau} u_{nt}^2\nu_t ds + \int_{\partial D_\tau} G(g;u_n)\nu_t ds$$

$$= \frac{1}{2}\int_{\omega_\tau} u_{nt}^2 dx - \frac{1}{2}\int_{\omega_0}\psi_n^2 dx + \int_{\omega_\tau} G(g;u_n)dx - \int_{\omega_0} G(g;\varphi_n)dx,$$

$$(2.8)\qquad \begin{aligned} -\int_{D_\tau} u_{nxx}u_{nt}dxdt &= \int_{D_\tau}[u_{nx}u_{ntx} - (u_{nx}u_{nt})_x]dxdt = \frac{1}{2}\int_{D_\tau}(u_{nx}^2)_t dxdt \\ -\int_{\partial D_\tau} u_{nx}u_{nt}\nu_x ds &= \frac{1}{2}\int_{\partial D_\tau} u_{nx}^2\nu_t ds + \int_{\Gamma_{1,\tau}} u_{nx}u_{nt}dt - \int_{\Gamma_{2,\tau}} \beta u_n u_{nt}dt \\ &= \frac{1}{2}\int_{\omega_\tau} u_{nx}^2 dx - \frac{1}{2}\int_{\omega_0}\varphi_{nx}^2 dx + \int_{\Gamma_{1,\tau}} u_{nx}u_{nt}dt - \frac{1}{2}\beta(\tau)u_n^2(l,\tau) \\ &\quad + \frac{1}{2}\beta(0)\varphi_n^2(l) + \frac{1}{2}\int_{\Gamma_{2,\tau}}\beta' u_n^2 dt, \end{aligned}$$

where $\Gamma_{i,\tau} = \Gamma_i \cap \{t \le \tau\}$, $i = 1, 2$.

In view of (2.8), the equality (2.6) we can write in the form:

$$2\int_{D_\tau} f_n u_{nt} dxdt = 2\int_{\Gamma_{1,\tau}} u_{nx}u_{nt}dt - \beta(\tau)u_n^2(l,\tau) + \beta(0)\varphi_n^2(l) + \int_{\Gamma_{2,\tau}}\beta' u_n^2 dt$$

$$(2.9)\qquad + \int_{\omega_\tau}(u_{nx}^2 + u_{nt}^2)dx + 2\int_{\omega_\tau} G(g;u_n)dx - \int_{\omega_0}(\varphi_{nx}^2 + \psi_n^2)dx - 2\int_{\omega_0} G(g;\varphi_n)dx.$$

Since $u_n \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma_1, \Gamma_2)$, we have

$$\int_{\Gamma_{1,\tau}} u_{nx}u_{nt}dt = \int_0^\tau F[u_n(0,t)]du_n(0,t) = \int_{\varphi_n(0)}^{u_n(0,\tau)} F(s)ds$$

$$(2.10)\qquad = \int_{\varphi_n(0)}^0 F(s)ds + \int_0^{u_n(0,\tau)} F(s)ds.$$

In view of (2.1), (2.2) and (2.10), from (2.9) we obtain

$$w_n(\tau) := \int_{\omega_\tau}(u_{nx}^2 + u_{nt}^2)dx \le 2\int_{D_\tau} f_n u_{nt}dxdt - \beta(0)\varphi_n^2(l) + \int_{\omega_0}(\varphi_{nx}^2 + \psi_n^2)dx$$

$$(2.11)\qquad + 2\int_{\omega_0} G(g;\varphi_n)dx + 2M_1\int_{\omega_\tau} u_n^2 dx + 2\int_0^{\varphi_n(0)} F(s)ds + 2(M_2 l + M_3).$$

Next, since by (2.5)

$$u_n(x,\tau)=\varphi_n(x)+\int_0^\tau u_{nt}(x,t)dt, \tag{2.12}$$

we have

$$|u_n(x,\tau)|^2\le 2\varphi_n^2(x)+2\left(\int_0^\tau u_{nt}(x,t)dt\right)^2\le 2\varphi_n^2(x)+2\tau\int_0^\tau u_{nt}^2(x,t)dt,$$

implying that

$$\int\limits_{\omega_\tau} u_n^2dx\le 2\|\varphi_n\|^2_{L_2(\omega_0)}+2T\int_0^\tau w_n(t)dt, \tag{2.13}$$

where w_n is as in (2.11).

Taking into account (2.13) and the following inequalities

$$2f_nu_{nt}\le u_{nt}^2+f_n^2,\quad \|f_n\|^2_{L_2(D_\tau)}\le lT\|f_n\|^2_{C(\overline{D}_T)},$$

$$\int\limits_{D_\tau} u_{nt}^2dxdt=\int_0^\tau\left[\int\limits_{\omega_t} u_{nt}^2dx\right]dt\le\int_0^\tau w_n(t)dt,$$

$$\int\limits_{\omega_0}(\varphi_{nx}^2+\psi_n^2)dx+2\int\limits_{\omega_0}G(g;\varphi_n)dx\le l\|\varphi_n'\|^2_{C(\omega_0)}+l\|\psi_n\|^2_{C(\omega_0)}+2l\|G(|g|;|\varphi_n|)\|_{C(\omega_0)},$$

$$2\int_0^{\varphi_n(0)}F(s)ds\le 2|\varphi_n(0)|\|F\|_{C([-|\varphi_n(0)|,|\varphi_n(0)|])}\le\varphi_n^2(0)+\|F\|^2_{C([-|\varphi_n(0)|,|\varphi_n(0)|])},$$

$$4M_1\|\varphi_n\|^2_{L_2(\omega_0)}+\varphi_n^2(0)-\beta(0)\varphi_n^2(l)+l\|\varphi_n'\|^2_{C(\omega_0)}\le(4M_1l+1+|\beta(0)|)\|\varphi_n\|^2_{C(\omega_0)}$$

$$+l\|\varphi_n'\|^2_{C(\omega_0)}\le l_0(\|\varphi_n\|^2_{C(\omega_0)}+\|\varphi_n'\|^2_{C(\omega_0)})\le l_0\|\varphi_n\|^2_{C^1(\omega_0)},$$

$$l_0:=max\,(4M_1l+1+|\beta(0)|,l),$$

from (2.11) we get

$$w_n(\tau)\le(4M_1T+1)\int\limits_0^\tau w_n(t)dt+lT\|f_n\|^2_{C(\overline{D}_T)}+l_0\|\varphi_n\|^2_{C^1(\omega_0)}+l\|\psi_n\|^2_{C(\omega_0)}$$

$$+2l\|G(|g|;|\varphi_n|)\|_{C(\omega_0)}+\|F\|^2_{C([-|\varphi_n(0)|,|\varphi_n(0)|])}+2(M_2l+M_3).$$

Therefore, in view of Gronwall's lemma, we obtain

$$w_n(\tau)\le\Big[lT\|f_n\|^2_{C(\overline{D}_T)}+l_0\|\varphi_n\|^2_{C^1(\omega_0)}+l\|\psi_n\|^2_{C(\omega_0)}+2l\|G(|g|;|\varphi_n|)\|_{C(\omega_0)}$$

$$+\|F\|^2_{C([-|\varphi_n(0)|,|\varphi_n(0)|])}+2(M_2l+M_3)\Big]\exp\big[T(4M_1T+1)\big]. \tag{2.14}$$

For $(x,t)\in D_T$, by integrating with respect to variable $\xi\in[0,l]$ the following obvious inequality

$$|u_n(x,t)|^2=\left|u_n(\xi,t)+\int_\xi^x u_{nx}(x_1,t)dx_1\right|^2\le 2|u_n(\xi,t)|^2+2l\int_0^l u_{nx}^2(x,t)dx,$$

we obtain

$$|u_n(x,t)|^2\le\frac{2}{l}\int_0^l|u_n(\xi,t)|^2d\xi+2lw_n(t). \tag{2.15}$$

By similar arguments, in view of (2.12), we obtain

$$\int_0^l|u_n(x,t)|^2dx\le 2\|\varphi_n\|^2_{L_2(\omega_0)}+2l\int_0^l dx\int_0^t u_{nt}^2(x,\sigma)d\sigma$$

$$\le 2\|\varphi_n\|^2_{L_2(\omega_0)}+2l\int_0^t w_n(\sigma)d\sigma.$$

Hence, taking into account (2.15), we get

$$|u_n(x,t)|^2\le\frac{4}{l}\|\varphi_n\|^2_{L_2(\omega_0)}+4\int_0^t w_n(\sigma)d\sigma+2lw_n(t)$$

$$\le\frac{4}{l}\|\varphi_n\|^2_{L_2(\omega_0)}+6l\max_{\sigma\in[0,T]}w_n(\sigma)\le 4\|\varphi_n\|^2_{C(\omega_0)}+6l\max_{\sigma\in[0,T]}w_n(\sigma). \tag{2.16}$$

Next, taking into account (2.14), (2.16) and the obvious inequality $\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}\le\sum_{i=1}^n|a_i|$, we obtain

$$\|u_n\|_{C(\overline{D}_T)}\le 2\|\varphi_n\|_{C(\omega_0)}+\Big[l\sqrt{6T}\|f_n\|_{C(\overline{D}_T)}+\sqrt{6ll_0}\|\varphi_n\|_{C^1(\omega_0)}$$

$$+l\sqrt{6}\|\psi_n\|_{C(\omega_0)}+2l\sqrt{3}\|G(|g|;|\varphi_n|)\|^{\frac{1}{2}}_{C(\omega_0)}+\sqrt{6l}\|F\|_{C([-|\varphi_n(0)|,|\varphi_n(0)|])}$$

$$+2\sqrt{3l(M_2l+M_3)}\Big]\exp\left[2^{-1}T(4M_1T+1)\right].$$

Finally, by (1.7), (1.8) and (2.5), passing to the limit (as $n\to\infty$) in the last inequality we get

$$\|u\|_{C(\overline{D}_T)}\le 2\|\varphi\|_{C(\omega_0)}+\Big[l\sqrt{6T}\|f\|_{C(\overline{D}_T)}+\sqrt{6ll_0}\|\varphi\|_{C^1(\omega_0)}+l\sqrt{6}\|\psi\|_{C(\omega_0)}$$

$$+2l\sqrt{3}\|G(|g|;|\varphi|)\|^{\frac{1}{2}}_{C(\omega_0)}+\sqrt{6l}\|F\|_{C([-|\varphi(0)|,|\varphi(0)|])}$$

$$+2\sqrt{3l(M_2l+M_3)}\Big]\exp\left[2^{-1}T(4M_1T+1)\right]. \tag{2.17}$$

Lemma 2.1 is proved.

Remark 2.1. It follows from (2.17) that the constants c_i, $1 \le i \le 6$, in the estimate (2.3) are given by

$$c_1 = l\sqrt{6T}c_0, \quad c_2 = 2 + \sqrt{6ll_0}c_0, \quad c_3 = l\sqrt{6}c_0, \quad c_4 = 2l\sqrt{3}c_0, \quad c_5 = \sqrt{6}lc_0,$$
$$c_6 = 2\sqrt{3l(M_2 l + M_3)}c_0, \quad \text{where } c_0 := \exp\left[2^{-1}T(4M_1T+1)\right]. \tag{2.18}$$

Remark 2.2. We give examples of classes of functions, which appears frequently in applications and for which the conditions in (2.1) are fulfilled:

1. $g(s) = g_0(s)sgn s + as + b$, where $g_0 \in C(\mathbb{R})$, $g_0 \ge 0$; $a, b, s \in \mathbb{R}$;
2. $F(s) = F_0(s)sgn s + as + b$, where $F_0 \in C(\mathbb{R})$, $F_0 \ge 0$; $a, b, s \in \mathbb{R}$, $a > 0$;
3. $g \in C(\mathbb{R})$, $g\big|_{(-\infty,0)} \in L_1(-\infty, 0)$; $g\big|_{(0,+\infty)} \ge 0$ (for instance, $g(s) = \exp s$, $s \in \mathbb{R}$).

3. Reduction of the problem (1.1)-(1.3) to a system of Volterra type nonlinear integral equations

We first represent the solution in the domain D_l of the following mixed linear problem

$$\square w = w_{tt} - w_{xx} = \widetilde{f}(x,t), \quad (x,t) \in D_l, \tag{3.1}$$

$$w(x,0) = \varphi(x), \quad w_t(x,0) = \psi(x), \quad 0 \le x \le l, \tag{3.2}$$

$$w_x(0,t) = \widetilde{\alpha}(t), \quad w_x(l,t) = \widetilde{\gamma}(t), \quad 0 \le t \le l, \tag{3.3}$$

in quadratures in a convenient form, where

$$\widetilde{f} \in C^1(\overline{D}_l), \quad \varphi \in C^2([0,l]), \quad \psi \in C^1([0,l]), \quad \widetilde{\alpha}, \widetilde{\gamma} \in C^1([0,l]) \tag{3.4}$$

are given functions satisfying the following second order consistency conditions:

$$\varphi'(0) = \widetilde{\alpha}(0), \quad \psi'(0) = \widetilde{\alpha}'(0), \quad \varphi'(l) = \widetilde{\gamma}(0), \quad \psi'(l) = \widetilde{\gamma}'(0), \tag{3.5}$$

and $w \in C^2(\overline{D}_l)$ is the unknown function.

Below the solution of the problem (3.1)-(3.3) we represent in the form:

$$w(x,t) = A_1(\widetilde{f}, \widetilde{\alpha}, \widetilde{\gamma})(x,t) + B_1(\varphi, \psi)(x,t), \quad (x,t) \in D_l, \tag{3.6}$$

with operators A_1 and B_1, which will be constructed in explicit form.

To this end, the domain D_l, being a square with vertices at the points $O(0,0)$, $A(0,l)$, $B(l,l)$ and $C(l,0)$, we split into four right triangles $\Delta_1 := \Delta OO_1C$, $\Delta_2 := \Delta OO_1A$, $\Delta_3 := \Delta CO_1B$ and $\Delta_4 := \Delta O_1AB$, where the point $O_1(\frac{l}{2}, \frac{l}{2})$ is the center of the square D_l. It is known that the solution of the problem (3.1)-(3.3) in the triangle Δ_1 is given by the following formula (see [1], p. 59):

$$w(x,t) = \frac{1}{2}\left[\varphi(x-t) + \varphi(x+t)\right]$$

(3.7) $$+\frac{1}{2}\int_{x-t}^{x+t} \psi(\tau)d\tau + \frac{1}{2}\int_{\Omega^1_{x,t}} \widetilde{f}(\xi,\tau)d\xi d\tau, \quad (x,t)\in\Delta_1,$$

where $\Omega^1_{x,t}$ denotes the triangle with vertices at the points (x,t), $(x-t,0)$ and $(t+x,0)$.

To obtain the solution of the problem (3.1)-(3.3) in the other triangles Δ_2, Δ_3 and Δ_4, we use the following equality (see [10], p. 173):

(3.8) $$w(P) = w(P_1) + w(P_2) - w(P_3) + \frac{1}{2}\int_{PP_1P_2P_3} \widetilde{f}(\xi,\tau)d\xi d\tau,$$

which is true for any characteristic (for equation (3.1)) rectangle $PP_1P_2P_3 \subset \overline{D}_l$, where P and P_3, as well as, P_1 and P_2 are the opposite vertices of that rectangle, and the ordinate of the point P is greater than the ordinates of the other points.

Now let $(x,t)\in\Delta_2$. Then setting

(3.9) $$\widetilde{\mu}_1 := w|_{\Gamma_{1,l}}$$

and applying the equality (3.8) for characteristic rectangle with vertices at the points $P(x,t)$, $P_1(0,t-x)$, $P_2(t,x)$ and $P_3(t-x,0)$, the formula (3.7) for point $P_2(t,x)\in\Delta_1$, and using (3.9), we can write

$$w(x,t) = w(P_1) + w(P_2) - w(P_3) + \frac{1}{2}\int_{PP_1P_2P_3} \widetilde{f}(\xi,\tau)d\xi d\tau = \widetilde{\mu}_1(t-x) - \varphi(t-x)$$

$$+\frac{1}{2}\left[\varphi(t-x)+\varphi(t+x)\right]+\frac{1}{2}\int_{t-x}^{t+x}\psi(\tau)d\tau+\frac{1}{2}\int_{\Omega^1_{t,x}}\widetilde{f}(\xi,\tau)d\xi d\tau+\frac{1}{2}\int_{PP_1P_2P_3}\widetilde{f}(\xi,\tau)d\xi d\tau =$$

(3.10)
$$\widetilde{\mu}_1(t-x)+\frac{1}{2}\left[\varphi(t+x)-\varphi(t-x)\right]+\frac{1}{2}\int_{t-x}^{t+x}\psi(\tau)d\tau+\frac{1}{2}\int_{\Omega^2_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau, \quad (x,t)\in\Delta_2.$$

Here $\Omega^2_{x,t}$ denotes the quadrangle $P\overline{P}_2P_3P_1$, where $\overline{P}_2 = \overline{P}_2(t+x,0)$.

Taking into account that for $(x,t)\in\Delta_2$

$$\int_{\Omega^2_{x,t}} \widetilde{f}(\xi,\tau)d\xi d\tau = \int_0^{t-x} d\tau \int_{-x+t-\tau}^{x+t-\tau} \widetilde{f}(\xi,\tau)d\xi + \int_{t-x}^{t} d\tau \int_{x-t+\tau}^{x+t-\tau} \widetilde{f}(\xi,\tau)d\xi,$$

in view of (3.10) we obtain

$$w_x(x,t) = -\widetilde{\mu}_1'(t-x) + \frac{1}{2}\left[\varphi'(t+x) + \varphi'(t-x) + \psi(t+x) + \psi(t-x)\right]$$

$$+\frac{1}{2}\int_0^{t-x}\big[\widetilde{f}(x+t-\tau,\tau)+\widetilde{f}(-x+t-\tau,\tau)\big]d\tau$$

$$+\frac{1}{2}\int_{t-x}^{t}\big[\widetilde{f}(x+t-\tau,\tau)-\widetilde{f}(x-t+\tau,\tau)\big]d\tau. \tag{3.11}$$

Similarly, for $(x,t)\in\Delta_2$ we get

$$w_t(x,t)=\widetilde{\mu}_1'(t-x)+\frac{1}{2}\big[\varphi'(t+x)-\varphi'(t-x)+\psi(t+x)-\psi(t-x)\big]$$

$$+\frac{1}{2}\int_0^{t-x}\big[\widetilde{f}(x+t-\tau,\tau)-\widetilde{f}(-x+t-\tau,\tau)\big]d\tau$$

$$+\frac{1}{2}\int_{t-x}^{t}\big[\widetilde{f}(x+t-\tau,\tau)+\widetilde{f}(x-t+\tau,\tau)\big]d\tau. \tag{3.12}$$

Setting $x=0$ in the equality (3.11), and taking into account the first boundary condition in (3.3), for unknown function $\widetilde{\mu}_1$ we obtain the equality:

$$-\widetilde{\mu}_1'(t)+\varphi'(t)+\psi(t)+\int_0^t\widetilde{f}(t-\tau,\tau)d\tau=\widetilde{\alpha}(t),\quad 0\le t\le l.$$

Integrating the last equality and taking into account the initial condition $\widetilde{\mu}_1(0)=\varphi(0)$, we get

$$\widetilde{\mu}_1(t)=A_2(\widetilde{f},\widetilde{\alpha},\widetilde{\gamma})(t)+B_2(\varphi,\psi)(t):=\varphi(t)-\int_0^t\widetilde{\alpha}(\tau)d\tau+\int_0^t\psi(\tau)d\tau$$

$$+\int_0^t d\tau_1\int_0^{\tau_1}\widetilde{f}(\tau_1-\tau,\tau)d\tau,\quad 0\le t\le l. \tag{3.13}$$

Now, in view of (3.10) and (3.13), the solution of the problem (3.1)-(3.3) in the domain Δ_2 can be represented in the form:

$$w(x,t)=-\int_0^{t-x}\widetilde{\alpha}(\tau)d\tau+\int_0^{t-x}\psi(\tau)d\tau$$

$$+\int_0^{t-x}d\tau_1\int_0^{\tau_1}\widetilde{f}(\tau_1-\tau,\tau)d\tau+\frac{1}{2}\big[\varphi(t+x)+\varphi(t-x)\big]+\frac{1}{2}\int_{t-x}^{t+x}\psi(\tau)d\tau$$

$$+\frac{1}{2}\int_{\Omega^2_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau,\quad (x,t)\in\Delta_2. \tag{3.14}$$

Next, to obtain representations for the solution of the problem (3.1)-(3.3) in the domains Δ_3 and Δ_4, we set

$$\widetilde{\mu}_2:=w|_{\Gamma_2} \tag{3.15}$$

and use the above arguments, applied to obtain the equality (3.10), to conclude that

$$w(x,t)=\widetilde{\mu}_2(x+t-l)+\frac{1}{2}\big[\varphi(x-t)-\varphi(2l-x-t)\big]$$

$$+\frac{1}{2}\int_{x-t}^{2l-x-t}\psi(\tau)d\tau+\frac{1}{2}\int_{\Omega^2_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau,\quad (x,t)\in\Delta_3. \tag{3.16}$$

and

$$w(x,t)=\widetilde{\mu}_1(t-x)+\widetilde{\mu}_2(x+t-l)-\frac{1}{2}\left[\varphi(t-x)+\varphi(2l-t-x)\right]$$

$$+\frac{1}{2}\int_{t-x}^{2l-t-x}\psi(\tau)d\tau+\frac{1}{2}\int_{\Omega^4_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau,\quad (x,t)\in\Delta_4. \tag{3.17}$$

Here $\Omega^3_{x,t}$ denotes the quadrangle with vertices $P^3(x,t)$, $P^3_1(l,x+t-l)$, $P^3_2(x-t,0)$, $P^3_3(2l-x-t,0)$, and $\Omega^4_{x,t}$ denotes the pentagon with vertices $P^4(x,t)$, $P^4_1(0,t-x)$, $P^4_2(t-x,0)$, $P^4_3(2l-x-t,0)$ and $P^4_4(l,x+t-l)$.

Taking into account that for $(x,t)\in\Delta_3$

$$\int_{\Omega^2_{x,t}}\widetilde{f}(\xi,\tau)d\xi d\tau=\int_0^{x+t-l}d\tau\int_{x-t+\tau}^{2l-x-t+\tau}\widetilde{f}(\xi,\tau)d\xi+\int_{x+t-l}^{t}d\tau\int_{x-t+\tau}^{x+t-\tau}\widetilde{f}(\xi,\tau)d\xi,$$

and differentiating the equality (3.16) by x, we obtain

$$w_x(x,t)=\widetilde{\mu}'_2(x+t-l)+\frac{1}{2}\left[\varphi'(x-t)+\varphi'(2l-x-t)\right]$$

$$-\frac{1}{2}\left[\psi(2l-x-t)+\psi(x-t)\right]-\frac{1}{2}\int_0^{x+t-l}\left[\widetilde{f}(2l-x-t+\tau,\tau)+\widetilde{f}(x-t+\tau,\tau)\right]d\tau$$

$$+\frac{1}{2}\int_{x+t-l}^{t}\left[\widetilde{f}(x+t-\tau,\tau)-\widetilde{f}(x-t+\tau,\tau)\right]d\tau,\quad (x,t)\in\Delta_3. \tag{3.18}$$

Substituting the expression (3.18) with $x=l$ into the second boundary condition in (3.3), for unknown function $\widetilde{\mu}_2$ we obtain

$$\widetilde{\mu}'_2(t)-\psi(l-t)+\varphi'(l-t)-\int_0^t\widetilde{f}(l-t+\tau,\tau)d\tau=\widetilde{\gamma}(t),\quad 0\le t\le l. \tag{3.19}$$

And, in view of (3.2) and (3.15), we have

$$\widetilde{\mu}_2(0)=\varphi(l). \tag{3.20}$$

Finally, from (3.19) and (3.20) we obtain

$$\widetilde{\mu}_2(t)=A_3(\widetilde{f},\widetilde{\alpha},\widetilde{\gamma})(t)+B_3(\varphi,\psi)(t):=\varphi(l-t)+\int_0^t\widetilde{\gamma}(\tau)d\tau+\int_{l-t}^{l}\psi(\tau)d\tau$$

$$+\int_0^t d\tau_1\int_0^{\tau_1}\widetilde{f}(l-\tau_1+\tau,\tau)d\tau,\quad 0\le t\le l. \tag{3.21}$$

Remark 3.1. If w is a solution of the problem (3.1)-(3.3), then in view of equalities (3.6), (3.13) and (3.21), for the triple of functions $(w, \widetilde{\mu}_i := w|_{\Gamma_i}, i = 1, 2)$ the following integral representation holds:

$$(3.22) \qquad (w, \widetilde{\mu}_1, \widetilde{\mu}_2) = A(\widetilde{f}, \alpha, \gamma) + B(\varphi, \psi),$$

where the actions of operators $A := (A_1, A_2, A_3)$, $B := (B_1, B_2, B_3)$ are specified by formulas (3.6), (3.7), (3.14), (3.16), (3.17), (3.13) and (3.21).

Remark 3.2. It is easy to check that in the case $\widetilde{f} \in C(D_l)$, $\varphi \in C^1([0, l])$, $\psi \in C([0, l])$, $\widetilde{\alpha}$, $\widetilde{\gamma} \in C([0, l])$, if the first order consistency conditions $\varphi'(0) = \widetilde{\alpha}(0)$, $\varphi'(l) = \widetilde{\gamma}(0)$ are satisfied, then in view of formulas (3.11) and (3.12) for every w_x, w_t in the domain Δ_2, and also in the other domains Δ_1, Δ_3 and Δ_4, the triple of functions $(w, \widetilde{\mu}_1, \widetilde{\mu}_2)$, defined by equality (3.22), belongs to the class $C^1(\overline{D}_l) \times C^1([0, l]) \times C^1([0, l])$. Moreover, the linear operator

$$(3.23) \qquad A : C(\overline{D}_l) \times C([0, l]) \times C([0, l]) \to C^1(\overline{D}_l) \times C^1([0, l]) \times C^1([0, l])$$

in (3.22) is continuous. A similar remark holds also for operator B in the corresponding spaces of functions.

Remark 3.3. Similar to Remark 3.2, it can be shown that if the smoothness condition (3.4) and the second order consistency condition (3.5) are satisfied, then according to (3.6), the function w, constructed by means of equalities (3.7), (3.14), (3.16), (3.17), (3.13), (3.21), belongs to the class $C^2(\overline{D}_l)$, and is the classical solution of the problem (3.1)-(3.3).

Remark 3.4. Notice that in the case where the problem (3.1)-(3.3) is considered in the domain D_T for $T \leq l$, then for the triple of functions $(w, \widetilde{\mu}_i := w|_{\Gamma_i}, i = 1, 2)$, the integral representation (3.22) remains valid.

Now we proceed to reduce the problem (1.1)-(1.3) to a system of Volterra type nonlinear integral equations. Let u be a strong generalized solution of this problem of the class C in the domain D_T, $T \leq l$, that is, $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in C^2(\overline{D}_T)$, such that the equalities (1.7)-(1.10) are satisfied. Consider the function u_n as a classical solution of the problem (3.1)-(3.3) for

$$\widetilde{f} = -g(u_n) + f_n,\ \varphi = \varphi_n,\ \psi = \psi_n,\ \widetilde{\alpha} = F(\mu_{1n}) + \alpha_n,\ \widetilde{\gamma} = \beta\mu_{2n} + \gamma_n,$$

where

$$f_n = Lu_n,\ \varphi_n := u_n|_{\omega_0},\ \psi_n = u_{nt}|_{\omega_0},$$

$$\mu_{1n} = u_n|_{\Gamma_1}, \ \alpha_n = u_{nx}|_{\Gamma_1} - F(\mu_{1n}), \ \gamma_n = u_{nx}|_{\Gamma_2} - \beta\mu_{2n}.$$

Then, by equality (3.22), for function u_n and its truncations $\mu_{in} := u_n|_{\Gamma_i}, i = 1, 2$, the following equalities hold:

$$u_n = A_1\big(-g(u_n) + f_n, F(\mu_{1n}) + \alpha_n, \beta\mu_{2n} + \gamma_n\big) + B_1(\varphi_n, \psi_n),$$

(3.24) $$\mu_{in} = A_{i+1}\big(-g(u_n) + f_n, F(\mu_{1n}) + \alpha_n, \beta\mu_{2n} + \gamma_n\big) + B_{i+1}(\varphi_n, \psi_n),$$

$$i = 1, 2.$$

Taking into account Remark 3.2, the equalities (1.7)-(1.10) and (3.22), and passing to the limit in equations (3.24) as $n \to \infty$, we conclude that the triple of functions $(u, \mu_i := u|_{\Gamma_i}, i = 1, 2)$ satisfies the nonlinear operator equation:

(3.25) $$(u, \mu_1, \mu_2) = A_0(u, \mu_1, \mu_2),$$

where

(3.26) $$A_0(u, \mu_1, \mu_2) = A(-g(u) + f, F(\mu_1) + \alpha, \beta\mu_2 + \gamma) + B(\varphi, \psi).$$

Remark 3.5. In view of Remark 3.2, the operator A_0 defined in (3.26) acts continuously from the space $C(D_T) \times C([0,T]) \times C([0,T])$ to the space $C^1(D_T) \times C^1([0,T]) \times C^1([0,T])$, $T \le l$. Hence, taking into account that the space $C^1(D_T) \times C^1([0,T]) \times C^1([0,T])$ is compactly embedded into the space $C(D_T) \times C([0,T]) \times C([0,T])$ (see [11], p. 135)], we conclude that the operator

(3.27) $$A_0: \ C(\bar{D}_T) \times C([0,T]) \times C([0,T]) \to C(D_T) \times C([0,T]) \times C([0,T])$$

is compact.

Remark 3.6. It is easy to see that if $(\xi, \tau) \in \Omega^i_{x,t}$, $1 \le i \le 4$, then $\tau \le t$, which in view of formulas (3.7), (3.14), (3.16), (3.17), (3.13), (3.21), permits to consider (3.25) as a system of Volterra type nonlinear integral equations with respect to variable t. Notice that in the linear case, for this system can be applied a converging method of Picard's successive approximations in the corresponding spaces of functions.

Remark 3.7. Similar to Remark 3.3, in view of (3.25) we can conclude that if u is a strong generalized solution of the problem (1.1)-(1.3) of the class C in the domain $D_T, T \le l$, and the following smoothness conditions

(3.28) $$f \in C^1(\bar{D}_T), \quad g, \ F \in C^1(\mathbb{R}),$$
$$\varphi \in C^2([0,l]), \quad \psi \in C^1([0,l]), \quad \alpha, \ \beta, \ \gamma \in C^1([0,T])$$

and the second order consistency condition (1.4) are satisfied, then u will be the classical solution of this problem from the space $C^2(D_T)$.

Remark 3.8. From the above presented arguments it follows that if the smoothness condition (1.5) and the first order consistency condition (1.6) are satisfied, and if a function u is a strong generalized solution of the problem (1.1)-(1.3) of the class C in the domain D_T in the sense of Definition 1.1, then the triple of functions $(u, \mu_i = u|_{\Gamma_i}, i = 1, 2)$ is a continuous solution of the system of Volterra type nonlinear integral equations (3.25). Using arguments similar to those applied in [9], it can easily be shown that the converse assertion also holds.

4. Local solvability in t of the problem (1.1)-(1.3)

Theorem 4.1. *Let the functions $f \in C(\overline{D}_l)$, $g, F \in C(\mathbb{R})$, $\varphi \in C^1([0,l])$, ψ, α, β, $\gamma \in C([0,l])$ satisfy the consistency condition (1.6). Then a positive number $T_0 = T_0(f, g, F, \varphi, \psi, \alpha, \beta, \gamma) \leq l$ can be found such that for $T \leq T_0$ the problem (1.1)-(1.3) in the domain D_T will have at least one strong generalized solution u of the class C.*

Proof. In Section 3, the problem (1.1)-(1.3) in the space $C(D_T) \times C([0,T]) \times C([0,T])$, $T \leq l$, was reduced to the equivalent equation (3.25), where by Remark 3.5 the operator A_0 is continuous and compact, acting in the space $C(\overline{D}_T) \times C([0,T]) \times C([0,T])$. Hence, according to Schauder theorem, for solvability of equation (3.25) it is enough to show that the operator A_0 transfers some ball $B_{R_0}(u^0, \mu_1^0, \mu_2^0)$ with center at point (u^0, μ_1^0, μ_2^0) and of radius $R_0 > 0$ of the Banach space $C(\overline{D}_T) \times C([0,T]) \times C([0,T])$ to itself. We show that this is the case for small enough $T \leq l$. Indeed, in view of Remark 3.1 and equality (3.22), the operator equation (3.25) can be written in the form:

$$(u, \mu_1, \mu_2) = A_0(u, \mu_1, \mu_2) = (u^0, \mu_1^0, \mu_2^0) + A\big(-g(u), F(\mu_1), \beta\mu_2\big), \tag{4.1}$$

where

$$u^0 = A_1(f, \alpha, \gamma) + B_1(\varphi, \psi), \quad \mu_i^0 = A_{i+1}(f, \alpha, \gamma) + B_{i+1}(\varphi, \psi), \ i = 1, 2.$$

It is easy to see that if $(\tilde{u}, \tilde{\mu}_1, \tilde{\mu}_2)$ belongs to the ball $B_{R_0}(u^0, \mu_1^0, \mu_2^0)$ and, according to Remark 3.6, the linear operator A from (3.23) is a Volterra type integral operator by the variable $t \leq T$, then

$$\|A\big(-g(u), F(\mu_1), \beta\mu_2\big)\|_{C(\overline{D}_T)\times C([0,T])\times C([0,T])} \leq TM, \tag{4.2}$$

where

$$0 < M := M(\|g\|_{C([-R,R])}, \|F\|_{C([-R,R])}, \|\beta\|_{C([0,l])}R) < \infty,$$

$$R := \|(u^0, \mu_1^0, \mu_2^0)\|_{C(\overline{D}_l)\times C([0,l])\times C([0,l])} + R_0,$$

and R_0 is an arbitrary fixed positive number, and the function $M = M(s_1, s_2, s_3)$ is continuous and nondecreasing by each of the argument $s_i \geq 0$, $i = 1, 2, 3$.

Taking $T \leq T_0$, where $T_0 := \frac{R_0}{M}$, from (4.1) and (4.2) for $(\overline{u}, \overline{\mu}_1, \overline{\mu}_2) \in B_{R_0}(u^0, \mu_1^0, \mu_2^0)$, we obtain

$$\|A_0(\overline{u}, \overline{\mu}_1, \overline{\mu}_2) - (u^0, \mu_1^0, \mu_2^0)\|_{C(\overline{D}_T)\times C([0,T])\times C([0,T])} \leq R_0,$$

implying that $A_0 : B_{R_0}(u^0, \mu_1^0, \mu_2^0) \to B_{R_0}(u^0, \mu_1^0, \mu_2^0)$, and the result follows. Theorem 4.1 is proved.

5. Uniqueness of a solution of problem (1.1) – (1.3)

Theorem 5.1. *Problem (1.1) – (1.3) cannot have more than one strong generalized solution of the class C in the domain $D_T, T \leq l$ in the sense of Definition 1.1, if in (1.5) it is assumed additionally that $g, F \in C^1(\mathbb{R})$.*

Proof. Assume that problem (1.1) – (1.3) has two distinct strong generalized solutions u^1 and u^2 of the class C in the domain $D_T, T \leq l$. Then, according to Remark 3.8, the triples of functions $(u^1, \mu_1^1 := u^1|_{\Gamma_1}, \mu_2^1 := u^1|_{\Gamma_2})$ and $(u^2, \mu_1^2 := u^2|_{\Gamma_1}, \mu_2^2 := u^2|_{\Gamma_2})$ are continuous solutions of the system of nonlinear integral equations (3.25). Setting $u^0 := u^2 - u^1$, $\mu_i^0 := \mu_i^2 - \mu_i^1$, $i = 1, 2$, and taking into account (3.13), (3.14) and Remark 3.4, we can write

$$\mu_1^0(t) = -\int_0^t [F(\mu_1^2) - F(\mu_1^1)](\tau)d\tau$$
$$-\int_0^t d\tau_1 \int_0^{\tau_1} [g(u^2) - g(u^1)](\tau_1 - \tau, \tau)d\tau, \; 0 \leq t \leq T,$$

(5.1)
$$u^0(x,t) = -\int_0^{t-x} [F(\mu_1^2) - F(\mu_1^1)](\tau)d\tau$$
$$-\int_0^{t-x} d\tau_1 \int_0^{\tau_1} [g(u^2) - g(u^1)](\tau_1 - \tau, \tau)d\tau$$
$$-\frac{1}{2}\int_{G_{x,t}^2} [g(u^2) - g(u^1)](\xi, \tau)d\xi d\tau, \; (x,t) \in \Delta_2 \cap \{t < T\}.$$

Next, since

$$
\begin{aligned}
F(\mu_1^2) - F(\mu_1^1) &= \left[\int_0^1 F'[\mu_1^1 + (\mu_1^2 - \mu_1^1)s]ds\right]\mu_1^0, \\
g(u^2) - g(u^1) &= \left[\int_0^1 g'[u^1 + (u^2 - u^1)s]ds\right]u^0,
\end{aligned} \tag{5.2}
$$

then assuming $u^i, \mu_1^i, i = 1, 2$ to be fixed functions and setting

$$\overline{u}(t) = \max_{0 \le x \le l} |u^0(x,t)|, \quad 0 \le t \le T,$$

by (5.1) and (5.2), we obtain

$$
\begin{gathered}
|u^0(x,t)| \le M_0 \int_0^t \left[|\mu_1^0(\tau)| + \overline{u}(\tau)\right]d\tau \\
\le M_0 \int_0^t \left[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\right]d\tau, \quad (x,t) \in \Delta_2 \cap \{t < T\}, \\
|\mu_2^0(t)| \le M_0 \int_0^t \left[|\mu_1^0(\tau)| + \overline{u}(\tau)\right]d\tau \\
\le M_0 \int_0^t \left[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\right]d\tau, \quad 0 \le t \le T,
\end{gathered} \tag{5.3}
$$

where M_0 is a positive constant depending on g, F and on fixed functions $u^i, \mu_j^i, i, j = 1, 2$. Similar arguments, carried out in the other domains $\Delta_j \cap \{t < T\}$, and possibly, by enlarging M_0, allow to obtain the following inequalities:

$$
\begin{gathered}
|u^0(x,t)| \le M_0 \int_0^t \left[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\right]d\tau, \\
(x,t) \in \Delta_j \cap \{t < T\}, \ j = 1, 3, 4, \\
|\mu_1^0(t)| \le M_0 \int_0^t \left[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\right]d\tau, \quad 0 \le t \le T.
\end{gathered} \tag{5.4}
$$

It follows from (5.3) and (5.4) that

$$|\mu_1^0(t)| + |\mu_2^0(t)| + \overline{u}(t) \le 2M_0 \int_0^t \left[|\mu_1^0(\tau)| + |\mu_2^0(\tau)| + \overline{u}(\tau)\right]d\tau, \quad 0 \le t \le T.$$

Therefore, in view of Gronwall's lemma, we conclude that $\overline{u}(t) = 0$, $0 \le t \le T$, that is, $u^1 = u^2$. The obtained contradiction completes the proof of the theorem. Theorem 5.1 is proved.

6. The solvability of problem (1.1) – (1.3) in domain D_T for any $T \leq l$ in the case $\alpha = \gamma = 0$

Let $\tau \in [0, 1]$, and let $u = u_\tau$ be a strong generalized solution of the class C in the domain $D_T, T \leq l$ of the following problem

$$u_{tt} - u_{xx} = \tau[-g(u) + f(x,t)], \quad (x,t) \in D_T,$$

(6.1) $$u(x,0) = \tau\varphi(x), \quad u_t(x,0) = \tau\psi(x), \quad 0 \leq x \leq l,$$

$$u_x(0,t) = \tau F[u(0,t)], \quad u_x(l,t) = \tau\beta(t)u(l,t), \quad 0 \leq t \leq T,$$

provided that the smoothness condition (1.5) and the following consistency condition (an analog of condition (1.6)):

$$\varphi'(0) = F[\tau\varphi(0)], \ \varphi'(l) = \tau\beta(0)\varphi(l)$$

are satisfied. It is easy to see that these conditions will be satisfied for any $\tau \in [0, 1]$ if, for instance,

(6.2) $$\varphi(0) = 0, \ \varphi'(0) = F(0), \ \varphi(l) = 0, \ \varphi'(l) = 0.$$

Similar arguments show that if $u = u_\tau$ is a classical solution of the problem (6.1) for any $\tau \in [0, 1]$, then according to Remark 3.7, it is natural to require that the smoothness condition (3.28) and the following equalities (instead of (1.4)) be fulfilled:

$$\varphi'(0) = F[\tau\varphi(0)], \quad \psi'(0) = \tau F'[\tau\varphi(0)]\psi(0),$$

$$\varphi'(l) = \tau\beta(0)\varphi(l), \quad \psi'(l) = \tau\beta'(0)\varphi(l) + \tau\beta(0)\psi(l).$$

It is easy to see that these conditions will be satisfied for any $\tau \in [0, 1]$, if, for instance, along with (6.2) will be satisfied the following conditions:

(6.3) $$\psi(0) = 0, \ \psi'(0) = 0, \ \psi(l) = 0, \ \psi'(l) = 0.$$

Remark 6.1. Note that for $\tau = 1$, the problems (6.1) and (1.1)-(1.3) coincide, and similar to Definition 1.1, it can be defined the notion of strong generalized solution of problem (6.1) of the class C in domain D_T, provided that the consistency condition (6.2) is satisfied.

Remark 6.2. In view of Remark 3.8, problem (6.1) in the class of continuous functions can be reduced the following equivalent nonlinear operator equation:

(6.4) $$(u, \mu_1, \mu_2) = \tau A_0(u, \mu_1, \mu_2),$$

where the operator A_0 is as in (3.27) and, by Remark 3.5, is compact.

As a consequence of Remarks 6.1, 6.2 and Leray-Schauder theorem (see [12], p. 375), we can state the following result.

Lemma 6.1. *Let conditions (1.5) and (6.2) be fulfilled. If for any strong generalized solution $u = u_\tau$ of problem (6.1) of the class C in the domain D_T for any $\tau \in [0,1]$ the following a priori estimate holds:*

$$\|u\|_{C(\overline{D}_T)} \leq M_*, \tag{6.5}$$

where $M_* = M_*(g, f, \varphi, \psi, F, \alpha, \beta, \gamma)$ is a nonnegative constant independent of τ, then problem (1.1)-(1.3) has at least one strong generalized solution of the class C in the domain D_T.

Proof. Observe first that in view of Remarks 6.1 and 6.2, a function $u \in C(\bar{D}_T)$ is a strong generalized solution of problem (1.1)-(1.3) of the class C in the domain D_T if and only if it is a continuous solution of the nonlinear operator equation (6.4) for $\tau = 1$. On the other hand, according to conditions of the lemma, for any solution $u \in C(\bar{D}_T)$ of equation (6.4) with compact operator A_0, for any $\tau \in [0,1]$ the a priori estimate (6.5) holds, and hence, according to Leray-Schauder theorem, equation (6.4) for $\tau = 1$ has at least one solution $u \in C(D_T)$, which is also a strong generalized solution of problem (1.1)-(1.3) of the class C in the domain D_T.

Lemma 6.1 is proved.

As a consequence of Lemmas 2.1 and 6.1 and Theorem 5.1, we have the following result.

Theorem 6.1. *Let $T \leq l$, and let (1.5), (6.2) and the conditions of Lemma 2.1 be fulfilled. Then problem (1.1)-(1.3) has at least one strong generalized solution of the class C in the domain D_T, which in the case $g, F \in C^1(\mathbb{R})$ is unique. Moreover, if the smoothness condition (3.28) and equalities (6.2), (6.3) are also satisfied, then this solution will also be classical.*

Proof. Observe first that if the given functions g, f, φ, ψ, F of problem (1.1)-(1.3) we replace by the functions $\tau g, \tau f, \tau\varphi, \tau\psi, \tau F$, $\tau \in [0,1]$, then by (2.3) and (2.18), for any strong generalized solution $u = u_\tau$ of the class C in the domain D_T of the obtained problem the following a priori estimate holds:

$$\|u\|_{C(D_T)} \leq c_1\tau\|f\|_{C(\overline{D}_T)} + c_2\tau\|\varphi\|_{C^1(\omega_0)} + c_3\tau\|\psi\|_{C(\omega_0)} + c_4\|G(|g|;|\tau\varphi|)\|_{C(\omega_0)}$$

$$+c_5\tau\|F\|_{C([-|\varphi(0)|,|\varphi(0)|])} + c_6$$

$$\leq c_1\|f\|_{C(\overline{D}_T)} + c_2\|\varphi\|_{C(\omega_0)} + c_3\|\psi\|_{C(\omega_0)} + c_4\|G(|g|;|\varphi|)\|^{\frac{1}{2}}_{C(\omega_0)}$$
$$+c_5\|F\|_{C([-|\varphi(0)|,|\varphi(0)|])} + c_0.$$

Hence, the first assertion of the theorem follows from Lemma 6.1 and Theorem 5.1. The assertion that under conditions (3.28) and (6.3) the solution is classical, follows from Remark 3.7. Theorem 6.1 is proved.

Remark 6.3. Notice that the existence of the unique classical solution in the domain $D_{l,k} := \{(x,t) \in \mathbb{R}^2 : 0 < x < l,\ (k-1)l < t < kl\}$, $k \in \mathbb{N}$, $k \geq 2$, of the mixed problem

$$Lu = f(x,t), \quad (x,t) \in D_{l,k},$$
$$u|_{t=(k-1)l} = \varphi, \quad u_t|_{t=(k-1)l} = \psi,$$
$$u_x(0,t) = F[u(0,t)] + \alpha(t), \quad u_x(l,t) = \beta(t)u(l,t) + \gamma(t), \quad (k-1)l \leq t \leq kl,$$

can be proved exactly in the same way as in the case $k = 1$, that is, in the domain $D_{l,1} = D_l$. Therefore, all the constructions of structural nature, given in the previous sections in the domain D_T with $T \leq l$ (such us the representations (3.7), (3.10), (3.16), (3.17) of a solution of the linear problem (3.1)-(3.3) and the nonlinear operator equations of type (3.25) as a system of Volterra type nonlinear integral equations with respect to variable t) analogously can be transferred to the case of domain D_T for any $T \geq l$. Hence, if the conditions of Lemma 2.1, the smoothness condition (3.28) for $T = \infty$, and the consistency conditions (6.2), (6.3) are satisfied, then for any $T > 0$ (in particular, for $T = \infty$) in the domain D_T there exists a unique classical solution $u \in C^2(\overline{D}_T)$ of the problem (1.1)-(1.3). Thus, we have the following result.

Theorem 6.2. *Let the conditions of Lemma 2.1, the smoothness condition (3.28) for $T = \infty$, and the consistency conditions (6.2), (6.3) be satisfied. Then for $T = \infty$ problem (1.1)-(1.3) has a unique global classical solution $u \in C^2(\overline{D}_\infty)$.*

7. The existence of blow-up solution of problem (1.1)-(1.3)

In this section, in a special case, we show that if the conditions in (2.1), imposed on the nonlinear functions g and F are violated, then the solution u of the problem (1.1)-(1.3) can turn out to be blow-up. That is, a number $T^* \in (0,l]$ can be found such that for $T < T^*$ problem (1.1)-(1.3) has a unique classical solution u, and

$$\lim_{T \to T^*-0} \|u\|_{C(\overline{D}_T)} = \infty. \tag{7.1}$$

This, in particular, implies that the considered problem has no a classical solution in the domain D_T for $T \geq T^*$.

Indeed, consider the following special case of problem (1.1)-(1.3)

$$
(7.2) \qquad \begin{gathered} u_{tt} - u_{xx} = 0, \quad (x,t) \in D_T, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 \leq x \leq l, \\ u_x(0,t) = F[u(0,t)], \quad u_x(l,t) = 0, \quad 0 \leq t \leq T, \end{gathered}
$$

where $\varphi \in C^2([0,l])$, $\varphi(0) > 0$, $\psi \in C^1([0,l])$ and $F(s) = -\delta|s|^\lambda s$, $\delta := const > 0$, $\lambda := const > 0$, $s \in \mathbb{R}$, and the corresponding consistency conditions, similar to (1.4), are satisfied. It is easy to check that in the case $\psi = -\varphi'$, the classical solution u of this problem in the domain D_T for $T = T^*$ is given by formula:

$$
(7.3) \qquad u(x,t) = \begin{cases} \varphi(x-t), & (x,t) \in \Delta_1 \cap \{t < T^*\}, \\ \mu_1(t-x), & (x,t) \in \Delta_2 \cap \{t < T^*\}, \\ \varphi(2l - x - t) - \varphi(l) + \varphi(x-t), & (x,t) \in \Delta_3 \cap \{t < T^*\}, \\ \mu_1(t-x) + \varphi(2l - x - t) - \varphi(x+t-l), \\ (x,t) \in \Delta_4 \cap \{t < T^*\}, \end{cases}
$$

where

$$
(7.4) \qquad \mu_1(t) = \frac{\varphi(0)}{[1 - \delta\lambda\varphi^\lambda(0)t]^{\frac{1}{\lambda}}}, \quad 0 \leq t < T^* := \frac{1}{\delta\lambda\varphi^\lambda(0)} < l.
$$

It follows from (7.3) and (7.4) that the solution of problem (7.2) is blow-up, that is, equality (7.1) is satisfied. Therefore, in the considered case, in the statement of this problem it should be required that $T < T^*$.

Remark 7.1. In fact, formula (7.3) allows to continue the solution of the considered problem from the domain D_{T^*} to domain $D_l \cap \{t < x + T^*\}$, and this solution $u(x,t)$ will unboundedly increase when the point (x,t) from the domain $D_l \cap \{t < x + T^*\}$ approaches to the characteristic $t - x = T^*$, to which border on this domain by the part of its boundary.

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MEROMORPHIC SOLUTIONS FOR A CLASS OF DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

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Abstract. In this note, we study the admissible meromorphic solutions for algebraic differential equation $f^n f' + P_{n-1}(f) = R(z)e^{\alpha(z)}$, where $P_{n-1}(f)$ is a differential polynomial in f of degree $\leq n-1$ with small function coefficients, R is a non-vanishing small function of f, and α is an entire function. We show that this equation does not possess any meromorphic solution $f(z)$ satisfying $N(r,f) = S(r,f)$ unless $P_{n-1}(f) \equiv 0$. Using this result, we generalize a well-known result by Hayman.

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1. Introduction and main results

Let f denote a transcendental meromorphic function. We assume that the reader is familiar with the fundamental results of Nevanlinna theory and its standard notation such as $m(r,f)$, $N(r,f)$, $T(r,f)$, $S(r,f)$, etc. (see [8] and [24]). Recall that a nonconstant meromorphic function α is said to be a small function of f if $T(r,\alpha) = S(r,f)(= o(1)T(r,f))$ as $r \to \infty$, possibly outside a set of r values of finite linear measure. Also, a polynomial in f and its derivatives with small functions of f being the coefficients is called a differential polynomial in f. By $P_n(f)$ we will denote a differential polynomial in f with the total degree in f and its derivatives $\leq n$. By $\rho(f)$ and $\lambda(f)$ we will denote the order and the exponent of convergence of zeros of f, respectively. We will need the following concept of admissibility (see, e.g., [14], [15]).

Definition 1.1. *Let $R(z,\omega)$ be rational in ω with meromorphic coefficients. A meromorphic solution ω of equation $(\omega')^n = R(z,\omega)$ is called admissible if $T(r,a) = S(r,\omega)$ for all coefficients $a(z)$ of $R(z,\omega)$.*

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It is clear that admissibility makes sense relative to any family of meromorphic functions, without any reference to differential equations.

In 1980, Gackstatter and Laine [6] conjectured that the following algebraic differential equation:

$$(f')^n = p_m(f),$$

where $p_m(f)$ is a polynomial in f and n is a positive integer, does not possess any admissible solution when $m \leq n - 1$. In 1990, He and Laine [12] gave a positive answer to this conjecture. Recently, Zhang and Liao [25] proved that if the following algebraic differential equation with polynomial coefficients:

$$P_n(f) = 0 \tag{1.1}$$

has only one dominant term (highest-degree term), then the equation (1.1) has no admissible transcendental meromorphic solutions with a few poles. Liu et al. [18] considered the possible admissible solutions for the following algebraic differential equation:

$$f^n f^{(k)} + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = Re^{\alpha}, \tag{1.2}$$

where $a_j\ (j = 0, 1, \cdots, n-1)$ are small functions of f, R is a nonzero small function and α is an entire function. They have obtained a simple expression for meromorphic solutions of equation (1.2) provided that the solutions satisfy $N(r, f) = S(r, f)$. This also means that the solutions have finitely many zeros determined by the term Re^{α} in the differential equation. Further, this result can be viewed as a generalization of the following well-known result due to Hayman [9] in 1959, which is a prototype of the studies of the zeros of certain special type of differential polynomials.

Theorem A. *Let f be a transcendental meromorphic function, and $n \geq 3$ be an integer. Then $f^n f'$ assumes all finite values, except possibly zero, infinitely many times.*

Later, Hayman [10] conjectured that Theorem A remains valid for $n = 1$ and 2. Then, Hayman's conjecture was confirmed by Mues [20] in the case $n = 2$, and independently by Bergweiler and Eremenko [2] and Chen and Fang [3] in the case $n = 1$. For the related results we refer to [1], [5], [7], [13], [16], [21], [22], and references therein.

It is clear now that distributions of zeros of differential polynomials $P(f)$ of the form $P(f) = f^n f^{(k)} - b$, with $n \geq 1$, $k = 1$ and b a nonzero constant, have been dealt with. In this paper, we study similar problems for such differential polynomials when $n = 1$ and $k \geq 2$, as well as for more general differential polynomials when $n \geq 2$.

Before proceeding further, we recall two known results from [17] and [18].

Theorem B ([17]). *Let $Q_d(z, f)$ be a differential polynomial in f of degree d with rational function coefficients. Suppose that u is a nonzero rational function and v is a nonconstant polynomial. If $n \geq d + 1$ and the differential equation*

$$f^n f' + Q_d(z, f) = u(z)e^{v(z)} \tag{1.3}$$

has a meromorphic solution f with finitely many poles, then f has the following form:

$$f(z) = s(z)e^{v(z)/(n+1)} \text{ and } Q_d(z, f) = 0,$$

where $s(z)$ is a rational function satisfying $s^n((n + 1)s' + v's) = (n + 1)u$.

Theorem C ([18]). *Let f be a transcendental meromorphic function and α be an entire function, and let q and R be small functions of f with $q \not\equiv 0$.. Then the differential equation $ff' - q - Re^{\alpha}$ has no transcendental meromorphic solutions.*

Remark 1.1. *In [19], the authors of the present paper proved the following result. Let α and β be entire functions, and let p, q, R_1 and R_2 be non-vanishing rational functions. Then the system of equations: $pff^{(k)} - q - R_1e^{\alpha}$, $pff^{(l)} - q - R_2e^{\beta}$ has no transcendental solutions for integers l and k with $l > k \geq 2$.*

Now we are in position to state our first main result, which extends Theorem B, proved in [17]. Note that our proof is different and much simple than that of applied [17]. For related recent results we refer the papers [17] – [19]).

Theorem 1.1. *Let $P_{n-1}(f)$ be a differential polynomial in f with coefficients being small functions, and let $\deg P_{n-1}(f) \leq n - 1$. Then for any positive integer n, any entire function α and any small function R, the equation*

$$f^n f' + P_{n-1}(f) = Re^{\alpha} \tag{1.4}$$

does not possess any transcendental meromorphic solution $f(z)$ with $N(r, f) = S(r, f)$ unless $P_{n-1}(f) \equiv 0$. Moreover, if the equation (1.4) possesses a meromorphic solution f with $N(r, f) = S(r, f)$, then (1.4) will become $f^n f' = Re^{\alpha}$ and $f(z)$ has the form $f(z) = u\exp(\alpha/(n + 1))$ as the only possible admissible solution of (1.4), where u is a small function of f.

Corollary 1.1. *Let f be a transcendental meromorphic function with $N(r, f) = S(r, f)$. and let $P_{n-1}(f)$ be a differential polynomial in f with small functions as its* *coefficients, such that $P_{n-1}(0) \not\equiv 0$ and $\deg P_{n-1}(f) \leq n-1$. Then for any positive integer n, the differential form $f^n f' + P_{n-1}(f)$ has infinitely many zeros.*

Based on Corollary 1.1, we pose the following more general conjecture.

Conjecture 1.1. *Let f be a transcendental meromorphic function with $N(r,f) = S(r,f)$, and let $P_{n-1}(f)$ be a differential polynomial in f with small functions as its coefficients, such that $\deg P_{n-1}(f) \leq n-1$ and $P_{n-1}(0) \not\equiv 0$. Then for any positive integers n and k, the differential form $f^n f^{(k)} + P_{n-1}(f)$ has infinitely many zeros.*

Remark 1.2. *The condition $N(r,f) = S(r,f)$ in Corollary 1.1 is necessary. For example, let $f(z) = \frac{e^z}{e^z-1}$. Then $f^2 f' + \frac{3}{2} f'' + \frac{3}{2} f' + f - 1 = -\frac{1}{(e^z-1)^3}$ has no zeros.*

Also, the condition $P_{n-1}(0) \not\equiv 0$ is necessary. For instance, if $f(z) = z^2 e^z$, then $z^2 f^3 f' + z^2 f f' - (2+z) z f^2 = (2+z) z^9 e^{4z}$ has finitely many zeros. The conclusion of Corollary 1.1 becomes invalid, if we replace the condition $\deg P_{n-1}(f) \leq n-1$ by the condition $\deg P_n(f) \leq n$. Indeed, to see this, take $f(z) = e^z - 1$, and observe that $P_2(f) = 2f^2 + 3f + 1$ and $f^2 f' + P_2(f) = e^{3z}$ has no zeros.

Remark 1.3. *(see [18]). Let f be an admissible meromorphic solution of equation (1.2), and let $a_0 = 0$. Then for $n \geq 2$ and $k \geq 1$, the other coefficients $a_1, \cdots, a_{n-1}$ must be identically zero. In this case, (1.2) becomes $f^n f^{(k)} = Re^{\alpha}$ and f has the form $f(z) = u\exp(\alpha/(n+1))$ as the only possible admissible solution of the equation (1.2), where u is a small function of f.*

In view of Theorem 1.1 and Remark 1.3, we obtain the following result, which improves the corresponding result from [17].

Theorem 1.2. *Let f be a transcendental meromorphic function with $N(r,f) = S(r,f)$, and $q_m(f) = b_m f^m + \cdots + b_1 f + b_0$ be a polynomial of degree m with coefficients being small functions of f, and let n be an integer with $n \geq m+1$. Then the differential form $f' f^n + q_m(f)$ assumes every small function γ infinitely many times, except for a possible small function $b_0 = q_m(0)$. On the other hand, if $f' f^n + q_m(f)$ assumes the small function $b_0 = q_m(0)$ finitely many times, then $q_m(z) = b_0$.*

2. Proof of Theorem 1.1

The following lemma is crucial in the proof of our theorem (see [4, 23]).

Lemma 2.1. *(see [4, 23]). Let f be a transcendental meromorphic solution of the equation:*

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda | \lambda \in I\}$ such that $m(r,a_\lambda) = S(r,f)$ for all $r \in I$. If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is at most n, then

$$m(r, P(r,f)) = S(r,f).$$

Proof of Theorem 1.1. We first show that $f^n f' + P_{n-1}(f)$ can not be a small function of f. Indeed, assuming the opposite, from $N(r,f) = S(r,f)$ and Lemma 2.1, we get $m(r,f') = S(r,f)$, and then $T(r,f') = S(r,f)$. A contradiction $T(r,f) = S(r,f)$ now follows by relying to a Theorem from [11] and combining it with the proof of Proposition E from [12]. Thus, for any transcendental meromorphic function f under the condition $N(r,f) = S(r,f)$, we have

$$T(r, f^n f' + P_{n-1}(f)) \neq S(r,f), \tag{2.1}$$

showing that Re^α is not a small function of f.

In view of Theorem C, without loss of generality, we can assume that $n \geq 2$. Let $P_{n-1}(f) \not\equiv 0$. From (1.4) and a result of Milloux (see, e.g., [8]), we obtain

$$T(r, e^\alpha) \leq (n+1)T(r,f) + S(r,f),$$

which and the equality $T(r,\alpha) + T(r,\alpha') = S(r,e^\alpha)$ lead to $T(r,\alpha) + T(r,\alpha') = S(r,f)$.

By taking the logarithmic derivative on both sides of (1.4), we get

$$\frac{nf^{n-1}(f')^2 + f^n f'' + P'_{n-1}(f)}{f^n f' + P_{n-1}(f)} = \frac{R'}{R} + \alpha',$$

implying that

$$-(\frac{R'}{R} + \alpha')f^n f' + nf^{n-1}(f')^2 + f^n f''$$

$$= (\frac{R'}{R} + \alpha')P_{n-1}(f) - P'_{n-1}(f). \tag{2.2}$$

Next, we set

$$\varphi = -(\frac{R'}{R} + \alpha')ff' + n(f')^2 + ff'', \tag{2.3}$$

and use (2.2) to obtain

$$f^{n-1}\varphi = (\frac{R'}{R} + \alpha')P_{n-1}(f) - P'_{n-1}(f) := Q_{n-1}(f). \tag{2.4}$$

Clearly, $Q_{n-1}(f)$ is a differential polynomial in f with $\deg Q_{n-1}(f) \leq n-1$. We claim $\varphi \not\equiv 0$. Indeed, if $\varphi \equiv 0$. then in view of $Q_{n-1}(f) \equiv 0$, and (2.4), with some constant B we have $BP_{n-1}(f) \equiv Re^{\alpha}$. Since f is a transcendental meromorphic function, (1.4) shows that $B \neq 1$. and

$$f^n f' = (B-1)P_{n-1}(f).$$

which together with Lemma 2.1 implies $m(r, f') = S(r, f)$. Thus, by $N(r, f) = S(r, f)$ we have $T(r, f') = S(r, f)$, yielding a contradiction. Hence $\varphi \not\equiv 0$. Moreover, applying Lemma 2.1 to (2.4) again, we can conclude that $m(r, \varphi) = S(r, f)$ and $T(r, \varphi) = S(r, f)$.

From (2.3), we get $m(r, \frac{\varphi}{f^2}) = S(r, f)$, and hence

$$m(r, \frac{1}{f}) = S(r, f). \tag{2.5}$$

It follows from (2.3) that

$$N_{(2}(r, \frac{1}{f}) \leq N(r, \frac{1}{\varphi}) + S(r, f)$$
$$\leq T(r, \varphi) + S(r, f) = S(r, f),$$

implying that the zeros of f are mainly simple zeros. Thus, by (2.5), we obtain

$$T(r, f) = N(r, \frac{1}{f}) + S(r, f) = N_{1)}(r, \frac{1}{f}) + S(r, f), \tag{2.6}$$

where $N_{1)}(r, 1/f)$ involves only the simple zeros of f.

Let z_0 be a simple zero of f such that $R(z_0) \neq 0$. Then in view of (2.3) we have

$$n(f')^2(z_0) = \varphi(z_0). \tag{2.7}$$

Now, we show that $\varphi' \not\equiv 0$. Suppose, contrary to our assertion, that $\varphi' \equiv 0$, that is, φ is a constant. If z_0 is a zero of $f'(z) - \sqrt{\varphi/n}$, then we set

$$h(z) = \frac{f'(z) - \sqrt{\frac{\varphi}{n}}}{f(z)}, \tag{2.8}$$

and observe that $h \not\equiv 0$. It follows by (2.5), (2.7) and (2.8) that

$$m(r, h) = S(r, f). \tag{2.9}$$

From (2.6) and (2.8), we get $N(r, h) = S(r, f)$, which together with (2.9) show that $T(r, h) = S(r, f)$, and

$$f' = hf + \sqrt{\frac{\varphi}{n}}, \quad f'' = (h^2 + h')f + h\sqrt{\frac{\varphi}{n}}. \tag{2.10}$$

By (2.10) and (2.3), we obtain

$$[(n+1)h^2 + h' - h(\frac{R'}{R} + \alpha')]f + [(2n+1)h - (\frac{R'}{R} + \alpha')]\sqrt{\frac{\varphi}{n}} = 0.$$

Therefore, we must have

$$(n+1)h^2 + h' - h(\frac{R'}{R} + \alpha') \equiv 0, \quad (2n+1)h - (\frac{R'}{R} + \alpha') \equiv 0,$$

which implies $(2n+1)\frac{h'}{h} = n(\frac{R'}{R} + \alpha')$, and thus $(Re^{\alpha})^n = Ch^{2n+1}$ with a constant C. This, however, contradicts (2.1) and $T(r,h) = S(r,f)$, and thus $\varphi' \not\equiv 0$.

Using the above arguments, it can be shown that $\varphi' \not\equiv 0$. In this case we set

$$h(z) = \frac{f'(z) + \sqrt{\varphi/n}}{f(z)}$$

and assume that $f'(z_0) + \sqrt{\varphi/n} = 0$.

Again, from (2.3), we get

$$\varphi' = -t'ff' - t(f')^2 - tff'' + (2n+1)f'f'' + ff''', \tag{2.11}$$

where $t = \frac{R'}{R} + \alpha'$. In view of (2.11) and (2.7), we see that a simple zero z_0 of $f(z)$ such that $R(z_0) \neq 0$, is a zero of $(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z)$.

If $(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z) \not\equiv 0$, we set

$$g(z) = \frac{(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z)}{f(z)}.$$

It is clear that g is a small function of f. Therefore, we have

$$\begin{aligned} f'' &= \frac{g}{(2n+1)\varphi}f + \frac{t\varphi + n\varphi'}{(2n+1)\varphi}f' \\ &:= s_1 f + s_2 f', \end{aligned} \tag{2.12}$$

and

$$f''' = (s_1' + s_1 s_2)f + (s_1 + s_2' + s_2^2)f'. \tag{2.13}$$

Next, it follows from (2.13), (2.12), (2.11) and (2.3) that

$$\begin{aligned} &(2n+1 - t' - ts_2 + s_1 + s_2' + s_2^2 + t\frac{\varphi'}{\varphi} - s_2\frac{\varphi'}{\varphi})f' \\ &+(s_1' + s_1 s_2 - ts_1 - s_1\frac{\varphi'}{\varphi})f = 0. \end{aligned} \tag{2.14}$$

In this case, (2.14) and (2.6) imply

$$s_1' + s_1 s_2 - ts_1 - s_1\frac{\varphi'}{\varphi} \equiv 0.$$

Therefore, we have $(2n+1)\log s_1 = 2n(\log R+\alpha)+(3n+1)\log\varphi+B$ with a constant B, which implies that $(Re^{\alpha})^{2n}e^{B}\varphi^{3n+1} = s_1^{2n+1}$. Thus, Re^{α} is a small function of f, which contradicts (2.1). Therefore, $(2n+1)\varphi f''(z)-(t\varphi+n\varphi')f'(z)\equiv 0$, and we have

$$f''=\beta f' \tag{2.15}$$

with $\beta = \frac{n\varphi'}{(2n+1)\varphi}+\frac{t}{2n+1}$. From (2.15) we obtain

$$f'''=(\beta'+\beta^2)f'. \tag{2.16}$$

It follows from (2.16), (2.15) and (2.11) that

$$(\beta'+\beta^2)f'=(t'-t\frac{\varphi'}{\varphi})f'+(t+\frac{\varphi'}{\varphi})\beta f'.$$

Therefore, we have

$$\beta'-t'\equiv-\beta(\beta-t)+(\beta-t)\frac{\varphi'}{\varphi}. \tag{2.17}$$

If $\beta-t\equiv 0$, then by the definitions of t and β, we see that $(Re^{\alpha})^2=C\varphi$, where C is a constant. So, Re^{α} is a small function of f, which contradicts (2.1). Hence, we have $\beta-t\not\equiv 0$. In this case, again, by (2.17), we obtain $(2n+1)\log(\beta-t)=n\log\varphi+\log R+\alpha+D$ with a constant D, showing that Re^{α} is a small function of f, which also contradicts (2.1).

This completes the proof of the theorem, namely the equation $f^nf'+P_{n-1}(f)=Re^{\alpha}$ does not possess any meromorphic solution f with $N(r,f)=S(r,f)$ unless $P_{n-1}(f)\equiv 0$.

3. Conclusions

Using different and much simpler proofs, this paper provides two main results, extending the main results of the paper [17] to more general differential polynomials. Some examples are discussed showing that the imposed conditions are necessary. For further study, a general conjecture is posed.

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EIGENFUNCTIONS OF COMPOSITION OPERATORS ON BLOCH-TYPE SPACES

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Abstract. Suppose φ is a holomorphic self map of the unit disk and C_φ is a composition operator with symbol φ that fixes the origin and $0 < |\varphi'(0)| < 1$. This paper explores sufficient conditions that ensure all the holomorphic solutions of Schröder equation for the composition operator C_φ to belong to a Bloch-type space $\mathcal{B}_\alpha$ for some $\alpha > 0$. In the second part of the paper, the results obtained for composition operators are extended to the case of weighted composition operators.

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1. Introduction

Let $\mathcal{D}$ be the unit disk of the complex plane $\mathbb{C}$, and let $\mathcal{H}(\mathcal{D})$ denote the space of holomorphic functions defined on the unit disk $\mathcal{D}$. Recall that a holomorphic function f defined on $\mathcal{D}$ is said to be in the Bloch-type space $\mathcal{B}_\alpha$ for some $\alpha > 0$ if

$$\sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

Notice that under the Bloch-type norm:

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha|f'(z)|, \tag{1.1}$$

the space $\mathcal{B}_\alpha$ becomes a Banach space. From the definition of Bloch-type spaces, it immediately follows that $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$ for $\alpha \le \beta$ and $\mathcal{B}_\alpha \subset H^\infty$ for $\alpha < 1$.

The Bloch type spaces have been studied extensively by many authors (see [1], [8], and references therein). In [8], it has been shown that the Bloch-type norm for $\alpha > 1$ is equivalent to the $\alpha - 1$ Lipschitz-type norm:

$$\|f\|_{\mathcal{B}_\alpha} \approx \sup_{z\in\mathcal{D}}(1-|z|^2)^{\alpha-1}|f(z)|, \qquad f \in \mathcal{B}_\alpha,\ \alpha > 1. \tag{1.2}$$

Composing functions f in $\mathcal{H}(\mathcal{D})$ with any holomorphic self-map φ of $\mathcal{D}$, induces a linear transformation, denoted by C_φ and called a ***composition operator*** on $\mathcal{H}(\mathcal{D})$:

$$C_\varphi f = f \circ \varphi.$$

For any $u \in \mathcal{H}(\mathcal{D})$ we define the ***weighted composition operator*** uC_φ on $\mathcal{H}(\mathcal{D})$ as follows:

$$uC_\varphi(f) = (u)(f \circ \varphi).$$

In this paper, we study holomorphic solutions f of the following Schröder's equation:

$$(C_\varphi)f(z) = \lambda f(z), \tag{1.3}$$

and of the corresponding weighted Schröder's equation:

$$uC_\varphi f = \lambda f, \tag{1.4}$$

where λ is a complex constant.

Assuming that φ fixes the origin and satisfies $0 < |\varphi'(0)| < 1$, Königs [5] showed that the set of all holomorphic solutions of equation (1.3) (the eigenfunctions of the operator C_φ acting on $\mathcal{H}(\mathcal{D})$) is exactly $\{\sigma^n\}_{n=0}^{\infty}$, where σ, the principal eigenfunction of C_φ, is called ***Königs function of φ***.

Following the Königs work, Hosokawa and Nguyen [4] showed that the set of all eigenfunctions of the weighted operator uC_φ acting on $\mathcal{H}(\mathcal{D})$ is exactly $\{v\sigma^n\}_{n=0}^{\infty}$, where v is the principal eigenfunction of uC_φ and σ is the Konigs function.
According to a general result of Hammond [2], if uC_φ is compact on any Banach space of holomorphic functions on $\mathcal{D}$ containing polynomials, then all the eigenfunctions $v\sigma^n$ belong to a Banach space. Under somewhat strong restrictions on the growths of u and φ near the boundary of the unit disk, Hosokawa and Nguyen [4] showed that all the eigenfunctions $v\sigma^n$ are eigenfunctions of uC_φ acting on the Bloch space $\mathcal{B}$.

Our goal in this paper is to obtain conditions under which all the eigenfunctions $v\sigma^n$ belong to a Bloch-type space $\mathcal{B}_\alpha$.

The rest of the paper is organized as follows. Section 2 contains some preliminary results. In Section 3 we present our main results concerning composition operators. Theorem 3.1 provides sufficient conditions ensuring all the eigenfunctions σ^n to belong to Bloch type spaces $\mathcal{B}_\alpha$ for $\alpha < 1$. Similar results for $\alpha = 1$ and $\alpha > 1$ are presented in Theorems 3.2 and 3.3, respectively. In Section 4 we prove results concerning the weighted composition operators.

2. Preliminaries

We recall the following criterion for boundedness of the operator uC_φ on the Bloch-type spaces $\mathcal{B}_\alpha$ (see [6, Theorem 2.1]).

Theorem 2.1. *Let u be an analytic function on $\mathbb{D}$, φ be an analytic self-map of $\mathbb{D}$, and let α be a positive real number. Then the following assertions hold.*

1. *If $0 < \alpha < 1$, then uC_φ is bounded on $\mathcal{B}_\alpha$ if and only if $u \in \mathcal{B}_\alpha$ and*
$$\sup_{z\in\mathbb{D}} |u(z)| \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty.$$
2. *The operator uC_φ is bounded on $\mathcal{B}$ if and only if the following conditions are satisfied.*
   (a) $\sup_{z\in\mathbb{D}} |u'(z)|(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2} < \infty$,
   (b) $\sup_{z\in\mathbb{D}} |u(z)|\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)}|\varphi'(z)| < \infty$.
3. *If $\alpha > 1$, then uC_φ is bounded on $\mathcal{B}_\alpha$ if and only if the following conditions are satisfied.*
   (a) $\sup_{z\in\mathbb{D}} |u'(z)|\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty$,
   (b) $\sup_{z\in\mathbb{D}} |u(z)|\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)| < \infty$.

The following theorem provides a compactness criterion for the operator uC_φ acting on $\mathcal{B}_\alpha$ (see [6, Theorem 3.1]).

Theorem 2.2. *Let u be a holomorphic function on $\mathbb{D}$ and let φ be a holomorphic self-map of $\mathbb{D}$. Let α be a positive real number, and let uC_φ be bounded on $\mathcal{B}_\alpha$. Then the following assertions hold.*

1. *If $0 < \alpha < 1$, then uC_φ is compact on $\mathcal{B}_\alpha$ if and only if*
$$\lim_{|\varphi(z)|\to 1^-} |u(z)| \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.$$
2. *The operator uC_φ is compact on $\mathcal{B}$ if and only if the following conditions are satisfied.*
   (a) $\lim_{|\varphi(z)|\to 1^-} |u'(z)|(1-|z|^2)\log\frac{1}{(1-|\varphi(z)|^2)} = 0$,
   (b) $\lim_{|\varphi(z)|\to 1^-} |u(z)|\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)}|\varphi'(z)| = 0$.
3. *If $\alpha > 1$, then uC_φ is compact on $\mathcal{B}_\alpha$ if and only if the following conditions are satisfied.*
   (a) $\lim_{|\varphi(z)|\to 1^-} |u'(z)|\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0$,

(b) $\lim_{|\varphi(z)|\to 1^-} |u(z)|\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)| = 0.$

Remark 2.1. *If in Theorems 2.1 and 2.2 we assume $u = 1$, then they provide a criterion for boundedness and compactness of composition operators C_φ acting on the Bloch-type spaces $\mathcal{B}_\alpha$.*

The following two theorems are fundamental for our work. Theorem 2.3 is the famous Konigs theorem about the solutions of Schröder equations (see [5] and [7, Chapter 6]).

Theorem 2.3 (Konigs theorem (1884)). *Assume that φ is a holomorphic self-map of $\mathcal{D}$ such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Then the following assertions hold.*

(i) The sequence of functions

$$\sigma_k(z) = \frac{\varphi_k(z)}{\varphi'(0)^k},$$

where φ_k is the k^{th} iteration of φ, converges uniformly on a compact subset of $\mathcal{D}$ to a non-constant function σ that satisfies (1.3) with $\lambda = \varphi'(0)$.

(ii) f and λ satisfy (1.3) if and only if there is a positive integer n such that $\lambda = \varphi'(0)^n$ and f is a constant multiple of σ^n.

The next theorem characterizes all the eigenfunctions of a weighted composition operator under some restriction on the symbol (see [4]).

Theorem 2.4. *Assume that φ is a holomorphic self-map of $\mathcal{D}$ and u is a holomorphic map of $\mathcal{D}$ such that $u(0) \neq 0$, $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Then the following statements hold.*

(i) The sequence of functions

$$v_k(z) = \frac{u(z)u(\varphi(z))\ldots u(\varphi_{k-1}(z))}{u(0)^k},$$

where φ_k is the k^{th} iteration of φ, converges to a non-constant holomorphic function v of $\mathcal{D}$ that satisfies (1.4) with $\lambda = u(0)$.

(ii) f and λ satisfy (1.4) if and only if $f = v\sigma^n$ and $\lambda = u(0)\varphi'(0)^n$, where n is a nonnegative integer and σ is a solution of the Schroder equation (1.3) $\sigma \circ \varphi = \varphi'(0)\sigma$.

3. Composition operators

In this section, we obtain sufficient conditions that ensure all the eigenfunctions σ^n of a composition operator to belong to $\mathcal{B}_\alpha$ for some positive number α and for all positive integers n.

Definition 3.1. Given a number $\alpha > 0$, the *Hyperbolic α-derivative* of a function φ at $z \in \mathcal{D}$ is defined by

$$\varphi^{(h_\alpha)}(z) = \frac{(1-|z|^2)^\alpha\ \varphi'(z)}{(1-|\varphi(z)|^2)^\alpha}.$$

For $\alpha = 1$, it simply is called the Hyperbolic derivative of φ at z, and is denoted by $\varphi^{(h)}(z)$.

Definition 3.2. Let φ be a holomorphic self-map of $\mathcal{D}$ such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let φ_m be the m^{th} iteration of φ for some fixed nonnegative integer m. Then we say that φ satisfies condition (A) if there exists a nonnegative integer m such that

$$\text{(A)} \qquad |\varphi^{(h_\alpha)}(\varphi_m(z))| = \frac{(1-|\varphi_m(z)|^2)^\alpha\ |\varphi'(\varphi_m(z))|}{(1-|\varphi_{m+1}(z)|^2)^\alpha} \le |\varphi'(0)|,$$

for all $z \in \mathcal{D}$ and for some fixed $\alpha > 0$.

Remark 3.1. *If condition* (A) *is satisfied for some m, then it also is satisfied for all nonnegative integers greater than m.*

The following example provides a family of maps that satisfies condition (A). The example is borrowed from [3].

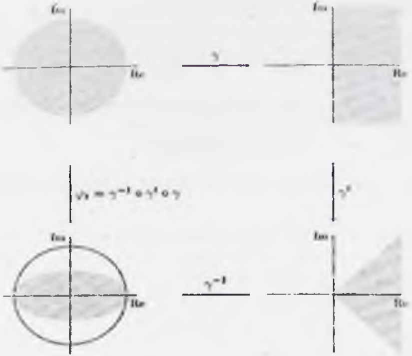
Example 3.1. Consider a map γ that maps the unit disk univalently to the right half plane. This map is given by formula:

$$\gamma(z) = \frac{1+z}{1-z}.$$

For any $t \in (0,1)$, define

$$\varphi_t(z) = \frac{\gamma(z)^t - 1}{\gamma(z)^t + 1}.$$

It is well known that φ_t maps the unit disk into itself for each $t \in (0,1)$ (see [7]). These maps are known as *lens maps*.



Claim 3.1. The map φ_t satisfies the condition (A) for $\alpha = 1$ and $m = 0$. that is, $|\varphi_t^{(h)}(z)| \leq |\varphi_t'(0)|$ for all $t \in (0, 1)$ and for all $z \in \mathcal{D}$.

Proof. Clearly, we have $\varphi_t(0) = 0$ and

$$|\varphi_t'(z)| = \frac{2t\,|\gamma(z)^{t-1}|\,|\gamma'(z)|}{|\gamma(z)^t + 1|^2}.$$

Since $\gamma'(z) = \dfrac{2}{(1-z)^2}$, we see that $|\varphi_t'(0)| = t$. It is known that the image of φ_t touches the boundary of the unit disk non-tangentially at 1 and -1. Now we put $w = \gamma(z) = re^{i\theta}$ to obtain

$$\begin{aligned}|\varphi_t^{(h)}(z)| &= \frac{1 - |z|^2}{1 - \left|\frac{w^t - 1}{w^t + 1}\right|^2}\,\frac{2t\,|w^{t-1}|\,|w'|}{|w^t + 1|^2}\\ &= \frac{1 - |z|^2}{|w^t + 1|^2 - |w^t - 1|^2}\,2t\,|w^{t-1}|\,|w'|.\end{aligned}$$

On the other hand, we have

$$|w^t + 1|^2 - |w^t - 1|^2 = (w^t + 1)\overline{(w^t + 1)} - (w^t - 1)\overline{(w^t - 1)} =$$

$$=(w^t + 1)(w^t + 1) - (w^t - 1)(\overline{w}^t - 1) = 2(w^t + w^t) = 2\, r^t (e^{it\theta} + e^{-it\theta}) = 4\, r^t \cos t\theta.$$

Also, we have $w' = \gamma'(z) = \dfrac{2}{(1-z)^2}$, and

$$|\varphi_t^{(h)}(z)| = \frac{1-|z|^2}{|1-z|^2}\, \frac{t\, r^{t-1} |e^{i(t-1)\theta}|}{r^t \cos t\theta}.$$

Using $z = \dfrac{w-1}{w+1}$, we get

$$|\varphi_t^{(h)}(z)| = \frac{1 - \left|\frac{w-1}{w+1}\right|^2}{\left|1 - \frac{w-1}{w+1}\right|^2}\, \frac{t\, r^{t-1}}{r^t \cos t\theta} =$$

$$= \frac{|w+1|^2 - |w-1|^2}{4}\, \frac{t\, r^{t-1}}{r^t \cos t\theta} = \frac{4\, r \cos\theta}{4}\, \frac{t\, r^{t-1}}{r^t \cos t\theta} = \frac{t \cos\theta}{\cos t\theta}.$$

If $z \in (-1, 1)$, then $\gamma(z) \in \mathbb{R}_+$. Therefore $\theta = 0$ and so $|\varphi_t^{(h)}(z)| = t$. On the other hand, if $z \in \mathcal{D} \setminus (-1,1)$, then $|\theta| \in (0, \pi/2)$. Hence $\cos t\theta > \cos\theta > 0$, and so $|\varphi_t^{(h)}(z)| < t$. This completes the proof. □

Remark 3.2. *From the proof of Claim 3.1, we see that* $|\varphi_t^{(h)}(z)| \nrightarrow 0$ *as* z *approaches the boundary of the unit disk along the real-axis. Hence the composition operator with symbol* φ_t *is a non-compact operator on* $\mathcal{B}$.

The following proposition, which provides a sufficient condition for Konigs function to belong to Bloch-type spaces, plays an important role in the proofs of our main results.

Proposition 3.1. *Assume that the operator* C_φ *is bounded on* $\mathcal{B}_\alpha$, *and* φ *satisfies condition* (A) *for some* $\alpha > 0$ *and for some fixed nonnegative integer* m. *Then* σ *belongs to* $\mathcal{B}_\alpha$.

Proof. Since the operator C_φ is bounded on $\mathcal{B}_\alpha$, there exists a positive number M such that

$$(1-|z|^2)^\alpha |\varphi'(z)| \le M(1-|\varphi(z)|^2)^\alpha \qquad \text{for } z \in \mathcal{D}. \tag{3.1}$$

For m given by the assumption, choose a nonnegative integer k such that $k > m$. For $z \in \mathcal{D}$, we have

$$(1-|z|^2)^\alpha|\varphi_k'(z)| = (1-|z|^2)^\alpha\ |\varphi'(\varphi_{k-1}(z))\varphi'(\varphi_{k-2}(z))...\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))...\varphi'(z)|$$
$$= (1-|z|^2)^\alpha\ |\varphi'(z)\varphi'(\varphi(z))\ ...\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))...\varphi'(\varphi_{k-2}(z))\ \varphi'(\varphi_{k-1}(z)|.$$

By using (3.1), we obtain

$$(1-|z|^2)^\alpha|\varphi_k'(z)| \leq$$
$$\leq M(1-|\varphi(z)|^2)^\alpha\ |\varphi'(\varphi(z))\ ...\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))...\varphi'(\varphi_{k-2}(z))\ \varphi'(\varphi_{k-1}(z)|.$$

Again using (3.1) repeatedly, we get

$$(1-|z|^2)^\alpha|\varphi_k'(z)| \leq M^m\ (1-|\varphi_m(z)|^2)^\alpha|\varphi'(\varphi_m(z))...\varphi'(\varphi_{k-1}(z)|$$

Now using condition (A) repeatedly, we get

$$(1-|z|^2)^\alpha|\varphi_k'(z)| \leq M^m|\varphi'(0)^{k-m}|\ (1-|\varphi_k(z)|^2)^\alpha.$$

Thus, we have

$$\lim_{k\to\infty}(1-|z|^2)^\alpha\ \left|\frac{\varphi_k'(z)}{\varphi'(0)^k}\right| \leq \frac{M^m}{|\varphi'(0)^m|}\overline{\lim}_{k\to\infty}\ (1-|\varphi_k(z)|^2)^\alpha \leq \frac{M^m}{|\varphi'(0)^m|},$$

implying that $(1-|z|^2)^\alpha|\sigma'(z)| \leq \dfrac{M^m}{|\varphi'(0)^m|}$. Hence, $\sigma \in \mathcal{B}_\alpha$. Proposition 3.1 is proved. □

The following corollary provides a sufficient condition that ensures all the integer powers of the Konigs function to belong to Bloch-type spaces $\mathcal{B}_\alpha$ for $\alpha < 1$.

Theorem 3.1. *Suppose $\alpha < 1$. If operator C_φ is bounded on $\mathcal{B}_\alpha$ and φ satisfies the condition* (A), *then $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n.*

Proof. From Proposition 3.1, we see that $\sigma \in \mathcal{B}_\alpha$. Let $\mathbb{H}^\infty$ denote the space of bounded holomorphic functions on the unit disk $\mathcal{D}$. Since $\mathcal{B}_\alpha \subset \mathbb{H}^\infty$ for $\alpha < 1$, there exists a positive constant C such that $\|\sigma\|_{\mathbb{H}^\infty} \leq C$, and

$$\begin{aligned}(1-|z|^2)^\alpha|(\sigma^n(z))'| &= (1-|z|^2)^\alpha\ |n\ \sigma^{n-1}(z)\ \sigma'(z)| \\ &\leq \|\sigma\|_{\mathcal{B}_\alpha}\ n\ |\sigma^{n-1}(z)| \\ &\leq n\ \|\sigma\|_{\mathcal{B}_\alpha}\ C^{n-1}.\end{aligned}$$

Hence, $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n. □

The following theorem gives a sufficient condition that ensures all the integer powers of Konigs function to belong to the Bloch space.

Theorem 3.2. *Let φ be a holomorphic self-map of $\mathcal{D}$ such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Also, assume that*

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}\frac{\log\frac{2}{1-|z|}}{\log\frac{2}{1-|\varphi(z)|}}|\varphi'(z)| \le |\varphi'(0)| \quad \text{for all } z \in \mathcal{D}. \tag{3.2}$$

Then operator C_φ is bounded on $\mathcal{B}$ and $\sigma^n \in \mathcal{B}$ for all positive integers n.

Proof. The boundedness of C_φ on the Bloch space follows from Schwarz-Pick theorem. From the hypothesis of the theorem, we have

$$(1-|z|^2)\log\frac{2}{1-|z|}|\varphi'(z)| \le |\varphi'(0)|(1-|\varphi(z)|^2)\log\frac{2}{1-|\varphi(z)|} \quad \text{for all } z \in \mathcal{D}. \tag{3.3}$$

Let k be a positive integer, then we have

$$\begin{aligned}(1-|z|^2)|\varphi_k'(z)|\log\frac{2}{1-|z|} &= (1-|z|^2)|\varphi'(z)\varphi'(\varphi(z))......\varphi'(\varphi_{k-1}(z))|\log\frac{2}{1-|z|}\\ &= (1-|z|^2)\log\frac{2}{1-|z|}|\varphi'(z)\varphi'(\varphi(z))......\varphi'(\varphi_{k-1}(z))|.\end{aligned}$$

By using (3.3), we see that

$$(1-|z|^2)|\varphi_k'(z)|\log\frac{2}{1-|z|} \le |\varphi'(0)|(1-|\varphi(z)|^2)\log\frac{2}{1-|\varphi(z)|}|\varphi'(\varphi(z))......\varphi'(\varphi_{k-1}(z))|.$$

And using (3.3) repeatedly, we get

$$\begin{aligned}(1-|z|^2)|\varphi_k'(z)|\log\frac{2}{1-|z|} &= |\varphi'(0)|^k(1-|\varphi_k(z)|^2)\log\frac{2}{1-|\varphi_k(z)|}\\ &\le 2|\varphi'(0)|^k(1-|\varphi_k(z)|)\log\frac{2}{1-|\varphi_k(z)|}.\end{aligned}$$

Since $\log x \le x$ for $x > 1$, we have

$$(1-|z|^2)|\varphi_k'(z)|\log\frac{2}{1-|z|} \le 4|\varphi'(0)|^k.$$

Hence,

$$\lim_{k\to\infty}(1-|z|^2)\left|\frac{\varphi_k'(z)}{\varphi'(0)^k}\right|\log\frac{2}{1-|z|} = (1-|z|^2)|\sigma'(z)|\log\frac{2}{1-|z|} \le 4, \quad z \in \mathcal{D},$$

showing that

$$|\sigma'(z)| \le \frac{4}{(1-|z|^2)\log\frac{2}{1-|z|}}. \tag{3.4}$$

Recall that $\sigma(0) = 0$. Now we obtain an estimate for σ. We have

$$|\sigma(z)| = \left|\int_o^1 \sigma'(tz)d(tz)\right| \le \int_o^1 |\sigma'(tz)d(t|z|) \le \int_0^1 \frac{4}{\log\frac{2}{1-|tz|}}\frac{1}{1-|tz|^2}d(t|z|) \le$$

$$\leq 4\left[\log\left(\log\frac{2}{1-t|z|}\right)\right]_0^1 = 4\left[\log\left(\log\frac{2}{1-|z|}\right) - \log(\log 2)\right]. \tag{3.5}$$

Next, by using (3.4) and the above obtained estimate for σ, we get

$$\begin{aligned}(1-|z|^2)(\sigma^n(z))' &= (1-|z|^2)\, n\, |\sigma^{n-1}(z)\, \sigma'(z)| \\ &\leq 4^n n\left(\log\log\frac{2}{1-|z|} - \log\log 2\right)^{n-1} \frac{1}{\log\frac{2}{1-|z|}}.\end{aligned}$$

Finally, it is easy to see that the right-hand side of the last expression teds to zero as $|z| \to 1$. Hence $\sigma^n \in \mathcal{B}$ for all positive integers n. □

Let us recall the Lipschitz-type norm, which is equivalent to the usual norm, defined for function $f \in \mathcal{B}_\alpha$, $\alpha > 1$ by

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathcal{D}}(1-|z|^2)^{\alpha-1}|f(z)|.$$

Next, we present results for the Bloch-type spaces $\mathcal{B}_\alpha$ for $\alpha > 1$. We start with the following definition.

Definition 3.3. Suppose $f \in \mathcal{B}_\alpha$ for some $\alpha > 0$, then we define the *Bloch number* of f by $b_f = \inf_\alpha \{\alpha : f \in \mathcal{B}_\alpha\}$.

Proposition 3.2. *Suppose $\beta > 0$. Then $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n if and only if b_f is at most 1.*

Proof. Suppose $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n. We have to show that $b_f \leq 1$. On the contrary, assume $b_f > 1$. Then there exists a positive integer n_0 such that $1 < 1 + \frac{\beta}{n_0} < b_f$. Now, in view of definition of Lipschitz-type norm, we see that for any fixed positive integer M there exists $z \in \mathcal{D}$ such that

$$M \leq (1-|z|^2)^{\beta/n_0}|f(z)| \leq \{(1-|z|^2)^{\beta/n_0}|f(z)|\}^{n_0} = (1-|z|^2)^{\beta}|f(z)|^{n_0},$$

showing that

$$M \leq \sup_{z\in\mathcal{D}}(1-|z|^2)^{\beta}|f(z)|^{n_0} = \|f^{n_0}\|_{\mathcal{B}_{\beta+1}}.$$

Since M is an arbitrary positive integer, we have $f^{n_0} \notin \mathcal{B}_{\beta+1}$. Which is a contradiction.

Conversely, suppose that $b_f < 1$. Since $\mathcal{B}_\alpha \subset \mathcal{B}$ for all $\alpha < 1$, then clearly $f \in \mathcal{B}$. For any fixed $\beta > 0$ and for any fixed positive integer n, we have

$$\begin{aligned}(1-|z|^2)^{\beta+1}|(f^n)'(z)| &= (1-|z|^2)^{\beta+1}|nf^{n-1}(z)f'(z)| \\ &= n(1-|z|^2)|f'(z)|(1-|z|^2)^{\beta}|f^{n-1}(z)| \\ &\le n\|f\|_{\mathcal{B}}(1-|z|^2)^{\beta}\left(\|f\|_{\mathcal{B}}\log\frac{1}{1-|z|}\right)^{n-1} \\ &= n(\|f\|_{\mathcal{B}})^n(1-|z|^2)^{\beta}\left(\log\frac{1}{1-|z|}\right)^{n-1}.\end{aligned}$$

The last expression goes to zero as $|z| \to 1$, showing that $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n. □

Theorem 3.3. *Let φ be a holomorphic self-map of $\mathcal{D}$ such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let $\alpha > 1$. If $|\varphi^{(h)}(z)| \le |\varphi'(0)|$ for all $z \in \mathcal{D}$, then operator C_φ is bounded on $\mathcal{B}_\alpha$ and $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n.*

Proof. Since $|\varphi^{(h)}(z)| \le |\varphi'(0)|$ for all $z \in \mathcal{D}$, by Proposition 3.1 we have $\sigma \in \mathcal{B}$. So $b_f \le 1$. Therefore the result follows from Proposition 3.2. □

4. Weighted Composition operators

Recall that if u is a holomorphic function of the unit disk, and φ is a holomorphic self-map of the unit disk, then the Schroder equation for weighted composition operator is given by

$$u(z)f(\varphi(z)) = \lambda f(z). \tag{4.1}$$

where $f \in \mathcal{H}(\mathcal{D})$ and λ is a complex constant.

Also, recall that if $u(0) \ne 0$, $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, then the solutions of equation (4.1) are given by Theorem 2.4. The principal eigenfunction corresponding to the eigenvalue $u(0)$ we denote by v, and observe that all the other eigenfunctions are of the form $v\sigma^n$, where σ is the Konigs function of φ and n is a positive integer. Hosokawa and Nguyen [4] studied the equation (4.1) in the Bloch space and obtained the following result.

Theorem 4.1. *Let φ be a holomorphic self-map of $\mathcal{D}$ with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let u be a holomorphic map of $\mathcal{D}$ such that $u(0) \ne 0$. Assume that operator*

uC_φ is bounded on $\mathcal{B}$. Further, for $0 < r < 1$, we set

$$M_r(\varphi) = \sup_{|z|=r} |\varphi(z)|, \qquad a_r = \sup_{|z|=r} \left(|u'(z)\varphi(z)| + |u(z)\varphi'(z)|\right).$$

and assume that the following conditions are satisfied:

(i) $\lim_{r\to 1} \log(1-r)\log M_r(\varphi) = \infty$.

(ii) $\log|a_r| < \epsilon\log(1-r)\log M_r(\varphi)$,

where $\epsilon > 0$ is a constant satisfying $\epsilon \log\|\varphi\|_\infty > -1$.

Then $u\sigma^n \in \mathcal{B}$ for all nonnegative integers n.

Now we proceed to obtain conditions on the weight u and on the symbol φ of the weighted composition operators uC_φ that ensure $u\sigma^n$ to belong to Bloch-type spaces $\mathcal{B}_\alpha$ for some $\alpha > 0$ and for all nonnegative integers n. We begin with the following remark.

Remark 4.1. *Let f be a holomorphic function defined on $\mathbb{D}$. If $\|f'\|_\infty < M$ for some $M > 0$, then we have*

$$|f(z) - f(0)| = \left|\int_0^1 zf'(tz)dt\right| \leq \int_0^1 |z\, f'(tz)|dt \leq M\int_0^1 |z|dt.$$

If, in addition, f also satisfies $f(0) = 0$, then $\|f\|_\infty \leq M$.

Proposition 4.1. *Let φ be a univalent holomorphic self-map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let σ be the Konigs function of φ. Then σ is bounded if and only if there is a positive integer k such that $\|\varphi_k\|_\infty < 1$.*

Proof. Suppose that σ is bounded. Since φ is univalent, σ is also univalent (see [7], p. 91). Since σ is bounded univalent map, there is a positive integer k such that $\|\varphi_k\|_\infty < 1$ (see [7]).

Conversely, suppose there is a positive integer k such that $\|\varphi_k\|_\infty < 1$. Since $\sigma(\varphi(z)) = \varphi'(0)\sigma(z)$, we have

$$\sigma(\varphi_k(z)) = \sigma(\varphi(\varphi_{k-1}(z)) = \varphi'(0)\sigma(\varphi_{k-1}(z)) = \varphi'(0)^k\sigma(z).$$

Clearly the left-hand side of the last relation is bounded, and therefore σ is also bounded, which completes the proof. □

Theorem 4.2. *Let φ be a univalent holomorphic self-map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$ satisfying $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathbb{D}$ and for some fixed*

$\alpha < 1$. *If u is a holomorphic map of $\mathcal{D}$ such that $u(0) \neq 0$ and $\|u'\|_\infty < \infty$, then operator uC_φ is bounded on $\mathcal{B}_\alpha$ and $v\sigma^n \in \mathcal{B}_\alpha$ for all nonnegative integers n.*

Proof. Since $\|u\|_\infty < \|u'\|_\infty + |u(0)| < \infty$ and $|\varphi^{(h_\alpha)}(z)| \le |\varphi'(0)|$, the operator uC_φ is bounded on $\mathcal{B}_\alpha$ for some $\alpha < 1$.

Since $|\varphi^{(h_\alpha)}(z)| \le |\varphi'(0)|$ for some $\alpha < 1$, in view of Proposition 3.1, we see that $\sigma \in \mathcal{B}_\alpha$ for $\alpha < 1$, and hence is bounded. Next, since φ is univalent, σ is also univalent. Consequently, there exists a nonnegative integer k such that $\|\varphi_k\|_\infty < 1$. Composing φ_{k-1} on both sides of the Schroder equation (4.1) from right, we get

$$u(\varphi_{k-1}(z))f(\varphi_k(z)) = \lambda f(\varphi_{k-1}(z)). \tag{4.2}$$

The left-hand side of the above equation is bounded, and so is $f \circ \phi_{k-1}$. Hence, differentiating both side of (4.2), we get

$$u'(\varphi_{k-1}(z))\,\varphi'_{k-1}(z)\,f(\varphi_k(z)) + u(\varphi_{k-1}(z))\,f'(\varphi_k(z))\,\varphi'_k(z) = \lambda f'(\varphi_{k-1}(z))\,\varphi'_{k-1}(z).$$

Next, multiplying both sides of the last equation by $(1-|z|^2)^\alpha$, and using boundedness of $\|u'\|_\infty$, $\|u\|_\infty$, $f \circ \varphi_k$ and $f' \circ \varphi_k$, we see that there exists a constant M such that

$$(1-|z|^2)^\alpha|\lambda f'(\varphi_{k-1}(z))\varphi'_{k-1}(z)| \le M(1-|z|^2)^\alpha\left(|\varphi'_{k-1}(z)| + |\varphi'_k(z)|\right). \tag{4.3}$$

The right-hand side of the above inequality is uniformly bounded, and therefore the left-hand side is bounded. Again, we compose φ_{k-2} on (4.1), to get

$$u(\varphi_{k-2}(z))f(\varphi_{k-1}(z)) = \lambda f(\varphi_{k-2}(z)).$$

Now we differentiate the above equation, then multiply by both sides by $(1-|z|^2)^\alpha$, and use (4.2) and (4.3) to show that $(1-|z|^2)^\alpha|f'(\varphi_{k-2}(z))\varphi'_{k-2}(z)|$ is bounded.

Continuing this process, we see that that $\sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha|f'(z)|$ is bounded, and hence $f \in \mathcal{B}_\alpha$. By Theorem 2.4, any holomorphic f satisfying (4.1) is of the form $v\sigma^n$ for some positive integer n, implying that $v\sigma^n \in \mathcal{B}_\alpha$ for all nonnegative integers n. This completes the proof. Theorem 4.2 is proved. □

The following two theorems give sufficient conditions that ensure $v\sigma^n$ to belong to Bloch-type spaces $\mathcal{B}_\alpha$ for some $\alpha > 1$ and for all nonnegative integers n.

Theorem 4.3. *Let φ be a holomorphic self-map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let u be a holomorphic map of $\mathcal{D}$ such that $u(0) \neq 0$. Assume* *that for a fixed positive number β*

$$|u(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta} \le |u(0)| \quad \textit{for all } z \in \mathcal{D}.$$

Then the following statements hold.

(i) *If $|\varphi^{(h_n)}(z)| \le |\varphi'(0)|$ for all $z \in \mathcal{D}$ and for some $\alpha < 1$, then $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all nonnegative integers n.*

(ii) *If $|\varphi^{(h)}(z)| \le |\varphi'(0)|$ for all $z \in \mathcal{D}$, then $v\sigma^n \in \mathcal{B}_{p+1}$ for some $p > \beta$ and for all nonnegative integers n.*

Proof. We first prove the assertion (i). From the definition of v_k (see Theorem 2.4), we have

$$\begin{aligned}(1-|z|^2)^\beta|v_k(z)| =&(1-|z|^2)^\beta\frac{|u(z)u(\varphi(z))......u(\varphi_{k-1}(z))|}{|u(0)|^k}\\ \le&(1-|\varphi(z)|^2)^\beta\frac{|u(\varphi(z))......u(\varphi_{k-1}(z))|}{|v(0)|^{k-1}}... \le 1.\end{aligned}$$

Hence $(1-|z|^2)^\beta|v(z)| = \lim_{k\to\infty}(1-|z|^2)^\beta|v_k(z)| \le 1$. Since z is arbitrary, we have

$$\sup_{z\in\mathcal{D}}(1-|z|^2)^\beta|v(z)|.<\infty.$$

On the other hand, the assumption $|\varphi^{(h_n)}(z)| \le |\varphi'(0)|$ and Proposition 3.1 imply that $\sigma^n \in \mathcal{B}_\alpha \subset \mathbb{H}^\infty$ for all nonnegative integer n. Therefore,

$$\sup_{z\in\mathcal{D}}(1-|z|^2)^\beta|v(z)\sigma^n(z)| < \infty$$

for all nonnegative integers n. Considering the equivalent norm (see (1.2)), we conclude that $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all nonnegative integers n. This completes the proof of assertion (i).

To prove the assertion (ii), observe first that from the proof of part (*i*). we have

$$\sup_{z\in\mathcal{D}}(1-|z|^2)^\beta|v(z)| < \infty. \tag{4.4}$$

On the other hand, since $|\varphi^{(h)}(z)| \le |\varphi'(0)|$, Proposition 3.1 implies that $\sigma \in \mathcal{B}$, and hence there exists a number $M > 0$ such that

$$|\sigma(z)| \le M\log\frac{2}{1-|z|^2}. \tag{4.5}$$

Next, using equations (4.4) and (4.5), with some constant $C > 0$ we have

$$\begin{aligned}(1-|z|^2)^p|v(z)\sigma^n(z)| =&\{(1-|z|^2)^\beta|v(z)|\}\{(1-|z|^2)^{p-\beta}|\sigma^n(z)|\} \\ \le& CM(1-|z|^2)^{p-\beta}\left(\log\frac{2}{1-|z|^2}\right)^n.\end{aligned}$$

Finally, it is easy to see that the last expression goes to zero as $|z| \to 1$. Hence, $v\sigma^n \in \mathcal{B}_{p+1}$ for all nonnegative integers n. Theorem 4.3 is proved. □

Theorem 4.4. *Let φ be a holomorphic self-map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let u be a holomorphic map of $\mathcal{D}$ such that $u(0) \neq 0$. Suppose that β is a positive integer and the following conditions are satisfied:*

(i) $|u(z)|\dfrac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}\dfrac{\log\frac{2}{(1-|z|)^\beta}}{\log\frac{2}{(1-|\varphi(z)|)^\beta}} \le |u(0)|$ *for all* $z \in \mathcal{D}$

(ii) $|\varphi^{(k)}(z)|\dfrac{\log\frac{2}{1-|z|}}{\log\frac{2}{1-|\varphi(z)|}} \le |\varphi'(0)|$ *for all* $z \in \mathcal{D}$.

Then $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all nonnegative integers n.

Proof. In view of the definition of v_k (see Theorem 2.4) and the condition (*i*), we can write

$$\begin{aligned}&(1-|z|^2)^\beta\log\frac{2}{(1-|z|)^\beta}|v_k(z)| = (1-|z|^2)^\beta\log\frac{2}{(1-|z|)^\beta}\frac{|u(z)u(\varphi(z))......u(\varphi_{k-1}(z))|}{|u(0)|^k} \\ &\le(1-|\varphi(z)|^2)^\beta\log\frac{2}{(1-|\varphi(z)|)^\beta}\frac{|u(\varphi(z))......u(\varphi_{k-1}(z))|}{|u(0)|^{k-1}} \\ &\le(1-|\varphi_k(z)|^2)^\beta\log\frac{2}{(1-|\varphi_k(z)|)^\beta} \le 2^\beta(1-|\varphi_k(z)|)^\beta\log\frac{2}{(1-|\varphi_k(z)|)^\beta}.\end{aligned}$$

Since $\log x \le x$ for $x > 1$, we have

$$(1-|z|^2)^\beta\log\frac{2}{(1-|z|)^\beta}|v_k(z)| \le 2^{\beta+1}.$$

So taking limit as k approaches to ∞, we see that

$$(1-|z|^2)^\beta|v(z)| \le \frac{2^{\beta+1}}{\log\frac{2}{1-|z|}}. \tag{4.6}$$

On the other hand, since φ satisfies condition (*ii*), in view of equation (3.5), there exists $K > 0$ such that

$$|\sigma(z)| \le K\log\log\frac{2}{1-|z|}. \tag{4.7}$$

Now using (4.6) and (4.7), we get

$$(1-|z|^2)^\beta|v(z)\sigma^n(z)| \le \frac{2^{\beta+1}K^n}{\log\frac{2}{1-|z|}}\left(\log\log\frac{2}{1-|z|}\right)^n.$$

Clearly the right-hand side of the above equation goes to 0 as $|z| \to 1$. Using the norm defined in (1.2), we conclude that $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all nonnegative integers n. Theorem 4.4 is proved. □

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ON THE CONVERGENCE OF PARTIAL SUMS WITH RESPECT TO VILENKIN SYSTEM ON THE MARTINGALE HARDY SPACES

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Abstract. In this paper, we derive characterizations of boundedness of subsequences of partial sums with respect to Vilenkin system on the martingale Hardy spaces H_p when $0 < p < 1$. Moreover, we find necessary and sufficient conditions for the modulus of continuity of martingales $f \in H_p$, which provide convergence of subsequences of partial sums on the martingale Hardy spaces H_p. It is also proved that these results are the best possible in a special sense. As applications, some known and new results are pointed out.

MSC2010 numbers: 42C10.

Keywords: Vilenkin system; partial sums; martingale Hardy space; modulus of continuity.

1. Introduction

The notation and definitions, used in this section, will be given in the next section of the paper. It is well-known that (for details see [14]):

$$\|S_n f\|_p \le c_p \|f\|_p , \quad \text{when} \quad p > 1,$$

where $S_n f$ is the n-th partial sum with respect to bounded Vilenkin system.

Moreover, the following more stronger result is also known (see [11]):

$$\|S^* f\|_p \le c_p \|f\|_p , \quad \text{when} \quad f \in L_p, \quad p > 1,$$

where $S^* f = \sup_{n \in \mathbb{N}} |S_n f|$.

Lukomskii [13] obtained a two-sided estimate for Lebesgue constants L_n with respect to Vilenkin system. By using this result, we easily can show that for every

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integrable function f, the sequence $S_{n_k}f$ converges to f in L_1-norm if and only if

$$\sup_{k\in\mathbb{N}} L_{n_k} \leq c < \infty.$$

Pointwise and uniform convergence and some approximation properties of partial sums in L_1-norm were studied by a number of authors (see, e.g, the papers by Goginava [9], Goginava and Sahakian [10], Avdispahić and Memić [2], and references therein). Fine [4] obtained sufficient conditions for the uniform convergence which are in complete analogy with the Dini-Lipschitz conditions. Gulićev [12] has estimated the rate of uniform convergence of a Walsh-Fourier series by using Lebesgue constants and modulus of continuity. Uniform convergence of a subsequence of partial sums with respect to Walsh system was investigated also in [8]. This problem for a Vilenkin group G_m was considered by Blahota [3], Fridli [5] and Gát [7].

It is known (for details see, e.g., [18]) that the Vilenkin system does not form a basis in the space $L_1(G_m)$. Moreover, there is a function f in the martingale Hardy space $H_1(G_m)$ such that the sequence of partial sums of f is not bounded in $L_1(G_m)$-norm, but a subsequence S_{M_n} of partial sums is bounded from the martingale Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$, for all $p > 0$.

In [21] it was proved that if $0 < p \leq 1$ and $\{\alpha_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers such that

$$\sup_{k\in\mathbb{N}} \rho(\alpha_k) < \infty, \tag{1.1}$$

where $\rho(n) = |n| - \langle n\rangle$ and

$$\langle n\rangle = \min\{j \in \mathbb{N} : n_j \neq 0\}, \qquad |n| = \max\{j \in \mathbb{N} : n_j \neq 0\},$$

for $n = \sum_{k=0}^{\infty} n_j M_j$, $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$, then the restricted maximal operator

$$\widetilde{S}^{*,\Delta} f := \sup_{k\in\mathbb{N}} |S_{\alpha_k} f|$$

is bounded from the Hardy space H_p to the Lebesgue space L_p.

Moreover, if $0 < p < 1$ and $\{\alpha_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers satisfying the condition

$$\sup_{k\in\mathbb{N}} \rho(\alpha_k) = \infty, \tag{1.2}$$

then there exists a martingale $f \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \|S_{\alpha_k} f\|_{L_{p,\infty}} = \infty.$$

It immediately follows that for any $p > 0$ and $f \in H_p$, the following restricted maximal operator

$$\widetilde{S}_{\#}^{*} f := \sup_{n \in N} |S_{M_n} f| ,$$

where $M_0 := 1$, $M_{k+1} := \prod_{i=0}^{k} m_i$ and $m := (m_0, m_1, ...)$ is a sequence of positive integers not less than 2, which generates the Vilenkin system, is bounded from the Hardy space H_p to the space L_p :

$$\left\| \widetilde{S}_{\#}^{*} f \right\|_p \leq \|f\|_{H_p}, \quad f \in H_p. \tag{1.3}$$

For the Vilenkin system, Simon [15] proved that there is an absolute constant c_p, depending only on p, such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p , \tag{1.4}$$

for all $f \in H_p(G_m)$, where $0 < p < 1$. In [17] we proved that the sequence $\left\{1/k^{2-p} : k \in \mathbb{N}\right\}$ can not be improved.

A similar theorem for $p = 1$ with respect to the unbounded Vilenkin systems was proved in Gát [6].

In [18] we proved that if $0 < p < 1$, $f \in H_p(G_m)$ and

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = o\left(\frac{1}{M_n^{1/p-1}}\right) \quad \text{as } n \to \infty,$$

then

$$\|S_{n_k} f - f\|_{H_p} \to 0 \quad \text{as } k \to \infty. \tag{1.5}$$

Moreover, for every $p \in (0, 1)$ there exists a martingale $f \in H_p(G_m)$, for which

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = O\left(\frac{1}{M_n^{1/p-1}}\right) \quad \text{as } n \to \infty$$

and

$$\|S_k f - f\|_{L_{p,\infty}(G_m)} \nrightarrow 0 \quad \text{as } k \to \infty.$$

In [20] we investigated some (H_p, H_p), (H_p, L_p) and $(H_p, L_{p,\infty})$ type inequalities for subsequences of partial sums of Walsh-Fourier series for $0 < p \leq 1$.

In this paper, we derive characterizations of boundedness of subsequences of partial sums with respect to the Vilenkin system on the martingale Hardy spaces H_p when $0 < p < 1$. Moreover, we find necessary and sufficient conditions for the modulus of continuity of $f \in H_p$, which provide convergence of subsequences of partial sums on the martingale Hardy spaces H_p. It is also proved that these results are the best possible in a special sense. As applications, we point out some known and new results.

The paper is organized as follows: In Section 2 we present necessary notation and definitions, and state a number of auxiliary lemmas, needed in the proofs of the main results. Some of these lemmas are new and represent independent interest. The formulations and detailed proofs of the main results and some of their consequences are given in Sections 3 and 4.

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, and $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$. Let $m = (m_0, m_1, ...)$ denote a sequence of positive integers not less than 2. By $Z_{m_k} = \{0, 1, ..., m_k - 1\}$ we denote the additive group of integers modulo m_k, and define the group G_m to be the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures $\mu_k(\{j\}) := 1/m_k$ $(j \in Z_{m_k})$ is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence $m := (m_0, m_1, ...)$ is bounded, then the group G_m is called a bounded Vilenkin group, else it is called an unbounded Vilenkin group. The elements of the group G_m are represented by sequences $x := (x_0, x_1, ..., x_j, ...)$ $(x_k \in Z_{m_k})$.

It is easy to give a base for the neighborhoods of G_m:

$$I_0(x) = G_m, \quad I_n(x) = \{y \in G_m \mid y_0 = x_0, ..., y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$. It is clear that

$$\overline{I_N} = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1}. \tag{2.1}$$

If we define the so-called generalized number system based on m in the following way $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can uniquely be expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of n_j's differ from zero. For all $n \in \mathbb{N}$ we define

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\}, \ |n| := \max\{j \in \mathbb{N} : n_j \neq 0\}, \ \rho(n) = |n| - \langle n \rangle .$$

For a natural number $n = \sum_{j=1}^{\infty} n_j M_j$, we define the functions v and v^* as follows:

$$v(n) = \sum_{j=1}^{\infty} |\delta_{j+1} - \delta_j| + \delta_0, \quad v^*(n) = \sum_{j=1}^{\infty} \delta_j^*,$$

where $\delta_j = \operatorname{sign} n_j = \operatorname{sign}(\ominus n_j)$, $\delta_j^* = |\ominus n_j - 1| \delta_j$ and $\ominus$ is the inverse operation for $a_k \oplus b_k := (a_k + b_k) \operatorname{mod} m_k$. The norms (or quasi-norms) of the spaces $L_p(G_m)$ and $L_{p,\infty}(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p d\mu, \qquad \|f\|_{L_{p,\infty}} := \sup_{\lambda>0} \lambda \mu (f > \lambda)^{1/p}.$$

Next, on the group G_m we introduce an orthonormal system, which is called the Vilenkin system. To this end, we first define the complex-valued functions $r_k(x) : G_m \to \mathbb{C}$, called the generalized Rademacher functions, as follows:

$$r_k(x) := \exp(2\pi i x_k / m_k) \qquad (\iota^2 = -1, \ x \in G_m, \ k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in \mathbb{N}).$$

Notice that in the special case where $m = 2$, that is, $m_k = 2$ for all $k \in \mathbb{N}$, the above defined system is called the Walsh-Paley system. Observe that the Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see, e.g. [1, 22]). If $f \in L_1(G_m)$, then we can define the Fourier coefficients, the partial sums of the Fourier series, and the Dirichlet kernel for the Vilenkin system ψ in the usual manner as follows:

$$\widehat{f}(k) \ : \ = \int_{G_m} f \bar{\psi}_k d\mu. \qquad (k \in \mathbb{N})$$

$$S_n f \ : \ = \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \qquad D_n = \sum_{k=0}^{n-1} \psi_k, \ (n \in \mathbb{N}).$$

Recall that (see [1])

(2.2)
$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases}$$

and

(2.3)
$$D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{u=m_j-n_j}^{m_j-1} r_j^u \right).$$

Moreover, if $n \in \mathbb{N}$ and $x \in I_s \setminus I_{s+1}$, $0 \leq s \leq N-1$, then the following estimates hold (see Tephnadze [16, 19]):

(2.4)
$$|D_n(x)| = |D_{n-M_{|n|}}(x)| \geq M_{(n)}, \qquad |n| \neq (n)$$

and

(2.5)
$$\int_{I_N} |D_n(x-t)| \, d\mu(t) \leq \frac{cM_s}{M_N}.$$

The n-th Lebesgue constant L_n for the Vilenkin system ψ is defined by

$$L_n := \|D_n\|_1 .$$

It is known that for every $n = \sum_{i=1}^{\infty} n_i M_i$, the following two-sided estimate is true (see Lukomskii [13]):

$$\frac{1}{4\lambda}v(n) + \frac{1}{\lambda}v^*(n) + \frac{1}{2\lambda} \leq L_n \leq \frac{3}{2}v(n) + 4v^*(n) - 1, \tag{2.6}$$

where $\lambda := \sup_{n\in\mathbb{N}} m_n$.

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ we denote by F_n $(n \in \mathbb{N})$, and by $f = (f_n, n \in \mathbb{N})$ we denote a martingale with respect to F_n $(n \in \mathbb{N})$ (for details see Weisz [23]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbb{N}} \left|f^{(n)}\right|.$$

In the case where $f \in L_1(G_m)$, the maximal function can also be given by the following formula:

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{|I_n(x)|} \left|\int_{I_n(x)} f(u)\, d\mu(u)\right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales, for which $\|f\|_{H_p} := \|f^*\|_p < \infty$.

Let $X = X(G_m)$ denote either the space $L_1(G_m)$ or the space of continuous functions $C(G_m)$. The corresponding norm is denoted by $\|\cdot\|_X$. The modulus of continuity, when $X = C(G_m)$ and the integral modulus of continuity, when $X = L_1(G_m)$ are defined by

$$\omega\left(\frac{1}{M_n}, f\right)_X = \sup_{h\in I_n} \|f(\cdot + h) - f(\cdot)\|_X .$$

The modulus of continuity in the Hardy martingale spaces $H_p(G_m)$ $(0 < p \leq 1)$ can be defined as follows:

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} := \|f - S_{M_n} f\|_{H_p(G_m)} .$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n} f : n \in \mathbb{N})$ is a martingale. If $f = (f_n, n \in \mathbb{N})$ is a martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f_k(x)\, \overline{\psi}_i(x)\, d\mu(x).$$

Notice that the Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as the martingale $(S_{M_n} f : n \in \mathbf{N})$ obtained from f . A bounded measurable function a is called a p-atom if there exists an interval I such that

$$\int_I a\,d\mu = 0, \qquad \|a\|_\infty \leq \mu(I)^{-1/p}, \qquad \operatorname{supp}(a) \subset I.$$

Observe that for $0 < p \leq 1$, the martingale Hardy spaces $H_p\ (G_m)$ have atomic characterizations (for details see, e.g., Weisz [23, 24]):

Lemma 2.1. *A martingale $f = (f_n, n \in \mathbf{N})$ is in $H_p\,(0 < p \leq 1)$ if and only if there exist a sequence $(a_k, k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that, for every $n \in \mathbf{N}$*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f_n \quad \text{a.e., where} \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{2.7}$$

Moreover,

$$\|f\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of form (2.7).

By using the atomic decomposition of martingales $f \in H_p$, we can construct a counterexample, which plays a central role to prove the sharpness of our main results, and it will be used several times in this paper (for details see Tephnadze [21], Section 1.7., Example 1.48).

Lemma 2.2. *Let $0 < p \leq 1$, $\lambda = \sup_{n \in \mathbf{N}} m_n$, and $\{\lambda_k : k \in \mathbf{N}\}$ be a sequence of real numbers such that*

$$\sum_{k=0}^{\infty} |\lambda_k|^p \leq c_p < \infty. \tag{2.8}$$

Let $\{a_k : k \in \mathbf{N}\}$ be a sequence of p-atoms defined by

$$a_k := \frac{M_{|\alpha_k|}^{1/p-1}}{\lambda} \left(D_{M_{|\alpha_k|+1}} - D_{M_{|\alpha_k|}} \right),$$

where $|\alpha_k| := \max\{j \in \mathbf{N} : (\alpha_k)_j \neq 0\}$ and $(\alpha_k)_j$ denotes the j-th binary coefficient of $\alpha_k \in \mathbf{N}$. Then $f = (f_n : n \in \mathbf{N})$, where

$$f_n := \sum_{\{k:\ |\alpha_k| < n\}} \lambda_k a_k,$$

is a martingale, $f \in H_p$ for all $0 < p \leq 1$, and

$$\widehat{f}(j) = \begin{cases} \frac{\lambda_k M_{|\alpha_k|}^{1/p-1}}{\lambda}, & j \in \left\{M_{|\alpha_k|}, \ldots, M_{|\alpha_k|+1} - 1\right\}, \ k \in \mathbb{N}, \\ 0, & j \notin \bigcup\limits_{k=1}^{\infty} \left\{M_{|\alpha_k|}, \ldots, M_{|\alpha_k|+1} - 1\right\}. \end{cases} \tag{2.9}$$

Further, let $M_{|\alpha_l|} \leq j < M_{|\alpha_l|+1}$, $l \in \mathbb{N}$. Then

$$\begin{aligned} S_j f &= S_{M_{|\alpha_l|}} + \frac{\lambda_l M_{|\alpha_l|}^{1/p-1} \psi_{M_{|\alpha_l|}} D_{j-M_{|\alpha_l|}}}{\lambda} \\ &= \sum_{\eta=0}^{l-1} \frac{\lambda_\eta M_{|\alpha_\eta|}^{1/p-1}}{\lambda} \left(D_{M_{|\alpha_\eta|+1}} - D_{M_{|\alpha_\eta|}}\right) + \frac{\lambda_l M_{|\alpha_l|}^{1/p-1} \psi_{M_{|\alpha_l|}} D_{j-M_{|\alpha_l|}}}{\lambda}. \end{aligned} \tag{2.10}$$

Moreover, the following asymptotic relation holds:

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = O\left(\sum_{\{k: \, |\alpha_k| \geq n\}}^{\infty} |\lambda_k|^p\right)^{1/p} \quad \text{as} \quad n \to \infty. \tag{2.11}$$

There exists a close connection between the H_p and L_p norms of partial sums (see Tephnadze [21], Section 1.7., Example1.45):

Lemma 2.8. *Let $M_k \leq n < M_{k+1}$ and $S_n f$ be the n-th partial sum with respect to Vilenkin system, where $f \in H_p$ for some $0 < p \leq 1$. Then for every $n \in \mathbb{N}$ we have the following estimate:*

$$\|S_n f\|_p \leq \|S_n f\|_{H_p} \leq \left\| \sup_{0 \leq l < k} |S_{M_l} f| \right\|_p + \|S_n f\|_p \leq \left\| \widetilde{S}_{\#}^* f \right\|_p + \|S_n f\|_p.$$

3. Convergence of subsequences of partial sums on the martingale Hardy spaces

Our first main result in this paper is the following theorem.

Theorem 3.1. *The following assertions hold.*

a) Let $0 < p < 1$ and $f \in H_p$. Then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f\|_{H_p} \leq \frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} \|f\|_{H_p}.$$

b) Let $0 < p < 1$ and $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that condition (1.2) is satisfied, and let $\{\Phi_n : n \in \mathbb{N}\}$ be any nondecreasing *sequence, satisfying the condition:*

$$\overline{\lim_{k\to\infty}} \frac{M_{(n_k)}^{1/p-1}}{M_{(n_k)}^{1/p-1}\Phi_{n_k}} = \infty. \tag{3.1}$$

Then there exists a martingale $f \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{S_{n_k} f}{\Phi_{n_k}} \right\|_{L_{p,\infty}} = \infty.$$

Proof. We first prove assertion a). Suppose that

$$\left\| \frac{M_{(n)}^{1/p-1} S_n f}{M_{|n|}^{1/p-1}} \right\|_p \leq c_p \|f\|_{H_p}. \tag{3.2}$$

Then according to Lemma 2.3 and estimates (1.3) and (3.2) we get

$$\left\| \frac{M_{(n)}^{1/p-1} S_n f}{M_{|n|}^{1/p-1}} \right\|_{H_p} \leq \left\| \widetilde{S}_{\#}^{*} f \right\|_p + \left\| \frac{M_{(n)}^{1/p-1} S_n f}{M_{|n|}^{1/p-1}} \right\|_p \leq c_p \|f\|_{H_p}. \tag{3.3}$$

In view of Lemma 2.1 and (3.3), the proof of part a) of the theorem will be completed, if we show that

$$\int_{\overline{I}} \left| \frac{M_{(n)}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}} \right| d\mu \leq c_p < \infty, \tag{3.4}$$

for every p-atom a, with support I and $\mu(I) = M_N^{-1}$.

We may assume that this arbitrary p-atom a has support $I = I_N$. It is easy to see that $S_n a = 0$, when $M_N \geq n$. Therefore, we can suppose that $M_N < n$. According to $\|a\|_\infty \leq M_N^{1/p}$, we can write

$$\begin{aligned} \left| \frac{M_{(n)}^{1/p-1} S_n a(x)}{M_{|n|}^{1/p-1}} \right| &\leq \frac{M_{(n)}^{1/p-1} \|a\|_\infty}{M_{|n|}^{1/p-1}} \int_{I_N} |D_n(x-t)| \, d\mu(t) \\ &\leq \frac{M_{(n)}^{1/p-1} M_N^{1/p}}{M_{|n|}^{1/p-1}} \int_{I_N} |D_n(x-t)| \, d\mu(t). \end{aligned} \tag{3.5}$$

Let $x \in I_N$. Since $x - t \in I_N$, $t \in I_N$ and $v(n) + v^*(n) \leq c(|n| - \langle n \rangle) = c\rho(n)$, we can apply (2.6) to obtain

$$
\begin{aligned}
\left|\frac{M_{\langle n \rangle}^{1/p-1} S_n a(x)}{M_{|n|}^{1/p-1}}\right| &\leq \frac{M_{\langle n \rangle}^{1/p-1} M_N^{1/p}}{M_{|n|}^{1/p-1}} \int_{I_N} |D_n(t)| \, d\mu(t) \\
&\leq \frac{M_{\langle n \rangle}^{1/p-1} M_N^{1/p} (v(n) + v^*(n))}{M_{|n|}^{1/p-1}} \qquad (3.6) \\
&\leq \frac{c M_{\langle n \rangle}^{1/p-1} M_N^{1/p} (|n| - \langle n \rangle)}{M_{|n|}^{1/p-1}} \leq \frac{c M_N^{1/p} \rho(n)}{2^{\rho(n)(1/p-1)}}
\end{aligned}
$$

and

$$
\int_{I_N} \left|\frac{M_{\langle n \rangle}^{1/p-1} S_n a(x)}{M_{|n|}^{1/p-1}}\right|^p d\mu(x) \leq \frac{\rho^p(n)}{2^{\rho(n)(1-p)}} < c_p < \infty. \tag{3.7}
$$

Let $x \in I_s \setminus I_{s+1}$, $0 \leq s \leq N-1 < \langle n \rangle$ or $0 \leq s \leq \langle n \rangle \leq N-1$. Then $x - t \in I_s \setminus I_{s+1}$ for $t \in I_N$. Combining (2.2) and (2.3) we get $D_n(x-t) = 0$, and

$$
\left|\frac{M_{\langle n \rangle}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}}\right| = 0. \tag{3.8}
$$

Let $x \in I_s \setminus I_{s+1}$, $0 \leq \langle n \rangle < s \leq N-1$ or $0 \leq \langle n \rangle < s \leq N-1$. Then $x - t \in I_s \setminus I_{s+1}$ for $t \in I_N$. Hence, applying (2.5), we get

$$
\left|\frac{M_{\langle n \rangle}^{1/p-1} S_n a(x)}{M_{|n|}^{1/p-1}}\right| \leq \frac{c_p M_{\langle n \rangle}^{1/p-1} M_N^{1/p}}{M_{|n|}^{1/p-1}} \frac{M_s}{M_N} = c_p M_{\langle n \rangle}^{1/p-1} M_s. \tag{3.9}
$$

Combining (2.1), (3.8) and (3.9), we obtain

$$
\begin{aligned}
\int_{\overline{I_N}} \left|\frac{M_{\langle n \rangle}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}}\right|^p d\mu &= \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left|\frac{M_{\langle n \rangle}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}}\right|^p d\mu \qquad (3.10) \\
&\leq c_p \sum_{s=\langle n \rangle}^{N-1} \int_{I_s \setminus I_{s+1}} \left| M_{\langle n \rangle}^{1/p-1} M_s \right|^p d\mu = c_p \sum_{s=\langle n \rangle}^{N-1} \frac{c_p M_{\langle n \rangle}^{1-p}}{M_s^{1-p}} \leq c_p < \infty.
\end{aligned}
$$

This completes the proof of part a) of the theorem.

Now we proceed to prove part b) of the theorem. To this end, observe first that under the condition (3.1), there exists a sequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$ such that

$$
\sum_{\eta=0}^{\infty} \frac{M_{\langle \alpha_\eta \rangle}^{(1-p)/2} \Phi_{\alpha_\eta}^{p/2}}{M_{|\alpha_\eta|}^{(1-p)/2}} < \infty. \tag{3.11}
$$

We note that such increasing sequence $\{\alpha_k : k \in \mathbf{N}\}$, which satisfies condition (3.11), can be constructed.

Let $f = (f_n, n \in \mathbf{N})$ be the martingale from Lemma 2.2, where

$$\lambda_k = \frac{M_{|\alpha_k|}^{(1/p-1)/2}\Phi_{\alpha_k}^{1/2}}{M_{|\alpha_k|}^{(1/p-1)/2}}. \tag{3.12}$$

In view of (3.12) we conclude that (2.8) is satisfied, and hence, using Lemma 2.2, we obtain that $f \in H_p$.

Next, using (2.10) with λ_k defined by (3.12), we get

$$\begin{aligned}\frac{S_{\alpha_k} f}{\Phi_{\alpha_k}} &= \frac{1}{\Phi_{\alpha_k}}\sum_{\eta=0}^{k-1} M_{|\alpha_\eta|}^{(1/p-1)/2} M_{(\alpha_\eta)}^{(1/p-1)/2}\Phi_{\alpha_\eta}^{1/2}\left(M_{|\alpha_\eta|+1} - D_{M_{|\alpha_\eta|}}\right)\\ &+ \frac{M_{|\alpha_k|}^{(1/p-1)/2} M_{(\alpha_k)}^{(1/p-1)/2} D_{\alpha_k - M_{|\alpha_k|}}}{\Phi_{\alpha_k}^{1/2}} = I + II.\end{aligned}$$

Hence, according to (3.11), we can write

$$\begin{aligned}\|I\|_{L_{p,\infty}}^p &\leq \frac{1}{\Phi_{\alpha_k}^p}\sum_{\eta=0}^{\infty}\frac{M_{(\alpha_\eta)}^{(1-p)/2}\Phi_{\alpha_\eta}^{p/2}}{M_{|\alpha_\eta|}^{(1-p)/2}}\left\|M_{|\alpha_\eta|}^{(1/p-1)}\left(M_{|\alpha_\eta|+1} - D_{M_{|\alpha_\eta|}}\right)\right\|_{L_{p,\infty}}^p\\ &\leq \frac{1}{\Phi_{\alpha_k}^p}\sum_{\eta=0}^{\infty}\frac{M_{(\alpha_\eta)}^{(1-p)/2}\Phi_{\alpha_\eta}^{p/2}}{M_{|\alpha_\eta|}^{(1-p)/2}} < \frac{c}{\Phi_{\alpha_k}^p} \leq c < \infty. \end{aligned}\tag{3.13}$$

Let $x \in I_{(\alpha_k)} \setminus I_{(\alpha_k)+1}$. Then we can apply (2.4) to conclude that

$$\begin{aligned}|II| &= \frac{M_{|\alpha_k|}^{(1/p-1)/2} M_{(\alpha_k)}^{(1/p-1)/2} D_{\alpha_k - M_{|\alpha_k|}}}{\Phi_{\alpha_k}^{1/2}}\\ &\geq \frac{M_{|\alpha_k|}^{(1/p-1)/2} M_{(\alpha_k)}^{(1/p+1)/2}}{\Phi_{\alpha_k}^{1/2}}.\end{aligned}\tag{3.14}$$

Combining (3.13) and (3.14), for sufficiently large k, we can write

$$\begin{aligned}&\left\|\frac{S_{\alpha_k} f}{\Phi_{\alpha_k}}\right\|_{L_{p,\infty}}^p \geq \|II\|_{L_{p,\infty}}^p - \|I\|_{L_{p,\infty}}^p \geq \frac{1}{2}\|II\|_{L_{p,\infty}}^p\\ &\geq \frac{cM_{|\alpha_k|}^{(1-p)/2} M_{(\alpha_k)}^{(1+p)/2}}{\Phi_{\alpha_k}^{p/2}}\mu\left\{x \in G_m : |II| \geq \frac{cM_{|\alpha_k|}^{(1/p-1)/2} M_{(\alpha_k)}^{(1/p+1)/2}}{\Phi_{\alpha_k}^{1/2}}\right\}\\ &\geq \frac{cM_{|\alpha_k|}^{(1-p)/2} M_{(\alpha_k)}^{(1+p)/2}}{\Phi_{\alpha_k}^{p/2}}\mu\left\{I_{(\alpha_k)} \setminus I_{(\alpha_k)+1}\right\} \geq \frac{cM_{|\alpha_k|}^{(1-p)/2}}{M_{(\alpha_k)}^{(1-p)/2}\Phi_{\alpha_k}^{p/2}} \to \infty \text{ as } k \to \infty.\end{aligned}$$

This completes the proof of part b) of the theorem. □

The next corollary contains equivalent characterizations of boundedness of subsequences of partial sums with respect to the Vilenkin system of martingales $f \in H_p$ in terms of measurable properties of the Dirichlet kernel.

Corollary 3.1. *The following assertions hold.*

a) Let $0 < p < 1$ and $f \in H_p$. Then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f\|_{H_p} \leq c_p \left(n\mu\left\{supp\left(D_n\right)\right\}\right)^{1/p-1} \|f\|_{H_p}.$$

b) Let $0 < p < 1$ and $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that

$$\sup_{k\in\mathbb{N}} n_k \mu\left\{supp\left(D_{n_k}\right)\right\} = \infty, \tag{3.15}$$

and let $\{\Phi_n : n \in \mathbb{N}\}$ be any nondecreasing sequence, satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{\left(n_k\mu\left\{supp\left(D_{n_k}\right)\right\}\right)^{1/p-1}}{\Phi_{n_k}} = \infty.$$

Then there exists a martingale $f \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{S_{n_k} f}{\Phi_{n_k}} \right\|_{L_{p,\infty}} = \infty.$$

Remark 3.1. Corollary 3.1 shows that when $0 < p < 1$, the main reason of divergence of partial sums of a Vilenkin-Fourier series is the unboundedness of Fourier coefficients, but in the case where the measure of the support of n_k-th Dirichlet kernel tends to zero, then the divergence rate drops and in the case when it is maximally small, that is,

$$\mu\left(suppD_{n_k}\right) = O\left(\frac{1}{M_{|n_k|}}\right) \quad \text{as} \quad k \to \infty, \qquad \left(M_{|n_k|} < n_k \leq M_{|n_k|+1}\right),$$

then we have convergence.

Proof. Combining (2.2) and (2.3) we get $I_{\langle n\rangle} \backslash I_{\langle n\rangle+1} \subset \text{supp}D_n \subset I_{\langle n\rangle}$ and

$$\frac{1}{2M_{\langle n\rangle}} \leq \mu\left(\text{supp}D_n\right) \leq \frac{1}{M_{\langle n\rangle}}.$$

Since $M_{|n|} \leq n < M_{|n|+1}$, we immediately get

$$\frac{M_{|n|}}{2M_{\langle n\rangle}} \leq n\mu\left\{\text{supp}\left(D_n\right)\right\} \leq \frac{\lambda M_{|n|}}{M_{\langle n\rangle}},$$

where $\lambda = \sup_{n\in\mathbb{N}} m_n$.

It follows that

$$\frac{M_{|n|}^{1/p-1}}{2M_{\langle n\rangle}^{1/p-1}} \leq \left(n\mu\left\{\operatorname{supp}\left(D_n\right)\right\}\right)^{1/p-1} \leq \frac{\lambda^{1/p-1}M_{|n|}^{1/p-1}}{M_{\langle n\rangle}^{1/p-1}}.$$

The result follows by using these estimates in Theorem 3.1. □

As special cases of Theorem 3.1, we can infer a number of known and new results that are of particular interest. In Corollaries 3.2-3.4 that follow we list some of them.

Corollary 3.2. *Let $0 < p < 1$, $f \in H_p$ and $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers. Then*

$$\left\|S_{n_k} f\right\|_{H_p} \leq c_p \left\|f\right\|_{H_p}$$

if and only if condition (1.1) is satisfied.

Proof. It is easy to show that

$$2^{\rho(n_k)} \leq \frac{M_{|n_k|}}{M_{\langle n_k\rangle}} \leq \lambda^{\rho(n_k)},$$

where $\lambda = \sup_{n\in\mathbb{N}} m_n$. It follows that

$$\sup_{k\in\mathbb{N}} \frac{M_{|n_k|}^{1/p-1}}{M_{\langle n_k\rangle}^{1/p-1}} < c < \infty$$

if and only if (1.1) holds. Thus, the result follows from Theorem 3.1. □

Corollary 3.3. *Let $n \in \mathbb{N}$ and $0 < p < 1$. Then there exists a martingale $f \in H_p$ such that*

(3.16) $$\sup_{n\in\mathbb{N}} \left\|S_{M_n+1} f\right\|_{L_{p,\infty}} = \infty.$$

Proof. It is easy to check that

(3.17) $$|M_n + 1| = n, \quad \langle M_n + 1\rangle = 0$$

and

(3.18) $$\rho\left(M_n + 1\right) = n.$$

By using Corollary 3.2 we obtain that there exists a martingale $f \in H_p$ $(0 < p < 1)$ such that (3.16) holds. The proof is complete. □

Corollary 3.4. *Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $f \in H_p$. Then*

(3.19) $$\left\|S_{M_n+M_{n-1}} f\right\|_{H_p} \leq c_p \left\|f\right\|_{H_p}.$$

Proof. Similar to (3.17) and (3.18), we obtain

$$|M_n + M_{n-1}| = n, \quad \langle M_n + M_{n-1} \rangle = n - 1$$

and $\rho(M_n + M_{n-1}) = 1$. By using Corollary 3.2 we immediately get the inequality (3.19) for all $0 < p \leq 1$. The proof is complete. □

Corollary 3.5. *Let $n \in \mathbb{N}$, $0 < p < 1$ and $f \in H_p$. Then*

$$\|S_{M_n} f\|_{H_p} \leq c_p \|f\|_{H_p}. \tag{3.20}$$

Proof. Similar to (3.17) and (3.18) we obtain $|M_n| = n$, $\langle M_n \rangle = n$ and $\rho(M_n) = 0$. Using Corollary 3.2 we get the inequality (3.20) for all $0 < p \leq 1$. □

4. Necessary and sufficient condition for convergence of partial sums in terms of modulus of continuity

The main result of this section is the following theorem.

Theorem 4.1. *The following assertions hold.*

a) Let $0 < p < 1$, $f \in H_p$ and $M_k < n \leq M_{k+1}$. Then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f - f\|_{H_p} \leq \frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} \omega\left(\frac{1}{M_k}, f\right)_{H_p(G_m)}, \quad 0 < p < 1. \tag{4.1}$$

Moreover, if $\{n_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers such that

$$\omega\left(\frac{1}{M_{|n_k|}}, f\right)_{H_p(G_m)} = o\left(\frac{M_{\langle n_k \rangle}^{1/p-1}}{M_{|n_k|}^{1/p-1}}\right) \quad as \quad k \to \infty, \tag{4.2}$$

then

$$\|S_{n_k} f - f\|_{H_p} \to 0 \quad as \quad k \to \infty. \tag{4.3}$$

b) Let $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that the condition (1.2) is satisfied. Then there exist a martingale $f \in H_p$ and a subsequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$, for which

$$\omega\left(\frac{1}{M_{|\alpha_k|}}, f\right)_{H_p(G_m)} = O\left(\frac{M_{\langle \alpha_k \rangle}^{1/p-1}}{M_{|\alpha_k|}^{1/p-1}}\right) \quad as \quad k \to \infty \tag{4.4}$$

and

$$\overline{\lim_{k\to\infty}} \|S_{\alpha_k} f - f\|_{L_{p,\infty}} > c > 0 \quad as \quad k \to \infty. \tag{4.5}$$

Proof. Let $0 < p < 1$. Then, by Theorem 3.1, we get

$$\|S_n f - f\|_{H_p}^p \leq \|S_n f - S_{M_k} f\|_{H_p}^p + \|S_{M_k} f - f\|_{H_p}^p$$
$$= \|S_n (S_{M_k} f - f)\|_{H_p}^p + \|S_{M_k} f - f\|_{H_p}^p \leq \left(\frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} + 1 \right) \omega_{H_p}^p \left(\frac{1}{M_k}, f \right).$$

and

$$\|S_n f - f\|_{H_p} \leq \frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} \omega \left(\frac{1}{M_k}, f \right)_{H_p(G_m)}.$$

Next, it is easy to see that relation (4.3) immediately follows from (4.1) and (4.2). Thus, the assertion a) is proved. To prove part b) of the theorem, we first note that under the conditions of part b), there exists a subsequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$ such that

$$\frac{M_{|\alpha_k|}}{M_{\langle \alpha_k \rangle}} \uparrow \infty \quad \text{as} \quad k \to \infty \tag{4.6}$$

$$\frac{M_{|\alpha_k|}^{2(1/p-1)}}{M_{\langle \alpha_k \rangle}^{2(1/p-1)}} \leq \frac{M_{|\alpha_{k+1}|}^{1/p-1}}{M_{\langle \alpha_{k+1} \rangle}^{1/p-1}}. \tag{4.7}$$

Let $f = (f_n, n \in \mathbb{N})$ be the martingale from Lemma 2.2, where

$$\lambda_k = \frac{\lambda M_{\langle \alpha_k \rangle}^{1/p-1}}{M_{|\alpha_k|}^{1/p-1}}. \tag{4.8}$$

Applying (4.6) and (4.7) with λ_k as in (4.8), we conclude that (2.8) is satisfied, and hence by Lemma 2.2, we obtain that $f \in H_p$.

Using (4.8) with λ_k as in (4.8), we get

$$\omega(\frac{1}{M_{|\alpha_k|}}, f)_{H_p(G_m)} \leq \sum_{i=k}^{\infty} \frac{M_{\langle \alpha_i \rangle}^{1/p-1}}{M_{|\alpha_i|}^{1/p-1}} = O\left(\frac{M_{\langle \alpha_k \rangle}^{1/p-1}}{M_{|\alpha_k|}^{1/p-1}} \right) \quad \text{as} \quad k \to \infty. \tag{4.9}$$

Next, applying (2.10) with λ_k as in (4.8), we obtain

$$S_{\alpha_k} f = S_{M_{|\alpha_k|}} + M_{\langle \alpha_k \rangle}^{1/p-1} \psi_{M_{|\alpha_k|}} D_{j-M_{|\alpha_k|}}.$$

In view of (2.4) we conclude that $\left|D_{\alpha_k - M_{|\alpha_k|}}\right| \geq M_{\langle \alpha_k \rangle}$ for $I_{\langle \alpha_k \rangle} \setminus I_{\langle \alpha_k \rangle + 1}$, and

$$M_{\langle \alpha_k \rangle} \mu \left\{ x \in G_m : \; \left|D_{\alpha_k - M_{|\alpha_k|}}\right| \geq M_{\langle \alpha_k \rangle} \right\} \tag{4.10}$$
$$M_{\langle \alpha_k \rangle} \mu \left\{ I_{\langle \alpha_k \rangle} \setminus I_{\langle \alpha_k \rangle + 1} \right\} \geq M_{\langle \alpha_k \rangle}^{1-p}$$

Finally, in view of Corollary 3.5 and formula (4.10), for sufficiently large k, we can write

$$\begin{aligned}\|S_{\alpha_k} f - f\|_{L_{p,\infty}} &\geq M_{(\alpha_k)}^{1/p-1}\|D_{\alpha_k}\|_{L_{p,\infty}} - \|S_{M_{|\alpha_k|}} f - f\|_{L_{p,\infty}} \\ &\geq \frac{M_{(\alpha_k)}^{1/p-1}\|D_{\alpha_k}\|_{L_{p,\infty}}}{2} \geq c.\end{aligned}$$

This completes the proof of part b) of the theorem.

Theorem 4.1 is proved. □

Next, we present a simple consequence of Theorem 4.1, which was proved in Tephnadze [18]:

Corollary 4.1. *The following assertions hold.*

a) Let $0 < p < 1$, $f \in H_p$ *and*

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = o\left(\frac{1}{M_n^{1/p-1}}\right) \quad \text{as} \quad n \to \infty.$$

Then

$$\|S_k f - f\|_{H_p} \to 0 \quad \text{as} \quad k \to \infty.$$

b) For every $0 < p < 1$ *there exists a martingale* $f \in H_p$ *for which*

$$\omega\left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{M_n^{1/p-1}}\right)_{H_p(G_m)} \quad \text{as} \quad n \to \infty$$

and

$$\|S_k f - f\|_{L_{p,\infty}} \nrightarrow 0 \quad \text{as} \quad k \to \infty.$$

Finally, we present a result that contains equivalent conditions for the modulus of continuity in terms of measurable properties of the Dirichlet kernel, which provide boundedness of the subsequences of partial sums with respect to the Vilenkin system of martingales $f \in H_p$.

Corollary 4.2. *The following assertions hold.*

a) Let $0 < p < 1$, $f \in H_p$ *and* $M_k < n \leq M_{k+1}$. *Then there exists an absolute constant* c_p, *depending only on p, such that*

$$\|S_n f - f\|_{H_p} \leq c_p \left(n\mu\left(suppD_n\right)\right)^{1/p-1} \omega_{H_p}\left(\frac{1}{M_k}, f\right), \quad (0 < p < 1).$$

Moreover, if $\{n_k : k \in \mathbb{N}\}$ *is a sequence of nonnegative integers such that*

$$\omega\left(\frac{1}{M_{|n_k|}}, f\right)_{H_p(G_m)} = o\left(\frac{1}{\left(n_k\mu\left(suppD_{n_k}\right)\right)^{1/p-1}}\right) \quad \text{as} \quad k \to \infty,$$

then (4.3) holds.

b) Let $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that the condition (1.2) is satisfied. Then there exist a martingale $f \in H_p$ and a subsequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$, for which

$$\omega\left(\frac{1}{M_{|\alpha_k|}}, f\right)_{H_p(G_m)} = O\left(\frac{1}{(\alpha_k \mu (suppD_{\alpha_k}))^{1/p-1}}\right) \quad as \ k \to \infty$$

and

$$\overline{\lim_{k\to\infty}} \|S_{\alpha_k} f - f\|_{L_{p,\infty}} > c > 0 \quad as \ k \to \infty.$$

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