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THE PARTICLE STRUCTURE OF THE QUANTUM MECHANICAL BOSE AND FERMI GAS

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Abstract. In the framework of von Neumann's description of measurements of discrete quantum observable we establish a one-to-one correspondence between symmetric statistical operators W of quantum mechanical systems and classical point processes κ_W, thereby giving a particle picture of indistinguishable quantum particles. This holds true under irreducibility assumptions if we fix the underlying complete orthonormal system. The method of the Campbell measure is developed for such statistical operators: it is shown that the Campbell measure of a statistical operator W coincides with the Campbell measure of the corresponding point process κ_W. Moreover, again under irreducibility assumptions, a symmetric statistical operator is completely determined by its Campbell measure. The method of the Campbell measure then is used to characterize Bose-Einstein and Fermi-Dirac statistical operators. This is an elementary introduction into the work of Fichtner and Freudenberg [10, 11] combined with the quantum mechanical investigations of [2] and the corresponding point process approach of [30]. It is based on the classical work of von Neumann [22], Segal, Cook and Chaiken [28, 8, 7] as well as Moyal [18].

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1. Introduction

We consider quantum statistical states and ask for a precise particle picture of them. Under irreducibility assumptions we develop a one-to-one correspondence between symmetric statistical operators W of finite quantum mechanical systems and point processes κ_W, thereby giving a particle picture of indistinguishable quantum particles. This is done by developing a disintegration theory for such statistical operators in complete analogy to the decomposition of classical into conditional probabilities.

We also need the *method of the Campbell measure*, which is well known for point processes, and which is developed here for statistical operators. (This is inspired by the work of Fichtner, see for instance [12], and Liebscher [16].) We show that the Campbell measure of a symmetric statistical operator W coincides with the usual

Campbell measure of its law κ_W, moreover, under irreducibility assumptions, W is then completely determined by its Campbell measure.

We then present the point processes which correspond to the quantum statistical operators of Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac in the case of a fixed number of particles. Surprisingly, only the point process belonging to the Maxwell-Boltzmann statistical operator is really known and has been considered in probability theory until now.

We then extend our considerations to systems with a random number of particles and therefore work on Fock spaces. In this framework the Poisson point process belongs to the Maxwell-Boltzmann statistical operator. Next the symmetric Bose-Einstein and Fermi-Dirac statistical operators are constructed together with their associated point processes. Since these statistical operators are determined by their Campbell measures, and since the Campbell measures coincide for statistical operators and their point processes, we shall investigate the Campbell measure of these point processes.

As a result of the application of the method of Campbell measures we find that the point processes belonging to Bose-Einstein and Fermi-Dirac statistical operators respectively are given by Papangelou processes with explicitly given conditional intensity kernels. They are called here *Pólya sum* and *Pólya difference processes* respectively. The corresponding random fields are of first order and have independent increments. The distribution of the field variables, which represent the number of particles in a given region, are explicitly known. These results have been shown in [20]. Thus these processes have all characteristic properties of an ideal gas. In this way we obtain detailed informations about the point processes and thereby about the correponding statistical operators.

We stress here the point of view that for the developement of a full interacting theory of quantum gases one should start with the corresponding ideal gas and then modify this by means of a Boltzmann factor to include an interaction between the particles. (First steps in this direction can be found in [20].)

Historically the first attempts to unify quantum mechanics with point process theory can be found in the work of Fock [13], Segal [28], Cook [8] and Chaiken [7] and then, more systematically, in the work of Moyal [18]. For a more recent contribution to the construction of Bose and Fermi processes from the point of view of quantum mechanics we refer to Tamura and Ito [29].

Note added in February 2015. Unpublished versions of this work exist since 2008. We did not intend to publish it. But in the meantime several publications (see [20, 19, 26, 27] e.g.) referred to it so that it might be useful to make it available to the public.

2. Disintegration of statistical operators

We consider von Neumann's description of the measuring process of discrete quantum observables (cf. [22, 23]) and use it for a representation of statistical operators in terms of their conditional statistical operators and their laws.

Consider a countable set $Y \neq \emptyset$ together with an equivalence relation $\sim$ in Y. Represent $(Y, \sim)$ by means of (Γ, r) in such a way that Γ is a countable set and $r : Y \to \Gamma$ a surjective mapping satisfying

$$(2.1) \qquad \big(x \sim y \Longleftrightarrow r(x) = r(y)\big).$$

Given $\gamma \in \Gamma$ we set $Y_\gamma = \{r = \gamma\}$ for the associated equivalence class. In the sequel we assume always that

$$(2.2) \qquad 1 \leq \text{cd } Y_\gamma < +\infty \quad \text{for any } \gamma.$$

Let $\mathcal{H}$ be a complex separable Hilbert space of countable dimension $|Y|$. We identify the set Y with the complete othonormal system (*cons*) $\mathcal{Y} = \{e_y | y \in Y\}$ chosen in $\mathcal{H}$. Furthermore, we set $\mathcal{Y}_\gamma = \{e_y | y \in Y_\gamma\}$. The equivalence relation $\sim$ induces an equivalence relation in $\mathcal{Y}$ by means of $(e_x \sim e_y \Leftrightarrow x \sim y)$ with $\mathcal{Y}_\gamma$ as equivalence classes.

The set of *events of the system* described by the Hilbert space $\mathcal{H}$ can be identified with the collection of all orthogonal projections resp. all (closed) subspaces. The *state space* $\mathcal{S}(\mathcal{H})$ of the system is the collection of *(self-adjoint) bounded linear operators* $\mathcal{W}$ on $\mathcal{H}$ which are *positive* and have *trace one*, i.e. $\operatorname{tr} \mathcal{W} = 1$. Such $\mathcal{W}$ are called *statistical operators*. They form a convex set whose extremal points, the socalled *pure states*, are defined by

$$h \circ h = \langle h, . \rangle \cdot h, \qquad h \in \mathcal{H}, \|h\| = 1.$$

By the spectral theorem every state $\mathcal{W}$ admits a representation

$$\mathcal{W} = \sum_{n=1}^{\infty} p_n \cdot h_n \circ h_n,$$

where $(p_n)_n$ is a probability on $\mathbb{N}$ and $(h_n)_n$ some *cons* in $\mathcal{H}$. (For more details we refer to [9].)

Our problem is how to associate to a given statistical operator $\mathcal{W} \in \mathcal{S}(\mathcal{H})$, admitting a spectral resolution with respect to a given *cons* $\mathcal{Y}$, a law, and, in particular situations, a point process κ, and vice versa.

In the above situation we are given a complex separable Hilbert space $\mathcal{H}$ with fixed basis $\mathcal{Y}$, indexed by Y. We consider

$$\mathcal{H}_\gamma = sp\{e_y | y \in Y_\gamma\},$$

the smallest subspace of $\mathcal{H}$ containing $\{e_y | y \in Y_\gamma\}$. The collection $(\mathcal{H}_\gamma)_{\gamma\in\Gamma}$ is an orthogonal decomposition of $\mathcal{H}$; and $\mathcal{H}$ is the direct sum of it. We have

$$1 \leq \dim \mathcal{H}_\gamma = |Y_\gamma| = \mathrm{cd}\, Y_\gamma < \infty.$$

Here cd denotes *cardinality*. Finally we write

$$P_\gamma = P^{\mathcal{H}_\gamma}$$

for the orthogonal projection onto $\mathcal{H}_\gamma$.

We start with a statistical operator $\mathcal{W} \in \mathcal{S}(\mathcal{H})$ which admits the spectral resolution

$$\mathcal{W} = \sum_{y\in Y} P_y\, \varrho(y) \tag{2.3}$$

for some law ϱ on Y with respect to the chosen *cons* $\mathcal{Y}$. Here $P_y = e_y \diamond e_y$ with $e_y \diamond e_y = (e_y, .) \cdot e_y$. Thus $\mathcal{W}$ is diagonalized by the given *cons* $\mathcal{Y}$. Set

$$\mathcal{W}_\gamma = \sum_{y\in Y_\gamma} P_y\, \varrho(y). \tag{2.4}$$

This defines self-adjoint linear operators on $\mathcal{H}$, leaving $\mathcal{H}_\gamma$ invariant s.th.

$$\mathcal{W}_\gamma = P_\gamma\, \mathcal{W}\, P_\gamma, \qquad \mathcal{W}_\gamma\, \mathcal{H}_\gamma^\perp = \{0\}.$$

Decomposition (2.4) is unique. If $\mathrm{tr}\, \mathcal{W}_\gamma = tr(P_\gamma \mathcal{W})$ is strictly positive, we can normalize $\mathcal{W}_\gamma$ to obtain the following statistical operator on $\mathcal{H}$:

$$\mathcal{W}(.|\gamma) = \frac{P_\gamma \mathcal{W} P_\gamma}{tr(P_\gamma \mathcal{W})}. \tag{2.5}$$

This is called the *conditional statistical operator of* $\mathcal{W}$ *given* P_γ. The notion of conditional statistical operators has been studied systematically by Cassinelli, Zanghi and Ozawa (cf. [6, 23] and the literature cited there).

Theorem 2.1. *Given an equivalence relation in* Y *which can be represented by means of* (Γ, r) *in such a way that conditions (2.1) and (2.2) are satisfied, any statistical* *operator $\mathcal{W} \in \mathcal{S}(\mathcal{H})$, admitting a spectral resolution (2.3) with respect to $\mathcal{Y}$, can be represented as*

$$\mathcal{W} = \sum_{\gamma \in \Gamma} \mathcal{W}(.|\gamma) \cdot \kappa_{\mathcal{W}}(\gamma), \tag{2.6}$$

where $\mathcal{W}(.|\gamma) \in \mathcal{S}(\mathcal{H})$, leaving $\mathcal{H}_\gamma$ invariant with $\mathcal{W}(.|\gamma)\mathcal{H}_\gamma^\perp = \{0\}$, and where $\kappa_{\mathcal{W}}$ is a probability on Γ having the following properties:

$$\kappa_{\mathcal{W}}(\gamma) = tr(P_\gamma \mathcal{W}), \qquad \gamma \in \Gamma. \tag{2.7}$$

This decomposition is unique.

In formula (2.6) and also later we use the convention that $\mathcal{W}(.|\gamma) \cdot \kappa_{\mathcal{W}}(\gamma) = 0$ if $\kappa_{\mathcal{W}}(\gamma) = 0$. We call $\kappa_{\mathcal{W}}$ the *law of the statistical operator* $\mathcal{W}$. It is some kind of partial trace of $\mathcal{W}$ with respect to γ, and we also write $\kappa_{\mathcal{W}}(\gamma) = tr_\gamma(\mathcal{W})$. This means that $tr_\gamma(\mathcal{W}) = \sum_{y \in Y_\gamma} \langle e_y, \mathcal{W} e_y \rangle$. We observe that for the calculation of the law $\kappa_{\mathcal{W}}$ we can use the *cons* which is most convenient, because a trace does not depend on the choice of a *cons*. Decomposition (2.6) is completely analogous to the decomposition of classical probabilities into conditional probabilities; and it is the starting point for the solution of our problem.

3. Disintegration of symmetric statistical operators

Consider next a finite group $\mathcal{G}$ acting on Y together with the equivalence relation $\sim$ induced by $\mathcal{G}$ in Y by means of $x \sim y \Longleftrightarrow \exists g \in \mathcal{G} : y = g.x$. All orbits are finite, and $\mathcal{G}$ acts transitively on each of them. We assume also that $(Y, \sim)$ is represented by (Γ, r). As above $\mathcal{H}$ denotes a complex separable Hilbert space with a *cons* given by $\mathcal{Y}$. We consider then the unitary representation $\mathcal{U} = (\mathcal{U}_g)_{g \in \mathcal{G}}$ induced by $\mathcal{G}$ on $\mathcal{H}$ by means of

$$\mathcal{U}_g h = \sum_y \lambda_y \cdot e_{gy}, \quad h = \sum_y \lambda_y e_y.$$

It is obvious that $\mathcal{U}$ acts on $\mathcal{H}$ as well as on each $\mathcal{H}_\gamma$. Thus each $\mathcal{H}_\gamma$ as well as $\mathcal{H}_\gamma^\perp$ remains invariant under $\mathcal{U}$. The collection $\mathcal{U}_\gamma$ of restrictions of $\mathcal{U}_g, g \in \mathcal{G}$, to the subspaces $\mathcal{H}_\gamma$ is called an irreducible system, if *any closed subspace S of $\mathcal{H}_\gamma$ which remains invariant under $\mathcal{U}_\gamma$ is either $\{0\}$ or $\mathcal{H}_\gamma$*. This is equivalent to the condition that it does not commute with no non-trivial (self-adjoint) projection ([1], Exercise 1.3.D.) A statistical operator $\mathcal{W}$ is called *symmetric* (with respect to $\mathcal{G}$) if

$$\mathcal{U}_g \mathcal{W} \mathcal{U}_{g^{-1}} = \mathcal{W} \text{ for any } g \in \mathcal{G}. \tag{3.1}$$

In the sequel we consider symmetric $\mathcal{W}$ admitting a spectral resolution for *cons* $\mathcal{Y}$.

Lemma 3.1. *$\mathcal{W}$ is symmetric if and only if each $\mathcal{W}_\gamma$ is symmetric.*

Proof. By (3.1) combined with decomposition (2.6) $\mathcal{W}$ is symmetric iff

$$\sum_\gamma \mathcal{W}_\gamma = \sum_\gamma \mathfrak{U}_g \mathcal{W}_\gamma \mathfrak{U}_g^* \text{ , for any } g \in \mathcal{G}.$$

The uniqueness of the decomposition combined with the fact that each $\mathcal{H}_\gamma$ resp. $\mathcal{H}_\gamma^\perp$ remains invariant under $\mathfrak{U}$ immediately implies the result. □

We need also the following result which in our context is Schur's lemma ([4], Satz 7.1 b.):

Lemma 3.2. *Let $\mathcal{W}$ be symmetric. If the collection $\mathfrak{U}_\gamma$ is irreducible then $\mathcal{W}_\gamma$ is of the form $\mathcal{W}_\gamma = \kappa^*_{\mathcal{W}}(\gamma) \cdot P_\gamma$. Here $\kappa^*_{\mathcal{W}}$ are non-negative functions on Γ, determined by the equation $\kappa^*_{\mathcal{W}}(\gamma) = (e_y, \mathcal{W} e_y)$, $\quad y \in Y_\gamma$.*

The positivity of $\kappa^*_{\mathcal{W}}$ follows from the positivity of the statistical operator $\mathcal{W}$. Thus we obtain the following disintegration of a symmetric statistical operator $\mathcal{W}$.

Corollary 3.1. *If $\mathcal{W}$ is symmetric and if each $\mathfrak{U}_\gamma$ is irreducible then*

$$\mathcal{W} = \sum_{\gamma\in\Gamma} \kappa^*_{\mathcal{W}}(\gamma) P_\gamma \quad \text{and} \quad \sum_{\gamma\in\Gamma} \kappa^*_{\mathcal{W}}(\gamma) \dim \mathcal{H}_\gamma = 1.$$

To summarize we have the following result.

Theorem 3.1. *Under the assumption that each $\mathfrak{U}_\gamma, \gamma \in \Gamma$, is irreducible the equation*

$$\mathcal{W} = \sum_{\gamma\in\Gamma} \frac{1}{\dim \mathcal{H}_\gamma} P_\gamma \cdot \kappa(\gamma) \tag{3.2}$$

induces a one-to-one correspondence between symmetric statistical operators $\mathcal{W}$ on $\mathcal{H}$, admitting a spectral resolution with respect to $\mathcal{Y}$, and probabilities κ on Γ.

This correspondence will be the main device in the sequel.

Corollary 3.2. *If $\mathcal{W}$ is a symmetric statistical operator on $\mathcal{H}$, admitting a spectral resolution with respect to $\mathcal{Y}$, and if $\mathfrak{U}_\gamma$ is irreducible then the conditional statistical operator $\mathcal{W}(.|\gamma)$, if well defined, coincides with the normalized projection onto $\mathcal{H}_\gamma$:*

$$\mathcal{W}(.|\gamma) = \frac{1}{\dim \mathcal{H}_\gamma} \cdot P_\gamma. \tag{3.3}$$

Moreover, $\kappa_{\mathcal{W}}(\gamma) = \dim \mathcal{H}_\gamma \cdot \kappa^*_{\mathcal{W}}(\gamma), \gamma \in \Gamma$, the law of $\mathcal{W}$, determines the operator $\mathcal{W}$ completely.

From now on the underlying group $\mathcal{G}$ is given by a finite symmetric group $\mathcal{S}(E)$ of all permutations σ of some finite set E. In this case we consider the following operators:

$$\Pi_{\pm} = \frac{1}{|E|!} \cdot \sum_{\sigma \in \mathcal{S}(E)} sgn_{\pm}(\sigma) \cdot \mathfrak{U}_{\sigma}.$$

Here $sgn(\sigma) \in \{-1, +1\}$ denotes the sign of σ where sgn_+ is the identity and $sgn_- = sgn$. Both operators are orthogonal projections onto subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{H}$ and satisfy

$$\text{(3.4)} \qquad \mathfrak{U}_{\sigma}\Pi_+ = \Pi_+, \; \mathfrak{U}_{\sigma}\Pi_- = sgn(\sigma) \cdot \Pi_- \quad \text{for any} \quad \sigma \in \mathcal{S}(E).$$

In particular the operators Π_+ and Π_- are *symmetric*. The elements of $\mathcal{H}_+$ are also called *symmetric*; the elements of $\mathcal{H}_-$ *antisymmetric*.

4. Examples

We consider the following standard finite setting (cf. [2, 24]). X is a finite, nonempty set of cardinality d; and $Y = X^n$. According to the convention of quantum mechanics the 1-particle space of a particle in X is given by $\mathbb{C}^X$, whereas the n-particle system is described by the complex Hilbert space $\mathcal{H} = \bigotimes^n \mathbb{C}^X$, i.e. the n-th tensor power of the 1-particle space. Note that $\mathcal{H}$ coincides with $\mathbb{C}^Y$, and if $n = 0$ then $\mathcal{H}$ is the one-dimensional complex plane. In $\mathbb{C}^X$ we choose some *cons* $(e_x)_{x \in X}$ conveniently. $\mathcal{Y} = \{e_y = \otimes_{j=1}^n e_{x_j} | y = (x_1, \ldots, x_n) \in Y\}$ then is a *cons* in $\mathcal{H}$ indexed by $\mathcal{Y}$. If $n = 0$ then $\mathcal{Y}$ is a singleton consisting of some unit vector $\mathbf{1}$ in $\mathbb{C}$ fixed once and for all. The underlying symmetric group is given by the collection $\mathcal{S}_n$ of bijections σ on $E = [n] = \{1, \ldots, n\}$. $\mathcal{S}_n$ acts on Y by means of

$$\sigma \longmapsto \big((x_1, \ldots, x_n) \longmapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})\big).$$

It operates on $\mathcal{H}$ by means of the collection of unitary representations consisting of

$$\mathfrak{U}_{\sigma} : e_{x_1} \otimes \cdots \otimes e_{x_n} \longmapsto e_{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma^{-1}(n)}},$$

and is then extended by linearity. We shall be interested in statistical operators which are symmetric, i.e. commute with the above representation of $\mathcal{S}_n$, and which admit a spectral resolution with respect to $\mathcal{Y}$. Every observation W of a system of identical particles has this property. The Hilbert spaces $\mathcal{H}_+, \mathcal{H}_-$, appropriate for the description of particles obeying quantum statistics, are constructed by means of the projections Π_+, Π_- induced by the group $\mathcal{S}_n$.

A representation (Γ, r) of the equivalence relation induced by $\mathcal{S}_n$ on Y is given by

$$\Gamma = \mathcal{M}_n(X) := \{\delta_{x_1} + \cdots + \delta_{x_n} | (x_1, \ldots, x_n) \in Y\},$$
$$r : (x_1, \ldots, x_n) \longmapsto \delta_{x_1} + \cdots + \delta_{x_n}.$$

4.1. **The Maxwell-Boltzmann statistical operator.** In $\mathcal{H}$ we choose a *cons* indexed by Y in the following way: We are given a statistical operator w on the 1-particle space $\mathcal{H}_1 := \mathbb{C}^X$. Denote by ϱ the probability on X appearing in the spectral resolution of w, which at the same time gives a *cons* $(e_x)_{x\in X}$ in $\mathcal{H}_1$. This basis will be fixed also in the following examples and enables one to define the *cons* $\mathcal{Y}$ in $\mathcal{H}$ as above. Moreover, we always assume that ϱ is not a Dirac measure. This implies that $d = \text{cd } X \geq 2$. The *Maxwell-Boltzmann statistical operator for* w is defined by the tensor product of w: $\mathrm{M}_w^n = w^n$. Here w^n denotes the $n-$fold tensor product of w. Using proposition 16.3. in [24] this statistical operator can be expressed explicitly by

$$\mathrm{M}_w^n = \sum_{y\in Y} P_y \cdot \varrho^n(y), \tag{4.1}$$

where $P_y = e_y \diamond e_y$, and ϱ^n is the product law $\varrho \otimes \cdots \otimes \varrho$ on Y. (4.1) is nothing else than the spectral resolution of M_w^n with respect to $\mathcal{Y}$. M_w^n is symmetric with respect to $\mathcal{S}_n$. By Theorem 2.1 there is associated the following law on $\mathcal{M}_n(X)$, which thus is a point process in X, namely

$$\kappa(\gamma) = \binom{n}{\gamma} \cdot \prod_{x\in X} \varrho(x)^{\gamma(x)}, \qquad \gamma \in \mathcal{M}_n(X). \tag{4.2}$$

Here

$$\binom{n}{\gamma} = \frac{n!}{\prod_{x\in X} \gamma(x)!}, \qquad \gamma \in \mathcal{M}_n(X).$$

(4.2) follows from the fact that $\dim \mathcal{H}_\gamma^n = \binom{n}{\gamma}$ and that, for $y = (x_1, \ldots, x_n) \in Y_\gamma$ and thereby $\gamma = \delta_{x_1} + \cdots + \delta_{x_n}$, by formula (4.1),

$$\kappa^*(\gamma) = \langle e_{x_1} \otimes \cdots \otimes e_{x_n}, \mathrm{M}_w^n \, e_{x_1} \otimes \cdots \otimes e_{x_n} \rangle = \prod_{j=1}^n \varrho(x_j).$$

The point process κ is called *Maxwell-Boltzmann process* for the parameters (ϱ, n), and will be denoted by P_ϱ^n.

4.2. **The Bose-Einstein statistical operator.** We start with the following observations: We are given a particle number $n \geq 0$. One can construct by means of $\mathcal{Y}$, as chosen above, a *cons* $\mathcal{Y}_+$ in $\mathcal{H}_+$ and $\mathcal{Y}_-$ in $\mathcal{H}_-$ respectively as follows:

$$\mathcal{Y}_+ = \left\{ e_+(\gamma) = \sqrt{\binom{n}{\gamma}} \cdot \Pi_+ \otimes_{a \in supp\,\gamma} e_a^{\otimes \gamma(a)} | \gamma \in \mathcal{M}_n(X) \right\},$$

$$\mathcal{Y}_- = \left\{ e_-(\gamma) = \sqrt{n!} \cdot \Pi_- \otimes_{a \in supp\,\gamma} e_a | \gamma \in \mathcal{M}_n(X) \right\}.$$

Here the tensor product is taken along a fixed numeration of X, and

$$\mathcal{M}_n^{\cdot}(X) = \left\{ \delta_{x_1} + \cdots + \delta_{x_n} | (x_1, \ldots, x_n) \in Y^{\cdot} \right\}.$$

$Y^{\cdot}$ is the collection of all $y \in Y$ with pairwise distinct components.

We work separately in each of the spaces $\mathcal{H}_\pm$ with these *cons*. In terms of $\mathcal{Y}_\pm$ the projections $\Pi_\pm$ can be written as $\Pi_\pm = \sum_{e \in \mathcal{Y}_\pm} Q_e^\pm$, where the one-dimensional projections are given by $Q_e^\pm = e \circ e$, $e \in \mathcal{Y}_\pm$. Since there is a bijection between $\mathcal{M}_n(X)$ and $\mathcal{Y}_+$ resp. $\mathcal{M}_n^{\cdot}(X)$ and $\mathcal{Y}_-$ we see immediately that (recall that $d = trX$)

$$cd\,\mathcal{Y}_+ = \binom{d+n-1}{n}; \quad cd\,\mathcal{Y}_- = \binom{d}{n}, \quad \text{if} \quad n \leq d; \quad cd\,\mathcal{Y}_- = 0, \quad \text{if} \quad n > d.$$

The *Bose-Einstein statistical operator for* w is given by the conditional Maxwell-Boltzmann statistical operator given the projection Π_+. This is an operator on $\mathcal{H}_+$ defined by $\mathbf{E}_w^n = \frac{1}{tr(\Pi_+ \mathbf{M}_w^n)} \cdot \Pi_+ \mathbf{M}_w^n$. Note that

$$tr(\Pi_+ \mathbf{M}_w^n) = \sum_{\mu \in \mathcal{M}_n(X)} \prod_{a \in X} \varrho(a)^{\mu(a)} > 0,$$

because the ϱ is assumed not to be a Dirac measure.

We choose a *cons* in $\mathcal{H}_+$ for which $\mathbf{E}_w^n$ can be diagonalized, namely $\mathcal{Y}_+$, which is indexed by the finite set $\mathcal{M}_n(X)$. The symmetric group now acts on the basis $\mathcal{Y}_+$ and is trivial, i.e. a singleton consisting of the identity. Thus the associated equivalence relation $\sim$ is given by the identity of elements in $\mathcal{Y}_+$; and the representation of $(\mathcal{Y}_+, \sim)$ is given by $\Gamma = \mathcal{M}_n(X)$ with $r : e_+(\gamma) \longrightarrow \gamma$. Theorem 3.1 then implies that the point process belonging to $\mathbf{E}_w^n$ is given by the following point process in X: For any $\gamma \in \mathcal{M}_n^{\cdot\cdot}(X)$

$$\mathsf{E}_\varrho^n(\gamma) = \frac{1}{\sum_{\mu \in \mathcal{M}_n(X)} \prod_{a \in X} \varrho(a)^{\mu(a)}} \cdot \prod_{a \in X} \varrho(a)^{\gamma(a)}. \tag{4.3}$$

Moreover, the Bose-Einstein statistical operator admits the representation

$$\mathbf{E}_w^n = \sum_{\gamma \in \mathcal{M}_n(X)} \mathsf{E}_\varrho^n(\gamma) \cdot Q_{e_+(\gamma)}^+.$$

We call E_ϱ^n the *Bose-Einstein point process in* X *for the parameters* (n, ϱ)

If ϱ is the uniform distribution on X, and thereby $w = \frac{1}{d} trI$, where I denotes the identity operator on $\mathbb{C}^X$, then

$$\mathbb{E}_w^n := \mathbb{E}_d^n := \frac{1}{\binom{d+n-1}{n}} \cdot \Pi_+,$$

and the Bose-Einstein process is then given by the uniform distribution on $\mathcal{M}_n^{\cdot\cdot}(X)$:

$$\mathsf{E}_d^n(\gamma) = \frac{1}{\binom{d+n-1}{n}}, \quad \gamma \in \mathcal{M}_n^{\cdot\cdot}(X).$$

4.3. **The Fermi-Dirac statistical operator.** For $n \le d = \text{cd}\, X$ the *Fermi-Dirac statistical operator for w* is given by the conditional Maxwell-Boltzmann statistical operator given the projection Π_-. This is a symmetric statistical operator on $\mathcal{H}_-$ defined by

$$\mathbb{D}_w^n = \frac{1}{tr(\Pi_- \mathbb{M}_w^n)} \cdot \Pi_- \mathbb{M}_w^n. \tag{4.4}$$

This operator admits a spectral resolution with respect to the *cons* $\mathcal{Y}_-$ in $\mathcal{H}_-$, where again the basis $(e_x)_{x\in X}$ is coming from the spectral resolution of w and ϱ is the corresponding law not being a Dirac measure. By Theorem 3.1 we then obtain as before the particle picture of the Fermi-Dirac statistical operator: It is given by the following simple point process, called *Fermi-Dirac process for* (n, ϱ) in X:

$$\mathsf{D}_\varrho^n(\gamma) = \frac{1}{Z}\, tr \prod_{a\in X} \varrho(a)^{\gamma(a)}, \qquad \gamma \in \mathcal{M}_n^{\cdot}(X), \textit{ and } 0 \textit{ otherwise}. \tag{4.5}$$

The partition function now is given by $Z = \sum_{\mu\in \mathcal{M}_n^{\cdot}(X)} \prod_{a\in X} \varrho(a)^{\mu(a)}$. Thus D_ϱ^n is the conditional law of E_ϱ^n given $\mathcal{M}_n(X)$, i.e. given that the realization γ of the particle process is simple. We again have a representation of the Fermi-Dirac statistical operator which is parallel to the one for the Bose-Einstein statistical operator, namely

$$\mathbb{D}_w^n = \sum_{\gamma\in\mathcal{M}_n(X)} \mathsf{D}_\varrho^n(\gamma) \cdot Q^-_{\nu_-(\gamma)}.$$

Note that in the special case where $w = \frac{1}{d} \cdot I$, thus ϱ being the uniform distribution on X, the Fermi-Dirac statistical operator is given by

$$\mathbb{D}_w^n := \mathbb{D}_d^n := \frac{1}{\binom{d}{n}} \cdot \Pi_-,$$

and the simple point process by the *Fermi-Dirac process* in X for the parameters (n, d). (Recall that $d = |X|$.)

$$\mathrm{D}_d^n(\gamma) = \frac{1}{\binom{d}{n}}, \qquad \gamma \in \mathcal{M}_n(X).$$

5. The method of the Campbell measure

In the situation of the last section we introduce the occupation number operator and the Campbell operator respectively Campbell measure of a statistical state.

The situation is the same as in the examples: $\mathcal{H}_1 = \mathbb{C}^X$ for some finite X; $(e_x)_{x \in X}$ is a *cons* in $\mathcal{H}_1$. Recall that $\Gamma = \mathcal{M}_n^{\cdot\cdot}(X)$, and $r : (x_1, \ldots, x_n) \longmapsto$ $\delta_{x_1} + \cdots + \delta_{x_n}$. Note that $r = M \circ \iota$, where $\iota : (x_1, \ldots, x_n) \longmapsto e_{x_1} \otimes \cdots \otimes e_{x_n}$ and $M(e_{x_1} \otimes \cdots \otimes e_{x_n}) = \delta_{x_1} + \cdots + \delta_{x_n}$.

We define for $x \in X$ the *occupation number operator in x* on $\mathcal{H} = \mathcal{H}_1^{\otimes n}$ as follows: If I is the identity operator on $\mathcal{H}_1$, let

$$\mathcal{N}_x = \sum_{j=1}^{n} I \otimes \cdots \otimes \underbrace{e_x \circ e_x}_{j} \otimes \cdots \otimes I. \tag{5.1}$$

(In the case $n = 0$ we set $\mathcal{N}_x^{(0)} = 0 \cdot I$.) And, more generally, $\mathcal{N}_B = \sum_{x \in B} \mathcal{N}_x$ the occupation number operator in $B \subset X$. It is evident that $\mathcal{N}_B = \zeta_B(M) I^n$, where for $x_1, \ldots, x_n \in X$ we set

$$\zeta_B(\delta_{x_1} + \cdots + \delta_{x_n}) = (\delta_{x_1} + \cdots + \delta_{x_n})(B).$$

Extend $\mathcal{N}_{(.)}$ linearly to an operator-valued measure on $X \times \mathcal{M}_n^{\cdot\cdot}(X)$ by $\mathcal{N}_h = \zeta_h(M) \cdot I^n$, $h \in F_+(X \times \mathcal{M}_n^{\cdot\cdot}(X))$. Here $\zeta_h(\mu) = \int h(x, \mu)\, \mu(\mathrm{d}x)$, and F_+ denotes the collection of non-negative, measurable functions on the underlying domain. Thus in particular $\mathcal{N}_{B \times C} = \zeta_B(M) \cdot 1_C(M) \cdot I^{\otimes n}$. This shows: Any element $e_y = e_{y_1} \otimes \cdots \otimes e_{y_n}$ of the basis is an eigenvector of $\mathcal{N}_{B \times C}$ with eigenvalue $\zeta_B(M(e_y)) \cdot 1_C(M(e_y))$.

We are now in the position to define the Campbell measure for statistical operators on $\mathcal{H}$. Given a statistical operator W we call $W\mathcal{N}_{(.)}$ on $\mathcal{H}$ the *Campbell operator measure* of W. Its trace $\mathcal{C}_W(.) = tr(W\mathcal{N}_{(.)})$ is called the *Campbell measure of W on $X \times \mathcal{M}_n^{\cdot\cdot}(X)$*. Recall that the Campbell measure of the law κ_W of W is defined by

$$\mathcal{C}_{\kappa_W}(a, \gamma) = \gamma(a)\kappa_W(\gamma), \qquad a \in X, \gamma \in \mathcal{M}_n^{\cdot\cdot}(X).$$

It is obvious that such a Campbell measure is supported by the set $\{(a, \gamma) : \gamma(a) \geq 1\}$. Moreover, we see that the law κ_W of W is determined by its Campbell measure.

Proposition 5.1. *For any statistical operator* $\mathcal{W}$ *on the space* $\mathcal{H}$ *its Campbell measure coincides with the Campbell measure of its law, i.e.* $\mathcal{C}_{\mathcal{W}} = \mathcal{C}_{\kappa_{\mathcal{W}}}$. *The law of* $\mathcal{W}$ *is completely determined by* $\mathcal{C}_{\mathcal{W}}$. *If* $\mathcal{W}$ *is also symmetric then, under the additional irreducibility assumptions of Theorem 2. even* $\mathcal{W}$ *is completely determined by its Campbell measure.*

Proof.

$$tr(\mathcal{W}\mathcal{N}_h) = \sum_y \langle e_y, \mathcal{W}\mathcal{N}_h(e_y)\rangle = \sum_y \zeta_h(r(y))\langle e_y, \mathcal{W}(e_y)\rangle = \sum_\gamma \zeta_h(\gamma) \sum_{y\in Y_\gamma} \langle e_y, \mathcal{W}(e_y)\rangle.$$

The assertion now follows from the definition (2.7) of $\kappa_{\mathcal{W}}$. The remaining statement follows immediately from Theorem 3.1. □

We remark for later use that Proposition 5.1 remains true for statistical operators $\mathcal{W}$ acting on subspaces of $\mathcal{H}$ because the occupation number operators $\mathcal{N}_H$ act on them by restriction.

6. States on Fock spaces and their Campbell measures

The above picture is now extended to systems with a random particle number.

Let X be a finite set of cardinality $d > 1$ and $\mathcal{H}_m = \bigotimes^m \mathbb{C}^X$, $m \geq 0$, with $\mathcal{H}_0 = \mathbb{C}$. The *cons* in $\mathbb{C}$ consists of some unit vector, denoted by $\mathbf{1}$. The direct sum of these Hilbert spaces is the Fock space over $\mathbb{C}^X$, denoted by $\mathbb{H}$. For each m the symmetric group $\mathcal{S}_m$ acts on X^m, and the corresponding unitary representation on $\mathcal{H}_m$ is denoted by $\mathcal{U}_m$. This family of representations gives rise to a unitary operator $\mathcal{U}$ on $\mathbb{H}$, defined by the direct sum $\mathcal{U} = \sum_{m=0}^{\infty} \mathcal{U}_m$. Thus $\mathcal{U}(g)h = \mathcal{U}_m(g)h$, if $g \in \mathcal{S}_m, h \in \mathcal{H}_m$. Given statistical operators $\mathcal{W}_m$ on $\mathcal{H}_m$ and scalars $p_m \geq 0, m \geq 0$, summing up to 1, then the direct sum

$$\mathcal{W} = \sum_{m=0}^{\infty} p_m \mathcal{W}_m \tag{6.1}$$

is a statistical operator on the Fock space $\mathbb{H}$. $\mathcal{W}$ is symmetric if and only if each $\mathcal{W}_m$ has this property. It is obvious that the point process belonging to this statistical operator is given by

$$\kappa_{\mathcal{W}} = \sum_{m=0}^{\infty} p_m \cdot \kappa_{\mathcal{W}_m}. \tag{6.2}$$

The simplest examples are obtained if $\mathcal{W}_m = w^m$ for some given statistical operator w on $\mathcal{H}_1 = \mathbb{C}^X$. Only them will be considered in the sequel in detail. In this framework the occupation number operator is given by the direct sum operator $\mathcal{N}_x = \sum_{m=0}^{\infty} \mathcal{N}_x^{(m)}$ on the Fock space over $\mathbb{C}^X$. Here $\mathcal{N}_x^{(m)}$ is the occupation number operator on $\mathcal{H}_m$

as defined above. And again $\mathcal{N}_B = \zeta_B(M) \cdot I, \qquad B \subset X$, where I now denotes the identity operator on $\mathbb{H}$. Extending $\mathcal{N}$ to an operator valued measure on $X \times \mathcal{M}^{\cdot\cdot}(X)$ as above by $\mathcal{N}_h = \zeta_h(M) \cdot I, \qquad h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$, we are now in the position to define the Campbell measure for statistical operators on $\mathbb{H}$ as we did already in a special situation. Recall that $\zeta_h(\mu) = \int h(x, \mu)\, \mu(dx)$.

Given a statistical operator $\mathcal{W}$ on $\mathbb{H}$ we call $\mathcal{W}\mathcal{N}_{(.)}$ the *Campbell operator measure* of $\mathcal{W}$. By Theorem 2.1 we know that $\mathcal{W}\mathcal{N}_h = \sum_{\gamma \in \Gamma} \zeta_h(\gamma) \kappa_{\mathcal{W}}(\gamma) \cdot \mathcal{W}(\,|\gamma)$, $h \in F_+$. Define $\mathcal{C}_{\mathcal{W}}(.) = tr(\mathcal{W}\mathcal{N}_{(.)})$. This object is called the *Campbell measure of* $\mathcal{W}$. Arguing as above we obtain

Theorem 6.1. *For any statistical operator $\mathcal{W}$ on the Fock space $\mathbb{H}$ one has $\mathcal{C}_{\mathcal{W}} = \mathcal{C}_{\kappa_{\mathcal{W}}}$. Thus the law of $\mathcal{W}$ is completely determined by $\mathcal{C}_{\mathcal{W}}$. If $\mathcal{W}$ is also symmetric then, under the additional irreducibility assumptions of Theorem 3.1, even $\mathcal{W}$ is completely determined by its Campbell measure.*

Consider now the direct sums $\Pi_\pm = \sum_{m=0}^{\infty} \Pi_\pm^{(m)}$, where $\Pi_\pm^{(m)}$ is the orthogonal projection onto the *BE*- resp. *FD symmetric subspace of* $\mathcal{H}_m$. $\Pi_\pm$ is then the orthogonal projection onto the *BE*- resp. *FD symmetric subspace* $\mathbb{H}_\pm$ *of* $\mathbb{H}$. It follows (see [2]) that $\Pi_\pm$ satisfy

$$\mathfrak{U}_\sigma \Pi_\pm = sgn_\pm(\sigma)\Pi_\pm, \quad \sigma \in \mathcal{S}_\infty := \bigcup_{m \geq 0} \mathcal{S}_m. \tag{6.3}$$

We are mainly interested in statistical operators $\mathcal{W}$ living on the symmetric subspaces $\mathbb{H}_\pm$. By this we mean that $\mathcal{W}$ satisfies the conditions $\mathcal{W} = \Pi_\pm \mathcal{W} \Pi_\pm$. In case + this is equivalent to say that $\mathcal{W}$ is Bose-Einstein symmetric, i.e. $\mathfrak{U}_\sigma \mathcal{W} = \mathcal{W}$, $\sigma \in \mathcal{S}_\infty$; and in case − that $\mathcal{W}$ is Fermi-Dirac symmetric, i.e. $\mathfrak{U}_\sigma \mathcal{W} = sgn(\sigma)\mathcal{W}$, $\sigma \in \mathcal{S}_\infty$. Moreover, these conditions imply the symmetry of the statistical operator. (All this can be found in [2])

Theorem 6.1 remains true for statistical operators acting on the Fock spaces $\mathbb{H}_\pm$ because the $\mathcal{N}_B$ act on $\mathbb{H}_\pm$ by restriction. Note also that one obtains by means of a basis in $\mathcal{H}_1$ a basis in the Fòckspaces $\mathbb{H}$, $\mathbb{H}_\pm$ by taking unions $\bigcup_{m \geq 1} \mathcal{Y}^{(m)}, \bigcup_{m \geq 1} \mathcal{Y}_\pm^{(m)}$, augmented in each case by the basis in $\mathcal{H}_0$, which consists of $\mathbf{1}$. Considered as an element of the Fock spaces $\mathbf{1}$ is called ground state and corresponds to the empty particle configuration.

7. States with random particle numbers

The method of second quantization is recalled which permits to lift an operator on a 1-particle space to a Fock space.

7.1. **The method of second quantization.** We recall the method of the so-called second quantization. The idea behind is to lift operators H on $\mathcal{H}$ to one of the Fock spaces. The method goes back to the work of Fock [13], Cook [8] and Berezin [3] (cf. also [5]). If H is a statistical operator on $\mathcal{H}$, one can define a operator H_m on the tensor product $\mathcal{H}_m$ by setting $H_0 \mathbf{1} = 0$ and

$$H_m(e_{a_1} \otimes \cdots \otimes e_{a_m}) = \sum_{j=1}^{m} e_{a_1} \otimes \cdots \otimes H\, e_{a_j} \otimes \cdots \otimes e_{a_m}, \qquad a_1, \ldots, a_m \in X.$$

Denoting by δ_{jk} the Kronecker symbol,

$$H_m = \sum_{j=1}^{m} H^{\delta_{j1}} \otimes \cdots \otimes H^{\delta_{jm}}.$$

The direct sum of the H_m is denoted by

$$d\Gamma(H) = \sum_{m=0}^{\infty} H_m.$$

Note that we used this method already for the operator $e_x \circ e_x$ and obtained in chapter 6 for the operator $d\Gamma(e_x \circ e_x)$ the occupation number operator $\mathbb{N}_x$ on the Fock space over $\mathbb{C}^X$.

If w is a statistical operator on $\mathcal{H}$, the *second quantization of* w then is defined by

$$\Gamma(w) = \sum_{m=0}^{\infty} \frac{1}{m!} w^m.$$

This is an operator on the full Fock space $\mathbb{H}$ having finite trace e.

An important observation is given in terms of such *trace class operators* These are multiples of statistical operators, i.e. operators of the form $w = z\bar{w}^m$, where $z > 0$ and $\bar{w}$ is some statistical operator. In this case

$$\Gamma(w) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \bar{w}^m \ \textit{with} \ tr\,\Gamma(w) = e^z.$$

Lemma 7.1. *Let H be a bounded, self adjoint operator such that $w = \exp(-\beta H)$ is a trace class operator with $\beta \in \mathbb{R}_+$. Then*

$$\exp\,(-\beta\, H)^m = \exp\left(-\beta \sum_{j=1}^{m} H^{\delta_{j1}} \otimes \cdots \otimes H^{\delta_{jm}}\right)$$

Recall here that the left hand side of this equation is given by $e^{-\beta H} \otimes \cdots \otimes e^{-\beta H}$. For a proof of the lemma we refer to Cook [8].

Lemma 7.2. *Let H be a self adjoint operator such that $w = \exp(-\beta H)$ is a trace class operator with $\beta \in \mathbb{R}$. Defining the associated Gibbs state*

$$G = \frac{1}{tr\,\exp(-\beta H)} \exp(-\beta H) \tag{7.1}$$

and $z = cd\,\exp(-\beta H)$ we obtain

$$\Gamma\left(\exp(-\beta H)\right) = \sum_{m=0}^{\infty} \frac{z^m}{m!} G^m \tag{7.2}$$

$\Gamma\left(\exp(-\beta H)\right)$ is trace class with trace e^z.

As a consequence we see that $\mathbf{M}_{zG} := e^{-z}\,\Gamma\left(\exp(-\beta H)\right)$ is a statistical operator on the Fock space.

According to Lemmas 7.1 and 7.2 there are two representations of this operator:

$$\mathbf{M}_{zG} = e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} G^m = e^{-z} \sum_{m=0}^{\infty} \frac{1}{m!} \exp\left(-\beta \sum_{j=1}^{m} H^{\delta_{j1}} \otimes \cdots \otimes H^{\delta_{jm}} \right).$$

To summarize in a slightly modified way: Given some trace class operator $w = z\widetilde{w}$ with corresponding spectral measure $\varrho = z\widetilde{\varrho}$, then w^m has trace $\mathrm{tr}\, w^m = z^m$. In this case the associated second quantization of w is given by

$$\mathbf{M}_w = \frac{1}{\Xi_w} \sum_{m=0}^{\infty} \frac{tr\, w^m}{m!} \cdot \frac{w^m}{cd\, w^m} = \frac{1}{\Xi_w} \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \widetilde{w^m}. \tag{7.3}$$

Ξ_w is the normalizing constant. In this way the trace class operator w is lifted to some symmetric statistical operator on the full Fock space $\mathbb{H}$.

The construction principle behind the *method of second quantization* is: Given m, the trace class operator w^m is normalized to some statistical operator $\widetilde{w^m}$, then weighted by the factor $\frac{cd\, w^m}{m!}$ and summed up; finally it is normalized so that the resulting operator becomes statistical.

One also uses this quantization method in a sligthly generalized form to lift the underlying w on the subspaces $\mathbb{H}_\pm$ and obtain the statistical operators

$$\mathbb{E}_w = \frac{1}{\Xi_w^+} \sum_{m=0}^{\infty} tr(\Pi_+^{(m)} w^m) \cdot \frac{\Pi_+^{(m)} w^m}{\mathrm{tr}(\Pi_+^{(m)} w^m)},$$

$$\mathbb{D}_w = \frac{1}{\Xi_w^-} \sum_{m=0}^{\infty} tr(\Pi_-^{(m)} w^m) \cdot \frac{\Pi_-^{(m)} w^m}{\mathrm{tr}(\Pi_-^{(m)} w^m)}.$$

Note here that the normalizing constants $\Xi_w^\pm = \sum_{m=0}^{\infty} tr(\Pi_\pm^{(m)} \mathbf{M}_w^m)$ are termwise strictly positive and convergent on account of the assumption that ϱ is not a Dirac measure. $\mathbb{E}_w$ is called the *Bose-Einstein operator for w*, $\mathbb{D}_w$ the *Fermi-Dirac operator* *for* w and $p_{\varrho}^{\pm} : m \longmapsto \frac{1}{\Xi^{\pm}} \cdot tr(\Pi_{\pm}^{(m)} \mathbf{M}_w^m)$ the *particle number distribution of* $\mathbb{E}_w$ *or* $\mathbb{E}_w$, *respectively*. Thus the operators $\mathbf{M}_w$, $\mathbb{E}_w$ and $\mathbb{D}_w$ are the second quantizations of w for the different Fock spaces $\mathbb{H}, \mathbb{H}_{\pm}$. One question then is to calculate the correponding laws and to characterize them.

7.2. **Maxwell-Boltzmann statistical operators with Poissonian random particle number.** The Maxwell-Boltzmann statistical operator is described as a solution of an integration-by-parts formula.

We are in the framework of section 4: $\bar{w}$ is a statistical operator on $\mathbb{C}^X$, X being a finite set of cardinality d. As above we choose a cons $e_x, x \in X$, the one coming from the spectral decomposition of $\bar{w}$ with law $\bar{\varrho}$. We are interested in the symmetric statistical operator given by the second quantization of the trace class operator $w = z\bar{w}$

$$\mathbf{M}_w = e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \mathbf{M}_{\bar{w}}^m. \tag{7.4}$$

This is the *Maxwell-Boltzmann statistical operator for* $z, \bar{w}$. We remark that, instead of the Poisson law, any law $(p_m)_m$ can be taken to get some statistical operator. By formula (6.2) the corresponding point process is the Poisson process P_{ϱ} with intensity $\varrho = z\bar{\varrho}$. Thus $\kappa_{\mathbf{M}_w} = P_{\varrho}$, where

$$P_{\varrho}(\varphi) = \kappa_{\mathbf{M}_w}(\varphi) = e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{(x_1,\dots,x_m)\in X^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_m})\ \bar{\varrho}(x_1)\dots\bar{\varrho}(x_m).$$

P_{ϱ} is supported by $\mathcal{M}^{\cdot\cdot}(X) = \bigcup_{n=0}^{\infty} \mathcal{M}_n^{\cdot\cdot}(X)$. Note that this formula is completely parallel to (7.4), namely

$$\kappa_{\mathbf{M}_w} = e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} (L_{\bar{\varrho}})^{*m}, \quad \text{where} \quad L_{\bar{\varrho}} = \sum_{x\in X} \delta_x\ \bar{\varrho}(x),$$

and $*$ denotes convolution of laws.

It is well-known by Mecke's characterization of the Poisson process (see [17]) that P_{ϱ} is characterized as the unique solution Q of the equation

$$\mathcal{C}_{\mathsf{Q}}(h) = \sum_{x\in X} \sum_{\gamma\in\mathcal{M}} h(x, \gamma + \delta_x)\,\varrho(x)\mathsf{Q}(\mathrm{d}\gamma), \quad h \in F_+. \tag{7.5}$$

To say it in another way, Q is the unique solution of the equation $\mathcal{C}_{\mathsf{Q}}(x,\gamma) = \varrho(x)\mathsf{Q}(\gamma - \delta_x)$, $x \in X$, $\gamma \in \mathcal{M}^{\cdot\cdot}(X)$, $\gamma(x) \geq 1$. Another very useful view to equation (7.6) is

$$\mathcal{C}_{\mathsf{Q}} = \mathcal{C}_{L_{\varrho}} \star \mathsf{Q}. \tag{7.6}$$

(Note that the operation $\star$ differs from the convolution operation $*$.) To summarize: The first part of Theorem 6.1 implies

Corollary 7.1. *Let $\tilde{w}$ be a statistical operator on $\mathbb{C}^X$ with spectral law $\tilde{\varrho}$ and $z > 0$ a parameter. Then $\mathbb{M}_{z\tilde{w}}$ is a solution $\mathbb{W}$ of the equation $\mathcal{C}_{\mathbb{W}} = \mathcal{C}_{L_{z\tilde{\varrho}}} \star \kappa_{\mathbb{W}}$.*

This result is a version of Lemma 4.12 of Liebscher [16].

7.3. **Bose-Einstein statistical operators with random particle number.** We consider the *Bose-Einstein statistical operator on the Fock space $\mathbb{H}_+$ with one-particle statistical operator w.* It is clear that $\mathbb{E}_w$ is symmetric and thereby also BE-symmetric. By the results obtained in Sub.4. $\mathbb{E}_w$ is given by the following direct sum

$$(7.7) \qquad \mathbb{E}_w = \frac{1}{\Xi_w^+} \sum_{m=0}^{\infty} tr(\Pi_+^{(m)} \mathbb{M}_w^m) \cdot \sum_{\gamma \in \mathcal{M}_m(X)} \mathsf{E}_\varrho^m(\gamma) \cdot Q_{e_+(\gamma)}^{+,m}.$$

Here we denote now the dependence on the particle number m in $Q_{e_+(\gamma)}^{+,m}$.

Example 7.1. *Consider a statistical operator w with ϱ being the uniform distribution on X, i.e. $\varrho \equiv \frac{1}{d}$. Recall that $d \geq 2$. In this case*

$$tr(\Pi_+^{(m)} \mathbb{M}_w^m) = \binom{d+m-1}{m} \cdot \frac{1}{d^m};$$

and $\Xi_w^+ = \Xi_w^+(d) = \frac{1}{(1-\frac{1}{d})^d}$. Thus the particle number distribution is given by the following negative binomial distribution

$$(7.8) \qquad p_d^+(m) = \binom{d+m-1}{m} \cdot \left(1 - \frac{1}{d}\right)^d \cdot \frac{1}{d^m}.$$

We want to calculate the Campbell measure $\mathcal{C}_{\mathbb{E}_w}$. Thus we first calculate its law: formulas (6.1) and (6.2) immediately imply that

$$(7.9) \qquad \kappa_{\mathbb{E}_w} = \mathsf{E}_\varrho := \frac{1}{\Xi_w^+(d)} \sum_{m=0}^{\infty} tr(\Pi_+^{(m)} \mathbb{M}_w^m) \cdot \mathsf{E}_\varrho^m.$$

This point process is called here the *Bose-Einstein process* and denoted by E_ϱ. This enables us to represent $\mathbb{E}_w$ as

$$\mathbb{E}_w = \sum_{\gamma \in \mathcal{M}(X)} \mathsf{E}_\varrho(\gamma) \cdot Q_{e_+(\gamma)}^+.$$

The Campbell measure of the Bose-Einstein statistical operator $\mathbb{E}_w$ is given by the usual Campbell measure of the Bose-Einstein process. Moreover, $\mathbb{E}_w$ is completely determined by the Campbell measure of its law E_ϱ. So we have to study the Campbell measure $\mathcal{C}_{\mathsf{E}_\varrho}$ which will be done in the Sub.8.

7.4. Fermi-Dirac statistical operators with random particle number. Consider now the *Fermi-Dirac statistical operator on* $\mathbb{H}_-$ *with one-particle statistical operator* w. Analogously to the case of the Bose-Einstein operator it is FD-symmetric and can be represented as

$$\mathbb{D}_w = \frac{1}{\Xi_w^-}\sum_{m=0}^{\infty} \mathrm{tr}(\Pi_-^{(m)}\mathbb{M}_w^m) \cdot \sum_{\gamma\in\mathcal{M}_m(X)} \mathsf{D}_\varrho^m(\gamma)\cdot Q^{-,m}_{e_-(\gamma)}. \tag{7.10}$$

Example 7.2. *Consider a statistical operator* w *with* ϱ *being the uniform distribution on* X, *i.e.* $\varrho \equiv \frac{1}{d}$ *with* $d \geq 2$. *Then*

$$\mathrm{tr}(\Pi_-^{(m)}\mathbb{M}_w^m) = \binom{d}{m}\cdot\frac{1}{d^m};$$

and $\Xi_w^- = \Xi_w^-(d) = (1+\frac{1}{d})^d$. *Thus the particle number distribution is given by the following binomial distribution*

$$p_d^-(m) = \binom{d}{m}\cdot\left(\frac{1}{d+1}\right)^m\cdot\left(1-\frac{1}{d+1}\right)^{d-m}. \tag{7.11}$$

Observe here the symmetry between Bose-Einstein and Fermi-Dirac statistical operators:

$$\Xi_w^-(d) = \Xi_w^+(-d).$$

We want to calculate its Campbell measure $\mathcal{C}_{\mathbb{D}_w}$. Again we calculate first its law: This is given by

$$\kappa_{\mathbb{D}_w} = \mathsf{D}_\varrho := \frac{1}{\Xi_w^-}\sum_{m=0}^{\infty}\mathrm{tr}(\Pi_-^{(m)}\mathbb{M}_w^m)\cdot\mathsf{D}_\varrho^m. \tag{7.12}$$

This point process is called the *Fermi-Dirac process* and is denoted by D_ϱ. Again we have a representation of the form

$$\mathbb{D}_w = \sum_{\gamma\in\mathcal{M}(X)} \mathsf{D}_\varrho(\gamma)\cdot Q^-_{e_-(\gamma)}.$$

Now we have the problem to study $\mathcal{C}_{\mathsf{D}_\varrho}$ and to analyze D_ϱ. This problem will be solved in Bʲb8 by using again the method of the Campbell measure.

8. Characterizations of Bose-Einstein and Fermi-Dirac processes

The question is, what are the properties of the Boson resp. Fermion point processes. The answer is given by means of the method of the Campbell measure. For this aim we derive integration-by-parts formulas for E_ϱ resp. D_ϱ in terms of its Campbell measures. The arguments are only sketched. For the details we refer to [15, 20, 21, 25].

8.1. **Bosons**. Recall that the law ϱ on X is not a Dirac measure. Recall that for a given $\mu \in \mathcal{M}\ (X)$

$$\mathsf{E}_\varrho(\mu) = \frac{1}{\Xi^+_w} \prod_{a \in X} \varrho(a)^{\mu(a)}.$$

If $\mu(X) = m$, this can be written as

$$\mathsf{E}_\varrho(\mu) = \frac{1}{\Xi^+_w} \frac{1}{\binom{m}{\mu}} \mathsf{P}^m_\varrho(\mu).$$

In terms of the Poisson process in X with intensity measure ϱ. which is defined by

$$\mathsf{P}_\varrho(\mu) = e^{-\varrho(X)} \frac{\varrho(X)^m}{m!} \mathsf{P}^m_\varrho(\mu),$$

we obtain a representation of E_ϱ in terms of P_ϱ:

$$\mathsf{E}_\varrho(\mu) = \frac{1}{\Xi^+_w} \frac{1}{\binom{m}{\mu}} \frac{m!}{\varrho(X)^m} e^{\varrho(X)} \mathsf{P}_\varrho(\mu). \tag{8.1}$$

Now we start to calculate the Campbell measure of E_ϱ, i.e.

$$\mathcal{C}_{\mathsf{E}_\varrho}(a, \mu) = \mu(a) \mathsf{E}_\varrho(\mu - \delta_a), \quad \mu(a) \geq 1.$$

Using representation (8.1) in combination with Mecke's characterization (7.5) of the latter yields a recurrence which immediately leads to

Lemma 8.1. *For* $(a, \mu) \in C = \{(a, \mu) : \mu(a) \geq 1\}$

$$\mathcal{C}_{\mathsf{E}_\varrho}(a, \mu) = \sum_{j=1}^{\mu(a)} \varrho(a)^j \cdot \mathsf{E}_\varrho(\mu - j\delta_a). \tag{8.2}$$

Observe that (8.2) is an equation for E_ϱ. To solve this equation we look at it in the following way:

Proposition 8.1. *For any* $h \in F_+$

$$\mathcal{C}_{\mathsf{E}_\varrho}(h) = \sum_{a \in X} \sum_{\gamma \in \mathcal{M}\ (X)} \sum_{j \geq 1} h(a, \gamma + j\delta_a) \varrho(a)^j \lambda(a) \mathsf{E}_\varrho(\gamma). \tag{8.3}$$

Here λ *denotes the counting measure on* X.

Equation (8.3) has the same structure as equation (7.6):

$$\mathcal{C}_{\mathsf{E}_\varrho} = \mathcal{C}_{L^+_\varrho} \star \mathsf{E}_\varrho, \tag{$\Sigma_{L^+_\varrho}$}$$

where the operation $\star$ is a version of a convolution operation defined by the right hand side of (8.3); and L^+_ϱ is given by the following positive measure on $\mathcal{M}^{\cdot}_f(X)$.

$$L^+_\varrho(\varphi) = \sum_{j \geq 1} \sum_{a \in X} \frac{1}{j} \varphi(j\delta_a) \varrho(a)^j, \quad \varphi \in F_+.$$

This implies that E_ϱ is the so called *random KMM measure in X for L_ϱ^+* in the sense of [21].

As Mathias Rafler [25] has shown in full generality E_ϱ then coincides with the *Polya sum process $\mathsf{S}_{\varrho,\lambda}$ for (ϱ,λ)*. This process is by definition a *Papangelou process with the kernel π^+* defined by

$$\pi^+(\mu,a) = \varrho(a)\cdot\big(\lambda(a)+\mu(a)\big), \qquad a\in X, \mu\in\mathcal{M}\ (X). \tag{8.4}$$

And this means that $\mathsf{S}_{\varrho,\lambda}$ is the unique solution S of the following integration by parts formula:

$$\mathcal{C}_{\mathsf{S}}(h) = \sum_\mu\sum_a h(a,\mu+\delta_a)\pi^+(\mu,a)\mathsf{S}(\mu), \qquad h\in F_+. \tag{8.5}$$

This process has been called in [20] the *Pólya sum process for the parameters (ϱ,λ)*. Thus we see that the characteristic properties of the Bose-Einstein process are twofold: It is a KMM process as well as a Pólya sum process.

The argument for the equality of E_ϱ and $\mathsf{S}=\mathsf{S}_{\varrho,\lambda}$ is as follows: If one iterates the last equation (8.5) one obtains for any $N\in\mathbb{N}$

$$\begin{aligned}
\mathcal{C}_{\mathsf{S}}(h) &= \sum_\mu\sum_a h(a,\mu+\delta_a)\varrho(a)(1+\mu(a))\mathsf{S}(\mu)\\
&= \sum_{j=1}^{N}\sum_\mu\sum_a \varrho(a)^j h(a,\mu+j\delta_a)\mathsf{S}(\mu) +\\
&+\sum_\mu\sum_a \varrho(a)^N h(a,\mu+N\delta_a)\mu(a)\mathsf{S}(\mu)\\
&\longrightarrow_{N\to+\infty} \sum_{j\geq 1}\sum_\mu\sum_a \varrho(a)^j h(a,\mu+j\delta_a)\mathsf{S}(\mu).
\end{aligned}$$

Here we used again that ϱ is not a Dirac measure and also that S is of first order. This shows that S solves equation (8.3) or equivalently $(\Sigma_{L_\varrho^+})$. One can show that this equation has only one solution. (Cf. [21]) To summarize we obtained the

Proposition 8.2. *Given a probability ϱ on X which is not a Dirac measure then the Bose-Einstein process E_ϱ coincides with the random KMM measure in X for L_ϱ^+ as well as the Polya sum process $\mathsf{S}_{\varrho,\lambda}$ for the parameters (ϱ,λ). Moreover, this process is infinitely divisible and uniquely determined as a solution of the integration-by-parts formula (8.3).*

We know also from [20] that the property of E_ϱ being a Papangelou process for π^+ allows to calculate explicitly its particle number distribution. In the case where ϱ is the uniform distribution on X this coincides with p_ϱ^+ which we calculated above by completely different quantum mechanical methods. This implies that the point process in this case is *of first order*, i.e. the mean particle number is finite. (All this can be found in [20].) This shows that E_ϱ has all properties of an ideal gas.

Moreover, equation $\left(\Sigma_{L_+^*}\right)$ implies that E_ϱ is a socalled permanental process. This means that its reduced density matrix has a permanental structure. A proof based on $\left(\Sigma_{L_+^*}\right)$ can be found in [21, 15] and the references therein.

Finally, using the above developed method of the Campbell measure, in particular Theorem 4, we obtain immediately characterizations of the Bose-Einstein statistical operator for w: The fact that $\kappa_{\mathbb{E}_w} = \mathsf{E}_\varrho$ solves equation $\left(\Sigma_{L_+^*}\right)$ immediately implies

Theorem 8.1. *Let w be a statistical operator on $\mathbb{C}^X$ with spectral law ϱ which is not a Dirac measure. A symmetric statistical operator $\mathbb{W}$ on the Fock space $\mathbb{H}_+$, admitting a spectral resolution with respect to $\mathcal{Y}_+$, coincides with $\mathbb{E}_w$ iff it is a solution of equation $\mathcal{C}_{\mathbb{W}} = \mathcal{C}_{L_+^*} \star \kappa_{\mathbb{W}}$.*

Moreover, $\kappa_{\mathbb{E}_w} = \mathsf{E}_\varrho$ being also a solution to equation (8.5), implies

Theorem 8.2. *Under the assumptions of Theorem 8.1, $\mathbb{W}$ coincides with $\mathbb{E}_w$ iff it is the solution of the equation*

$$\mathcal{C}_{\mathbb{W}}\, h = \sum_{(x,\gamma)} h(x, \gamma + \delta_x)\, \pi^+(\gamma, x) \kappa_{\mathbb{W}}(\gamma), \quad h \in F_+. \tag{8.6}$$

Statistical operators $\mathbb{W}$ which solve equation (8.6) can be called Pólya sum statistical operators specified by π_+.

8.2. Fermions. The Campbell measure of D_ϱ is concentrated on C and given there by

$$\mathcal{C}_{\mathsf{D}_\varrho}(a, \mu) = \varrho(a) \cdot \mathsf{D}_\varrho(\mu - \delta_a), \quad \mu(a) = 1.$$

This implies that D_ϱ is a Papangelou process for the kernel

$$\pi^-(a, \mu) = \varrho(a) \cdot \Big(\lambda(a) - \mu(a)\Big), \qquad \mu(a) \leq 1;$$

(and $\pi^- \equiv 0$ else.) Recall here that λ denotes the counting measure. In the terminology of [20], D_ϱ is a *Pólya difference process for* (λ, ϱ). As for Bosons the distribution of the particle number is explicitly known, and the process is of first order. Again D_ϱ is completely determined by its kernel π_-. D_ϱ is a simple process, i.e. concentrated on $\mathcal{M}^\cdot(X)$, and thus respects Pauli's exclusion principle. Furthermore, D_ϱ has independent increments. Thus it has all properties of an ideal gas. (For more details we refer to [20].) We observe here that the same reasoning we did above for the Papangelou process E_ϱ yields that

Proposition 8.3. *The Papangelou process D_ϱ is the unique solution of the following equation for simple point processes* Q.

$$\mathcal{C}_Q(h) = \sum_{j=1}^{+\infty}(-1)^{j-1}\sum_{a,\mu}\varrho(a)^j h(a,\mu+j\delta_a)\ Q(\mu), \quad h \in F_+. \tag{8.7}$$

(The proof is exactly the same as above.) Again equation (8.10), which has D_ϱ as a unique solution, is of the form

$$\left(\Sigma_{L_\varrho^-}\right) \qquad C_Q = \mathcal{C}_{L_\varrho^-} \star Q,$$

but now for the signed measure

$$L_\varrho^-(\varphi) = \sum_{j\geq 1}\sum_{a\in X}\frac{(-1)^{j-1}}{j}\varphi(j\delta_a)\varrho(a)^j, \quad \varphi \in F_+.$$

In this case one can show (see [21, 15]) that $\left(\Sigma_{L_\varrho^-}\right)$ implies that D_ϱ is a so called determinantal process.

As above for Bosons we obtain a characterization of symmetric statistical operators for Fermions: A symmetric statistical operator W, admitting a spectral resolution with respect to $\mathcal{Y}_-$, coincides with $\mathbb{D}_w$ iff it is the unique solution of the equation $\mathcal{C}_W = \mathcal{C}_{L_w^-} \star \kappa_W$; or equivalently, iff it is the solution of the equation

$$\mathcal{C}_W h = \sum_{(x,\gamma)} h(\gamma+\delta_x)\,\pi_-(\gamma,x)\kappa_W(\gamma), \quad h \in F_+.$$

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ТОНКИЕ СВОЙСТВА ФУНКЦИЙ ИЗ КЛАССОВ ХАЙЛАША-СОБОЛЕВА M_α^p ПРИ $p > 0$, II. АППРОКСИМАЦИЯ ЛУЗИНА

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Аннотация. В работе изучается аппроксимация Лузина функций из классов Хайлаша–Соболева $M_\alpha^p(X)$ при $p > 0$. Доказано, что для $f \in M_\alpha^p(X)$ и любого $\varepsilon > 0$ существуют открытое множество $O_\varepsilon \subset X$, мера которого меньше ε (в качестве меры можно взять соответствующие емкость или вместимость Хаусдорфа), и приближающая функция f_ε такие, что $f = f_\varepsilon$ на $X \setminus O_\varepsilon$. При этом исправляющая функция f_ε является регулярной (принадлежит исходному пространству $M_\alpha^p(X)$ классу и является локально гельдеровской) и приближает исходную функцию в метрике пространства $M_\alpha^p(X)$.

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Ключевые слова: Метрическое пространство с мерой, условие удвоения, класс Соболева, аппроксимация Лузина, емкость, внешняя мера, мера и размерность Хаусдорфа.

1. Введение

Наша работа является непосредственным продолжением работы [1]. Мы используем результаты из [1] для изучения свойства аппроксимации в смысле Лузина для классов Хайлаша–Соболева M_α^p при $p > 0$. При этом мы полностью придерживаемся обозначений и определений из [1].

Теорема Лузина утверждает, что любая измеримая на $\mathbb{R}^n$ функция f обладает C-свойством — f является непрерывной, если пренебречь множеством сколь угодно малой меры. Точнее, для любой измеримой на $\mathbb{R}^n$ функции f и любого $\varepsilon > 0$ существуют такие функция $\varphi \in C(\mathbb{R}^n)$ и открытое множество $O_\varepsilon \subset \mathbb{R}^n$, для которых

$$f(x) = \varphi(x) \quad \text{при} \quad x \in \mathbb{R}^n \setminus O_\varepsilon, \quad \mu(O_\varepsilon) < \varepsilon$$

(μ — мера Лебега на $\mathbb{R}^n$).

Если функция f является более регулярной в том или ином смысле, то исправляющая функция φ может обладать дополнительными свойствами гладкости и аппроксимирующими свойствами.

Для классов Хайлаша–Соболева M^p_α при $p \geq 1$ такие вопросы исследовались в [2]–[5]. Мы распространим результаты этих работ на случай $p > 0$.

2. Основная теорема

Приведем определения, необходимые для формулировки основного результата. Пусть (X, d, μ) — метрическое пространство с регулярной борелевской мерой μ и метрикой d, $B(x, r) = \{y \in X : d(x, y) < r\}$ — шар с центром в точке $x \in X$, радиуса $r > 0$.

Будем предполагать, что мера μ удовлетворяет условию удвоения с показателем $\gamma > 0$, т.е. для некоторой постоянной a_μ выполнено неравенство

$$\mu(B(x, R)) \leq a_\mu \left(\frac{R}{r}\right)^\gamma \mu(B(x, r)), \quad x \in X, \ 0 < r \leq R.$$

Для шара $B \subset X$ обозначаем r_B и x_B соответственно его радиус и центр, кроме того, λB — шар, концентрический с B, радиуса λr_B. Кроме того, пусть

$$f_B = \fint_B f \, d\mu = \frac{1}{\mu(B)} \int_B f \, d\mu$$

Через c всюду обозначаем различные положительные постоянные, зависящие, возможно, от определенных параметров, но эти зависимости для нас несущественны. Кроме того, запись $A \lesssim B$ всегда будет означать, что $A \leq cB$.

Неотрицательную функцию ν, определенную на σ-алгебре борелевских множеств из X, будем называть внешней мерой, если она монотонна и субаддитивна с некоторой постоянной a_ν, то есть для любой последовательности борелевских множеств E_k выполнено неравенство

$$\nu\left(\bigcup_k E_k\right) \leq a_\nu \sum_k \nu(E_k).$$

Кроме того, внешнюю меру будем называть регулярной в нуле, если для любого множества $E \subset X$ с $\nu(E) = 0$ и для любого $\varepsilon > 0$ существует открытое множество $O \supset E$, для которого $\nu(O) < \varepsilon$.

Пусть задана возрастающая функция $h : (0, 1] \to (0, 1]$, $h(+0) = 0$. Такие функции будем называть измеряющими. Мы будем использовать следующее условие, связывающее исходную меру μ и внешнюю меру ν: существует такая постоянная c_ν, что выполнено неравенство

$$\nu(B) \leq c_\nu \frac{\mu(B)}{h(r_B)} \quad \text{для всех шаров} \quad B \subset X, \ r_B \leq 1. \tag{2.1}$$

Напомним, что для измеряющей функции h и $0 < R \le 1$ классическая и модифицированная (h, R)-вместимость Хаусдорфа множества $E \subset X$ вводится как

$$H_R^h(E) = \inf \left\{ \sum_{i=1}^{\infty} h(r_i) : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \quad r_i < R \right\},$$

$$\mathcal{H}_R^h(E) = \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{h(r_i)} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \quad r_i < R \right\}$$

соответственно (точная нижняя грань берется по всевозможным покрытиям множества E счетными семействами шаров). Величины

$$H^h(E) = \lim_{R \to +0} H_R^h(E), \quad \mathcal{H}^h(E) = \lim_{R \to +0} \mathcal{H}_R^h(E)$$

называются классической и модифицированной h-мерой Хаусдорфа для E соответственно. Для $h(t) = t^\alpha$, $\alpha > 0$, пишем H^α и $\mathcal{H}^\alpha$ вместо H^{t^α} и $\mathcal{H}^{t^\alpha}$. В случае классических мер также можно определить размерность Хаусдорфа

$$\dim_H E = \inf \{ s : H_1^s(E) = 0 \}.$$

Введем класс Хайлаша–Соболева $M_\alpha^p(X)$, $0 < p < \infty$, $\alpha > 0$, как множество

$$M_\alpha^p(X) = \{ f \in L^p(X) : D_\alpha[f] \cap L^p(X) \ne \varnothing \},$$

$$\|f\|_{M_\alpha^p(X)} = \|f\|_{L^p(X)} + \inf \left\{ \|g\|_{L^p(X)} : g \in D_\alpha[f] \cap L^p(X) \right\},$$

где через $D_\alpha[f]$ обозначен класс всех неотрицательных μ-измеримых функций g, для каждой из которых существует такое множество $E \subset X$, $\mu(E) = 0$, что

$$|f(x) - f(y)| \le \left[d(x, y)\right]^\alpha [g(x) + g(y)], \quad x, y \in X \setminus E.$$

Классы $M_\alpha^p(X)$ порождают емкости

$$\mathrm{Cap}_{\alpha,p}(E) = \inf \left\{ \|f\|_{M_\alpha^p(X)}^p : f \in M_\alpha^p(X), f \ge 1 \text{ в окрестности } E \subset X \right\}.$$

Наконец, для $\alpha > 0$ определим классы Гельдера

$$H_\alpha(X) = \left\{ \phi \in C(X) : \|\phi\|_{H_\alpha(X)} = \sup_{x \ne y} \frac{|\phi(x) - \phi(y)|}{[d(x, y)]^\alpha} < +\infty \right\}.$$

Наш основной результат формулируется следующим образом.

Теорема 2.1. *Пусть $0 < \beta \le \alpha \le 1$, $0 < p < \gamma/\alpha$ и задана функция $f \in M_\alpha^p(X)$. Пусть также задана внешняя мера ν, регулярная в нуле и удовлетворяющая условию (2.1) с функцией $h(t) = t^{(\alpha-\beta)p}$. Тогда для любого $\varepsilon > 0$ существуют функция f_ε и открытое множество $O \subset X$, такие, что*

1) $\nu(O) < \varepsilon$.

2) $f = f_\varepsilon$ *на* $X \setminus O$,

3) $f_\varepsilon \in M^p_\alpha(X)$ *и* $f_\varepsilon \in H_\beta(B)$ *для любого шара* $B \subset X$,

4) $\|f - f_\varepsilon\|_{M^p_\alpha(X)} < \varepsilon$.

В качестве примеров внешних мер, удовлетворяющих условию теоремы, можно взять $\nu = \mathrm{Cap}_{(\alpha-\beta),p}$ и $\nu = \mathcal{H}_1^{(\alpha-\beta)p}$, а также $\nu = H_1^{\gamma-(\alpha-\beta)p}$ (при условии $\mu(X) < \infty$).

При $\beta = \alpha = 1$, $p > 1$ подобный результат был ранее получен П.Хайлашем в [2], где вместо 1) утверждалось, что $\mu(O) < \varepsilon$, а в 3) было $f_\varepsilon \in H_1(X)$. Случай $\beta < \alpha = 1$ теоремы 2.1 существенно сложнее, он был изучен в работе [3] при $p > 1$ и в [4] при $p = 1$. В работе [5] теорема 2.1 была доказана для $p > 1$, $0 < \alpha \leq 1$.

Следствие 2.1. Пусть $0 < \beta \leq \alpha \leq 1$, $0 < p < \gamma/\alpha$ *и задана функция* $f \in M^p_\alpha(X)$. *Тогда для любого* $\varepsilon > 0$ *существуют функция* f_ε *и открытое множество* $O \subset X$, *такие, что*

1) $\mathrm{Cap}_{(\alpha-\beta),p}(O) < \varepsilon$, $H_1^{\gamma-(\alpha-\beta)p}(O) < \varepsilon$, $\mathcal{H}_1^{(\alpha-\beta)p}(O) < \varepsilon$

2) $f = f_\varepsilon$ *на* $X \setminus O$,

3) $f_\varepsilon \in M^p_\alpha(X)$ *и* $f_\varepsilon \in H_\beta(B)$ *для любого шара* $B \subset X$,

4) $\|f - f_\varepsilon\|_{M^p_\alpha(X)} < \varepsilon$.

Во время подготовки нашей работы к печати появился препринт [6], в котором также доказано утверждение следствия 2.1 для модифицированной вместимости Хаусдорфа. Методы [6] отличны от наших.

Результаты нашей работы докладывались на семинаре "Функциональные пространства" Университета Фридриха Шиллера (Иена, Германия, 19 декабря 2014 г. и 3 декабря 2015 г.), на Международной конференции "Функциональные пространства и теория аппроксимации функций", посвященной 110-летию со дня рождения академика С.М. Никольского (Москва, 28 мая 2015 г., и на Международной конференции «Harmonic Analysis and Approximations, VI» (Цахкадзор, Армения, 13 сентября 2015 г.).

При доказательстве теоремы 2.1 мы следуем схеме работ [3, 5]. Работу с несуммируемыми функциями обеспечивают результаты работ [1] и [9]. По существу, задача разбивается на две части. С одной стороны нам нужны квалифицированные оценки массивности лебеговых множеств некоторых максимальных функций, а с другой — надо уметь продолжать функции с этих множеств, сохраняя определенные условия гладкости. Для оценки исключительных множеств будем использовать результаты из работ [10] и [1]. При этом важно использование некоторого аппроксимирующего аппарата — при $p \geq 1$ эту роль выполняют средние Стеклова, для $p \in (0,1)$ эта роль передается наилучшим L^p-приближениям постоянными на шарах $B \subset X$ [1]. Для продолжения функций применяется аналог конструкции Уитни, предложенный в [3] при рассмотрении аналогичной задачи для частного случая $\alpha = 1$.

3. Вспомогательные утверждения

Для доказательства основной теоремы нам понадобится ряд результатов, большинство из которых известны при $p > 1$.

Лемма 3.1 ([1], лемма 9). *Пусть $E \subset X$, $0 < \alpha \leq 1$, $\gamma > \alpha p$. Тогда:*

1) емкость $\mathrm{Cap}_{\alpha,p}$ является внешней мерой и

$$\mathrm{Cap}_{\alpha,p}(E) = \inf\left\{\mathrm{Cap}_{\alpha,p}(O) : E \subset O, O - \text{открыто}\right\}.$$

2) $\mathrm{Cap}_{\alpha,p}(B(x,r)) \lesssim r^{-\alpha p}\mu(B(x,r))$ для $x \in X$, $0 < r \leq 1$,

3) при $0 < \beta \leq \alpha$ из $\mathrm{Cap}_{\alpha,p}(E) = 0$ следует $\mathrm{Cap}_{\beta,p}(E) = 0$.

Пусть

$$A_p(f,B) = \inf_I \left(\fint_B |f(y) - I|^p \, d\mu(y) \right)^{1/p}, \quad p > 0, \tag{3.1}$$

тогда существует число $I_B^{(p)} f$ ([9, лемма 3]), реализующее точную нижнюю границу в (3.1).

Техническим средством для доказательства основной теоремы является максимальный оператор $A_{\alpha,R}^{(p)} f$. Определим его следующим образом

$$A_{\alpha,R}^{(p)} f(x) = \sup_{B \ni x, r_B < R} r_B^{-\alpha} A_p(f,B),$$

При $R = \infty$ вместо $A_{\alpha,\infty}^{(p)} f$ будем писать $A_\alpha^{(p)} f$.

Лемма 3.2 ([10]). *Пусть $p > 0$, $0 \leq \beta < \alpha$, а мера μ и внешняя мера ν связаны условием (2.1) с функцией $h(t) = t^{(\alpha-\beta)p}$. Тогда для $f \in L^p_{\mathrm{loc}}(X)$ справедливо неравенство*

$$\int_0^\infty \lambda^{p-1} \nu\{A_\beta^{(p)} f > \lambda\} \, d\lambda \lesssim \|A_\alpha^{(p)} f\|^p_{L^p(X)}.$$

Лемма 3.3 ([9], лемма 3). *Пусть $f \in L^p(X)$, $p > 0$, $B_1, B_2 \subset X$ — шары, причем $r_{B_1} < r_{B_2}$ и $B_1 \subset B_2$. Тогда*

$$|I^{(p)}_{B_1} f - I^{(p)}_{B_2} f| \lesssim A_p(f, B_1) + \left(\frac{r_{B_2}}{r_{B_1}}\right)^{\frac{Q}{p}} A_p(f, B_2).$$

Лемма 3.4. *Пусть $\beta, p > 0$, $f \in L^p(X)$ и точка $x \in X$ такова, что*

$$f(x) = \lim_{r \to +0} I^{(p)}_{B(x,r)} f. \tag{3.2}$$

Тогда

$$|f(x) - I^{(p)}_{B(x,r)} f| \lesssim r^\beta A^{(p)}_\beta f(x).$$

Доказательство. Пусть для краткости $I_k = I^{(p)}_{B(x, 2^{-k} r)} f$. Тогда

$$|f(x) - I_0| = \left| \lim_{n \to \infty} \sum_{k=0}^{n-1} [I_k - I_{k+1}] \right| \leq \sum_{k=0}^{\infty} |I_k - I_{k+1}|.$$

Далее используем лемму 3.3 для оценки каждого слагаемого в сумме

$$\sum_{k=0}^{\infty} |I_k - I_{k+1}| \lesssim \sum_{k=0}^{\infty} A_p(f, B(x, 2^{-k} r)) \lesssim A^{(p)}_\beta f(x) \sum_{k=0}^{\infty} (2^{-k} r)^\beta.$$

Утверждение доказано.

Лемма 3.5. [11, лемма 2.5] *Если $f \in M^p_\alpha(X)$, $\phi \in H_\alpha(X)$ и ограничена, то $f\phi \in M^p_\alpha(X)$. Кроме того, если $E \subset X$ и $\phi(x) = 0$ при $x \in X \setminus E$, то для любой функции $g \in D_\alpha(f) \cap L^p$*

$$\left(g\|\phi\|_\infty + |f| \cdot \|\phi\|_{H^\alpha(X)}\right) \chi_E \in D_\alpha(f\phi) \cap L^p.$$

Основным техническим средством для построения разбиений единицы и продолжения функции с сохранением гладкости является конструкция, изложенная в следующих двух леммах.

Лемма 3.6. [3, лемма 5.7] *Пусть $O \subset X$ — открытое множество, $O \neq X$, $\mu(O) < \infty$. Для заданного $C \geq 2$ обозначим $r(x) = \frac{\operatorname{dist}(x, X \setminus O)}{2C}$. Тогда существует $N \geq 1$ и последовательность $\{x_i\}$ точек из X такие, что*

1) шары $B(x_i, r_i/4)$ попарно не пересекаются, $r_i = r(x_i)$,

2) $\bigcup_{i=1}^\infty B(x_i, r_i) = O$,

3) $B(x_i, Cr_i) \subset O$,

4) если $x \in B(x_i, Cr_i)$, то $Cr_i \leq \operatorname{dist}(x, X \setminus O) \leq 3Cr_i$,

5) для любого i существует такое $y_i \in X \setminus O$, что $d(x_i, y_i) < 3Cr_i$,

6) $\sum_{i=1}^\infty \chi_{B(x_i, Cr_i)} \leq N$.

Лемма 3.7. [7, лемма 2.16] *Пусть $0 < \alpha \le 1$, O — открытое множество, $\{B(x_i, r_i)\}$ — покрытие O шарами из леммы 3.6 для $C = 5$. Тогда существует такая последовательность функций ϕ_i, что*

1) $\operatorname{supp} \phi_i \subset B(x_i, 2r_i)$, $0 \le \varphi_i(x) \le 1$,

2) $|\phi_i(x) - \varphi_i(y)| \le c r_i^{-\alpha} [d(x, y)]^\alpha$,

3) $\sum_{i=1}^\infty \varphi_i(x) = \chi_O(x)$.

4. Доказательство основной теоремы

Сначала сделаем дополнительное предположение – для некоторого $x_0 \in X$

$$\operatorname{supp} f \subset B(x_0, 1) \equiv B_0 \tag{4.1}$$

Доказательство утверждения 1). В силу [1, теорема 3] $\nu(\Lambda) = 0$, где Λ — множество точек $x \in X$, в которых не выполнено (3.2). Поэтому для $\varepsilon > 0$ существует такое открытое множество $L \supset \Lambda$, что $\nu(L) < \varepsilon$. Для $\lambda > 0$ обозначим

$$E_\lambda = \left\{ x \in X : \mathcal{A}_\alpha^{(p)} f(x) > \lambda \right\}.$$

Положим $O = E_\lambda \cup L$ и покажем, что при достаточно большом λ множество O обладает необходимыми свойствами. Легко видеть, что O открыто и $O \subset 2B_0$.

Из леммы 3.2

$$\int_0^\infty \lambda^{p-1} \nu(E_\lambda)\, d\lambda \lesssim \|\mathcal{A}_\alpha^{(p)} f\|_{L^p(X)}^p < \infty,$$

откуда следует, что $\nu(E_\lambda) \to 0$ при $\lambda \to +\infty$, кроме того, очевидно, что $\mu(E_\lambda) \to 0$ при $\lambda \to +\infty$.

Таким образом, утверждение 1) теоремы выполнено. Дополнительно выберем $\lambda > 0$ настолько большим, чтобы

$$\int_O |f|^p \, d\mu + \int_O [\mathcal{A}_\alpha^{(p)} f]^p \, d\mu < \varepsilon. \tag{4.2}$$

Доказательство утверждения 2). Пусть $\{B(x_i, r_i)\}$ — покрытие множества O из леммы 3.6 для $C = 5$. Тогда, применяя лемму 3.7, найдем набор функций $\{\phi_i\}_{i=1}^\infty$ таких, что

$$\operatorname{supp} \phi_i \subset B(x_i, 2r_i), \ 0 \le \phi_i(x) \le 1,$$

$$|\varphi_i(x) - \phi_i(y)| \lesssim r_i^{-\alpha} [d(x, y)]^\alpha, \quad \sum_{i=1}^\infty \phi_i(x) = \chi_O(x).$$

Определим функцию f_ε равенством

$$f_\varepsilon(x) = \begin{cases} f(x), & x \in X \setminus O, \\ \sum\limits_{i=1}^\infty \phi_i(x) I_{B(x_i, 2r_i)}^{(p)} f, & x \in O \end{cases} \tag{4.3}$$

Утверждение 2 теоремы следует непосредственно из этого определения.

Доказательство утверждения 3). Сперва проведем вспомогательное рассуждение. Пусть $x \in O$, тогда существует такая точка $x^* \in X \setminus O$, что $d(x, x^*) \leq 2\,\mathrm{dist}(x, X \setminus O)$. Поэтому

$$|f_\varepsilon(x^*) - f_\varepsilon(x)| = \left|\sum_{i=1}^{\infty} \phi_i(x)\left[f(x^*) - I^{(p)}_{B(x_i, 2r_i)} f\right]\right| \leq \sum_{i \in I_x} |f(x^*) - I^{(p)}_{B(x_i, 2r_i)} f|,$$

где $I_x = \{i : x \in \operatorname{supp} \phi_i\}$.

Заметим, что в точке $x^* \in X \setminus O$ выполнено соотношение (3.2), и $B(x_i, 2r_i) \subset B(x^*, 40r_i)$ для любого $i \in I_x$. Поэтому в силу лемм 3.3 и 3.4

$$|f(x^*) - I^{(p)}_{B(x_i, 2r_i)} f| \leq |f(x^*) - I^{(p)}_{B(x^*, 40r_i)} f| + |I^{(p)}_{B(x_i, 2r_i)} f - I^{(p)}_{B(x^*, 40r_i)} f| \lesssim r_i^\beta A^{(p)}_\beta f(x^*).$$

Так как в I_x не более N слагаемых и $A^{(p)}_\beta f(x^*) \leq \lambda$ (так как $x^* \in X \setminus O$), то

$$|f_\varepsilon(x^*) - f_\varepsilon(x)| \lesssim \sum_{i \in I_x} r_i^\beta A^{(p)}_\beta f(x^*) \lesssim [d(x, x^*)]^\beta A^{(p)}_\beta f(x^*) \lesssim \lambda\, [d(x, x^*)]^\beta. \tag{4.4}$$

Дальнейшее доказательство того, что $f_\varepsilon \in H^\beta(X)$, проводится точно так же, как и при $p > 1$ в работе [5], с той лишь разницей, что операторы $\mathcal{S}_\beta$ заменяются на операторы $A^{(p)}_\beta$. Для полноты повторим это рассуждение здесь. Рассмотрим три возможных случая расположения точек $x, y \in X$.

Случай 1. Пусть $x, y \in X \setminus O$. Запишем очевидное неравенство

$$|f_\varepsilon(x) - f_\varepsilon(y)| \leq |f(y) - I^{(p)}_{B(y, d(x,y))} f| + |f(x) - I^{(p)}_{B(x, 2d(x,y))} f| +$$
$$+ |I^{(p)}_{B(y, d(x,y))} f - I^{(p)}_{B(x, 2d(x,y))} f|.$$

Из леммы 3.4 следует, что первые два слагаемых мажорируются величиной

$$c\,[d(x, y)]^\beta \left[A^{(p)}_\beta f(x) + A^{(p)}_\beta f(y)\right].$$

Третье слагаемое также оценивается сверху этой же величиной в силу леммы 3.3. Таким образом, для $x, y \in X \setminus O$ выполнено

$$|f_\varepsilon(x) - f_\varepsilon(y)| \lesssim [d(x, y)]^\beta \left[A^{(p)}_\beta f(x) + A^{(p)}_\beta f(y)\right] \lesssim \lambda\, [d(x, y)]^\beta \tag{4.5}$$

Случай 2. Пусть $x, y \in O$. Введем обозначение

$$d_0 = \max\{\mathrm{dist}\,(x, X \setminus O), \mathrm{dist}\,(y, X \setminus O)\}. \tag{4.6}$$

Если $d(x, y) > d_0$, то подберем точки $x^*, y^* \in X \setminus O$ так, чтобы $d(x, x^*) \leq 2\,\mathrm{dist}(x, X \setminus O)$ и $d(y, y^*) \leq 2\,\mathrm{dist}(y, X \setminus O)$. Запишем очевидное неравенство

$$|f_\varepsilon(x) - f_\varepsilon(y)| \leq |f_\varepsilon(x) - f_\varepsilon(x^*)| + |f_\varepsilon(y) - f_\varepsilon(y^*)| + |f_\varepsilon(x^*) - f_\varepsilon(y^*)|$$

Оценим первые два слагаемых с помощью (4.4), а третье, используя (4.5). Получим

$$|f_\varepsilon(x) - f_\varepsilon(y)| \lesssim \lambda\, [d(x,y)]^\beta.$$

Пусть теперь $d(x,y) \le d_0$. Как и прежде, выберем $x^* \in X \setminus O$ так, чтобы $d(x,x^*) < 2\,\mathrm{dist}(x, X\setminus O)$.

Поэтому, учитывая пункт 4) леммы 3.6, получаем

$$|f_\varepsilon(x) - f_\varepsilon(y)| = \left|\sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_i(y)] \left[I^{(p)}_{B(x_i,2r_i)} f - f(x^*)\right]\right| \lesssim$$

$$\lesssim [d(x,y)]^\alpha \sum_{I_x \cup I_y} \frac{1}{r_i^\alpha} |I^{(p)}_{B(x_i,2r_i)} f - f(x^*)|.$$

Заметим, что для $i \in I_x \cup I_y$ справедливо включение $B(x_i, 2r_i) \subset B(x^*, 100r_i)$, следовательно, в силу лемм 3.3 и 3.4

$$|f(x^*) - I^{(p)}_{B(x_i,2r_i)} f| \le |f(x^*) - I^{(p)}_{B(x^*,100r_i)} f| +$$

$$+ |I^{(p)}_{B(x_i,2r_i)} f - I^{(p)}_{B(x^*,100r_i)} f| \lesssim r_i^\beta A^{(p)}_\beta f(x^*).$$

Таким образом, мы приходим к неравенству

$$|f_\varepsilon(x) - f_\varepsilon(y)| \lesssim [d(x,y)]^\beta \sum_{I_x \cup I_y} \frac{[d(x,y)]^{\alpha-\beta}}{r_i^{\alpha-\beta}} A^{(p)}_\beta f(x^*) \lesssim \lambda\, [d(x,y)]^\beta.$$

Случай 3. Пусть $x \in O$, а $y \in X \setminus O$. Выберем $x^* \in X \setminus O$ так, чтобы $d(x,x^*) \le 2\,\mathrm{dist}(x, X \setminus O)$. Тогда из (4.4) и уже доказанного пункта 1 следует

$$|f_\varepsilon(x) - f_\varepsilon(y)| \le |f_\varepsilon(x) - f_\varepsilon(x^*)| + |f_\varepsilon(y) - f_\varepsilon(x^*)| \lesssim \lambda\, [d(x,y)]^\beta$$

Таким образом, показано, что $f_\varepsilon \in H_\beta(X)$, если носитель функции f сосредоточен в единичном шаре.

Осталось показать, что $f_\varepsilon \in M^p_\alpha(X)$. Докажем сначала, что $f_\varepsilon \in L^p(X)$. Для этого оценим сверху $|I^{(p)}_{B(x_i,2r_i)} f|$: из (3.1) получаем

$$|I^{(p)}_{B(x_i,2r_i)} f|^p \lesssim A^p_p(f, B(x_i,2r_i)) + \fint_{B(x_i,2r_i)} |f|^p\, d\mu \lesssim \fint_{B(x_i,2r_i)} |f|^p\, d\mu.$$

Используя это неравенство и (4.3), имеем

$$\int_O |f_\varepsilon|^p\, d\mu \lesssim \sum_{i=1}^{\infty} \int_{B(x_i,2r_i)} |I^{(p)}_{B(x_i,2r_i)} f|^p\, d\mu =$$

$$= c \sum_{i=1}^{\infty} \mu(B(x_i,2r_i)) |I^{(p)}_{B(x_i,2r_i)} f|^p \lesssim \sum_{i=1}^{\infty} \int_{B(x_i,2r_i)} |f|^p\, d\mu = c \int_O |f|^p\, d\mu. \tag{4.7}$$

Так как $f_\varepsilon = f$ на $X \setminus O$, то доказано, что $f_\varepsilon \in L^p(X)$.

Чтобы доказать, что $D_\alpha(f_\varepsilon)\cap L^p \neq \varnothing$, покажем, что для некоторой постоянной c будет выполнено $c\mathcal{A}_\alpha^{(p)} f \in D_\alpha(f_\varepsilon)\cap L^p$. Снова рассмотрим три различных случая расположения точек x, y.

1. Пусть $x, y \in X \setminus O$. Тогда, так как $f_\varepsilon = f$ на $X \setminus O$, то

$$|f_\varepsilon(x) - f_\varepsilon(y)| = |f(x) - f(y)| \lesssim [d(x,y)]^\alpha \left[\mathcal{A}_\alpha^{(p)} f(x) + \mathcal{A}_\alpha^{(p)} f(y)\right].$$

2. Пусть $x, y \in O$. Предположим сначала, что $d(x,y) \le d_0$. Так как, в силу леммы 3.7 $\|\phi_i\|_{H^\alpha(X)} \le c r_i^{-\alpha}$, то

$$|f_\varepsilon(x) - f_\varepsilon(y)| = \left|\sum_{i=1}^{\infty} [\phi_i(x) - \phi_i(y)] \left[I_{B(x_i,2r_i)}^{(p)} f - f(x)\right]\right| \lesssim$$

$$\lesssim \sum_{I_x\cup I_y} \frac{[d(x,y)]^\alpha}{r_i^\alpha} |I_{B(x_i,2r_i)}^{(p)} f - f(x)|. \tag{4.8}$$

Так как при $i \in I_x \cup I_y$ выполняется включение $B(x_i, 2r_i) \subset B(x, 100r_i)$, то имеет место оценка

$$|I_{B(x_i,2r_i)}^{(p)} f - f(x)| \le |I_{B(x,100r_i)}^{(p)} f - f(x)| + |I_{B(x_i,2r_i)}^{(p)} f - I_{B(x,100r_i)}^{(p)} f| \lesssim r_i^\alpha \mathcal{A}_\alpha^{(p)} f(x)$$

(см. леммы 3.3 и 3.4). Подставляя полученную оценку в (4.8) и используя условие 6) леммы 3.6, получим

$$|f_\varepsilon(x) - f_\varepsilon(y)| \lesssim [d(x,y)]^\alpha \mathcal{A}_\alpha^{(p)} f(x).$$

Теперь рассмотрим случай $d(x,y) > d_0$. Тогда

$$|f_\varepsilon(x) - f_\varepsilon(y)| \le \sum_{i\in I_x} \left|\phi_i(x)[I_{B(x_i,2r_i)}^{(p)} f - f(x)]\right| +$$

$$+ \sum_{i\in I_y} |\phi_i(y)[I_{B(x_i,2r_i)}^{(p)} f - f(y)]| + |f(x) - f(y)| \le$$

$$\lesssim \sum_{i\in I_x} r_i^\alpha \mathcal{A}_\alpha^{(p)} f(x) + \sum_{i\in I_y} r_i^\alpha \mathcal{A}_\alpha^{(p)} f(y) + [d(x,y)]^\alpha \left[\mathcal{A}_\alpha^{(p)} f(x) + \mathcal{A}_\alpha^{(p)} f(y)\right] \lesssim$$

$$\lesssim [\operatorname{dist}(x, X\setminus O)]^\alpha \mathcal{A}_\alpha^{(p)} f(x) + [\operatorname{dist}(y, X\setminus O)]^\alpha \mathcal{A}_\alpha^{(p)} f(y) +$$

$$+ [d(x,y)]^\alpha \left[\mathcal{A}_\alpha^{(p)} f(x) + \mathcal{A}_\alpha^{(p)} f(y)\right] \lesssim [d(x,y)]^\alpha \left[\mathcal{A}_\alpha^{(p)} f(x) + \mathcal{A}_\alpha^{(p)} f(y)\right].$$

3. Пусть $x \in O, y \in X \setminus O$. В этом случае получаем

$$|f_\varepsilon(x) - f_\varepsilon(y)| \le \sum_{i\in I_x} |f(y) - I_{B(x_i,2r_i)}^{(p)} f| \lesssim$$

$$\lesssim |f(y) - f(x)| + \sum_{i\in I_x} |f(x) - I_{B(x_i,2r_i)}^{(p)} f| \lesssim [d(x,y)]^\alpha [\mathcal{A}_\alpha^{(p)} f(x) + \mathcal{A}_\alpha^{(p)} f(y)].$$

Таким образом, $c\mathcal{A}_\alpha^{(p)} f \in D_\alpha(f_\varepsilon) \cap L^p$.

Доказательство утверждения 4). В силу (4.3), (4.7) и (4.2) $\|f - f_\varepsilon\|_{L^p} \lesssim \varepsilon$. Кроме того, если $g \in D_\alpha(f) \cap L^p$, то $c[A_\alpha^{(p)} f]\chi_O \in D_\alpha(f - f_\varepsilon) \cap L^p$. Тогда

$$\|f - f_\varepsilon\|_{M_\alpha^p} \le \|f - f_\varepsilon\|_{L^p} + \|c[A_\alpha^{(p)} f]\chi_O\|_{L^p} \lesssim \varepsilon.$$

Избавимся теперь от предположения (4.1). Это делается точно так же, как и в случае $p > 1$ (см. [3] и [5]). Действительно, существует не более чем счетный набор точек $\{x_i\}$, такой, что

$$X \subset \bigcup_i B(x_i, 1/2), \quad B(x_i, 1/4) \cap B(x_j, 1/4) = \varnothing \quad (i \neq j),$$

для которого можно построить другое разбиение единицы (см. [3]) — набор функций $\{\varphi_i\} \subset H_\alpha(X)$ со свойствами

$$0 \le \varphi_i(x) \le 1, \quad \operatorname{supp}\varphi_i \subset B(x_i, 1), \quad \|\varphi_i\|_{H_\alpha(X)} = c, \quad \sum_i \varphi_i(x) = 1.$$

В силу леммы 3.5 $f\varphi_i \in M_\alpha^p(X)$, а так как $\operatorname{supp} f\varphi_i \subset B(x_i, 1)$, то в силу доказанного существует набор функций $\{f_\varepsilon^i\}$, удовлетворяющий условиям

$$f_\varepsilon^i \in M_\alpha^p(X) \cap H_\beta(X), \quad \operatorname{supp} f_\varepsilon^i \subset B(x_i, 2), \quad \|f_\varepsilon^i - f\varphi_i\| < \varepsilon/2^i.$$

При этом также

$$\nu\left\{x \in X : f_\varepsilon^i(x) \neq f\varphi_i(x)\right\} < \varepsilon/2^i.$$

Легко проверяется, что функция $f_\varepsilon = \sum_i f_\varepsilon^i$ удовлетворяет всем необходимым условиям. Теорема 2.1 доказана.

Abstract. The present paper is devoted to the Lusin's approximation of functions from Hajlasz Sobolev classes $M_\alpha^p(X)$ for $p > 0$. It is proved that for any $f \in M_\alpha^p(X)$ and any $\varepsilon > 0$ there exist an open set $O_\varepsilon \subset X$ with measure less than ε (as a measure can be taken the corresponding capacity or Hausdorff content) and an approximating function f_ε such that $f = f_\varepsilon$ on $X \setminus O_\varepsilon$. Moreover, the correcting function f_ε is regular (that is, it belongs to the underlying space $M_\alpha^p(X)$ and it is a locally Holder function), and it approximates the original function in the metric of the space $M_\alpha^p(X)$.

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HYPERSURFACES OF A FINSLER SPACE WITH A SPECIAL (α, β)-METRIC

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Abstract. In the present paper we study the Finslerian hypersurfaces of a Finsler space with a special (α, β) metric, and examine the hypersurfaces of this special metric as a hyperplane of first, second and third kinds.

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1. Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L)$, that is, a pair consisting of an n-dimensional differentiable manifold M^n equipped with a Fundamental function L. The concept of an (α, β) metric, denoted by $L(\alpha, \beta)$, was introduced by M. Matsumoto [5], and later on has been studied by many authors (see [1 - 5, 8 - 9] and references therein). Well-known examples of (α, β) metrics are the Rander's metric $(\alpha + \beta)$, the Kropina metric $\frac{\alpha^2}{\beta}$ and the generalized Kropina metric $\frac{\alpha^{m+1}}{\beta^m} (m \neq 0, -1)$. Recall that a Finsler metric $L(x, y)$ is called an (α, β) metric if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x) y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is an $1-$form on M^n.

We consider a special Finsler Space $F^n = \{M^n, L(\alpha, \beta)\}$ with the metric $L(\alpha, \beta)$ given by

$$L(\alpha, \beta) = \alpha + \beta + \frac{\alpha^2}{(\alpha - \beta)}. \tag{1.1}$$

Differentiating equation (2.1) partially with respect to α and β, we get

$$L_\alpha = \frac{2\alpha^2 + \beta^2 - 4\alpha\beta}{(\alpha-\beta)^2}, \quad L_\beta = \frac{2\alpha^2 + \beta^2 - 2\alpha\beta}{(\alpha-\beta)^2},$$

$$L_{\alpha\alpha} = \frac{2\beta^2}{(\alpha-\beta)^3}, \quad L_{\beta\beta} = \frac{2\alpha^2}{(\alpha-\beta)^3}, \quad L_{\alpha\beta} = \frac{-2\alpha\beta}{(\alpha-\beta)^3},$$

where

$$L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}, \quad L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}.$$

In the Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ the normalized element of the support $l_i = \partial_i L$ and the angular metric tensor h_{ij} are given by the following formulas (see [5]):

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i,$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j,$$

where $Y_i = a_{ij} y^j$. For the fundamental function (2.1) the constants p, q_0, q_{-1} and q_{-2} in the last equation are given by the following formulas:

(1.2)
$$p = L L_\alpha \alpha^{-1} = \frac{4\alpha^4 - \beta^4 - 8\alpha^3\beta + 4\alpha\beta^3}{\alpha(\alpha - \beta)^3},$$

$$q_0 = L L_{\beta\beta} = \frac{4\alpha^4 - 2\alpha^2\beta^2}{(\alpha - \beta)^4}, \quad q_{-1} = L L_{\alpha\beta}\alpha^{-1} = \frac{2\beta^3 - 4\alpha^2\beta}{(\alpha - \beta)^4},$$

$$q_{-2} = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \frac{-4\alpha^5 - 2\alpha^2\beta^3 + 8\alpha^4\beta + \alpha\beta^4 - \beta^5}{\alpha^3(\alpha - \beta)^4}.$$

The fundamental metric tensor $g_{ij} = \frac{1}{2}\partial_i \partial_j L^2$ for $L = L(\alpha, \beta)$ is given by the following formula (see [4, 5]):

(1.3)
$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j,$$

where

(1.4)
$$p_0 = q_0 + L_\beta^2 = \frac{8\alpha^4 + \beta^4 + 6\alpha^2\beta^2 - 8\alpha^3\beta - 4\alpha\beta^3}{(\alpha - \beta)^4},$$

$$p_{-1} = q_{-1} + L^{-1} p L_\beta = \frac{2\alpha\beta^3 - 4\alpha^3\beta + (2\alpha^2 + \beta^2 - 2\alpha\beta)^2}{\alpha(\alpha - \beta)^4},$$

$$p_{-2} = q_{-2} + p^2 L^{-2} = \frac{2\beta^4 + 8\alpha^2\beta^2 - 6\alpha\beta^3 + \frac{\beta^5}{\alpha}}{\alpha^2(\alpha - \beta)^4}.$$

The reciprocal tensor g^{ij} of g_{ij} is given by the following formula (see [4, 5]):

(1.5)
$$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j,$$

where $b^i = a^{ij} b_j$, $b^2 = a_{ij} b^i b^j$, and

(1.6)
$$s_0 = \frac{1}{\tau p}\{p p_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\},$$

$$s_{-1} = \frac{1}{\tau p}\{p p_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\},$$

$$s_{-2} = \frac{1}{\tau p}\{p p_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2\},$$

$$\tau = p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2).$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by formula (see [10]):

$$2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{1.7}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \quad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{1.8}$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{^i_{jk}\}$ be the component of the Christoffel symbol of the associated Riemannian space R^n, and let ∇_k be the covariant derivative with respect to x^k relative to this Christoffel symbol. Define

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji}, \tag{1.9}$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^i_{jk})$ be the Cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^i_{jk}\}$ of the special Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} = {} & B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ & - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + \\ & C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji}, \\ B_{ij} = \frac{1}{2}\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}, \quad B^k_i = g^{kj} B_{ji}, \\ A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\ \lambda^m = B^m E_{00} + 2B_0 F^m_0, \quad B_0 = B_i y^i, \end{aligned} \tag{1.11}$$

and '0' denotes the contraction with y^i except for the quantities p_0, q_0 and s_0.

2. Induced Cartan connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i(u^\alpha)$, where $\alpha = 1, 2, 3 \ldots (n-1)$. The element of the support y^i of F^n is taken to be tangential to F^{n-1}, that is, it is given by formula (see [6]):

$$y^i = B^i_\alpha(u) v^\alpha. \tag{2.1}$$

The metric tensor $g_{\alpha\beta}$ and the hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \quad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma,$$

and at each point (u^α) of F^{n-1}, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u,v),y(u,v)\}B^i_\alpha N^j = 0, \quad g_{ij}\{x(u,v),y(u,v)\}N^iN^j = 1.$$

The angular metric tensor $h_{\alpha\beta}$ of the hypersurface is determined by formulas:

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \quad h_{ij}B^i_\alpha N^j = 0, \quad h_{ij}N^iN^j = 1. \tag{2.2}$$

The inverse (B^α_i, N_i) of (B^i_α, N^i) is given by

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0,$$

$$N_i = g_{ij}N^j, \quad B^k_i = g^{kj}B_{ji}, \quad B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j.$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{*i}_{jk})$ is given by formulas (see [6]):

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma,$$

$$G^\alpha_\beta = B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta), \quad C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma,$$

where

$$M_{\beta\gamma} = N_iC^i_{jk}B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \quad H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta),$$

and

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}, \quad B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha.$$

The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental v-tensor and the normal curvature vector, respectively (see [6]). The second fundamental h-tensor $H_{\beta\gamma}$ is defined as follows (see [6]):

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{2.3}$$

where

$$M_\beta = N_iC^i_{jk}B^j_\beta N^k. \tag{2.4}$$

The relative $h-$ and $v-$covariant derivatives of the projection factor B^i_α with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \quad B^i_\alpha|_\beta = M_{\alpha\beta}N^i.$$

It easily follows form equation (3.3) that $H_{\beta\gamma}$ generally is not symmetric and satisfies the equation

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{2.5}$$

implying that

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{2.6}$$

The following lemmas, due to Matsumoto [6], will be used in Section 4

Lemma 2.1. *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.*

Lemma 2.2. *A hypersurface $F^{(n-1)}$ is a hyperplane of the first kind with respect to the connection CΓ if and only if $H_\alpha = 0$.*

Lemma 2.3. *A hypersurface $F^{(n-1)}$ is a hyperplane of the second kind with respect to the connection CΓ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 2.4. *A hypersurface $F^{(n-1)}$ is a hyperplane of the third kind with respect to the connection CΓ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.*

3. A hypersurface $F^{(n-1)}(c)$ of a special Finsler space

Let us consider a Finsler space with the metric $L = \alpha + \beta + \frac{\alpha^2}{\alpha-\beta}$, where the vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{(n-1)}(c)$ given by the equation $b(x) = c$, where c is a constant (see [10]). From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$ we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0,$$

$$\frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0,$$

$$b_i B^i_\alpha = 0,$$

showing that $b_i(x)$ is a covariant component of a normal vector field of the hypersurface $F^{n-1}(c)$. Further, we have

$$b_i B^i_\alpha = 0 \quad \textit{and} \quad b_i y^i = 0, \quad \textit{that is,} \quad \beta = 0, \tag{3.1}$$

and the induced matric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u,v) = a_{\alpha\beta} v^\alpha v^\beta, a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta, \tag{3.2}$$

which is a Riemannian metric.

Taking $\beta = 0$ in the equations (2.2), (2.3) and (2.5) we get

$$p = 4, \quad q_0 = 4, \quad q_{-1} = 0, \quad q_{-2} = -4\alpha^{-2}, \tag{3.3}$$

$$p_0 = 8, \quad p_{-1} = 4\alpha^{-1}, \quad p_{-2} = 0, \quad \tau = 16(1+b^2),$$

$$s_0 = \frac{1}{4(1+b^2)}, \quad s_{-1} = \frac{1}{4\alpha(1+b^2)}, \quad s_{-2} = \frac{-b^2}{4\alpha^2(1+b^2)}.$$

From (2.4) we get

$$g^{ij} = \frac{1}{4}a^{ij} - \frac{1}{4(1+b^2)}b^i b^j - \frac{1}{4\alpha(1+b^2)}(b^i y^j + b^j y^i) + \frac{b^2}{4\alpha^2(1+b^2)}y^i y^j. \tag{3.4}$$

Thus, from (4.1) and (4.4), along $F^{n-1}(c)$ we obtain

$$g^{ij}b_i b_j = \frac{b^2}{4(1+b^2)}.$$

Therefore we have

$$b_i(x(u)) = \sqrt{\frac{b^2}{4(1+b^2)}}N_i, \quad b^2 = a^{ij}b_i b_j. \tag{3.5}$$

where b is the length of the vector b^i.

Next, from (4.4) and (4.5) we get

$$b^i = a^{ij}b_j = \sqrt{\frac{4b^2(1+b^2)}{\{1+b^2(1-\alpha^2)\}^2}}N^i + \frac{\alpha b^2 y^i}{1+b^2(1-\alpha^2)}. \tag{3.6}$$

Thus, we have the following result.

Theorem 3.1. *In a special Finsler hypersurface $F^{(n-1)}(c)$, the induced Riemannian metric is given by (4.2) and the scalar function $b(x)$ is given by (4.5) and (4.6).*

Now, observe that the angular metric tensor h_{ij} and the metric tensor g_{ij} of F^n are given by formulas:

$$h_{ij} = 4a_{ij} + 4b_i b_j - \frac{4}{\alpha^2}Y_i Y_j \quad and \quad g_{ij} = 4a_{ij} + 8b_i b_j + \frac{4}{\alpha}(b_i Y_j + b_j Y_i. \tag{3.7}$$

From equations (4.1), (4.7) and (3.2) it follows that if $h^{(a)}_{\alpha\beta}$ denotes the angular metric tensor of the Riemannian $a_{ij}(x)$, then along $F^{n-1}_{(c)}$ we have $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. Thus, along $F^{n-1}_{(c)}$ we have $\frac{\partial p_0}{\partial \beta} = \frac{2l}{\alpha}$, and hence from equation (2.6) we get

$$Y_1 = \frac{48}{\alpha}, \qquad m_i = b_i.$$

Therefore, in the special Finsler hypersurface $F^{(n-1)}_{(c)}$, the hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{2\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + \frac{6}{\alpha}b_i b_j b_k. \tag{3.8}$$

Next, it follows from (3.2), (3.3), (3.5), (4.1) and (4.8) that

$$M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{4(1+b^2)}}h_{\alpha\beta} \quad and \quad M_\alpha = 0. \tag{3.9}$$

Therefore, it follows from equation (3.6) that $H_{\alpha\beta}$ is symmetric. Thus, we have the following result.

Theorem 3.2. *The second fundamental v-tensor of the special Finsler hypersurface $F^{(n-1)}_{(c)}$ is given by (4.9) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now, from (4.1) we have $b_i B^i_\alpha = 0$, and hence

$$b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0.$$

Therefore, using the equality $b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_j N^j H_\beta$, from (3.5) we obtain

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0. \tag{3.10}$$

Since $b_i|_j = -b_h C^h_{ij}$, we get $b_{i|j}B^i_\alpha N^j = 0$. Therefore, taking into account that $b_{i|j}$ is symmetric, from equation (4.10) we have

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0. \tag{3.11}$$

Next, contracting (4.11) with v^β and using (3.1), we get

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0. \tag{3.12}$$

Again contracting by v^α the equation (4.12) and using (3.1), we have

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_0 + b_{i|j}y^i y^j = 0. \tag{3.13}$$

It follows from Lemmas 3.1 and 3.2 that the hypersuface $F^{(n-1)}_{(c)}$ is a hyperplane of first kind if and only if $H_0 = 0$. Thus, in view of (4.13), it is obvious that $F^{n-1}_{(c)}$ is a hyperplane of first kind if and only if $b_{i|j}y^i y^j = 0$. On the other hand, $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n is defined on y^i, but $b_{ij} = \nabla_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^i_{jk}\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i.

Below we consider the difference $b_{i|j} - b_{ij}$, where $b_{ij} = \nabla_j b_i$. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^i_{jk}\}$ is given by (2.10), and since b_i is a gradient vector, then from (2.9) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$ *and* $F^i_j = 0$. Thus, (2.10) reduces to the following

$$\begin{aligned} D^i_{jk} = {} & B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - \\ & C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} & B_i = 8b_i + 4\alpha^{-1}Y_i, \quad B^i = (\frac{1}{1+b^2})b^i + \frac{1}{\alpha(1+b^2)}y^i, \\ & \lambda^m = B^m b_{00}, \quad B_{ij} = \frac{2}{\alpha}(a_{ij} - \frac{Y_i Y_j}{\alpha^2}) + \frac{12}{\alpha}b_i b_j, \\ & B^i_j = \frac{1}{2\alpha}(\delta^i_j - \alpha^{-1}y^i Y_j) + \frac{5}{2\alpha(1+b^2)}b^i b_j - \\ & \frac{(1+6b^2)}{2\alpha^2(1+b^2)}b_j Y^i, \quad A^m_k = B^m_k b_{00} + B^m b_{k0}. \end{aligned} \tag{3.15}$$

In view of (4.3) and (4.4), the relation in (2.11) becomes to by virtue of (4.15) we have $B^i_0 = 0$, $B_{i0} = 0$ which leads $A^m_0 = B^m b_{00}$.

Now contracting (4.14) by y^k we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}.$$

Again contracting the above equation with respect to y^j we obtain

$$D^i_{00} = B^i b_{00} = \{(\tfrac{1}{1+b^2})b^i + \tfrac{1}{\alpha(1+b^2)} y^i\} b_{00}.$$

In view of (4.1), along $F^{(n-1)}_{(c)}$ we get

$$(3.16) \qquad b_i D^i_{j0} = \frac{b^2}{(1+b^2)} b_{j0} + \frac{(1+6b^2)}{2\alpha(1+b^2)} b_j b_{00} + \frac{1}{(1+b^2)} b_i b^m C^i_{jm} b_{00}.$$

Now we contract (4.16) by y^j to obtain

$$(3.17) \qquad b_i D^i_{00} = \frac{1}{(1+b^2)} b_{00}.$$

From (3.3), (4.5), (4.6), (4.9) and $M_\alpha = 0$ we obtain

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Thus, the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and the equations (4.16), (4.17) give

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \tfrac{1}{1+b^2} b_{00}.$$

Consequently, the equations (4.12) and (4.13) can be written as follows:

$$(3.18) \qquad \sqrt{\frac{b^2}{4(1+b^2)}} H_\alpha + \frac{1}{1+b^2} b_{i0} B^i_\alpha = 0,$$

$$\sqrt{\frac{b^2}{4(1+b^2)}} H_0 + \frac{1}{1+b^2} b_{00} = 0.$$

Thus, the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact that $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Therefore, we can write

$$(3.19) \qquad 2b_{ij} = b_i c_j + b_j c_i.$$

Now from (4.1) and (4.19) we get

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0.$$

It follows from (4.18) that $H_\alpha = 0$, and hence in view of (4.15) and (4.19) we get $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{2}{\alpha} h_{\alpha\beta}$.

Next, we use the equations (3.3), (4.4) (4.6), (4.9) and (4.14) to obtain

$$(3.20) \qquad b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^2(4+3b^2)}{16\alpha(1+b^2)^2} h_{\alpha\beta}.$$

Thus, the equation (4.11) reduces to the following

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_{\alpha\beta} + \frac{b^2(4+3b^2)}{16\alpha(1+b^2)^2}h_{\alpha\beta} = 0. \tag{3.21}$$

and hence the hypersurface $F^{n-1}_{(c)}$ is umbilic. Thus, we have the following result.

Theorem 3.3. *A necessary and sufficient condition for $F^{(n-1)}_{(c)}$ to be a hyperplane of first kind is (4.19). In this case the second fundamental tensor of $F^{n-1}_{(c)}$ is proportional to its angular metric tensor.*

Now, taking into account that by Lemma 3.3, $F^{(n-1)}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$, from (4.20) we get $c_0 = c_i(x)y^i = 0$. Therefore, there exists a function $\psi(x)$ such that $c_i(x) = \psi(x)b_i(x)$, and, in view of (4.19), we get $2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$. The last equation can also be written as follows $b_{ij} = \psi(x)b_ib_j$. Thus, we have the following result.

Theorem 3.4. *A necessary and sufficient condition for a hypersurface $F^{(n-1)}_{(c)}$ to be a hyperplane of second kind is (4.21).*

Putting together Lemma 3.4 and formula (4.9), we conclude that $F^{n-1}_{(c)}$ is not a hyperplane of third kind. Thus, we have the following result.

Theorem 3.5. *The hypersurface $F^{(n-1)}_{(c)}$ is not a hyperplane of the third kind.*

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О ЛОКАЛЬНОЙ ЭКВИВАЛЕНТНОСТИ МАЖОРАНТЫ ЧАСТИЧНЫХ СУММ И ФУНКЦИИ ПЭЛИ ДЛЯ РЯДОВ ФРАНКЛИНА

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Аннотация. Доказывается, что мажоранта частичных сумм и функция Пэли ряда Франклина имеют эквивалентные нормы в пространстве $L_p(I)$, $p > 0$, если интервалы "пика" функций Франклина с ненулевыми коэффициентами лежат в I. Приводятся примеры рядов указывающие на существенность этого условия.

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1. Введение

Для формулировки полученных результатов, напомним определение системы Франклина. Пусть $n = 2^\mu + \nu$, где $\mu = 0, 1, 2, \dots$ и $1 \leq \nu \leq 2^\mu$. Обозначим

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}}, & \text{для } 0 \leq i \leq 2\nu, \\ \frac{i-\nu}{2^\mu}, & \text{для } 2\nu < i \leq n. \end{cases} \tag{1.1}$$

Через S_n обозначим пространство функций, непрерывных и кусочно линейных на $[0;1]$ с узлами $\{s_{n,i}\}_{i=0}^n$, т.е. $f \in S_n$, если $f \in C[0;1]$ и линейная на каждом отрезке $[s_{n,i-1}; s_{n,i}]$, $i = 1, 2, \dots, n$. Ясно, что $\dim S_n = n + 1$ и множество $\{s_{n,i}\}_{i=0}^n$ получается добавлением точки $s_{n,2\nu-1}$ к множеству $\{s_{n-1,i}\}_{i=0}^{n-1}$. Поэтому, существует единственная, с точностью до знака, функция $f_n \in S_n$, которая ортогональна S_{n-1} и $\|f_n\|_2 = 1$. Полагая $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x - 1)$, $x \in [0;1]$, получим ортонормированную систему $\{f_n(x)\}_{n=0}^\infty$, которая эквивалентным образом определена в работе [1] и называется системой Франклина.

Для $n = 2^\mu + \nu$, где $\mu = 0, 1, 2, \dots$, и $1 \leq \nu \leq 2^\mu$, обозначим (см. (1.1)) $\{n\} := [s_{n,2\nu-2}, s_{n,2\nu}]$ и $[n] = \mu$. Отрезок $\{n\}$ иногда называют интервалом пика функции

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f_n в связи с тем, что функции f_n достигает своего наименьшего и наибольшего значений на этом отрезке. Число $[n]$ называют рангом числа n и отрезка $\{n\}$.

Систематическое изучение системы Франклина началось с работ [2], [3], где в частности доказано, что если $f \in L_p[0;1], 1 < p < \infty$, и $\sum_{n=0}^{\infty} a_n f_n(x)$ ее ряд Фурье-Франклина, то

$$(1.2)\quad S^*(f,\cdot) \in L_p[0;1], \quad \text{где} \quad S^*(f,x) = \sup_n |S_n(f,x)| \quad \text{и} \quad S_n(f,x) = \sum_{k=0}^{n} a_k f_k(x).$$

С. В. Бочкаревым [1] была доказана, что система Франклина является безусловным базисом в пространстве $L_p[0;1]$, $1 < p < \infty$. Для этого он доказал, что оператор Пэли для системы Франклина имеет слабый тип (1,1), т.е. существует постоянная $C > 0$, такая что если $f \in L[0;1]$ и $\sum_{n=0}^{\infty} a_n f_n(x)$ ее ряд Фурье-Франклина, то

$$(1.3)\qquad \operatorname{mes}\{x \in [0,1] : P(f,x) > \lambda\} \le \frac{C}{\lambda} \int_0^1 |f(x)|dx,$$

где $P(f,x) = \left\{\sum_{n=0}^{\infty} a_n^2 f_n^2(x)\right\}^{1/2}$.

Так как P имеет сильный тип (2,2), т.е. $\|P(f,\cdot)\|_2 \le C\|f\|_2$, из (1.3), в силу известной интерполяционной теоремы Марцинкевича (см. напр. [5] стр. 485), следует, что для всех $p \in (1,\infty)$ имеет место $\|P(f,\cdot)\|_p \le C_p\|f\|_p$. Следовательно, с учетом (1.2), для любого $p > 1$ имеем

$$(1.4)\qquad \int_0^1 \sup_n \left|\sum_{k=0}^{n} a_k f_k(x)\right|^p dx \sim_p \int_0^1 \left(\sum_{k=0}^{\infty} a_k^2 f_k^2(x)\right)^{p/2} dx,$$

где запись $a \sim_\gamma b$ означает, что существуют постоянные c_γ и C_γ, зависящие только от γ, такие что $c_\gamma \cdot a \le b \le C_\gamma \cdot a$.
Из результатов работ [6]–[8] следует, что (1.4) верно также для $p \in (0,1]$.

Напомним, что пространство $L_0(E)$ – метрическое пространство п.в. конечных и измеримых на E функций с метрикой, сходимость по которой совпадает со сходимостью по мере на множестве E.

В работе [9] доказан не только аналог соотношения (1.4) в случае $p = 0$, но и ее локализация на множества положительной меры. А именно, доказана следующая (см. теоремы 2.1, 2.2 и 2.3 работы [9])

Теорема 1.1. *Для ряда* $\sum\limits_{k=0}^{\infty} a_k f_k(x)$ *следующие условия эквивалентны:*

(1) *ряд почти всюду сходится на* E,

(2) *ряд по мере безусловно сходится на* E,

(3) $\sup_n |\sum_{k=0}^{n} a_k f_k(x)| < +\infty$ *п.в. на* E,

(4) $\sum_{k=0}^{\infty} a_k^2 f_k^2(x) < +\infty$ *п.в. на* E.

В настоящей работе мы докажем, что невозможно получить локализацию соотношения (1.4) даже на двоичных интервалах и получим такую локализацию при некотором дополнительном условии.

2. Основные результаты

Теорема 2.1. *Для любого двоичного отрезка* $I = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right]$, *любого ряда* $\sum a_n f_n(x)$ *и любого* $p > 0$ *имеет место*

$$\left\| \sup_N \left| \sum_{\{n\} \subset I, n \leq N} a_n f_n(x) \right| \right\|_{L_p(I)} \sim_p \left\| \left\{ \sum_{\{n\} \subset I} a_n^2 f_n^2(x) \right\}^{\frac{1}{2}} \right\|_{L_p(I)}.$$

Теорема 2.2. *Для любого отрезка* $I = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right] \neq [0,1]$, *любых* $p > 0$ *и* $C > 0$ *существуют ряды* $\sum_{n=0}^{\infty} a_n f_n(x)$ *и* $\sum_{n=0}^{\infty} b_n f_n(x)$ *такие, что*

$$\left\| \sup_N \left| \sum_{n \leq N} a_n f_n(x) \right| \right\|_{L_p(I)} > C \cdot \left\| \left\{ \sum_{n=0}^{\infty} a_n^2 f_n^2(x) \right\}^{\frac{1}{2}} \right\|_{L_p(I)}, \tag{2.1}$$

$$\left\| \left\{ \sum_{n=0}^{\infty} b_n^2 f_n^2(x) \right\}^{\frac{1}{2}} \right\|_{L_p(I)} > C \cdot \left\| \sup_N \left| \sum_{n \leq N} b_n f_n(x) \right| \right\|_{L_p(I)}. \tag{2.2}$$

Теорема 2.2 указывает на то, что условие $\{n\} \subset I$ в теореме 2.1 существенно. Мы убедимся, что для системы Хаара не верен аналог теоремы 2.2, а аналог теоремы 2.1 верен и без условия $\{n\} \subset I$. Однако в этом случае постоянные эквивалентности также зависят от I. А именно верна следующая

Теорема 2.3. *Для любого двоичного отрезка* $I = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right]$, *любого* $p > 0$ *и любого ряда* $\sum a_n \chi_n(x)$ *имеет место*

$$\left\| \sup_N \left| \sum_{n \leq N} a_n \chi_n(x) \right| \right\|_{L_p(I)} \sim_{p,I} \left\| \left\{ \sum_n a_n^2 \chi_n^2(x) \right\}^{\frac{1}{2}} \right\|_{L_p(I)}.$$

Учитывая соотношение (1.4), для доказательства теоремы 2.1, достаточно доказать следующую лемму.

Лемма 2.1. *Для любого двоичного отрезка* $I = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right]$, *любого* $p > 0$ *и любого ряда* $\sum a_n f_n(x)$ *имеют место*

$$\int_0^1 \sup_N \left| \sum_{\{n\}\subset I, n\le N} a_n f_n(x) \right|^p dx \le C_p \cdot \int_I \sup_N \left| \sum_{\{n\}\subset I, n\le N} a_n f_n(x) \right|^p dx, \tag{2.3}$$

$$\int_0^1 \left\{ \sum_{\{n\}\subset I} a_n^2 f_n^2(x) \right\}^{\frac{p}{2}} dx \le C_p \cdot \int_I \left\{ \sum_{\{n\}\subset I} a_n^2 f_n^2(x) \right\}^{\frac{p}{2}} dx. \tag{2.4}$$

Для доказательства вышесформулированных утверждений напомним некоторые свойства системы Франклина.

В исследованиях рядов по системе Франклина важную роль играют так называемые экспоненциальные оценки, полученные З. Чисельским [3]

$$C_1 \cdot 2^{\frac{[\mu]}{2}} \cdot (2-\sqrt{3})^{|i-2\nu-1|} < (-1)^{i+1} f_n(s_{n,i}) < C_2 \cdot 2^{\frac{[\mu]}{2}} \cdot (2-\sqrt{3})^{|i-2\nu-1|},$$

где $C_1 = \frac{2+\sqrt{3}}{3\sqrt{3}}$ и $C_2 = 4 \cdot \sqrt{3}(2+\sqrt{3})$.

Для доказательства леммы 2.1 удобнее использовать оценки, полученные в [10]. Их сформулируем в виде Предложений 2.1 и 2.2.

Предложение 2.1. *Для любого* $n = 2^\mu + \nu$ *имеют место следующие неравенства:*

$$\frac{1}{4}|f_n(s_{n,i+1})| < |f_n(s_{n,i})| < \frac{2}{7}|f_n(s_{n,i+1})| \quad \text{когда} \quad 1 \le i < 2\nu - 2, \tag{2.5}$$

$$\frac{1}{4}|f_n(s_{n,i-1})| < |f_n(s_{n,i})| < \frac{2}{7}|f_n(s_{n,i-1})| \quad \text{когда} \quad 2\nu < i < n, \tag{2.6}$$

$$f_n(s_{n,1}) = -2f_n(s_{n,0}), \quad f_n(s_{n,n-1}) = -2f_n(s_{n,n}), \quad f_n(s_{n,i}) \cdot f_n(s_{n,i+1}) < 0. \tag{2.7}$$

Предложение 2.2. *Для* $n = 2^k + \nu$, *с условием* $1 < \nu < 2^k$, *выполняются*

$$\frac{97}{48} < \frac{|f_n(s_{n,2\nu-1})|}{|f_n(s_{n,2\nu})|} < \frac{95}{42} \quad и \quad \frac{107}{66} < \frac{|f_n(s_{n,2\nu-1})|}{|f_n(s_{n,2\nu-2})|} < \frac{98}{60}. \tag{2.8}$$

Из Предложения 2.1 и линейности функции f_n на $[s_{n,i-1}, s_{n,i}]$ простыми вычислениями легко выводится

Предложение 2.3. *Для* $n = 2^k + \nu$, *с условием* $1 < \nu < 2^k$, *любого* $p > 0$ *выполняются*

$$\int_{s_{n,i-2}}^{s_{n,i-1}} |f_n(x)|^p dx < \left(\frac{2}{7}\right)^p \cdot \int_{s_{n,i-1}}^{s_{n,i}} |f_n(x)|^p dx, \quad \text{когда} \quad i \le 2\nu - 2 \tag{2.9}$$

и

$$\int_{s_{n,i-1}}^{s_{n,i}} |f_n(x)|^p dx < \left(\frac{2}{7}\right)^p \cdot \int_{s_{n,i-2}}^{s_{n,i-1}} |f_n(x)|^p dx, \quad \textit{когда} \quad i \geq 2\nu + 2.$$

Предложение 2.4. *Пусть $n_1 = 2^\mu + \nu_1$, $n_2 = 2^\mu + \nu_2$ и $\nu_1 < \nu_2$. Тогда существуют числа α, β (зависящие от μ, ν_1 и ν_2), такие, что*

$$f_{n_1}(x) = \alpha \cdot f_{n_2}(x), \quad \textit{когда} \quad x \leq s_{n_1, 2\nu_1 - 2} = \frac{\nu_1 - 1}{2^\mu}$$

и

$$f_{n_1}(x) = \beta \cdot f_{n_2}(x), \quad \textit{когда} \quad x \geq s_{n_2, 2\nu_2} = \frac{\nu_2}{2^\mu}.$$

Это Предложение впервые было применено в [8]. Из этого Предложения немедленно следует

Предложение 2.5. *Для любых a_n, $n = 2^\mu + \nu$, $1 \leq \nu \leq 2^\mu$ и любого ν_0: $1 \leq \nu_0 \leq 2^\mu$ имеют место*

$$\sum_{\nu=1}^{\nu_0} a_{2^\mu+\nu} f_{2^\mu+\nu}(x) = \alpha \cdot f_{2^\mu+\nu_0}(x), \quad \textit{когда} \quad x \geq \frac{\nu_0}{2^\mu}$$

и

$$\sum_{\nu=\nu_0}^{2^\mu} a_{2^\mu+\nu} f_{2^\mu+\nu}(x) = \beta \cdot f_{2^\mu+\nu_0}(x), \quad \textit{когда} \quad x \leq \frac{\nu_0 - 1}{2^\mu},$$

где α и β некоторые числа зависящие от a_n.

Условимся через $C, C_1, C_\gamma, \ldots$ обозначать постоянные, зависящие только от своих индексов. Значения этих постоянных в разных формулах могут быть разными. Длину отрезка I обозначим через $|I|$.

Доказательство леммы 2.1. Сначала докажем соотношение (2.3). Вместо $\sum_{\{n\} \subset I} a_n f_n(x)$ будем писать $\sum_n a_n f_n(x)$, предполагая $a_n = 0$, когда $\{n\} \not\subset I$.

Допустим

(2.10)
$$\int_I (S^*(x))^p dx = 1, \quad \text{где} \quad S^*(x) = \sup_N \left| \sum_{n \leq N} a_n f_n(x) \right|,$$

и докажем, что

(2.11)
$$\int_0^{\frac{x}{2^k}} (S^*(x))^p dx \leq C.$$

Положим для $n_m = 2^m + æ \cdot 2^{m-k} + 1$,

$$\sigma_m(x) = |a_{n_m} f_{n_m}(x)|, \quad \text{и} \quad S^*_m(x) = \max_N \left| \sum_{[n]=m, n \leq N} a_n f_n(x) - a_{n_m} f_{n_m}(x) \right|.$$

Очевидно, что

$$\sigma_m(x) \le 2 \cdot S^*(x), \quad S^*_m(x) \le 2 \cdot S^*(x), \quad \text{когда } x \in [0,1] \tag{2.12}$$

и

$$S^*(x) \le \sum_{m=k}^{\infty} \sigma_m(x) + \sum_{m=k}^{\infty} S^*_m(x) =: \Sigma_1(x) + \Sigma_2(x), \quad \text{когда } x \in [0,1].$$

Мы докажем, что

$$\int_0^{\frac{\mathfrak{x}}{2^k}} \Sigma_i^p(x)dx \le C_p, \quad i = 1, 2. \tag{2.13}$$

Пусть I_m- слева первый интервал ранга m, который содержится в I, а I^*_m-его правая половина. Заметим, что отрезки I^*_m не пересекаются и их объединение есть отрезок I. Заметим также, что функции $\sigma_m(x)$ и $S^*_m(x)$, $m \ge k$, являются модулями линейных на отрезке I^*_m функций. Действительно, $\sigma_m(x)$-модуль от линейной на I^*_m функции $a_{n_m} f_{n_m}(x)$. А для $S^*_m(x)$, в силу Предложения 2.5 имеем, что для любого N существует α_N, такое что

$$\sum_{|n|=m, n\le N} a_n f_n(x) - a_{n_m} f_{n_m}(x) = \alpha_N \cdot f_{n_m+1}(x) \quad \text{для} \quad x \le \frac{\mathfrak{x}}{2^k} + \frac{1}{2^m}. \tag{2.14}$$

Отсюда имеем

$$S^*_m(x) = |f_{n_m+1}(x)| \max_N |\alpha_N|, \quad \text{для} \quad x \le \frac{\mathfrak{x}}{2^k} + \frac{1}{2^m}. \tag{2.15}$$

Через A_m и B_m обозначим интегральные средние на I^*_m функций $\sigma_m(x)$ и $S^*_m(x)$, соответственно, т.е.

$$A_m = \frac{1}{|I^*_m|} \int_{I^*_m} \sigma_m(x)dx, \quad \text{и} \quad B_m = \frac{1}{|I^*_m|} \int_{I^*_m} S^*_m(x)dx, \quad m \ge k. \tag{2.16}$$

Из того, что $\sigma_m(x)$ и $S^*_m(x)$, $m \ge k$, являются модулями линейных на отрезке I^*_m функций, получаются

$$A_m \sim \max_{x \in I^*_m} \sigma_m(x) \quad \text{и} \quad B_m \sim \max_{x \in I^*_m} S^*_m(x). \tag{2.17}$$

Из (2.16), (2.12) и (2.10), с применением неравенства Гелдера, имеем

$$\sum_{m\ge k} A^p_m |I^*_m| = \sum_{m\ge k} |I^*_m| \cdot |I^*_m|^{-p} \left(\int_{I^*_m} \sigma_m(x)dx \right)^p \le$$

$$\sum_{m\ge k} |I^*_m|^{1-p} \int_{I^*_m} \sigma^p_m(x)dx \cdot |I^*_m|^{p-1} \le 2^p \sum_{m\ge k} \int_{I^*_m} (S^*(x))^p dx = 2^p.$$

Аналогично получаем

$$\sum_{m\ge k} B^p_m |I^*_m| \le 2^p. \tag{2.18}$$

Сначала докажем неравенство (2.13) для $i = 2$ и $p \leq 1$. В этом случае, последовательно применяя (2.9), (2.12) и (2.10) получим

$$\int_0^{\frac{\pi}{2^k}} \Sigma_2^p(x)dx = \int_0^{\frac{\pi}{2^k}} \left(\sum_m S_m^*(x)\right)^p dx \leq \int_0^{\frac{\pi}{2^k}} \sum_m (S_m^*(x))^p\, dx \leq$$

$$C_p \cdot \sum_m \int_{I_m^*} (S_m^*(x))^p\, dx \leq C_p \cdot \int_I (S^*(x))^p\, dx \leq C_p.$$

Аналогично получается неравенство (2.13), когда $i = 1$ и $p \leq 1$.

Перейдем к получению оценок (2.13), когда $p > 1$. Без ограничения общности можем считать, что суммы Σ_i, $i = 1, 2$, конечны, т. е. $\Sigma_1 = \sum_{m=k}^{k_1} \sigma_m$ и $\Sigma_2 = \sum_{m=k}^{k_1} S_m^*$.

Через I_m^j, $k \leq m \leq k_1$, обозначим двоичные отрезки ранга $m+1$, где правый конец отрезка I_m^1 совпадает с левым концом отрезка I, а для $j > 1$ правый конец отрезка I_m^j совпадает с левым концом I_m^{j-1}.

Для $x \in [0, \frac{\pi}{2^k}]$ и $m \geq k$ обозначим

$$\varphi_m(x) := B_{m,j} := \frac{1}{|I_m^j|} \int_{I_m^j} S_m^*(t)dt, \quad \text{когда } x \in I_m^j. \tag{2.19}$$

Поскольку $S_m^*(x)$-модуль от линейной на I_m^i функции, то (здесь и далее полагается $q = 2/7$) из (2.15), (2.17) и (2.9) имеем

$$S_m^*(x) \leq C \cdot \varphi_m(x) \leq C \cdot B_{m,j} \leq C \cdot q^{j-1} \cdot B_{m,1} \leq C \cdot q^{j-1} \cdot B_m, \quad \text{когда } x \in I_m^j. \tag{2.20}$$

Обозначим $J_m = I_m^1 \backslash I_{m+1}^1$, когда $m = k, k+1, \ldots, k_1 - 1$, и $J_{k_1} = I_{k_1}^1$. Тогда

$$|J_m| = \frac{1}{2}|I_m^1| = 2^{-m-1} = |I_m^*|. \tag{2.21}$$

Нетрудно заметить, что если $x \in J_m$, то из (2.20) следует

$$\Sigma_2(x) \leq C \cdot \sum_{m=k}^{k_1} \varphi_m(x) \leq C \cdot (B_k + \cdots + B_m + B_{m+1} \cdot q + \cdots + B_{k_1} \cdot q^{k_1 - m}).$$

Поэтому

$$\int_{I_k^1} (\Sigma_2(x))^p dx = \sum_{j=k}^{k_1} \int_{J_m} (\Sigma_2(x))^p dx \leq \tag{2.22}$$

$$C_p \cdot \sum_{m=k}^{k_1} |J_m| (\sum_{\nu=k}^{m} B_\nu)^p + C_p \cdot \sum_{m=k}^{k_1} |J_m| (\sum_{\nu=m+1}^{k_1} B_\nu \cdot q^{\nu-m})^p =: C_p(\Gamma_1 + \Gamma_2).$$

Пусть $q_1 \in (0, 1)$, такое что $q_1^{-p} < 2$. Тогда для Σ_1 получим

$$\Gamma_1 = C_p \sum_{m=k}^{k_1} |J_m| \left(\sum_{\nu=k}^{m} B_\nu q_1^{\nu-k} q_1^{k-\nu}\right)^p \leq C_p \sum_{m=k}^{k_1} |J_m| \sum_{\nu=k}^{m} B_\nu^p q_1^{p(\nu-k)} \left(\sum_{\nu=k}^{m} q_1^{\frac{p(k-\nu)}{p-1}}\right)^{p-1}$$

Учитывая, что в последней сумме $\nu \geq k$ и $q_1 < 1$, имеем

$$\left(\sum_{\nu=k}^{m} q_1^{\frac{p(k-\nu)}{p-1}}\right)^{p-1} \leq C_p \cdot q_1^{p(k-m)}.$$

Поэтому, для Γ_1 имеем

$$\Gamma_1 \leq C_p \sum_{m=k}^{k_1} |J_m| \sum_{\nu=k}^{m} B_\nu^p \cdot q_1^{p(\nu-m)} = C_p \sum_{\nu=k}^{k_1} B_\nu^p \sum_{m=\nu}^{k_1} |J_m| \cdot q_1^{p(\nu-m)}. \tag{2.23}$$

Так как $|J_{m+1}| = \frac{1}{2}|J_m|$ и $q_1^{-p} < 2$, то $|J_{m+1}| \cdot q_1^{p(\nu-m-1)} < \gamma \cdot |J_m| \cdot q_1^{p(\nu-m)}$, для некоторого $\gamma < 1$. Поэтому из (2.23) и (2.18) имеем

$$\Gamma_1 \leq C_p \sum_{\nu=k}^{k_1} B_\nu^p |J_\nu| < C_p, \quad \text{когда} \quad p > 1. \tag{2.24}$$

Оценим Γ_2, когда $p > 1$. Обозначим $q_1 = 0.9$ и напомним, что $q = 2/7$. Тогда, учитывая что $q \cdot q_1^{-1} < 1$, из (2.22) с применением неравенства Гелдера получим

$$\Gamma_2 = C_p \sum_{m=k}^{k_1} |J_m| \left(\sum_{\nu=m+1}^{k_1} B_\nu \cdot (q_1^{-1} \cdot q)^{\nu-m} \cdot q_1^{\nu-m}\right)^p \leq \tag{2.25}$$

$$C_p \sum_{m=k}^{k_1} |J_m| \left(\sum_{\nu=m+1}^{k_1} B_\nu^p \cdot (q_1^{-1} \cdot q)^{p(\nu-m)}\right) \cdot \left(\sum_{\nu=m+1}^{k_1} q_1^{(\nu-m)\frac{p}{p-1}}\right)^{p-1} \leq$$

$$C_p \sum_{m=k}^{k_1} |J_m| \sum_{\nu=m+1}^{k_1} B_\nu^p \cdot (q_1^{-1} \cdot q)^{p(\nu-m)} = C_p \sum_{\nu=k+1}^{k_1} B_\nu^p \sum_{m=k}^{\nu-1} |J_m| (q_1^{-1} \cdot q)^{p(\nu-m)}$$

Из (2.25) и (2.21) следует

$$\Gamma_2 \leq C_p \sum_{\nu=k+1}^{k_1} B_\nu^p \cdot |J_\nu| \sum_{m=k}^{\nu-1} 2^{\nu-m} (q_1^{-1} \cdot q)^{p(\nu-m)}.$$

Учитывая, что $2 \cdot (q_1^{-1} \cdot q)^p < 2 \cdot \frac{10}{9} \cdot \frac{2}{7} < 1$, из (2.25), (2.21) и (2.18) получим

$$\Gamma_2 \leq C_p \sum_{\nu=k}^{k_1} B_\nu^p |I_\nu^*| \leq C_p. \tag{2.26}$$

Из (2.26), (2.24) и (2.22) получаем

$$\int_{I_k^1} (\Sigma_2(x))^p dx \leq C_p, \quad \text{когда} \quad p > 1. \tag{2.27}$$

Очевидно, из (2.20) имеем

$$\Sigma_2(x) \leq C \cdot \sum_{m=k}^{k_1} \varphi_m(x) \leq C \cdot q^{j-1} \sum_{m=k}^{k_1} \varphi_m\left(x + \frac{j}{2^{k+1}}\right), \quad \text{когда} \quad x \in I_k^j.$$

Следовательно, получаем

$$\int_{I_k^2} (\Sigma_2(x))^p dx \le C_p \cdot q^{j-1} \int_{I_k^1} (\Sigma_2(x))^p dx.$$

Отсюда и из (2.27) получим (2.13) для $i = 2$.

В случае $i = 1$ неравенство (2.13) доказывается чуть проще. В этом случае нет необходимости соотношений типа (2.14), (2.15), поскольку $\sigma_m(x)$ одна функция о которой известно, что (см. (2.5))

$$\max_{x \in I_m^j} \sigma_m(x) \le \left(\frac{2}{7}\right)^{j-1} \cdot \sigma_m\left(\frac{æ}{2^k}\right) \le C \cdot \left(\frac{2}{7}\right)^{j-1} \cdot A_m,$$

где отрезки I_m^j те же, что в соотношении (2.20). Второе неравенство следует из того, что $æ > 0$ (в противном случае нечего доказывать) и поэтому (см. Предложение 2.2 и (2.16), (2.17))

$$\sigma_m\left(\frac{æ}{2^k}\right) \le \max_{x \in I_m} \sigma_m(x) = \max_{x \in I_m^*} \sigma_m(x) \le C \cdot A_m.$$

Отсюда, рассуждениями, аналогичными рассуждениям при доказательстве (2.27), получим

$$\int_0^{\frac{æ}{2^k}} (\Gamma_1(x))^p dx \le C_p, \quad \text{для} \quad p > 0.$$

Тем самым доказано неравенство (2.11). Аналогично доказывается, что

$$\int_{\frac{æ+1}{2^k}}^1 (S^*(x))^p dx < C_p,$$

которое вместе с (2.11) доказывают неравенство (2.3). Неравенство (2.4) доказывается аналогично. Лемма 2.1 доказана.

Доказательство теоремы 2.2. Сначала убедимся в справедливости неравенства (2.1). Поскольку $\left[\frac{æ}{2^k}, \frac{æ+1}{2^k}\right] \ne [0, 1]$, то либо $æ > 0$ либо $æ < 2^k$. Рассмотрим случай, когда $æ > 0$. Для $m > k$ и $l \le æ \cdot 2^{m-k}$ выберем a_{2^m+l} так, чтобы

(2.28) $$a_{2^m+l} f_{2^m+l}\left(\frac{æ}{2^k}\right) = \frac{1}{l}, \quad \text{когда} \quad l = 1, 2, \dots, æ \cdot 2^{m-k}.$$

Убедимся, что для любого $C > 0$ при достаточно большом m ряд $\sum_n a_n f_n(x) := \sum_{l=1}^{æ \cdot 2^{m-k}} a_{2^m+l} f_{2^m+l}(x)$ удовлетворяет (2.1). Во первых

(2.29) $$\left\| \sup_N \left| \sum_{n \le N} a_n f_n(x) \right| \right\|_{L_p(I)}^p > \int_{\frac{æ}{2^k}}^{\frac{æ}{2^k} + \frac{1}{2^m}} \left| \sum_{l=1}^{æ \cdot 2^{m-k}} a_{2^m+l} f_{2^m+l}(x) \right|^p dx > C_p \frac{m-k}{2^m}.$$

С другой стороны (см. (2.28) и (2.6))

$$(2.30)\qquad \left\|\left\{\sum_n a_n^2 f_n^2(x)\right\}^{\frac{1}{2}}\right\|_{L_p(I)}^p \le \int_{\frac{æ}{2^k}}^{\frac{æ}{2^k}+\frac{1}{2^m}} \left(\sum_l a_{2^m+l}^2 f_{2^m+l}^2(x)\right)^{\frac{p}{2}} dx \sum_{\eta=0}^{\infty} q^{\eta p} \le$$

$$C_p \cdot 2^{-m}\left(\sum_l a_{2^m+l}^2 f_{2^m+l}^2\left(\frac{æ}{2^k}\right)\right)^{\frac{p}{2}} < C_p \cdot 2^{-m}.$$

При достаточно большом m из (2.29) и (2.30) следует (2.1).

Случай $æ < 2^k$ доказывается аналогично. Только в сумме $\sum_n a_n f_n(x)$ будут участвовать n с условиями $[n] = m$, $\{n\}$ находится правее I.

Докажем соотношение (2.2). Опять рассмотрим только случай $æ > 0$. Для $m > k$ и $l \le æ \cdot 2^{m-k}$ выберем a_{2^m+l} так, чтобы

$$(2.31)\qquad a_{2^m+l} f_{2^m+l}\left(\frac{æ}{2^k}\right) = (-1)^l, \quad \text{когда} \quad l = 1, 2, ..., æ \cdot 2^{m-k}.$$

Убедимся, что для любого $C > 0$ при достаточно большом m ряд $\sum_n a_n f_n(x) := \sum_{l=1}^{æ \cdot 2^{m-k}} a_{2^m+l} f_{2^m+l}(x)$ удовлетворяет (2.2).

Из Предложения 2.4 и (2.31) следует, что

$$\sup_{n \le N}\left|\sum_{n \le N} a_n f_n(x)\right| = |a_{2^m+1} f_{2^m+1}(x)|, \quad \text{когда} \quad x \ge \frac{æ}{2^k}.$$

Поэтому

$$(2.32)\qquad \left\|\sup_N\left|\sum_{n\le N} a_n f_n(x)\right|\right\|_{L_p(I)}^p < \int_I |a_{2^m+1} f_{2^m+1}(x)|^p dx < C_p \cdot 2^{-m}.$$

С другой стороны

(2.33)

$$\int_I \left(\sum_n a_n^2 f_n^2(x)\right)^{\frac{p}{2}} dx > \int_{\frac{æ}{2^k}}^{\frac{æ}{2^k}+\frac{1}{2^m}} \left(\sum_{l=1}^{æ 2^{m-k}} a_{2^m+l}^2 f_{2^m+1}^2(x)\right)^{\frac{p}{2}} dx > C_p(æ 2^{m-k})^{\frac{p}{2}} 2^{-m}.$$

Из (2.32) и (2.33), при достаточно большом m, следует (2.2). Теорема 2.2 доказана.

Доказательство теоремы 2.3. Допустим

$$I = \left[\frac{æ}{2^k}, \frac{æ+1}{2^k}\right] \quad \text{и} \quad \|S^*(\cdot)\|_{L_p(I)} = 1, \quad \text{где} \quad S^*(x) = \sup_N \left|\sum_{n \le N} a_n \chi_n(x)\right|$$

Пусть $n_1 < n_2 < ... < n_k$, те номера, для которых

$$\Delta_{n_i} \bigcap I \ne \emptyset, \quad \text{и} \quad \Delta_{n_i} \not\subset I, \quad \text{где} \quad \Delta_n = \operatorname{supp}\chi_n.$$

Отметим, что таких n_i ровно $k+1$ штук. Если учесть, что функции $a_{n_i}\chi_{n_i}(x)$ принимают постоянные значения на I, то получим

$$\max_{x\in I}|a_{n_i}\chi_{n_i}| \le 2\min_{x\in I} S^*(x) \le 2S^*(x) \quad \text{и} \quad \max_{l}\left|\sum_{i=1}^{l} a_{n_i}\chi_{n_i}(x)\right| \le S^*(x), \quad \text{для} \quad x\in I.$$

Следовательно, получаем

$$\text{(2.34)} \qquad \sum_{n=1}^{\infty} a_n^2 f_n^2(x) \le 4(k+1)\cdot(S^*(x))^2 + \sum_{n:\Delta_n\subset I} a_n^2 f_n^2(x) \quad x\in I$$

и

$$\text{(2.35)} \qquad S_1^*(x) := \sup_N \left|\sum_{\Delta_n\subset I: n\le N} a_n\chi_n(x)\right| \le 2\cdot S^*(x), \quad x\in I.$$

Учитывая, что

$$\left\|\sum_{n:\Delta_n\subset I} a_n^2\chi_n^2(\cdot)\right\|_{L_p(I)} \sim_p \|S_1^*(\cdot)\|_{L_p(I)},$$

из (2.34), (2.35) получим $\int_I\left(\sum_{n=1}^{\infty} a_n^2\chi_n^2(x)\right)^{\frac{p}{2}}dx \le C_{p,k}\int_I (S^*(x))^p dx$.

Аналогично доказывается неравенство $\int_I (S^*(x))^p dx \le C_{p,k}\int_I\left(\sum_{n=1}^{\infty} a_n^2\chi_n^2(x)\right)^{\frac{p}{2}}dx$. Тем самым доказали следующее соотношение

$$\int_I (S^*(x))^p dx \sim_{p,k} \int_I\left(\sum_{n=1}^{\infty} a_n^2\chi_n^2(x)\right)^{\frac{p}{2}}dx,$$

причем постоянные эквивалентности зависят не от самой I, а от ранга I. Теорема доказана.

В теоремах 2.1 и 2.3, вообще говоря, мажоранту ряда нельзя заменить суммой ряда. Действительно, известно, что в пространстве $L[0,1]$ не существует безусловных базисов (см. [11]). Отсюда следует, что существуют функции $\phi,\psi\in L_1$, такие, что

$$\left(\sum_{n=0}^{\infty} a_n^2 f_n^2(\cdot)\right)^{\frac{1}{2}} \notin L_1, \quad \text{где} \quad a_n = \int_0^1 \phi(x) f_n(x)dx.$$

и

$$\left(\sum_{n=1}^{\infty} b_n^2\chi_n^2(\cdot)\right)^{\frac{1}{2}} \notin L_1, \quad \text{где} \quad b_n = \int_0^1 \psi(x)\chi_n(x)dx.$$

Следовательно

$$\|\sum_n a_n\varphi_n\|_1 \not\sim \|(\sum_n a_n^2\varphi_n^2)^{\frac{1}{2}}\|_1,$$

где $\{\varphi_n\}$-система Хаара или Франклина.

В случае $p > 1$, из безусловной базисности системы Хаара в пространстве L_p и конструкции функций Хаара, для любого двоичного интервала I имеем

$$\left\| \sum_{\Delta_n \subset I} a_n \chi_n \right\|_{L_p(I)} \sim_p \left\| \left(\sum_{\Delta_n \subset I} a_n^2 \chi_n^2 \right)^{\frac{1}{2}} \right\|_{L_p(I)} . \tag{2.36}$$

Интересно было бы выяснить, имеет ли аналог соотношения (2.36) для системы Франклина. Для установления такой эквивалентности необходимо (и достаточно) установить аналог соотношений (2.3), (2.4) для суммы $\sum_{\langle n \rangle \subset I} a_n f_n(x)$.

Abstract. In this paper we prove that the majorant of partial sums and the Paley function of Franklin series have equivalent norms in the space $L_p(I)$, $p > 0$, provided that the "peak" intervals of Franklin functions with non-vanishing coefficients lie in I. Examples of series emphasizing that this condition is essential are also given.

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ON L^p-INTEGRABILITY OF A SPECIAL DOUBLE SINE SERIES FORMED BY ITS BLOCKS

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Abstract. In this paper we deal with a special double sine trigonometric series formed by its blocks. This type of trigonometric series is of particular interest since its blocks always are bounded, that is, under some additional assumptions the sum-function of such series always exists. We give some conditions under which such sum-function is integrable of power $p \in \{2, 3, \dots\}$, as well as is integrable with some natural weight.

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Keywords: Sine series; function of bounded variation; series by their blocks.

1. Introduction

Let $\Lambda_1 = \{n_i\}$ and $\Lambda_2 = \{r_j\}$ be two strictly increasing sequences of natural numbers $1 = n_1 < n_2 < n_3 < \cdots$ and $1 = r_1 < r_2 < r_3 < \cdots$ satisfying the conditions:

$$\sum_{i=1}^{\infty} \frac{1}{n_i} < +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{r_j} < +\infty.$$

Considering the special double sine series

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{\sin kx \sin \ell y}{k\ell},$$

we form the following series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left| \sum_{k=n_i}^{n_{i+1}-1} \sum_{\ell=r_j}^{r_{j+1}-1} \frac{\sin kx \sin \ell y}{k\ell} \right|. \tag{1.1}$$

According to the well-known estimate

$$\left| \sum_{k=v}^{V} \frac{\sin kx}{k} \right| \le \frac{\pi}{vx}, \quad v \le V \le \infty, \quad 0 < x \le \pi, \tag{1.2}$$

the series (1.1) converges for all (x, y) and its sum $G_{\Lambda_1,\Lambda_2}(x, y)$ is a continuous function on $(0, \pi] \times (0, \pi]$. This fact is of particular interest and therefore this is the main reason why we have formed the series (1.1).

In the one-dimensional case such series has been considered by Telyakovskii [1] and Trigub [3]. In particular, Telyakovskii [2] has considered the question: when the sum-function $g_{\Lambda_1}(x)$ of the series

$$\sum_{k=1}^{\infty}\left|\sum_{k=n_i}^{n_{i+1}-1}\frac{\sin kx}{k}\right|$$

belongs to the spaces $L^p[0,\pi]$ for $p = 2,3,\dots$?

Specifically, in [2] was proved the following theorem.

Theorem 1.1. *For any natural $p = 2,3,\dots$ the function $g_{\Lambda_1}(x)$ belongs to the space $L^p[0,\pi]$ if the series $\sum_{i=1}^{\infty}\frac{1}{n_i}m_i^{1-\frac{1}{p}}$ is convergent, where $m_i = \min(n_i, n_{i+1}-n_i+1)$.*

In the same paper was considered the problem of integrability of the function $g_{\Lambda_1}(x)$ with weight $x^{-\gamma}$ under natural condition $0<\gamma<1$. Among others, the following result was proved in [2].

Theorem 1.2. *If for $\gamma\in(0,1)$ the series*

$$\sum_{i=1}^{\infty}\frac{1}{n_i}m_i^{\gamma}$$

is convergent, then the integral $\int_0^{\pi}\frac{1}{x^{\gamma}}g_{\Lambda_1}(x)dx$ converges.

Note that questions pertaining to trigonometric series formed by their blocks were considered in [4] - [6], and still receive considerable attention. The main aim of this paper is to extend the above results to two-dimensional case. In order to do this we will use the technique developed in [2], the estimate (1.2) and the following inequality (see [2], page 818):

$$u_i(x) := \left|\sum_{k=n_i}^{n_{i+1}-1}\frac{\sin kx}{k}\right| \le \frac{A}{n_i}\min\left(\frac{1}{x}, m_i\right), \quad 0<x\le\pi. \tag{1.3}$$

where A is an absolute constant. Here and in the sequel we write $G_{\Lambda_1,\Lambda_2}\in L^p$, $p\ge 1$, if the integral $\int_0^{\pi}\int_0^{\pi}|G_{\Lambda_1,\Lambda_2}(x,y)|^p dxdy$ is finite.

2. The main results

In this section we state and prove the main results of the paper. We first prove the following result.

Theorem 2.1. *For any natural* $p = 2, 3, \dots$ *the function* G_{Λ_1,Λ_2} *belongs to the space* $L^p([0,\pi] \times [0,\pi])$ *if the series*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(m_i s_j)^{1-\frac{1}{p}}}{n_i r_j}$$

is convergent, where $m_i = \min(n_i, n_{i+1} - n_i + 1)$ *and* $s_j = \min(r_j, r_{j+1} - r_j + 1)$.

Proof. For arbitrary natural numbers M and N we have

$$\int_0^\pi \int_0^\pi \left(\sum_{i=1}^{M}\sum_{j=1}^{N} u_i(x) u_j(y) \right)^p dxdy$$

$$= \int_0^\pi \int_0^\pi \sum_{i_1=1}^{M} u_{i_1}(x) \cdots \sum_{i_p=1}^{M} u_{i_p}(x) \sum_{j_1=1}^{N} u_{j_1}(y) \cdots \sum_{j_p=1}^{N} u_{j_p}(y) dxdy$$

$$= \sum_{i_1=1}^{M} \cdots \sum_{i_p=1}^{M} \sum_{j_1=1}^{N} \cdots \sum_{j_p=1}^{N} \int_0^\pi \int_0^\pi u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dxdy. \tag{2.1}$$

Next, we split the square $[0,\pi] \times [0,\pi]$ into the rectangles $[0,\alpha] \times [0,\beta]$, $[0,\alpha] \times [\pi,\beta]$, $[\alpha,\pi] \times [0,\beta]$ and $[\alpha,\pi] \times [\beta,\pi]$, where α and β will be determined later in an appropriate way. Using the estimates (1.3) we can write

$$\int_0^\alpha \int_0^\beta u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dxdy$$

$$\le A^{2p} \int_0^\alpha \int_0^\beta \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} dxdy = A^{2p} \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \alpha\beta, \tag{2.2}$$

$$\int_0^\alpha \int_\beta^\pi u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dxdy \tag{2.3}$$

$$\le A^{2p} \int_0^\alpha \int_\beta^\pi \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{1}{r_{j_1} \cdots r_{j_p}} \frac{dxdy}{y^p} < \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{A^{2p}}{r_{j_1} \cdots r_{j_p}} \frac{\alpha\beta^{1-p}}{p-1},$$

$$\int_\alpha^\pi \int_0^\beta u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dxdy \tag{2.4}$$

$$\le A^{2p} \int_\alpha^\pi \int_0^\beta \frac{1}{n_{i_1} \cdots n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \frac{dxdy}{x^p} < \frac{A^{2p}}{n_{i_1} \cdots n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \frac{\alpha^{1-p}\beta}{p-1},$$

$$\int_\alpha^\pi \int_\beta^\pi u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dxdy \tag{2.5}$$

$$\le A^{2p} \int_\alpha^\pi \int_\beta^\pi \frac{1}{n_{i_1} \cdots n_{i_p}} \frac{1}{r_{j_1} \cdots r_{j_p}} \frac{dxdy}{(xy)^p} < \frac{A^{2p}}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}} \frac{(\alpha\beta)^{1-p}}{(p-1)^2}$$

Inserting the estimates (2.2)-(2.5) into (2.1), we obtain

$$\int_0^\pi\int_0^\pi\left(\sum_{i=1}^{M}\sum_{j=1}^{N}u_i(x)u_j(y)\right)^p dxdy < A^{2p}\sum_{i_1=1}^{M}\cdots\sum_{i_p=1}^{M}\sum_{j_1=1}^{N}$$

$$\cdots\sum_{j_p=1}^{N}\left(\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\alpha\beta+\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{1}{r_{j_1}\cdots r_{j_p}}\frac{\alpha\beta^{1-p}}{p-1}\right.$$

$$\left.+\frac{1}{n_{i_1}\cdots n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{\alpha^{1-p}\beta}{p-1}+\frac{1}{n_{i_1}\cdots n_{i_p}r_{j_1}\cdots r_{j_p}}\frac{(\alpha\beta)^{1-p}}{(p-1)^2}\right). \tag{2.6}$$

Whence, choosing in (2.6) $\alpha=(m_{i_1}\cdots m_{i_p})^{-\frac{1}{p}}$ and $\beta=(s_{j_1}\cdots s_{j_p})^{-\frac{1}{p}}$, we find that

$$\int_0^\pi\int_0^\pi\left(\sum_{i=1}^{M}\sum_{j=1}^{N}u_i(x)u_j(y)\right)^p dx\,dy < 4A^{2p}\sum_{i_1=1}^{M}\cdots\sum_{i_p=1}^{M}\sum_{j_1=1}^{N}\cdots$$

$$\sum_{j_p=1}^{N}\frac{(m_{i_1}\cdots m_{i_p}s_{j_1}\cdots s_{j_p})^{1-\frac{1}{p}}}{n_{i_1}\cdots n_{i_p}r_{j_1}\cdots r_{j_p}} < 4A^{2p}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(m_is_j)^{1-\frac{1}{p}}}{n_ir_j}\right)^p.$$

Consequently, since the last series converges by assumption, the integrals

$$\int_0^\pi\int_0^\pi\left(\sum_{i=1}^{M}\sum_{j=1}^{N}u_i(x)u_j(y)\right)^p dx\,dy$$

are bounded by a quantity that is independent of M, N. Therefore, based on the double version of the Levi's theorem, we conclude that the function G_{Λ_1,Λ_2} belongs to the space $L^p([0,\pi]\times[0,\pi])$. □

The next result gives an answer to the following question: under what conditions the function G_{Λ_1,Λ_2} belongs to the space $L^p([0,\pi]\times[0,\pi])$ with weight $x^{-\gamma_1}y^{-\gamma_2}$, $\gamma_1,\gamma_2\in(0,1)$?

Theorem 2.2. *If for $\gamma_1,\gamma_2\in(0,1)$, the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n_ir_j}m_i^{\gamma_1}s_j^{\gamma_2}$ is convergent, then the following integral converges*

$$\int_0^\pi\int_0^\pi\frac{G_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy$$

Proof. Based on the uniform convergence of the series (1.1) we have

$$\int_0^\pi\int_0^\pi\frac{G_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^\pi\int_0^\pi\frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy.$$

Splitting the square $[0,\pi]\times[0,\pi]$ into the rectangles $[0,\alpha_i]\times[0,\beta_j]$, $[0,\alpha_i]\times[\pi,\beta_j]$, $[\alpha_i,\pi]\times[0,\beta_j]$ and $[\alpha_i,\pi]\times[\beta_j,\pi]$, where α_i and β_j are determined by

$$\alpha_i=\frac{1}{m_i}\quad\text{and}\quad\beta_j=\frac{1}{s_j}, \tag{2.7}$$

we find that

$$\int_0^{\alpha_i}\int_0^{\beta_j}\frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy \le A^2\int_0^{\alpha_i}\int_0^{\beta_j}\frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{m_is_j}{n_ir_j}dxdy = \frac{A^2}{(1-\gamma_1)(1-\gamma_2)}\frac{m_is_j}{n_ir_j}\alpha_i^{1-\gamma_1}\beta_j^{1-\gamma_2}.$$

$$\int_{\alpha_i}^{\pi}\int_0^{\beta_j}\frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy \le A^2\int_{\alpha_i}^{\pi}\int_0^{\beta_j}\frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{s_j}{n_ixr_j}dxdy < \frac{A^2}{\gamma_1(1-\gamma_2)}\frac{s_j}{n_ir_j}\alpha_i^{-\gamma_1}\beta_j^{1-\gamma_2}.$$

$$\int_0^{\alpha_i}\int_{\beta_j}^{\pi}\frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy \le A^2\int_0^{\alpha_i}\int_{\beta_j}^{\pi}\frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{m_i}{n_ir_jy}dxdy < \frac{A^2}{(1-\gamma_1)\gamma_2}\frac{m_i}{n_ir_j}\alpha_i^{1-\gamma_1}\beta_j^{-\gamma_2}.$$

$$\int_{\alpha_i}^{\pi}\int_{\beta_j}^{\pi}\frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy \le A^2\int_{\alpha_i}^{\pi}\int_{\beta_j}^{\pi}\frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{1}{n_ir_jxy}dxdy < \frac{A^2}{\gamma_1\gamma_2}\frac{1}{n_ir_j}\alpha_i^{-\gamma_1}\beta_j^{-\gamma_2}.$$

Finally, using (2.7) and the latest estimates, we obtain

$$\int_0^{\pi}\int_0^{\pi}\frac{G_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy < \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\Big(\frac{A^2}{(1-\gamma_1)(1-\gamma_2)}\frac{m_is_j}{n_ir_j}\alpha_i^{1-\gamma_1}\beta_j^{1-\gamma_2}$$
$$+\frac{A^2}{\gamma_1(1-\gamma_2)}\frac{s_j}{n_ir_j}\alpha^{-\gamma_1}\beta^{1-\gamma_2}+\frac{A^2}{(1-\gamma_1)\gamma_2}\frac{m_i}{n_ir_j}\alpha^{1-\gamma_1}\beta^{-\gamma_2}$$
$$+\frac{A^2}{\gamma_1\gamma_2}\frac{1}{n_ir_j}\alpha^{-\gamma_1}\beta^{-\gamma_2}\Big) = C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n_ir_j}m_i^{\gamma_1}s_j^{\gamma_2} < +\infty.$$

where $C = A^2\cdot\max\left\{\frac{1}{(1-\gamma_1)(1-\gamma_2)},\frac{1}{\gamma_1(1-\gamma_2)},\frac{1}{(1-\gamma_1)\gamma_2},\frac{1}{\gamma_1\gamma_2},\right\}$. □

The next statement supplements Theorem 2.1, and gives conditions under which the integral

$$\int_0^{\pi}\int_0^{\pi}\frac{G^p_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy$$

is convergent for $\gamma_1,\gamma_2\in(0,1)$ and $p = 2,3,\dots$.

Theorem 2.3. *If $p = 2,3,\dots$ and $\gamma_1,\gamma_2\in(1-p,1)$, then the integral*

$$\int_0^{\pi}\int_0^{\pi}\frac{G^p_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy$$

is convergent provided that the series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n_ir_j}m_i^{1-\frac{1}{p}(1-\gamma_1)}s_j^{1-\frac{1}{p}(1-\gamma_2)}$$

is convergent.

Proof. Using a similar technique as in the proof Theorem 2.1 we have

$$\begin{aligned}
&\int_0^\pi\int_0^\pi \frac{1}{x^{\gamma_1}y^{\gamma_2}}\left(\sum_{i=1}^{M}\sum_{j=1}^{N} u_i(x)u_j(y)\right)^p dx\,dy\\
&= \sum_{i_1=1}^{M}\cdots\sum_{i_p=1}^{M}\sum_{j_1=1}^{N}\cdots\sum_{j_p=1}^{N}\int_0^\pi\int_0^\pi \frac{1}{x^{\gamma_1}y^{\gamma_2}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy,
\end{aligned}\tag{2.8}$$

for all $p = 2, 3, \ldots$ and natural numbers M, N.

Again we split the square $[0,\pi]\times[0,\pi]$ into the rectangles $[0,\alpha]\times[0,\beta]$, $[0,\alpha]\times[\pi,\beta]$, $[\alpha,\pi]\times[0,\beta]$ and $[\alpha,\pi]\times[\beta,\pi]$, where α and β are determined as in Theorem 2.1.

Using the estimates (1.3) and taking into account that $\gamma_1,\gamma_2 \in (1-p, 1)$, we can write

$$\begin{aligned}
&\int_0^\alpha\int_0^\beta \frac{1}{x^{\gamma_1}y^{\gamma_2}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy\\
&\qquad\le A^{2p}\int_0^\alpha\int_0^\beta \frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}dxdy\\
&\qquad= A^{2p}\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{\alpha^{1-\gamma_1}\beta^{1-\gamma_2}}{(1-\gamma_1)(1-\gamma_2)},
\end{aligned}\tag{2.9}$$

$$\begin{aligned}
&\int_0^\alpha\int_\beta^\pi \frac{1}{x^{\gamma_1}y^{\gamma_2}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy\\
&\qquad\le A^{2p}\int_0^\alpha\int_\beta^\pi \frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{1}{r_{j_1}\cdots r_{j_p}}\frac{dxdy}{x^{\gamma_1}y^{\gamma_2+p}}\\
&\qquad< \frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{A^{2p}}{r_{j_1}\cdots r_{j_p}}\frac{\alpha^{1-\gamma_1}\beta^{1-\gamma_2-p}}{(1-\gamma_1)(\gamma_2+p-1)},
\end{aligned}\tag{2.10}$$

$$\begin{aligned}
&\int_\alpha^\pi\int_0^\beta \frac{1}{x^{\gamma_1}y^{\gamma_2}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy\\
&\qquad\le A^{2p}\int_\alpha^\pi\int_0^\beta \frac{1}{n_{i_1}\cdots n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{dxdy}{x^{\gamma_1+p}y^{\gamma_2}}\\
&\qquad< \frac{A^{2p}}{n_{i_1}\cdots n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{\alpha^{1-\gamma_1-p}\beta^{1-\gamma_2}}{(\gamma_1+p-1)(1-\gamma_2)},
\end{aligned}\tag{2.11}$$

and

$$\int_\alpha^\pi \int_\beta^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y)\,dx\,dy$$

$$\leq A^{2p} \int_\alpha^\pi \int_\beta^\pi \frac{1}{n_{i_1} \cdots n_{i_p}} \frac{1}{r_{j_1} \cdots r_{j_p}} \frac{dx\,dy}{x^{\gamma_1+p} y^{\gamma_2+p}}$$

$$< \frac{A^{2p}}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}} \frac{\alpha^{1-\gamma_1-p} \beta^{1-\gamma_2-p}}{(\gamma_1+p-1)(\gamma_2+p-1)}. \tag{2.12}$$

The above estimates along with

$$\alpha = \frac{1}{(m_{i_1} \cdots m_{i_p})^{\frac{1}{p}}} \quad \text{and} \quad \beta = \frac{1}{(s_{j_1} \cdots s_{j_p})^{\frac{1}{p}}}$$

imply

$$\int_0^\pi \int_0^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} \left(\sum_{i=1}^M \sum_{j=1}^N u_i(x) u_j(y) \right)^p dx\,dy$$

$$< A(p,\gamma_1,\gamma_2) \sum_{i_1=1}^M \cdots \sum_{i_p=1}^M \sum_{j_1=1}^N \cdots \sum_{j_p=1}^N \frac{(m_{i_1} \cdots m_{i_p})^{1-\frac{1}{p}(1-\gamma_1)} (s_{j_1} \cdots s_{j_p})^{1-\frac{1}{p}(1-\gamma_2)}}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}},$$

where $A(p,\gamma_1,\gamma_2)$ is a constant that depends only on p, γ_1, and γ_2.

Hence,

$$\int_0^\pi \int_0^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} \left(\sum_{i=1}^M \sum_{j=1}^N u_i(x) u_j(y) \right)^p dx\,dy$$

$$< A(p,\gamma_1,\gamma_2) \left(\sum_{i=1}^\infty \sum_{j=1}^\infty \frac{m_i^{1-\frac{1}{p}(1-\gamma_1)} s_j^{1-\frac{1}{p}(1-\gamma_2)}}{n_i r_j} \right)^p.$$

Finally, the use of the double version of the Levi's theorem implies the statement of the theorem. □

It is clear that the conditions $\gamma_1, \gamma_2 > 1-p$ in Theorem 2.3 are essential, therefore in the next theorem we examine the boundary case $\gamma_1, \gamma_2 = 1-p$.

Theorem 2.4. *If $p = 2, 3, \ldots$ and $\gamma_1, \gamma_2 = 1-p$, then the integral*

$$\int_0^\pi \int_0^\pi \frac{G^p_{\Lambda_1,\Lambda_2}(x,y)}{(xy)^{1-p}} dx\,dy$$

is convergent provided that the series

$$\sum_{i=1}^\infty \sum_{j=1}^\infty \frac{1}{n_i r_j} (\log m_i)(\log s_j)$$

is convergent.

Proof. Observe first that in this boundary case the equality (2.8) reduces to the following:

$$\int_0^\pi\int_0^\pi \frac{1}{(xy)^{1-p}}\left(\sum_{i=1}^{M}\sum_{j=1}^{N} u_i(x)u_j(y)\right)^p dxdy \tag{2.13}$$
$$=\sum_{i_1=1}^{M}\cdots\sum_{i_p=1}^{M}\sum_{j_1=1}^{N}\cdots\sum_{j_p=1}^{N}\int_0^\pi\int_0^\pi \frac{1}{(xy)^{1-p}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy.$$

Also, for $\gamma_1, \gamma_2 = 1-p$ the estimates (2.9)-(2.12) take the following forms:

$$\int_0^\alpha\int_0^\beta \frac{1}{(xy)^{1-p}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy$$
$$\leq A^{2p}\int_0^\alpha\int_0^\beta \frac{1}{(xy)^{1-p}}\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}dxdy = A^{2p}\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{(\alpha\beta)^p}{p^2},$$

$$\int_0^\alpha\int_\beta^\pi \frac{1}{(xy)^{1-p}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy$$
$$\leq A^{2p}\int_0^\alpha\int_\beta^\pi \frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{1}{r_{j_1}\cdots r_{j_p}}\frac{dxdy}{(xy)^{1-p}} < \frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{A^{2p}}{r_{j_1}\cdots r_{j_p}}\frac{\alpha^p\log\frac{\pi}{\beta}}{p},$$

$$\int_\alpha^\pi\int_0^\beta \frac{1}{(xy)^{1-p}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy$$
$$\leq A^{2p}\int_\alpha^\pi\int_0^\beta \frac{1}{n_{i_1}\cdots n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{dxdy}{(xy)^{1-p}} < \frac{A^{2p}}{n_{i_1}\cdots n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{\beta^p\log\frac{\pi}{\alpha}}{p},$$

$$\int_\alpha^\pi\int_\beta^\pi \frac{1}{(xy)^{1-p}}u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy$$
$$\leq A^{2p}\int_\alpha^\pi\int_\beta^\pi \frac{1}{n_{i_1}\cdots n_{i_p}}\frac{1}{r_{j_1}\cdots r_{j_p}}\frac{dxdy}{(xy)^{1-p}} < \frac{A^{2p}}{n_{i_1}\cdots n_{i_p}r_{j_1}\cdots r_{j_p}}\log\frac{\pi}{\alpha}\log\frac{\pi}{\beta},$$

respectively.

Next, specifying $\alpha = (m_{i_1}\cdots m_{i_p})^{-\frac{1}{p}}$ and $\beta = (s_{j_1}\cdots s_{j_p})^{-\frac{1}{p}}$ we obviously have

$$\log\frac{\pi}{\alpha} = \log\pi + \frac{1}{p}\log(m_{i_1}\cdots m_{i_p}) \quad \text{and} \quad \log\frac{\pi}{\beta} = \log\pi + \frac{1}{p}\log(s_{j_1}\cdots s_{j_p}).$$

Using these equalities, the above estimates and the equality (2.13)), we obtain

$$\int_0^\pi \int_0^\pi \frac{1}{(xy)^{1-p}} \left(\sum_{i=1}^{M} \sum_{j=1}^{N} u_i(x) u_j(y) \right)^p dxdy$$

$$< \sum_{i_1=1}^{M} \cdots \sum_{i_p=1}^{M} \sum_{j_1=1}^{N} \cdots \sum_{j_p=1}^{N} \frac{A^{2p}}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}} \left\{ \left(\frac{1}{p} + \log \pi \right)^2 \right.$$

$$\left. + \left(\frac{1}{p^2} + \log \pi \right) \left[\sum_{\nu=1}^{p} \log(m_{i_\nu}) + \sum_{\mu=1}^{p} \log(s_{j_\mu}) \right] + \frac{1}{p^2} \sum_{\nu=1}^{p} \sum_{\mu=1}^{p} \log(m_{i_\nu}) \log(s_{j_\mu}) \right\}.$$

Therefore, we have

$$\int_0^\pi \int_0^\pi \frac{G^p_{\Lambda_1,\Lambda_2}(x,y)}{(xy)^{1-p}} dxdy$$

$$\leq K A^{2p} \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \sum_{j_1=1}^{\infty} \cdots \sum_{j_p=1}^{\infty} \frac{1}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}} \sum_{\nu=1}^{p} \sum_{\mu=1}^{p} \log(m_{i_\nu}) \log(s_{j_\mu})$$

$$\leq K A^{2p} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{n_i r_j} \right)^{p-1} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\log(m_i) \log(s_j)}{n_i r_j},$$

where K is an absolute positive constant. The proof is completed. □

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BRUCK CONJECTURE FOR A LINEAR DIFFERENTIAL POLYNOMIAL

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Abstract. In the paper we study the Brück Conjecture for a linear differential polynomial.

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1. Introduction, definitions and results

Let f, g be nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f, g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicities, and we say that f, g share the value a IM (ignoring multiplicities) if f, g have the same a-points but the multiplicities are not taken into account.

The monograph [7] is a good source of standard notations and definitions of the value distribution theory. We now introduce some notation and a definition.

Definition 1.1. *Given a meromorphic function f, a number $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer k,*

(i) $N_{(k}(r, a; f)$ $(\overline{N}_{(k}(r, a; f))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than k;*

(ii) $N_{k)}(r, a; f)$ $(\overline{N}_{k)}(r, a; f))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than k;*

Definition 1.2. *A meromorphic function $a = a(z)$ is called a small function of a meromorphic function f if $T(r, a) = S(r, f)$.*

In [5], R. Brück considered the uniqueness problem of an entire function when it shares a single value CM with its first derivative, and proposed the following conjecture, which inspired a number of people to work on the topic.

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Brück Conjecture: *Let f be a nonconstant entire function satisfying $\nu(f) < \infty$, and let $\nu(f)$ be not a positive integer, where $\nu(f)$ is the hyper-order of f. If f and f' share one finite value a CM, then $f' - a = c(f - a)$ for some constant $c \neq 0$.*

R. Brück [5] himself proved the following result.

Theorem A ([5]). *Let f be a nonconstant entire function. If f and f' share the value 1 CM and $N(r, 0; f') = S(r, f)$, then $f - 1 = c(f' - 1)$, where c is a nonzero constant.*

Considering entire functions of finite order, L. Z. Yang [9] proved the following theorem.

Theorem B ([9]). *Let f be a nonconstant entire function of finite order, and let $a(\neq 0)$ be a finite constant. If f and $f^{(k)}$ share the value a CM, then $f - a = c(f^{(k)} - a)$, where c is a nonzero constant and $k \geq 1$ is an integer.*

In 2005, A. H. H. Al-khaladi [2] extended Theorem A to the class of meromorphic functions and proved the following result.

Theorem C ([2]). *Let f be a nonconstant meromorphic function satisfying $N(r, 0; f') = S(r, f)$. If f and f' share the value 1 CM, then $f - 1 = c(f' - 1)$ for some nonzero constant c.*

Also, in [2] were considered the following examples, showing that the value sharing cannot be relaxed from CM to IM, and the condition $N(r, 0; f') = S(r, f)$ is essential.

Example 1.1. Let $f = 1 + \tan z$. Then $f' - 1 = (f - 1)^2$ and $N(r, 0; f') \equiv 0$. Clearly f and f' share the value 1 IM but the conclusion of Theorem C does not hold.

Example 1.2. Let $f = \frac{z}{1 + e^{-z}}$. Then f and f' share the value 1 CM and $N(r, 0; f') \neq S(r, f)$. It is easy to verify that $f' - 1 = \frac{1}{1 + e^{z}}(f - 1)$.

A. H. H. Al-khaladi [1] also observed by the following example that in Theorem A the shared value cannot be replaced by a shared small function.

Example 1.3. Let $f = 1 + e^{e^z}$ and $a = \frac{1}{1 - e^{-z}}$. Then a is a small function of f and $f - a$, $f' - a$ share the value 0 CM and $N(r, 0; f') \equiv 0$. Also, we see that $f - a = \frac{1}{e^z}(f' - a)$.

Considering the sharing of small functions, A. H. H. Al-khaladi [1] proved the following result.

Theorem D ([1]). *Let f be a nonconstant entire function satisfying $N(r,0;f') = S(r,f)$, and let $a\,(\not\equiv 0,\infty)$ be a meromorphic small function of f. If $f-a$ and $f'-a$ share the value 0 CM, then $f-a = \left(1+\frac{c}{a}\right)(f'-a)$, where $1+\frac{c}{a} = e^{\beta}$, c is a constant and β is an entire function.*

For higher order derivatives, A. H. H. Al-khaladi [3] proved the following theorem.

Theorem E ([3]). *Let f be a nonconstant entire function satisfying $\overline{N}(r,0;f^{(k)}) = S(r,f)$ $(k>1)$, and let $a\,(\not\equiv 0,\infty)$ be a meromorphic small function of f. If $f-a$ and $f^{(k)}-a$ share the value 0 CM, then $f-a = \left(1+\frac{P_{k-1}}{a}\right)(f^{(k)}-a)$, where P_{k-1} is a polynomial of degree at most $k-1$ and $1+\frac{P_{k-1}}{a} \not\equiv 0$.*

Recently A. H. H. Al-khaladi [4] extended Theorem E to meromorphic functions. A natural extension of a derivative is a linear differential polynomial. For a transcendental meromorphic function f we denote by $L = L(f^{(k)})$ a linear differential polynomial of the form

$$L = L(f^{(k)}) = a_0 f^{(k)} + a_1 f^{(k+1)} + \cdots + a_p f^{(k+p)}, \tag{1.1}$$

where $a_0, a_1, \ldots, a_p\,(\neq 0)$ are constants, and $k\,(\geq 1)$ and $p(\geq 0)$ are integers such that $p=0$ if $k=1$ and $0 \leq p \leq k-2$ if $k \geq 2$.

In the present paper we consider the problem of sharing a small function by a meromorphic function and a linear differential polynomial in conformity with Brück conjecture. The following theorem is the main result of the paper.

Theorem 1.1. *Let f be a transcendental meromorphic function and let the differential polynomial $L = L(f^{(k)})$, given by (1.1), be nonconstant. Suppose that $f-a$ and $L-a$ share 0 CM, where $a\,(\not\equiv 0,\infty)$ is a small function of f. If $N(r,0;f^{(k)}) = S(r,f)$, then*

$$f-a = \left(1+\frac{P_{k-1}}{a}\right)(L-a),$$

where P_{k-1} is a polynomial of degree at most $k-1$ and $1+\frac{P_{k-1}}{a} \not\equiv 0$.

The following example shows that the condition $N(r,0;f^{(k)}) = S(r,f)$ is essential in Theorem 1.1.

Example 1.4. Let P be a nonconstant polynomial, and let $f = \dfrac{Pe^z}{1+e^z}$. Then $f' = \dfrac{e^z(P+P'+P'e^z)}{(1+e^z)^2}$, and hence $N(r,0;f') \neq S(r,f)$. Also $f - P'$ and $f' - P'$ share 0 CM but $f' - P' = \dfrac{1}{1+e^z}(f - P')$, where $T(r,P') = S(r,f)$.

2. Lemmas

In this section we present some necessary lemmas to be used in the proof of Theorem 1.1.

Lemma 2.1. *Let f be a nonconstant meromorphic function and let $L = L(f^{(k)})$, given by (1.1), be nonconstant. If $f-a$ and $L-a$ share 0 CM, where $a = a(z)(\not\equiv 0, \infty)$ is a small function of f, then one of the following assertions holds:*

(i) $f - a = \left(1 + \frac{P_{k-1}}{a}\right)(L-a)$, *where P_{k-1} is a polynomial of degree at most $k-1$ and $1 + \dfrac{P_{k-1}}{a} \not\equiv 0$,*

(ii) $T(r,f^{(k)}) \leq (k+p+1)\overline{N}(r,\infty;f) + \overline{N}(r,0;f^{(k)}) + N(r,0;f^{(k)}) + S(r,f)$.

Proof. Let $h = \dfrac{f-a}{L-a}$. Then h is an entire function and the poles of f are precisely the zeros of h. Now differentiating

$$f - a = hL - ah \tag{2.1}$$

k-times we get

$$f^{(k)} - a^{(k)} = (hL)^{(k)} - (ha)^{(k)}. \tag{2.2}$$

We now consider the following cases.

Case I. Let $a^{(k)} \not\equiv 0$. We put

$$W = \frac{(hL)^{(k)}}{hf^{(k)}} - \frac{(ha)^{(k)}}{ha^{(k)}}. \tag{2.3}$$

Since $W' = \dfrac{(hL)^{(k)}}{hL}\cdot\dfrac{L}{f^{(k)}} - \dfrac{(ha)^{(k)}}{ha}\cdot\dfrac{a}{a^{(k)}}$, we have $m(r,W) = S(r,f)$.

We first suppose that $W \not\equiv 0$. Let z_0 be a zero of $f^{(k)} - a^{(k)}$ and $a^{(k)}(z_0) \neq 0, \infty$. Then from (2.2) we see that z_0 is a zero of $(hL)^{(k)} - (ha)^{(k)}$. Hence $W(z_0) = 0$ and we have

$$\begin{aligned} \overline{N}(r,0;f^{(k)} - a^{(k)}) &\leq N(r,0;W) + S(r,f) \\ &\leq T(r,W) + S(r,f) \\ &= N(r,W) + S(r,f). \end{aligned} \tag{2.4}$$

Also

$$N(r,W) \le (k+p)\overline{N}(r,\infty;f) + N(r,0;f^{(k)}) + S(r,f). \tag{2.5}$$

By Nevanlinna's three small functions theorem (see [7], p. 47), and formulas (2.4) and (2.5), we get

$$T(r,f^{(k)}) \le (k+p+1)\overline{N}(r,\infty;f) + \overline{N}(r,0;f^{(k)}) + N(r,0;f^{(k)}) + S(r,f),$$

which is (ii).

Now let $W \equiv 0$. Then from (2.2) and (2.3) we get

$$(f^{(k)} - a^{(k)})a^{(k)} \equiv (ha)^{(k)}(f^{(k)} - a^{(k)}).$$

Since $f^{(k)} - a^{(k)} \not\equiv 0$, we obtain $(ha)^{(k)} \equiv a^{(k)}$. Integrating the last equality k-times we get $ha = a + P_{k-1}(z)$, where $P_{k-1}(z)$ is a polynomial of degree at most $k-1$. So $h = 1 + \dfrac{P_{k-1}}{a}$ and hence $f - a = \left(1 + \dfrac{P_{k-1}}{a}\right)(L-a)$, which is (i).

Case II. Let $a^{(k)} \equiv 0$. Then a is a polynomial of degree at most $k-1$. From (2.2) we get $f^{(k)} = (hL)^{(k)} - (ah)^{(k)}$, and hence

$$\frac{1}{h} = \frac{(hL)^{(k)}}{hf^{(k)}} - \frac{(ah)^{(k)}}{hf^{(k)}}. \tag{2.6}$$

Putting $F = f^{(k)}$, $G = \dfrac{(hL)^{(k)}}{hf^{(k)}}$ and $b = \dfrac{(ah)^{(k)}}{h}$, from (2.6) we get

$$\frac{1}{h} = G - \frac{b}{F}. \tag{2.7}$$

Differentiating (2.7) we obtain

$$-\frac{1}{h}\cdot\frac{h'}{h} = G' - \frac{b'}{F} + \frac{b}{F}\cdot\frac{F'}{F}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\frac{A}{F} = G' + G\cdot\frac{h'}{h}, \tag{2.9}$$

where $A = b\cdot\dfrac{h'}{h} + b' - b\cdot\dfrac{F'}{F}$.

We first suppose that $G \equiv 0$. Then by integration we get $hL = Q_{k-1}$, where $Q_{k-1} = Q_{k-1}(z)$ is a polynomial of degree at most $k-1$. Putting $h = \dfrac{f-a}{L-a}$ we get

$$(f-a)L = (L-a)Q_{k-1}. \tag{2.10}$$

Since a is a polynomial, from (2.10) we see that f is an entire function. Hence h is an entire function having no zeros. We put $h = e^{\alpha}$, where α is an entire function, and so $f = a + h(L-a) = a + Q_{k-1} - ae^{\alpha}$ and $L = Q_{k-1}e^{-\alpha}$. It follows from the definition of L that $L = R(a', \alpha')e^{\alpha}$, where $R(a', \alpha')$ is a differential polynomial in a' and α'. Hence

$$R(a', \alpha')e^{2\alpha} = Q_{k-1}. \tag{2.11}$$

From (2.11) we see that $T(r, e^{\alpha}) = S(r, e^{\alpha})$, yielding a contradiction. Therefore $G \not\equiv 0$.

If h is a constant, say c, then $f - a = c(L - a)$, which is (i).

Now we suppose that h is nonconstant and $b \equiv 0$. Then by integration we get $ah = P_{k-1}$, where $P_{k-1} = P_{k-1}(z)$ is a polynomial of degree at most $k - 1$.

Since h is an entire function and a is a polynomial of degree at most $k - 1$, the equality $h = \frac{P_{k-1}}{a}$ implies that a is a factor of P_{k-1}, and hence

$$h = Q^*_{k-t}, \tag{2.12}$$

where $Q^*_{k-t} = Q^*_{k-t}(z)$ is a polynomial of degree at most $k - t$ $(t \geq 1)$.

If z_0 is a pole of f, then z_0 is a zero of h with multiplicity $k + p$, which is impossible by (2.12). So, f is an entire function, and hence h is an entire function having no zeros. Therefore from (2.12) we see that h is a constant, which is impossible.

Now we suppose that $b \not\equiv 0$. Let $A = 0$, then from (2.9) we get $\frac{G'}{G} + \frac{h'}{h} \equiv 0$. By integration we obtain $Gh = K$ and hence

$$(hL)^{(k)} = Kf^{(k)}, \tag{2.13}$$

where K is a nonzero constant.

Again, $\frac{A}{b} = \frac{h'}{h} + \frac{b'}{b} - \frac{F'}{F} = 0$ implies by integration $hb = MF$, and so

$$(ah)^{(k)} = Mf^{(k)}, \tag{2.14}$$

where M is a nonzero constant.

Since a is a polynomial and h is an entire function, we see from (2.14) that f is an entire function. So, h is an entire function having no zeros and we can put $h = e^{\alpha}$, where α is an entire function.

Integrating (2.13) k-times we get

$$hL = Kf + P^*_{k-1}, \tag{2.15}$$

where $P^*_{k-1} = P^*_{k-1}(z)$ is a polynomial of degree at most $k - 1$.

Since $hL = f - a + ah$, from (2.15) we get

$$(1 - K)f = a(1 - e^{\alpha}) + P^*_{k-1}. \tag{2.16}$$

If $K = 1$, from (2.16) we see that $e^{\alpha} = 1 + \frac{P^*_{k-1}}{a}$, which is impossible. Hence $K \neq 1$. Now from (2.16) we get

$$f = \frac{ae^{\alpha}}{K-1} - \frac{a + P^*_{k-1}}{K-1}. \tag{2.17}$$

Therefore from (1.1) we have

$$L = R(\alpha')e^{\alpha}, \tag{2.18}$$

where $R(\alpha')(\not\equiv 0)$ is a differential polynomial in α' with polynomial coefficients.

From (2.15) we obtain

$$L = \frac{Ka}{K-1} - \frac{Ka + P^*_{k-1}}{K-1}e^{-\alpha}. \tag{2.19}$$

It follows from (2.18) and (2.19) that

$$R(\alpha')e^{2\alpha} = \frac{Ka}{K-1}e^{\alpha} - \frac{Ka + P^*_{k-1}}{K-1}.$$

This implies $T(r, e^{\alpha}) = S(r, e^{\alpha})$, yielding a contradiction. Therefore $A \not\equiv 0$.

Now observe that $A = b\left(\frac{h'}{h} + \frac{b'}{b} - \frac{F'}{F}\right)$ implies $m(r, A) = S(r, f)$. Also, the poles of A are contributed by: (i) the poles of $b = \frac{(ah)^{(k)}}{h}$, (ii) the poles of $\frac{h'}{h}$ and (iii) the poles of $\frac{F'}{F} = \frac{f^{(k+1)}}{f^{(k)}}$. Since h is entire and the zeros of h are precisely the poles of f, and each zero of h is of multiplicity $k + p$, we get

$$N(r, A) \leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + S(r, f).$$

Therefore

$$T(r, A) \leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + S(r, f). \tag{2.20}$$

From (2.9) and (2.20) we get

$$\begin{aligned} m(r, \frac{1}{F}) &\leq m(r, \frac{1}{A}) + m(r, G' + G\frac{h'}{h}) \leq T(r, A) + S(r, f) \\ &\leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + S(r, f). \end{aligned}$$

So, by the first fundamental theorem, we obtain

$$T(r, f^{(k)}) \leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + N(r, 0; f^{(k)}) + S(r, f),$$

which implies (ii). This completes the proof of Lemma 2.1. □

Lemma 2.2 ([2]). *Let k be a positive integer and f be a meromorphic function such that $f^{(k)}$ is not constant. Then either $\left(f^{(k+1)}\right)^{k+1} = c\left(f^{(k)} - \lambda\right)^{k+2}$ for some nonzero constant c or*

$$kN_{1)}(r,\infty;f) \le \overline{N}_{(2}(r,\infty;f) + N_{1)}(r,\lambda;f^{(k)}) + \overline{N}(r,0;f^{(k+1)}) + S(r,f),$$

where λ is a constant.

Lemma 2.3 ([10], p.39). *Let f be a nonconstant meromorphic function in the complex plane and let k be a positive integer. Then*

$$N(r,0;f^{(k)}) \le N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.4 ([8]). *Given a transcendental meromorphic function f and a constant $K > 1$. Then there exists a set $M(K)$ whose upper logarithmic density is at most*

$$\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e(K-1)\exp(e(1-K)))\}$$

such that for every positive integer k,

$$\limsup_{r\to\infty, r\notin M(K)} \frac{T(r,f)}{T(r,f^{(k)})} \le 3eK.$$

Lemma 2.5. *Let f be a transcendental meromorphic function such that $N(r,0;f^{(1)}) = S(r,f)$. If $f-a$ and $a_1 f^{(1)} - a$ share 0 CM, where $a = a(z)(\not\equiv 0,\infty)$ is a small function of f and a_1 is a nonzero constant, then*

$$N_{1)}(r,0;f^{(2)}) \le \overline{N}_{(2}(r,\infty;f) + S(r,f).$$

Proof. If $a+a' \equiv 0$, then using the method of [4] (pp. 349 - 351), we get $N_{1)}(r,0;f^{(2)}) = S(r,f)$, and the result follows. If $a + a' \not\equiv 0$, then again using the method of [4] (pp. 351 - 354), we get $N_{1)}(r,\infty;f) = S(r,f)$. Now by Lemma 2.3 we obtain

$$\begin{aligned} N(r,0;f^{(2)}) &\le N(r,0;f^{(1)}) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}_{(2}(r,\infty;f) + S(r,f). \end{aligned}$$

Since $N_{1)}(r,0;f^{(2)}) \le N(r,0;f^{(2)})$, the lemma is proved. □

Lemma 2.6 ([6]). *Let f be a transcendental meromorphic function and k be a positive integer. Then*

$$k\overline{N}(r,\infty;f) \le N(r,0;f^{(k)}) + (1+\varepsilon)N(r,\infty;f) + S(r,f),$$

where ε is any fixed positive number.

3. Proof of Theorem 1.1

Proof. First we verify that

$$\left(f^{(k+1)}\right)^{k+1} \neq c\left(f^{(k)}\right)^{k+2}, \tag{3.1}$$

where $c \neq 0$ is a constant. Indeed, if (3.1) does not hold, then we get

$$\left(\frac{f^{(k+1)}}{f^{(k)}}\right)^{k+1} = cf^{(k)}. \tag{3.2}$$

Differentiating (3.2) and then using (3.2) we obtain

$$\left(\frac{f^{(k+1)}}{f^{(k)}}\right)^{-2}\left(\frac{f^{(k+1)}}{f^{(k)}}\right)' = \frac{1}{k+1}.$$

Integrating twice we get

$$f^{(k)} = \frac{1}{\{Cz + D(k+1)\}^{k+1}},$$

where $C \neq 0$ and D are constants. This is impossible because f is transcendental.

Let $k \geq 2$. We suppose that

$$T(r, f^{(k)}) \leq (k+p+1)\overline{N}(r,\infty; f) + \overline{N}(r,0; f^{(k)}) + N(r,0; f^{(k)}) + S(r,f).$$

Since $N(r,0; f^{(k)}) = S(r,f)$, we get from above

$$T(r, f^{(k)}) \leq (k+p+1)\overline{N}(r,\infty; f) + S(r,f). \tag{3.3}$$

Also, from Lemma 2.6 we obtain for $0 < \varepsilon < \frac{k}{p+1} - 1$,

$$k\overline{N}(r,\infty; f) \leq (1+\varepsilon)N(r,\infty; f) + S(r,f).$$

Hence from (3.3) we obtain

$$m(r, f^{(k)}) + N(r,\infty; f) \leq \frac{p+1}{k}(1+\varepsilon)N(r,\infty; f) + S(r,f)$$

and so $m(r, f^{(k)}) + N(r,\infty; f) = S(r,f)$. Therefore

$$T(r, f^{(k)}) = S(r,f). \tag{3.4}$$

Let $M(K)$ be defined as in Lemma 2.4. By (3.4) we can choose a sequence $r_n \to \infty$ such that $r_n \notin M(K)$ and $\lim_{n\to\infty} \frac{T(r_n, f^{(k)})}{T(r_n, f)} = 0$. This contradicts Lemma 2.4.

Next, let $k = 1$. We suppose

$$T(r, f^{(1)}) \leq 2\overline{N}(r,\infty; f) + \overline{N}(r,0; f^{(1)}) + N(r,0; f^{(1)}) + S(r,f).$$

Since $N(r,0; f^{(1)}) = S(r,f)$, we obtain

$$m(r, f^{(1)}) + N(r,\infty; f) \leq \overline{N}(r,\infty; f) + S(r,f)$$

and so

(3.5) $$m(r, f^{(1)}) + N_{(2}(r, \infty; f) = S(r, f).$$

By the second fundamental theorem we get in view of (3.5)

$$T(r, f^{(1)}) \le N(r, 1; f^{(1)}) + N(r, 0; f^{(1)}) + \overline{N}(r, \infty; f) - N(r, 0; f^{(2)}) + S(r, f)$$

and so

(3.6) $$m(r, 1; f^{(1)}) + N(r, 0; f^{(2)}) \le N_{1)}(r, \infty; f) + S(r, f).$$

Now by Lemma 2.2 and (3.5) we get for $\lambda = 0$

(3.7) $$N_{1)}(r, \infty; f) \le \overline{N}(r, 0; f^{(2)}) + S(r, f).$$

From (3.6) and (3.7) we get

(3.8) $$N_{(2}(r, 0; f^{(2)}) = S(r, f).$$

By (3.5), (3.8) and Lemma 2.5 we obtain

(3.9) $$N(r, 0; f^{(2)}) = S(r, f).$$

Hence by (3.5), (3.7) and (3.9) we get $N(r, \infty; f) = S(r, f)$, and so by (3.5) we have $T(r, f^{(1)}) = S(r, f)$, which is (3.4) for $k = 1$. Similarly using Lemma 2.4 we arrive at a contradiction. Therefore by Lemma 2.1 we obtain

$$f - a = \left(1 + \frac{P_{k-1}}{a}\right)(L - a).$$

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FIXED POINTS OF MIXED MONOTONE OPERATORS FOR EXISTENCE AND UNIQUENESS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems by using new fixed point results of mixed monotone operators on cones.

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1. Introduction

In recent years, boundary value problems for nonlinear fractional differential equations with a variety of boundary conditions have been investigated by many researchers. Fractional differential equations appear naturally in various fields of science and engineering, and thus constitute an important field of research (see [1 – 3]). As a matter of fact, fractional derivatives provide a powerful tool for the description of memory and hereditary properties of various materials and processes. A significant feature of a fractional order differential operator, in contrast to its counterpart in classical calculus, is its nonlocal behavior, meaning that the future state of a dynamical system or process based on the fractional differential operator depends on its current state as well its past states. In other words, differential equations of arbitrary order are capable of describing memory and hereditary properties of certain important materials and processes. This aspect of fractional calculus has contributed towards the growing popularity of the subject. Mixed monotone operators were introduced by Guo and Lakshmikantham in [4]. Their study has wide applications in the applied sciences such as engineering, biological chemistry technology, nuclear physics and in mathematics (see [6 – 8]). Various existence and uniqueness theorems of fixed points for mixed monotone operators have been obtained by a number of authors (see [9] – [12]). Bhaskar and Lakshmikantham, [9], established some coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces and discussed a question of existence and uniqueness of a solution for a periodic boundary value problem. Recently Y. Sang, [13], proved some new existence and uniqueness theorems of a fixed point of mixed monotone operators with perturbations.

In this paper, by applying Sang's results, we obtain some new results on the existence and uniqueness of positive solutions for some nonlinear fractional differential equations via given boundary value problems.
We first introduce some notations, definitions and known results to be used in the paper.

Definition 1.1 ([1, 2]). *For a continuous function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined by*

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n-1 < \alpha < [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 1.2 ([1, 2]). *The Riemann-Liouville fractional derivative of order α for a continuous function f is defined by*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n-1}} ds, \qquad n = [\alpha]+1.$$

where the right-hand side is defined pointwise on $(0,\infty)$.

Definition 1.3 ([1, 2]). *Let $[a,b]$ be an interval in $\mathbb{R}$ and $\alpha > 0$. The Riemann-Liouville fractional order integral of a function $f \in L^1([a,b],\mathbb{R})$ is deined by*

$$I_a^{\alpha} f(t) = \frac{1}{\gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

whenever the integral exists.

Let $(E, \|\cdot\|)$ be a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \neq y$, then we denote $x < y$ or $x > y$. Also, the zero element of E we denote by θ. Recall that a non-empty closed convex set $P \subset E$ is called a cone if it satisfies the conditions: (i) $x \in P$, $\lambda \geq 0 \Longrightarrow \lambda x \in P$, (ii) $x \in P$, $-x \in P \Longrightarrow x = \theta$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$. Also, we define the order interval $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ for all $x_1, x_2 \in E$. We say that an operator $A : E \to E$ is increasing whenever $x \leq y$ implies $Ax \leq Ay$.

Definition 1.4 ([4, 5]). *Let $D \subset E$. An operator $A : D \times D \to D$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, that is, $u_i, v_i \in D\,(i = 1, 2)$, $u_1 \leq u_2, v_1 \geq v_2$ implies $A(u_1, v_1) \leq A(u_2, v_2)$.*

An element $x^* \in D$ is called a fixed point of A if it satisfies $A(x^*, x^*) = x^*$. For $h > \theta$ we define $P_h = \{x \in E| \quad \exists \lambda, \mu > 0;\ \lambda h \leq x \leq \mu h\}$.

In this paper, using the existence and uniqueness results for the solution of the following operator equation

$$A(x, x) + Bx = x, \tag{1.1}$$

where A is a mixed monotone operator, B is sublinear and E is a real ordered Banach space, obtained in [13] by the partial ordering theory and monotone iterative technique, we study a question of existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems.

Theorem 1.1 ([13]). *Let P be a normal cone in E. $A : P \times P \to P$ be a mixed monotone operator, and let $B : E \to E$ be sublinear. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t)$ and $\varphi(t, x, y)$ defined on an interval (a, b) such that:*

(H_1) $\tau : (a, b) \to (0, 1)$ is surjection;

(H_2) $\varphi(t, x, y) > \tau(t)$ for all $t \in (a, b)$, $x, y \in P$;

(H_3) $A(\tau(t)x, \frac{1}{\tau(t)}y) \geq \varphi(t, x, y)A(x, y)$ for all $t \in (a, b)$, $x, y \in P$;

(H_4) $(I - B)^{-1} : E \to E$ exists and is an increasing operator.

Furthermore, for any $t \in (a, b)$ the function $\phi(t, x, y)$ is nonincreasing in x for fixed y, and nondecreasing in y for fixed x. In addition, suppose that there exist $h \in P - \{\theta\}$ and $t_0 \in (a, b)$ such that

$\tau(t_0)h \leq (I - B)^{-1}A(h, h) \leq \frac{\varphi(t_0, \frac{1}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h.$

Then the following assertions hold:

(i) there are $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$ and

$$u_0 \leq (I - B)^{-1}A(u_0, v_0) \leq (I - B)^{-1}A(v_0, u_0) \leq v_0;$$

(ii) the equation (1.1) has a unique solution x^* in $[u_0, v_0]$;

(iii) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = (I - B)^{-1}A(x_{n-1}, y_{n-1}), \quad y_n = (I - B)^{-1}A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $\| x_n - x^* \| \to 0$ and $\| y_n - x^* \| \to 0$ as $n \to \infty$.

2. Main results

We study the existence and uniqueness of a solution of a fractional differential equation on a partially ordered Banach space with two types of boundary conditions and two types of fractional derivatives. We first study the existence and uniqueness of a positive solution for the following fractional differential equation:

$$\frac{D^\alpha}{Dt}u(t) = f(t, u(t), u(t)), \qquad t \in [0, 1],\ 3 < \alpha \leq 4, \tag{2.1}$$

subject to conditions

$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{2.2}$$

where D^α is the Riemann-Liouville fractional derivative of order α.

Consider the Banach space of continuous functions on $[0,1]$ with sup norm and set $P = \{y \in C[0,1] : \min_{t\in[0,1]} y(t) \geq 0\}$. Then P is a normal cone. The next two lemmas were proved in [14].

Lemma 2.1 ([14]). *Given $y \in C[0,1]$ and a number α such that $3 < \alpha \leq 4$. Then the unique solution of the following fractional differential equation boundary value problem*

$$\begin{aligned} &\frac{D^\alpha}{Dt} u(t) = f(t, y(t)), \qquad t \in [0,1],\ 3 < \alpha \leq 4, \\ &u(0) = u'(0) = u(1) = u'(1) = 0, \end{aligned} \tag{2.3}$$

is given by

$$u(t) = \int_0^1 G(t,s) f(s, y(s))ds,$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-2} t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.4}$$

If $f(t, u(t)) = 1$, then the unique solution of (2.3) is given by
$u_0(t) = \int_0^1 G(t,s)ds = \frac{1}{\Gamma(\alpha+1)} t^{\alpha-2}(1-t)^2$.

Lemma 2.2 ([14]). *The Green's function $G(t,s)$ has the following properties:*
(1) $G(t,s) > 0$ and $G(t,s)$ is continuous for $t, s \in [0,1]$;
(2) $\qquad \frac{(\alpha-2)h(t)k(s)}{\Gamma(\alpha)} \leq G(t,s) \leq \frac{M_0 k(s)}{\Gamma(\alpha)}$,
where $M_0 = \max\{\alpha-1, (\alpha-2)^2\}$, $\quad h(t) = t^{\alpha-2}(1-t)^2$, $\quad k(s) = s^2(1-s)^{\alpha-2}$.

Now we are ready to state and prove our first main result.

Theorem 2.1. *Let $f(t, u(t), v(t)) \in C([0,1] \times [0,\infty) \times [0,\infty))$ be an increasing in u and decreasing in v function. Assume that for all $a < t < b$ there exist two positive-valued functions $\tau(t)$ and $\varphi(t,u,v)$ defined on an interval (a,b) such that:*
(H_1) $\tau : (a,b) \to (0,1)$ is surjection;
(H_2) $\varphi(t,u,v) > \tau(t)$ for all $t \in (a,b)$, $u, v \in P$;
(H_3) $\int_0^1 G(t,s) f(s, \tau(t)u(s), \frac{1}{\tau(t)}v(s)) \geq \varphi(t,u,v) \int_0^1 G(t,s) f(s, u(s), v(s))ds$.
Furthermore, for any $t \in (a,b)$ the function $\varphi(t,u,v)$ is nonincreasing in u for fixed v, and nondecreasing in v for fixed u. In addition, suppose that there exist $h \in P - \{\theta\}$

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Furthermore, for any $t \in (a, b)$ the function $\varphi(t, x, y)$ is nonincreasing in x for fixed y, and nondecreasing in y for fixed x. In addition, suppose that there exist $h \in P - \{\theta\}$ and $t_0 \in (a, b)$ such that
$\tau(t_0)h \leq (I - B)^{-1}A(h, h) \leq \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h.$
Then the following assertions hold:
(i) there are $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$ and

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where $M_0 = \max\{\alpha - 1, (\alpha-2)^2\}$, $\quad h(t) = t^{\alpha-2}(1-t)^2$, $\quad k(s) = s^2(1-s)^{\alpha-2}$.

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Furthermore, for any $t \in (a,b)$ the function $\varphi(t,u,v)$ is nonincreasing in u for fixed v, and nondecreasing in v for fixed u. In addition, suppose that there exist $h \in P - \{\theta\}$

and $t_0 \in (a,b)$ such that

$$\tau(t_0)h \leq \int_0^1 G(t,s)f(s,h(s),h(s))ds \leq \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h. \tag{2.5}$$

Then the following assertions hold:

(i) there are $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 \leq v_0$, and $u_0 \leq \int_0^1 G(t,s)f(s,u_0(s),v_0(s))ds \leq \int_0^1 G(t,s)f(s,v_0(s),u_0(s))ds \leq v_0$,

(ii) the problem (2.1), (2.2) has a unique solution x^* in $[u_0, v_0]$,

(iii) for any initial values $u_0, v_0 \in P_h$ and $n = 1, 2, \ldots$, constructing successively the sequences

$$u_n = \int_0^1 G(t,s)f(s,u_{n-1}(s),v_{n-1}(s))ds, \quad v_n = \int_0^1 G(t,s)f(s,v_{n-1}(s),u_{n-1}(s))ds,$$

we have $\| u_n - u^* \| \to 0$ and $\| v_n - v^* \| \to 0$ as $n \to \infty$.

Proof. By Lemma 2.1, the problem is equivalent to equation $u(t) = \int_0^1 G(t,s)f(s,y(s))ds$, where $G(t,s)$ is defined by (2.4). Define the operator $A : P \times P \to E$ as follows: $A(u(t), v(t)) = \int_0^1 G(t,s)f(s,u(s),v(s))ds$, and observe that u is a solution for the problem if and only if $u = A(u,u)$.

Next, it is easy to see that the operator A is increasing in u and decreasing in v on P. Hence, under the assumptions of the theorem we have $A(\tau(t)u, \frac{1}{\tau(t)}v) \geq \varphi(t,u,v)A(u,v)$ for all $t \in (a,b)$, $u, v \in P$ and $\tau(t_0)h \leq A(h,h) \leq \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h$. Thus, the operator A satisfies all the conditions of Theorem 1.1, and hence A has a unique positive solution (u^*, u^*) such that $A(u^*, u^*) = u^*$, $u^* \in [u_0, v_0]$. □

Example 2.1. Consider the following periodic boundary value problem:

$$D^{\frac{7}{2}}u(t) = f(t,u(t),u(t)) = g(t) + u(t) + \frac{1}{\sqrt{u(t)}}, \qquad t \in [0,1], \tag{2.6}$$

$$u(0) = u'(0) = u(1) = u'(1) = 0,$$

where $g(t)$ is continuous on $[0,1]$ with $388.625 \leq g(t) \leq 63728$.

For every $\lambda \in (0,1)$ and $u, v \in P$ we have
$\int_0^1 G(t,s)[g(s) + \lambda u(s) + \frac{1}{\sqrt{\frac{1}{\lambda}v(s)}}]ds = \lambda \int_0^1 G(t,s)[\frac{g(s)}{\lambda} + u(s) + \frac{1}{\sqrt{\lambda v(s)}}]ds$
$\leq \lambda \frac{\frac{g(s)}{\lambda} + u(s) + \frac{1}{\sqrt{\lambda v(s)}}}{g(s) + u(s) + \frac{1}{\sqrt{v(s)}}} \int_0^1 G(t,s)[g(s) + u(s) + \frac{1}{\sqrt{v(s)}}]ds.$

We note that

$$\lambda < \varphi(\lambda, u, v) = \lambda \frac{\frac{g(s)}{\lambda} + u(s) + \frac{1}{\sqrt{\lambda v(s)}}}{g(s) + u(s) + \frac{1}{\sqrt{v(s)}}} \int_0^1 G(t,s)[g(s) + u(s) + \frac{1}{\sqrt{v(s)}}]ds < 1.$$

By means of some calculations, we can conclude that for any $\lambda \in (0,1)$ the function φ is nonincreasing in u for fixed v and nondecreasing in v for fixed u.

So, it is enough to verify that the condition (2.6) of Theorem 2.1 is satisfied. Putting $u = v = h = 1$, and taking into account that $M_1 = \min_{t\in[0,1]} \int_0^1 G(t,s)ds = 0.00001$ and $M_2 = \max_{t\in[0,1]} \int_0^1 G(t,s)ds = 0.004$, we can easily get
$2^{-8} \leq 0.00001 \times 390.625 \leq \int_0^1 G(t,s)[g(s) + u(s) + \frac{1}{\sqrt{v(s)}}]ds$
$\leq 0.004 \times 63730 \leq \frac{2^8(g(s)+2)}{g(s)+2^8+2^4} = \frac{s(2^{-8},2^8,2^{-8})}{2^{-8}}, \quad s \in [0,1]$,
implying that the condition (2.5) of Theorem 2.1 holds. Therefore, we can apply this theorem to conclude that the problem in the example has a unique solution.

Now, we study the existence and uniqueness of a positive solution for the following fractional differential equation:

$$ {}^cD^\alpha y(t) = h(t), \qquad t \in [0,T], \quad T \geq 1. \tag{2.7}$$

subject to

$$ y(0) + \int_0^T y(s)ds = y(T). \tag{2.8}$$

Lemma 2.3 ([15]). *Let $0 < \alpha \leq 1$ and let $h \in C([0,T],\mathbb{R})$ be a given function. Then the boundary value problem (2.7), (2.8) has a unique solution given by*

$$ y(t) = \int_0^T G(t,s)h(s)ds, $$

where $G(t,s)$ is the Green's function given by

$$ G(t,s) = \begin{cases} \frac{-(T-s)^\alpha + \alpha T(t-s)^{\alpha-1}}{T\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \leq s < t, \\ \frac{-(T-s)^\alpha}{T\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & t \leq s < T. \end{cases} $$

By using arguments similar to those applied in the proof of Theorem 2.1, it can easily be verified that Theorem 2.1 remains true for Green function defined in Lemma 2.3.

Example 2.2. Consider the following boundary value problem

$$ {}^cD^{\frac{1}{2}}u(t) = f(t,u(t),u(t)) = g(t)u(t)^{\frac{1}{2}} + u(t)^{\frac{-1}{3}}, \qquad t \in [0,1], $$

$$ u(0) + \int_0^1 u(s)ds = u(1), $$

where $g(t)$ is continuous on $[0,1]$ with $0.6378 \leq g(t) \leq 2.89967$.

For every $\lambda \in (0,1)$ and $u, v \in P$ we have

$$ \int_0^1 G(t,s)[g(s)((\lambda u(s))^{\frac{1}{2}} + (\frac{1}{\lambda}v(s))^{\frac{-1}{3}})]ds \geq \lambda \int_0^1 G(t,s)[\frac{g(s)}{\lambda} + \tag{2.9}$$

$$ +u(s) + \frac{1}{\sqrt{v(s)}}]ds \geq \lambda \int_0^1 G(t,s)[(g(s)(\frac{u(s)}{\lambda})^{\frac{1}{2}} + (\lambda v(s))^{\frac{-1}{3}})]ds $$

$$ \geq \lambda \frac{g(s)(\frac{u(s)}{\lambda})^{\frac{1}{2}} + (\lambda v(s))^{\frac{-1}{3}}}{g(s)u(s)^{\frac{1}{2}} + v(s)^{\frac{-1}{3}}} \int_0^1 G(t,s)[g(s)u(s)^{\frac{1}{2}} + v(s)^{\frac{-1}{3}}]ds. $$

Note that

$$\lambda < \varphi(\lambda, u, v) = \lambda \frac{g(s)(\frac{u(s)}{\lambda})^{\frac{1}{2}} + (\lambda v(s))^{\frac{-1}{3}}}{g(s)u(s)^{\frac{1}{2}} + v(s)^{\frac{-1}{3}}} < 1.$$

By means of some calculations, we can conclude that for any $\lambda \in (0, 1)$ the function φ is nonincreasing in u for fixed v and nondecreasing in v for fixed u.

So, it is enough to verify that the condition (2.5) of Theorem 2.1 is satisfied. Putting $u = v = h = 1$, and taking into account that $M_1 = \min_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{3}$, and $M_2 = \max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{80}{51}$, we can easily get $2^{-6} \leq \frac{1}{3} \times 1.6378 \leq \int_0^1 G(t,s)[g(s)u(s)^{\frac{1}{2}} + u(s)^{\frac{-1}{3}}]ds \leq \frac{80}{51} \times 3.8967 \leq \frac{16g(s)+8}{2g(s)+1} = \frac{\varphi(2^{-6},2^{6},2^{-6})}{2^{-6}}$, $s \in [0, 1]$, implying that the condition (2.5) of Theorem 2.1 holds. Therefore, we can apply this theorem to conclude that the problem in the example has a unique solution.

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Contents

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ИЗВЕСТИЯ НАН АРМЕНИИ: МАТЕМАТИКА

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Содержание

IZVESTIYA NAN ARMENII: MATEMATIKA

Vol. 52, No. 1, 2017

Contents