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ОПЕРАТОРЫ ТИПА БЕРГМАНА НА ПРОСТРАНСТВАХ СО СМЕШАННОЙ НОРМОЙ В ШАРЕ ИЗ $\mathbb{C}^n$

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Аннотация. В статье рассмотрены введенные Шилдсом и Вильямсом операторы типа Бергмана, зависящие от нормальной пары весовых функций. Доказано, что существуют значения параметра β, при которых эти операторы ограничены на пространствах $L(p,q,\beta)$ со смешанной нормой в единичном шаре из $\mathbb{C}^n$.

MSC2010 number: 47B38, 32A37.

Ключевые слова: единичный шар из $\mathbb{C}^n$; нормальная весовая функция; нормальная пара; пространства со смешанной нормой; операторы Бергмана[1].

1. Введение и обозначения

Пусть $B = B_n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z| < 1\}$ — открытый единичный шар в $\mathbb{C}^n$, и $S := \partial B$ — его граница, единичная сфера. Скалярное произведение в $\mathbb{C}^n$ обозначим через $\langle z, w\rangle := z_1\overline{w}_1 + \dots + z_n\overline{w}_n$, $z, w \in \mathbb{C}^n$. Всюду далее будем полагать $z = r\zeta$, $w = \rho\eta \in B$, $0 \le r, \rho < 1$, $\zeta, \eta \in S$, $r = |z| = \sqrt{\langle z, z\rangle}$.

Множество всех голоморфных функций в шаре B обозначим через $H(B)$. Для функции $f(z) = f(r\zeta)$, заданной в шаре B, ее интегральные средние порядка p на сфере $|z| = r$ обозначены как обычно, через

$$M_p(f;r) = \|f(r\cdot)\|_{L^p(S;d\sigma)}, \qquad 0 \le r < 1,\ 0 < p \le \infty,$$

где $d\sigma$ — $(2n-1)$-мерная поверхностная мера Лебега на сфере S, нормированная так, что $\sigma(S) = 1$. Класс функций $f \in H(B)$ с "нормой" $\|f\|_{H^p} = \sup\limits_{0<r<1} M_p(f;r)$ есть обычное пространство Харди $H^p(B)$ в единичном шаре B.

Определим пространство $L(p,q,\beta)$ $(0 < p, q \le \infty,\ \beta \in \mathbb{R})$ со смешанной нормой как пространство тех измеримых функций $f(z) = f(r\zeta)$ в шаре B, для которых

[1]Настоящее исследование первого автора выполнено при финансовой поддержке Центра Математических Исследований Ереванского Государственного Университета

конечна квазинорма

$$\|f\|_{L(p,q,\beta)} = \|f\|_{p,q,\beta} := \begin{cases} \left(\int_0^1 (1-r)^{\beta q-1} M_p^q(f;r)\,dr\right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}_{0<r<1} (1-r)^\beta M_p(f;r), & q = \infty. \end{cases}$$

Подпространства $L(p,q,\beta)$, состоящие из голоморфных функций, обозначим через $H(p,q,\beta) := H(B)\cap L(p,q,\beta)$, $\beta > 0$. Если $1 \le p,q \le \infty$, то $L(p,q,\beta), H(p,q,\beta)$ являются банаховыми пространствами с нормой $\|\cdot\|_{p,q,\beta}$. При $p = q < \infty$ пространства $H(p,p,\beta)$ совпадают с весовыми классами Бергмана, а при $q = \infty$ их часто называют весовыми пространствами Харди.

Пространства со смешанной нормой для голоморфных в единичном круге функций были введены Харди и Литтлвудом в [1], [2] и развиты в дальнейшем Флеттом [3]. Смотри также монографии [4], [5], посвященные весовым пространствам Бергмана $H(p,p,\beta)$ в единичном круге.

Много работ посвящено пространствам $L(p,q,\beta)$ со смешанной нормой или их подпространствам, состоящим из голоморфных, плюригармонических или гармонических функций в круге, шаре из $\mathbb{C}^n$ или $\mathbb{R}^n$. Пространства $H(p,q,\beta)$ для голоморфных функций в единичном шаре $B \subset \mathbb{C}^n$ и бергмановские операторы на них подробно исследованы, в работах [6]–[10], а для голоморфных и n-гармонических функций в поликруге из $\mathbb{C}^n$ смотри, например [11].

Символы $C(\alpha,\beta,\dots)$, c_α и т.п. всюду будут обозначать положительные постоянные, различные в разных местах и зависящие только от указанных индексов $\alpha,\beta,\dots$. Через dV обозначим лебегову меру на B, нормированную так, что $V(B) = 1$. В полярных координатах будем иметь $dV(z) = 2n\,r^{2n-1}dr d\sigma(\zeta)$.

Вместо стандартных степенных весовых функций Шилдс и Вильямс [12] впервые предложили использовать более общие нормальные весовые функции. Фактически это те весовые функции, которые имеют степенные миноранты и мажоранты с положительными показателями.

Определение 1.1. *(Нормальная весовая функция, [12]) Положительная непрерывная функция $\varphi(r)$, $0 \le r < 1$, называется нормальной, если найдутся постоянные $0 < a < b$ и $0 \le r_0 < 1$ такие, что имеют место*

$$(1.1)\qquad \frac{\varphi(r)}{(1-r)^a} \searrow 0 \quad \text{и} \quad \frac{\varphi(r)}{(1-r)^b} \nearrow +\infty \quad \text{при} \quad r \to 1^-, \quad r_0 \le r < 1.$$

Здесь и далее монотонность функций всегда будем подразумевать в широком, нестрогом смысле. Индексы a и b для нормальной функции φ определяются неоднозначно.

Типичными примерами нормальных функций являются функции вида

$$\varphi_{c,d}(r) = (1-r)^c \left(\log \frac{e}{1-r}\right)^d, \qquad c > 0,\ d \in \mathbb{R},$$

причем при $c = 0$ функция $\varphi_{0,d} = \left(\log \frac{e}{1-r}\right)^d$ уже не будет нормальной.

Определение 1.2. *(Нормальная пара, [12]) Скажем, что пара функций $\{\varphi, \psi\}$ составляет нормальную пару, если функция φ нормальна, и существует число $\alpha > b - 1$ (индекс пары) такое, что*

$$\varphi(r)\,\psi(r) = (1-r^2)^\alpha, \qquad 0 \le r < 1. \tag{1.2}$$

Ввиду условия $\alpha > b - 1$ функция ψ будет интегрируемой на интервале $(0, 1)$. Как показано в [12], для нормальной функции φ всегда найдется ее нормальная пара, а при более строгом условии $\alpha > b$ функция ψ сама также будет нормальной с индексами $\alpha - b$ и $\alpha - a$.

Расширим область определения таких радиальных весовых функций до шара B, положив $\varphi(z) := \varphi(|z|) = \varphi(r)$, $\psi(z) := \psi(|z|) = \psi(r)$.

Посредством нормальных весовых функций Шилдс и Вильямс [12] в единичном круге $\mathbb{D} = B_1$ предложили обобщения операторов Бергмана, которые для шара B определены в работах А. И. Петросяна [13], [14] в виде

$$Q_{\varphi,\psi}(f)(z) := \int_B \frac{\psi(z)\,\varphi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, f(w)\, dV(w), \qquad z \in B. \tag{1.3}$$

$$\widetilde{Q}_{\varphi,\psi}(f)(z) := \int_B \frac{\psi(z)\,\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\, f(w)\, dV(w), \qquad z \in B. \tag{1.4}$$

В частном случае $\varphi(r) = (1-r^2)^\alpha$, $\psi \equiv 1$ операторы (1.3), (1.4) сводятся к классическим проекторам Бергмана, см. [4]–[10]. В случае $\varphi(r) = (1-r^2)^c$, $\psi(r) = (1-r^2)^d$, $c + d = \alpha$ операторы типа Бергмана (1.3), (1.4) также хорошо известны, см. [7]–[11].

В настоящей статье мы доказываем, что существуют значения параметра β, при которых общие операторы (1.3), (1.4) ограничены на пространствах $L(p, q, \beta)$ со смешанной нормой в шаре B.

Основным результатом статьи является следующая теорема.

Теорема 1.1. *Пусть $1 \le p, q \le \infty$, $\beta \in \mathbb{R}$, $\{\varphi, \psi\}$ — нормальная пара функций с индексами a и b $(0 < a < b)$ и с индексом пары α $(\alpha > b-1)$ в смысле Определений 1.1–1.2.*

Если $b - \alpha < \beta < 1 + a$, то операторы $Q_{\varphi,\psi}$ и $\widetilde{Q}_{\varphi,\psi}$ ограниченно действуют из пространства $L(p, q, \beta)$ в себя, т.е.

$$Q_{\varphi,\psi} : L(p, q, \beta) \longrightarrow L(p, q, \beta), \tag{1.5}$$

$$\widetilde{Q}_{\varphi,\psi} : L(p, q, \beta) \longrightarrow L(p, q, \beta). \tag{1.6}$$

Замечание 1.1. *В частном случае $1 \le p = q = 1/\beta < \infty$, т.е. для невесового класса $L(p, p, 1/p)$, Теорема 1.1 установлена в [13], [14], но другим методом с использованием так называемого теста Шура ([4]–[7]), который не подходит в нашем случае. Еще более частные случаи операторов Бергмана со степенными весами изучены в [5]–[10].*

Замечание 1.2. *В Теореме 1.1 мы фактически обобщаем результат из [13], [14] в трех направлениях: во-первых, предполагаем все значения $1 \le p \le \infty$, во-вторых, рассматриваем весовые пространства, в-третьих, вместо пространств Бергмана рассматриваем более общие пространства $L(p, q, \beta)$ со смешанной нормой. При этом вместо неподходящего теста Шура мы применяем обобщения неравенства Харди.*

2. Неравенства Харди и другие интегральные неравенства

Широко известны классические неравенства Харди (см. [3], [15]):

$$\int_0^1 x^{-\beta-1} \left(\int_0^x h(t)\, dt \right)^p dx \le C(p, \beta) \int_0^1 x^{p-\beta-1}\, h^p(x)\, dx, \tag{2.1}$$

$$\int_0^1 (1-r)^{\beta-1} \left(\int_0^r h(t)\, dt \right)^p dr \le C(p, \beta) \int_0^1 (1-r)^{p+\beta-1}\, h^p(r)\, dr, \tag{2.2}$$

$$\int_0^1 (1-r)^{-\beta-1} \left(\int_r^1 h(t)\, dt \right)^p dr \le C(p, \beta) \int_0^1 (1-r)^{p-\beta-1}\, h^p(r)\, dr, \tag{2.3}$$

где $1 \le p < \infty$, $\beta > 0$, $h(r) \ge 0$.

Отметим, что неравенство (2.3) выводится из (2.1) линейной заменой переменных интегрирования.

Для последующих доказательств нам понадобятся также обобщения неравенств (2.2) и (2.3).

Лемма 2.1. *Пусть* $1 \le p < \infty$, $\gamma > 0$, $h(r) \ge 0$. *Для положительной непрерывной функции* $\varphi(r)$, $0 \le r < 1$, *найдутся постоянные* $b \in \mathbb{R}$, $\gamma - pb > 0$, *и* $0 \le r_0 < 1$ *такие, что*

$$(2.4) \qquad \frac{\varphi(r)}{(1-r)^b} \nearrow \qquad \text{при} \quad r_0 \le r < 1.$$

Тогда

$$(2.5) \qquad \int_0^1 \frac{(1-r)^{\gamma-1}}{\varphi^p(r)} \left(\int_0^r h(t)\,dt \right)^p dr \le C(p,\gamma,b,r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)} h^p(r)\,dr.$$

Доказательство. Применим неравенство Харди (2.2) по отношению к функции $\frac{(1-r)^b}{\varphi(r)} h(r)$ и с индексом $\beta = \gamma - pb > 0$,

$$\int_0^1 (1-r)^{\gamma-pb-1} \left(\int_0^r \frac{(1-t)^b}{\varphi(t)} h(t)\,dt \right)^p dr \le C \int_0^1 (1-r)^{p+\gamma-pb-1} \left(\frac{(1-r)^b}{\varphi(r)} h(r) \right)^p dr,$$

где постоянная C зависит лишь от p, γ, b. Ввиду условия (2.4), монотонного убывания функции $\frac{(1-r)^b}{\varphi(r)}$ на интервале $(r_0, 1)$ и непрерывности на $[0, 1)$ получаем

$$\int_0^1 (1-r)^{\gamma-pb-1} \frac{(1-r)^{pb}}{\varphi^p(r)} \left(\int_0^r h(t)\,dt \right)^p dr \le C(p,\gamma,b,r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)} h^p(r)\,dr,$$

что совпадает с (2.5). □

Замечание 2.1. *Схожее с неравенством* (2.5) *другое неравенство типа Харди с участием нормальных весовых функций можно найти в* [10].

Нам понадобится также другая разновидность неравенства (2.5).

Лемма 2.2. *Пусть* $1 \le p < \infty$, $h(r) \ge 0$. *Для положительной непрерывной функции* $\varphi(r)$, $0 \le r < 1$, *найдутся постоянные* $a \in \mathbb{R}$, $\gamma - pa < 0$, *и* $0 \le r_0 < 1$ *такие, что*

$$(2.6) \qquad \frac{\varphi(r)}{(1-r)^a} \searrow \qquad \text{при} \quad r_0 \le r < 1.$$

Тогда

$$(2.7) \qquad \int_0^1 \frac{(1-r)^{\gamma-1}}{\varphi^p(r)} \left(\int_r^1 h(t)\,dt \right)^p dr \le C(p,\gamma,a,r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)} h^p(r)\,dr.$$

Доказательство. Применим неравенство Харди (2.3) по отношению к функции $\frac{(1-r)^a}{\varphi(r)} h(r)$ и с индексом $-\beta = \gamma - pa < 0$,

$$\int_0^1 (1-r)^{\gamma-pa-1} \left(\int_r^1 \frac{(1-t)^a}{\varphi(t)} h(t)\,dt \right)^p dr \le C \int_0^1 (1-r)^{p+\gamma-pa-1} \left(\frac{(1-r)^a}{\varphi(r)} h(r) \right)^p dr.$$

где постоянная C зависит лишь от p, γ, a. Ввиду условия (2.6), монотонного возрастания функции $\frac{(1-r)^a}{\varphi(r)}$ на интервале $(r_0, 1)$ и непрерывности на $[0,1)$ получаем

$$\int_0^1 (1-r)^{\gamma-pa-1} \frac{(1-r)^{pa}}{\varphi^p(r)} \left(\int_r^1 h(t)\, dt \right)^p dr \le C(p,\gamma,a,r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)} h^p(r)\, dr,$$

что совпадает с (2.7). □

Лемма 2.3. *([6], [7]) Для $\alpha > 0$ справедлива оценка*

$$\int_S \frac{d\sigma(\xi)}{|1-\langle z,\xi\rangle|^{n+\alpha}} \le \frac{C(\alpha,n)}{(1-|z|)^\alpha}, \qquad z \in B.$$

Лемма 2.4. *([12]) При $m > \beta > 0$ справедливо неравенство*

$$\int_0^1 \frac{(1-\rho)^{\beta-1}}{(1-r\rho)^m}\, d\rho \le \frac{C(\beta,m)}{(1-r)^{m-\beta}}, \qquad 0 \le r < 1.$$

Следующая лемма является вариантом схожих оценок из [9], [12]–[14].

Лемма 2.5. *Пусть φ — нормальная функция с индексами a и b $(0 < a < b)$ и с индексом пары α $(\alpha > b-1)$ в смысле Определений 1.1–1.2.*

Если $b - \alpha < \beta < 1 + a$, то

$$\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho \le C(\alpha,\beta,a,b,r_0)\, \frac{\varphi(r)}{(1-r)^{\alpha+\beta}}, \qquad 0 \le r < 1. \tag{2.8}$$

Доказательство. Условие $\beta < 1 + a$ обеспечивает сходимость интеграла (2.8). Достаточно доказать неравенство (2.8) для r, близких к 1. Возьмем r, $r_0 < r < 1$, и разобьем интеграл (2.8) на три части,

$$J := \int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho =$$
$$= \left(\int_0^{r_0} + \int_{r_0}^r + \int_r^1 \right) \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho =: J_1 + J_2 + J_3.$$

Интеграл J_1 ограничен некоторой постоянной $C(\alpha,\beta,r_0)$. Для оценок интегралов J_2 и J_3 используем условия нормальности (1.1) и Лемму 2.4,

$$J_2 = \int_{r_0}^r \frac{\varphi(\rho)}{(1-\rho)^b} \frac{(1-\rho)^b}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho \le \frac{\varphi(r)}{(1-r)^b} \int_{r_0}^r \frac{(1-\rho)^{b-\beta}}{(1-r\rho)^{1+\alpha}}\, d\rho \le$$
$$\le C(\alpha,\beta,b)\, \frac{\varphi(r)}{(1-r)^b} \frac{1}{(1-r)^{\alpha+\beta-b}} = C(\alpha,\beta,b)\, \frac{\varphi(r)}{(1-r)^{\alpha+\beta}}.$$

Поскольку $\beta > b - \alpha > a - \alpha$, то аналогичным образом получаем

$$J_3 = \int_r^1 \frac{\varphi(\rho)}{(1-\rho)^a} \frac{(1-\rho)^a}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho \le \frac{\varphi(r)}{(1-r)^a} \int_r^1 \frac{(1-\rho)^{a-\beta}}{(1-r\rho)^{1+\alpha}}\, d\rho \le$$
$$\le C(\alpha,\beta,a)\, \frac{\varphi(r)}{(1-r)^a} \frac{1}{(1-r)^{\alpha+\beta-a}} = C(\alpha,\beta,a)\, \frac{\varphi(r)}{(1-r)^{\alpha+\beta}}.$$

что завершает доказательство Леммы 2.5. □

3. Ограниченность операторов типа Бергмана на пространствах со смешанной нормой

Лемма 3.1. *Пусть* $1 \le p \le \infty$, $\alpha > -1$, $\{\varphi, \psi\}$ — *пара положительных весовых функций. Тогда имеет место оценка*

$$M_p(\widetilde{Q}_{\varphi,\psi}(f);r) \le C(p,n,\alpha)\,\psi(r)\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}}\,M_p(f;\rho)\,d\rho, \quad 0 \le r < 1. \tag{3.1}$$

Доказательство. Перейдем к полярным координатам в интегральном представлении $\widetilde{Q}_{\varphi,\psi}(f)(z)$,

$$\begin{aligned}\left|\widetilde{Q}_{\varphi,\psi}(f)(z)\right| &\le \psi(z)\int_B \frac{\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,|f(w)|\,dV(w) = \\ &= 2n\,\psi(z)\int_0^1\left[\int_S \frac{|f(\rho\eta)|}{|1-\langle z,\rho\eta\rangle|^{n+1+\alpha}}\,d\sigma(\eta)\right]\varphi(\rho)\,\rho^{2n-1}\,d\rho,\end{aligned}$$

и перепишем его в виде

$$\begin{aligned}\left|\widetilde{Q}_{\varphi,\psi}(f)(r\zeta)\right| &\le 2n\,\psi(r)\int_0^1\left[\int_S \frac{|f(\rho\eta)|}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\,d\sigma(\eta)\right]\varphi(\rho)\,\rho^{2n-1}\,d\rho = \\ &= 2n\,\psi(r)\int_0^1 g(r,\rho,\zeta)\,\varphi(\rho)\,\rho^{2n-1}\,d\rho,\end{aligned} \tag{3.2}$$

где обозначено

$$g(r,\rho,\zeta) = \int_S \frac{|f(\rho\eta)|}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\,d\sigma(\eta).$$

Если $p=\infty$, то немедленно получаем из (3.2) по Лемме 2.3.

$$\begin{aligned}M_\infty(\widetilde{Q}_{\varphi,\psi}(f);r) &\le 2n\,\psi(r)\int_0^1 M_\infty(f;\rho)\sup_{\zeta\in S}\left[\int_S \frac{d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right]\varphi(\rho)\,\rho^{2n-1}\,d\rho \le \\ &\le C(n,\alpha)\,\psi(r)\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}}\,M_\infty(f;\rho)\,d\rho.\end{aligned}$$

Если $p=1$, то интегрированием (3.2) немедленно получаем требуемое неравенство (3.1), воспользовавшись теоремой Фубини и Леммой 2.3.

Если $1 < p < \infty$, то из неравенства Гельдера и Леммы 2.3 получаем

$$\begin{aligned}g(r,\rho,\zeta) &\le \left(\int_S \frac{|f(\rho\eta)|^p\,d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right)^{1/p}\left(\int_S \frac{d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right)^{1/p'} \le \\ &\le \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)/p'}}\left(\int_S \frac{|f(\rho\eta)|^p\,d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right)^{1/p},\end{aligned}$$

где p' — сопряженный индекс, $1/p + 1/p' = 1$. Далее проинтегрируем по переменной ζ на сфере S и вновь воспользуемся Леммой 2.3,

$$\|g(r,\rho,\cdot)\|^p_{L^p(S;d\sigma)} \le \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p/p'}} \int_S \left(\int_S \frac{d\sigma(\zeta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}}\right) |f(\rho\eta)|^p\, d\sigma(\eta) \le$$
$$\le \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p/p'}(1-r\rho)^{1+\alpha}} \int_S |f(\rho\eta)|^p\, d\sigma(\eta) =$$
$$= \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p}} M_p^p(f;\rho),$$

или

$$\|g(r,\rho,\cdot)\|_{L^p(S;d\sigma)} \le \frac{C(p,n,\alpha)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho). \tag{3.3}$$

Теперь вернемся к неравенству (3.2) и применим неравенство Минковского, а затем — оценку (3.3),

$$M_p(\tilde{Q}_{\varphi,\psi}(f);r) \le 2n\,\psi(r) \int_0^1 \|g(r,\rho,\cdot)\|_{L^p(S;d\sigma)}\, \varphi(\rho)\, \rho^{2n-1}\, d\rho \le$$
$$\le C(p,n,\alpha)\,\psi(r) \int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\, d\rho,$$

что завершает доказательство Леммы 3.1. □

Доказательство Теоремы 1.1. *Поскольку* $|Q_{\varphi,\psi}(f)(z)| \le \tilde{Q}_{\varphi,\psi}(|f|)(z)$, *то достаточно доказать лишь ограниченность* (1.6) *оператора* $\tilde{Q}_{\varphi,\psi}(|f|)$.

Вначале положим $1 \le q < \infty$. *По Лемме 3.1 имеем*

$$M_p(\tilde{Q}_{\varphi,\psi}(f);r) \le C(p,n,\alpha)\,\psi(r) \int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\, d\rho, \quad 0 \le r < 1. \tag{3.4}$$

Проинтегрируем теперь по радиальной переменной так, чтобы получить смешанную норму,

$$\|\tilde{Q}_{\varphi,\psi}(f)\|^q_{L(p,q,\beta)} = \int_0^1 (1-r)^{\beta q-1} M_p^q(\tilde{Q}_{\varphi,\psi}(f);r)\, dr \le$$
$$\le C \int_0^1 (1-r)^{\beta q-1} \psi^q(r) \left[\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\, d\rho\right]^q dr.$$

Далее воспользуемся условием (1.2) *Определения 1.2 нормальной пары, а затем разобьем интеграл на две части,*

$$\|\tilde{Q}_{\varphi,\psi}(f)\|^q_{L(p,q,\beta)} \le C \int_0^1 \frac{(1-r)^{\alpha q+\beta q-1}}{\varphi^q(r)} \left[\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\, d\rho\right]^q dr =$$
$$\le C \int_0^1 \frac{(1-r)^{\alpha q+\beta q-1}}{\varphi^q(r)} \left[\left(\int_0^r + \int_r^1\right) \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\, d\rho\right]^q dr \le$$
$$\le I_1 + I_2. \tag{3.5}$$

Интегралы I_1 и I_2 оценим по отдельности, используя неравенства типа Харди из Лемм 2.1 и 2.2.

По отношению к интегралу I_1 можно применить неравенство (2.5), ибо условие $\alpha q + \beta q - bq > 0$ равносильно $\beta > b - \alpha$,

$$I_1 = C\int_0^1 \frac{(1-r)^{\alpha q+\beta q-1}}{\varphi^q(r)}\left[\int_0^r \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}}\, M_p(f;\rho)\, d\rho\right]^q dr \le$$

$$\le C\int_0^1 \frac{(1-r)^{\alpha q+\beta q-1+q}}{\varphi^q(r)}\left[\frac{\varphi(r)}{(1-r^2)^{1+\alpha}}\, M_p(f;r)\right]^q dr \le$$

$$\le C\int_0^1 (1-r)^{\beta q-1} M_p^q(f;r)\, dr =$$

$$= C(n,p,q,\beta,\alpha,b,r_0)\, \|f\|^q_{L(p,q,\beta)}. \tag{3.6}$$

По отношению к интегралу I_2 можно применить неравенство (2.7), ибо условие $\beta q - q - aq < 0$ равносильно $\beta < 1 + a$.

$$I_2 = C\int_0^1 \frac{(1-r)^{\alpha q+\beta q-1}}{\varphi^q(r)}\left[\int_r^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}}\, M_p(f;\rho)\, d\rho\right]^q dr \le$$

$$\le C\int_0^1 \frac{(1-r)^{\beta q-q-1}}{\varphi^q(r)}\left[\int_r^1 \varphi(\rho)\, M_p(f;\rho)\, d\rho\right]^q dr \le$$

$$\le C\int_0^1 \frac{(1-r)^{\beta q-q-1+q}}{\varphi^q(r)}\left[\varphi(r)\, M_p(f;r)\right]^q dr =$$

$$= C\int_0^1 (1-r)^{\beta q-1} M_p^q(f;r)\, dr =$$

$$= C(n,p,q,\beta,\alpha,a,r_0)\, \|f\|^q_{L(p,q,\beta)}. \tag{3.7}$$

Неравенства (3.5) – (3.7) вместе дают требуемое неравенство

$$\|\widetilde{Q}_{\varphi,\psi}(f)\|_{L(p,q,\beta)} \le C\, \|f\|_{L(p,q,\beta)},$$

где постоянная $C = C(n,p,q,\beta,\alpha,a,b,r_0) > 0$ зависит лишь от указанных параметров.

Теперь положим $q = \infty$. Из неравенства (3.4) с применением Леммы 2.5 получаем

$$M_p(\widetilde{Q}_{\varphi,\psi}(f);r) \le C(p,n,\alpha)\,\psi(r)\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\,(1-\rho)^\beta M_p(f;\rho)\, d\rho \le$$

$$\le C\,\psi(r)\,\|f\|_{L(p,\infty,\beta)}\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\, d\rho \le$$

$$\le C\,\psi(r)\,\|f\|_{L(p,\infty,\beta)}\frac{\varphi(r)}{(1-r)^{\alpha+\beta}} \le$$

$$\le C(p,n,\alpha,\beta,a,b,r_0)\,\|f\|_{L(p,\infty,\beta)}\frac{1}{(1-r)^\beta}.$$

Отсюда

$$\left\|\tilde{Q}_{\varphi,\psi}(f)\right\|_{L(p,\infty,\beta)} \leq C\,\|f\|_{L(p,\infty,\beta)},$$

где постоянная $C = C(p, n, \alpha, \beta, a, b, r_0) > 0$ *зависит лишь от указанных параметров. Теорема 1.1 доказана.*

Abstract. *The paper considers Bergman type operators introduced by Shields and Williams depending on normal pair of weight functions. We prove that there exist values of parameters* β *for which these operators are bounded on mixed norm spaces* $L(p, q, \beta)$ *on the unit ball in* $\mathbb{C}^n$.

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ЗАДАЧА РИМАНА В ВЕСОВЫХ ПРОСТРАНСТВАХ $L^1(\rho)$

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Аннотация. В единичном круге ограниченной окружностью $T = \{z, |z| = 1\}$ рассматривается граничная задача Римана в весовом пространстве $L^1(\rho)$, где $\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}$, $t_k \in T$, $k = 1, 2, ..., m$, и α_k, $k = 1, 2, ..., m$ действительные числа. Требуется определить аналитическую вне окружности T функцию $\Phi(z)$, $\Phi(\infty) = 0$, такую, чтобы имело место $\lim_{r\to 1-0} \|\Phi^+(rt) - a(t)\Phi^-(r^{-1}t) - f(t)\|_{L^1(\rho_r)} = 0$, где $f \in L^1(\rho)$, $a(t) \in C^\delta(T)$, $\delta > 0$, а ρ_r некоторое продолжение функции ρ во внутрь окружности. Устанавливается нормальная разрешимость этой задачи.

MSC2010 number: 35J25

Ключевые слова: Задача Римана; весовое пространство; интеграл типа Коши; факторизация.

1. Введение

Пусть Γ простая замкнутая кривая Ляпунова в комплексной плоскости z. G^+ внутренняя, а G^- внешняя области ограниченные кривой Γ. Хорошо известна граничная задача Римана или задача сопряжения (см. [1] – [6]).

$$\Phi^+(t) - a(t)\Phi^-(t) = f(t), t \in \Gamma,$$

где $a(t)$ заданная кусочно непрерывная на Γ функция из класса C^α, а f принадлежит классу $C^\alpha(\Gamma)$ или $L^p(\Gamma)(1 < p < \infty)$. Искомые функции $\Phi^\pm$ аналитические $G^\pm$ соответственно функции из класса E^p (см. [12]). При исследовании этой задачи важную роль играет тот факт, что интеграл типа Коши является ограниченным оператором в соответствующих пространствах. Исследование граничной задачи Римана когда $f \in L^1(\Gamma)$ связана с определенными трудностями поскольку интеграл типа Коши не является ограниченным оператором в этом пространстве. В работе [7] предложена новая постановка задачи Римана в этом пространстве. В случае, когда Γ-единичная окружность, эта задача формулируется в виде: Определить аналитические в $D^\pm$, где $D^+ = \{z, |z| < 1\}$, $D^- = \{z, |z| > 1\}$, функции

$\Phi^{\pm}$ так, чтобы имело место

$$\lim_{r\to 1-0}||\Phi^{+}(rt)-a(t)\Phi^{-}(r^{-1}t)-f(t)||_1=0,\quad \Phi^{-}(\infty)=0, \tag{1.1}$$

где $||\cdot||_1$ – норма пространства $L^1(T)$, $T=\{z,|z|=1\}$.

В настоящей работе граничная задача Римана исследуется в весовом пространстве $f\in L^1(\rho)$ (функция $f\in L^1(\rho)$ если

$$||f||_{L^1(\rho)}=\int_T|f(t)|\rho(t)|dt|<\infty)$$

где $\rho(t)=\prod\limits_{k=1}^{m}|t-t_k|^{\alpha_k}$, $t_k\in T$, $k=1,2,\dots,m$, и α_k, $k=1,2,\dots,m$ действительные числа. Для формулировки задачи приведем некоторые обозначения. Пусть

$$\rho_r(t)=\rho^*(t)\prod_{k=1}^{m}|r^{\delta_k}t-t_k|^{n_k},\quad \rho^*(t)=\prod_{k=1}^{m}|t-t_k|^{\beta_k}, \tag{1.2}$$

$$\delta_k=\begin{cases}1, & \text{если } \alpha_k\le -1\ ,\\ 0, & \text{если } \alpha_k>-1\ ,\end{cases}\qquad n_k=\begin{cases}[\alpha_k]+1, & \text{если } \alpha_k \text{ нецелое},\\ \alpha_k, & \text{если } \alpha_k \text{ целое},\end{cases}$$

$\beta_k=\alpha_k-n_k$. Ясно что $\beta_k\in(-1,0]$ и $\rho^*(t)\in L^1(T)$.

Рассмотрим граничную задачу Римана в следующей постановке:

Задача R. Пусть f произвольная измеримая на T функция из класса $L^1(\rho)$. Следует определить аналитическую в $D^+\cup D^-$ функцию $\Phi(z)$, $\Phi(\infty)=0$, так чтобы имело место граничное условие

$$\lim_{r\to 1-0}||\Phi^{+}(rt)-a(t)\Phi^{-}(r^{-1}t)-f(t)||_{L^1(\rho_r)}=0, \tag{1.3}$$

где $a(t)$, $a(t)\neq 0$ произвольная функция из класса $C^{\delta}(T)$, $\delta>0$, $\Phi^{\pm}$ сужения функции Φ на $D^{\pm}$ соответственно. Задача R при $a\equiv 1$ и задача Дирихле в аналогичной постановке когда $\alpha_k\ge 0$ исследована в работах [8] и [10]. Задача R в случае когда $\rho(t)=|1-t|^{\alpha}$ исследована в работе [9], а задача Римана в весовых пространствах $L^p, p>1$ в работе [11].

2. Предварительные леммы

В этом параграфе мы доказываем несколько лемм, которые будут использованы при доказательстве основных результатов.

Лемма 2.1. *Пусть* $t_k\in T$, $t_i\neq t_j$, $i\neq j$ *и* $\lambda_k\in[0,1)$, $k=1,\dots,m$, $\delta>0$ *произвольные действительные числа. Тогда*

$$а)\ \sup_{r\in(0,1)} \prod_{k=1}^{m} |t_k-\tau|^{\lambda_k} \int_T \frac{(1-r)^\delta |dt|}{\prod_{k=1}^{m} |t_k-t|^{\lambda_k} |\tau - rt|} < C,$$

$$б)\ \sup_{r\in(0,1)} \prod_{k=1}^{m} |t_k-\tau|^{\lambda_k} \int_T \frac{(1-r^2) |dt|}{\prod_{k=1}^{m} |t_k-t|^{\lambda_k} |\tau - rt|^2} < C,$$

$$в)\ \sup_{r\in(0,1)} |t_k-\tau|^{\lambda_k} \int_T \frac{(1-r^2) |dt|}{|t_k - rt|^{1+\lambda_k} |\tau - rt|} < C,$$

где $C < \infty$ не зависит от τ.

Доказательство. Выберем полуинтервалы $T_k \in T$, так чтобы $T = \bigcup_{k=1}^{m} T_k$, $t_k \in T_k$, $T_i \bigcap T_j = \emptyset, i \neq j$. Для любого $r \in (0,1)$, $\tau \in T$ будем иметь

$$\prod_{k=1}^{m} |t_k-\tau|^{\lambda_k} \int_T \frac{(1-r)^\delta |dt|}{\prod_{k=1}^{m} |t_k-t|^{\lambda_k} |\tau - rt|} < C \sum_{k=1}^{m} |t_k-\tau|^{\lambda_k} \int_{T_k} \frac{(1-r)^\delta |dt|}{|t_k-t|^{\lambda_k} |\tau - rt|}.$$

Достаточно установить, что для любого $t_0 \in T$

$$\sup_{r\in(0,1)} |t_0-\tau|^{\lambda} \int_T \frac{(1-r)^\delta |dt|}{|t_0-t|^{\lambda} |\tau - rt|} < \infty.$$

Ввиду того, что

$$\sup_{r\in(0,1)} \int_T \frac{(1-r)^\delta |dt|}{|\tau - rt|} = \sup_{r\in(0,1)} (1-r)^\delta ln|1-r|^{-1} < \infty,$$

получаем

$$\sup_{r\in(0,1)} \int_T (1-r)^\delta \frac{||t_0-\tau|^\lambda - |t_0-t|^\lambda| |dt|}{|t_0-t|^{\lambda} |\tau - rt|} < \infty.$$

Поскольку

$$||t_0-\tau|^\lambda - |t_0-t|^\lambda| < C|\tau - t|^\lambda,$$

то достаточно доказать, что

$$\sup_{r\in(0,1)} \int_T \frac{(1-r)^\delta |\tau - t|^\lambda |dt|}{|t_0-t|^{\lambda} |\tau - rt|} < \infty.$$

Действительно, имеем

$$\int_T \frac{(1-r)^\delta |dt|}{|t_0-t|^{\lambda} |\tau - rt|^{1-\lambda}} \leq \int_T \frac{(1-r)^{\delta-\delta_1} |dt|}{|t_0-t|^{\lambda} |\tau - rt|^{1-\lambda-\delta_1}} \leq$$

$$\int_T \frac{(1-r)^{\delta-\delta_1} |dt|}{|t_0-t|^{1-\delta_1}} + \int_T \frac{(1-r)^{\delta-\delta_1} |dt|}{|\tau - rt|^{1-\delta_1}} < C,$$

где $\delta_1 \in (0,\delta)$. Утверждение а) леммы доказано.

Для доказательства утверждения б), по аналогии, достаточно установить, что

$$\sup_{r\in(0,1)} |t_0-\tau|^\lambda \int_T \frac{(1-r^2)|dt|}{|t_0-t|^\lambda|\tau-rt|^2} < \infty, \lambda \in (0,1)$$

Так как

$$|t_0-\tau|^\lambda \int_T \frac{(1-r^2)|dt|}{|t_0-t|^\lambda|\tau-rt|^2} \leq C(\int_T \frac{(1-r^2)|dt|}{|t_0-t|^\lambda|\tau-rt|^{2-\lambda}} + 1),$$

то положим

$$I_1(r,t) = \int_{|t_0-t|\geq 1-r} \frac{(1-r^2)|dt|}{|t_0-t|^\lambda|\tau-rt|^{2-\lambda}},$$

$$I_2(r,t) = \int_{|t_0-t|<1-r} \frac{(1-r^2)|dt|}{|t_0-t|^\lambda|\tau-rt|^{2-\lambda}}$$

Учитывая, что при $|t_0-t|\geq 1-r$, $|t_0-t|>2^{-1}|t_0-rt|$, получим

$$I_1(r,t) \leq 2\int_T \frac{(1-r^2)|dt|}{|t_0-rt|^\lambda|\tau-rt|^{2-\lambda}} \leq$$

$$\leq 2(\int_T \frac{(1-r^2)|dt|}{|t_0-rt|^2} + \int_T \frac{(1-r^2)|dt|}{|\tau-rt|^2}) = 8\pi.$$

Поскольку

$$|I_2(r,t)| \leq \int_{|\theta|<1-r} \frac{d\theta}{(1-r)^{1-\lambda}|\theta|^\lambda} = C < \infty$$

получаем доказательство утверждения б) леммы.

Далее имеем

$$|t_k-\tau|^\lambda \int_T \frac{(1-r^2)|dt|}{|t_k-rt|^{1+\lambda}|\tau-rt|} \leq$$

$$\int_T \frac{(1-r^2)||t_k-\tau|^\lambda - |t_k-rt|^\lambda||dt|}{|t_k-rt|^{1+\lambda}|\tau-rt|} + \int_T \frac{(1-r^2)|dt|}{|t_k-rt||\tau-rt|}.$$

Так как

$$\int_T \frac{(1-r^2)|dt|}{|t_k-rt||\tau-rt|} \leq \int_T \frac{(1-r^2)|dt|}{|t_k-rt|^2} + \int_T \frac{(1-r^2)|dt|}{|\tau-rt|^2} = 4\pi,$$

то, применяя неравенства Гельдера получаем

$$\int_T \frac{(1-r^2)||t_k-\tau|^\lambda - |t_k-rt|^\lambda||dt|}{|t_k-rt|^{1+\lambda}|\tau-rt|} \leq C\int_T \frac{(1-r^2)|dt|}{|t_k-rt|^{1+\lambda}|\tau-rt|^{1-\lambda}} \leq$$

$$\leq C(\int_T \frac{(1-r^2)|dt|}{|t_k-rt|^2} + \int_T \frac{(1-r^2)|dt|}{|\tau-rt|^2}) < \infty.$$

Лемма 2.1 доказана. □

Пусть $\kappa = \operatorname{ind} a(t), t\in T$. Хорошо известно, что функция a допускает представление (см. [1], [2])

$$a(t) = S^+(t)(S^-(t))^{-1},$$

где

$$S^+(z) = exp\{\frac{1}{2\pi i}\int_T \frac{ln(t^{-\kappa}a(t))dt}{t-z}\},\ z \in D^+,$$

$$S^-(z) = z^{-\kappa}exp\{\frac{1}{2\pi i}\int_T \frac{ln(t^{-\kappa}a(t))dt}{t-z}\},\ z \in D^-,$$

$S^+ \in C^\delta(\overline{D^\pm})$ и $|S^-(z)| = O(|z|^{-\kappa})$ при $z \to \infty$.

Лемма 2.2. *Пусть* $\alpha_k > -1$, $k = 1, ..., m$, $N = \sum_{k=1}^m n_k$, $f \in L^1(\rho)$ *и* $\Phi(z)$ *некоторое решение задачи R. Тогда ее можно представить в виде*

а) Если $N + \kappa \geq 0$, *то*

$$\Phi(z) = \frac{S(z)}{2\pi i \Pi(z)}\int_T \frac{f(t)\Pi(t)dt}{S^+(t)(t-z)} + \frac{S(z)P(z)}{\Pi(z)}, \tag{2.1}$$

где $z \in D^+ \cup D^-$, $P(z)$ *некоторый полином порядка* $N + \kappa - 1$,

$$\Pi(z) = \prod_{k=1}^m (t_k - z)^{n_k},$$

б) Если $N+\kappa < 0$, *то* $\Phi(z)$ *имеет представление (2.1), где* $P(z) \equiv 0$ *и функция* f *удовлетворяет условиям*

$$\int_T \frac{f(t)\Pi(t)}{S^+(t)} t^k dt = 0,\ \ k = 0, 1, ... - (N+\kappa) - 1. \tag{2.2}$$

Доказательство. Пусть $N + \kappa \geq 0$. Так как $-1 < \alpha_k - n_k \leq 0$, то

$$\lim_{r\to 1-0}\int_T \left|\frac{\Phi^+(rt)\Pi(t)}{S^+(t)} - \frac{\Phi^-(r^{-1}t)\Pi(t)}{S^-(t)} - \frac{f(t)\Pi(t)}{S^+(t)}\right||dt| = 0$$

Обозначим

$$\Psi_r^+(z) = \frac{\Phi^+(rz)\Pi(z)}{S^+(z)},\ \ \Psi_r^-(z) = \frac{\Phi^-(r^{-1}z)\Pi(z)}{S^+(z)},$$

$$f_r(t) = \frac{\Phi^+(rt)\Pi(t)}{S^+(t)} - \frac{\Phi^-(r^{-1}t)\Pi(t)}{S^+(t)},\ |t| = 1.$$

Получим граничную задачу Гильберта $\Psi_r^+(t) - \Psi_r^-(t) = f_r(t)$ относительно функции Ψ_r. Так как функция Ψ^- имеет полюс порядка $N + \kappa - 1$ в бесконечности, то

$$\Psi_r^\pm(z) = \frac{1}{2\pi i}\int_T \frac{f_r(t)dt}{t-z} + P_r(z),\ z \in D^\pm,$$

где $P_r(z)$ полином порядка $N + \kappa - 1$. Так как

$$f_r(t) \to \frac{f(t)\Pi(t)}{S^+(t)},\ \Psi_r(z) \to \frac{\Phi(z)\Pi(z)}{S^+(z)}$$

при $r \to 1-0$, то, переходя к пределу получаем доказательство утверждения а). Если $N+\kappa<0$, то $P_r(z)\equiv 0$ и

$$\Phi(z)=\frac{S(z)}{2\pi i\Pi(z)}\int_T\frac{f(t)\Pi(t)dt}{S^+(t)(t-z)}$$

Поскольку функция $S(z)(\Pi(z))^{-1}$ имеет полюс порядка $-(N+\kappa)$ в бесконечности, то получаем утверждение б) леммы. □

Лемма 2.3. *Пусть* $f\in L^1(\rho)$ *и*

(2.3) $$K(f,z)=\frac{S(z)}{2\pi i\Pi(z)}\int_T\frac{f(t)\Pi(t)dt}{S^+(t)(t-z)},\ z\in D^+\cup D^-,$$

тогда

$$\|K^+(f,rt)-a(t)K^-(f,r^{-1}t)\|_{L^1(\rho_r)}\le C\|f\|_{L^1(\rho)}$$

Доказательство. Положим $K^+(f,rt)-a(t)K^-(f,r^{-1}t)=I_1(r,t)+I_2(r,t)$, где

$$I_1(r,t)=\frac{S^+(rt)-a(t)S^-(r^{-1}t)}{2\pi i\Pi(rt)}\int_T\frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau-rt)},$$

$$I_2(r,t)=\frac{a(t)S^-(r^{-1}t)}{2\pi i}\Big(\frac{1}{\Pi(rt)}\int_T\frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau-rt)}-\frac{1}{\Pi(r^{-1}t)}\int_T\frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau-r^{-1}t)}\Big).$$

Так как $|S^+(rt)-a(t)S^-(r^{-1}t)|<C(1-r)^\delta$, $\delta>0$, то получаем

$$|I_1(r,t)|\le\frac{C(1-r)^\delta}{|\Pi(rt)|}\int_T\frac{|f(\tau)||\Pi(\tau)||d\tau|}{|\tau-rt|}.$$

Далее имеем $I_2(r,t)=I_2^{(1)}(r,t)+I_2^{(2)}(r,t)$, где

$$I_2^{(1)}(r,t)=\frac{a(t)S^-(r^{-1}t)}{2\pi i\Pi(rt)}\int_T\frac{f(\tau)\Pi(\tau)(1-r^2)dt}{S^+(\tau)|\tau-rt|^2},$$

$$I_2^{(2)}(r,t)=\frac{a(t)S^-(r^{-1}t)}{2\pi i}((\Pi(rt))^{-1}-(\Pi(r^{-1}t))^{-1})\int_T\frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau-rt)}.$$

Так как функция $a(t)S^-(r^{-1}t)$ равномерно ограничена при $r\to 1-0$, то

$$|I_2^{(1)}(r,t)|<\frac{C}{|\Pi(rt)|}\int_T\frac{|f(\tau)||\Pi(\tau)|(1-r^2)|d\tau|}{|\tau-rt|^2},$$

$$|I_2^{(2)}(r,t)|<C\Big|\frac{1}{\Pi(rt)}-\frac{1}{\Pi(r^{-1}t)}\Big|\int_T\frac{|f(\tau)||\Pi(\tau)||d\tau|}{|\tau-rt|}.$$

Далее, предполагая $\alpha_k>-1$ будем иметь

$$\|I_1(r,t)\|_{L^1(\rho)}\le C\int_T\frac{(1-r)^\delta\rho(t)}{|\Pi(rt)|}\int_T\frac{|f(\tau)||\Pi(\tau)||d\tau||dt|}{|\tau-rt|}\le$$

$$\le C\int_T|f(\tau)|\rho(\tau)\rho^*(\tau)\int_T\frac{(1-r)^\delta|dt||d\tau|}{|\tau-rt|\rho^*(t)},$$

где $\rho^*(t)$ определена в (1.2). В силу утверждения а) леммы 2.1

$$\sup_{r\in(0,1)} \rho^*(\tau)\int_T \frac{(1-r)^\delta |dt|}{|\tau - rt|\rho^*(t)} < \infty,$$

где $\delta \in (0,1)$, $\tau \in T$. Следовательно

$$\|I_1(r,t)\|_{L^1(\rho)} \le C\|f\|_{L^1(\rho)}.$$

Далее имеем

$$\|I_2^{(1)}(r,t)\|_{L^1(\rho)} \le C\int_T |f(\tau)|\rho(\tau)\rho^*(\tau)\int_T \frac{(1-r^2)|dt|}{\rho^*(t)|\tau - rt|^2}|d\tau|.$$

В силу утверждения б) леммы 2.1

$$\sup \rho^*(\tau)\int_T \frac{(1-r^2)|dt|}{|\tau - rt|^2\rho^*(t)} < \infty.$$

Поэтому

$$\|I_2^{(1)}(r,t)\|_{L^1(\rho)} \le C\|f\|_{L^1(\rho)}.$$

Далее имеем

$$\left|\frac{1}{\Pi(rt)} - \frac{1}{\Pi(r^{-1}t)}\right| = \left|\frac{1}{|t_k - rt|^{n_k}}\frac{1}{\Pi_k(rt)} - \frac{1}{|t_k - r^{-1}t|^{n_k}}\frac{1}{\Pi_k(r^{-1}t)}\right| \le$$

$$\le \frac{1}{|t_k - rt|^{n_k}}\left|\frac{1}{\Pi_k(rt)} - \frac{1}{\Pi_k(r^{-1}t)}\right| +$$

$$+\frac{1}{|\Pi_k(r^{-1}t)|}\left|\frac{1}{|t_k - rt|^{n_k}} - \frac{1}{|t_k - r^{-1}t|^{n_k}}\right| \le C\left(\frac{(1-r)}{|t_k - rt|^{n_k}} + \frac{(1-r)}{|t_k - rt|^{n_k+1}}\right),$$

где $\Pi_k(t) = \Pi(t)(t-t_k)^{-n_k}$. Следовательно

$$\|I_2^{(2)}(r,t)\|_{L^1(\rho)} \le C\int_T |f(\tau)||\rho(\tau)||\rho^*(\tau)|\int_T \frac{|\rho(t)||((\Pi(rt))^{-1} - (\Pi(r^{-1}t))^{-1}||dt|}{|\tau - rt|}|d\tau|$$

Учитывая что

$$\sup_{r\in(0,1)} \rho^*(\tau)\int_T \frac{\rho(t)|(\Pi(rt))^{-1} - (\Pi(r^{-1}t))^{-1}||dt|}{|\tau - rt|} \le \sum_{k=1}^m \int_{T_k} \frac{(1-r)\rho^*(\tau)|dt|}{|\tau - rt||t_k - rt|^{n_k - \alpha_k}} +$$

$$+\sum_{k=1}^m \int_{T_k} \frac{(1-r)\rho^*(\tau)|dt|}{|\tau - rt||t_k - rt|^{n_k - \alpha_k + 1}},\ \tau \in T_k$$

и применяя утверждение в) леммы 2.1 получаем

$$\sup_{r\in(0,1)} \rho^*(\tau)\int_T \frac{\rho(t)|(\Pi(rt))^{-1} - (\Pi(r^{-1}t))^{-1}||dt|}{|\tau - rt|} < \infty.$$

Следовательно

$$\|I_1(r,t)\|_{L^1(\rho_r)} \le C\int_T \frac{(1-r)^\delta \rho_r(t)}{|\Pi(rt)|}\int_T \frac{|f(\tau)||\Pi(\tau)||d(\tau)|}{|\tau - rt|}|dt| \le$$

$$\le C\int_T |f(\tau)|\rho(\tau)\rho^*(\tau)\int_T \frac{(1-r)^\delta \rho_r(t)|dt|}{|\tau - rt|\Pi(rt)}|d\tau|.$$

Учитывая, что

$$\rho^*(\tau)\int_T \frac{(1-r)^\delta \rho_r(t)|dt|}{|\tau - rt||\Pi(rt)|} \le \rho^*(\tau)\int_T \frac{(1-r)^\delta |dt|}{\rho^*(t)|\tau - rt|}$$

и применяя лемму 2.1 завершаем доказательство леммы. □

Функцию $a(t)$ отнесем к классу R^α ($a(t) \in R^\alpha$), где $\alpha = \{\alpha_1, ..., \alpha_m\}$, если

(2.4) $$\lim_{r\to 1-0} \|S^+(rt) - a(t)S^-(r^{-1}t)\|_{L^1(\rho_r)} = 0.$$

К примеру $a(t) \in R^\alpha$, если $\lambda_k > -n_k - 1$ и

$$a(t) = \cos^\gamma(\frac{\pi}{2}(\prod_{k=1}^m (1 - \frac{t_k}{t})^{\lambda_k})), \ \ t \in T$$

где γ целое число, λ_k неотрицательные целые числа. Тогда $\gamma = \operatorname{ind} a(t)$. Очевидно, что если $\gamma \ge 0$, то $S^+(z) \equiv 1$,

$$S^-(z) = \cos^{-\gamma}(\frac{\pi}{2}(\prod_{k=1}^m (1 - \frac{t_k}{z})^{\lambda_k})).$$

Имеем

$$|S^+(rt) - a(t)S^-(r^{-1}t)| \le C|\prod_{k=1}^m (t_k - r^{-1}t)^{\lambda_k} - \prod_{k=1}^m (t_k - t)^{\lambda_k}|$$

Следовательно при $t \in T_k$

$$|S^+(rt) - a(t)S^-(r^{-1}t)|\rho_r(t) \le C(1-r)|t_k - rt|^{n_k+\lambda_k-1}|t_k - t|^{\alpha_k - n_k}$$

Имея в виду что $\lambda_k + n_k > -1$ получаем $a(t) \in R^\alpha$. Аналогично можно доказать что $a(t) \in R^\alpha$, когда $\gamma < 0$.

Лемма 2.4. *Пусть $a(t) \in R^\alpha$ и $\alpha_j \le -2$ для некоторого $j \in \overline{1,m}$. Тогда если*

$$P(z) = A_1(t_j - z) + A_2(t_j - z)^2 + ... + A_{-n_j-1}(t_j - z)^{-n_j-1}$$

удовлетворяет условию

$$\lim_{r\to 1-0}\int_{|t_j - t|<\delta} |S^+(rt)P(rt) - a(t)S^-(r^{-1}t)P(r^{-1}t)|\rho_r(t)|dt| = 0,$$

для некоторого $0 < \delta < \min|t_j - t_k|$, $j \ne k$, то $P(z) \equiv 0$.

Доказательство. Так как

$$S^+(rt)P(rt) - a(t)S^-(r^{-1}t)P(r^{-1}t) = I_1(r,t) + I_2(r,t),$$

где

$$I_1(r,t) = (S^+(rt) - a(t)S^-(r^{-1}t)P(rt),$$

$$I_2(r,t) = a(t)S^-(r^{-1}t)(P(rt) - P(r^{-1}t)),$$

то из условия $a(t) \in R^\alpha$ следует что $\|I_1(r,t)\|_{L^1(\rho_r)} \to 0$. Пусть $A_1 \neq 0$, тогда имеем

$$|P(rt) - P(r^{-1}t)| > (1-r),$$

где $|t_j - t| < C(1-r)$, $C > 0$. Поэтому

$$\int_T |I_2(rt)|\rho_r(t)|dt| \geq \int_{|t_j - rt| < C(1-r)} \frac{(1-r)|dt|}{|t_j - rt|^{-n_j}|t_j - t|^{n_j - \alpha_j}}$$

Так как $|t_j - rt| = O(1-r)$ при $|t_j - t| < C(1-r)$, то

$$\int_{|t_j - t| < C(1-r)} |I_2(rt)|\rho_r(t)|dt| \geq (1-r)^{n_j+1} \int_0^{C(1-r)} \frac{d\theta}{\theta^{n_j - \alpha_j}} \geq C(1-r)^{2+\alpha_j}.$$

Следовательно $A_1 = 0$.

Предположим $A_k \neq 0$. Пусть k нечетное число и $A_1 = A_2 = ... = A_{k-1} = 0$. Тогда при $|t_j - t| < C(1-r)$ имеем

$$|P(rt) - P(r^{-1}t)| > (1-r)|t_j - t|^{k-1}.$$

Следовательно

$$\int_{|t_j - t| < \delta} |I_2(rt)|\rho_r(t)|dt| \geq \int_{|t_j - t| < C(1-r)} \frac{(1-r)|t_j - t|^{k+\alpha_j - n_j - 1}}{|t_j - rt|^{-n_j}}|dt|$$

$$> (1-r)^{n_k+1} \int_0^{C(1-r)} \theta^{k+\alpha_j - n_j - 1} d\theta = C(1-r)^{k+\alpha_j+1}.$$

Поскольку $k + \alpha_j + 1 \leq 0$, то

$$\lim_{r\to 1-0} \int_{|t_j - t| < \delta} |I_2(r,t)|\rho_r(t)|dt| > 0.$$

Если k четное число, то

$$\int_{|t_j - t| < \delta} |I_2(rt)|\rho_r(t)|dt| > (1-r)^{n_j+1} \int_{c(1-r)^2}^{C(1-r)} \theta^{k+\alpha_j - n_j - 1} d\theta =$$

$$(1-r)^{k+\alpha_j+1}(A - B(1-r)^{k+\alpha_j - n_j}).$$

Так как $k + \alpha_j - n_j > 0$, то получаем доказательство леммы. □

В дальнейшем будем предполагать, что

(2.5) $$\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_m, \ \alpha_{m_0} \leq -1, \alpha_{m_0+1} > -1$$

Лемма 2.5. *Пусть* $k_j \geq -n_j$, $j = 1, 2, ..., m_0$, *тогда*

$$\lim_{r\to 1-0} \|S^+(rt) \prod_{j=1}^{m_0} (t_j - rt)^{k_j} - a(t)S^-(r^{-1}t) \prod_{j=1}^{m_0} (t_j - r^{-1}t)^{k_j}\|_{L^1(\rho_r)} = 0.$$

Доказательство. Так как

$$S^+(rt)\prod_{\alpha_j\le -1}(t_j-rt)^{k_j}-a(t)S^-(r^{-1}t)\prod_{\alpha_j\le -1}(t_j-r^{-1}t)^{k_j}=I_1(r,t)+I_2(r,t),$$

где

$$I_1(r,t)=(S^+(rt)-a(t)S^-(r^{-1}t)\prod_{\alpha_j\le -1}(t_j-rt)^{k_j},$$

$$I_2(r,t)=a(t)S^-(r^{-1}t)(\prod_{\alpha_j\le -1}(t_j-rt)^{k_j}-\prod_{\alpha_j\le -1}(t_j-r^{-1}t)^{k_j}),$$

то учитывая, что $|S^+(rt)-a(t)S^-(r^{-1}t)|<C(1-r)^\delta$ и $\delta>0$ получаем

$$\int_T |I_1(r,t)|\rho_r(t)|dt|\le C(1-r)^\delta(\sum_{j=1}^m\int_{T_j}\frac{|t_j-rt|^{k_j}|dt|}{|t_j-rt|^{-n_j}|t_j-t|^{n_j-\alpha_j}})$$

и

$$\lim_{r\to 1-0}\int_T |I_1(r,t)||dt|=0.$$

Далее, так как при $t\in T_j$

$$|\prod_{\alpha_j\le -1}(t_j-rt)^{k_j}-\prod_{\alpha_j\le -1}(t_j-rt)^{k_j}|\le C|(t_j-rt)^k-(t_j-r^{-1}t)^k|<C(1-r)|t_j-rt|^{k-1},$$

то

$$\int_T |I_2(r,t)|\rho_r(t)|dt|\le C(1-r)(\sum_{j-1}^m\int_{T_j}\frac{|dt|}{|t_j-rt||t_j-t|^{n_j-\alpha_j}}).$$

Учитывая что $n_j-\alpha_j<1$, получаем

$$\lim_{r\to 1-0}\int_T |I_2(r,t)||dt|=0.$$

Лемма 2.5 доказана. □

Лемма 2.6. *Пусть $\alpha_k\le -2$ для некоторого k, $1\le k\le m$, $a(t)\in R^\alpha$ и $\Phi_k(z)=(t_k-z)^{-n_k}\prod_{j=1}^m(t_j-z)^{n_j}$. Если функция*

$$\varphi_k(z)=\left(\frac{A_1}{(t_k-z)}+\ldots+\frac{A_{-n_k}}{(t_k-z)^{-n_k}}\right)\Phi_k(z),$$

удовлетворяет условию

$$\int_T |S^+(rt)\varphi_k^+(rt)-a(t)S^-(r^{-1}t)\varphi_k^-(rt)|\rho_r(t)|dt|\to 0,$$

то $\varphi_k(z)$ представима в виде $\varphi_k(z)=A_0+(t_k-z)^{-n_k}\Phi_0(z)$, где $\Phi_0(z)$ аналитическая функция в точке t_k, A_0 некоторое комплексное число.

Доказательство. В силу леммы 2.5

$$\int_{|t_k-t|<\varepsilon} |(t_k-rt)^{-n_k}\varphi_k(rt)S^+(rt)-a(t)(t_k-r^{-1}t)^{-n_k}\varphi_k(r^{-1}t)S^-(r^{-1}t)|\rho_r(t)|dt| \to 0,$$

когда $r \to 1-0$, то функция

$$\phi_k(z) = \varphi_k'(t_k)(t_k - z) + \ldots + \frac{\varphi_k^{(-n_k-1)}(t_k)(t_k-z)^{-n_k-1}}{(-n_k)!}$$

также удовлетворяет условию

$$\int_{|t_k-t|<\varepsilon} |S^+(rt)\phi_k(rt) - a(t)S^-(r^{-1}t)\phi_k(r^{-1}t)|\rho_r(t)|dt| \to 0.$$

В силу леммы 2.4 числа $A_1, \ldots, A_{-n_k}$ однозначно определяются из условий

$$\begin{cases} A_{-n_k}F'(t_k) + A_{-n_k-1}F(t_k) = 0 \\ A_{-n_k}F''(t_k) + A_{-n_k-1}F'(t_k) + A_{-n_k-2}F(t_k) = 0 \\ \ldots \\ A_{-n_k}F^{(-n_k)}(t_k) + A_{-n_k-1}F^{(-n_k-1)}(t_k) + \ldots + A_1F(t_k) = 0. \end{cases}$$

Выбирая A_{-n_k} произвольным образом из этой системы однозначно определяем числа $A_1, A_2, \ldots, A_{-n_k-1}$ Учитывая леммы 2.4 и 2.5 завершаем доказательство леммы 2.6. □

3. Основная теорема

Теорема 3.1. *Пусть $f \in L^1(\rho)$. Тогда*

$$\lim_{r\to 1-0} \|K^+(f, rt) - a(t)K^-(f, r^{-1}t) - f(t)\|_{L^1(\rho_r)} = 0,$$

где $K(f, z)$ определяется формулой (2.3). Таким образом $K(f, z)$ решение задачи R, когда $N + \kappa \geq 0$. Если $N + \kappa < 0$, то $K(f, z)$ является решением задачи R тогда и только тогда когда f удовлетворяет условиям (2.2).

Доказательство. Пусть $f \equiv 0$ в некоторых непересекающихся окрестностях $T_{1k} \in T$ точек t_k. Обозначим через T_1 объединение этих интервалов. Тогда функция

$$\Phi_1(z) = \int_T \frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau - z)}$$

аналитична в T_1. Ряд Тейлора этой функции в точках t_i, $i = 1, \ldots, m$ имеет вид

$$\Phi_1(z) = A_0^{(i)} + A_1^{(i)}(z - t_i) + \ldots + A_k^{(i)}(z - t_i)^k + \ldots,$$

где $A_k^{(i)} = \dfrac{k!}{2\pi i}\displaystyle\int_T \frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau - t_i)^{k+1}}$, $i = 1, \ldots, m$.

Имеем

$$K^+(f,rt) - a(t)K^-(f,r^{-1}t) = I_1(r,t) + I_2(r,t),$$

где

$$I_1(r,t) = \frac{S^+(rt)\Phi_1(rt) - a(t)S^-(r^{-1}t)\Phi_1(r^{-1}t)}{\Pi(rt)},$$

$$I_2(r,t) = a(t)S^-(r^{-1}t)\Phi_1(r^{-1}t)((\Pi(rt))^{-1} - (\Pi(r^{-1}t))^{-1}).$$

Так как

$$|\Phi_1(rt) - \Phi_1(r^{-1}t)| < A(1-r), \ t \in T_1,$$

то учитывая что

$$|S^+(rt)\Phi_1(rt) - a(t)S^-(r^{-1}t)\Phi_1(r^{-1}t)| < C(1-r)^\delta, \ \delta > 0, t \in T_1,$$

получаем

$$|\int_{T_1} I_1(rt)\rho_r(t)dt| \le C\int_{T_1} \frac{(1-r)^\delta \rho_r(t)|dt|}{\Pi(rt)} \le C(1-r)^\delta \int_{T_1} |\rho^*(t)| \to 0, r \to 1-0.$$

Для $t \in T_1$ имеем

$$|\int_{T_1} I_2(r,t)\rho_r(t)dt| \le C\sum_{k=1}^{m}\int_{T_1^k}(1-r)(\frac{1}{|t_k - rt|^{n_k}} + \frac{1}{|t_k - rt|^{n_k+1}})|\rho_r(t)||dt| \le$$

$$\le C(1-r)\sum_{k=1}^{m}(\int_{T_1^k} |t-t_k|^{\alpha_k - n_k}|dt| + \int_{T_1} |t-t_k|^{\alpha_k - n_k - 1}|dt|) \to 0, r \to 1.$$

Далее, пусть $t \in T \setminus T_1$. Так как $f(t)\Pi(t) \in L^1(T)$ то функция

$$\Phi_2(z) = \frac{S(z)}{2\pi i}\int_T \frac{f(\tau)\Pi(\tau)d\tau}{S^+(\tau)(\tau - z)}$$

удовлетворяет условию

$$\|\Phi_2^+(rt) - a(t)\Phi_2^-(r^{-1}t) - f(t)\Pi(t)\|_{L^1} \to 0, \ r \to 1-0.$$

Следовательно, теорема доказана для функции обращающейся в нуль в окрестностях точек t_k, $k = 1, \dots, m$ Пусть теперь f есть произвольная функция из $L^1(\rho)$. Для любого $\varepsilon > 0$ можно найти функцию $f_\varepsilon \in L^1(\rho)$ такую, что $f_\varepsilon \equiv 0$ в T_1 и $\|f - f_\varepsilon\|_{L^1(\rho)} < \varepsilon$. Из леммы 2.3 следует

$$\|K^+(f,rt) - a(t)K^-(f,r^{-1}t) - f(t)\|_{L^1(\rho_r)} \le$$

$$\le \|K^+(f - f_\varepsilon, rt) - a(t)K^-(f - f_\varepsilon, r^{-1}t)\|_{L^1(\rho_r)} +$$

$$\|K^+(f_\varepsilon, rt) - a(t)K^-(f_\varepsilon, r^{-1}t) - f_\varepsilon\|_{L^1(\rho_r)} + \|f - f_\varepsilon\|_{L^1(\rho)} \le$$

$$\le C\|f - f_\varepsilon\|_{L^1(\rho)} + \|K^+(f_\varepsilon, rt) - a(t)K^-(f_\varepsilon, r^{-1}t) - f_\varepsilon\|_{L^1(\rho_r)}.$$

Так как

$$\|K^+(f_\varepsilon, rt) - a(t)K^-(f_\varepsilon, r^{-1}t) - f_\varepsilon\|_{L^1(\rho_r)} \to 0, \ r \to 1-0$$

получаем доказательство теоремы 3.1. □

4. Исследование задачи R в случае $\alpha_k > -1$, $k = 1, 2, ..., m$

Теорема 4.1. *Справедливы следующие утверждения:*

а) Если $\kappa \geq 0$, то общее решение однородной задачи R можно представить в виде

$$\Phi_0(z) = S(z)\left(\sum_{k=1}^{m}\sum_{j=1}^{n_k}\frac{A_j^{(k)}}{(t_k - z)^j} + P(z)\right), \tag{4.1}$$

где $P(z)$ полином порядка $\kappa - 1$ при $\kappa > 0$ и $P(z) \equiv 0$ при $\kappa = 0$.

б) Если $\kappa < 0$ и $N + \kappa > 0$, то общее решение однородной задачи R можно представить в виде

$$\Phi_0(z) = \frac{S(z)P(z)}{\Pi(z)}, \tag{4.2}$$

где $P(z)$ некоторый полином порядка $N + \kappa - 1$.

в) Если $N + \kappa \leq 0$, то однородная задача имеет только нулевое решение.

Доказательство. Пусть $P(z) = \sum_{k=0}^{\kappa-1} A_k z^k$. Поскольку $|S^+(rt) - a(t)S^-(r^{-1}t))| < A(1-r)^\delta$, то будем иметь

$$|S^+(rt)P(rt) - a(t)S^-(r^{-1}t)P(r^{-1}t)| < C(1-r)^\delta.$$

Поэтому

$$\|S^+(rt)P(rt) - a(t)S^-(r^{-1}t)P(r^{-1}t)\|_{L^1(\rho)} \to 0, r \to 1-0.$$

Положим

$$\Phi_{jk}(z) = S(z)(t_k - z)^{-j},\ j < n_k,\ k = 1, ..., m.$$

Получим

$$|\Phi_{jk}^+(rt) - a(t)\Phi_{jk}^-(r^{-1}t)| \leq |S^+(rt)(t_k - rt)^{-j} - a(t)S^-(r^{-1}t)(t_k - r^{-1}t)^{-j}|$$
$$\leq \frac{|S^+(rt) - a(t)S^-(r^{-1}t)|}{|t_k - rt|^j} + |a(t)S^-(r^{-1}t)|\left|\frac{1}{(t_k - rt)^j} - \frac{1}{(t_k - r^{-1}t)^j}\right|$$
$$\leq C\left(\frac{(1-r)^\delta}{|t_k - rt|^j} + \left|\frac{1}{(t_k - rt)^j} - \frac{1}{t_k - r^{-1}t|^j}\right|\right) \leq C\left(\frac{(1-r)^\delta}{|t_k - rt|^j} + \frac{(1-r)}{|t_k - rt|^{j+1}}\right).$$

Поэтому

$$\|\Phi_{jk}^+(rt) - a(t)\Phi_{jk}^-(r^{-1}t)\|_{L^1(\rho)} \leq$$
$$C\left(\int_T \frac{(1-r)^\delta \rho(t)|dt|}{|t_k - rt|^j} + \int_T \frac{(1-r)\rho(t)|dt|}{|t_k - rt|^{j+1}}\right).$$

Поскольку $j < n_k$ и $n_k - \alpha_k < 1$ то последние интегралы стремятся к нулю при $r \to 1-$. Утверждение а) теоремы доказано. Аналогично доказываются утверждения б) и в). □

Теорема 4.2. *Справедливы следующие утверждения*

а) если $N + \kappa \geq 0$, то общее решение задачи R можно представить в виде

$$\Phi(z) = K(f, z) + \Phi_0(z), \tag{4.3}$$

где $K(f, z)$ определяется формулой (2.1), а $\Phi_0(z)$ общее решение однородной задачи R,

б) если $N + \kappa < 0$, то задача R разрешима тогда и только тогда когда f удовлетворяет условиям (2.2). При этом решение можно представить в виде (2.1).

5. Исследование задачи R в случае произвольных α_k, $k = 1, 2, ..., m$

Теорема 5.1. *Пусть $a(t) \in R^\alpha$. Тогда*

а) Если $N + \kappa \geq 0$, то решение задачи R можно представить в виде

$$\Phi(z) = K(f, z) + S(z)(A_0 + P(z)(\Pi(z))^{-1}, \tag{5.1}$$

где A_0 произвольное комплексное число при $\kappa > 0$ и $A_0 = 0$ при $\kappa \leq 0$, $P(z)$ полином порядка $N + \kappa - 1$.

б) Если $N + \kappa < 0$ и $\kappa > 0$, то $\Phi(z)$ имеет представление (4.3), где $\Phi_0(z) \equiv 0$ и f удовлетворяет следующим условиям:

$$\int_T \frac{f(t)}{S^+(t)} \Pi(t) t^k dt = 0, \ k = 0, 1, ..., -(N + \kappa) - 1 \tag{5.2}$$

в) Если $N < 0$ и $\kappa \leq 0$, то задача R имеет единственное решение $\Phi(z) = K(f, z) + A_0 S(z)$, где

$$A_0 = -\frac{1}{2\pi i} \int_T \frac{f(t)\Pi(t)dt}{S^+(t)t^{N+1}} \tag{5.3}$$

и f удовлетворяет условиям (5.2), если $\kappa \neq -N - 1$.

г) Если $N + \kappa < 0$ и $N \geq 0$, то то задача R имеет одно решение $\Phi(z) = K(f, z)$ и f удовлетворяет условиям (5.2).

Доказательство. Если Φ решение задачи R, то будем иметь

$$\lim_{r \to 1-0} \int_T \left| \frac{\Phi^+(rt)}{S^+(t)} - \frac{\Phi^-(r^{-1}t)}{S^-(t)} - \frac{f(t)}{S^+(t)} \right| |P_{jr}(t)| |\widetilde{P_j}(t)| |dt| = 0,$$

где

$$P_{jr}(z) = \prod_{k=1}^{j}(t_k - rz)^{n_k}, \ \widetilde{P_j}(z) = \prod_{k=j+1}^{m}(t_k - z)^{n_k}.$$

Обозначив

$$\Psi_r^+(z) = \frac{\Phi^+(rz)}{S^+(z)}P_{jr}(z)\widetilde{P_j}(z), \ \Psi_r^-(z) = \frac{\Phi^-(r^{-1}z)}{S^-(z)}P_{jr}(z)\widetilde{P_j}(z),$$

получаем задачу Гильберта для функции $\Psi_r(z)$

$$\Psi_r^+(t) - \Psi_r^-(t) = f_r(t),$$

где $f_r(t) = \dfrac{f(t)}{S^+(t)}P_{jr}(t)\widetilde{P_j}(t)$. Так как функция $\Psi_r(z)$ имеет полюс порядка $-n_k$ в точке $t = r^{-1}t_k$, $k = 1, 2, ..., j$ и в бесконечности $|\Psi_r^+(z)| < C|z|^{N+\kappa-1}$, то обозначив

$$Q_{kr}(z) = \frac{A_1^k(r)}{(t_k - rz)} + ... + \frac{A_{n_k}^k(r)}{(t_k - rz)^{-n_k}}$$

главную часть разложения Лорана функции $\Psi_r^-(z)$ в точке $r^{-1}t_k$, будем иметь

$$\Psi_r^+(t) - (\Psi_r^-(t) - \sum_{k=1}^{m_0} Q_{kr}(t)) = f_r(t) + \sum_{k=1}^{m_0} Q_{kr}(t).$$

Далее положив $N + \kappa \geq 0$, получаем

$$\Psi_r(z) = \frac{1}{2\pi i}\int_T \frac{f_r(t)}{(t - z)}dt + \sum_{k=1}^{m_0} Q_{kr}(z) + P_r(z),$$

где $P_r(z)$ некоторый полином порядка $N + \kappa - 1$ при $N + \kappa > 0$ и $P_r(z) \equiv 0$ при $N + \kappa = 0$. Так как $f_r(t) \to f(t)\Pi(t)(S^+(t))^{-1}$ в L^1 получаем

$$\Phi(z) = K(f, z) + \frac{S(z)(\sum_{k=1}^{m_0} Q_k(z) + P(z))}{\Pi(z)},$$

где

$$Q_k(z) = \frac{C_1^k}{(t_k - z)} + ... + \frac{C_{-n_k}^k}{(t_k - z)^{-n_k}}, \ k = 1, ..., m_0.$$

Очевидно, что функция $\dfrac{S(z)P(z)}{\Pi(z)}$ удовлетворяет однородному условию (1.3). Поэтому функция

$$\frac{\sum_{k=1}^{m_0} Q_k(z)}{\Pi(z)},$$

тоже удовлетворяет условию (1.3). Учитывая лемму 2.6 получаем следующее представление функции $\Phi(z)$:

$$\Phi(z) = K(f, z) + S(z)(A_0 + P(z)(\Pi(z))^{-1}).$$

Предположим $N + \kappa < 0$, $\kappa > 0$. Поскольку $a(t) \in R^{\alpha}$, то $A_0S(z)$ является решением однородной задачи R, следовательно общее решение задачи R можно представить в виде:

$$\Phi(z) = K(f, z) + S(z)A_0,$$

где A_0 есть произвольное комплексное число.

Если $N > 0$, $\kappa \leq 0$. То функция $S(z)$ имеет полюс порядка $-\kappa$ в бесконечности, следовательно $A_0S(z)$ не решение однородной задачи R. Чтобы имело место условие $\Phi(\infty) = 0$, необходимо найти A_0 из (5.1) и потребовать чтобы f удовлетворяла условиям (5.2) при $\kappa \neq -N - 1$.

При $N + \kappa < 0$, $N \leq 0$ очевидно что $A_0 = 0$ и f удовлетворяет условиям (11). □

Теорема 5.2. *Пусть $a(t) \in R^{\alpha}$. Тогда а) Если $N + \kappa > 0$, то общее решение однородной задачи R можно представить в виде:*

$$\Phi(z) = S(z)(A_0 + P(z)(\Pi(z))^{-1})$$

где A_0 произвольное комплексное число при $\kappa > 0$ и $A_0 = 0$ при $\kappa \leq 0$, $P(z)$ полином порядка $N + \kappa - 1$.

б) Если $N + \kappa \leq 0$ и $\kappa > 0$, то $\Phi(z) = A_0S(z)$, где A_0 произвольное комплексное число.

в) Если $N + \kappa \leq 0$ и $\kappa \leq 0$, то однородная задача R имеет только нулевое решения $\Phi(z) \equiv 0$.

Abstract. In the unit disc bounded by the circle $T = \{z, |z| = 1\}$ we consider the Riemann boundary value problem in the weighted space $L^1(\rho)$, where $\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}$, $t_k \in T$, $k = 1, 2, \ldots, m$, and α_k, $k = 1, 2, \ldots, m$ are real numbers. The question of interest is to determine an analytic outside the circle T function $\Phi(z)$, $\Phi(\infty) = 0$ to satisfy $\lim_{r \to 1-0} \|\Phi^+(rt) - a(t)\Phi^-(r^{-1}t) - f(t)\|_{L^1(\rho_r)} = 0$, where $f \in L^1(\rho)$, $a(t) \in C^{\delta}(T)$, $\delta > 0$, and ρ_r are some continuations of function ρ inside the circle. The normal solvability of this problem is established.

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MAGNETIC BI-HARMONIC DIFFERENTIAL OPERATORS ON RIEMANNIAN MANIFOLDS AND THE SEPARATION PROBLEM

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Abstract. In this paper we obtain sufficient conditions for the bi-harmonic differential operator $A = \Delta_E^2 + q$ to be separated in the space $L^2(M)$ on a complete Riemannian manifold (M, g) with metric g, where Δ_E is the magnetic Laplacian on M and $q > 0$ is a locally square integrable function on M. Recall that, in the terminology of Everitt and Giertz, the differential operator A is said to be separated in $L^2(M)$ if for all $u \in L^2(M)$ such that $Au \in L^2(M)$ we have $\Delta_E^2 u \in L^2(M)$ and $qu \in L^2(M)$.

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1. Introduction

The separation problem for differential operators was first introduced in 1971 by Everitt and Giertz [10], then this problem for different differential expressions was studied by many authors, such as Boimatove [4], Brown [6-7], Mohamed and Atia [16-17], Zayed et al [19], Atia et al [1-3], etc. In [14], Milatovic has studied the separation property for Schrodinger operators on the Riemannian manifolds. Recently Milatovic [15], has introduced the magnetic Schrodinger operator of the form $L = \Delta_E + q$ on a complete Riemannian manifold (M, g) with metric g, where Δ_E is the magnetic Laplacian on M and $q \geq 0$ is a locally square integrable function on M. Sufficient conditions for the operator L to be separated in $L^2(M)$ were obtained in [15]. Atia et al [2] have studied the separation property for the bi-harmonic differential expression of the form $\Delta_E^2 + q$ with $E = 0$.

In this paper we generalize the results of [2] to the magnetic bi-harmonic differential expression of the form $A = \Delta_E^2 + q$, where $E \neq 0$. Let (M, g) be a Riemannian manifold without boundary (that is, M is a C^∞-manifold without boundary and

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(g_{jk}) is a Riemannian metric on M), and let $dimM = n$. We will assume that M is connected.

Throughout the paper we use the following notation. By $d\mu$ we will denote the Riemannian volume element of M, by $L^2(M)$ we denote the space of complex-valued square integrable functions on M with the inner product:

$$(u, v) = \int_M (u\bar{v})\, d\mu, \tag{1.1}$$

and $\|\cdot\|$ denotes the norm in $L^2(M)$ corresponding to the inner product (1.1). We will use the notation $L^2(\Lambda^1 T^* M)$ for the space of complex-valued square integrable 1-forms on M with the inner product:

$$(W, \Psi)_{L^2(\Lambda^1 T^* M)} = \int_M \langle W, \overline{\Psi} \rangle\, d\mu, \tag{1.2}$$

where for 1-forms $W = W_j dx^j$ and $\Psi = \Psi_k dx^k$, we define $\langle W, \Psi \rangle = g^{jk} W_j \Psi_k$, where (g^{jk}) stands for the inverse of the matrix (g_{jk}), and $\overline{\Psi} = \overline{\Psi_k} dx^k$. (We use the standard Einstein summation convention). By $\|\cdot\|_{L^2(\Lambda^1 T^* M)}$ we denote the norm in $L^2(\Lambda^1 T^* M)$ corresponding to the inner product (1.2). By $C^\infty(M)$ we denote the space of smooth functions on M, by $C_c^\infty(M)$ – the space of smooth compactly supported functions on M, by $\Omega^1(M)$ – the space of smooth 1-forms on M and by $\Omega_c^1(M)$ – the space of smooth compactly supported 1-forms on M.

Recall that a magnetic potential is a real valued 1-form $E \in \Omega^1(M)$, and note that in any local coordinates $x^1, x^2, \ldots, x^n$, the form E can be written as $E = E_j dx^j$, where $E_j = E_j(x)$ are real valued C^∞–functions of the local coordinates. The operator $d : C^\infty(M) \to \Omega^1(M)$ stands for the usual differential and by $d_E : C^\infty(M) \to \Omega^1(M)$ we denote the deformed differential defined by $d_E(u) = du + iuE$, for every $u \in C^\infty(M)$, where $i = \sqrt{-1}$. We denote the formal adjoint of d_E by $d_E^* : \Omega^1(M) \to C^\infty(M)$ which is defined by the identity: $(d_E u, w)_{L^2(\Lambda^1 T^* M)} = (u, d_E^* w)$, $\forall\ u \in C_c^\infty(M)$, $w \in \Omega^1(M)$. By $\Delta_E = d_E^* d_E : C^\infty(M) \to C^\infty(M)$ we denote the magnetic Laplacian on M, with magnetic potential E. In this paper we consider the bi-harmonic differential expression:

$$A = \Delta_E^2 + q, \tag{1.3}$$

where $q \in L_{loc}^2(M)$ is a real-valued function, called the electric potential. Also, we use the notation

$$D_1 = \{u \in L^2(M) : Au \in L^2(M)\}. \tag{1.4}$$

Definition 1.1. Using the terminology of Everitt and Giertz [11], we say that the differential expression $A = \Delta_L^2 + q$ is separated in the space $L^2(M)$ if for all $u \in D_1$ we have $\Delta_L^2 u \in L^2(M)$ and $qu \in L^2(M)$.

Definition 1.2. Let A be as in (1.3). We define the minimal operator S in $L^2(M)$ associated with A by the formula $Su = Au$ with domain $Dom(S) = C_c^\infty(M)$.

Remark 1.1. Since S is a symmetric operator, it follows that S is closable (see [12], Section V.3.3). In what follows, we will denote by $\bar{S}$ and S^* the closure and the adjoint of the operator S in $L^2(M)$, respectively.

Lemma 1.1. *If (M, g) is a complete Riemannian manifold with a metric g and a positive smooth measure $d\mu$ and if $0 \le q \in L^2_{loc}(M)$, then the operator S is essentially self-adjoint in $L^2(M)$ (see [5, 8, 13, 18]). Moreover, in this case we have $\bar{S} = S^*$ (see [9, 12]).*

Definition 1.3. The set of admissible parameters $P \subset R^3$ is defined to be the set of parameters $(\alpha, \beta, \gamma) \in R^3$ satisfying the following three conditions: (1) $\gamma > 0$; (2) $\beta > 0$; (3) $0 < \alpha\beta < 1$ or $\alpha = \beta = 1$.

2. The main result

The main result of the present paper is the following theorem.

Theorem 2.1. *Let (M, g) be a complete and connected C^∞−Riemannian manifold without boundary, with a positive smooth measure $d\mu$ and a metric g satisfying the following conditions:*

$$0 \le q(x) \in L^2_{loc}(M), \quad dq(x), \; d^2q(x) \in L^2_{loc}(M), \tag{2.1}$$

$$\left\| d^2q(x)\; u(x) \right\| \le C_1 \left\| q^{\frac{3}{2}}(x)\; u(x) \right\|, \tag{2.2}$$

$$\left\| dq(x)\; du(x) \right\| \le C_2 \left\| q^{\frac{3}{2}}(x)\; u(x) \right\| \tag{2.3}$$

for every $x \in M$ and $u \in C_c^\infty(M)$, where $C_1 \ge 0$ and $C_2 \ge 0$ are constants with $C_1 + 2C_2 \in [0, 2)$. Then the differential expression A defined by (1.3) is separated in the space $L^2(M)$.

Proof. Let $(\alpha, \beta, \gamma) \in P$ and $u \in C_c^\infty(M)$. By the definition (1.3) of the expression A, for every $u \in C_c^\infty(M)$, we have

$$
\begin{aligned}
\|Au\|^2 &= (\Delta_E^2 u + qu, \Delta_E^2 u + qu) \\
&= \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(\Delta_E^2 u, qu\right) + \|qu\|^2 \\
&= \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(\Delta_E^2 u, Au - \Delta_E^2 u\right) + \|qu\|^2 \\
&= -\left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(\Delta_E^2 u, Au\right) + \|qu\|^2 \\
&= (2\beta^{-1} - 1)\left\|\Delta_E^2 u\right\|^2 - 2\beta^{-1}\left(\Delta_E^2 u, \Delta_E^2 u\right) + 2\mathrm{Re}\left(\Delta_E^2 u, Au\right) + \|qu\|^2 \\
&= (2\beta^{-1} - 1)\left\|\Delta_E^2 u\right\|^2 - 2\beta^{-1}\mathrm{Re}\left(\Delta_E^2 u, Au - qu\right) + 2\mathrm{Re}\left(\Delta_E^2 u, Au\right) + \|qu\|^2 \\
&= (2\beta^{-1} - 1)\left\|\Delta_E^2 u\right\|^2 + 2\beta^{-1}\mathrm{Re}\left(\Delta_E^2 u, qu\right) \\
&\quad + 2(1 - \beta^{-1})\mathrm{Re}\left(\Delta_E^2 u, Au\right) + \|qu\|^2. \qquad (2.4)
\end{aligned}
$$

For any imaginary number z, we have

$$
-|z| \leq \mathrm{Re} z \leq |z|. \qquad (2.5)
$$

Also, for any two positive real numbers a and b, we have

$$
ab \leq \frac{k}{2}a^2 + \frac{1}{2k}b^2, \qquad (2.6)
$$

where k is a positive real number. Hence, applying the inequalities (2.6), (2.7) and the Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
2 \quad & (1 - \beta^{-1})\mathrm{Re}\left(\Delta_E^2 u, Au\right) \geq -2\left|(1 - \beta^{-1})\left(\Delta_E^2 u, Au\right)\right| \\
&= -2\left|1 - \beta^{-1}\right| \left|\left(\Delta_E^2 u, Au\right)\right| \geq -2\left|1 - \beta^{-1}\right| \left\|\Delta_E^2 u\right\| \|Au\| \\
&\geq -\left|1 - \beta^{-1}\right| \left(k\left\|\Delta_E^2 u\right\|^2 + k^{-1}\|Au\|^2\right). \qquad (2.7)
\end{aligned}
$$

Since $q \geq 0$, by the definitions of d_E and d_E^*, we obtain

$$
\begin{aligned}
\mathrm{Re}\left(\Delta_E^2 u, qu\right) &= \mathrm{Re}\left(qu, \Delta_E^2 u\right) = \mathrm{Re}\left(\Delta_E(qu), \Delta_E u\right) \\
&= \mathrm{Re}\left((d^2 q)u + 2(dq)(du) + q(\Delta_E u), \Delta_E u\right) \\
&= \mathrm{Re}\left((d^2 q)u, \Delta_E u\right) + 2\mathrm{Re}\left((dq)(du), \Delta_E u\right) + \mathrm{Re}\left(q(\Delta_E u), \Delta_E u\right) \\
&= \mathrm{Re}\left((d^2 q)u, \Delta_E u\right) + 2\mathrm{Re}\left((dq)(du), \Delta_E u\right) + \left\|q^{\frac{1}{2}}\Delta_E u\right\|^2. \qquad (2.8)
\end{aligned}
$$

Next, by using the Cauchy-Schwartz inequality, the inequalities (2.5) and (2.6) with $k=\gamma$, and the conditions (2.2) and (2.3), we can write

$$\begin{aligned} \operatorname{Re}\left((d^2q)u,\triangle_E u\right) &\geq -\left|\left((d^2q)u,\triangle_E u\right)\right| \geq -\left\|(d^2q)u\right\|\left\|\triangle_E u\right\| \\ &\geq -C_1\left\|q^{\frac{3}{2}}u\right\|\left\|\triangle_E u\right\| = -C_1\left\|qu\right\|\left\|q^{\frac{1}{2}}\triangle_E u\right\| \\ &\geq -C_1\frac{\gamma}{2}\left\|qu\right\|^2 - C_1\frac{\gamma^{-1}}{2}\left\|q^{\frac{1}{2}}\triangle_E u\right\|^2, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} 2\operatorname{Re}\left((dq)(du),\triangle_E u\right) &\geq -2\left|(dqdu,\triangle_E u)\right| \geq -2\left\|dqdu\right\|\left\|\triangle_E u\right\| \\ &\geq -2C_2\left\|q^{\frac{3}{2}}u\right\|\left\|\triangle_E u\right\| = -2C_2\left\|qu\right\|\left\|q^{\frac{1}{2}}\triangle_E u\right\| \\ &\geq -C_2\gamma\left\|qu\right\|^2 - C_2\gamma^{-1}\left\|q^{\frac{1}{2}}\triangle_E u\right\|^2. \end{aligned} \tag{2.10}$$

Taking into account (2.9) and (2.10), from (2.8) we obtain

$$\operatorname{Re}\left(\triangle_E^2 u,qu\right) \geq -\left(C_1\frac{\gamma}{2}+C_2\gamma\right)\left\|qu\right\|^2 + \left(1-C_1\frac{\gamma^{-1}}{2}-C_2\gamma^{-1}\right)\left\|q^{\frac{1}{2}}\triangle_E u\right\|^2,$$

implying that

$$\begin{aligned} 2\beta^{-1}\operatorname{Re}\left(\triangle_E^2 u,qu\right) &\geq -\beta^{-1}\gamma\left(C_1+2C_2\right)\left\|qu\right\|^2 \\ &\quad +\beta^{-1}\left(2-C_1\gamma^{-1}-2C_2\gamma^{-1}\right)\left\|q^{\frac{1}{2}}\triangle_E u\right\|^2. \end{aligned} \tag{2.11}$$

Taking into account (2.7) and (2.11), from (2.4) we get

$$\left(1+\left|1-\beta^{-1}\right|k^{-1}\right)\left\|Au\right\|^2 \geq \left(1-\beta^{-1}C_1\gamma-2\beta^{-1}C_2\gamma\right)\left\|qu\right\|^2 \tag{2.12}$$

$$+\beta^{-1}\left(2-C_1\gamma^{-1}-2C_2\gamma^{-1}\right)\left\|q^{\frac{1}{2}}\triangle_E u\right\|^2 + \left(2\beta^{-1}-1-\left|1-\beta^{-1}\right|k\right)\left\|\triangle_E^2 u\right\|^2.$$

If $\alpha\beta<1$, we choose $k>0$ such that $1+\left|1-\beta^{-1}\right|k^{-1}=(\alpha\beta)^{-1}$, and multiply both sides of (2.12) by $\alpha\beta$ to obtain

$$\begin{aligned} \left\|Au\right\|^2 &\geq \alpha\left(\beta-C_1\gamma-2C_2\gamma\right)\left\|qu\right\|^2 + \alpha\left(2-C_1\gamma^{-1}-2C_2\gamma^{-1}\right)\left\|q^{\frac{1}{2}}\triangle_E u\right\|^2 \\ &\quad +\alpha\beta\left(2\beta^{-1}-1-\left|1-\beta^{-1}\right|k\right)\left\|\triangle_E^2 u\right\|^2. \end{aligned} \tag{2.13}$$

Next, we show that the following equality holds:

$$\alpha\beta\left(2\beta^{-1}-1-\left|1-\beta^{-1}\right|k\right) = 1-(1-\alpha)^2(1-\alpha\beta)^{-1}. \tag{2.14}$$

Indeed, since $1 + |1-\beta^{-1}|\,k^{-1} = (\alpha\beta)^{-1}$, we have $k = \frac{\alpha\beta|1-\beta^{-1}|}{1-\alpha\beta}$. Hence, we can write

$$\begin{aligned}
\alpha\beta\left(2\beta^{-1}-1-|1-\beta^{-1}|\,k\right) &= \alpha\beta\left(2\beta^{-1}-1-\frac{\alpha\beta\,|1-\beta^{-1}|^2}{1-\alpha\beta}\right)\\
&= \alpha\beta\left(\frac{2\beta^{-1}-2\alpha-1+\alpha\beta-\alpha\beta\,|1-\beta^{-1}|^2}{1-\alpha\beta}\right)\\
&= \frac{2\alpha-2\alpha^2\beta-\alpha\beta+\alpha^2\beta^2-\alpha^2\beta^2\left(1-2\beta^{-1}+\beta^{-2}\right)}{1-\alpha\beta}\\
&= \frac{-\alpha\beta-\alpha^2+2\alpha}{1-\alpha\beta} = \frac{(1-\alpha\beta)-(1-\alpha)^2}{1-\alpha\beta} = 1-(1-\alpha)^2(1-\alpha\beta)^{-1}.
\end{aligned}$$

Now from (2.13) and (2.14), we get

$$\begin{aligned}
\|Au\|^2 \;\geq\;& \alpha\left(\beta-C_1\gamma-2C_2\gamma\right)\|qu\|^2+\alpha\left(2-C_1\gamma^{-1}-2C_2\gamma^{-1}\right)\left\|q^{\frac12}\Delta_E u\right\|^2\\
&+\left(1-(1-\alpha)^2(1-\alpha\beta)^{-1}\right)\left\|\Delta_E^2 u\right\|^2.
\end{aligned}\tag{2.15}$$

If $\alpha=\beta=1$, we can take any $k>0$ in (2.13) to obtain

$$\begin{aligned}
\|Au\|^2 \;\geq\;& \left(1-C_1\gamma-2C_2\gamma\right)\|qu\|^2+\left\|\Delta_E^2 u\right\|^2\\
&+\left(2-C_1\gamma^{-1}-2C_2\gamma^{-1}\right)\left\|q^{\frac12}\Delta_E u\right\|^2.
\end{aligned}\tag{2.16}$$

From (2.15) and (2.16), we obtain

$$a\,\|qu\|^2+b\left\|q^{\frac12}\Delta_E u\right\|^2+c\left\|\Delta_E^2 u\right\|^2 \leq \|Au\|^2, \tag{2.17}$$

where $a=\alpha\left(\beta-(C_1+2C_2)\gamma\right)$, $b=\alpha\left(2-(C_1+2C_2)\gamma^{-1}\right)$ and

$$c=\begin{cases}1-(1-\alpha)^2(1-\alpha\beta)^{-1} & \text{if} \quad \alpha\beta<1\\ 1 & \text{if} \quad \alpha=\beta=1.\end{cases}$$

If $C_1+2C_2\in(0,2)$, then there exists an admissible triple of parameters $(\alpha,\beta,\gamma)\in P$ satisfying the inequalities:

$$\beta\geq(C_1+2C_2)\gamma,\quad 2\gamma\geq C_1+2C_2,\quad \textit{and}\quad \alpha+\beta\leq 2, \tag{2.18}$$

which implies that $a\geq 0$, $b\geq 0$, and $c\geq 0$. If $C_1=C_2=0$, then from (7) and (8) we get $d^2q(x)\;u(x)=0$ and $dq(x)\;du(x)=0$, for every $x\in M$ and $u\in C_c^\infty(M)$. Consequently, we have $\Delta_E(qu)=q\left(\Delta_E u\right)$, and for every $u\in C_c^\infty(M)$, we can write

$$\|Au\|^2=(Au,Au)=\left(\Delta_E^2u+qu,\Delta_E^2u+qu\right) \tag{2.19}$$

$$= \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(\Delta_E^2 u, qu\right) + \|qu\|^2 = \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(qu, \Delta_E^2 u\right) + \|qu\|^2$$
$$= \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(\Delta_E (qu), \Delta_E u\right) + \|qu\|^2$$
$$= \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(q(\Delta_E u), \Delta_E u\right) + \|qu\|^2$$
$$= \left\|\Delta_E^2 u\right\|^2 + 2\left\|q^{\frac{1}{2}}\Delta_E u\right\|^2 + \|qu\|^2.$$

It follows from (2.17) – (2.19) that the inequality (2.17) holds for all $u \in C_c^\infty(M)$, with $a \geq 0$, $b \geq 0$, and $c \geq 0$ if $C_1 + 2C_2 \in [0, 2)$.

Now we proceed to prove that under the hypotheses of the theorem the inequality (2.17) holds for all $u \in D_1$. To this end, observe first that from the completeness of (M, g), it follows that the operator S is essentially self-adjoint and $Dom(\bar{S}) = D_1$. Let $u \in D_1$, then there exists a sequence $\{u_n\}$ in $C_c^\infty(M)$ such that $u_n \to u$ and $Au_n \to \bar{S}u$ in $L^2(M)$ as $n \to \infty$. Applying (2.17) with $u = u_n - u_m$, we conclude that the sequences $\{qu_n\}$, $\{\Delta_E^2 u_n\}$ and $\{q^{\frac{1}{2}}\Delta_E u_n\}$ are Cauchy sequences in $L^2(M)$. Since $\|\Delta_E u_n\|^2 = (\Delta_E u_n, \Delta_E u_n) = (\Delta_E^2 u_n, u_n) \leq \left\|\Delta_E^2 u_n\right\| \|u_n\|$, and $\{\Delta_E^2 u_n\}$ and $\{u_n\}$ are Cauchy sequences in $L^2(M)$, it follows that $\{\Delta_E u_n\}$ is also a Cauchy sequence in $L^2(M)$. Taking into account that the operator Δ_E is essentially self-adjoint on $C_c^\infty(M)$ (see [17]), we have

$$\Delta_E u_n \to \Delta_E u, \tag{2.20}$$

implying that

$$q^{\frac{1}{2}}\Delta_E u_n \to q^{\frac{1}{2}}\Delta_E u \quad \text{and} \quad \Delta_E^2 u_n \to \Delta_E^2 u, \tag{2.21}$$

in $L^2(M)$ as $n \to \infty$.

Finally, taking into account that $\{qu_n\}$ is a Cauchy sequence in $L^2(M)$ and $C_c^\infty(M)$ is dense in $L^2(M)$, it follows that

$$qu_n \to qu, \tag{2.22}$$

in $L^2(M)$ as $n \to \infty$. Now, replacing u by u_n in (2.17), passing to the limit as $n \to \infty$ in all terms, and using (2.20) – (2.22), we conclude that the inequality (2.17) holds for all $u \in D_1$. This means that the differential expression A defined by (1.3) is separated in the space $L^2(M)$. This completes the proof of the theorem. □

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THE RICCI FLOW AS A GEODESIC ON THE MANIFOLD OF RIEMANNIAN METRICS

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Abstract. The Ricci flow is an evolution equation in the space of Riemannian metrics. A solution for this equation is a curve on the manifold of Riemannian metrics. In this paper we introduce a metric on the manifold of Riemannian metrics such that the Ricci flow becomes a geodesic. We show that the Ricci solitons introduce a special slice on the manifold of Riemannian metrics.

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1. Introduction

The collection $\mathfrak{M}$ of all smooth Riemannian metrics on a compact smooth n-dimensional manifold M is an infinite dimensional Fréchet manifold. Geometry of this space has been studied at first by D. Ebin [8], where he proved the existence of a slice in the space of Riemannian metrics. The basic facts about the manifold of Riemannian metrics $\mathfrak{M}$ can be found in [8, 10, 12]. In [4, 5], B. Clarke proved that the geodesic distance for the natural metric is a positive topological metric on $\mathfrak{M}$, and determined the metric completion of $\mathfrak{M}$. The existence of a vanishing geodesic distance for some infinite dimensional manifolds has been established in [3, 16, 17].

The Ricci flow is a valuable geometric flow introduced by R. Hamilton in the early 1980 [14]. Following the paper by J. Eells and J. Sampson [9], he introduced an evolution equation for a family of Riemannian metrics as follows:

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)), \quad g(0) = g_0,$$

where $Ric(g(t))$ denotes the Ricci curvature of the metric $g(t)$.
The short time existence of solutions of the above evolution equation has been proved by R. Hamilton [13, 14], by using Nash and Moser implicit function theorem. Later D. DeTurck [7] gave a shorter proof based on linearization of differential operators.

In [11], the authors have proved this result by considering geometry of the manifold of Riemannian metrics $\mathfrak{M}$.

Ricci solitons are special solutions of the Ricci flow (see [1]). Namely, a solution $g(t)$ of the Ricci flow on M is a Ricci soliton (or self-similar solution) if there exist a positive time-dependent function $\sigma(t)$ with $\sigma(0) = 1$, and an 1−parameter family of time-dependent diffeomorphisms $\varphi_t : M \longrightarrow M$ with $\varphi_0 = id$, such that

$$g(t) = \sigma(t)\varphi(t)^* g(0).$$

It is a very useful tool in the sdudy of the differential geometry and physics (see, e.g., [6, 18, 19, 21]). Observe that the solution of Ricci flow is a curve in the space of Riemannian metrics. In this paper, guided by the results of [2], we show that the Ricci flow can be considered as a geodesic of a Riemannian metric on $\mathfrak{M}$. Also, we show that the Ricci solitons are applicable to give a special slice on $\mathfrak{M}$.

The paper is organized as follows. In Section 2, we present the necessary notation and some preliminary facts. In Section 3, we recall some results of D. Ebin [8] on the manifold of Riemannian metrics, and prove a useful lemma concerning Levi-Civita connection. In Section 4, we prove the main results of the paper (Theorems 4.1 and 4.2), giving a Riemannian metric on $\mathfrak{M}$ such that the Ricci flow is a geodesic on $\mathfrak{M}$. In Section 5, the relation between Ricci solitons and slices on $\mathfrak{M}$ is described. We show that Ricci soliton is equivalent to existence of a finite dimensional slice for $\mathfrak{M}$.

2. Notation

2.1. A metric on tensor spaces. A Riemannian metric $g : TM \times_M TM \to \mathbb{R}$ will equivalently be interpreted as musical isomorphisms:

$$\flat = g : TM \to T^*M \qquad \sharp = g^{-1} : T^*M \to TM$$

The metric g can be extended to the cotangent bundle $T^*M = T^0_1 M$ by setting

$$g^{-1}(\alpha, \beta) = g^0_1(\alpha, \beta) = \alpha(\sharp\beta)$$

for $\alpha, \beta \in T^*M$, and the product metric

$$g^r_s = \bigotimes^r g \otimes \bigotimes^s g^{-1}$$

extends g to all tensor spaces $T^r_s M$. A useful formula is

$$g^0_2(h, k) = Tr(g^{-1}hg^{-1}k)$$

for symmetric $h, k \in T^0_2 M$.

2.2. **A metric on tensor fields.** A metric on the space of tensor fields can be defined by integrating the appropriate metric on the tensor space with respect to the volume density:

$$\tilde{g}^r_s(h,k) = \int_M g^r_s(h(x),k(x))vol(g)(x)$$

for $h,k \in \Gamma(T^r_s M)$, where $vol(g)$ is the volume density $\sqrt{det(g_{ij})}dx^1 \wedge ... \wedge dx^n$ in local coordinates $\{x^i\}$ for M. According to Section 2.1, if h and k are tensor fields of type $\binom{0}{2}$ and h or k is symmetric, then we have

$$\tilde{g}^0_2(h,k) = \int_M Tr(g^{-1}h(x)g^{-1}k(x))vol(g)(x).$$

2.3. **Directional derivatives of functions.** We use the following ways to denote directional derivatives of functions, in particular in infinite dimensions. Given a function $F(x,y)$, for instance, we will write:

$$D_{(x,h)}F \quad or \quad dF(x)(h) \quad \text{as shorcut for} \quad \partial_t|_0 F(x+th,y).$$

Here (x,h) in the subscript denotes the tangent vector with foot point x and direction h. Here the calculus in infinite dimensions as explained in [15] has been applied.

3. The manifold of Riemannian metrics

In this section we recall some fundamentals on the manifold of Riemannian metrics and the natural L^2 metric. The manifold of Riemannian metrics $\mathfrak{M}$ is the subset of all sections in S^2T^*M of symmetric rank-2 covariant tensor fields that are positive definite on each T^*_pM for $p \in M$, and $\mathfrak{M}$ is an open convex positive cone in $\Gamma(S^2T^*M)$, which is an infinite-dimensional Fréchet manifold (see [13]).

We first recall some results of Ebin [8]. Let $\mathfrak{D}$ be the group of smooth functions on M, and let

$$\Psi : \mathfrak{M} \times \mathfrak{D} \to \mathfrak{M}, \qquad (g,f) \mapsto f^*g$$

denote the usual "pull-back"action of $\mathfrak{D}$ on $\mathfrak{M}$. For $g \in \mathfrak{M}$, let

$$\Psi_g : \mathfrak{D} \to \mathfrak{M}, \qquad f \mapsto f^*g$$

denote the orbit map at g. Then Ψ_g is a smooth map with derivative at the identity $e \in \mathfrak{D}$ given by

$$\alpha_g = T_e\Psi_g : \mathfrak{X}(M) \to S_2(M), \qquad X \mapsto L_Xg,$$

where L_X is the Lie derivative with respect to the vector field X. We can describe the canonical splitting of $S_2(M)$. Let $O_g = \{f^*g | f \in \mathfrak{D}\} \subseteq \Psi_g(\mathfrak{D}) \subseteq \mathfrak{M}$ be the orbit through g. Then O_g is a smooth closed sub-manifold of $\mathfrak{M}$, with tangent space at g given by T_gO_g =rangα_g. Observe that there exists, orthogonal to O_g, a slice $S_g \subseteq \mathfrak{M}$, which is also a smooth closed manifold of $\mathfrak{M}$ with tangent space at g given by $T_gS_g = S_2^0(g)$. Here $S_2^0(g) = \{h \in S_2(M) | \delta_g h = 0\}$ is the space of C^∞ divergence free two-covariant symmetric tensor fields on M. Thus, the canonical splitting of $S_2(M)$ can be written as follows:

$$T_g\mathfrak{M} = T_gS_g \oplus T_gO_g.$$

The curvature and the geodesic spaces in $\mathfrak{M}$ relative to the canonical metric were studied in [10, 12]. Other weak Riemannian metrics on $\mathfrak{M}$ have been introduced in [20], and formulas for covariant derivative, curvature tensor, sectional curvature and geodesics have been obtained. Metrics on $\mathfrak{M}$ that are stronger than L^2-metric has recently been described in [2]. Using a pseudo-differential operator they introduced the following general metrics:

$$G_g^p(h,k) = \int_M g_2^0(P_g h, k) vol(g) = \int_M Tr(g^{-1}.P_g(h).g^{-1}.k) vol(g),$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is a positive, symmetric, bijective pseudo-differential operator of order $2p, p \geq 0$, depending smoothly on the metric g. They obtained a geodesic equation for the general metric and all particular cases, and among other results, they showed that under certain conditions on the operator P_g, the geodesic equation is well-posed.

The next lemma will be used in Section 4, in the proofs of the main results of the paper.

Lemma 3.1. *The Levi-Civita connection induced by the Sobolev metric G_g^p on the manifold of Riemannian metrics is given by the following formula:*

$$\begin{aligned}\nabla_h k &= \frac{1}{2}P_g^{-1}[-hg^{-1}P_gk - P_gkg^{-1}h + D_{(g,h)}P_gk + D_{(g,k)}P_gh - (D_{(g,.)}P_gh)^*(k))] \\ &+ \frac{1}{4}[Tr(g^{-1}h)k + Tr(g^{-1}k)h - Tr(g^{-1}P_ghg^{-1}k)P_g^{-1}g].\end{aligned}$$

Proof. The Levi-civita connection on any Riemannian manifold is determined by the following six terms formula:

$$\begin{aligned}2G_g^p(\nabla_h k, m) &= hG_g^p(k,m) + kG_g^p(h,m) - mG_g^p(h,k) \\ &\quad -G_g^p(h,[k,m]) - G_g^p(k,[m,h]) + G_g^p(m,[h,k]).\end{aligned}$$

It suffices to look at constant vector fields h and k satisfying $[h,k]=0$. So, we can write

$$\begin{aligned} 2G^P_g(\nabla_h k, m) &= hG^P_g(k,m) + kG^P_g(h,m) - mG^P_g(h,k) \\ &= \int_M [-Tr(g^{-1}hg^{-1}P_g kg^{-1}m) - Tr(g^{-1}kg^{-1}P_g hg^{-1}m) \\ &\quad -Tr(g^{-1}mg^{-1}P_g hg^{-1}k) - Tr(g^{-1}P_g kg^{-1}hg^{-1}m) \\ &\quad -Tr(g^{-1}P_g hg^{-1}kg^{-1}m) + Tr(g^{-1}P_g hg^{-1}mg^{-1}k) \\ &\quad +Tr(g^{-1}D_{(g,h)}(P_g k)g^{-1}m) + Tr(g^{-1}D_{(g,k)}(P_g h)g^{-1}m) \\ &\quad -Tr(g^{-1}D_{(g,m)}(P_g h)g^{-1}k) + \frac{1}{2}Tr(g^{-1}P_g kg^{-1}m)Tr(g^{-1}h) \\ &\quad +\frac{1}{2}Tr(g^{-1}P_g hg^{-1}m)Tr(g^{-1}k) - \frac{1}{2}Tr(g^{-1}P_g hg^{-1}k)Tr(g^{-1}m)]vol(g). \end{aligned}$$

Notice that some terms in the last formula cancel out because for symmetric h, k, m one has

$$Tr(hkm) = Tr((hkm)^T) = Tr(m^T k^T h^T) = Tr(h^T m^T k^T) = Tr(hmk).$$

Therefore, we have

$$\begin{aligned} 2G^P_g(\nabla_h k, m) &= \int_M [Tr(g^{-1}hg^{-1}P_g kg^{-1}m) - Tr(g^{-1}P_g kg^{-1}hg^{-1}m) \\ &\quad +Tr(g^{-1}D_{(g,h)}(P_g k)g^{-1}m) + Tr(g^{-1}D_{(g,k)}(P_g h)g^{-1}m) \\ &\quad -Tr(g^{-1}D_{(g,m)}(P_g h)g^{-1}k) + \frac{1}{2}Tr(g^{-1}P_g kg^{-1}m)Tr(g^{-1}h) \\ &\quad +\frac{1}{2}Tr(g^{-1}P_g hg^{-1}m)Tr(g^{-1}k) - \frac{1}{2}Tr(g^{-1}P_g hg^{-1}k)Tr(g^{-1}m)]vol(g) \\ &= -G^P_g(P_g^{-1}(hg^{-1}P_g k), m) - G^P_g(P_g^{-1}(P_g kg^{-1}h), m) \\ &\quad +G^P_g(P_g^{-1}(D_{(g,h)}(P_g k)), m) + G^P_g(P_g^{-1}(D_{(g,k)}(P_g h)), m) \\ &\quad -\int_M Tr(g^{-1}D_{(g,m)}(P_g h)g^{-1}k)vol(g) + \frac{1}{2}[G^P_g(Tr(g^{-1}h)k, m) \\ &\quad +G^P_g(Tr(g^{-1}k)h, m) - G^P_g(Tr(g^{-1}P_g hg^{-1}k)P_g^{-1}g, m)]. \end{aligned}$$

We assume that there exists an adjoint in the following sense

$$\int_M g^0_2((D_{(g,m)}P)h, k)vol(g) = \int_M g^0_2(m, (D_{(g,.)}Ph)^*(k))vol(g),$$

which is smooth in (g,h,k) and is bilinear in (h,k).

Thus, we can write

$$\begin{aligned} 2G_g^p(\nabla_h k, m) &= -G_g^P(P_g^{-1}(hg^{-1}P_g k), m) - G_g^P(P_g^{-1}(P_g k g^{-1} h), m) \\ &+G_g^P(P_g^{-1}(D_{(g,h)}(P_g k)), m) + G_g^P(P_g^{-1}(D_{(g,k)}(P_g h)), m) \\ &-G_g^P((D_{(g,.)}P_g h)^*(k), m) + \frac{1}{2}[G_g^P(Tr(g^{-1}h)k, m) \\ &+G_g^P(Tr(g^{-1}k)h, m) - G_g^P(Tr(g^{-1}P_g h g^{-1} k)P_g^{-1} g, m)]. \end{aligned}$$

Finally, we have

$$\begin{aligned} \nabla_h k &= \frac{1}{2}P_g^{-1}[-hg^{-1}P_g k - P_g k g^{-1} h + D_{(g,h)}P_g k + D_{(g,k)}P_g h - (D_{(g,.)}P_g h)^*(k))] \\ &+ \frac{1}{4}[Tr(g^{-1}h)k + Tr(g^{-1}k)h - Tr(g^{-1}P_g h g^{-1} k)P_g^{-1} g]. \end{aligned}$$

Lemma 3.1 is proved. □

Remark 3.1. The above formula, applied to the geodesic equation $\nabla_{g'}g' = g''$ yields:

$$\begin{aligned} g'' &= P_g^{-1}[-\frac{1}{2}(D_{(g,.)}P_g g')^*(g')) - \frac{1}{2}g' g^{-1} P_g g' - \frac{1}{2}P_g g' g^{-1} g' \\ &+\frac{1}{2}Tr(g^{-1}g')P_g g' - \frac{1}{4}Tr(g^{-1}P_g g' g^{-1} g')g + (D_{(g,g')}P_g)g'], \end{aligned}$$

which coincides with the geodesic equation obtained in [2] using minimizing energy function.

4. A variant of Riemannian metric using a pseudo-differential operator

Let g_0 be a fixed Riemannian metric. The formula

$$\begin{aligned} G_{g_0}^p(h,k) &= \int_M g_2^0(P_g h, k) vol(g_0) \\ &= \int_M Tr(g^{-1}.P_g(h).g^{-1}.k) vol(g_0), \end{aligned} \tag{4.1}$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is a positive, symmetric and bijective pseudo-differential operator of order $2p, p \geq 0$ depending smoothly on the metric g, defines a Riemannian metric on the manifold of Riemannian metrics.

Theorem 4.1. *The geodesic equation for $G_{g_0}^p$ metrics defined on the manifold of Riemannian metrics $\mathfrak{M}$ is given by the following formula:*

$$\begin{aligned} g'' = P_g^{-1}[-\frac{1}{2}(D_{(g,.)}P_g g')^*(g')) - \frac{1}{2}g' g^{-1} P_g g' \\ - \frac{1}{2}P_g g' g^{-1} g' + (D_{(g,g')}P_g)g'] \end{aligned} \tag{4.2}$$

Proof. In view of Lemma 3.1 and Remark ??, for $G^p_{g_0}$ we have

$$\begin{aligned} 2G^p_{g_0}(\nabla_h k, m) &= hG^p_{g_0}(k,m) + kG^p_{g_0}(h,m) - mG^p_{g_0}(h,k) \\ &= \int_M [-Tr(g^{-1}hg^{-1}P_g kg^{-1}m) - Tr(g^{-1}kg^{-1}P_g hg^{-1}m) \\ &\quad -Tr(g^{-1}mg^{-1}P_g hg^{-1}k) - Tr(g^{-1}P_g kg^{-1}hg^{-1}m) \\ &\quad -Tr(g^{-1}P_g hg^{-1}kg^{-1}m) + Tr(g^{-1}P_g hg^{-1}mg^{-1}k) \\ &\quad +Tr(g^{-1}D_{(g,h)}(P_g k)g^{-1}m) + Tr(g^{-1}D_{(g,k)}(P_g h)g^{-1}m) \\ &\quad -Tr(g^{-1}D_{(g,m)}(P_g h)g^{-1}k)]vol(g_0). \end{aligned}$$

Some terms in the last formula cancel out because $vol(g_0)$ is fixed for $G^p_{g_0}$ metric. The rest of the proof is similar to that of Lemma 3.1, and so is omitted. □

In the following, as (non-linear) mappings at the base point g, we assume that $P_g h, (Pg)^{-1}h, (D_{(g,.)}Ph)^*(m)$ are compositions of operators of the following type (see [2]):

(a) non-linear differential operators of order $l \leq 2p$, that is,

$$A(g)(x) = A(x, g(x), (\nabla g)(x), ..., (\nabla^l g)(x)),$$

(b) linear pseudo-differential operators of order $\leq 2p$, such that the total (top) order of the composition is $\leq 2p$.

Now consider P_g as a pseudo-differential operator defined on $\Gamma(S^2T^*M)$ such that it has the following forms for special tensors.

$$P_g(Ric) := e^g Ric \tag{4.3}$$

$$P_g(g^{pq}\nabla^2_{q,i}R_{jp}) := -2Ricg^{-1}e^g Ric \tag{4.4}$$

$$P_g(\nabla^2_{i,j}R) := -4(Ric)e^g Ric \tag{4.5}$$

$$P_g(\Delta R_{ij}) := ge^g Ricg^{-1}Ric \tag{4.6}$$

Now we can state the following result.

Theorem 4.2. *There is a pseudo-differential operator on $\Gamma(S^2T^*M)$ such that the Ricci flow is a geodesic of $G^p_{g_0}$ metric on the manifold of Riemannian metrics.*

Proof. The result can easily be deduced from equation (4.2) with $g' = -2Ric$ and formulas (4.3)-(4.6), since

$$P_g(\frac{\partial Ric}{\partial t}) = -(D_{(g,.)}P_g(Ric))^*(Ric))$$

$$-Ricg^{-1}P_g(Ric) - P_g(Ric)g^{-1}Ric - 2(D_{(g,Ric)}P_g)Ric,$$

and for the adjoint operator $P_g(Ric)$ we have

$$\int_M g_2^0((D_{(g,m)}P)Ric, Ric)vol(g_0) = \int_M g_2^0(me^q Ric, Ric)vol(g_0)$$

$$= \int_M Tr(G^{-1}me^q Ricg^{-1}Ric)vol(g_0) = \int_M (m, ge^q Ricg^{-1}Ric)vol(g_0)$$

$$= \int_M g_2^0(m, (D_{(g,.)}P_g(Ric))^*(Ric))vol(g_0).$$

Theorem 4.2 is proved. □

Other Riemannian metrics. The Ricci flow as a curve is not a geodesic of Riemannian metrics on $\mathfrak{M}$ defined in [8, 20]. In [11], we have shown that the Ricci flow is not a geodesic of the known Riemannian metric on $\mathfrak{M}$. Let g_0 be a fixed Riemannian metric on M. In fact we have the following.

(1) For the metric defined as follows:

$$< h, k >_g: \;=\; \int_M Tr(g_0^{-1}hg_0^{-1}k)vol(g_0)$$

for $h, k \in T_g\mathfrak{M}$, the geodesics with initial conditions $(\bar{g}, a)$ are of the form $g(t) = \bar{g} + ta$ (see [20]). It is obvious that the Ricci flow is not a geodesic of this metric. Since the velocity vector for geodesic is constant, that is, $\frac{\partial g}{\partial t} = a$. For more general metric defined by

$$< h, k >_0^\alpha: \;=\; \int_M Tr(g_0^{-1}hg_0^{-1}k)vol(g_0) + \alpha\int_M Tr(g_0^{-1}h)Tr(g_0^{-1}k)vol(g_0),$$

where $\alpha > -\frac{1}{n}$, the geodesics are the same as above (see [20]). Therefore the Ricci flow is not a geodesic of this general metric, too.

(2) Consider the following Riemannian metric on $\mathfrak{M}$:

$$< h, k >_g \;=\; \int_M Tr(g_0^{-1}hg_0^{-1}k)vol(g).$$

The geodesics are solutions of the following second-order equation (see [20]):

$$\frac{d}{dt}\left(\rho(g)\frac{dg}{dt}\right) \;=\; kd\psi,$$

where $vol(g) = \rho(g)vol(g_o)$, $k = k(x)$ is a positive function on M and $\psi = ln\rho(g)$. It can be shown that the Ricci flow is not a geodesic of this metric, too. Indeed, since

$$\begin{aligned}\frac{d\rho}{dt}\frac{\partial g}{\partial t}+\rho(g)\frac{\partial^2 g}{\partial t^2} &= \frac{1}{2}k(x)g_0g^{-1}g_0\\ 2scal\rho(g)Ric+\rho(g)\frac{\partial Ric}{\partial t} &= \frac{1}{2}k(x)g_0g^{-1}g_0,\end{aligned}$$

we have

$$\frac{\partial Ric}{\partial t} = \frac{1}{\rho(g)}\frac{1}{2}k(x)g_0g^{-1}g_0 - 2RRic$$

But under the Ricci flow we have (see [1]):

$$\frac{\partial}{\partial t}R_{ik} = g^{pq}(-\nabla^2_{q,k}R_{ip}+\nabla^2_{i,k}R_{pq}-\nabla^2_{q,i}h_{kp}+\nabla^2_{q,p}R_{ik}).$$

Therefore the Ricci flow is not a geodesic on $\mathfrak{M}$.

(3) For the metric defined by

$$<h,k>_g: = \int_M Tr(g^{-1}hg^{-1}k)vol(g_0),$$

the geodesics with initial condition $(\bar{g},a)$ have the form $g(t)=\bar{g}e^{tA}$, where $A=\bar{g}^{-1}a$ (see [20]). The velocity vector of geodesic is

$$\frac{\partial g}{\partial t} = A\bar{g}e^{tA} = -2\bar{g}^{-1}\bar{Ric}\bar{g}e^{tA},$$

which does not coincide with the Ricci flow $\frac{\partial g}{\partial t}=-2Ric$. For more general metric defined on $\mathfrak{M}$ by

$$<h,k>_{g,\alpha}: = \int_M Tr(g^{-1}hg^{-1}k)vol(g_0)+\alpha\int_M Tr(g^{-1}h)Tr(g^{-1}k)vol(g_0)$$

the geodesics are exactly the same as above, that is, $g(t)=\bar{g}e^{tA}$. Thus, the Ricci flow can not be a geodesic.

(4) Consider the following metric on $\mathfrak{M}$

$$<h,k>^{\alpha}_g: = \int_M Tr(g^{-1}hg^{-1}k)vol(g)+\alpha\int_M Tr(g^{-1}h)Tr(g^{-1}k)vol(g).$$

The geodesics of this metric coincide with the geodesics of $<h,k>^0_g$, that is with those of the canonical metric on $\mathfrak{M}$ (see [20]). In [11], we have shown that the Ricci flow is not a geodesic of the canonical metric on $\mathfrak{M}$.

Remark 4.1. The Ricci flow is not a geodesic of the three special metrics defined by pseudo-differential operators in [2].

5. Slice and Ricci solitons

The existence of a slice for the manifold of Riemannian metrics at first have been studied by Ebin in [8]. He proved that the manifold of Riemannian metrics has a slice such that it is infinite dimensional sub-manifold of $\mathfrak{M}$. Observing that the Ricci flow and Ricci solitons are curves on $\mathfrak{M}$, we show that for every manifold M with Riemannian metric g_0 which has Ricci solitons, the manifold of Riemannian metrics has a slice such that it is a finite dimensional sub-manifold of $\mathfrak{M}$.

Indeed, as we know the Ricci solitons $g(t) = \sigma(t)\varphi(t)^* g(0)$ with initial metric g_0 are equivalent to existence of a vector field X and a scalar λ, such that

$$-2Ric = \lambda g_0 - 2L_X g_0$$

On the other hand, according to canonical splitting around g_0, there exist a slice $S_{g_0} \subsetneq \mathfrak{M}$ such that

$$T_{g_0}\mathfrak{M} = T_{g_0}S_{g_0} \oplus T_{g_0}O_{g_0}.$$

Combining the above equations and discussion in Section 3, we obtain the following result.

Theorem 5.1. *Ricci soliton is equivalent to existence of a finite dimensional slice for $\mathfrak{M}$ and then the tangent space at initial metric is λg_0, where λ is a real scalar.*

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ON THE ROBUSTNESS TO SMALL TRENDS OF PARAMETER ESTIMATION FOR CONTINUOUS-TIME STATIONARY MODELS WITH MEMORY

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Abstract. The paper deals with a question of robustness of inferences, carried out on a continuous-time stationary process contaminated by a small trend, to this departure from stationarity. We show that a smoothed periodogram approach to parameter estimation is highly robust to the presence of a small trend in the model. The obtained result is a continuous version of that of Hede and Dai (Journal of Time Series Analysis, 17, 141-150, 1996) for discrete time processes.

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Keywords: trend; robust inference; short, intermediate and long memory; smoothed periodogram; parameter estimation[1].

1. Introduction

Much of statistical inferences about unknown spectral parameters is concerned with the discrete-time stationary models, in which case it is assumed that the model is centered, or has a constant mean (see Beran et al. [7], Dzhaparidze [12], Giraitis et al. [24], Taniguchi and Kakizawa [28], and references therein). In this paper we are concerned with the robustness of inferences, carried out on a continuous-time stationary process contaminated by a small trend, to this departure from stationarity.

Specifically, let $\{Y(t),\ t \in \mathbb{R}\}$ be a a zero mean stationary process with spectral density $f(\lambda, \theta)$, where $\theta := (\theta_1, \ldots, \theta_p) \in \Theta \subset \mathbb{R}^p$ is an *unknown* vector parameter. We want to make inferences about θ in the case where the actual observed data are in the contaminated form:

$$X(t) = Y(t) + M(t), \quad 0 \le t \le T, \tag{1.1}$$

where $M(t)$ is a deterministic trend.

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We assume that the trend $M(t)$ is small, that is, we consider the situation in which the major trend is removed from the model and a certain component that remains in the model has only minor effect. In these cases standard inferences can be carried on the basis of the stationary model $Y(t)$, and we are interested in question whether the conclusions are robust against this kind of departure from the stationary model.

A sufficiently developed inferential theory is now available for a continuous-time stationary model $Y(t)$. For instance, in Anh et al. [3, 4], Avram et al. [5], Casas and Gao [9], Gao [14], Gao et al. [15, 16], Leonenko and Sakhno [26] were obtained sufficient conditions ensuring consistency and asymptotic normality of various statistical estimators of θ, including quasi maximum likelihood (Whittle) and minimum contrast estimators of θ constructed on the basis of a finite realization $\mathbf{Y}_T := \{Y(t), 0 \leq t \leq T\}$ of the process $Y(t)$.

In this paper we show that under some conditions on the process $Y(t)$ and the deterministic trend $M(t)$ the above asymptotic properties of Whittle and minimum contrast estimators remains valid for the model $X(t)$, that is, the estimating procedure is relatively robust against replacing the stationary model $Y(t)$ by the non-stationary model $X(t)$ of the form (1.1). We will be concerned with this question for models which may exhibit long memory, short memory or intermediate memory.

Throughout the paper the letters C and c are used to denote positive constants, the values of which can vary from line to line.

The paper is structured as follows. In Section 2 we describe the statistical model. Section 3 contains the approach and the main result of the paper - Theorem 3.1. Section 4 is devoted to the proof of Theorem 3.1.

2. The model: long memory, short memory and intermediate memory processes

Let $\{Y(t),\ t \in \mathbb{R}\}$ be a centered, real-valued, continuous-time second-order stationary process with covariance function $r(t)$, possessing a spectral density $f(\lambda)$, $\lambda \in \mathbb{R}$, that is, $E[|Y(t)|^2] < \infty$, $E[Y(t)] = 0$, $r(t) = E[Y(t+u)Y(u)]$ $(u, t \in \mathbb{R})$, and $r(t)$ and $f(\lambda)$ are connected by the Fourier integral:

$$r(t) = \int_{\mathbb{R}} e^{i\lambda t} f(\lambda) d\lambda, \quad t \in \mathbb{R}. \tag{2.1}$$

There are several possible definitions of the notion of "memory" of a stationary process, and they are not necessarily identical (see Beran et al. [7], Gao [14], Giraitis et al. [24], Heyde and Dai [25], Taniguchi and Kakizawa [28]). In this paper, we define the memory concept basing on the integrability property of covariance function $r(t)$, and depending on the memory structure we will distinguish the following types of stationary models: (a) short memory or short-range dependent, (b) long memory or long-range dependent, (c) intermediate memory or anti-persistent.

We will say that the process $Y(t)$ displays *short memory (SM)* or *short-range dependence (SRD)* if the covariance function $r(t)$ is integrable: $r \in L^1(\mathbb{R})$ and $\int_{-\infty}^{+\infty} r(t)dt \neq 0$. In this case the spectral density $f(\lambda)$ is bounded away from zero and infinity at frequency $\lambda = 0$, that is, $0 < f(0) < \infty$.

A typical continuous-time short memory model example is the stationary continuous-time autoregressive moving average (CARMA) process whose spectral density is a rational function (see, e.g., Brockwell [8]).

Much of statistical inference is concerned with the short memory stationary models. However, data in many fields of science (economics, hydrology, telecommunications, etc.) is well modeled by a stationary process with *unbounded* or *vanishing* at the origin spectral density (see Beran et al. [7], Casas and Gao [9], Gao [14], Tsai and Chan [29] and references therein).

The process $Y(t)$ is said to be *anti-persistent* or exhibits *intermediate memory (IM)* if the covariance function $r(t)$ is integrable: $r \in L^1(\mathbb{R})$ and $\int_{-\infty}^{+\infty} r(t)dt = 0$. In this case the spectral density $f(\lambda)$ vanishes at frequency zero: $f(0) = 0$.

We say that the process $Y(t)$ displays *long memory (LM)* or *long-range dependence (LRD)* if the covariance function $r(t)$ is not integrable: $r \notin L^1(\mathbb{R})$. In this case the spectral density $f(\lambda)$ has a pole at frequency zero, that is, it is unbounded at the origin.

The memory property of a stationary process can also be characterized by the behavior of spectral density $f(\lambda)$ in the neighborhood of zero, or by the behavior of covariance function $r(t)$ at infinity (see Beran et al. [7], Section 1.3.4).

An example of a continuous-time model that displays the above defined memory structures is the continuous-time autoregressive fractionally integrated moving-average (CARFIMA) process (see Chambers [10], Tsai and Chan [29]).

In the continuous context, a basic process which has commonly been used to model LRD is fractional Brownian motion (fBm) $B_H(t)$ with Hurst index H. This is a Gaussian process with stationary increments and spectral density of the form

$$f(\lambda) \sim c|\lambda|^{1-2H}, \quad c > 0, \quad 1/2 < H < 1, \tag{2.2}$$

as $\lambda \to 0$, and covariance function:

$$r(t) \sim ct^{2H-2}, \ 1/2 < H < 1, \tag{2.3}$$

as $t \to \infty$, where the symbol " $\sim$ " indicates that the ratio of left- and right-hand sides tends to 1. Notice that the form (2.2) can be understood in a limiting sense, since the fBm B_H is a nonstationary process (see, e.g., Solo [27], Gao et al. [15]).

A proper stationary model in lieu of fBm is the fractional Riesz-Bessel motion (fRBm), introduced in Anh et al. [1], and then extensively discussed in a number of papers (see Anh et al. [2], Gao et al. [15], Leonenko and Sakhno [26], and references therein). The fRBm is defined to be a continuous-time Gaussian stationary process with spectral density of the form

$$f(\lambda) = \frac{c}{|\lambda|^{2u}(1+\lambda^2)^v}, \quad \lambda \in \mathbb{R}, 0 < c < \infty, 0 < u < 1/2, \ v > 0. \tag{2.4}$$

Observe that the spectral density (2.4) behaves as $O(|\lambda|^{-2u})$ as $|\lambda| \to 0$ and as $O(|\lambda|^{-2(u+v)})$ as $|\lambda| \to \infty$. Thus, under the conditions $0 < u < 1/2$, $v > 0$ and $u + v > 1/2$ the function $f(\lambda)$ in (2.4) is well-defined for both $|\lambda| \to 0$ and $|\lambda| \to \infty$ due to the presence of the component $(1 + \lambda^2)^{-v}$, which is the Fourier transform of the Bessel potential. Note that in the spacial case $0 < u < 1/2, v > 1/2$ the condition $u + v > 1/2$ holds automatically. The exponent u determines the LRD, while the exponent v indicates the second-order intermittency of the fRBm (see Anh et al. [2] and Gao et al. [15]).

Comparing (2.2) and (2.4), we observe that the spectral density of fBm is the limiting case as $v \to 0$ that of fRBm with Hurst index $H = u + 1/2$. Thus, the form (2.4) means that fRBm may exhibit both LRD and second-order intermittency.

The next result, which was proved in Ginovyan and Sahakyan [22], gives an asymptotic formula for covariance function of an fRBm: Let $f(\lambda)$ be as in (2.4) with $0 < u < 1/2$ and $v > 1/2$, and let $r(t) := \hat{f}(t)$ be the Fourier transform of $f(\lambda)$, then

$$r(t) = Ct^{2u-1} \sin(\pi\alpha)\Gamma(1 - 2u) \cdot (1 + o(1)) \quad \text{as} \quad t \to \infty. \tag{2.5}$$

3. The approach and results

The basic approach in estimating unknown spectral parameters, originated by Whittle [30], is based on the smoothed periodogram analysis on a frequency domain, involving approximation of the likelihood function and asymptotic distributions of empirical spectral functionals.

The Whittle estimation procedure, originally devised for discrete-time short memory stationary processes, has played a major role in the parametric estimation in the frequency domain, and was the focus of interest of many statisticians. Their aim was to weaken the conditions needed to guarantee the validity of the Whittle approximation for short memory models, to find analogues for long and intermediate memory models, and to show that the Whittle estimator is asymptotically equivalent to exact maximum likelihood estimator (see Dahlhaus [11], Dzhaparidze [12], Fox and Taqqu [13], Giraitis and Surgailis [23], Giraitis et al. [24] and references therein). In particular, it was shown that for Gaussian and linear stationary models the Whittle approach leads to consistent and asymptotically normal estimators with the standard rate of convergence under short, intermediate and long memory assumptions.

Continuous versions of Whittle estimation procedure have been considered, for example, in Anh et al. [3, 4], Avram et al. [5], Casas and Gao [9], Gao [14], Gao et al. [15, 16], Leonenko and Sakhno [26].

The procedure of estimation of a parameter θ involved in the spectral density $f(\lambda, \theta)$ of the model, based on a finite realization $\mathbf{Y}_T := \{Y(t), 0 \leq t \leq T\}$ of the centered stationary process $Y(t)$, is to choose the estimator $\hat{\theta}_W$ to minimize the weighted Whittle functional:

$$U_{w,T}(\theta) := \frac{1}{4\pi} \int_{-\infty}^{+\infty} \left[\log f(\lambda, \theta) + \frac{I_{T,Y}(\lambda)}{f(\lambda, \theta)} \right] \cdot w(\lambda)\, d\lambda, \tag{3.1}$$

where

$$I_{T,Y}(\lambda) = \frac{1}{2\pi T} \left| \int_0^T e^{i\lambda t} Y(t)\, dt \right|^2 \tag{3.2}$$

is the "continuous" periodogram of $Y(t)$, and $w(\lambda)$ is an even weight function (that is, $w(-\lambda) = w(\lambda)$, $w(\lambda) \geq 0$, and $w(\lambda) \in L^1(\mathbb{R})$) for which the integral in (3.1) is well defined. The choice of an appropriate weight function depends on the specific form of the spectral density (see Anh et al. [4]). An example of common used weight function is $w(\lambda) = 1/(1 + \lambda^2)$.

Thus, the Whittle estimator $\hat{\theta}_W$ with weight function $w(\lambda)$ is defined to be a solution of the following estimating equation

$$\int_{-\infty}^{+\infty} \left[I_{T,Y}(\lambda) - f(\lambda, \theta) \right] \frac{\partial}{\partial \theta} f^{-1}(\lambda, \theta) \cdot w(\lambda)\, d\lambda = 0, \tag{3.3}$$

obtained by differentiating under the integral sign in (3.1).

The asymptotic properties of the Whittle estimator $\hat{\theta}_W$ then can be obtained using the standard Taylor expansion methods based on the following smoothed periodogram convergence results:

$$\int_{-\infty}^{+\infty} g(\lambda,\theta) I_{T,Y}(\lambda) d\lambda \xrightarrow{P} \int_{-\infty}^{+\infty} g(\lambda,\theta) f(\lambda,\theta) d\lambda \quad \text{as} \quad T\to\infty, \tag{3.4}$$

and

$$T^{1/2}\int_{-\infty}^{+\infty} g(\lambda,\theta)\left[I_{T,Y}(\lambda) - f(\lambda,\theta)\right] d\lambda \xrightarrow{d} \xi \sim N(0,\sigma^2) \quad \text{as} \quad T\to\infty, \tag{3.5}$$

where $g(\lambda,\theta) = \frac{\partial}{\partial\theta} f^{-1}(\lambda,\theta) w(\lambda)$, $I_{T,Y}(\lambda)$ is the periodogram of $Y(t)$ given by (3.2), $N(0,\sigma^2)$ denotes the normal law with mean zero and variance σ^2, and $\xrightarrow{d}$ and $\xrightarrow{P}$ stand for convergence in distribution and in probability, respectively.

Using this approach, statistical properties of Whittle minimum contrast estimators for continuous-time stationary processes were studied in Anh et al. [3], Avram et al. [5], Casas and Gao [9], Gao [14], Gao et al. [15, 16], Leonenko and Sakhno [26]. In particular, consistency and asymptotic normality of Whittle minimum contrast estimator $\hat{\theta}_W$ was established for some classes of stationary models, including the fractional Riesz-Bessel motion model, specified by spectral density $f(\lambda) = f(\lambda;\theta)$ given by (2.4) with $\theta = (u, v, c)$.

In our analysis we will use a general even integrable smoothing function $g(\lambda;\theta)$ rather than the specific form $g(\lambda,\theta) = \frac{\partial}{\partial\theta} f^{-1}(\lambda,\theta) w(\lambda)$ which is suggested by the Whittle procedure in (3.3). The general estimator $\hat{\theta}_G$ of θ is then obtained as a solution of the estimating equation

$$\int_{-\infty}^{+\infty} \left[I_{T,Y}(\lambda) - f(\lambda,\theta)\right] g(\lambda,\theta) d\lambda = 0. \tag{3.6}$$

Then the asymptotic properties of the estimator $\hat{\theta}_G$ can be obtained from smoothed periodogram convergence results of type (3.4) and(3.5) with general smoothing function $g(\lambda;\theta)$.

Notice that in the continuous context the basic tool for derivation of limit theorems for empirical spectral functionals of the form

$$J(g) := \int_{-\infty}^{+\infty} g(\lambda,\theta) I_{T,Y}(\lambda) d\lambda \tag{3.7}$$

is a central limit theorem for Toeplitz type quadratic functionals of stationary processes (see Ginovyan [18, 19], Ginovyan and Sahakyan [21] for Gaussian processes, and Bai et al. [5, 6] for linear processes).

It can be shown that the standard Taylor expansion methods based on the smoothed periodogram convergence results of type (3.4) and(3.5) with a general smoothing function $g(\lambda;\theta)$ and with the contaminated periodogram $I_{T,X}(\lambda)$ instead of $I_{T,Y}(\lambda)$, lead consistent and asymptotically normally distributed estimators of θ. We will not pursue this matter here (the details will be reported elsewhere), however, notice that in the special case of Whittle procedure, where $g(\lambda;\theta) = \frac{\partial}{\partial}\frac{1}{f(\lambda,\theta)} \cdot w(\lambda)$ the results of Anh et al. [3], Avram et al. [5], Casas and Gao [9], Gao [14], Gao et al. [15, 16], Leonenko and Sakhno [26] concerning consistency and asymptotic normality of the Whittle minimum contrast estimators constructed on the basis of the periodogram $I_{T,Y}(\lambda)$, continue to hold without change for estimators calculated on the basis of the contaminated periodogram $I_{T,X}(\lambda)$, under appropriate assumptions imposed on the model $Y(t)$ on the smoothing function $g(\lambda,\theta)$ and on the trend $M(t)$.

In Theorem 3.1 that follows we show that a small trend of the form $|M(t)| \le C|t|^{-\beta}$ does not effect the asymptotic properties (3.4) and (3.5) of the smoothed periodogram, and hence, the asymptotic properties of the estimator $\hat{\theta}_G$, even if $I_{T,Y}(\lambda)$ is replaced by the contaminated periodogram $I_{T,X}(\lambda)$.

Theorem 3.1. *Suppose that the stationary mean zero process $\{Y(t), t \in \mathbb{R}\}$ in (1.1) is such that the asymptotic relations (3.4) and (3.5) are satisfied with general even integrable smoothing function $g(\lambda)$ and σ^2 as in (??). If the trend $M(t)$ and the Fourier transform $a(t) := \widehat{g}(t)$ of smoothing function $g(\lambda)$ are such that $M(t)$ is locally integrable on $\mathbb{R}$ and*

$$(3.8)\qquad |M(t)| \le C|t|^{-\beta}, \qquad |a(t)| \le C|t|^{-\gamma}, \quad t \in \mathbb{R}, \quad 2\beta + \gamma > \frac{3}{2},$$

with some constants $C > 0$, $\gamma > 0$ and $\beta > 1/4$, then

$$(3.9)\qquad T^{1/2} \int_{-\infty}^{+\infty} g(\lambda,\theta)\left[I_{T,X}(\lambda) - I_{T,Y}(\lambda)\right] d\lambda \xrightarrow{P} 0 \quad \text{as} \quad T \to \infty,$$

and hence the asymptotic relations (3.4) and (3.5) are satisfied with $I_{T,Y}(\lambda)$ replaced by the contaminated periodogram $I_{T,X}(\lambda)$, provided that one of the following conditions holds:

(i) *the process $Y(t)$ has SM or IM, that is, the covariance function $r(t)$ of $Y(t)$ satisfies $r \in L^1(\mathbb{R})$, and $\beta + \gamma > 1$.*

(ii) *the process $Y(t)$ has LM with covariance function $r(t)$ satisfying*

$$(3.10)\qquad |r(t)| \le C|t|^{-\alpha}, \quad t \in \mathbb{R}, \quad \alpha + \gamma \ge \frac{3}{2}$$

with some constants $C > 0$, $0 < \alpha \le 1$, and $\alpha + 2\beta > 1$ if $\beta < 1 < \gamma$.

Remark 3.1. It is easy to check that the statement of Theorem 3.1 holds, in particular, if the parameters α, β and γ satisfy the following conditions:

in the case *(i)*: $\beta > 1/2$, $\gamma \geq 1/2$,

in the case *(ii)*: $\alpha \geq 3/4$, $\beta > 3/8$, $\gamma \geq 3/4$.

Remark 3.2. The discrete version of Theorem 3.1 (with additional conditions $\gamma = 1$ in the case *(i)*, and $\gamma > 1$, $\alpha < 1/2$ in the case *(ii)*), was proved by Heyde and Dai [25] (see also Taniguchi and Kakizawa [28], Theorems 6.4.1 and 6.4.2). Using the same arguments applied in the proof of Theorem 3.1 one can prove that the complete discrete analog of Theorem 3.1 is also true.

Remark 3.3. Convergence results of type (3.4) and (3.5) holds under broad circumstances of SM, IM and LM . For detailed conditions see, for example, Avram et al. [5], Ginovyan [17] – [20], Ginovyan and Sahakyan [21], and Leonenko and Sakhno [26].

Remark 3.4. The conditions imposed on the Fourier transform of generating function $g(t)$ in (3.8) and on the covariance function $r(t)$ in (3.10) ensure central limit theorem for empirical functionals of Gaussian and linear long memory processes. This can be seen from the considerations of Theorem 5 of Ginovyan and Sahakyan [21] (for Gaussian processes), and Theorem 2.1 and Corollary 2.1 of Bai et al. [6] (for linear processes).

4. Proof of the main result

Proof of Theorem 3.1. In view of (1.1) and (3.2) we can write

$$\begin{aligned} I_{T,X}(\lambda) - I_{T,Y}(\lambda) &= \frac{1}{2\pi T}\left(\left|\int_0^T e^{i\lambda t}X(t)\,dt\right|^2 - \left|\int_0^T e^{i\lambda t}Y(t)\,dt\right|^2\right) \\ &= \frac{1}{2\pi T}\left(\left|\int_0^T e^{i\lambda t}[Y(t)+M(t)]\,dt\right|^2 - \left|\int_0^T e^{i\lambda t}Y(t)\,dt\right|^2\right) \\ &= \frac{1}{2\pi T}\int_0^T\int_0^T e^{i\lambda(t-s)}\left[Y(t)M(s)+Y(s)M(t)+M(t)M(s)\right]dt\,ds \end{aligned}$$

and

$$\int_{-\infty}^{+\infty} g(\lambda,\theta)\left[I_{T,X}(\lambda) - I_{T,Y}(\lambda)\right]d\lambda$$

$$= \frac{1}{T}\int_0^T\int_0^T \left[Y(t)M(s)+Y(s)M(t)+M(t)M(s)\right]a(t-s)\,dt\,ds.$$

Thus, to complete the proof it is enough to prove that under the conditions of the theorem we have

$$T^{-1/2}\int_0^T\int_0^T M(t)M(s)a(t-s)\,dtds \to 0 \quad \text{as} \quad T\to\infty \tag{4.1}$$

and

$$T^{-1/2}\int_0^T\int_0^T Y(t)M(s)a(t-s)\,dtds \xrightarrow{P} 0 \quad \text{as} \quad T\to\infty. \tag{4.2}$$

Proof of (4.1). For $T>2$ we set

$$I(T)=\int_0^T\int_0^T |M(t)M(s)a(t-s)|\,dtds \tag{4.3}$$

$$=\int_0^1\int_0^2+\int_0^1\int_2^T+\int_1^T\int_0^{1/2}+\int_1^T\int_{1/2}^T =: I_1(T)+I_2(T)+I_3(T)+I_4(T),$$

and estimate the integrals $I_i(T)$, $i=1,2,3,4$, separately.

Observe first that the Fourier transform $a(t):=\hat{g}(t)$ is a bounded function on $\mathbb{R}$, since g is integrable on $\mathbb{R}$. Hence, taking into account that by assumption the trend $M(t)$ is locally integrable on $\mathbb{R}$, for $I_1(T)$ we obtain the estimate

$$I_1(T)\le C\|a\|_\infty\int_0^1|M(s)|\,ds\int_0^2|M(t)|\,dt\le C<\infty, \qquad T>2. \tag{4.4}$$

Next, in view of (3.8), for $0<s<1$ and $t>2$ we have $|a(t-s)|\le C(t-s)^{-\gamma}\le Ct^{-\gamma}$, and hence, taking into account that $\beta+\gamma>1$, $I_2(T)$ can be estimated as follows

$$I_2(T)\ \le\ C\int_0^1|M(s)|\,ds\int_2^T\frac{1}{t^{\beta+\gamma}}\,dt\le C<\infty, \qquad T>2. \tag{4.5}$$

Similarly, for $I_3(T)$ we have

$$I_3(T)\le C<\infty, \qquad T>2. \tag{4.6}$$

To estimate $I_4(T)$ observe first that, in view of (3.8), for $1<s<T$ we can write

$$h(s)=\int_{1/2}^T|M(t)a(t-s)|\,dt\le C\left[\int_{(s-1)}^{s+1}\frac{|a(t-s)|}{t^\beta}dt+\int_{s+1}^{2s}\frac{1}{t^\beta(t-s)^\gamma}dt\right.$$

$$\left.+\int_{2s}^T\frac{1}{t^\beta(t-s)^\gamma}dt+\int_{1/2}^{s/2}\frac{1}{t^\beta(s-t)^\gamma}dt+\int_{s/2}^{s-1}\frac{1}{t^\beta(s-t)^\gamma}dt\right]$$

$$\le C\left[\|a\|_\infty\cdot s^{-\beta}+s^{-\beta}\int_1^s\frac{1}{\tau^\gamma}d\tau+\int_{2s}^T\frac{1}{t^{\beta+\gamma}}dt+s^{-\gamma}\int_{1/2}^{s/2}\frac{1}{t^\beta}dt+s^{-\beta}\int_1^{s/2}\frac{1}{\tau^\gamma}d\tau\right]$$

$$\le C\left[s^{-\beta}+L(\gamma,T)s^{1-\beta-\gamma}+L(\beta+\gamma,T)\left(T^{1-\beta-\gamma}+s^{1-\beta-\gamma}\right)\right.$$

$$\left.+L(\beta,T)\left(s^{1-\beta-\gamma}+s^{-\gamma}\right)+L(\gamma,T)s^{1-\beta-\gamma}\right]$$

$$\le C\log T\cdot\left(T^{1-\beta-\gamma}+s^{1-\beta-\gamma}+s^{-\beta}+s^{-\gamma}\right). \tag{4.7}$$

where the function $L(u,T)$ is defined by

$$L(u,T)=\begin{cases}\log T & \text{if} \quad u=1,\\ 1 & \text{otherwise.}\end{cases}$$

Taking into account that $\beta+\gamma>1$, from (4.7) we get

$$h(s)\le C\log T\cdot\left(s^{1-\beta-\gamma}+s^{-\beta}+s^{-\gamma}\right),\qquad 1<s<T, \tag{4.8}$$

and hence for $T>2$, $I_4(T)$ can be estimated as follows

$$\begin{aligned}
I_4(T) &= \int_1^T |M(s)h(s)|\,ds\\
&\le C\log T\left[\int_1^T \frac{1}{s^{2\beta+\gamma-1}}\,ds+\int_1^T \frac{1}{s^{2\beta}}\,ds+\int_1^T \frac{1}{s^{\beta+\gamma}}\,ds\right]\\
&\le C\log T\left[1+L(2\beta+\gamma-1,T)\,T^{2-2\beta-\gamma}+L(2\beta,T)\,T^{1-2\beta}+T^{1-\beta-\gamma}\right]\\
&\le C\log^2 T\left(1+T^{2-2\beta-\gamma}+T^{1-2\beta}\right). \qquad (4.9)
\end{aligned}$$

Finally, taking into account that by assumption $2\beta+\gamma>3/2$ and $\beta>1/4$, from (4.3)-(4.6) and (4.9) we obtain

$$T^{-1/2}\cdot I(T)\le C\log^2 T\left(T^{-1/2}+T^{3/2-2\beta-\gamma}+T^{1/2-2\beta}\right)\to 0 \quad \text{as} \quad T\to\infty,$$

which implies (4.1).

Proof of (4.2). Observe first that the inequality

$$\beta+\gamma>1 \tag{4.10}$$

holds also in the case *(ii)*, since by (3.8) and (3.10), we have $2\beta+2\gamma\ge 2\beta+\gamma+3/2-\alpha>3/2+1/2=2$.

Denote

$$\nu(s)=\nu(T,s):=\int_0^T M(t)a(t-s)\,dt,\quad 0<s<T,$$

and observe that

$$\int_0^{1/2}|M(t)a(t-s)|\,dt\le C\int_0^{1/2}|M(t)|\frac{1}{(s-1/2)^{\gamma}}\,dt\le C\cdot s^{-\gamma},\qquad 1<s<T,$$

and by (4.8),

$$|\nu(s)|\le C\log T\cdot\left(s^{1-\beta-\gamma}+s^{-\beta}+s^{-\gamma}\right),\qquad 1<s<T. \tag{4.11}$$

On the other hand, by (3.8), for $T>2$ and $0<s<1$ we have

$$|\nu(s)|\le C\left[\|a\|_\infty\int_0^2|M(t)|\,dt+\int_2^T\frac{1}{t^{\beta}(t-1)^{\gamma}}dt\right]\le C\log T. \tag{4.12}$$

Observe that from (3.8) and (4.11) it follows that (4.12) holds for $0<s<T$.

Now, we denote

$$Q(T) := T^{-1/2}\int_0^T\int_0^T Y(s)M(t)a(t-s)\,dtds = T^{-1/2}\int_0^T Y(s)\nu(s)ds,$$

and observe that

$$\begin{aligned} E\{Q^2(T)\} &= T^{-1}\int_0^T\int_0^T E\{Y(s)Y(\tau)\}\nu(s)\nu(\tau)dsd\tau \\ &= T^{-1}\int_0^T\int_0^T \nu(s)\nu(\tau)r(s-\tau)dsd\tau. \end{aligned}$$

Hence, to prove (4.2) it is enough show that

$$J(T) := \int_0^T\int_0^T |\nu(s)\nu(\tau)r(s-\tau)|\,dsd\tau = o(T) \quad \text{as} \quad T\to\infty. \tag{4.13}$$

In the case *(i)*, when the process $Y(t)$ has SM or IM, and hence $r \in L^1(\mathbb{R})$, from (4.12) for $T > 2$ we get

$$|J(T)| \le C\log T\int_0^T |\nu(s)|\int_0^T |r(s-\tau)|\,d\tau ds \le C\log T\int_0^T |\nu(s)|\,ds. \tag{4.14}$$

In view of (4.11), the last integral in (4.14) can be estimated as follows:

$$\begin{aligned} \int_0^T |\nu(s)|\,ds &\le C\log^2 T\left[\int_0^1 ds + \int_1^T \left(s^{1-\beta-\gamma} + s^{-\beta} + s^{-\gamma}\right) ds\right] \\ &\le C\log^2 T\left[1 + L(\beta+\gamma-1,T)\,T^{2-\beta-\gamma} + L(\beta,T)\,T^{1-\beta} + L(\gamma,T)\,T^{1-\gamma}\right] \\ &\le C\log^3 T\left(1 + T^{1-\beta} + T^{1-\gamma} + T^{2-\beta-\gamma}\right). \end{aligned} \tag{4.15}$$

Hence, taking into account that $\beta+\gamma > 1$, from (4.14) and (4.15) we obtain

$$J(T) = o(T) \quad \text{as} \quad T\to\infty.$$

In the case *(ii)*, when the process $Y(t)$ has LM, using (3.10), (4.10) - (4.12), for $1 < \tau < T$ we obtain

$$\begin{aligned} q(\tau) &:= \int_0^T |\nu(s)r(s-\tau)|ds \le C\log T\int_0^{1/2}\left(\tau-\frac{1}{2}\right)^{-\alpha} ds \\ &+ C\log T\left[\int_{1/2}^T \frac{|r(s-\tau)|}{s^{\beta+\gamma-1}}ds + \int_{1/2}^T \frac{|r(s-\tau)|}{s^{\beta}}ds + \int_{1/2}^T \frac{|r(s-\tau)|}{s^{\gamma}}ds\right]. \end{aligned} \tag{4.16}$$

Taking into account that r is bounded ($|r(t)| \le r(0) = E|Y(t)|^2 < \infty$, $t \in \mathbb{R}$), and using similar arguments as in (4.7), from (3.10) we obtain that for any $\eta > 0$

$$\begin{aligned} \int_{1/2}^T \frac{|r(t-\tau)|}{t^{\eta}}\,dt &\le C\log T\left(T^{1-\alpha-\eta} + \tau^{1-\alpha-\eta} + \tau^{-\alpha} + \tau^{-\eta}\right) \\ &\le C\log T\left(1 + T^{1-\alpha-\eta}\right), \qquad 1 < \tau < T. \end{aligned}$$

Applying this inequality for $\eta = \beta + \gamma - 1$, $\eta = \beta$ and $\eta = \gamma$, from (4.16) we obtain

$$q(\tau) \le C \log^2 T \left(1 + T^{2-\alpha-\beta-\gamma} + T^{1-\alpha-\beta}\right), \qquad 1 < \tau < T, \tag{4.17}$$

since $\alpha + \gamma > 1$. On the other hand, by (4.11) and (4.12) for $T > 2$ and $0 < \tau < 1$, we have

$$\begin{aligned} q(\tau) &\le C \left[\log T \int_0^2 |r(s-\tau)|\, ds + \int_2^T \frac{|\nu(s)|}{(s-1)^\alpha} ds\right] \\ &\le C \log T \left(1 + T^{2-\alpha-\beta-\gamma} + T^{1-\alpha-\beta}\right) \le C \log T \left(1 + T^{1-\beta}\right), \end{aligned} \tag{4.18}$$

$0 < \tau < 1$, since $\alpha + \gamma > 1$ and $\alpha > 0$.

Next, we denote

$$J(T) \;=\; \int_0^T |\nu(\tau)|\, q(\tau)\, d\tau = \int_0^1 + \int_1^T =: J_1(T) + J_2(T), \tag{4.19}$$

and estimate $J_1(T)$ and $J_2(T)$. By (4.12) and (4.18), for $J_1(T)$ we have

$$J_1(T) \le C \log^2 T \left(1 + T^{1-\beta}\right) = o(T) \quad \text{as} \quad T \to \infty, \tag{4.20}$$

since $\beta > 0$.

To estimate $J_2(T)$ we consider three cases, and use conditions (3.8), (3.10), (4.10) and inequalities (4.11), (4.17).

Case 1. If $\beta \ge 1$, then we have

$$|\nu(\tau)| \le C \log T \left(\tau^{-\beta} + \tau^{-\gamma}\right), \quad q(\tau) \le C \log^2 T, \qquad 1 < \tau < T,$$

and hence

$$J_2(T) \le C \log^3 T \left(1 + T^{1-\beta} + T^{1-\gamma}\right) = o(T) \quad \text{as} \quad T \to \infty. \tag{4.21}$$

Case 2. If $\beta < 1 < \gamma$, then we have

$$|\nu(\tau)| \le C \log T \cdot \tau^{-\beta}, \qquad q(\tau) \le C \log^2 T \left(1 + T^{1-\alpha-\beta}\right) \qquad 1 < \tau < T.$$

and hence

$$\begin{aligned} J_2(T) &\le C \log^3 T \left(1 + T^{1-\beta}\right)\left(1 + T^{1-\alpha-\beta}\right) \\ &\le C \log^3 T \left(1 + T^{1-\alpha-\beta} + T^{1-\beta} + T^{2-\alpha-2\beta}\right) = o(T) \quad \text{as} \quad T \to \infty \end{aligned} \tag{4.22}$$

since in this case by assumption $\alpha + 2\beta > 1$.

Case 3. If $\beta < 1$ and $\gamma \le 1$, then we have

$$|\nu(\tau)| \le C \log T \cdot \tau^{1-\beta-\gamma}, \qquad q(\tau) \le C \log^2 T \left(1 + T^{2-\alpha-\beta-\gamma}\right), \qquad 1 < \tau < T,$$

and hence

$$(4.23)\qquad J_2(T) \le C\log^3 T\left(1+T^{2-\beta-\gamma}\right)\left(1+T^{2-\alpha-\beta-\gamma}\right) \le$$

$$\le C\log^3 T\left(1+T^{2-\alpha-\beta-\gamma}+T^{2-\beta-\gamma}+T^{4-\alpha-2\beta-2\gamma}\right)=o(T)$$

as $T\to\infty$, since $\beta+\gamma>1$ and $\alpha+2\beta+2\gamma=(2\beta+\gamma)+(\alpha+\gamma)>3$.

From (4.19)–(4.23) we obtain $J(T)=o(T)$ as $T\to\infty$. Thus, the relation (4.13) and hence (4.2) are proved. □

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ANALYTIC SUMMABILITY OF REAL AND COMPLEX FUNCTIONS

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Abstract. Gamma-type functions satisfying the functional equation $f(x+1) = g(x)f(x)$ and limit summability of real and complex functions were introduced by Webster (1997) and Hooshmand (2001). However, some important special functions are not limit summable, and so other types of such summability are needed. In this paper, by using Bernoulli numbers and polynomials $B_n(z)$, we define the notions of analytic summability and analytic summand function of complex or real functions, and prove several criteria for analytic summability of holomorphic functions on an open domain D. As consequences of our results, we give some criteria for absolute convergence of the functional series $\sum_{n=0}^{\infty} c_n\sigma(z^n)$, where $\sigma(z^n) = S_n(z) = \frac{B_{n+1}(z+1)-B_{n+1}(1)}{n+1}$. Finally, we state some open problems for future study of analytic and limit summability of functions.

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1. Introduction and preliminaries

The notion of *limit summability of real functions* was introduced and studied in [3,4] as a generalization of the Gamma-type functions satisfying the functional equation $f(x+1) = g(x)f(x)$ from [6]. Below we summarize some definitions and results from [3,4]. Let f be a real or complex function with domain $D_f \supseteq \mathbb{N}^* := \{1, 2, 3, \cdots\}$. Put

$$\Sigma_f = \{x | x + \mathbb{N}^* \subseteq D_f\},$$

and then for any $x \in \Sigma_f$ and $n \in \mathbb{N}^*$ set

$$R_n(f, x) = R_n(x) := f(n) - f(x+n),$$

$$f_{\sigma_n}(x) = f_{\sigma_{f,n}}(x) := xf(n) + \sum_{k=1}^{n} R_k(x).$$

The function f is called limit summable at $x_0 \in \Sigma_f$ if the functional sequence $\{f_{\sigma_n}(x)\}$ is convergent at $x = x_0$. The function f is called limit summable on a set $S \subseteq \Sigma_f$ if it is limit summable at all points of S.

Now, put

$$f_\sigma(x) = f_{\sigma_\ell}(x) = \lim_{n\to\infty} f_{\sigma_n}(x)\ ,\ R(x) = R(f,x) = \lim_{n\to\infty} R_n(f,x),$$

and observe that $D_{f_\sigma} = \{x \in \Sigma_f | f \text{ is limit summable at } x\}$, and $f_{\sigma_\ell} = f_\sigma$ is the same limit function f_{σ_n} with domain D_{f_σ}.

The function f is called limit summable if it is summable on Σ_f, $R(1) = 0$ and $D_f \subseteq D_f - 1$. In this case the function f_σ is referred to as the limit summand function of f. Notice that if f is limit summable, then $D_{f_\sigma} = D_f - 1$ and

$$f_\sigma(x) = f(x) + f_\sigma(x-1)\ ;\ \forall x \in D_f.$$

Therefore, if f is limit summable, then its limit summand function f_σ satisfies the well-known difference functional equation $\varphi(x) - \varphi(x-1) = f(x)$ (see [2 - 4]). Hence, we have

$$f_\sigma(m) = f(1) + \cdots + f(m) = \sum_{j=1}^{m} f(j)\ ;\ \forall m \in \mathbb{N}^*.$$

If f is limit summable, then one may use the notation $\sigma_\ell(f(x))$ instead of $f_{\sigma_\ell}(x)$.
In [3,4] were obtained some criteria for existence of unique solutions of the above functional equation. For instance, if $|a| < 1$, then the complex (resp. real) exponential function a^x is limit summable and $\sigma_\ell(a^x) = \frac{a}{a-1}(a^x - 1)$.
Often if a real function f is limit summable on an interval of length 1 and $R(1) = 0$, then f is limit summable (see [3,4]).

Example 1.1. If $0 < b \neq 1$ and $0 < a < 1$, then the real function $f(x) = ca^x + \log_b x$ is limit summable and

$$f_\sigma(x) = \frac{ca}{a-1}(a^x - 1) + \log_b \Gamma(x+1).$$

However, some important special functions, such as nonconstant polynomials and trigonometric functions are not limit summable according to the above definition. So, we need to introduce other types of summability. To this end, we first recall the Bernoulli polynomials and numbers.

The Bernoulli polynomial $B_n(x)$ is generated by the identity

$$\frac{te^{zt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} t^n\ ;\ |t| < 2\pi,\ z \in \mathbb{C}.$$

Denote by $B_n := B_n(0)$ and $b_n := B_n(1)$ the first and second Bernoulli numbers, respectively. Recall that $b_n = B_n$ for all $n \geq 2$, and $b_n = (-1)^n B_n$, $|b_n| = |B_n|$ for all

$n \geq 0$ ($b_{2k+1} = B_{2k+1} = 0$ for all $k \geq 1$ and $b_1 = -B_1 = \frac{1}{2}$, $b_0 = B_0 = 1$).

Also, we have

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}, \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n = \frac{te^t}{e^t - 1} \ ; \ |t| < 2\pi.$$

We refer the readers to [1, 5] for more properties of Bernoulli polynomials and numbers. Now, put

(1.1) $$\sigma_A(z^n) = \sigma(z^n) := S_n(z) = \frac{B_{n+1}(z+1) - b_{n+1}}{n+1} \ ; \ z \in \mathbb{C}, n \geq 0.$$

Note that the notation $S_n(x)$ was used in many references (see, e.g., [1, 5], and references therein).

Since $B_n(z+1) - B_n(z) = nz^{n-1}$ ($z \in \mathbb{C}$, $n \geq 1$), then $S_n(m) = \sum_{k=1}^{m} k^n$ for all $m \in \mathbb{N}^*$, and

(1.2) $$\sigma(z^n) = z^n + \sigma((z-1)^n) \ ; \ z \in \mathbb{C}, n \geq 0.$$

On the other hand, we can write

(1.3) $$\sigma(z^n) = \sum_{k=1}^{n+1} \beta_{nk} z^k \ ; \ z \in \mathbb{C}, n \geq 0,$$

where

(1.4)
$$\beta_{nk} = \beta_{n,k} = \binom{n+1}{k} \frac{b_{n+1-k}}{n+1} = \frac{n!}{k!(n+1-k)!} b_{n+1-k} \ ; \ n \geq 0, 1 \leq k \leq n+1.$$

Note that we can define $\beta_{nk} = 0$ for all $k \geq n+2$, but β_{n0} is not defined. Simple calculations show that $\beta_{n,n+1} = \frac{1}{n+1}$, $\beta_{n,n} = b_0 = \frac{1}{2}$, $\beta_{n,1} = b_n$, $\beta_{n,k} = \frac{n}{k}\beta_{n-1,k-1}$ and $\sum_{k=1}^{n+1} \beta_{nk} = 1$. Also, if $n-k$ is an even number ≥ 2, then $\beta_{nk} = 0$. Hence we have

(1.5)
$$\sigma(z^n) = \sum_{k=1}^{n+1} \beta_{nk} z^k = \sum_{k=1}^{n+1} \frac{n!}{k!(n+1-k)!} b_{n+1-k} z^k = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} b_k z^{n+1-k}.$$

2. Analytic summability and analytic summand functions

Now, we are ready to introduce the notion of analytic summability of complex and real functions. For simplicity, we define the analytic summability for analytic functions around $c = 0$, the case $c \neq 0$ is similar.

Definition 2.1. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a complex or real analytic function defined on an open domain D. We call f "analytic summable at z_0"(resp. "absolutely analytic summable at z_0"), if the functional series

$$f_{\sigma_A}(z_0) = f_\sigma(z_0) = \sum_{n=0}^{\infty} c_n \sigma(z_0^n)$$

is convergent (resp. is absolutely convergent). We call f "analytic summable on a set $E \subseteq D$ if it is analytic summable at every point of E. The function $f_{\sigma_A} = f_\sigma$ (with the largest possible domain) is called "analytic summand (function) of f". If f is analytic summable on the whole $\mathbb{C}$, then we call f "entire analytic summable".

Remark 2.1. In the cases where we use both concepts (analytic and limit summable functions), we will use the symbols f_{σ_l} and f_{σ_A} to denote the limit summand and the analytic summand functions of f, respectively.

We will use the following identity for iterated series of double complex sequences, which represents the sum of all arrays of the lower triangle of the $(N+1)\times(N+1)$ matrix $[C_{nk}]$ by two different ways:

$$\sum_{n=0}^{N}\sum_{k=1}^{n+1} C_{nk} = \sum_{n=1}^{N+1}\sum_{k=n-1}^{N} C_{kn}. \tag{2.1}$$

It is known that the natural exponential function e^z is not limit summable. Indeed, the function a^z is limit summable if and only if $|a| \leq 1$ (see [3,4]). The following example shows that e^z is analytic summable.

Example 2.1. The exponential function $\exp(z) = e^z$ is entire analytic summable and

$$\exp_\sigma(z) = \frac{e}{e-1}(e^z - 1) \quad : \quad z \in \mathbb{C}.$$

Indeed, using (2.1) we can write

$$\exp_\sigma(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\sigma(z^n) = \lim_{N\to\infty} \sum_{n=0}^{N}\sum_{k=1}^{n+1} \frac{1}{n!}\frac{n!}{k!(n+1-k)!} b_{n+1-k} z^k$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N+1}\sum_{k=n-1}^{N} \frac{1}{n!}\frac{b_{k+1-n}}{(k+1-n)!} z^n = \sum_{n=1}^{\infty}\frac{1}{n!}\Big(\sum_{j=0}^{\infty}\frac{b_j}{j!}\Big)z^n$$

$$= \frac{e}{e-1}\sum_{n=1}^{\infty}\frac{z^n}{n!} = \frac{e}{e-1}(e^z - 1).$$

For the last equality, we used the identity $\sum_{j=0}^{\infty}\frac{b_j}{j!} = \sum_{j=0}^{\infty}(-1)^j\frac{B_j}{j!} = \frac{e}{e-1}$.

Now we are in position to state some basic properties of analytic summability of real and complex functions. One can see that these properties are similar to that of limit summability of functions.

Theorem 2.1. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and $g(z) = \sum_{n=0}^{\infty} d_n z^n$ be analytic functions defined on an open domain D. The following assertions hold.*
(a) If $z, z-1 \in D$, then f is analytic summable at z if and only it is analytic summable at $z-1$. So, if f is analytic summable on D, then

(2.2) $$f_\sigma(z) = f(z) + f_\sigma(z-1) \ ; \ \forall z \in D \cap (D+1).$$

(b) If f is analytic summable on D and $D \subseteq D+1$, then

(2.3) $$f_\sigma(z) = f(z) + f_\sigma(z-1) \ ; \ \forall z \in D.$$

(c) If f and g are analytic summable at z (resp. on D), then every linear combination of f and g is also analytic summable, and we have $(af+bg)_\sigma(z) = af_\sigma(z) + bg_\sigma(z)$ (resp. for all $z \in D$).

Proof. Put $f_{\sigma_N}(z) := \sum_{n=0}^{N} c_n \sigma(z^n)$. If $z, z-1 \in D$, then by using (1.2) we have

$$f_{\sigma_N}(z) = \sum_{n=0}^{N} c_n z^n + f_{\sigma_N}(z-1).$$

Also, a simple calculation shows that

$$(af+bg)_{\sigma_N}(z) = af_{\sigma_N}(z) + bg_{\sigma_N}(z).$$

Now, one easily can get the results. □

3. Some upper bounds for $\sigma_A(z^n)$

Since the analytic summand function is generated by the sequence $\{\sigma_A(z^n)\}_{n=1}^{\infty}$, upper bounds for $\sigma_A(z^n)$ should be useful in establishing criteria about analytic summability. We first consider the following bounds for Bernoulli numbers:

(3.1) $$\frac{1}{1-2^{-2r}} \cdot \frac{2(2r)!}{(2\pi)^{2r}} < |B_{2r}| = |b_{2r}| < \frac{2(2r)!}{(2\pi)^{2r}} \cdot \frac{1}{1-2^{1-2r}} \ ; \ r = 1, 2, 3, \ldots.$$

The inequality (3.1) together with $B_{2r+1} = b_{2r+1} = 0$ (for all $r \geq 1$) imply

(3.2) $$|B_n| = |b_n| < \frac{2n!}{(2\pi)^n} \cdot \frac{1}{1-2^{1-n}} \ ; \ n = 2, 3, 4, 5, \ldots.$$

Applying the identity (1.4), for every positive integer n and $1 \leq k \leq n-1$, we obtain

$$|\beta_{nk}| < \frac{n!}{k!(n-k+1)!} \cdot \frac{2(n-k+1)!}{(2\pi)^{n-k+1}} \cdot \frac{1}{1-2^{k-n}} = \frac{n!}{k!\pi^{n-k+1}} \cdot \frac{1}{2^{n-k}-1}.$$

Since $n-k \geq 1$, then $\frac{1}{2^{n-k}-1} \leq 1$, and hence we have

(3.3) $$|\beta_{nk}| \leq \frac{n!}{k!\pi^{n-k+1}} \cdot \frac{1}{2^{n-k}-1} \leq \frac{n!}{k!\pi^{n-k+1}} \ ; \ 1 \leq k \leq n-1 \text{ or } k = n+1$$

Observe that the inequality (3.3) does not hold for $k = n$, but the next inequality holds for all $1 \leq k \leq n+1$

$$|\beta_{nk}| \leq \frac{2n!}{k!\pi^{n-k+1}} \; ; \; 1 \leq k \leq n+1, \tag{3.4}$$

and for $k = n$ we have $\beta_{nn} = \frac{1}{2} \leq \frac{2n!}{n!\pi^{n-n+1}} = \frac{2}{\pi}$.
Now, using (1.5) and (3.3), we obtain

$$|\sigma(z^n)| \leq \frac{|z|^{n+1}}{n+1} + \frac{|z|^n}{2} + \sum_{k=1}^{n-1} \frac{n!}{k!\pi^{n-k+1}}|z|^k = \frac{|z|^{n+1}}{n+1} + \frac{|z|^n}{2} + \frac{n!}{\pi^{n+1}} \sum_{k=1}^{n-1} \frac{(\pi|z|)^k}{k!}$$

$$= \frac{\pi-2}{2\pi}|z|^n + \frac{n!}{\pi^{n+1}} \sum_{k=1}^{n+1} \frac{(\pi|z|)^k}{k!}.$$

Therefore

$$|\sigma(z^n)| \leq \frac{\pi-2}{2\pi}|z|^n + \frac{n!}{\pi^{n+1}} \sum_{k=1}^{n+1} \frac{(\pi|z|)^k}{k!} \leq \frac{\pi-2}{2\pi}|z|^n + \frac{n!}{\pi^{n+1}}(e^{\pi|z|} - 1). \tag{3.5}$$

In similar way, by using (3.4), we can derive the following inequality

$$|\sigma(z^n)| \leq \frac{2n!}{\pi^{n+1}} \sum_{k=1}^{n+1} \frac{(\pi|z|)^k}{k!} \leq \frac{2n!}{\pi^{n+1}}(e^{\pi|z|} - 1). \tag{3.6}$$

4. Some criteria for analytic summability of complex and real functions

The inequalities for $\sigma_A(z^n)$, stated in Section 3, together with some previous results allow to prove a number of criteria for analytic summability.

Theorem 4.1. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic function defined on an open domain D. If $\sum_{n=0}^{\infty} \frac{n!}{\pi^n} c_n$ is absolutely convergent (for example if $\limsup_{n\to\infty} \sqrt[n]{n!|c_n|} < \pi$), then f is absolutely analytic summable on D. Moreover, by putting $\sigma_{n,N} := \sum_{k=n-1}^{N} \beta_{kn} c_k$, $Abs(f(z)) := \sum_{n=0}^{\infty} |c_n||z|^n$ and $Abs_{1/\pi}(f) := \sum_{n=0}^{\infty} \frac{n!}{\pi^n}|c_n|$, we have the following assertions.*

(a) The analytic summand function f_σ is analytic on D. Indeed, the limit $\sigma_n := \lim_{N\to\infty} \sigma_{n,N}$ exists (for all n).

$$|\sigma_n| \leq \frac{2\pi^{n-1}}{n!} Abs_{1/\pi}(f) \; ; \; \forall n. \tag{4.1}$$

and f_σ admits the representation:

$$f_\sigma(z) = \sum_{n=1}^{\infty} \sigma_n z^n = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{j=0}^{\infty} \frac{(j+n-1)!}{j!} b_j c_{j+n-1}\right) z^n \; ; z \in D. \tag{4.2}$$

(which provides an important explicit formula for computing the analytic summand function $f_{\sigma_A}(z)$.)

(b) The following upper bounds for f_{σ_A} hold:

$$|f_\sigma(z)| \le \frac{2}{\pi}(e^{\pi|z|}-1)Abs_{1/\pi}(f) \ ; \ z \in D \tag{4.3}$$

and

$$|f_\sigma(z)| \le \frac{1}{\pi}\{(\frac{\pi}{2}-1)Abs(f(z)) + (e^{\pi|z|}-1)Abs_{1/\pi}(f)\} \ ; \ z \in D. \tag{4.4}$$

Proof. By using (3.6), for every $z \in D$ and a positive integer N, we can write

$$\sum_{n=0}^{N}|c_n\sigma(z^n)| \le \sum_{n=0}^{N}|c_n|\frac{2n!}{\pi^{n+1}}(e^{\pi|z|}-1) = \frac{2}{\pi}(e^{\pi|z|}-1)\sum_{n=0}^{N}\frac{n!}{\pi^n}|c_n|$$

$$\Rightarrow |f_{\sigma_N}(z)| \le \sum_{n=0}^{N}|c_n\sigma(z^n)| \le \frac{2}{\pi}(e^{\pi|z|}-1)\sum_{n=0}^{\infty}\frac{n!}{\pi^n}|c_n|.$$

Therefore, f is absolutely analytic summable on D and (4.3) holds. Similarly, using (3.5), we can obtain (4.4).

Next, by applying (2.1), we can write

$$f_{\sigma_N}(z) = \sum_{n=0}^{N}c_n\sigma(z^n) = \sum_{n=0}^{N}c_n\sum_{k=1}^{n+1}\beta_{nk}z^k = \sum_{n=0}^{N}\sum_{k=1}^{n+1}\beta_{nk}c_nz^k = \sum_{n=1}^{N+1}\sum_{k=n-1}^{N}\beta_{kn}c_kz^n.$$

Therefore

$$f_{\sigma_N}(z) = \sum_{n=1}^{N+1}\sigma_{n,N}z^n = \sum_{n=1}^{N+1}\sum_{k=n-1}^{N}\beta_{kn}c_kz^n = \sum_{n=0}^{N}c_n\sigma(z^n) = \sum_{n=0}^{N}\sum_{k=1}^{n+1}\beta_{nk}c_nz^k. \tag{4.5}$$

Taking into account that

$$|\sigma_{n,N}| \le \sum_{k=n-1}^{N}|\beta_{kn}||c_k| \le \frac{2\pi^{n-1}}{n!}\sum_{k=n-1}^{N}\frac{k!}{\pi^k}|c_k| \le \frac{2\pi^{n-1}}{n!}\sum_{k=0}^{N}\frac{k!}{\pi^k}|c_k|,$$

we obtain (4.1), and conclude that $\lim_{N\to\infty}\sigma_{n,N}$ exists (for all n). Since the series $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}c_n$ is absolutely convergent, in view of (4.5), we get

$$f_\sigma(z) = \lim_{N\to\infty}f_{\sigma_N}(z) = \lim_{N\to\infty}\sum_{n=1}^{N+1}\sigma_{n,N}z^n = \sum_{n=1}^{\infty}\sigma_nz^n.$$

Finally, noting that

$$\sigma_n = \lim_{N\to\infty}\sigma_{n,N} = \sum_{k=n-1}^{\infty}\beta_{kn}c_k = \frac{1}{n!}\sum_{j=0}^{\infty}\frac{(j+n-1)!}{j!}b_jc_{j+n-1},$$

we complete the proof. Theorem 4.1 is proved. □

Corollary 4.1. *Let $f(z) = \sum_{n=0}^{\infty}c_nz^n$ be an analytic function defined on an open domain D, and let $z_0 \in D$. If the iterated series $\sum_{n=1}^{\infty}\sum_{k=n-1}^{\infty}\beta_{kn}c_kz^n$ is absolutely* *convergent on D (resp. at z_0), then f is absolutely analytic summable on D (resp. at z_0).*

Proof. Since the iterated series $\sum_{n=1}^{\infty}\sum_{k=n-1}^{\infty}\beta_{kn}c_k z^n$ is absolutely convergent, then $\sum_{k=n-1}^{\infty}\beta_{kn}c_k$ is convergent (for all n), and

$$\lim_{N\to\infty}\sum_{n=1}^{N+1}\sum_{k=n-1}^{N}\beta_{kn}c_k z^n=\sum_{n=1}^{\infty}\sum_{k=n-1}^{\infty}\beta_{kn}c_k z^n$$

(note that the absolute convergence is enough for the above equality). Hence, we can apply the identity (4.5) to conclude that f is analytic summable at z, and

$$f_\sigma(z)=\sum_{n=1}^{\infty}\sum_{k=n-1}^{\infty}\beta_{kn}c_k z^n.$$

Corollary 4.2. *Analytic summand function of every polynomial of degree n exists, and it is a polynomial of degree $n+1$ without a constant term.*

Corollary 4.3. *If the series $\sum_{n=0}^{\infty}\frac{n!}{\pi^n}c_n$ is absolutely convergent, then*

$$\lim_{n\to\infty}\sum_{j=0}^{\infty}(-1)^j\frac{(j+n-1)!}{j!n!}B_j c_{j+n-1}=0.$$

Theorem 4.2. *Let $f(z)=\sum_{n=0}^{\infty}c_n z^n$ be an analytic function defined on an open domain D. If $\sqrt[n]{n!|c_n|}\le\delta<\pi$ for all n, then f is absolutely analytic summable on D, and the following inequalities hold:*

$$|f_\sigma(z)|\le\frac{1}{\pi-\delta}(e^{\pi|z|}-1)+\frac{\pi-2}{2\pi}e^{\delta|z|}\ ;\ z\in D \tag{4.6}$$

and

$$|f_\sigma(z)|\le\frac{2}{\pi-\delta}(e^{\pi|z|}-1)\ ;\ z\in D. \tag{4.7}$$

Proof. By applying Theorem 4.1 and (3.5), for all $z\in D$ we can write

$$|f_\sigma(z)|\le\sum_{n=0}^{\infty}|c_n||\sigma(z^n)|\le\sum_{n=0}^{\infty}|c_n|\Big(\frac{\pi-2}{2\pi}|z|^n+\frac{n!}{\pi^{n+1}}(e^{\pi|z|}-1)\Big)$$

$$\le\frac{\pi-2}{2\pi}\sum_{n=0}^{\infty}\frac{(\delta|z|)^n}{n!}+\frac{e^{\pi|z|}-1}{\pi}\sum_{n=0}^{\infty}\Big(\frac{\delta}{\pi}\Big)^n$$

$$=\frac{\pi-2}{2\pi}e^{\delta|z|}+\frac{1}{\pi-\delta}(e^{\pi|z|}-1),$$

and thus (4.6) is proved. The proof of (4.7) is similar. □

Example 4.1. If $f(z)=e^z$, then $\delta=1$, and we have

$$|exp_\sigma(z)|=\Big|\frac{e}{e-1}(e^z-1)\Big|\le\frac{1}{\pi-1}(e^{\pi|z|}-1)+\frac{\pi-2}{2\pi}e^{|z|}\ ;\ z\in\mathbb{C}.$$

Hence

$$\left|\frac{e}{e-1}(e^z-1)\right| \leq \frac{2}{\pi-1}(e^{\pi|z|}-1) \ ; \ z\in\mathbb{C}.$$

5. Analytic summand of exponential and trigonometric functions

As it was mentioned before, polynomials of degree at least one, the trigonometric functions (sin and cos) and the exponential functions a^z with $|a|>1$ are not limit summable. However, they are analytic summable. Indeed, observe first that in view of Corollary 4.3 and Example 2.2, we have

$$\sigma_{\mathcal{A}}\Big(\sum_{n=0}^{N} c_n z^n\Big) = \sum_{n=1}^{N+1} \sigma_{n,N} z^n \quad \text{and} \quad \sigma_{\mathcal{A}}(e^z) = \frac{e}{e-1}(e^z-1).$$

Next, we consider analytic summability of functions a^z, $\sin(z)$, $\cos(z)$, etc.
Note that $a^z = \exp(z\ln a)$ is an entire function for a fixed value of $\ln a$, and $f(z) = a^z = \sum_{n=0}^{\infty}\frac{(\ln a)^n}{n!}z^n$. If $|\ln a|<\pi$, then in view of Theorem 4.1, a^z is (absolutely) entire analytic summable and

$$\sigma_n = \frac{1}{n!}\sum_{j=0}^{\infty}\frac{(j+n-1)!}{j!}b_j\frac{(\ln a)^{j+n-1}}{(j+n-1)!} = \frac{1}{n!}(\ln a)^{n-1}\sum_{j=0}^{\infty}\frac{b_j}{j!}(\ln a)^j$$

$$= \frac{1}{n!}(\ln a)^{n-1}\cdot\frac{(\ln a)a}{a-1} = \frac{a}{a-1}\cdot\frac{(\ln a)^n}{n!}.$$

Therefore, $\sigma_{\mathcal{A}}(a^z) = \sum_{n=1}^{\infty}\frac{a}{a-1}\cdot\frac{(\ln a)^n}{n!}z^n$, and hence

$$\sigma_{\mathcal{A}}(a^z) = \frac{a}{a-1}(a^z-1) \ ; \ |\ln a|<\pi, \ z\in\mathbb{C}.$$

To determine the analytic summand function of $\sin(z)$, let $\{\epsilon_n\}$ be a sequence such that $\epsilon_n=0$ if n is even and $\epsilon_n=1$ if n is odd. Then, we have

$$\sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^{[\frac{n}{2}]}\epsilon_n}{n!}z^n = \sum_{n=0}^{\infty}c_n z^n,$$

and hence

$$\sigma_n = \frac{1}{n!}\sum_{j=0}^{\infty}(-1)^{[\frac{j+n-1}{2}]}\epsilon_{j+n-1}\frac{b_j}{j!} = \begin{cases}\frac{(-1)^{\frac{n-1}{2}}}{2.n!}, & n\in 2\mathbb{Z}+1\\ \frac{(-1)^{\frac{n-2}{2}}}{n!}\sum_{k=0}^{\infty}(-1)^k\frac{b_{2k}}{(2k)!}, & n\in 2\mathbb{Z}.\end{cases}$$

Taking into account that

$$\sum_{k=0}^{\infty}(-1)^k\frac{b_{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k\frac{B_{2k}}{(2k)!} = \frac{\frac{1}{2}\sin(1)}{1-\cos(1)},$$

we can write

$$\sin_\sigma(z) = \sum_{n=1}^{\infty}\sigma_n z^n = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!} - \frac{\frac{1}{2}\sin(1)}{1-\cos(1)}\sum_{k=1}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!}$$

$$= \frac{1}{2}\sin(z) + \frac{\frac{1}{2}\sin(1)}{1-\cos(1)}(1-\cos(z)) = \frac{\sin(z)+\sin(1)-\sin(z+1)}{2-2\cos(1)}.$$

The function $\cos_\sigma(z)$ can be calculated analogously, or by using the identity

$$\frac{e^i}{e^i-1}(e^{iz}-1) = \cos_\sigma(z) + i\sin_\sigma(z).$$

Finally, we have

$$\sin_\sigma(z) = \frac{\sin(z)+\sin(1)-\sin(z+1)}{2-2\cos(1)}, \quad \cos_\sigma(z) = \frac{\cos(z)+\cos(1)-\cos(z+1)-1}{2-2\cos(1)}$$

Using the properties of analytic summability, some trigonometric identities and the above results, we obtain

$$\sigma_A(\sin(az+b)) = \frac{\sin(az+b)+\sin(a+b)-\sin(az+a+b)-\sin(b)}{2-2\cos(a)}$$

$$\sigma_A(\cos(az+b)) = \frac{\cos(az+b)+\cos(a+b)-\cos(az+a+b)-\cos(b)}{2-2\cos(a)},$$

where a, b are real or complex constants and $a \neq 0$.

Now, we pose a number of questions that are very important for future study of analytic and limit summability of functions.

Open problem I. Let f be an analytic function defined on an open domain $D = D_f$ with the property $\mathbb{N}^* \subsetneq D \subsetneq \Sigma_f$. If f is both limit and analytic summable, then is it true that $f_{\sigma_l} = f_{\sigma_A}$ on D?

Open problem II. If f is analytic summable on $D = D_f$, then under what conditions is it a unique solution of the functional equation $f_\sigma(z) = f(z) + f_\sigma(z-1)$ on D with the initial condition $f_\sigma(0) = 0$? Compare with the uniqueness Theorem 3.1, Corollary 3.4 of [3] and Theorem A, Corollary 3.4 of [3]).

Open problem III. Is $f(z) = \sum_{n=0}^{\infty} c_n z^n$ absolutely analytic summable (on D) whenever $\pi \leq \limsup_{n\to\infty} \sqrt[n]{n!|c_n|} < 2\pi$? A special interest represents the case when it is equal to π.

Open problem IV. Is the inequality (4.3) (or (4.4)) sharp? If no, find a sharp upper bound for the analytic summand of f.

Finally, as another direction of research, one may study intersection of the spaces of limit and analytic summable functions.

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О МЕТРИЧЕСКОМ ТИПЕ ИЗМЕРИМЫХ ФУНКЦИЙ И СХОДИМОСТИ ПО РАСПРЕДЕЛЕНИЮ

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Аннотация. Изучаются последовательности действительных измеримых функций f_n на пространстве с мерой $([0,1],\ \mu)$, где μ-мера Лебега. Доказано, что если f_n сходится к f по распределению то существует последовательность автоморфизмов S_n пространства $([0,1],\ \mu)$ такая, что $f_n(S_n(t))$ сходится к $f(t)$ по мере на $[0,\ 1]$. Обсуждается связь данного утверждения с другими известными результатами.

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Ключевые слова: Изоморфизм; метрический тип; сходимость по распределению.

1. Введение

Пусть f_n, $n = 1,\ 2,\ \dots$ и f -действительные измеримые функции, заданные на измеримом пространстве (Ω,μ), $\mu(\Omega) = 1$. Пусть, далее, F_n, $n = 1,\ 2,\ \dots$ и F-их функции распределения, т.е.

$$F_n(x) = \mu(\omega \in \Omega:\ f_n(\omega) \leq x),\ -\infty < x < \infty,$$

$$F(x) = \mu(\omega \in \Omega:\ f(\omega) \leq x),\ -\infty < x < \infty.$$

Говорят, что последовательность f_n сходится к f по распределению, если $F_n(x) \to F(x)$ при $n \to \infty$ в каждой точке непрерывности F. Записывается это так: $f = D - \lim\limits_{n\to\infty} f_n$. Известно ([1], стр. 31), что из сходимости по мере следует сходимость по распределению. Обратное, очевидно, не верно.

Нам понадобятся еще понятия изоморфизма пространств с мерой и метрического типа измеримых функций, введенных В. А. Рохлиным ([2], [3]).

Отображение одного пространства с мерой на другое называется изоморфным, если оно взаимно однозначно, и как оно, так и обратное ему отображение переводит всякое измеримое множество в измеримое множество той же меры. В том случае, когда оба пространства совпадают, изоморфизм называется автоморфизмом.

Два пространства, допускающие изоморфные отображения друг на друга, называются изоморфными. Две функции f и g, определенные, соответственно, на пространствах M и N, называются изоморфными, если существуют такие множества $M_1 \subset M$ и $N_1 \subset N$ меры нуль и такое изоморфное отображение T пространства $M \setminus M_1$ на пространство $N \setminus N_1$, что для всякого $t \in M \setminus M_1$ $f(t) = g(T(t))$. В этом случае говорят также, что функции f и g принадлежат одному метрическому типу.

Из очевидной цепочки равенств

$$\mu\{t \in [0,1]: \ f(t) \le x\} = \mu\{t \in [0,1]: \ g(T(t)) \le x\} = \mu\{(g \circ T)^{-1}(-\infty, x]\} =$$

$$= \mu\{T^{-1}(g^{-1}(-\infty, x])\} = \mu\{t \in [0,1]: \ g(t) \le x\}$$

следует, что функции, принадлежащие к одному метрическому типу, одинаково распределены. Обратное не верно. Вот простой пример

$$f(t) = t, \ 0 \le t \le 1, \ g(t) = \begin{cases} 2t & \text{если } 0 \le t \le 1/2, \\ 2(1-t) & \text{если } 1/2 \le t \le 1 \ . \end{cases}$$

Необходимые и достаточные условия для того, чтобы две функции принадлежали одному метрическому типу, получены В. А. Рохлиным в его квалификационной теореме ([2]).

В настоящей работе доказывается следующая

Теорема 1.1. *Пусть f и f_n, $n = 1, 2, \ldots$ -измеримые функции, заданные на $[0,1]$ и $f = D - \lim\limits_{n\to\infty} f_n$. Тогда существует последовательность S_n, $n = 1, 2, \ldots$ автоморфизмов пространства $([0,1], \mu)$ такая, что*

$$\lim_{n\to\infty} f_n(S_n(t)) = f(t) \text{ по мере на } [0,1]. \tag{1.1}$$

В работе [4], с помощью упомянутой выше довольно сложной квалификационной теоремы Рохлина, доказывается следующая

Теорема 1.2. *Если последовательность измеримых функций $f_1, f_2, \ldots$, определенных на $[0,1]$, сходится по мере к функции f, то существует последовательность $S_1, S_2, \ldots$ автоморфизмов $[0,1]$ такая, что*

$$\lim_{n\to\infty} f_n(S_n(t)) = f(t) \text{ почти всюду на } [0,1], \tag{1.2}$$

$$\lim_{n\to\infty} \mu\{t \in [0,1] : S_n(t) \ne t\} = 0. \tag{1.3}$$

Отметим, что в теореме 1.1 условие (1.3) гарантировать нельзя.

Комбинируя теорему 1.2 с нашей теоремой 1.1, мы получим следующий результат.

Теорема 1.3. *Если последовательность f_n, $n = 1,\ 2,\ \ldots$ -измеримых функций, определенных на $[0,1]$ сходится по распределению к функции f, то существует последовательность $S_1, S_2, \ldots$ автоморфизмов $[0,1]$ такая, что*

$$\lim_{n\to\infty} f_n(S_n(t)) = f(t) \text{ почти всюду на } [0,1]. \tag{1.4}$$

Так как функции, принадлежащие к одному метрическому типу, одинаково распределены, то из теоремы 1.3 следует известная теорема Скорохода о представлении ([5]).

Следствие 1.1. *(Скороход) Пусть X_n, $n = 1,\ 2,\ \ldots$ и X -случайные величины, заданные на вероятностном пространстве $([0,1],\ \mu)$ и пусть $X = D - \lim\limits_{n\to\infty} X_n$. Тогда существует последовательность случайных величин Y_n, $n = 1,\ 2,\ \ldots$ такая, что*

a) при любом $n = 1,\ 2,\ \ldots$ случайные величины X_n и Y_n одинаково распределены;

b) $\lim_{n\to\infty} Y_n(t) = X(t)$ почти наверное на $[0,1]$.

Сформулируем еще одно очевидное следствие из теоремы 1.3, которое показывает, что одинаково распределенные функции в определенном смысле близки по метрическому типу.

Следствие 1.2. *Если f и g -одинаково распределенные измеримые функции на $[0,1]$, то существует последовательность S_n автоморфизмов $[0,1]$ такая, что*

$$\lim_{n\to\infty} f(S_n(t)) = g(t) \text{ почти всюду на } [0,1].$$

2. Вспомогательные утверждения

Нам понадобятся два вспомогательных утверждения, которые приводятся здесь в виде лемм.

Лемма 2.1. *Пусть A и B - измеримые множества, содержащиеся в $[0,1]$, причем $\mu(A) = \mu(B) > 0$. Тогда пространства (A, μ) и (B, μ) изоморфны.*

Доказательство. Достаточно доказать, что (A, μ) изоморфно $([0, \mu(A)], \mu)$. Можно считать, что все точки A являются точками плотности. Рассмотрим функцию

$$f(t) = \int_0^t \chi_A d\mu = \mu([0, t] \cap A),$$

где χ_A-характеристическая функция множества A.

Функция f возрастающая и абсолютно непрерывная на $[0, 1]$. Следовательно, образ $f(E)$ любого измеримого множества $E \subset [0, 1]$ будет измеримым. Очевидно, f взаимно однозначно на A. Докажем, что f сохраняет меру подмножеств A. Проверим это для A.

Пусть $\varepsilon > 0$-произвольное число. Так как $f'(x) = 1$ на A, то существует счетная система интервалов Δ_k, $k = 1, 2, \ldots$ такая, что

$$A \subset \bigcup_{k=1}^{\infty} \Delta_k, \tag{2.1}$$

$$\mu(A) \le \sum_{k=1}^{\infty} \mu(\Delta_k) < \mu(A) + \varepsilon . \tag{2.2}$$

$$(1 - \varepsilon)\mu(\Delta_k) < \mu(f(\Delta_k \cap A)) < (1 + \varepsilon)\mu(\Delta_k) , \quad k = 1, 2, \ldots \tag{2.3}$$

Суммируя по k все части (2.3), получим

$$(1 - \varepsilon) \sum_{k=1}^{\infty} \mu(\Delta_k) \le \mu(f(A)) \le (1 + \varepsilon) \sum_{k=1}^{\infty} \mu(\Delta_k).$$

откуда, в силу (2.2), будем иметь

$$(1 - \varepsilon)\mu(A) \le \mu(f(A)) \le \mu(A) + \varepsilon(\mu(A) + 1 + \varepsilon). \tag{2.4}$$

Наконец, из (2.4), в силу произвольности ε, получим $\mu(f(A)) = \mu(A)$, что и требовалось.

Нам понадобится еще следующее совсем очевидное утверждение

Лемма 2.2. *Пусть $m > 1$-натуральное число и $\varepsilon > 0$. Тогда для любых двух систем положительных чисел $a_1, \ldots, a_m$ и $b_1, \ldots, b_m$, удовлетворяющих условиям*

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i = 1 \ \text{и} \ |a_i - b_i| \le \varepsilon, \ i = 1, \ldots, m.$$

Справедливо неравенство

$$\sum_{i=1}^{m} \min\{a_i, b_i\} \ge 1 - m\varepsilon.$$

3. Доказательство теоремы 1.1

Пусть F_n и F-функции распределения f_n и f соответственно. Обозначим через $C(F)$ множество точек непрерывности F. По условию теоремы

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ для всех } x \in C(F). \tag{3.1}$$

Сначала рассмотрим случай, когда последовательность f_n равномерно ограничена на $[0,1]$. Возьмем отрезок $[a,b]$ так, чтобы $a,\ b \in C(F)$ и

$$a < f_n(t) \le b \text{ для всех } n = 1,\ 2,\dots \text{ и } t \in [0,1]. \tag{3.2}$$

Построим последовательность разбиений отрезка $[a,b]$:

$$Q_k = \{a = x_{k,0} < x_{k,1} < \dots < x_{k,m_k} = b\},\ k = 1,\ 2,\dots$$

такую, что для всех $k = 1,\ 2,\dots$ и $i = 0,\ 1,\dots,m_k$ выполняются условия

$$Q_k \subset Q_{k+1} \subset C(F) \tag{3.3}$$

и

$$\max\{(x_{k,i+1} - x_{k,i}) : 0 \le i \le m_k - 1\} < \frac{1}{k}. \tag{3.4}$$

Теперь построим последовательность S_n автоморфизмов пространства $([0,1],\mu)$. Сначала возьмем числа $\varepsilon_k > 0$ так, чтобы

$$m_k\varepsilon_k \to 0 \text{ при } k \to \infty. \tag{3.5}$$

Затем возьмем последовательность натуральных чисел $1 < n_1 < n_2 < \dots < n_k < \dots$ такую, что для всех $k = 1,\ 2,\dots;\ n \ge n_k$ и $i = 0,\ 1,\dots,m_k$ выполнялись неравенства

$$|F_n(x_{k,i}) - F(x_{k,i})| < \varepsilon_k. \tag{3.6}$$

Автоморфизмы S_n будут построены группами. Сначала для $1 \le n < n_1$, затем для $n_1 \le n < n_2$ и т.д. Для $1 \le n < n_1$ положим $S_n = I$, где I-тождественный автоморфизм пространства $([0,1],\mu)$.

Пусть построены автоморфизмы $S_1,\dots,S_{n_k-1}$ и пусть $n_k \le n < n_{k+1}$. Введем обозначения

$$E^n_{k,i} = \{t \in [0,1] : x_{k,i} < f_n(t) \le x_{k,i+1}\}\ i = 0,\ 1,\dots,m_k - 1\ ,$$

$$E_{k,i} = \{t \in [0,1] : x_{k,i} < f(t) \le x_{k,i+1}\}\ i = 0,\ 1,\dots,m_k - 1\ .$$

Так как $\mu(E^n_{k,i}) = F_n(x_{k,i+1}) - F_n(x_{k,i})$ и $\mu(E_{k,i}) = F(x_{k,i+1}) - F(x_{k,i})$, то, в силу (3.6) имеем

$$|\mu(E^n_{k,i}) - \mu(E_{k,i})| < 2\varepsilon_k, \ n \geq n_k, \quad i = 0, \ 1, \ldots, m_k - 1. \tag{3.7}$$

Для каждой пары $E^n_{k,i}$ и $E_{k,i}$ возьмем множества $A^n_{k,i} \subset E^n_{k,i}$ и $A_{k,i} \subset E_{k,i}$ такие, что

$$\mu(A^n_{k,i}) = \mu(A_{k,i}) = \min\{\mu(E^n_{k,i}), \mu(E_{k,i})\} \ .$$

Тогда, по лемме 2.1, существует изоморфизм множества $A_{k,i}$ на $A^n_{k,i}$. Обозначим этот изоморфизм через $S^n_{k,i}$, $n_k \leq n < n_{k+1}$, $i = 0, \ 1, \ldots, m_k - 1$.
Теперь для каждого n такого, что $n_k \leq n < n_{k+1}$ построим автоморфизм S^n_k пространства $([0,1], \mu)$ следующим образом. Положим

$$S^n_k(t) = S^n_{k,i}(t) \text{ при } t \in A^n_{k,i}; \ i = 0, \ 1, \ldots, m_k - 1 \ .$$

На дополнительном множестве $[0,1] \setminus \bigcup_{i=0}^{m_k-1} A^n_{k,i}$ за S^n_k возьмем по той же лемме 1.1 произвольный изоморфизм $[0,1] \setminus \bigcup_{i=0}^{m_k-1} A^n_{k,i}$ на $[0,1] \setminus \bigcup_{i=0}^{m_k-1} S^n_k(A^n_{k,i})$.

Далее, для каждого n из промежутка $n_k \leq n < n_{k+1}$ положим $S_n = S^n_k$. Продолжив этот процесс неограниченно, мы построим последовательность автоморфизмов S_n пространства $([0,1], \mu)$. Докажем, что $f_n(S_n(t))$ сходится к $f(t)$ по мере на $[0,1]$.

Действительно, если $n_k \leq n < n_{k+1}$, то согласно построению S_n и в силу леммы 2.2, имеем

$$\mu\left\{t \in [0,1] : |f_n(t) - f(t)| \geq \frac{1}{k}\right\} < 2m_k\varepsilon_k \tag{3.8}$$

Из (3.5) и (3.8) следует, что $f_n \circ S_n \to f$ по мере.

Теперь рассмотрим общий случай. Пусть f_n, $n = 1, 2, \ldots$-произвольная (не обязательно равномерно ограниченная) последовательность измеримых функций.

Возьмем произвольные точки $a, b \in C(F)$, $a < b$, и пусть φ-непрерывная, строго возрастающая функция, отображающая $(-\infty, \infty)$ на (a, b). Можно взять, например,

$$\varphi(x) = \frac{b-a}{2} \cdot \frac{x}{1+|x|} + \frac{a+b}{2}.$$

Таким образом, φ и обратное отображение φ^{-1} непрерывны и сохраняют порядок. Пусть, как и выше, F_n и F -функции распределения f_n и f соответственно.

Докажем, что суперпозиции $\varphi \circ f_n$ сходятся по распределению к $\varphi \circ f$. Пусть G_n и G-это функции распределения $\varphi \circ f_n$ и $\varphi \circ f$ соответственно. Тогда имеем

$$G_n(x) = \mu\{t \in [0,1] : \varphi(f_n(t)) \leq x\} =$$

$$= \mu\left\{f_n^{-1}\left(\varphi^{-1}\left((-\infty, x]\right)\right)\right\} = \mu\left\{f_n^{-1}(a, \varphi^{-1}(x)]\right\} =$$

$$= \mu\left\{t \in [0,1] : a < f(t) \leq \varphi^{-1}(x)\right\} = F_n(\varphi^{-1}(x)) - F_n(a). \tag{3.9}$$

Аналогично,

$$G(x) = F(\varphi^{-1}(x)) - F(a). \tag{3.10}$$

Если $x \in C(G)$, то, в силу непрерывности φ^{-1}, $\varphi^{-1}(x) \in C(F)$. Отсюда, в силу (3.9) и (3.10) и условия $a \in C(F)$ получим

$$\lim_{n\to\infty} G_n(x) = G(x) \text{ для всех } x \in C(G). \tag{3.11}$$

Тогда, согласно доказанному случаю, существует последовательность автоморфизмов S_n такая, что

$$\varphi \circ f_n \circ S_n \to \varphi \circ f \text{ по мере }. \tag{3.12}$$

Известно, что взятие непрерывной функции сохраняет сходимость по мере ([1], стр. 39). В силу этого, из (3.12) следует

$$f_n \circ S_n = \varphi^{-1}(\varphi \circ f_n \circ S_n) \to \varphi^{-1}(\varphi \circ f) = f \text{ по мере.}$$

Теорема 1.1 полностью доказана. □

Abstract. In the present paper, sequences of real measurable functions defined on a measure space $([0,1],\ \mu)$, where μ is the Lebesgue measure, are studied. It is proved that for every sequence f_n that converges to f in distribution, there exists a sequence of automorphisms S_n of $([0,1],\ \mu)$ such that $f_n(S_n(t))$ converges to $f(t)$ in measure on $[0, 1]$. Connection with some known results is also discussed.

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