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КОВАРИОГРАММА ЦИЛИНДРА

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Аннотация. В статье получена формула связывающая ковариограмму и функцию распределения длины хорды в направлении для цилиндра с теми же функциями его основания. С помощью формулы получены выражения для ковариограммы и функции распределения длины хорды в направлении для цилиндров с круговыми, эллиптическими и треугольными основаниями.

MSC2010 numbers: 60D05; 52A22; 53C65

Keywords: ограниченная выпуклая область; ковариограмма; зависящая от направления функция распределения длины хорды.

1. Введение

Пусть $\mathbb{R}^n$ n-мерное евклидово пространство, $D \subset \mathbb{R}^n$ – ограниченная выпуклая область с внутренними точками, S^{n-1} – $(n-1)$-мерная единичная сфера с центром в начале координат, а $L_n(\cdot)$ – n-мерная мера Лебега в $\mathbb{R}^n$.

В [1] Матерон сформулировал гипотезу, что ковариограмма выпуклого тела определяет ее в классе всех выпуклых тел, с точностью до параллельных переносов и отражений. Эта гипотеза известна как гипотеза Матерона (см. [17]).

В [2] Г. Бианчи и Г. Аверков доказали гипотезу Матерона для $n = 2$. На плоскости положительный ответ для гипотезы Матерона в классе выпуклых многоугольников получил В. Нагель (см. [3]). Бианчи так же доказал, что в случае $n \geq 4$ гипотеза не верна (см. [6]). Очень мало известно относительно гипотезы Матерона, когда размерность пространства больше двух. Известно, что центрально-симметричные выпуклые тела любой размерности единственным образом определяются по ковариограмме, с точностью до параллельных переносов (см. [4]). В случае 3-мерного пространства вопрос остается открытым. Несмотря на это, в случае ограниченного выпуклого многогранника при n=3 гипотеза

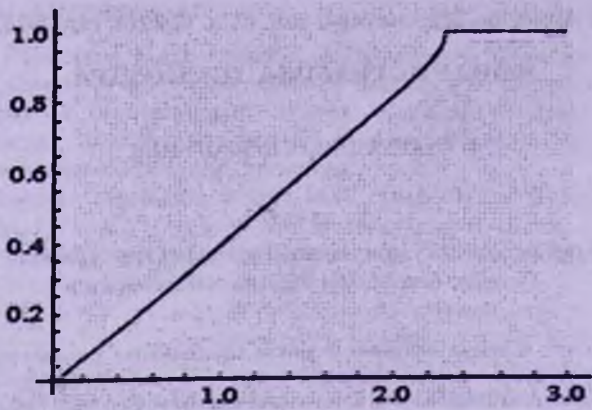

Рис. 1. Функция распределения длины хорды в направлении $\theta = \frac{\pi}{6}$ для кругового цилиндра с радиусом 1 и высотой 2.

Матерона получила положительный ответ (см. [5], [17] и [8]). Отметим, что выпуклость существена в этом круге вопросов. Авторы работы [7] построили пример двух не конгруэнтных и не выпуклых многоугольников с одной и той же ковариограммой.

Чтобы найти соответствующий подход для решения задачи в $\mathbb{R}^3$, нужно понять характер поведения ковариограммы в случае пространственных тел. Явный вид ковариограммы для тел в $\mathbb{R}^3$ известен только в случае шара.

Ковариограмма $C_D(\cdot)$ области D определяется как

$$C_D(x) = L_n(D \cap \{D + x\}), \quad x \in \mathbb{R}^n. \tag{1.1}$$

$C_D(\cdot)$ инвариантна относительно параллельных переносов и отражений. Г. Матерон (см. [1]) доказал. что для любого $t > 0$ и $\varphi \in S^{n-1}$

$$\frac{\partial C_D(t\varphi)}{\partial t} = -L_{n-1}(\{y \in \varphi^\perp : \ L_1(D \cap (l_\varphi + y)) \geq t\}), \tag{1.2}$$

где $l_\varphi + y$ есть прямая, параллельная направлению φ и проходящая через точку y, а $\varphi^\perp$ – ортогональное дополнение к φ, то есть – гиперплоскость в $\mathbb{R}^n$ с нормальным направлением $\varphi \in S^{n-1}$.

Пусть G пространство прямых на евклидовой плоскости $\mathbb{R}^2$ $g \in$ G, (p, φ)= полярные координаты перпендикуляра, опущенного из начала координат на прямую g; $p \geq 0$, $\varphi \in S^1$.

Для замкнутой ограниченной выпуклой области $D \subset \mathbb{R}^2$ обозначим через $S_D(\varphi)$ опорную функцию в направлении $\varphi \in S^1$, определяемую следующим образом

$$S_D(\varphi) = \max\{p \in \mathbb{R}^+ : \ g(p, \varphi) \cap D \neq \emptyset\},$$

где $\mathbb{R}^+$ – множество неотрицательных действительных чисел.

Для ограниченной выпуклой области $D \subset \mathbb{R}^2$ обозначим через $b_D(\varphi)$ функцию ширины в направлении $\varphi \in S^1$, т.е. расстояние между опорными прямыми к границе D, которые перпендикулярны направлению φ. Имеем

$$b_D(\varphi) = S_D(\varphi) + S_D(\varphi + \pi).$$

Функция $b_D(\varphi)$ есть периодическая функция с периодом π (см. [18]).

Для области D зависящая от направления функция распределения длины хорды $F_D(x, \varphi)$ определяется как вероятность, что случайная хорда $\chi(g) = g \cap D$, где g из пучка прямых параллельных направлению φ, будет иметь длину не превосходящую x. Случайная прямая которая перпендикулярна направлению φ и пересекает D имеет пересечение (обозначим точку пересечения через y) с прямой параллельной направлению φ и проходящей через начало координат. Точка пересечения y равномерно распределена в интервале $[0, b_D(\varphi)]$. Таким образом, имеем

$$F_D(x, \varphi) = \frac{L_1\{y : \chi(l_\varphi + y) \leq x\}}{b_D(\varphi)}. \tag{1.3}$$

Нетрудно убедиться, что для $n = 2$ формула (1.2) эквивалентна

$$-\frac{\partial C_D(t\varphi)}{\partial t} = b_D(\varphi)\left(1 - F_D(t, \varphi)\right). \tag{1.4}$$

В случае $n = 2$ явный вид функции распределения длины хорды в направлении, а так же ковариограммы, известны только в случае круга, треугольника, правильного многоугольника, параллелограмма и эллипса (см. [11]– [13]). Практическое применение этих результатов в кристалографии можно найти в [9] (см. также [10], [14] и [15]).

Обозначим через Γ пространство прямых γ в $\mathbb{R}^3$. Обозначим через $\Pi_D(\omega)$ проекцию области $D \subset \mathbb{R}^3$ в направлении $\omega \in S^2$, а через $s_D(\omega)$ – площадь $\Pi_D(\omega)$. Каждая прямая, параллельная направлению ω и пересекающая D имеет пересечение с $\Pi_D(\omega)$. Обозначим эту точку через y а прямую – через $l_\omega + y$. Точка пересечения y равномерно распределена в $\Pi_D(\omega)$. Функция распределения длины

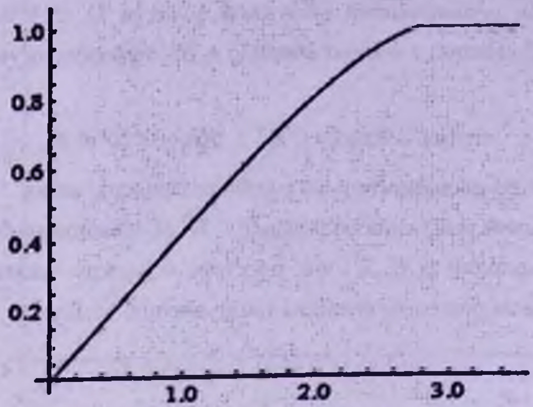

Рис. 2. Функция распределения длины хорды в направлении $\theta = \frac{\pi}{4}$ для кругового цилиндра с радиусом 1 и высотой 2.

хорды области D в направлении $\omega \in S^2$ определяется как

$$F_D(x,\omega) = \frac{L_2\{y : \chi(l_\omega + y) \leq x\}}{s_D(\omega)}. \tag{1.5}$$

Нетрудно убедиться, что для $n = 3$ формула (1.2) эквивалентна

$$-\frac{\partial C_D(t\omega)}{\partial t} = s_D\,(\omega)\,(1 - F_D\,(t,\omega))\,. \tag{1.6}$$

В статье получены следующие результаты:

(1) Формула связывающая ковариограмму и функцию распределения длины хорды (в направлении) цилиндра с теми же функциями его основания,

(2) Ковариограмма и функция распределения длины хорды в направлении для цилиндров с круговыми, эллиптическими и треугольными основаниями.

Мы также приводим графики зависящих от направления функций распределения длины хорды для цилиндров с круговыми, эллиптическими и треугольными основаниями для некоторых значений ω и замечаем, что эти графики или непрерывны всюду, или имеют скачок только в одной точке.

2. Основная формула

Рассмотрим цилиндр U с основанием B (не обязательно выпуклым) и высотой h. Очевидно, что область $U \cap \{U + x\} \neq \emptyset$ тоже является цилиндром. Если обозначить через t длину вектора x а через $\omega = (\varphi, \theta)$ ((φ, θ) – цилиндрические

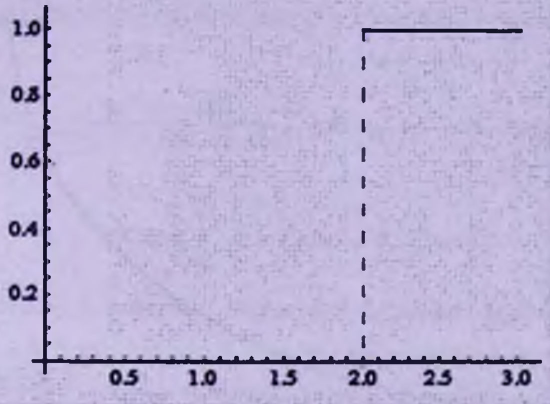

Рис. 3. Функция распределения длины хорды в направлении $\theta = \frac{\pi}{2}$ для кругового цилиндра с радиусом 1 и высотой 2.

координаты точки ω, $\varphi \in S^1$, $\theta \in [-\pi/2, \pi/2]$) направление x, то основанием цилиндра $U \cap \{U + x\}$ будет область $B \cap \{B + y\}$, где y вектор на плоскости длины $t\cos\theta$ и направлением φ, а высота цилиндра будет $h - t\sin\theta$ (ввиду симметрии мы будем рассматривать только случай $\theta \in [0, \pi/2]$). Таким образом, из (1.1) получаем

$$C_U(x) = C_U(t\omega) = L_3(U \cap \{U + t\omega\}) = L_2(B \cap \{B + (t\cos\theta)\varphi\}) \cdot (h - t\sin\theta),$$

следовательно,

$$C_U(t\omega) = (h - t\sin\theta) \cdot C_B((t\cos\theta)\varphi). \tag{2.1}$$

Формула (2.1) дает возможность найти ковариограмму цилиндра высоты h в терминах ковариограммы его основания. Тривиальный случай – это случай кругового цилиндра (обычный цилиндр). В [11] – [13] получены ковариограммы эллипса, правильного многоугольника, треугольника и прямоугольника, следовательно используя (2.1) мы можем найти ковариограммы цилиндров с круговыми, эллиптическими, правильно-многоугольными, треугольными и параллелограммными основаниями.

Дифференцируя по t обе части уравнения (2.1), получаем

$$\frac{\partial C_U(t\omega)}{\partial t} = -\sin\theta C_B((t\cos\theta)\varphi) + (h - t\sin\theta)\frac{\partial C_B((t\cos\theta)\varphi)}{\partial t}. \tag{2.2}$$

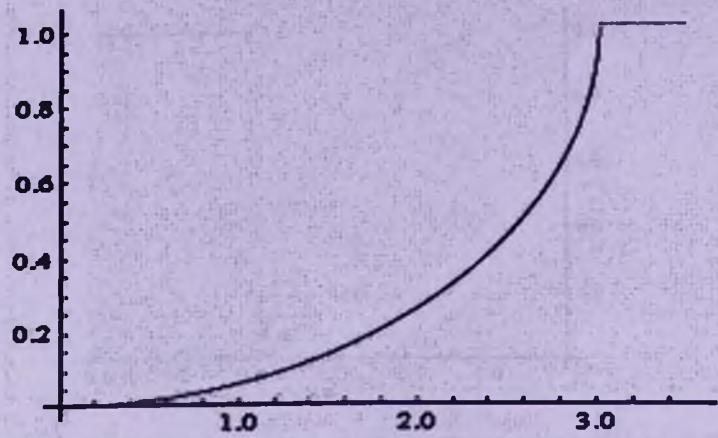

Рис. 4. Функция распределения длины хорды в направлении $\omega = \left(\frac{\pi}{6}, 0\right)$ для эллиптического цилиндра с полуосями 2 и 1 и высотой 3.

Очевидно, что функция распределения длины хорды цилиндра U в направлении ω равна 0, если $t < 0$, и 1, если $t \geq \chi_{\max}(\omega)$, где $\chi_{\max}(\omega)$ максимальная хорда цилиндра в направлении ω. Пусть $t \in [0, \chi_{\max}(\omega))$. Используя (1.2) и (1.6) получаем

(2.3)

$$s_U(\omega)(1 - F_U(t,\omega)) = \sin\theta C_B((t\cos\theta)\varphi) + (h - t\sin\theta)\cos\theta b_B(\varphi)(1 - F_B(t\cos\theta, \varphi)).$$

Интегрируя (1.2) в случае $n = 2$ относительно t, получаем

$$C_B(t\varphi) = ||B|| - b_B(\varphi)\int_0^t (1 - F_B(u, \varphi))\,du, \tag{2.4}$$

где $||B||$ площадь области B.

Из (2.3) и (2.4) выводим

$$s_U(\omega)(1 - F_U(t,\omega)) = \sin\theta\left[||B|| - \cos\theta b_B(\varphi)\int_0^t (1 - F_B(u\cos\theta, \varphi))\,du\right] +$$

$$+(h - t\sin\theta)\cos\theta b_B(\varphi)(1 - F_B(t\cos\theta, \varphi)) = ||B||\sin\theta + \cos\theta b_B(\varphi)\times$$

$$\times\left[(h - t\sin\theta)(1 - F_B(t\cos\theta, \varphi)) - \sin\theta\int_0^t (1 - F_B(u\cos\theta, \varphi))\,du\right]. \tag{2.5}$$

Нетрудно убедиться, что $s_U(\omega) = ||B|| \sin\theta + b_B(\varphi) h \cos\theta$, поэтому из (2.5) получаем

$$F_U(t,\omega) = \frac{b_B(\varphi)\cos\theta}{||B||\sin\theta + b_B(\varphi)h\cos\theta}\Big[t\sin\theta + (h - t\sin\theta)F_B(t\cos\theta,\varphi) +$$

$$+\sin\theta\int_0^t (1 - F_B(u\cos\theta,\varphi))\,du\Big] \tag{2.6}$$

Очевидно, $F_U(t,\omega) = 0$, если $t = 0$. Более того, нетрудно убедиться, что $F_U(t,\omega) = 1$ когда $t = \chi_{\max}(\omega)$ для любого $\theta \leq \arctan\frac{h}{b_B(\varphi)}$ (в этом случае $\chi_{\max}(\omega)\cdot\cos\theta$ максимальная хорда основания в направлении φ) и $F_U(t,\omega) < 1$ в противном случае (это связано с фактом, что все хорды с концами на основании цилиндра равны по длине).

Окончательно, получаем формулу между $F_U(t,\omega)$ и $F_B(t,\varphi)$:

(2.7)

$$F_U(t,\omega) = \begin{cases} 0, & \text{если} \quad t < 0, \\ \frac{b_B(\varphi)\cos\theta}{||B||\sin\theta + b_B(\varphi)h\cos\theta}\Big[(h - t\sin\theta)F_B(t\cos\theta,\varphi) + \\ +t\sin\theta + \sin\theta\int_0^t (1 - F_B(u\cos\theta,\varphi))\,du\Big], & \text{если} \quad 0 \leq t < \chi_{\max}(\omega), \\ 1, & \text{если} \quad t \geq \chi_{\max}(\omega). \end{cases}$$

Таким образом, зависящая от направления функция распределения длины хорды $F_U(t,\omega)$ цилиндра U может иметь скачок, зависящий от направления ω (для $\theta = \frac{\pi}{2}$ скачок равен 1). Такой скачок возникает в точке $\chi_{\max}(\omega)$; функция распределения $F_U(t,\omega)$ непрерывна всюду за исключением точки $\chi_{\max}(\omega)$. Мы иллюстрируем это свойство на Рисунках 1–8, где для некоторых значений направления ω функция распределения $F_U(t,\omega)$ имеет скачок в точке $\chi_{\max}(\omega)$, в то время как для других значений функция $F_U(t,\omega)$ непрерывна всюду. Эту закономерность мы изучаем в следующем параграфе.

3. Частные случаи

3.1. Случай кругового цилиндра. Пусть L_r круговой цилиндр с радиусом основания r и высотой h. Ковариограмма круга радиуса r равна

$$C_r(t,\varphi) = \begin{cases} 2r^2\arccos\frac{t}{2r} - \frac{t}{2}\sqrt{4r^2 - t^2}, & \text{если} \quad 0 \leq t \leq 2r, \\ 1, & \text{в противном случае,} \end{cases}$$

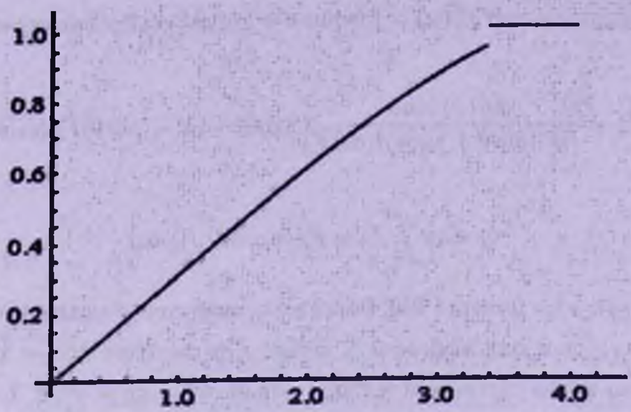

РИС. 5. Функция распределения длины хорды в направлении $\omega = \left(\frac{\pi}{2}, \frac{\pi}{3}\right)$ для эллиптического цилиндра с полуосями 2 и 1 и высотой 3.

следовательно из (2.1) для ковариограммы L_r получаем ([16], Приложение K)

$$C_{L_r}(t,\omega) = \begin{cases} (h - t\sin\theta)\left(2r^2 \arccos\frac{t\cos\theta}{2r} - \right. \\ \left. -\frac{t\cos\theta}{2}\sqrt{4r^2 - t^2\cos^2\theta}\right), & \text{если} \quad 0 \le t \le \chi_{\max}(\omega), \\ 1, & \text{otherwise}, \end{cases} \tag{3.1}$$

где

$$\chi_{\max}(\omega) = \chi_{\max}(\varphi,\theta) = \begin{cases} \frac{2r}{\cos\theta}, & \text{если} \quad \theta \in \left[0, \arctan\frac{h}{2r}\right] \\ \frac{h}{\sin\theta}, & \text{если} \quad \theta \in \left[\arctan\frac{h}{2r}, \frac{\pi}{2}\right]. \end{cases} \tag{3.2}$$

Для функции распределения длины хорды в направлении получаем

$$F_r(t,\varphi) = \begin{cases} 0, & \text{если} \quad t < 0, \\ 1 - \sqrt{1 - \frac{t^2}{4r^2}}, & \text{если} \quad 0 \le t < 2r, \\ 1, & \text{если} \quad t > 2r, \end{cases}$$

следовательно из (2.7) получаем

(3.3)

$$F_{L_r}(t,\omega) = \begin{cases} 0, & \text{если} \quad t < 0, \\ \frac{2}{\pi r\sin\theta + 2h\cos\theta}\left[r\sin\theta \arcsin\frac{t\cos\theta}{2r} + h\cos\theta + \right. \\ \left. + \left(\frac{3t\sin 2\theta}{4} - h\cos\theta\right)\sqrt{1 - \frac{t^2\cos^2\theta}{4r^2}}\right], & \text{если} \quad 0 \le t < \chi_{\max}(\omega), \\ 1, & \text{если} \quad t \ge \chi_{\max}(\omega). \end{cases}$$

Как видно из формулы (3.3) функция распределения $F_{L_r}(t,\omega)$ зависит от ω только через θ-координату направления ω. Это следует из симметрии кругового цилиндра относительно φ-координаты направления ω. По этой причине

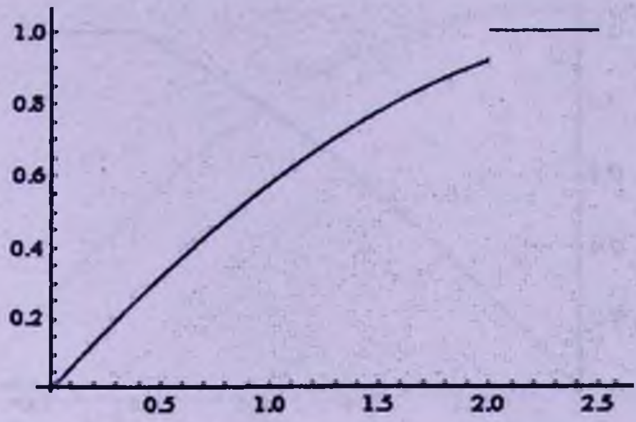

Рис. 6. Функция распределения длины хорды в направлении $\omega = \left(\frac{\pi}{3}, \frac{\pi}{6}\right)$ для правильной призмы со стороной 3 и высотой 1.

на рисунках 1–3 мы рассматриваем только значения θ. Рисунки 1–2 показывают, что функция распределения $F_{L_r}(t,\omega)$ всюду непрерывна для $\omega = (\cdot, \pi/6)$ и $\omega = (\cdot, \pi/4)$, так как для этих случаев $\arctan \frac{h}{2r} = \pi/4$ и $\pi/6 \leq \pi/4$ и $\pi/4 \leq \pi/4$. Когда же θ-координата направления ω превосходит $\pi/4$ возникает скачок в точке $\chi_{\max}(\omega)$, который возрастает при стремлении θ к $\pi/2$. В случае $\theta = \pi/2$, функция распределения $F_{L_r}(t,\omega)$ имеет скачок равный 1 в точке 2 (см. Рис. 3).

3.2. Случай эллиптического цилиндра. Рассмотрим цилиндр L_e высоты h и эллипсом с полуосями a и b в основании. Ковариограмма эллипса с полуосями a и b имеет вид (см. [12]):

$$C_e(t\varphi) = \begin{cases} 2ab\left(\frac{\pi}{2} - \frac{t}{\chi_{\max}(\varphi)}\sqrt{1 - \frac{t^2}{\chi^2_{\max}(\varphi)}} - \arcsin\frac{t}{\chi_{\max}(\varphi)}\right), & \text{если } 0 \leq t \leq \chi_{\max}(\varphi), \\ 0, & \text{в противном случае,} \end{cases}$$

где $\chi_{\max}(\varphi) = \frac{2ab}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}$ максимальная хорда в направлении φ (см. [12]).

Из (2.1) имеем

$$C_{L_e}(t\omega) = \begin{cases} 2ab(h - t\sin\theta)\left(\frac{\pi}{2} - \frac{t\cos\theta}{\chi_{\max}(\varphi)}\sqrt{1 - \frac{t^2\cos^2\theta}{\chi^2_{\max}(\varphi)}}\right. \\ \left. - \arcsin\frac{t\cos\theta}{\chi_{\max}(\varphi)}\right), & \text{если } 0 \leq t \leq \chi_{\max}(\omega), \\ 0, & \text{в противном случае,} \end{cases}$$

где

$$\chi_{\max}(\omega) = \chi_{\max}(\varphi,\theta) = \begin{cases} \frac{\chi_{\max}(\varphi)}{\cos\theta}, & \text{если } \theta \in \left[0, \arctan\frac{h}{\chi_{\max}(\varphi)}\right] \\ \frac{h}{\sin\theta}, & \text{если } \theta \in \left[\arctan\frac{h}{\chi_{\max}(\varphi)}, \frac{\pi}{2}\right]. \end{cases} \tag{3.4}$$

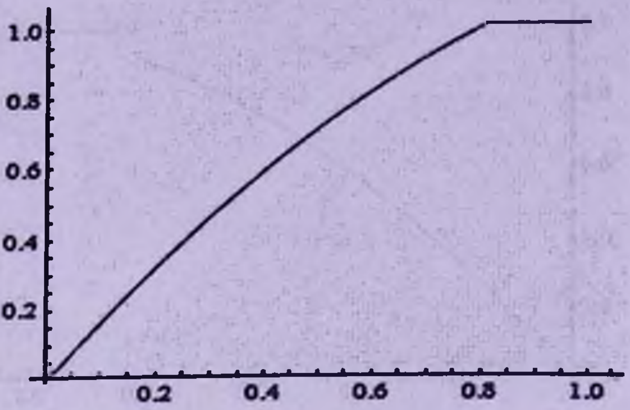

Рис. 7. Функция распределения длины хорды в направлении $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$ для призмы высоты 2 и прямоугольным треугольником с катстом 1 и углом $\frac{\pi}{6}$ в основапии.

Для функции распределения в направлении имеем (см. [12])

$$F_e(t,\varphi) = \begin{cases} 0, & \text{если} \quad t \le 0, \\ 1 - \sqrt{1 - \frac{t^2}{\chi^2_{max}(\varphi)}}, & \text{если} \quad 0 \le t \le \chi_{max}(\varphi), \\ 1, & \text{если} \quad t \ge \chi_{max}(\varphi), \end{cases} \tag{3.5}$$

следовательно из (2.7) получаем

(3.6)

$$F_{L_e}(t,\omega) = \begin{cases} 0, & \text{если} \quad t < 0, \\ \frac{b_e(\varphi)}{\pi ab \sin\theta + h b_e(\varphi)\cos\theta} \left[\frac{1}{2}\chi_{max}(\varphi)\sin\theta \arcsin\frac{t\cos\theta}{\chi_{max}(\varphi)} + \right. & \\ \left. + h\cos\theta + \left(\frac{3t\sin 2\theta}{4} - h\cos\theta\right)\sqrt{1 - \frac{t^2\cos^2\theta}{\chi^2_{max}(\varphi)}}\right], & \text{если} \quad 0 \le t < \chi_{max}(\omega), \\ 1, & \text{если} \quad t \ge \chi_{max}(\omega), \end{cases}$$

где $\chi_{max}(\omega)$ определяется уравнением (3.4) и $b_e(\varphi) = \sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}$. Полагая $a = b = r$ в (3.6) получаем (3.3).

Дальнейшая иллюстрация свойства "скачка" для зависящей от направления функции распределения длины хорды представлена на Рисунках 4–5. На Рис. 4 нет скачков, потому что

$$\theta = 0 < \arctan\frac{3}{\sqrt{2^2\sin^2\pi/6 + 1^2\cos^2\pi/6}} \approx 66.2^\circ,$$

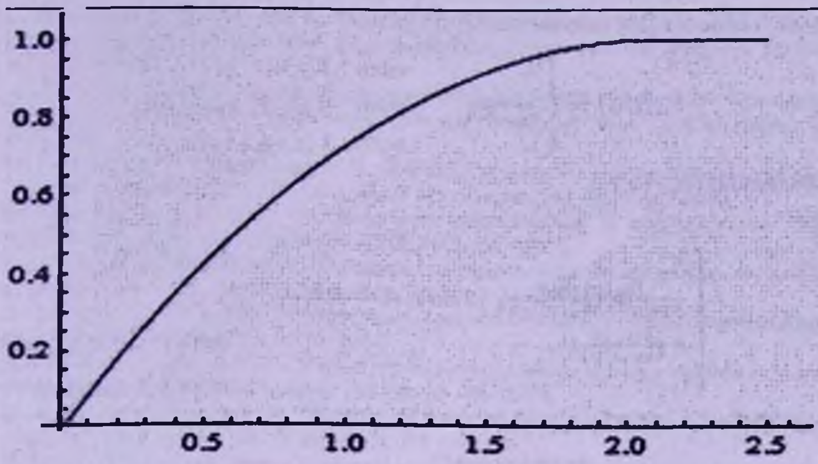

Рис. 8. Функция распределения длины хорды в направлении $(\pi, \frac{\pi}{3})$ для призмы высоты 2 и треугольным основанием со стороной 1 и прилегающими к нему углами $\frac{2\pi}{3}$, $\frac{\pi}{4}$.

а на Рис. 5 имеется скачок в точке $\chi_{\max}(\pi/2, /\pi/3) = \frac{3}{\sin \pi/3} = 2\sqrt{3}$, так как

$$\frac{\pi}{3} > \arctan \frac{3}{\sqrt{2^2 \sin^2 \pi/2 + 1^2 \cos^2 \pi/2}} = \arctan \frac{3}{2}.$$

3.3. Случай призмы. Рассмотрим призму L_Δ с треугольным основанием Δ. Мы полагаем что одна из сторон Δ лежит на оси X. Пусть a длина этой стороны, а α и β – прилегающие к нему углы. В [11] показано, что ковариограмма Δ имеет вид

$$C_\Delta(t\varphi) = \begin{cases} S_\Delta \left(1 - \frac{t}{\chi_{\max}(\varphi)}\right)^2, & \text{если} \quad 0 \le t \le \chi_{\max}(\varphi), \\ 0, & \text{в противном случае,} \end{cases} \tag{3.7}$$

где S_Δ площадь треугольника Δ, а $\chi_{\max}(\varphi)$ определяется следующим образом

$$\chi_{\max}(\varphi) = \begin{cases} \frac{a \sin \beta}{\sin(\varphi+\beta)}, & \text{если} \quad \varphi \in [0, \alpha], \\ \frac{a \sin \alpha \sin \beta}{\sin(\alpha+\beta) \sin \varphi}, & \text{если} \quad \varphi \in [\alpha, \pi - \beta], \\ \frac{a \sin \alpha}{\sin(\varphi - \alpha)}, & \text{если} \quad \varphi \in [\pi - \beta, \pi]. \end{cases} \tag{3.8}$$

Имея в виду (2.1) получаем

$$C_{L_\Delta}(t\omega) = \begin{cases} S_\Delta(h - t \sin \theta) \left(1 - \frac{t \cos \theta}{\chi_{\max}(\varphi)}\right)^2 & \text{если} \quad 0 \le t \le \chi_{\max}(\omega), \\ 0, & \text{в противном случае,} \end{cases} \tag{3.9}$$

где $\chi_{\max}(\omega)$ определяется формулой (3.4).

Далее, снова из [11] имеем

$$F_\Delta(t,\varphi)=\begin{cases}0, & \text{если} \quad t\le 0,\\ \frac{t}{\chi_{\max}(\varphi)}, & \text{если} \quad 0\le t\le \chi_{\max}(\varphi),\\ 1, & \text{если} \quad t\ge \chi_{\max}(\varphi),\end{cases} \tag{3.10}$$

следовательно (см. (2.7))

(3.11)

$$F_{L_\Delta}(t,\omega)=\begin{cases}0, & \text{если } t<0,\\ \frac{tb_\Delta(\varphi)\cos\theta}{S_\Delta\sin\theta+hb_\Delta(\varphi)\cos\theta}\left[2\sin\theta+\frac{h\cos\theta}{\chi_{\max}(\varphi)}-\right. & \\ \left.-\frac{3\sin 2\theta}{4\chi_{\max}(\varphi)}t\right], & \text{если } 0\le t<\chi_{\max}(\omega),\\ 1, & \text{если } t\ge \chi_{\max}(\omega),\end{cases}$$

где (см. [11])

$$b_\Delta(\varphi)=\begin{cases}\frac{a\sin\alpha\sin(\varphi+\beta)}{\sin(\alpha+\beta)}, & \text{если} \quad \varphi\in[0,\alpha],\\ a\sin\varphi, & \text{если} \quad \varphi\in[\alpha,\pi-\beta],\\ \frac{a\sin\beta\sin(\varphi-\alpha)}{\sin(\alpha+\beta)}, & \text{если} \quad \varphi\in[\pi-\beta,\pi].\end{cases}$$

На рисунках 6–8 изображены графики функции распределения $F_{L_\Delta}(t,\omega)$ для некоторых типов треугольников и для конкретных направлений ω. Можно заметить, что общее свойство непрерывности зависящей от направления функции распределения длины хорды, которое было описано в параграфах 2 и 3 также имеет место в этих случаях.

Abstract. In this paper we establish relationships between the covariogram and the orientation-dependent chord length distribution function of a cylinder and those of its base. Also, we obtain explicit expressions for the covariogram and the orientation-dependent chord length distribution function of a cylinder with cyclic, elliptical and triangular bases.

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ON A q-FRACTIONAL VARIANT OF NONLINEAR LANGEVIN EQUATION OF DIFFERENT ORDERS

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Abstract. In this paper we introduce a q-fractional variant of nonlinear Langevin equation of different orders with q-fractional antiperiodic boundary conditions. The nonlinearity in the proposed problem involves an integral term (a Riemann-Liouville type q-integral) and a non-integral term. Some existence results for solutions of the given problem are established by means of classical tools of fixed point theory. An illustrative example is also presented.

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Keywords: Langevin equation; q-fractional; antiperiodic; existence; fixed point.

1. Introduction

Nonlinear boundary value problems of fractional differential equations have received considerable attention in the preceding decades. In the literature one can easily find a variety of results on the topic, ranging from theoretical analysis to asymptotic behavior and numerical methods for fractional equations.

An important feature of a fractional order differential operator, distinguishing it from an integer-order differential operator, is that it is of nonlocal nature and takes into account memory and hereditary properties of some important and useful materials and processes.

The fractional calculus has evolved as an effective mathematical modeling tool in several real world phenomena occurring in physical and technical sciences (see [1]). More details and examples on the topic can be found in the papers [2] and [3] and references therein.

The subject of q-difference equations has gained considerable attention over the years since its inception by Jackson [4]. One of the advantages to consider q-difference equations is that these equations are always completely controllable and appear in the q-optimal control problems (see [5]). For further details, we refer the reader to references [6] and [7].

Fractional q-difference (or q-fractional) equations, regarded as fractional counterparts of q-difference equations, have been studied by a number of authors (see [8] - [10]). For some earlier work on the topic, we refer to [11] and [12], whereas the basic concepts on q-fractional calculus can be found in the recent text [7].

Antiperiodic boundary conditions occur in the mathematical modeling of a variety of problems of applied nature. An account of classical and fractional antiperiodic boundary conditions can be found in the papers [13] and [14] and references therein. However, the concept of fractional q-difference antiperiodic boundary conditions has not been introduced yet.

The Langevin equation involving fractional derivatives of different non-integer orders provides a more flexible model for fractal processes. Some recent results on Langevin equation can be found in the papers [15] and [16]. We recall that the ordinary Langevin equation does not provide correct description of the dynamics of systems in complex media. Notice that Langevin equation involving q-fractional derivatives of different orders has not been studied so far.

The objective of the present paper is to study a new boundary value problem for the q-fractional nonlinear Langevin equation of different orders involving an integral term (a Riemann-Liouville type q-integral) and a non-integral term, with q-fractional antiperiodic boundary conditions. More precisely, for given numbers $0 < \beta < 1$ and $0 < \gamma < 1$, we consider a full q-fractional antiperiodic boundary value problem for the Langevin equation given by

$$(1.1)\qquad {}^cD_q^{\beta}({}^cD_q^{\gamma} + \lambda)x(t) = \rho f(t, x(t)) + \delta I_q^{\zeta} g(t, x(t)),\ \ 0 \le t \le 1,\ 0 < q < 1,$$

$$(1.2)\qquad x(0) = -x(1),\ \ {}^cD_q^{\gamma}x(0) = -{}^cD_q^{\gamma}x(1),$$

where ${}^cD_q^{\beta}$ and ${}^cD_q^{\gamma}$ denote the Caputo type fractional q-derivative, $I_{q,0}^{\zeta}(\cdot) = I_q^{\zeta}(\cdot)$ denotes the Riemann-Liouville integral with $0 < \zeta < 1$, f, g are given continuous functions, $\lambda \neq 0$, and ρ, δ are real constants.

The paper is organized as follows. Section 2 deals with some general concepts and results from q-fractional calculus, as well as an auxiliary lemma for a linear variant of the problem (1.1), (1.2). In Section 3, we present some existence results for solutions of the problem (1.1), (1.2) by applying Krasnoselskii's fixed point theorem, Leray-Schauder alternative and Banach's contraction mapping principle.

2. Preliminaries on fractional q-calculus

In this section we discuss some general concepts and results from q-fractional calculus. We first recall the necessary notation and definitions, and introduce the terminology of q-fractional calculus (see [7, 17, 18]).

For a real parameter $q \in \mathbf{R}^+ \setminus \{1\}$, a q-real number denoted by $[a]_q$ is defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbf{R}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i), \quad k \in \mathbf{N} \cup \{\infty\}.$$

The q-analogue of the exponent $(x-y)^k$ is defined as

$$(x-y)^{(0)} = 1, \quad (x-y)^{(k)} = \prod_{j=0}^{k-1}(x-yq^j), \quad k \in \mathbf{N}, \quad x, y \in \mathbf{R}.$$

The q-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y) = \frac{(1-q)^{(y-1)}}{(1-q)^{y-1}},$$

where $y \in \mathbf{R} \setminus \{0, -1, -2, \ldots\}$. Observe that $\Gamma_q(y+1) = [y]_q\Gamma_q(y)$.

Definition 2.1. *Let f be a function defined on $[0, b]$, $b > 0$ and let $a \in (0, b)$ be an arbitrary fixed point. The Riemann-Liouville type fractional q-integral is defined by*

$$(I_{q,a}^{\beta} f)(t) = \int_a^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) d_q(s), \quad \beta > 0,$$

provided that the integral exists.

Remark 2.1. *The q-fractional integration possesses the semigroup property:*

$$(I_{q,a}^{\gamma} I_{q,a}^{\beta} f)(t) = (I_{q,a}^{\beta+\gamma} f)(t); \quad \gamma, \beta \in \mathbf{R}^+, \quad a \in (0, b).$$

Before giving the definition of fractional q-derivative, we recall the concept of q-derivative. We know that the q-derivative of a function $f(t)$ is defined as

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \quad t \neq 0, \quad (D_q f)(0) = \lim_{t \to 0}(D_q f)(t).$$

Furthermore, we define $D_q^0 f = f$, $D_q^n f = D_q(D_q^{n-1} f)$, $\quad n = 1, 2, \ldots$.

Definition 2.2. *([17]) The Riemann-Liouville type fractional q-derivative of order β of a function $f(t)$ is defined by*

$$(2.1)\qquad (D_{q,a}^{\beta}f)(t)=\begin{cases}(I_{q,a}^{-\beta}f)(t), & \beta<0,\\ f(x), & \beta=0,\\ (D_q^{\lceil\beta\rceil}I_{q,a}^{\lceil\beta\rceil-\beta}f)(t), & \beta>0,\end{cases}$$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Remark 2.2. *The following relations hold (see [18], Lemma 6):*

(i) $(D_{q,a}^{\beta}I_{q,a}^{\beta}f)(t)=f(t),\ 0<a<t.$

(ii) $I_{q,a}^{\beta}\left((x-a)^{(\lambda)}\right)=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)}(x-a)^{(\beta+\lambda)},\ 0<a<x<b,\beta\in\mathbf{R}^+,\lambda\in(-1,\infty).$

Definition 2.3. *([17]) The Caputo type fractional q-derivative of order $\beta\in\mathbf{R}^+$ of a function $f(t)$ is defined by $({}^cD_{q,a}^{\beta}f)(t)=(I_{q,a}^{\lceil\beta\rceil-\beta}D_q^{\lceil\beta\rceil}f)(t)$.*

Remark 2.3. *For $0<a<t$ and $\beta\in\mathbf{R}\setminus\mathbf{N}$, the following relations hold (see [17]):*

(a): $({}^cD_{q,a}^{\beta+1}f)(t)=({}^cD_{q,a}^{\beta}D_qf)(t)$;

(b): $({}^cD_{q,a}^{\beta}I_{q,a}^{\beta}f)(t)=f(t)$;

(c): $(I_{q,a}^{\beta}\,{}^cD_{q,a}^{\beta}f)(t)=f(t)-\sum_{k=0}^{\lceil\beta\rceil-1}\frac{(D_q^kf)(a)}{\Gamma_q(k+1)}t^k(a/t;q)_k$;

To define the solution of the problem (1.1), (1.2), we need the following lemma.

Lemma 2.1. *For a given $h\in C([0,1],\mathbf{R})$, the unique solution of the boundary value problem*

$$(2.2)\qquad \begin{cases}{}^cD_q^{\beta}({}^cD_q^{\gamma}+\lambda)x(t)=h(t),\ \ 0\le t\le 1,\ 0<q<1,\\ x(0)=-x(1),\ \ {}^cD_q^{\gamma}x(0)=-{}^cD_q^{\gamma}x(1)\end{cases}$$

is given by

$$(2.3)\qquad \begin{aligned} x(t)&=\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}h(m)d_qm-\lambda x(u)\Big)d_qu\\ &+\frac{(1-2t^{\gamma})}{4\Gamma_q(\gamma+1)}\int_0^1\frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)}h(u)d_qu\\ &-\frac{1}{2}\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}h(m)d_qm-\lambda x(u)\Big)d_qu.\end{aligned}$$

Proof. Applying the operator I_q^{β} to the q-fractional Langevin equation in (2.2), we get

$$(2.4)\qquad {}^cD_q^{\gamma}x(t)=I_q^{\beta}h(t)-\lambda x(t)-b_0.$$

Next, we apply the operator I_q^{γ} to the both sides of (2.4) with $t \in [0,1]$ to obtain

$$x(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} h(m)d_qm - \lambda x(u)\Big)d_qu - \frac{t^{\gamma}}{\Gamma_q(\gamma+1)}b_0 - b_1. \tag{2.5}$$

Using the boundary conditions (2.2) in (2.5) and solving the resulting system of equations for b_0 and b_1, we get

$$b_0 = \frac{1}{2}\int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} h(u)d_qu,$$

$$b_1 = \frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} h(m)d_qm - \lambda x(u)\Big)d_qu - \frac{1}{4\Gamma_q(\gamma+1)}\int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} h(u)d_qu.$$

Substituting the obtained values of b_0 and b_1 into (2.5) we get (2.3). This completes the proof of Lemma 2.1. □

3. The main results

Let $\mathfrak{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$, endowed with the usual norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$. We use Lemma 2.1 to define an operator $\mathfrak{U} : \mathfrak{C} \to \mathfrak{C}$ by

$$\begin{aligned}(\mathfrak{U}x)(t) &= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\rho\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm \\ &+\delta\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu \\ &+\frac{(1-2t^{\gamma})}{4\Gamma_q(\gamma+1)}\Big(\rho\int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} f(u,x(u))d_qu \\ &+\delta\int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} g(u,x(u))d_qu\Big) \\ &-\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\rho\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm \\ &+\delta\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu.\end{aligned} \tag{3.1}$$

Observe that the problem (1.1), (1.2) has solutions if and only if the operator equation $x = \mathfrak{U}x$ has fixed points. In the sequel, we need the following assumptions:

(A_1) $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that $|f(t,x) - f(t,y)| \le L_1|x-y|$ and $|g(t,x) - g(t,y)| \le L_2|x-y|$, $\forall t \in [0,1]$, $L_1, L_2 > 0$, $x,y \in \mathbb{R}$;

(A_2) There exist $\vartheta_1, \vartheta_2 \in C([0,1], \mathbf{R}^+)$ with $|f(t,x)| \leq \vartheta_1(t)$, $|g(t,x)| \leq \vartheta_2(t)$, $\forall (t,x) \in [0,1] \times \mathbf{R}$, where $\|\vartheta_i\| = \sup_{t\in[0,1]} |\vartheta_i(t)|$, $i = 1,2$.

For the sake of brevity, we introduce the following quantities:

$$\begin{aligned} \mu_1 &= \frac{3}{2\Gamma_q(\beta+\gamma+1)} + \frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+1)}, \\ \mu_2 &= \frac{3}{2\Gamma_q(\beta+\zeta+\gamma+1)} + \frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+\zeta+1)}, \\ \mu_3 &= \frac{3}{2\Gamma_q(\gamma+1)}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Omega = L\Big[|\rho|\Big(&\frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+1)} + \frac{1}{2\Gamma_q(\beta+\gamma+1)}\Big) \\ +|\delta|\Big(&\frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+\zeta+1)} + \frac{1}{2\Gamma_q(\beta+\zeta+\gamma+1)}\Big)\Big] + \frac{|\lambda|}{2\Gamma_q(\gamma+1)}, \end{aligned} \tag{3.3}$$

where $L = \max\{L_1, L_2\}$.

Our first existence result is based on the Krasnoselskii's fixed point theorem ([19]).

Lemma 3.1 (Krasnoselskii). *Let Y be a closed, convex, bounded and nonempty subset of a Banach space X, and let $\mathcal{S}_1, \mathcal{S}_2$ be operators such that*

(i) $\mathcal{S}_1 x + \mathcal{S}_2 y \in Y$ whenever $x, y \in Y$;

(ii) $\mathcal{S}_1$ is compact and continuous;

(iii) $\mathcal{S}_2$ is a contraction mapping.

Then there exists $z \in Y$ such that $z = \mathcal{S}_1 z + \mathcal{S}_2 z$.

Theorem 3.1. *Let $f, g : [0,1]\times\mathbf{R} \to \mathbf{R}$ be continuous functions satisfying assumptions (A_1) and (A_2). Furthermore let $\Omega < 1$, where Ω is given by (3.3). Then the problem (1.1), (1.2) has at least one solution on $[0,1]$.*

Proof. With μ_1, μ_2, μ_3 given by (3.2), we consider the set $B_r = \{x \in \mathfrak{C} : \|x\| \leq r\}$, where

$$r \geq \frac{|\rho|\|\vartheta_1\|\mu_1 + |\delta|\|\vartheta_2\|\mu_2}{1 - |\lambda|\mu_3}.$$

Now we show that the conditions of Lemma 3.1 are satisfied. To this end, we define the operators $\mathcal{U}_1$ and $\mathcal{U}_2$ on B_r by

$$\begin{aligned} (\mathcal{U}_1 x)(t) &= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\rho \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm \\ &+\delta \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu, \quad t \in [0,1], \\ (\mathcal{U}_2 x)(t) &= \frac{(1-2t^\gamma)}{4\Gamma_q(\gamma+1)}\Big(\rho \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} f(u,x(u))d_qu \end{aligned}$$

$$+\delta\int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}g(u,x(u))d_qu\Big)$$

$$-\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\rho\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_qm$$

$$+\delta\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}g(m,x(m))d_qm-\lambda x(u)\Big)d_qu,\quad t\in[0,1].$$

For $x,y\in B_r$, we find that

$$\|\mathcal{U}_1x+\mathcal{U}_2y\|\le|\rho|\|\vartheta_1\|\mu_1+|\delta|\|\vartheta_2\|\mu_2+|\lambda|r\mu_3\le r,$$

implying that $\mathcal{U}_1x+\mathcal{U}_2y\in B_r$. It is clear that the continuity of the operator $\mathcal{U}_1$ follows from that of f and g. Also, observe that $\mathcal{U}_1$ is uniformly bounded on B_r since

$$\|\mathcal{U}_1x\|\le\frac{|\rho|\|\vartheta_1\|}{\Gamma_q(\beta+\gamma+1)}+\frac{|\delta|\|\vartheta_2\|}{\Gamma_q(\beta+\zeta+\gamma+1)}+\frac{|\lambda|r}{\Gamma_q(\gamma+1)}.$$

Next, we show the compactness of the operator $\mathcal{U}_1$. In view of (A_1), we set

$$\sup_{(t,x)\in[0,1]\times B_r}|f(t,x)|=f_1,\qquad \sup_{(t,x)\in[0,1]\times B_r}|g(t,x)|=g_1.$$

Hence, we have

$$\|(\mathcal{U}_1x)(t_2)-(\mathcal{U}_1x)(t_1)\|\le\int_0^{t_1}\frac{(t_2-qu)^{(\gamma-1)}-(t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|\rho|f_1\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm$$

$$+|\delta|g_1\int_0^u\frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}d_qm+|\lambda|r\Big)d_qu+\int_{t_1}^{t_2}\frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\times$$

$$\times\Big(|\rho|f_1\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm+|\delta|g_1\int_0^u\frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}d_qm+|\lambda|r\Big)d_qu,$$

which is independent of x and tends to zero as $t_2\to t_1$. Thus, $\mathcal{U}_1$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{U}_1$ is compact on B_r.

Now, we show that $\mathcal{U}_2$ is a contraction. In view of (A_1), for $x, y \in B_r$ we can write

$$\|\mathcal{U}_2 x - \mathcal{U}_2 y\| \leq \sup_{t\in[0,1]} \Bigg\{ \frac{1}{4\Gamma_q(\gamma+1)} \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(u,x(u)) - f(u,y(u)) \Big| d_q u$$
$$+ |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(u,x(u)) - g(u,y(u)) \Big| d_q u \Big)$$
$$+ \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m,x(m)) - f(m,y(m)) \Big| d_q m$$
$$+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(m,x(m)) - g(m,y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Bigg\}$$
$$\leq \sup_{t\in[0,1]} \Bigg\{ \frac{1}{4\Gamma_q(\gamma+1)} \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} L_1 |x(u) - y(u)| d_q u$$
$$+ |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} L_2 |x(u) - y(u)| d_q u \Big)$$
$$+ \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} L_1 |x(m) - y(m)| d_q m$$
$$+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} L_2 |x(m) - y(m)| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Bigg\}$$
$$\leq \Bigg[L \Big[|\rho| \Big(\frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+1)} + \frac{1}{2\Gamma_q(\beta+\gamma+1)} \Big)$$
$$+ |\delta| \Big(\frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+\zeta+1)} + \frac{1}{2\Gamma_q(\beta+\zeta+\gamma+1)} \Big) \Big] + \frac{|\lambda|}{2\Gamma_q(\gamma+1)} \Bigg] \|x - y\|$$
$$= \Omega \|x - y\|,$$

where we have used (3.3). Hence, taking into account that by our assumption $\Omega < 1$, we conclude that $\mathcal{U}_2$ is a contraction mapping. Thus all the assumptions of Lemma 3.1 are satisfied. So the conclusion of Lemma 3.1 applies and the problem (1.1), (1.2) has at least one solution on $[0, 1]$. This completes the proof of Theorem 3.1. □

The second existence result is based on the Leray-Schauder alternative (see [20]).

Lemma 3.2. *(A nonlinear alternative for single valued maps). Let E be a Banach space, C be a closed, convex subset of E, and V be an open subset of C with $0 \in V$. Suppose that $\mathcal{U} : \overline{V} \to C$ is a continuous, compact (that is, $\mathcal{U}(\overline{V})$ is a relatively compact subset of C) map. Then either $\mathcal{U}$ has a fixed point in $\overline{V}$, or there is a $x \in \partial V$ (the boundary of V in C) and $\kappa \in (0,1)$ with $x = \kappa\mathcal{U}(x)$.*

Theorem 3.2. *Let $f, g : [0,1] \times \mathbf{R} \to \mathbf{R}$ be continuous functions and let the following assumptions be fulfilled:*

(A_3) *There exist functions $\nu_1, \nu_2 \in C([0,1], \mathbf{R}^+)$, and nondecreasing functions $\psi_1, \psi_2 : \mathbf{R}^+ \to \mathbf{R}^+$ such that $|f(t,x)| \leq \nu_1(t)\psi_1(\|x\|)$ and $|g(t,x)| \leq \nu_2(t)\psi_2(\|x\|)$ $\forall (t,x) \in [0,1] \times \mathbf{R}$.*

(A_4) *There exists a constant $\omega > 0$ such that*

$$\omega > \frac{|\rho|\|\nu_1\|\psi_1(\omega)\mu_1 + |\delta|\|\nu_2\|\psi_2(\omega)\mu_2}{1 - |\lambda|\mu_3}, \quad \text{where} \quad |\lambda| \neq \frac{1}{\mu_3}.$$

Then the boundary value problem (1.1), (1.2) *has at least one solution on* $[0,1]$.

Proof. Consider the operator $\mathcal{U} : \mathcal{C} \to \mathcal{C}$ defined by (3.1). It is easy to show that $\mathcal{U}$ is continuous. We complete the proof in the following steps.

(i) $\mathcal{U}$ maps bounded sets into bounded sets in $C([0,1], \mathbf{R})$. Indeed, for a positive number ε, let $B_\varepsilon = \{x \in \mathcal{C} : \|x\| \leq \varepsilon\}$ be a bounded set in $C([0,1], \mathbf{R})$. Since $|f(m, x(m))| \leq \nu_1(m) \cdot \psi_1(\|x\|)$ and $|g(m, x(m))| \leq \nu_2(m) \cdot \psi_2(\|x\|)$, we have

$$\|(\mathcal{U}x)\| \leq \sup_{t \in [0,1]} \Bigg\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m,x(m))| d_q m$$

$$+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} |g(m,x(m))| d_q m + |\lambda||x(u)| \Big) d_q u + \frac{1}{4\Gamma_q(\gamma+1)} \times$$

$$\times \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} |f(u,x(u))| d_q u + |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} |g(u,x(u))| d_q u \Big) +$$

$$\text{(3.4)} \qquad \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m,x(m))| d_q m$$

$$+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} |g(m,x(m))| d_q m + |\lambda||x(u)| \Big) d_q u \Bigg\}$$

$$\leq |\rho|\|\nu_1\|\psi_1(\|x\|)\mu_1 + |\delta|\|\nu_2\|\psi_2(\|x\|)\mu_2 + |\lambda|\|x\|\mu_3,$$

and the result follows.

(ii) $\mathcal{U}$ maps bounded sets into equicontinuous sets of $C([0,1], \mathbf{R})$.

Indeed, let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\varepsilon$, where B_ε is a bounded set of

$C([0,1],\mathbb{R})$. Then we can write

$$
\begin{aligned}
&\|(\mathcal{U}x)(t_2)-(\mathcal{U}x)(t_1)\| \\
\leq\ & \Bigg| \int_0^{t_1} \frac{(t_2-qu)^{(\gamma-1)}-(t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \nu_1(m)\psi_1(\varepsilon) d_q m \\
+\ & |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \nu_2(m)\psi_2(\varepsilon) d_q m + |\lambda|\varepsilon \Big) d_q u \\
+\ & \int_{t_1}^{t_2} \frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \nu_1(m)\psi_1(\varepsilon) d_q m \\
+\ & |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \nu_2(m)\psi_2(\varepsilon) d_q m + |\lambda|\varepsilon \Big) d_q u \Bigg| \\
+\ & \frac{(t_2^\gamma - t_1^\gamma)}{2\Gamma_q(\gamma+1)} \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} \nu_1(u)\psi_1(\varepsilon) d_q u \\
+\ & |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \nu_2(u)\psi_2(\varepsilon) d_q u \Big).
\end{aligned}
$$

It is clear that the right hand side of the above inequality tends to zero independently of $x \in B_\varepsilon$ as $t_2 - t_1 \to 0$. Since $\mathcal{U}$ satisfies the above assumptions, it follows from the Arzelá-Ascoli theorem that $\mathcal{U} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

(iii) Let x be a solution and $x = \kappa \mathcal{U} x$ for $\kappa \in (0,1)$. Using the arguments of the proof of boundedness of $\mathcal{U}$, for $t \in [0,1]$ we can write

$$|x(t)| = |\kappa(\mathcal{U}x)(t)| \leq |\rho| \|\nu_1\| \psi_1(\|x\|)\mu_1 + |\delta| \|\nu_2\| \psi_2(\|x\|)\mu_2 + |\lambda| \|x\| \mu_3.$$

Consequently, we have

$$\|x\| \leq \frac{|\rho| \|\nu_1\| \psi_1(\|x\|)\mu_1 + |\delta| \|\nu_2\| \psi_2(\|x\|)\mu_2}{1 - |\lambda|\mu_3}.$$

In view of (A_4), there exists ω such that $\|x\| \neq \omega$. We set

$$V = \{x \in \mathcal{C} : \|x\| < \omega\},$$

and observe that the operator $\mathcal{U} : \overline{V} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of V, there is no $x \in \partial V$ such that $x = \kappa \mathcal{U}(x)$ for some $\kappa \in (0,1)$. Consequently, we can apply Lemma 3.2, a nonlinear Leray-Schauder type alternative, to conclude that $\mathcal{U}$ has a fixed point $x \in \overline{V}$ which is a solution of the problem (1.1), (1.2). This completes the proof of Theorem 3.2. □

Now we are going to prove the uniqueness of solutions of problem (1.1), (1.2), using Banach's contraction principle (that is, Banach fixed point theorem).

Theorem 3.3. *Suppose that the assumption (A_1) holds and that*

$$\overline{\Omega} = (L\Lambda + |\lambda|\mu_3) < 1, \quad \Lambda = |\rho|\mu_1 + |\delta|\mu_2, \tag{3.5}$$

where μ_1, μ_2 and μ_3 are given by (3.2) and $L = \max\{L_1, L_2\}$. Then the boundary value problem (1.1), (1.2) has a unique solution.

Proof. Define $N = \max\{N_1, N_2\}$, where N_1 and N_2 are finite numbers given by $N_1 = \sup_{t\in[0,1]} |f(t,0)|$ and $N_2 = \sup_{t\in[0,1]} |g(t,0)|$. Selecting $\sigma \geq \frac{N\Lambda}{1-\overline{\Omega}}$, we show that $\mathcal{U}B_\sigma \subset B_\sigma$, where $B_\sigma = \{x \in \mathcal{C} : \|x\| \leq \sigma\}$. Indeed, using the inequalities

$$|f(s,x(s)) \leq |f(s,x(s)) - f(s,0)| + |f(s,0)| \leq L_1\sigma + N_1,$$

and

$$|g(s,x(s))| \leq |g(s,x(s)) - g(s,0)| + |g(s,0)| \leq L_2\sigma + N_2$$

for $x \in B_\sigma$, we can write (see (3.4))

$$\begin{aligned}
\|(\mathcal{U}x)\| \leq\ & |\rho|(L_1\sigma + N_1) \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_q m \right) d_q u \right. \\
& + \frac{1}{4\Gamma_q(\gamma+1)} \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} d_q u \\
& \left. + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_q m \right) d_q u \right\} \\
& + |\delta|(L_2\sigma + N_2) \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} d_q m \right) d_q u \right. \\
& + \frac{1}{4\Gamma_q(\gamma+1)} \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} d_q u \\
& \left. + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} d_q m \right) d_q u \right\} \\
& + |\lambda|\sigma \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u \right\} \\
\leq\ & (L\sigma + N)\Lambda + |\lambda|\sigma\mu_3 \leq \sigma,
\end{aligned}$$

showing that $\mathcal{U}B_\sigma \subset B_\sigma$.

Next, for $x, y \in \mathfrak{C}$, we have

$$\begin{aligned}
&\|\mathfrak{U}x - \mathfrak{U}y\| \\
&\leq \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m \\
&+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(m, x(m)) - g(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \\
&+ \frac{1}{4\Gamma_q(\gamma+1)} \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(u, x(u)) - f(u, y(u)) \Big| d_q u \\
&+ |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(u, x(u)) - g(u, y(u)) \Big| d_q u \Big) \\
&+ \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m \\
&+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(m, x(m)) - g(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Bigg\} \\
&\leq \overline{\Omega} \|x - y\|.
\end{aligned}$$

Taking into account that by our assumption $\overline{\Omega} < 1$,, we conclude that the operator $\mathfrak{U}$ is a contraction. Therefore, by Banach's contraction principle, the problem (1.1), (1.2) has a unique solution. This completes the proof of Theorem 3.3. □

4. An Example

Consider a boundary value problem for integro-differential equations of fractional order given by

$$(4.1) \quad \begin{cases} {}^cD_q^{1/2}({}^cD_q^{1/2} + \frac{1}{16})x(t) = \frac{1}{3}f(t, x(t)) + \frac{1}{7}I_q^{1/2}g(t, x(t)), \quad t, q \in (0,1), \\ x(0) = -x(1), \quad {}^cD_q^{\gamma}x(0) = -{}^cD_q^{\gamma}x(1), \end{cases}$$

where $f(t,x) = \frac{1}{(4+t^2)^2}\Big(\sin t + \frac{|x|}{1+|x|} + |x|\Big)$ and $g(t,x) = \frac{1}{4}\tan^{-1} x + t^3 + 6$.

It is clear that

$$|f(t,x) - f(t,y)| \leq \frac{1}{8}|x-y|, \quad |g(t,x) - g(t,y)| \leq \frac{1}{4}|x-y|.$$

With $\beta = \gamma = \zeta = q = 1/2$, $\lambda = 1/16$, $p = 1/3$, $k = 1/7$, $L_1 = 1/8$, $L_2 = 1/4$, we find that $\overline{\Omega} \simeq 0.2905925472 < 1$.

Clearly $L = \max\{L_1, L_2\} = 1/4$. Thus all the assumptions of Theorem 3.3 are satisfied. Hence, by the conclusion of Theorem 3.3, the problem (4.1) has a unique solution.

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ОБ ОДНОМ ОБОБЩЕНИИ ОБЩЕЙ СИСТЕМЫ ФРАНКЛИНА НА R

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Аннотация. Определяется общая система Франклина на R, порожденная допустимой последовательностью $\mathcal{T}$. Для таких систем доказываются теоремы о локально равномерной сходимости рядов Фурье-Франклина на R и безусловная базисность системы в пространстве $L^p(R)$, $1 < p < \infty$.

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1. Введение

Классическая система Франклина, как первый пример ортонормированного базиса в $C[0;1]$, была определена в [5].

Определение 1.1. *Последовательность (разбиение)* $\mathcal{T} = \{t_n : n \geq 0\}$ *называется допустимой на* $[0;1]$, *если* $t_0 = 0$, $t_1 = 1$, $t_n \in (0;1)$, $n \geq 2$, $\mathcal{T}$ *всюду плотно в* $[0;1]$ *и каждая точка* $t \in (0;1)$ *встречается в* $\mathcal{T}$ *не более чем два раза.*

Пусть $\mathcal{T} = \{t_n : n \geq 0\}$ допустимая последовательность. Для $n \geq 2$ обозначим $\mathcal{T}_n = \{t_i : 0 \leq i \leq n\}$. Допустим π_n получается из $\mathcal{T}_n$ неубывающей перестановкой: $\pi_n = \{\tau_i^n : \tau_i^n \leq \tau_{i+1}^n, 0 \leq i \leq n-1\}$, $\pi_n = \mathcal{T}_n$. Через S_n обозначим пространство функций определенных на $[0;1]$, непрерывных слева и линейных на $(\tau_i^n; \tau_{i+1}^n)$ и непрерывных в τ_i^n, если $\tau_{i-1}^n < \tau_i^n < \tau_{i+1}^n$. Ясно, что $\dim S_n = n+1$ и $S_{n-1} \subset S_n$. Следовательно существует единственная (с точностью до знака) функция $f \in S_n$, ортогональная S_{n-1} и $\|f\|_2 = 1$. Эту функцию называют n-ой функцией Франклина, соответствующей разбиению $\mathcal{T}$. Известно, что $f(t_n) \neq 0$. Поэтому полагается $f(t_n) > 0$.

Определение 1.2. *Общая система Франклина* $\{f_n(x) : n \geq 0\}$ *соответствующая разбиению* $\mathcal{T}$ *определяется по правилу* $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x-1)$ *и*

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для $n \geq 2$ функция $f_n(x)$ есть n-ая функция Франклина, соответствующая разбиению $\mathcal{T}$.

При $t_n = \frac{2m-1}{2^k}$, где $n = 2^k + m$, $k = 0, 1, \ldots$, $m = 1, \ldots, 2^k$, получается классическая система Франклина [5]. Общая система Франклина исследовалась во многих работах. О некоторых свойствах этой системы, полученных в работах [1]-[3], мы укажем по мере необходимости.

В настоящей работе мы введем и изучим одно обобщение этой системы на R. При определении этой системы мы используем те же буквы, что и при определении общей системы Франклина. Однако это не приведет к путанице, поскольку об общей системы Франклина на $[0;1]$ в буквенных обозначениях мы больше говорить не будем.

Определение 1.3. *Последовательность (разбиение) $\mathcal{T} = \{t_n : n \geq 0\}$ назовем допустимой на R, если $\mathcal{T}$ всюду плотно в R и каждая точка $t \in R$ встречается в $\mathcal{T}$ не более одного раза.*

Пусть $\mathcal{T} = \{t_n : n \geq 0\}$ допустимая последовательность. Для $n \geq 2$ обозначим $\mathcal{T}_n = \{t_i : 0 \leq i \leq n+1\}$. Допустим π_n получается из $\mathcal{T}_n$ неубывающей перестановкой: $\pi_n = \{\tau_i^n : \tau_i^n < \tau_{i+1}^n, 0 \leq i \leq n\}$, $\pi_n = \mathcal{T}_n$. Через S_n обозначим пространство функций определенных и непрерывных на R, линейных на $[\tau_i^n; \tau_{i+1}^n]$ и равных нулю вне $(\tau_0^n; \tau_{n+1}^n)$. Ясно, что $\dim S_n = n$ и $S_{n-1} \subset S_n$. Следовательно, существует единственная (с точностью до знака) функция $f \in S_n$, ортогональная S_{n-1} и $\|f\|_2 = 1$. Эту функцию назовем n-ой функцией Франклина на R, соответствующей разбиению $\mathcal{T}$. Для фиксированного n через N_i^n, $0 \leq i \leq n+1$, обозначим B-сплайны соответствующие π_n, т.е.

$$N_0^n(t) = \begin{cases} 1, \text{ когда } & t = \tau_0^n, \\ 0, \text{ когда } & t \in (-\infty; \tau_0^n) \cup [\tau_1^n; \infty), \\ \text{линейная на } & [\tau_0^n; \tau_1^n], \end{cases}$$

$$N_i^n(t) = \begin{cases} 1, \text{ когда } & t = \tau_i^n, \\ 0, \text{ когда } & t \in (-\infty; \tau_{i-1}^n] \cup [\tau_{i+1}^n; \infty), \\ \text{линейная на } & [\tau_{i-1}^n; \tau_i^n] \text{ и } [\tau_i^n; \tau_{i+1}^n], \end{cases} \qquad 1 \leq i \leq n.$$

$$N_{n+1}^n(t) = \begin{cases} 1, \text{ когда } & t = \tau_{n+1}^n, \\ 0, \text{ когда } & t \in (-\infty; \tau_n^n] \cup (\tau_{n+1}^n; \infty), \\ \text{линейная на } & [\tau_n^n; \tau_{n+1}^n]. \end{cases}$$

Ясно, что $N_i^n \in S_n$, $1 \leq i \leq n$, образуют базис этого пространства, причем $N_i^n \notin S_n$, когда $i = 0$ или $n + 1$.

Определение 1.4. *Общая система Франклина $\{f_n(x) : n \geq 1\}$ соответствующая разбиению $\mathcal{T}$ определяется по правилу $f_1(x) = \frac{1}{\|N_1^1\|_2} N_1^1(x)$ и для $n \geq 2$ функция $f_n(x)$ есть n-ая функция Франклина, соответствующая разбиению $\mathcal{T}$.*

При исследовании общей системы Франклина на $[0;1]$ важную роль сыграли понятия регулярности последовательности $\mathcal{T}$. Эти понятия нам нужны также при изучении системы Франклина на R. Введем понятия регулярности на R.

Определение 1.5. *Допустимая последовательность $\mathcal{T}$ называется сильно регулярной с параметром γ, если*

$$\gamma^{-1} \leq \frac{\lambda_{i+1}^n}{\lambda_i^n} \leq \gamma, \quad \text{для всех} \ \ n \geq 2, \quad i = 1, ..., n,$$

здесь и далее $\lambda_i^n = \tau_i^n - \tau_{i-1}^n$.

Определение 1.6. *Допустимая последовательность $\mathcal{T}$ называется регулярной по парам с параметром γ, если*

$$\gamma^{-1} \leq \frac{\lambda_{i+2}^n + \lambda_{i+1}^n}{\lambda_{i+1}^n + \lambda_i^n} \leq \gamma, \quad \text{для всех} \ \ n \geq 2, \quad i = 1, 2, ..., n-1.$$

Так как последовательность $\mathcal{T}$ всюду плотна на R, то система $\{f_n\}_{n=1}^{\infty}$ является полной ортонормированной системой в $L^2(R)$. Введем обозначения.

Через $c, C, C_1, C_\gamma, ...$, обозначаются постоянные, зависящие только от своих индексов. Значения этих постоянных в разных формулах могут быть разными.

$|I|$ – длина отрезка I.

$\rho(t, I)$ – расстояние между точкой t и интервалом I.

Через $d_n(t, x)$ обозначим количество точек из $\mathcal{T}_n$, которые находятся между точками t и x. В случае $x = t_n$, вместо $d_n(t, t_n)$ будем писать $d_n(t)$, т.е. $d_n(t)$-количество точек из $\mathcal{T}_n$, которые находятся между точками t_n и t. Запись $a \sim b$ означает, что существуют положительные постоянные c и C, такие что $c \cdot a \leq b \leq C \cdot a$, а запись $a \sim_\gamma b$ означает, что эти постоянные могут зависеть от γ.

Через $\chi_A(x)$, $\mu(A)$ и A^c будем обозначать характеристическую функцию, лебеговую меру и дополнение множества A, соответственно.

Статья имеет следующую структуру: В §2 получены некоторые оценки для ядра Дирихле общей системы Франклина. В §3 доказаны теоремы о локально равномерной сходимости рядов Фурье-Франклина. В §4 изучаются свойства функции Франклина и безусловная базисность системы Франклина.

2. Оценки для ядра Дирихле общей системы Франклина

Через $K_n(t,\tau)$ обозначим ядро Дирихле общей системы Франклина, т.е. $K_n(t,\tau) = \sum_{k=1}^{n} f_n(t)f_n(\tau)$. В этом разделе изучим ядро $K_n(t,\tau)$ и применяя полученные оценки для $K_n(t,\tau)$, докажем теоремы о равномерной локальной сходимости рядов Фурье-Франклина. Нам пригодятся следующие свойства, легко проверяемые прямыми вычислениями.

Свойство 2.1. *Для всех n, $0 \le i \le j \le n+1$ имеем*

$$\int_R N_i^n(t)N_j^n(t)dt = \begin{cases} 0, & \text{когда } j > i+1, \\ \frac{\lambda_{i+1}^n}{6}, & \text{когда } j = i+1, \\ \frac{\lambda_i^n + \lambda_{i+1}^n}{3}, & \text{когда } j = i, \end{cases}$$

где полагается $\lambda_0^n = \lambda_{n+2}^n = 0$.

Свойство 2.2. *Пусть $f \in S_n$ и $f(t) = \sum_{i=1}^{n} a_i N_i^n(t)$. Тогда для всех $1 \le p < \infty$ выполняются*

$$\left(\frac{1}{p+1}\right)^{1/p} \cdot \left(\sum_{i=1}^{n} |a_i|^p \nu_i^n\right)^{1/p} \le \|f\|_p \le \left(\sum_{i=1}^{n} |a_i|^p \nu_i^n\right)^{1/p},$$

где $\nu_i^n = (\lambda_i^n + \lambda_{i+1}^n)/2$.

Прежде чем изучать n-ую функцию Франклина f_n, получим основные свойства ядра $K_n(t,\tau)$.

Лемма 2.1. *Для всех n и t имеем $\int_R |K_n(t,\tau)|d\tau \le 3$.*

Доказательство. Дословно применим метод из [6] для оценивания нормы проектора из $L^\infty(R)$ в S_n. Пусть $g \in L^\infty(R)$ и $P \in S_n$ ее проекция, т.е. $P(t) = \int_R K_n(t,\tau)g(\tau)d\tau$. Допустим $P(t) = \sum_{i=1}^{n} b_i N_i^n(t)$ и $|b_j| = \max_{1\le i\le n}|b_i| = \|P\|_\infty$. Тогда

$$\frac{\lambda_j^n + \lambda_{j+1}^n}{2} \cdot \|g\|_\infty \ge \left|\int_R g(t)N_j^n(t)dt\right| = \left|\int_R P(t)N_j^n(t)dt\right| =$$

$$\left|b_{j-1}\frac{\lambda_j^n}{6} + b_j\frac{\lambda_j^n + \lambda_{j+1}^n}{3} + b_{j+1}\frac{\lambda_{j+1}^n}{6}\right| \ge \frac{\lambda_j^n + \lambda_{j+1}^n}{6} \cdot \|P\|_\infty.$$

Отсюда, учитывая, что норма проектора равна $\sup_t \int_R |K_n(t,\tau)|d\tau$, получим утверждение леммы 2.1.

Ясно, что если $g \in S_n$, то $g(t) = \int_R K_n(t,\tau)g(\tau)d\tau$. Поэтому имеем

$$\int_R K_n(t,\tau)N_i^n(\tau)d\tau = N_i^n(t), \; i = 1, 2, ..., n,$$

которое, с учетом линейности функции $K_n(t,\tau)$ по каждой переменной на отрезках $[\tau_i^n;\tau_{i+1}^n]$, равносильно

$$\int_R K_n(\tau_k^n,\tau)N_i^n(\tau)d\tau = N_i^n(\tau_k^n) = \delta_{ik}, \text{ когда } i=1,2,...,n,\ k=1,2,...,n. \tag{2.1}$$

Учитывая линейность функции $K_n(\cdot,\tau)$ на интервалах $[\tau_i^n;\tau_{i+1}^n]$, получим

$$K_n(t,\tau) = \sum_{k=1}^n N_k^n(t)K_n(\tau_k^n,\tau).$$

В доказательстве следующей леммы вместо K_n, τ_i^n, λ_i^n будем писать K, τ_i, λ_i.

Лемма 2.2. *Для фиксированного k обозначим $\alpha_i = K(\tau_k,\tau_i)$. Тогда для α_i имеют место*

$$\alpha_i\cdot\alpha_{i+1} < 0, \ \textit{для } i=1,2,...,n-1, \tag{2.2}$$

$$|\alpha_i|\left(2+\frac{3}{2}\cdot\frac{\lambda_i}{\lambda_{i+1}}\right) < |\alpha_{i+1}| < |\alpha_i|\left(2+2\frac{\lambda_i}{\lambda_{i+1}}\right), \ \textit{для } i=1,...,k-1, \tag{2.3}$$

$$|\alpha_i|\left(2+\frac{3}{2}\frac{\lambda_{i+1}}{\lambda_i}\right) < |\alpha_{i-1}| < |\alpha_i|\left(2+2\frac{\lambda_{i+1}}{\lambda_i}\right), \ \textit{для } i=k+1,...,n-1 \tag{2.4}$$

$$\frac{3}{\lambda_k+\lambda_{k+1}} < \alpha_k < \frac{4}{\lambda_k+\lambda_{k+1}}. \tag{2.5}$$

Доказательство. Из (2.1), с применением свойства 2.1 получаем

$$\frac{\alpha_1(\lambda_1+\lambda_2)}{3}+\frac{\alpha_2\lambda_2}{6}=0, \tag{2.6}$$

$$\frac{\alpha_n(\lambda_n+\lambda_{n+1})}{3}+\frac{\alpha_n\lambda_{n+1}}{6}=0, \tag{2.7}$$

$$\frac{\alpha_{i-1}\lambda_i}{6}+\frac{\alpha_i(\lambda_i+\lambda_{i+1})}{3}+\frac{\alpha_{i+1}\lambda_{i+1}}{6}=0, \text{ для } i=2,...,k-1,k+1,...,n-1, \tag{2.8}$$

$$\frac{\alpha_{k-1}\lambda_k}{6}+\frac{\alpha_k(\lambda_k+\lambda_{k+1})}{3}+\frac{\alpha_{k+1}\lambda_{k+1}}{6}=1. \tag{2.9}$$

Из (2.6) и (2.8) для $i=2,...,k-1$ получим

$$\alpha_2 = -\alpha_1\left(2+2\frac{\lambda_1}{\lambda_2}\right),$$

$$\alpha_{i+1} = -\alpha_i\left(2+2\frac{\lambda_i}{\lambda_{i+1}}\right)-\alpha_{i-1}\frac{\lambda_i}{\lambda_{i+1}}.$$

Отсюда, применяя математическую индукцию, получим (2.3) и (2.2) для $i = 1, ..., k-1$. Аналогично, применяя (2.7), (2.8) для $i = k+1, ..., n-1$, получим (2.4) и (2.2) для $i = k, ..., n-1$. В частности, $|\alpha_k| > 2|\alpha_{k-1}|$ и $|\alpha_k| > 2|\alpha_{k+1}|$. Отсюда и из (2.9) следует (2.5). Лемма 2.2 доказана.

Из (2.3) для $i = 1, ..., k-1$ имеем $|\alpha_i| < q \cdot |\alpha_{i+1}| \cdot \frac{\lambda_{i+1}}{\lambda_i + \lambda_{i+1}}$. Здесь и далее $q = \frac{2}{3}$. Отсюда, с учетом (2.5), для $i = 1, ..., k-1$ получим

$$|\alpha_i| < q^{k-i} \cdot \frac{\lambda_{i+1}}{\lambda_i + \lambda_{i+1}} \cdot \frac{\lambda_{i+2}}{\lambda_{i+1} + \lambda_{i+2}} \cdot \ldots \cdot \frac{\lambda_k}{\lambda_{k-1} + \lambda_k} \cdot \frac{4}{\lambda_k + \lambda_{k+1}}. \tag{2.10}$$

Последовательно применяя очевидное неравенство $\frac{b}{(a+b)(b+c)} \leq \frac{1}{a+b+c}$, из (2.10) получим

$$|\alpha_i| < q^{k-i} \frac{4}{\lambda_i + \lambda_{i+1} + \ldots + \lambda_{k+1}}. \tag{2.11}$$

Из (2.11) вытекает следующее утверждение.

Лемма 2.3. *Для всех n и $1 \leq i \leq k \leq n$ имеют место неравенства*

$$|K(\tau_k^n, \tau_i^n)| \leq q^{|k-i|} \frac{4}{|\tau_{k+1}^n - \tau_{i-1}^n|}. \tag{2.12}$$

Лемма 2.4. *Верны следующие оценки*

$$\|K_n(\tau_k^n, \cdot)\|_p \sim (\nu_k^n)^{\frac{1}{p}-1}, \quad \text{для} \ \ 1 \leq p < \infty, \quad k = 1, 2, ..., n, \tag{2.13}$$

существует $\epsilon \in (0; 1)$ такое, что

$$\int_{\tau_{i-1}^n}^{\tau_i^n} |K_n(\tau_k^n, \tau)|^p d\tau \leq \epsilon^p \int_{\tau_i^n}^{\tau_{i+1}^n} |K_n(\tau_k^n, \tau)|^p d\tau, \quad \text{когда} \ \ i = 1, ..., k-1, \tag{2.14}$$

$$\int_{\tau_i^n}^{\tau_{i+1}^n} |K_n(\tau_k^n, \tau)|^p d\tau \leq \epsilon^p \int_{\tau_{i-1}^n}^{\tau_i^n} |K_n(\tau_k^n, \tau)|^p d\tau, \quad \text{когда} \ \ i = k+1, ..., n-1. \tag{2.15}$$

Доказательство. Неравенства (2.14) и (2.15) доказываются аналогично. Докажем (2.14). Для фиксированных n и k, как и при доказательстве леммы 2.2 полагается $\alpha_i = K_n(\tau_k^n, \tau_i^n)$, $\lambda_i = \lambda_i^n$. С учетом $\alpha_i \cdot \alpha_{i+1} < 0$, получим

$$m_{i,p} := \int_{\tau_{i-1}^n}^{\tau_i^n} |K_n(\tau_k^n, \tau)|^p d\tau = \frac{\lambda_i}{p+1} \frac{|\alpha_{i-1}|^{p+1} + |\alpha_i|^{p+1}}{|\alpha_{i-1}| + |\alpha_i|}.$$

Поскольку $p \geq 1$, с учетом выпуклости функции x^p и $|\alpha_{i-1}| < 2|\alpha_i|$, имеем

$$\frac{|\alpha_{i-1}|^{p+1} + |\alpha_i|^{p+1}}{|\alpha_{i-1}| + |\alpha_i|} \geq \left(\frac{|\alpha_{i-1}|^2 + |\alpha_i|^2}{|\alpha_{i-1}| + |\alpha_i|} \right)^p \geq 2^p(\sqrt{2}-1)^p |\alpha_i|^p.$$

С другой стороны $m_{i,p} \leq \frac{\lambda_i}{p+1} |\alpha_i|^p$. Следовательно

$$\frac{m_{i+1,p}}{m_{i,p}} \geq 2^p(\sqrt{2}-1)^p \frac{\lambda_{i+1}}{\lambda_i} \frac{|\alpha_{i+1}|^p}{|\alpha_i|^p}.$$

Отсюда, с применением (2.3) для $\epsilon = \frac{\sqrt{2}+1}{3}$ получим

$$\frac{m_{i+1,p}}{m_{i,p}} \geq 2^p(\sqrt{2}-1)^p \frac{\lambda_{i+1}}{\lambda_i}\left(2+\frac{3}{2}\frac{\lambda_i}{\lambda_{i+1}}\right)^p \geq \frac{3}{4}4^p(\sqrt{2}-1)^p \geq \epsilon^{-p}.$$

Соотношение (2.13) следует из (2.14), (2.15) и (2.5). Лемма 2.4 доказана.

Замечание 2.1. *Пусть $\mathcal{T}$ сильно регулярная последовательность с параметром γ. Тогда нетрудно заметить, что из (2.3)-(2.5) следует, что для $q_\gamma = (2+2\gamma)^{-1}$, выполняются следующие неравенства:*

(2.16) $$|K_n(\tau_i^n, \tau_j^n)| \geq q_\gamma^{|i-j|} \frac{3}{(\lambda_i^n + \lambda_{i+1}^n) \bigwedge (\lambda_j^n + \lambda_{j+1}^n)},$$

где $a \wedge b = \min(a, b)$.

Из леммы 2.3 следует

Лемма 2.5. *Для любого $\delta > 0$ имеет место $\lim\limits_{n\to\infty} \int_{|t-\tau|>\delta} |K_n(t,\tau)|d\tau = 0$.*

3. О локально равномерной сходимости рядов Фурье-Франклина

С применением лемм 2.1 и 2.5 стандартными рассуждениями доказываются следующие теоремы.

Теорема 3.1. *Пусть $\mathcal{T}$ допустимая последовательность и $\{f_n(t)\}_{n=1}^\infty$ соответствующая ей общая система Франклина. Тогда для любой функции $f \in C(R)$, с компактным носителем, частичные суммы $S_n(f,x)$ ряда Фурье по системе $\{f_n(t)\}_{n=1}^\infty$ равномерно сходятся к $f(x)$ на R.*

Теорема 3.2. *Пусть $\mathcal{T}$ допустимая последовательность. Тогда соответствующая ей общая система Франклина $\{f_n(t)\}_{n=1}^\infty$ является базисом в $L^p(R)$, для $1 \leq p < \infty$.*

Стромберг [8] на действительной оси R построил систему из кусочно линейных функций следующим образом. Пусть $R_0 = \{n : n \in N\} \cup \{0\} \cup \{-n/2 : n \in N\}$ и $R_{1/2} = R_0 \cup \{1/2\}$, где N-множество натуральных чисел. Через S_0 и $S_{1/2}$ обозначим множества непрерывных и кусочно линейных функций из $L^2(\mathrm{R})$, соответственно, с узлами из R_0 и $R_{1/2}$. Существует единственная функция $f \in S_{1/2}$ со свойствами: f ортогональна S_0, $\|f\|_2 = 1$ и $f(1/2) > 0$. Далее полагается $f_{jk}(t) = 2^{j/2}f(2^jt-k)$, $j,k \in Z$, где Z -множество целых чисел. Стромберг [8] доказал, что система $\{f_{jk}(t)\}_{j,k\in Z}$ является безусловным базисом в $L^p(R)$, $1 < p < \infty$, и $H^1(R)$. Однако эта система не является базисом в $L^1(R)$, так как $\int_R f_{jk}(t)dt = 0$, $j,k \in Z$. Неизвестно также, существует ли одноиндексная нумерация системы $\{f_{jk}(t)\}_{j,k\in Z}$, при которой частичные суммы ряда Фурье-Стромберга непрерывной функции с компактным носителем равномерно сходятся. Пусть $\varphi(t)$ четная, положительная и возрастающая на $[0;\infty)$ функция. Обозначим

$$C_\varphi(R) = \{f \in C(R) : \ |f(t)| \le c\varphi(t), \text{ для некоторого } c > 0 \text{ и любого } t \in R\}.$$

Последовательность $\mathcal{T}^1$ построим следующим образом. Положим $t_0 = 0$, $t_1 = -1$, $t_2 = 1$. На втором шаге добавим точки $t_3 = -2$, $t_4 = -\frac{1}{2}$, $t_5 = \frac{1}{2}$, $t_6 = 2$. На n-ом шаге берем $t_{2^n-1} = -n$, а потом последовательно слева направо добавим средние точки интервалов полученных точками определенных до n-ого шага, и положим $t_{2^{n+1}-2} = n$. Бесконечно продолжая этот процесс, получим допустимую последовательность $\mathcal{T}^1$.

Пусть последовательность $\mathcal{T}^2$, построена тем же алгоритмом, что и последовательность $\mathcal{T}^1$, с той разницей, что на n-ом шаге добавлены точки $t_{2^n-1} = -\xi_n$, $t_{2^{n+1}-2} = \xi_n$, где $\xi_n \uparrow \infty$. Ясно, что $\mathcal{T}^2$ совпадает с $\mathcal{T}^1$, если $\xi_n = n$. Также нетрудно заметить, что $\mathcal{T}^2$ будет сильно регулярной, тогда и только тогда, когда

$$0 < \inf_n \frac{\xi_{n+2} - \xi_{n+1}}{\xi_{n+1} - \xi_n} \quad \text{и} \quad \sup_n \frac{\xi_{n+2} - \xi_{n+1}}{\xi_{n+1} - \xi_n} < \infty.$$

В частности $\mathcal{T}^1$-сильно регулярна. Верна следующая теорема.

Теорема 3.3. *Пусть $\{f_n(x)\}_{n=1}^\infty$ система Франклина соответствующая последовательности $\mathcal{T}^2$ и функция φ удовлетворяет условию*

$$\lim_{n\to\infty} \frac{\ln \varphi(\xi_n)}{2^n} = 0. \tag{3.1}$$

Тогда для любой функции $f \in C_\varphi(R)$ частичные суммы $S_n(f,t)$ ряда Фурье по системе $\{f_n(t)\}_{n=1}^\infty$ локально равномерно сходятся к $f(t)$.

Теорема 3.4. *Пусть последовательность $\mathcal{T}^2$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=1}^\infty$ система Франклина соответствующая последовательности $\mathcal{T}^2$. Тогда если функция φ удовлетворяет условию*

$$\limsup_{n\to\infty} \frac{\ln \varphi(\xi_n)}{2^n} > 0, \tag{3.2}$$

то существует функция $f \in C_\varphi(R)$ для которой $S_n(f,t)$ не сходится к $f(t)$ в некоторых точках.

Доказательство теоремы 3.3. Оценим $S_n(f,t) - f(t)$. Пусть k любое натуральное и ε любое положительное числа. Выберем $\delta > 0$, такое что

$$|f(t) - f(\tau)| < \varepsilon, \quad \text{как только} \quad |t - \tau| < \delta \text{ и } t, \tau \in [-\xi_k; \xi_k]. \tag{3.3}$$

Тогда

$$S_n(f,t)-f(t)=\int_R K_n(t,\tau)f(\tau)d\tau-f(t)= \tag{3.4}$$

$$=\int_{|t-\tau|<\delta}K_n(t,\tau)(f(\tau)-f(t))d\tau+\int_{\delta<|t-\tau|<A}K_n(t,\tau)(f(\tau)-f(t))d\tau+$$

$$+\int_{A<|t-\tau|}K_n(t,\tau)(f(\tau)-f(t))d\tau+f(t)\left(\int_R K_n(t,\tau)d\tau-1\right)=$$

$$=I_\delta(f,t,n)+I_{\delta,A}(f,t,n)+I_A(f,t,n)+I(f,t,n).$$

Из (3.3) и леммы 2.1 следует

$$|I_\delta(f,t,n)|<3\varepsilon. \tag{3.5}$$

Из леммы 2.5 при фиксированном A получим

$$\lim_{n\to\infty}|I_{\delta,A}(f,t,n)|=0,\quad \text{причем равномерно, если}\quad t\in[-\xi_k;\xi_k]. \tag{3.6}$$

Для $I(f,t,n)$ при достаточно большом n имеем (см. (2.12))

$$|I(f,t,n)|\le|f(t)|\left|\sum_{i=0}^{n+1}\int_R K_n(t,\tau)N_i^n(\tau)d\tau-1\right|=$$

$$|f(t)|\left|\int_R K_n(t,\tau)N_0^n(\tau)d\tau+\int_R K_n(t,\tau)N_{n+1}^n(\tau)d\tau\right|\le$$

$$C|f(t)|\left(q^{d_n(t,\tau_0^n)}\frac{\tau_1^n-\tau_0^n}{t-\tau_0^n}+q^{d_n(t,\tau_{n+1}^n)}\frac{\tau_{n+1}^n-\tau_n^n}{\tau_{n+1}^n-t}\right).$$

Следовательно, получаем

$$\lim_{n\to\infty}|I(f,t,n)|=0,\quad \text{причем равномерно, если}\quad t\in[-\xi_k;\xi_k]. \tag{3.7}$$

Из (3.4)-(3.7) следует, что для доказательства теоремы 3.3 нужно доказать, что если $f\in C_\varphi(R)$ и $\varphi(t)$ удовлетворяет (3.1), то при достаточно большом A

$$\lim_{n\to\infty}\int_{A<|\tau|}|K_n(t,\tau)|\varphi(\tau)d\tau=0,\quad \text{причем равномерно, если}\quad t\in[-\xi_k;\xi_k]. \tag{3.8}$$

Для фиксированного k выберем $\vartheta>0$, а потом натуральное $m_0\ge k$ такие, что

$$\epsilon^{2^{-k-1}}\cdot e^\vartheta<1\quad\text{и}\quad\varphi(\xi_n)<\left(e^\vartheta\right)^{2^n},\ \text{когда}\ n\ge m_0. \tag{3.9}$$

Убедимся, что в качестве A можно взять ξ_{m_0}. Пусть $2^m-1\le n<2^{m+1}-1$, $m>m_0$. Нетрудно заметить, что количество точек из $\mathcal{T}_n$, которые принадлежат $[\xi_i;\xi_{i+1})$ не меньше 2^{m-i-1}. Поэтому, количество точек из $\mathcal{T}_n$, которые принадлежат $[\xi_k;\xi_i)$ не меньше 2^{m-k-1}. Следовательно, (см. (2.13), (2.15)) получаем

$$\int_{\xi_i}^{\xi_{i+1}}|K_n(t,\tau)|d\tau\le C\epsilon^{2^{m-k-1}},\quad\text{когда}\quad m_0\le i\quad\text{и}\quad t\in[-\xi_k,\xi_k].$$

Отсюда получим

$$(3.10)\quad \int_{\xi_{m_0}<|\tau|} |K_n(t,\tau)|\varphi(t)dt \le C\sum_{i=m_0}^{m-1}\varphi(\xi_i)\epsilon^{2^{m-k-1}} = C\epsilon^{2^{m-k-1}}\sum_{i=m_0}^{m-1}\varphi(\xi_i) \le$$

$$C\epsilon^{2^{m-k-1}}\left(e^{\vartheta}\right)^{2^m} = C\left(\epsilon^{2^{-k-1}}\cdot e^{\vartheta}\right)^{2^m}.$$

Из (3.10), (3.9) следует (3.8). Теорема 3.3 доказана.

Доказательство теоремы 3.4. Докажем, что если выполняется (3.2) и последовательность $\mathfrak{T}^2$ сильно регулярная с параметром γ, то найдется функция $f \in C_\varphi$, для которой $S_n(f,z)$ не сходится к $f(z)$ в некоторой точке z.

Учитывая (3.2), можем найти положительное число δ и возрастающую последовательность натуральных чисел n_k, такие что

$$(3.11)\qquad \ln\varphi(\xi_{n_k}) > \delta\cdot 2^{n_k}.$$

Учитывая, что $e^\delta > 1$, получим существование такого числа k_0, что

$$(3.12)\qquad e^\delta\cdot q_\gamma^{2^{-k_0}} > 1.$$

Обозначая $m_k = 2^{n_k+2}-2$, из определения $\mathfrak{T}^2$ имеем, что

$$t_{m_k-1}\in\mathfrak{T}^2_{m_k},\ t_{2^{n_k+1}-2}=\xi_{n_k}\in\mathfrak{T}^2_{m_k},\ \ t_{m_k}=\xi_{n_k+1}\in\mathfrak{T}^2_{m_k}$$

и других точек между ними из $\mathfrak{T}^2_{m_k}$ не существует. Следовательно,

$$K_{m_k}(\xi_{k_0},t_{m_k-1})\cdot K_{m_k}(\xi_{k_0},\xi_{n_k})<0 \text{ и кроме того } K_{m_k}(\xi_{k_0},\xi_{n_k+1})=0.$$

Пусть ξ_k^* такая точка из $[t_{m_k-1};\xi_{n_k}]$, что $K_{m_k}(\xi_{k_0},\xi_k^*)=0$. Положим

$$(3.13)\qquad \varrho_k(t)=\begin{cases}\varphi(\xi_{n_k})\cdot\operatorname{sgn}K_{m_k}(\xi_{k_0},\xi_{n_k}), & \text{когда } t=\xi_{n_k},\\ 0, & \text{когда } t\in(-\infty;\xi_k^*]\cup[\xi_{n_k+1};\infty),\\ \text{линейная на } [\xi_k^*;\xi_{n_k}] & \text{и } [\xi_{n_k};\xi_{n_k+1}].\end{cases}$$

Заметим также, что количество точек из $\mathfrak{T}^2_{m_k}$ между ξ_{k_0} и ξ_{n_k} равно $2^{n_k-k_0}$. Поэтому из (2.16) имеем

$$|K_{m_k}(\xi_{k_0},\xi_{n_k})| > 3\cdot q_\gamma^{2^{n_k-k_0}}\frac{1}{\xi_{k_0+1}-\xi_{k_0}}.$$

Отсюда и из (3.13), (3.11) следует

$$\int_R K_{m_k}(\xi_{k_0},\tau)\varrho_k(\tau)d\tau > \frac12|K_{m_k}(\xi_{k_0},\xi_{n_k})|\varphi(\xi_{n_k})(\xi_{n_k+1}-\xi_{n_k}) >$$

$$q_\gamma^{2^{n_k-k_0}}\cdot e^{\delta\cdot 2^{n_k}}\cdot\frac{1}{\gamma^{n_k-k_0}} = \left(q_\gamma^{2^{-k_0}}\cdot e^\delta\right)^{2^{n_k}}\cdot\frac{1}{\gamma^{n_k-k_0}}.$$

Следовательно, получаем

$$(3.14)\qquad S_{m_k}(\varrho_k,\xi_{k_0}) > \left(q_\gamma^{2^{-k_0}}\cdot e^\delta\right)^{2^{n_k}}\cdot\frac{1}{\gamma^{n_k-k_0}}.$$

Функцию f из $C_\varphi(R)$, для которой $S_n(f,\xi_{k_0})$ не сходится, будем искать в виде

$$f(t)=\sum_{\nu=1}^{\infty}\varrho_{k_\nu}(t),\ \text{где}\ \ k_\nu\uparrow\infty. \tag{3.15}$$

Ясно, что при любом выборе k_ν функция f принадлежит $C_\varphi(R)$ и при любом σ функция $\sum_{\nu=1}^{\sigma}\varrho_{k_\nu}(t)$ непрерывна и имеет компактный носитель. Учитывая это, нетрудно заметить, что по индукции можно выбрать числа k_ν, так чтобы выполнялись

$$\left|\sum_{l=1}^{\nu-1}S_n(\varrho_{k_l},\xi_{k_0})\right|<1,\ \ \text{когда}\ \ n\geq m_{k_\nu},\ \ \nu=2,3,.... \tag{3.16}$$

Нетрудно заметить, что если $k>k_\nu$, то $S_{m_{k_\nu}}(\varrho_k,\xi_{k_0})=0$. Поэтому из (3.15), (3.16), (3.14) и (3.12) будем иметь $\lim\limits_{\nu\to\infty}S_{m_{k_\nu}}(f,\xi_{k_0})=\infty$. Теорема 3.4 доказана.

Теоремы 3.3 и 3.4 указывают на то, что чем медленее растут ξ_n тем шире класс функций $C_\varphi(R)$, ряды Фурье-Франклина которых локально равномерно сходятся.

4. Свойства функций Франклина и безусловная базисность системы Франклина

Получим некоторые оценки для n-ой функции Франклина.

Лемма 4.1. *Пусть $f_n(t)=\sum_{k=1}^{n}\zeta_k N_k^n(t)$ n-ая функция Франклина и $\tau_i^n=t_n$. Тогда*

$$\|f_n\|_p\sim\mu_n^{1/2-1/p},\quad \text{для}\ \ 1\leq p\leq\infty, \tag{4.1}$$

где

$$\mu_n=\begin{cases}\frac{1}{\nu_{i-1}^n}+\frac{1}{\nu_i^n}+\frac{1}{\nu_{i+1}^n}, & \text{когда}\ \ 2\leq i\leq n-1,\\ \frac{1}{\nu_1^n}+\frac{1}{\nu_2^n}, & \text{когда}\ \ i=1,\\ \frac{1}{\nu_1^n}, & \text{когда}\ \ i=0,\\ \frac{1}{\nu_{n-1}^n}+\frac{1}{\nu_n^n}, & \text{когда}\ \ i=n,\\ \frac{1}{\nu_n^n}, & \text{когда}\ \ i=n+1.\end{cases}$$

$$\zeta_j=(-1)^{i+j}|\zeta_j|,\quad j=1,2,...,n, \tag{4.2}$$

$$|\zeta_j|\sim\frac{\mu_n^{1/2}}{\nu_j^n},\quad j=i-1,i,i+1, \tag{4.3}$$

$$|\zeta_{j-1}|(\frac{3}{2}\lambda_{j-1}^n+2\lambda_j^n)\leq|\zeta_j|\lambda_j^n\leq2|\zeta_{j-1}|(\lambda_{j-1}^n+\lambda_j^n),\quad \text{для}\ \ j\leq i-1, \tag{4.4}$$

(4.5) $\quad |\zeta_{j+1}|(2\lambda_{j+1}^n + \frac{3}{2}\lambda_{j+2}^n) \le |\zeta_j|\lambda_{j+1}^n \le 2|\zeta_{j+1}|(\lambda_{j+1}^n + \lambda_{j+2}^n), \quad$ *для* $j \ge i+1$.

Доказательство. Сначала рассмотрим случай $2 \le i \le n-1$. Пусть $\{N_j^{n-1}(t)\}_{j=1}^{n-1}$ базис пространства S_{n-1}. Нетрудно заметить, что

(4.6) $$N_j^{n-1} = N_j^n \text{ для } j \le i-2 \text{ и } N_j^{n-1} = N_{j+1}^n \text{ для } j \ge i+1$$

(4.7) $$N_{i-1}^{n-1} = N_{i-1}^n + \frac{\lambda_{i+1}^n}{\lambda_i^n + \lambda_{i+1}^n} N_i^n, \quad N_i^{n-1} = N_{i+1}^n + \frac{\lambda_i^n}{\lambda_i^n + \lambda_{i+1}^n} N_i^n.$$

Обозначим

(4.8) $$\omega(t) = -\frac{\lambda_{i+1}^n}{\lambda_i^n + \lambda_{i+1}^n} K_n(\tau_{i-1}^n, t) + K_n(\tau_i^n, t) - \frac{\lambda_i^n}{\lambda_i^n + \lambda_{i+1}^n} K_n(\tau_{i+1}^n, t).$$

Из (4.6), (4.7) и (2.1) имеем, что $\omega \in S_n(R)$ и ортогональна $S_{n-1}(R)$. Следовательно $\omega(t) = \alpha f_n(t)$, для некоторого α. Заметим, что из (2.2) следует

$$(-1)^{i+j}\omega(\tau_j^n) = |\omega(\tau_j^n)| =$$

$$\frac{\lambda_{i+1}^n}{\lambda_i^n + \lambda_{i+1}^n}|K_n(\tau_{i-1}^n, \tau_j^n)| + |K_n(\tau_i^n, \tau_j^n)| + \frac{\lambda_i^n}{\lambda_i^n + \lambda_{i+1}^n}|K_n(\tau_{i+1}^n, \tau_j^n)|.$$

Поэтому (4.2), (4.4) и (4.5) следуют из (2.2)-(2.4). Кроме того, отсюда и из (2.13) следует

$$\|\omega\|_p \sim \frac{\lambda_{i+1}^n}{\lambda_i^n + \lambda_{i+1}^n}\|K_n(\tau_{i-1}^n, \cdot)\|_p + \|K_n(\tau_i^n, \cdot)\|_p + \frac{\lambda_i^n}{\lambda_i^n + \lambda_{i+1}^n}\|K_n(\tau_{i+1}^n, \cdot)\|_p \sim$$

$$\frac{\lambda_{i+1}^n}{\lambda_i^n + \lambda_{i+1}^n}\nu_{i-1}^{\frac{1}{p}-1} + \nu_i^{\frac{1}{p}-1} + \frac{\lambda_i^n}{\lambda_i^n + \lambda_{i+1}^n}\nu_{i+1}^{\frac{1}{p}-1}.$$

Следовательно

(4.9) $$\|\omega\|_p \sim \mu_n^{1-\frac{1}{p}}, \quad 1 \le p \le \infty.$$

Учитывая, что $\|f_n\|_2 = 1$ из (4.9) получим $\alpha \sim \mu_n^{\frac{1}{2}}$ и $\|f_n\|_p \sim \mu_n^{\frac{1}{2}-\frac{1}{p}}$. Для завершения рассматриваемого случая простыми вычислениями нужно проверить (4.3), применяя (4.8), (2.3) – (2.5) и $\alpha \sim \mu_n^{\frac{1}{2}}$.

В случае $i = 0$ нетрудно заметить, что в $f_n(t) = K_n(\tau_1^n, t) \cdot \|K_n(\tau_1^n, \cdot)\|_2^{-1}$ и поэтому свойства (4.1)-(4.5) следуют из лемм 2.4, 2.2. В случае $i = n+1$ будет $f_n(t) = K_n(\tau_{n+1}^n, t) \cdot \|K_n(\tau_{n+1}^n, \cdot)\|_2^{-1}$.

В случае $i = 1$ полагая

$$\omega(t) = K_n(\tau_1^n, t) - \frac{\lambda_1^n}{\lambda_1^n + \lambda_2^n} K_n(\tau_2^n, t)$$

аналогичными вычислениями получим (4.1)-(4.5) для этого случая. В случае $i = n$ полагается

$$\omega(t) = -\frac{\lambda^n_{n+1}}{\lambda^n_n + \lambda^n_{n+1}} K_n(\tau^n_{n-1}, t) + K_n(\tau^n_n, t).$$

Лемма 4.1 доказана.

Теперь определим "канонический интервал" J_n, связанный с функцией f_n. Отметим, что понятие такого интервала в работах [3], [4], [10], [9] сыграла важную роль при исследовании системы Франклина на $[0; 1]$.

Пусть в последовательности π_n выполняется $\tau^n_i = t_n$ и $1 < i < n$. В этом случае интервал J_n определяется следующим образом. Во первых, пусть $j^* \in \{i-1, i, i+1\}$, такое что $\lambda^n_{j^*} + \lambda^n_{j^*+1} = \min_{i-1 \le j \le i+1}(\lambda^n_j + \lambda^n_{j+1})$. Далее, больший из интервалов $(\tau^n_{j^*-1}; \tau^n_{j^*})$, $(\tau^n_{j^*}; \tau^n_{j^*+1})$ обозначается через J_n. В случаях $i = 0$ и $i = n+1$ положим $J_n = (\tau^n_0; \tau^n_2)$ и $J_n = (\tau^n_{n-1}; \tau^n_{n+1})$, соответственно. Если $i = 1$, то $j^* \in \{1; 2\}$, такое что $\lambda^n_{j^*} + \lambda^n_{j^*+1} = \min(\lambda^n_1 + \lambda^n_2, \lambda^n_2 + \lambda^n_3)$, а J_n больший из интервалов $(\tau^n_{j^*-1}; \tau^n_{j^*})$, $(\tau^n_{j^*}; \tau^n_{j^*+1})$. Аналогично определяется J_n в случае $i = n$. Нетрудно заметить, что $|J_n| \sim \mu_n$. Применяя лемму 4.4 и анализируя определение интервала J_n можно доказать следующую лемму.

Лемма 4.2. *Пусть Δ-интервал линейности функции $f_n(t)$. Тогда*

$$|f_n(t)| \le Cq^{d_n(t)} \frac{|J_n|^{\frac{1}{2}}}{|J_n| + \rho(J_n, \Delta) + |\Delta|}, \quad \text{когда} \quad t \in \Delta. \tag{4.10}$$

Доказательство этой леммы излагать не будем, поскольку это повторение доказательства леммы 3.2 из [3]. На самом деле определение интервалов J_n связано только с последовательностью $\mathcal{T}$. Поэтому верна следующая лемма, доказанная в работе [3] для разбиений отрезка $[0; 1]$.

Лемма 4.3. *Пусть $\mathcal{T} = (t_n, n \ge 0)$ допустимая последовательность Тогда для любого n и $i = 0, 1, ..., n$ имеет место*

$$\#\left\{m : \quad J_m \subset [\tau^n_i; \tau^n_{i+1}] \quad u \quad |J_m| > \frac{\lambda^n_{i+1}}{2}\right\} \le 25.$$

Применяя леммы 4.2, 4.3 и другие оценки для функций Франклина и повторяя рассуждения из работ [3], [10] (см. [3] теорема 2.1 или [10] теорема 2) можно доказать следующую теорему.

Теорема 4.1. *Пусть $\mathcal{T}$ - допустимая последовательность. Тогда соответствующая ей система Франклина $\{f_n(t)\}_{n=1}^{\infty}$ является безусловным базисом в любом пространстве $L_p(R)$, $1 < p < \infty$.*

Отметим, что безусловную базисность классической системы Франклина в $L_p(0;1)$, $1 < p < \infty$ доказана С.В. Бочкаревым [11]. Безусловную базисность общей системы Франклина в $L_p(0;1)$, $1 < p < \infty$ доказана в [3]. А в работе [10] доказана безусловная базисность общей периодической системы Франклина в $L_p(0;1)$, $1 < p < \infty$.

Доказательство теоремы 4.1 достаточно длинное и почти повторяет доказательство теоремы 2.1 из [3]. Поэтому его приводить не будем.

Abstract. A general Franklin system on R, generated by an admissible sequence $\mathcal{T}$ is defined. We show that such defined system forms an unconditional basis in the space $L^p(R)$, $1 < p < \infty$, and prove theorems on locally uniform convergence of Fourier-Franklin series by this system.

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POWER SERIES WITH H.-O. GAPS; TAUBERIAN THEOREMS

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Abstract.[1] Let $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence 1, and let $s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ denote its partial sums. For a given triangular matrix $A = [\alpha_{n\nu}]$ we consider the A-transforms $\sigma_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_\nu(z)$, and prove two Tauberian theorems of the following type: from certain summability properties of $\{\sigma_n(z)\}$ outside the unit disk and a condition on the entries $\alpha_{n\nu}$ the convergence of a subsequence $\{s_{n_k}(z)\}$ is concluded.

MSC2010 numbers: 30B30, 40E05, 40E15, 30B40.
Keywords: Tauberian theorem; Hadamard-Ostrowski gap; overconvergence.

1. Introduction

1.1. Overconvergence and H.-O. gaps. Let be given a power series with radius of convergence 1:

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \qquad \overline{\lim_{\nu\to\infty}} |a_\nu|^{1/\nu} = 1, \tag{1.1}$$

which represents a holomorphic function in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. As usual, we denote its partial sums by

$$s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu. \tag{1.2}$$

Such a series is called overconvergent if there exists a domain G which is not contained in $\mathbb{D}$ and a subsequence $\{p_k\}$ of natural numbers such that $\{s_{p_k}(z)\}$ converges compactly on G. Then $\{s_{p_k}(z)\}$ is called an overconvergent subsequence of (1.1). If G intersects $\mathbb{D}$, then $\{s_{p_k}(z)\}$ generates an analytic continuation of f. (Note that there are other definitions of overconvergence.)

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The phenomenon of overconvergence was discovered by Porter [16] more than a century ago and thoroughly has been investigated by Ostrowski in [11] - [14]. For a good treatise on the theory of overconvergence we refer to Hille's book [5, Section 16.7]. One of Ostrowski's main results is an interdependence between overconvergence and existence of certain gaps in the sequence of coefficients $\{a_\nu\}$.

We say that the power series (1.1) has a sequence $\{p_k, q_k\}$ of H.-O. gaps (short for Hadamard-Ostrowski gaps) if p_k and q_k are natural numbers satisfying

$$p_1 < q_1 < p_2 < q_2 < \dots, \qquad \varliminf_{k\to\infty} \frac{q_k}{q_k} > 1$$

and

$$\varlimsup_{\substack{\nu\to\infty \\ \nu\in J}} |a_\nu|^{1/\nu} < 1 \qquad \text{for} \qquad J = \bigcup_{k=1}^{\infty} \{p_k, \dots, q_k\}.$$

We summarize the main results on overconvergence in the following theorem.

Theorem O (Ostrowski [11], [13]).

(a) *If the power series* (1.1) *possesses H.-O. gaps* $\{p_k, q_k\}$, *then any sequence* $\{s_{n_k}(z)\}$ *with* $n_k \in [p_k, q_k]$ *converges compactly in a domain which contains every point on* $|z| = 1$ *in which* f *is holomorphic.*

(b) *Every overconvergent power series possesses H.-O. gaps.*

1.2. Summability of power series. Let $A = [\alpha_{n\nu}]_{\nu,n=0}^{\infty}$ be an infinite triangular matrix with complex entries $\alpha_{n\nu}$, where $\alpha_{n\nu} = 0$ for $\nu > n$. Such a matrix generates a transformation of a power series. The A-transforms of the series (1.1) are given by

$$\sigma_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_\nu(z). \tag{1.3}$$

The matrix A is called p-regular ("regular for power series") if for all series of type (1.1) the sequence $\{\sigma_n(z)\}$ converges compactly in $\mathbb{D}$. This property can be characterized by the entries of A only. The following conditions are necessary and sufficient for p-regularity (see [7]):

$$\lim_{n\to\infty} \alpha_{n\nu} = 0 \qquad \text{for all } \nu \in \mathbb{N}_0, \tag{1.4}$$

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} \alpha_{n\nu} = 1, \tag{1.5}$$

$$\sup_{n} \sum_{\nu=0}^{n} |\alpha_{n\nu}| r^\nu < \infty \qquad \text{for all } r \in (0,1). \tag{1.6}$$

In common use of summability theory the matrix A is regular if and only if the conditions (1.4), (1.5) and instead of (1.6) the following stronger property

$$\sup_n \sum_{\nu=0}^{n} |\alpha_{n\nu}| < \infty$$

hold. Observe that if A is regular, then A is also p-regular, but not conversely.

If A is p-regular, then the following properties of the sequence (1.2) can easily be obtained by straightforward estimates:

$$\overline{\lim_{n\to\infty}} \left\{ \max_{|z|=R} |\sigma_n(z)| \right\}^{1/n} \leq R \quad \text{for all } R \geq 1; \tag{1.7}$$

if (1.1) has H.-O. gaps $\{p_k, q_k\}$, then

$$\overline{\lim_{k\to\infty}} \left\{ \max_{|z|=R} |\sigma_{q_k}(z)| \right\}^{1/q_k} < R \quad \text{for all } R > 1. \tag{1.8}$$

2. A Tauberian Theorem

The following Theorem is our main result.

Theorem 2.1. *Suppose that $A = [\alpha_{n\nu}]$ is a p-regular matrix with the property that there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ and a constant $\gamma \in (0,1)$ such that*

$$\lim_{\substack{\nu\to\infty \\ \nu\in J}} \left| \sum_{\mu=\nu}^{n_k} \alpha_{n_k\mu} \right|^{1/\nu} = 1, \quad \textit{where} \quad J = \bigcup_{k=1}^{\infty} \{[\gamma n_k], \ldots, n_k\}. \tag{2.1}$$

Let a power series of type (1.1) be given and $\sum_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_\nu(z)$ be its A-transforms. Suppose that for an $R > 1$ there exists a closed arc $\Gamma \subset \{z : |z| = R\}$ with

$$\overline{\lim_{k\to\infty}} \left\{ \max_{\Gamma} |\sigma_{n_k}(z)| \right\}^{1/n_k} < R. \tag{2.2}$$

Then the considered power series has H.-O. gaps of the type $\{[\delta n_k], n_k\}$ for some $\delta \in (0,1)$. If f has an analytic continuation, then $\{s_{n_k}(z)\}$ is overconvergent.

Remark 2.1

(1) If the sequence $\{\sigma_{n_k}(z)\}$ converges compactly in a domain, which is not contained in $\mathbb{D}$, then (2.2) is trivially satisfied for suitably chosen $R > 1$ and arcs $\Gamma \subset \{z : |z| = R\}$. In the case where the matrix A has the property that there are constants $c > 0$ and $\gamma \in (0,1)$ with

$$\left| \sum_{\mu=\nu}^{n} \alpha_{n\nu} \right| \geq c \quad \text{for all } \nu \text{ with } [\gamma n] \leq \nu \leq n \tag{2.3}$$

and all sufficiently large n, then (2.1) is satisfied. In section 3 we list a number of well known summability methods which are generated by matrices and satisfy (2.3).

(2) Suppose that the condition (2.1) is satisfied. Then the matrix A is not efficient for the analytic continuation of *all* function elements of type (1.1). More precisely, there exist power series (1.1) such that $\sigma_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_\nu(z)$ converges compactly in $\mathbf{D}$, but not in any bigger domain (however, a subsequence $\{\sigma_{n_k}(z)\}$ may have this property). If $\lim_{n\to\infty} |a_n|^{1/n} = 1$, then $\{\sigma_n(z)\}$ and all its subsequences are compactly convergent only in $\mathbf{D}$. For a detailed discussion of this problem we refer to [1], [2], [7], [8] (see, also, [3], [4]).

(3) Theorem 2.1 may be considered as a Tauberian theorem: From summability properties (here condition (2.2)) together with a so-called Tauberian condition (here (2.1)) convergence properties are derived. However, in contrast to classical Tauberian results, in Theorem 2.1 convergence of a subsequence is concluded.

Proof of Theorem 2.1. Suppose that an $\varepsilon > 0$ is given and consider the circle $|z| = R$ and its closed subarc Γ. Then, by (1.7) there exists an n_0 such that for all $n \geq n_0$

$$\max_{|z|=1} \left| \frac{\sigma_n(z)}{z^n} \right| \leq e^{\varepsilon n}, \quad \max_{|z|=R} \left| \frac{\sigma_n(z)}{z^n} \right| \leq e^{\varepsilon n}.$$

In addition, by (2.2) there exists a $k_0 \geq n_0$ such that for all $k \geq k_0$

$$\max_{\Gamma} \left| \frac{\sigma_{n_k}(z)}{z^{n_k}} \right| \leq \frac{e^{\varepsilon n_k}}{R^{n_k}}.$$

Let r with $1 < r < R$ be given. Then, according to Nevanlinna's N-constants theorem (see Hille, [5, p. 409]), there exists a universal $\Theta \in (0,1)$, which depends only on the geometrical configuration, such that for all $k \geq k_0$

$$\max_{|z|=r} \left| \frac{\sigma_{n_k}(z)}{z^{n_k}} \right| \leq e^{(1-\Theta)\varepsilon n_k} \cdot \frac{e^{\Theta \varepsilon n_k}}{R^{\Theta n_k}} = \left(\frac{e^{\varepsilon}}{R^{\Theta}} \right)^{n_k}.$$

Therefore, if $\varepsilon > 0$ is chosen sufficiently small, we obtain for those k

$$\max_{|z|=r} |\sigma_{n_k}(z)| \leq (qr)^{n_k},$$

where $0 < q < 1$ (but $qr > 1$). We have

$$\sigma_{n_k}(z) = \sum_{\nu=0}^{n_k} a_\nu z^\nu \cdot \sum_{\mu=\nu}^{n_k} \alpha_{n_k \mu},$$

and Cauchy's inequality gives for $0 \leq \nu \leq n_k$ and all $k \geq k_0$

$$|a_\nu| \cdot \left| \sum_{\mu=\nu}^{n_k} \alpha_{n_k \mu} \right| \leq \left(q \cdot r^{1-\nu/n_k} \right)^{n_k}.$$

If we now choose δ with $\gamma \le \delta < 1$ so near to 1 that $r^{1-\delta^*} < \frac{1}{q}$, then for all ν with $[\delta n_k] \le \nu \le n_k$ and $k \ge k_0$ we get the estimate

$$|a_\nu|^{1/\nu} \cdot \Big|\sum_{\mu=\nu}^{n_k} \alpha_{n_k\mu}\Big|^{1/\nu} \le q \cdot r^{1-\delta} < 1.$$

But then (2.2) implies

$$\overline{\lim_{\substack{\nu\to\infty\\ \nu\in J}}} |a_\nu|^{1/\nu} < 1 \quad \text{for} \quad J = \bigcup_{k=1}^{\infty} \{[\delta n_k], \dots, n_k\}.$$

Therefore the power series under consideration has H.-O. gaps of the type $\{[\delta n_k], n_k\}$. In the case where f has an analytic continuation, Theorem 0 implies that $\{s_{n_k}(z)\}$ is an overconvergent subsequence of the series. Theorem 2.1 is proved. □

As a corollary of Theorem 2.1 we have the following result.

Theorem 2.2. *Let a power series of type* (1.1) *be given which has an analytic continuation. Suppose that $A = [\alpha_{n\nu}]$ is a p-regular triangular matrix and that for a sequence $\{p_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} \frac{p_{k+1}}{p_k} > 1$ transformations*

$$\tau_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_{p_\nu}(z)$$

are compactly convergent in a domain which is not contained in the unit disk. If

$$\lim_{\substack{\nu\to\infty\\ \nu\in J}} |\alpha_{kk}|^{1/\nu} = 1 \quad \text{for} \quad J = \bigcup_{k=1}^{\infty} \{p_k + 1, \dots, p_{k+1}\} \tag{2.4}$$

is satisfied, then $\{s_{p_k}(z)\}$ is an overconvergent subsequence of the considered power series.

Proof. Without loss of generality we can assume that $p_{k+1}/p_k \ge \lambda > 1$ for all $k \in \mathbb{N}_0$. We define a triangular matrix $B = [\beta_{n\nu}]$ in the following way:

$$\text{for } n \ne p_k: \qquad \beta_{n\nu} := \begin{cases} 0 & \text{if } \nu \ne n \\ 1 & \text{if } \nu = n \end{cases},$$

$$\text{for } n = p_k: \qquad \beta_{p_k\nu} := \begin{cases} 0 & \text{if } \nu \ne p_\mu \\ \alpha_{k\mu} & \text{if } \nu = p_\mu \end{cases} \qquad (\mu = 0, \dots, k).$$

The matrix B is obviously p-regular. We consider

$$\sigma_n(z) = \sum_{\nu=0}^{n} \beta_{n\nu} s_\nu(z)$$

and obtain

$$\sigma_{p_k}(z) = \sum_{\mu=0}^{k} \beta_{p_k p_\mu} s_{p_\mu}(z) = \sum_{\mu=0}^{k} \alpha_{k\mu} s_{p_\mu}(z) = \tau_k(z)$$

as well as $\sum_{\mu=\nu}^{p_k} \beta_{p_k\mu} = \alpha_{kk}$ for all ν with $p_{k-1} < \nu \le p_k$.

Therefore, the matrix B satisfies a condition of type (2.1), while the sequence $\{\sigma_{p_k}(z)\}$ has property (2.2) for a suitably chosen $R > 1$ and an arc $\Gamma \subset \{z : |z| = R\}$. It follows that $\{s_{p_k}(z)\}$ is an overconvergent subsequence. Theorem 2.2 is proved. □

3. Examples

We discuss some examples of well-known summability methods that are defined by triangular matrices and satisfy the requirements of Theorem 2.1. Especially we are interested whether the property (2.1), which acts as a Tauberian condition in this result, can be realized. Whenever in addition a power series (1.1) is considered, for which the corresponding transformations satisfy a property of type (2.2), then a Tauberian result as in Theorem 2.1 can be concluded for this series.

1. Nørlund means $\mathbf{N_c}$. Let $c = \{c_n\}$ be a sequence of real numbers with $c_0 > 0$ and $c_n \ge 0$ for $n \ge 1$, and let $C_n = \sum_{\nu=0}^{n} c_\nu$. Then the Nørlund means are generated by the triangular matrix $A = [\alpha_{n\nu}]$ given by

$$\alpha_{n\nu} = \frac{c_{n-\nu}}{C_n} \quad \text{if } 0 \le \nu \le n.$$

The condition $\lim_{n\to\infty} \frac{c_n}{C_n} = 0$ is necessary and sufficient for the regularity of N_c, and it is also well-known that all Nørlund methods are ineffective for analytic continuations of any power series. For $0 \le \nu \le n$ we get

$$\frac{c_0}{C_n} \le \sum_{\mu=\nu}^{n} \alpha_{n\nu} \le 1,$$

and by the regularity condition we have $\lim_{n\to\infty} \frac{C_{n-1}}{C_n} = 1$. Hence $\lim_{n\to\infty}(C_n)^{1/n} = 1$, which implies that for all Nørlund means the condition (2.1) is satisfied for all subsequences $\{n_k\}$ of $\mathbf{N}$.

Hence a Tauberian result as in Theorem 2.1 holds for all power series whose N_c transformations satisfy condition (2.2).

(Actually the N_c method was first introduced by Russian mathematician Voronoi in 1902 (see [17]); independently of Voronoi the definition was given by Nørlund in 1920 (see [10]).)

2. Cesàro means $\mathbf{C_\alpha}$. These are special regular Nørlund means which for $\alpha \ge 0$ are generated by the sequence $c_n = \binom{n+\alpha-1}{n}$.

3. Weighted means $\mathbf{R_c}$. (Also known as Riesz means or Nørlund-type means.) Let $c = \{c_n\}$ be again a sequence of real numbers with $c_0 > 0$ and $c_n \ge 0$ for $n \ge 1$, and let $C_n = \sum_{\nu=0}^{n} c_\nu$. Then the R_c means are generated by the triangular matrix $A = [\alpha_{n\nu}]$ given by

$$\alpha_{n\nu} = \frac{c_\nu}{C_n} \quad \text{if} \quad 0 \leq \nu \leq n.$$

Here the condition $\lim_{n\to\infty} C_n = \infty$ is necessary and sufficient for the regularity of R_c. As in the case of Nørlund means we have for $0 \leq \nu \leq n$

$$\frac{c_0}{C_n} \leq \sum_{\mu=\nu}^{n} \alpha_{n\nu} \leq 1.$$

Therefore condition (2.1) is satisfied if $\{C_n\}$ is not "too fast" increasing sequence, that is, if $\lim_{n\to\infty}(C_n)^{1/n} = 1$. In this case Theorem 2.1 applies also to this method.

4. Hausdorff means H_χ. This is a wide class of summability methods, containing many well-known methods as special cases.

Let χ be a real-valued function of bounded variation on $[0,1]$ satisfying

$$\chi(t) = \chi(t+) \quad \text{for all} \quad t \in [0,1).$$

The H_χ means are generated by the triangular matrix $A = [\alpha_{n\nu}]$ with

$$\alpha_{n\nu} = \binom{n}{\nu} \int_0^1 t^\nu (1-t)^{n-\nu} d\chi(t)$$

and the regularity conditions are $\chi(0) = 0$, $\chi(1) = 1$ (see [15]). The best known Hausdorff means are the Cesàro means C_α $(\alpha > 0)$, where

$$\chi(t) = 1 - (1-t)^\alpha,$$

the Hölder means H_α $(\alpha > 0)$, where (Γ denotes the Gamma function)

$$\chi(t) = \frac{1}{\Gamma(\alpha)} \cdot \int_0^t \left(\ln \frac{1}{s}\right)^{\alpha-1} ds,$$

and the Euler means E_r $(0 < r < 1)$, where

$$\chi(t) = \begin{cases} 0 & \text{for } 0 \leq t < r \\ 1 & \text{for } r \leq t \leq 1. \end{cases}$$

The (upper) order of a regular Hausdorff mean is defined as

$$\rho = \rho(\chi) = \inf\{s : \chi(t) = 1 \text{ for all } t \in [s,1]\}.$$

Obviously C_α and H_α have order $\rho = 1$, while $\rho = r < 1$ for the Euler means E_r.

If a power series has an analytic continuation, then all H_χ means with $\rho < 1$ are efficient for those series and also an estimate (depending on ρ) for the summability domain can be given. On the other hand, all H_χ means of order $\rho = 1$ are inefficient for analytic continuation (for details see [9, section 2], [15, chapter IV, 2]). Especially for Cesàro and Hölder means we have inefficiency for any power series.

If H_χ has order $\rho = 1$, then there exist constants $\gamma \in (0,1)$ and $c > 0$ such that for all sufficiently large n $\left|\sum_{\mu=\nu}^{n} a_{n\nu}\right| \geq c$ for all $\nu \in [\gamma n, n]$.

This estimate is a special case of a result on the distribution of Hausdorff elements (see [9], Lemma 1), which was proved by probabilistic methods.

It follows that H_χ means of order $\rho = 1$ satisfy condition (2.1) for any subsequence of N, and under the additional assumption (2.2) on the behavior of a power series a Tauberian result as in Theorem 2.1 holds.

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ON THE CONVERGENCE AND SUMMABILITY OF DOUBLE WALSH-FOURIER SERIES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

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Abstract. [1]The convergence of partial sums and Cesáro means of negative order of double Walsh-Fourier series of functions of bounded generalized variation is investigated.

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Keywords: Walsh function; bounded variation; Cesáro means.

1. Classes of functions of bounded generalized variation

In 1881 C. Jordan [17] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation etc. (see, e.g., [2, 18, 27, 29]). In two dimensional case the class of functions of bounded variation (BV) was introduced by G. Hardy [16].

Let f be a real and measurable function of two variables on the unit square. Given intervals $\Delta = (a, b)$, $J = (c, d)$ and points x, y from $I := [0, 1)$ we denote

$$f(\Delta, y) = f(b, y) - f(a, y), \qquad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(\Delta, J) = f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{\Delta_i\}$ be a collection of nonoverlapping intervals from I ordered in arbitrary way and let Ω be the set of all such collections E. Denote by Ω_n the set of all collections of n nonoverlapping intervals $I_k \subset I$.

For the sequences of positive numbers

$$\Lambda^1 = \{\lambda_n^1\}_{n=1}^{\infty}, \qquad \Lambda^2 = \{\lambda_n^2\}_{n=1}^{\infty}$$

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and $I^2 = [0,1)^2$ we denote

$$\Lambda^1 V_1(f; I^2) = \sup_y \sup_{E\in\Omega} \sum_i \frac{|f(\Delta_i, y)|}{\lambda_i^1} \qquad (E = \{\Delta_i\}),$$

$$\Lambda^2 V_2(f; I^2) = \sup_x \sup_{F\in\Omega} \sum_j \frac{|f(x, J_j)|}{\lambda_j^2} \qquad (F = \{J_j\}),$$

$$(\Lambda^1\Lambda^2)\, V_{1,2}(f; I^2) = \sup_{F, E\in\Omega} \sum_i \sum_j \frac{|f(\Delta_i, J_j)|}{\lambda_i^1 \lambda_j^2}.$$

Definition 1.1. *We say that a function f has bounded (Λ^1, Λ^2)-variation on I^2 and write $f \in (\Lambda^1, \Lambda^2)\, BV\,(I^2)$, if*

$$(\Lambda^1, \Lambda^2)\, V(f; I^2) := \Lambda^1 V_1(f; I^2) + \Lambda^2 V_2(f; I^2) + (\Lambda^1\Lambda^2)\, V_{1,2}(f; I^2) < \infty.$$

If $\Lambda^1 = \Lambda^2 = \Lambda$, then we say that f has bounded Λ-variation on I^2 and use the notation $\Lambda BV(I^2)$.

We say that a function f has bounded partial Λ-variation on I^2 and write $f \in P\Lambda BV\,(I^2)$, if

$$P\Lambda BV(f; I^2) := \Lambda V_1(f; I^2) + \Lambda V_2(f; I^2) < \infty.$$

If $\Lambda = \{\lambda_n\}$ with $\lambda_n \equiv 1$, or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \ldots$, the classes ΛBV and $P\Lambda BV$ coincide, respectively, with the Hardy class BV and with the class PBV functions of bounded partial variation introduced by Goginava [6]. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{\Delta_i\}$ are ordered arbitrarily, we can assume, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus, we assume that

$$(1.1) \qquad 1 < \lambda_1 \le \lambda_2 \le \ldots, \qquad \lim_{n\to\infty} \lambda_n = \infty, \qquad \sum_{n=1}^{\infty} (1/\lambda_n) = +\infty.$$

In the case where $\lambda_n = n$, $n = 1, 2\ldots$ we will use the term *harmonic variation* instead of Λ-variation and will write H instead of Λ, that is, HBV, $PHBV$, $HV(f)$, etc.

The notion of Λ-variation was introduced by Waterman [27] in one dimensional case, and by Sahakian [23] in two dimensional case; the notion of bounded partial Λ-variation ($P\Lambda BV$) was introduced by Goginava and Sahakian [12].

Dyachenko and Waterman [5] introduced another class of functions of generalized bounded variation. Denoting by Γ the set of finite collections of nonoverlapping rectangles $A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset I^2$, we define

$$\Lambda^* V_{1,2}(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.$$

Definition 1.2 (Dyachenko and Waterman, [5]). *Let f be a real function on I^2. We say that $f \in \Lambda^* BV$, if*

$$\Lambda^* V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^* V_{1,2}(f) < \infty.$$

In [13], the authors have introduced new classes of functions of generalized bounded variation and investigated the convergence of Fourier series of functions from that classes.

For the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we define

$$\Lambda^{\#} V_1(f) := \sup_{\{y_i\} \subset I} \sup_{\{I_i\} \in \Omega} \sum_i \frac{|f(I_i, y_i)|}{\lambda_i},$$

$$\Lambda^{\#} V_2(f) := \sup_{\{x_j\} \subset I} \sup_{\{J_j\} \in \Omega} \sum_j \frac{|f(x_j, J_j|}{\lambda_j}.$$

Definition 1.3. *We say that a function f belongs to the class $\Lambda^{\#} BV$, if*

$$\Lambda^{\#} V(f) = \Lambda^{\#} V_1(f) + \Lambda^{\#} V_2(f) < \infty.$$

The notion of continuity of functions in Λ-variation plays an important role in the study of convergence of Fourier series of functions of bounded Λ-variation.

Definition 1.4. *We say that a function f is continuous in (Λ^1, Λ^2)-variation on I^2 and write $f \in C(\Lambda^1, \Lambda^2)V$, if*

$$\lim_{n\to\infty} \Lambda_n^1 V_1(f) = \lim_{n\to\infty} \Lambda_n^2 V_2(f) = 0$$

and

$$\lim_{n\to\infty} (\Lambda_n^1, \Lambda^2) V_{1,2}(f) = \lim_{n\to\infty} (\Lambda^1, \Lambda_n^2) V_{1,2}(f) = 0,$$

where $\Lambda_n^i := \{\lambda_k^i\}_{k=n}^{\infty} = \{\lambda_{k+n}^i\}_{k=0}^{\infty}$, $i = 1, 2$.

Definition 1.5. *A function f is continuous in $\Lambda^{\#}$-variation on I^2 and write $f \in C\Lambda^{\#}V$, if*

$$\lim_{n\to\infty} \Lambda_n^{\#} V(f) = 0,$$

where $\Lambda_n = \{\lambda_k\}_{k=n}^{\infty}$.

Definition 1.6. *We say that a function f is continuous in Λ^*-variation on I^2 and write $f \in C\Lambda^*V$, if*

$$\lim_{n\to\infty} \Lambda_n V_1(f) = \lim_{n\to\infty} \Lambda_n V_2(f) = 0; \quad \lim_{n\to\infty} \Lambda_n^* V_{1,2}(f) = 0.$$

Now, we define

$$v_1^{\#}(n,f) := \sup_{\{y_i\}_{i=1}^n} \sup_{\{I_i\}\in\Omega_n} \sum_{i=1}^{n} |f(I_i, y_i)|, \quad n = 1, 2, \ldots,$$

$$v_2^{\#}(m,f) := \sup_{\{x_j\}_{j=1}^m} \sup_{\{J_k\}\in\Omega_m} \sum_{j=1}^{m} |f(x_j, J_j)|, \quad m = 1, 2, \ldots.$$

The following theorems hold.

Theorem 1.1 (Goginava, Sahakian [13]). $\left\{\frac{n}{\log n}\right\}^{\#} BV \subset HBV$.

Theorem 1.2 (Goginava, Sahakian [13]). *Suppose*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f;n)\log(n+1)}{n^2} < \infty, \quad s = 1, 2.$$

Then $f \in \left\{\frac{n}{\log(n+1)}\right\}^{\#} BV$.

Theorem 1.3 (Goginava [10]). *Let $\alpha, \beta \in (0,1)$, $\alpha + \beta < 1$ and*

$$\sum_{j=1}^{\infty} \frac{v_s^{\#}(f;2^j)}{2^{j(1-(\alpha+\beta))}} < \infty, \quad s = 1, 2.$$

Then $f \in C\left\{n^{1-(\alpha+\beta)}\right\}^{\#} V$.

Theorem 1.4 (Goginava [10]). *Let $\alpha, \beta \in (0,1)$ and $\alpha + \beta < 1$. Then*

$$C\left\{i^{1-(\alpha+\beta)}\right\}^{\#} V \subset C\left\{i^{1-\alpha}\right\}\left\{j^{1-\beta}\right\} V.$$

The next theorem shows, that for some sequences Λ the classes $\Lambda^{\#}V$ and $C\Lambda^{\#}V$ coincide.

Theorem 1.5. *Let the sequence $\Lambda = \{\lambda_n\}$ be as in (1.1) and*

$$\liminf_{n\to\infty} \frac{\lambda_{2n}}{\lambda_n} = q > 1. \tag{1.2}$$

Then $\Lambda^{\#}V = C\Lambda^{\#}V$.

Proof. Assume the opposite, that there exists a function $f \in \Lambda^{\#}V$ for which $\liminf_{n\to\infty} \Lambda_n^{\#}V(f) > 0$ (see Definition 1.5). Without loss of generality, we can assume that $\liminf_{n\to\infty} \Lambda_n^{\#}V_1(f) =$

$\delta > 0$ and that $\delta = 1$. Then, taking into account that the sequence $\{\Lambda_n^{\#}V_1(f)\}$ is decreasing, we have

$$\lim_{n\to\infty} \Lambda_n^{\#}V_1(f) = 1. \tag{1.3}$$

Let a natural k and the numbers $\varepsilon > 0$, $q_0 \in (1, q)$ be fixed.

According to (1.2) and (1.3) there exist a natural $N' > k$ such that

$$\frac{\lambda_{2n}}{\lambda_n} > q_0, \quad \Lambda_n^{\#}V_1(f) > 1-\varepsilon \quad \text{for} \quad n \geq N'. \tag{1.4}$$

Then for a natural $N > 2N'$ there are a set of points $\{y_i\}_{i=1}^{2i_0}$ and a set of nonoverlapping intervals $\{\delta_i\}_{i=1}^{2i_0} \in \Omega$ such that

$$I := \sum_{i=1}^{2i_0} \frac{|f(\delta_i, y_i)|}{\lambda_{N+i}} \geq 1-\varepsilon. \tag{1.5}$$

Adding, if necessary, new terms in (1.5) we can assume that

$$\bigcup_{i=1}^{2i_0} \delta_i = (0,1).$$

Denote

$$I_1 := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N+2i-1}}, \qquad I_2 := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N+2i}}. \tag{1.6}$$

Since $N > 2N'$ implies that $N + 2i - 1 \geq 2(N' + i)$, from (1.4) and (1.6) we have

$$I_1' := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N'+i}} = \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N+2i-1}} \cdot \frac{\lambda_{N+2i-1}}{\lambda_{N'+i}} > q_0 I_1 \tag{1.7}$$

and

$$I_2' := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N'+i}} = \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N+2i}} \cdot \frac{\lambda_{N+2i}}{\lambda_{N'+i}} > q_0 I_2. \tag{1.8}$$

Consequently, by (1.5) we get

$$I' := I_1' + I_2' \geq q_0(I_1 + I_2) = q_0 I \geq q_0(1-\varepsilon). \tag{1.9}$$

Now, we take a natural M to satisfy

$$M > N + 2(i_0 + 1) \quad \text{and} \quad \frac{2(2i_0+1)}{\lambda_M} \sup_{x\in[0,1]} |f(x)| < \varepsilon, \tag{1.10}$$

and hence using (1.4), we can find a set of points $\{z_j\}_{j=1}^{j_0}$ and a set of nonoverlapping intervals $\{\Delta_j\}_{j=1}^{j_0} \in \Omega$ such that

$$\sum_{j=1}^{j_0} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \geq 1-\varepsilon. \tag{1.11}$$

Denote by Q the set of indices $j = 1, 2, \cdots, j_0$ for which the corresponding interval Δ_j does not contain an endpoint of the intervals $\delta_i, i = 1, 2, \ldots, 2i_0$, that is, Δ_j lies in one of the intervals $\delta_i, i = 1, 2, \ldots, 2i_0$. Then the number of indices in $[1, j_0] \setminus Q$ does not exceed $2i_0 + 1$, and by (1.10) we get

$$\sum_{j\in[1,j_0]\setminus Q} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \leq \varepsilon.$$

Consequently, by (1.11) we have

$$J := \sum_{j\in Q} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \geq 1 - 2\varepsilon. \tag{1.12}$$

Denoting

$$Q_1 = \left\{ j \in Q : \Delta_j \subset \bigcup_{i=1}^{i_0} \delta_{2i-1} \right\}, \qquad Q_2 = \left\{ j \in Q : \Delta_j \subset \bigcup_{i=1}^{i_0} \delta_{2i} \right\}$$

and

$$J_1 := \sum_{j\in Q_1} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}}, \qquad J_2 := \sum_{j\in Q_2} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}},$$

from (1.9) and (1.12) we obtain

$$(I_1' + J_2) + (I_2' + J_1) = I' + J \geq q_0(1 - \varepsilon) + 1 - 2\varepsilon \geq q_0 + 1 - 3\varepsilon.$$

Theretofore

$$I_1' + J_2 \geq \frac{q_0 + 1 - 3\varepsilon}{2} \quad \text{or} \quad (I_2' + J_1) \geq \frac{q_0 + 1 - 3\varepsilon}{2},$$

which means that

$$\Lambda^{\#}_{N'} V_1(f) \geq \frac{q_0 + 1 - 3\varepsilon}{2},$$

and hence

$$\Lambda^{\#}_k V_1(f) \geq \frac{q_0 + 1}{2},$$

since ε is any positive number and $N' > k$. Taking into account that k is an arbitrary natural number, the last inequality implies

$$\lim_{n\to\infty} \Lambda^{\#}_n V_1(f) \geq \frac{q_0 + 1}{2} > 1,$$

which contradicts the assumption (1.3), and the result follows. Theorem 1.5 is proved.

It is easy to see that for any $\gamma > 0$ the sequence $\lambda_n = n^\gamma$, $n = 1, 2, \ldots$ satisfies the condition (1.2) with $q = 2^\gamma$. Hence Theorem 1.5 implies the following result.

Corollary 1.1. *If* $0 < \gamma \leq 1$, *then* $\{n^\gamma\}^{\#} V = C\{n^\gamma\}^{\#} V$.

This, combined with Theorem 1.4 implies the next result.

Corollary 1.2. *Let $\alpha,\beta\in(0,1)$ and $\alpha+\beta<1$. Then*

$$\left\{i^{1-(\alpha+\beta)}\right\}^{\#}V\subset C\left\{i^{1-\alpha}\right\}\left\{j^{1-\beta}\right\}V.$$

2. Walsh functions

Let $\mathbf{P}$ be the set of positive integers, and $\mathbf{N}=\mathbf{P}\cup\{0\}$. We denote the set of all integers by $\mathbf{Z}$ and the set of dyadic rational numbers in the unit interval $I=[0,1)$ by $\mathbf{Q}$. Each element of $\mathbf{Q}$ is of the form $\frac{p}{2^n}$ for some $p,n\in\mathbf{N}$, $0\leq p\leq 2^n$. By a dyadic interval in I we mean an interval of the form $I_N^l:=[l2^{-N},(l+1)\,2^{-N})$ for some $l\in\mathbf{N},0\leq l<2^N$. Given $N\in\mathbf{N}$ and $x\in I$, we denote by $I_N(x)$ the dyadic interval of length 2^{-N} that contains x. Finally, we set $I_N:=[0,2^{-N})$ and $\overline{I}_N:=I\backslash I_N$.

Let $r_0(x)$ be the following function

$$r_0(x)=\begin{cases}1, & \text{if } x\in[0,1/2)\\ -1, & \text{if } x\in[1/2,1)\end{cases},\qquad r_0(x+1)=r_0(x),\quad x\in\mathbb{R}.$$

The Rademacher system is defined by

$$r_n(x)=r_0(2^nx),\quad x\in I,\qquad n=1,2,\ldots.$$

The Walsh functions $w_0,w_1,\ldots$ are defined as follows. Denote $w_0(x)=1$ and if $k=2^{n_1}+\cdots+2^{n_s}$ is a positive integer with $n_1>n_2>\cdots>n_s\geq 0$, then

$$w_k(x)=r_{n_1}(x)\cdots r_{n_s}(x).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1}w_k(x),\quad n=1,2,\ldots$$

Recall that (see [15, 25])

$$D_{2^n}(x)=\begin{cases}2^n, & \text{if } x\in[0,2^{-n})\\ 0, & \text{if } x\in[2^{-n},1)\end{cases}\tag{2.1}$$

and

$$D_{2^n+m}(x)=D_{2^n}(x)+w_{2^n}(x)\,D_m(x),\quad 0\leq m<2^n,\quad n=0,1,\ldots\tag{2.2}$$

It is well known that (see [25])

$$D_n(t)=w_n(t)\sum_{j=0}^{\infty}n_jw_{2^j}(t)\,D_{2^j}(t),\quad \text{if}\quad n=\sum_{j=0}^{\infty}n_j2^j\tag{2.3}$$

and

$$|D_{q_n}(x)|\geq\frac{1}{4x},\qquad 2^{-2n-1}\leq x<1,\tag{2.4}$$

where

(2.5)
$$q_n := 2^{2n-2} + 2^{2n-4} + \cdots + 2^2 + 2^0.$$

Given $x \in I$, the expansion

(2.6)
$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)},$$

where each $x_k = 0$ or 1, is called a dyadic expansion of x. If $x \in I \backslash \mathbb{Q}$, then (2.6) is uniquely determined. For $x \in \mathbb{Q}$ we choose the dyadic expansion with $\lim_{k\to\infty} x_k = 0$. The dyadic sum of $x, y \in I$ in terms of the dyadic expansion of x and y is defined by

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

We say that $f(x, y)$ is continuous at (x, y) if

(2.7)
$$\lim_{h,\delta\to 0} f(x \dotplus h, y \dotplus \delta) = f(x, y).$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m \in \mathbb{N}\}$ on the unit square $I^2 = [0,1) \times [0,1)$.

If $f \in L^1(I^2)$, then

$$\hat{f}(n, m) = \int_{I^2} f(x, y)\, w_n(x) w_m(y) dx dy$$

is the (n, m)-th Walsh-Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}(x, y; f) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n)\, w_m(x) w_n(y).$$

The Cesàro $(C; \alpha, \beta)$-means of double Walsh-Fourier series are defined as follows

$$\sigma_{n,m}^{\alpha,\beta}(x, y; f) = \frac{1}{A_{n-1}^{\alpha} A_{m-1}^{\beta}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j}(x, y; f),$$

where

$$A_0^{\alpha} = 1, \quad A_n^{\alpha} = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}, \quad \alpha \neq -1, -2, \ldots.$$

It is well-known that (see [30])

(2.8)
$$A_n^{\alpha} = \sum_{k=0}^{n} A_{n-k}^{\alpha-1},$$

(2.9)
$$A_n^{\alpha} \sim n^{\alpha}$$

and

$$\sigma_{n,m}^{\alpha,\beta}(x,y;f)=\int_{I^2} f(s,t)\,K_n^{\alpha}(x+s)\,K_m^{\beta}(y+t)\,dsdt, \tag{2.10}$$

where

$$K_n^{\alpha}(x):=\frac{1}{A_{n-1}^{\alpha}}\sum_{k=1}^{n}A_{n-k}^{\alpha-1}D_k(x). \tag{2.11}$$

3. Convergence of two-dimensional Walsh-Fourier series

The well known Dirichlet-Jordan theorem (see [30]) states that the Fourier series of a function $f(x)$, $x\in T$ of bounded variation converges at every point x to the value $[f(x+0)+f(x-0)]/2$.

Hardy [16] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if a function $f(x,y)$ has bounded variation in the sense of Hardy ($f\in BV$), then $S[f]$ converges at any point (x,y) to the value $\frac{1}{4}\sum f(x\pm 0,y\pm 0)$. Here and below we consider the convergence of only rectangular partial sums of double Fourier series.

Convergence of d-dimensional trigonometric Fourier series of functions of bounded Λ-variation was investigated in details by Sahakian [23], Dyachenko [3, 4, 5], Bakhvalov [1], Sablin [22], Goginava, Sahakian [12, 13], and others.

For the d-dimensional Walsh-Fourier series the convergence of partial sums of functions of bounded harmonic variation and other bounded generalized variations were studied by Moricz [19, 20], Onnewer, Waterman [21], and Goginava [7].

In the two-dimensional case the following result is known.

Theorem 3.1 (Sargsyan [24]). *If $f\in HBV(I^2)$, then the double Walsh-Fourier series of f converges to $f(x,y)$ at any point $(x,y)\in I^2$ where f is continuous.*

The authors have investigated convergence of multiple Walsh-Fourier series of functions of partial Λ-bounded variation. In particular, the following result was proved in [14].

Theorem 3.2 (Goginava, Sahakian [14]). *The following assertions hold:*

a) *If $f\in P\{\frac{n}{\log^{1+\varepsilon}n}\}BV(I^2)$ for some $\varepsilon>0$, then the double Walsh-Fourier series of f converges to $f(x,y)$ at any point $(x,y)\in I^2$ where f is continuous.*

b) *There exists a continuous function $f\in P\{\frac{n}{\log n}\}BV(I^2)$ such that the quadratic partial sums of its Walsh-Fourier series diverge at some point.*

The next theorem contains a similar result for functions of bounded $\Lambda^{\#}$-variation.

Theorem 3.3. *The following assertions hold:*

a) *If* $f \in \left\{\frac{n}{\log n}\right\}^{\#} BV$, *then the double Walsh-Fourier series of* f *converges to* $f(x,y)$ *at any point* (x,y) *where* f *is continuous.*

b) *For an arbitrary sequence* $\alpha_n \to \infty$ *there exists a continuous function* $f \in \left\{\frac{n\alpha_n}{\log(n+1)}\right\}^{\#} BV$ *such that the quadratic partial sums of its Walsh-Fourier series diverge unboundedly at* $(0,0)$.

Proof. The assertion a) immediately follows from Theorems 1.1 and 3.1.

To prove the assertion b), observe first that for any sequence $\Lambda = \{\lambda_n\}$ satisfying (1.1) the class $C\left(I^2\right) \bigcap \Lambda^{\#}BV$ is a Banach space with the norm

$$\|f\|_{\Lambda^{\#}BV} = \|f\|_C + \Lambda^{\#}BV(f),$$

and $S_{N,N}(0,0,f)$, $n = 1,2,\ldots$, is a sequence of bounded linear functionals on that space. Denote

$$\varphi_{N,j}(x) = \begin{cases} 2^{2N+1}x - 2j, & \text{if } x \in \left[j2^{-2N}, (2j+1)2^{-2N-1}\right], \\ -\left(2^{2N+1}x - 2j - 2\right), & \text{if } x \in \left[(2j+1)2^{-2N-1}, (j+1)2^{-2N}\right], \\ 0, & \text{if } x \in I \setminus \left[j2^{-2N}, (j+1)2^{2N}\right], \end{cases}$$

$$\varphi_N(x) = \sum_{j=1}^{2^{2N}-1} \varphi_{N,j}(x), \qquad x \in I, \tag{3.1}$$

$$g_N(x,y) = \varphi_N(x)\varphi_N(y)\,\mathrm{sgn}D_{q_N}(x)\,\mathrm{sgn}D_{q_N}(y), \qquad x,y \in I,$$

where q_N is defined in (2.5).

Suppose $\Lambda = \left\{\lambda_n = \frac{n\alpha_n}{\log(n+1)}\right\}_{n=1}^{\infty}$, where $\alpha_n \to \infty$. It is easy to show that for $s = 1,2$

$$\Lambda^{\#}V_s(g_N) \le c \sum_{i=1}^{2^{2N}-1} \frac{\log(i+1)}{i\alpha_i} = o\left(N^2\right) \text{ as } N \to \infty.$$

Therefore $\|g_N\|_{\Lambda^{\#}BV} = o\left(N^2\right) = \eta_N N^2$, where $\eta_N \to 0$ as $N \to \infty$. Hence, denoting $G_N = \frac{g_N}{\eta_N N^2}$, we conclude that $G_N \in \Lambda^{\#}BV$ and

$$\sup_N \|G_N\|_{\Lambda^{\#}BV} < \infty. \tag{3.2}$$

By construction of the function G_N we have

$$S_{q_N,q_N}(0,0;G_N) = \iint_{I^2} G_N(x,y)\, D_{q_N}(x)\, D_{q_N}(y)\, dxdy$$

$$(3.3) \qquad = \frac{1}{N^2\eta_N} \iint_{I^2} \varphi_N(x)\,\varphi_N(y)\,|D_{q_N}(x)|\,|D_{q_N}(y)|\, dxdy$$

$$= \frac{1}{N^2\eta_N}\left(\int_I \varphi_N(x)\,|D_{q_N}(x)|\,dx\right)^2$$

Next, using (2.4), we can write

$$\int_I \varphi_N(x)\,|D_{m_N}(x)|\,dx = \sum_{j=1}^{2^{2N}-1} \int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j}(x)\,|D_{m_N}(x)|\,dx =$$

$$= \sum_{j=1}^{2^{2N}-1} \left| D_{m_N}\left(\frac{j}{2^{2N}}\right)\right| \int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j}(x)\,dx \geq \frac{1}{2^{2N+1}} \sum_{j=1}^{2^{2N}-1} \frac{2^{2N}}{4j} \geq cN.$$

Consequently, from (3.3) we obtain

$$(3.4) \qquad |S_{q_N,q_N}(0,0;G_N)| \geq \frac{c}{\eta_N} \to \infty \quad \text{as} \quad N \to \infty.$$

According to the Banach-Steinhaus Theorem, (3.2) and (3.4) imply that there exists a continuous function $f \in \left\{\frac{n\alpha_n}{\log(n+1)}\right\}^{\#} BV$ such that

$$\sup_N |S_{N,N}(0,0;f)| = +\infty.$$

Theorem 3.3 is proved. □

As an immediate consequence of Theorems 1.2 and 3.3 we have the following result.

Theorem 3.4. *Let the function $f(x,y)$, $(x,y) \in I^2$, satisfy the condition*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f,n)\log(n+1)}{n^2} < \infty, \quad s = 1,2.$$

Then the double Walsh-Fourier series of f converges to $f(x,y)$ at any point $(x,y) \in I^2$ where f is continuous.

4. Cesáro means of negative order for two-dimensional Walsh-Fourier series

The problem of summability of Cesáro means of negative order for one dimensional Walsh-Fourier series was studied in the papers [8], [26]. In the two-dimensional case the summability of Walsh-Fourier series by Cesáro method of negative order for functions of partial bounded variation was investigated by the first author in [9], [11]. In particular, the following results were obtained.

Theorem 4.1 (Goginava [9]). *Let $f \in C_w(I^2) \cap PBV$ and $\alpha, \beta > 0$, $\alpha+\beta < 1$. Then the double Walsh-Fourier series of the function f is uniformly $(C; -\alpha, -\beta)$ summable in the sense of Pringsheim.*

Theorem 4.2 (Goginava [9]). *Let $\alpha, \beta > 0$, $\alpha+\beta \geq 1$. Then there exists a continuous function $f_0 \in PBV$ such that the Cesàro $(C; -\alpha, -\beta)$ means $\sigma_{n,n}^{-\alpha,-\beta}(0,0; f_0)$ of the double Walsh-Fourier series of f_0 diverge.*

Theorem 4.3 (Goginava [11]). *Let $f \in C\left(\{i^{1-\alpha}\}, \{i^{1-\beta}\}\right) V(I^2)$ with $\alpha, \beta \in (0,1)$. Then the $(C, -\alpha, -\beta)$-means of double Walsh-Fourier series converge to $f(x,y)$, if f is continuous at (x,y).*

Theorem 4.4 (Goginava [11]). *Let $\alpha, \beta \in (0,1)$, $\alpha+\beta < 1$. The following assertions hold:*

a) *If $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log^{1+\varepsilon}(n+1)}\right\} BV(I^2)$ for some $\varepsilon > 0$, then the double Walsh-Fourier series of the function f is $(C; -\alpha, -\beta)$ summable to $f(x,y)$, if f is continuous at (x,y).*

b) *There exists a continuous function $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log(n+1)}\right\} BV(I^2)$ such that the means $\sigma_{2^n,2^n}^{-\alpha,-\beta}(0,0; f)$ diverge.*

In this paper we prove the following result.

Theorem 4.5. *The following assertions hold:*

a) *Let $\alpha, \beta \in (0,1)$, $\alpha+\beta < 1$ and $f \in \{n^{1-(\alpha+\beta)}\}^{\#} BV$. Then the means $\sigma_{n,m}^{-\alpha,-\beta}(x,y; f)$ converge to $f(x,y)$, if f is continuous at (x,y).*

b) *Let $\Lambda := \{n^{1-(\alpha+\beta)}\xi_n\}$, where $\xi_n \uparrow \infty$ as $n \to \infty$. Then there exists a function $f \in C(I^2) \cap C\Lambda^{\#}V$ for which the $(C; -\alpha, -\beta)$-means of double Walsh-Fourier series diverge unboundedly at $(0,0)$.*

Proof. The assertion a) immediately follows from Corollary 1.2 and Theorem 4.3. To prove part b) of the theorem, observe first that

$$\left\{n^{1-(\alpha+\beta)}\sqrt{\xi_n}\right\}^{\#} BV \subset C\left\{n^{1-(\alpha+\beta)}\xi_n\right\}^{\#} V,$$

and since $\xi_n \uparrow \infty$ is arbitrary, it is enough to show that there exists a continuous function $f \in \Lambda^{\#} BV$ for which $(C; -\alpha, -\beta)$-means of double Walsh-Fourier series diverge unboundedly at $(0,0)$.

Denote

$$h_N(x,y) := \varphi_N(x)\varphi_N(y)\operatorname{sgn}K_{2^{2N}}^{-\alpha}(x)\operatorname{sgn}K_{2^{2N}}^{-\beta}(y),$$

where φ_N is defined in (3.1), and the kernel K_n^{α} is defined in (2.11). It is easy to show that for $s=1,2$ and $N\to\infty$ we have

$$\left\{n^{1-(\alpha+\beta)}\xi_n\right\}^{\#} V_s(h_N) \le c(\alpha,\beta)\sum_{i=1}^{2^{2N}-1}\frac{1}{i^{1-(\alpha+\beta)}\xi_i} = o\left(2^{2N(\alpha+\beta)}\right).$$

Hence

$$\|h_N\|_{\Lambda^{\#}BV} = o\left(2^{2N(\alpha+\beta)}\right) = \eta_N 2^{2N(\alpha+\beta)},$$

where $\eta_N = o(1)$ as $N\to\infty$. Consequently, denoting

$$H_N(x,y) = \frac{h_N(x,y)}{\eta_N 2^{2N(\alpha+\beta)}},$$

we conclude that $H_N \in C(I^2)\cap\Lambda^{\#}BV$ and

$$\sup_N \|H_N\|_{\Lambda^{\#}BV} < \infty. \tag{4.1}$$

By construction of the function H_N, we have

$$\sigma_{2^{2N},2^{2N}}^{-\alpha,-\beta}(0,0;H_N) = \iint_{I^2} H_N(x,y)\,K_{2^{2N}}^{-\alpha}(x)\,K_{2^{2N}}^{-\beta}(y)\,dxdy$$

$$= \frac{1}{\eta_N 2^{2N(\alpha+\beta)}}\iint_{I^2} h_N(x,y)\,K_{2^{2N}}^{-\alpha}(x)\,K_{2^{2N}}^{-\beta}(y)\,dxdy \tag{4.2}$$

$$= \frac{1}{\eta_N 2^{2N(\alpha+\beta)}}\int_I \varphi_N(x)\left|K_{2^{2N}}^{-\alpha}(x)\right|dx\int_I \varphi_N(y)\left|K_{2^{2N}}^{-\beta}(y)\right|dy.$$

Now, using the following estimate from [26]:

$$\int_{2^{m-N-1}}^{2^{m-N}}\left|K_{2^N}^{-\alpha}(x)\right|dx \ge c(\alpha)\,2^{m\alpha},\quad N\in\mathbb{N},\quad m=1,\dots,N,\quad 0<\alpha<1,$$

we can write

$$\int_I \varphi_N(x)\left|K_{2^{2N}}^{-\alpha}(x)\right|dx = \sum_{j=1}^{2^{2N}-1}\int_{j2^{-2N}}^{(j+1)2^{-2N}}\varphi_{N,j}(x)\left|K_{2^{2N}}^{-\alpha}(x)\right|dx \tag{4.3}$$

$$= \sum_{j=1}^{2^{2N}-1}\left|K_{2^{2N}}^{-\alpha}\left(\frac{j}{2^{2N}}\right)\right|\int_{j2^{-2N}}^{(j+1)2^{-2N}}\varphi_{N,j}(x)\,dx = \frac{1}{2}\sum_{j=1}^{2^{2N}-1}\left|K_{2^{2N}}^{-\alpha}\left(\frac{j}{2^{2N}}\right)\right|\int_{j2^{-2N}}^{(j+1)2^{-2N}}dx$$

$$= \frac{1}{2}\sum_{j=1}^{2^{2N}-1}\int_{j2^{-2N}}^{(j+1)2^{-2N}}\left|K_{2^{2N}}^{-\alpha}(x)\right|dx = \frac{1}{2}\sum_{m=0}^{2N-1}\sum_{j=2^m}^{2^{m+1}-1}\int_{j2^{-2N}}^{(j+1)2^{-2N}}\left|K_{2^{2N}}^{-\alpha}(x)\right|dx$$

$$=\frac{1}{2}\sum_{m=0}^{2N-1}\int\limits_{2^{m-2N}}^{2^{m+1-2N}}\left|K_{2^{2N}}^{-\alpha}(x)\right|dx\geq c(\alpha)\sum_{m=0}^{2N-1}2^{m\alpha}\geq c(\alpha)\,2^{2N\alpha}.$$

Similarly, we can prove that

$$\int\limits_{I}\varphi_N(x)\left|K_{2^{2N}}^{-\beta}(x)\right|dx\geq c(\beta)\,2^{2N\beta},\quad N\in\mathbf{N},\quad 0<\beta<1. \tag{4.4}$$

Combining (4.3) and (4.4) we get

$$\left|\sigma_{2^{2N},2^{2N}}^{-\alpha,-\beta}(0,0;H_N)\right|\geq\frac{c(\alpha,\beta)}{\eta_N}\to\infty\text{ as }N\to\infty. \tag{4.5}$$

Applying the Banach-Steinhaus theorem, from (4.1) and (4.5) we infer that there exists a continuous function $f\in\Lambda^{\#}BV$ such that

$$\sup_N\left|\sigma_{N,N}^{-\alpha,-\beta}(0,0,;f)\right|=+\infty.$$

Theorem 4.5 is proved. □

Taking into account the embedding $\Lambda^{*}BV\subset\Lambda^{\#}BV$, from Theorem 4.5 we obtain the following result.

Corollary 4.1. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$ and $f\in\left\{n^{1-(\alpha+\beta)}\right\}^{*}BV$. Then the means $\sigma_{n,m}^{-\alpha,-\beta}(x,y;f)$ converge to $f(x,y)$, if f is continuous at (x,y).*

A combination of Theorems 1.3 and 4.5 yields the following result.

Theorem 4.6. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$ and*

$$\sum_{j=1}^{\infty}\frac{v_s^{\#}(f;2^j)}{2^{j(1-(\alpha+\beta))}}<\infty\quad\text{for}\quad s=1,2.$$

Then the means $\sigma_{n,m}^{-\alpha,-\beta}(x,y;f)$ converge to $f(x,y)$, if f is continuous at (x,y).

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INTEGRALS OF CHEBYSHEV POLYNOMIALS OF THIRD AND FOURTH KINDS: AN APPLICATION TO SOLUTION OF BOUNDARY VALUE PROBLEMS WITH POLYNOMIAL COEFFICIENTS

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Abstract. Two new formulae expressing explicitly the repeated integrals of Chebyshev polynomials of third and fourth kinds of arbitrary degree in terms of the same polynomials are derived. The method of proof is novel and essentially based on making use of the power series representation of these polynomials and their inversion formulae. Using the Galerkin spectral method, we show that those formulae can be used to solve some high-order boundary value problems with varying coefficients, and propose two Galerkin-type algorithms for solving the integrated forms of some high-order boundary value problems with polynomial coefficients. A numerical example is discussed, which shows that the proposed algorithms are more accurate and efficient compared with the analytical ones.

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1. Introduction

Spectral methods have been extensively used in applied mathematics and scientific computing to obtain numerical solutions of ordinary and partial differential equations (see Boyd [5] and Canuto et al. [6]). These numerical solutions are written as expansions in terms of certain "basis functions which may be expressed in terms of various orthogonal polynomials. Spectral methods have advantage that they take on a global approach, while finite-element methods use a local approach, and as a consequence, spectral methods have "good" error properties and converge exponentially.

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ play an important role in mathematical analysis and its applications (see Abramowitz and Stegun [3], Andrews et al. [4] and Boyd [5]). In particular, the Legendre, the Chebyshev and the ultraspherical polynomials, which are special classes of Jacobi polynomials, have already played an important role in the spectral methods for solving ordinary and partial differential equations.

The Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. It is well-known that there are four kinds of Chebyshev polynomials. They all are special cases of the classical Jacobi polynomials. A large number of books and research articles deal with the first and second kinds of Chebyshev polynomials $T_n(x)$ and $U_n(x)$ and their various applications (see Boyd [5], Doha et al. [14], Julien and Watson [17], and references therein). However, there is only a few number of publications devoted to the Chebyshev polynomials of third and fourth kinds $V_n(x)$ and $W_n(x)$ (see, e.g., Doha et al. [13] and Eslahchi et al. [16]). This motivates our interest to such polynomials.

The study of both high even-order and high odd-order boundary-value problems (BVP's) is of interest. For instance, the third order equations are of mathematical and physical interest, since they lack symmetry and, in addition, contain an important type of operators which appears in many commonly occurring partial differential equations, such as the Kortewege-de Vries equation. Another important example is the sixth-order boundary-value problem, which arise in astrophysics. Due to their great importance in various applications in many fields, high-order boundary-value problems have been extensively discussed by a number of authors (see, e.g., [15, 19, 22, 24, 25, 26]). In a series of papers [1, 2, 9, 10, 11, 13], the authors dealt with such equations by using the Galerkin or Petrov-Galerkin methods. Using compact combinations of various orthogonal polynomials, they have constructed suitable bases functions which satisfy the boundary conditions of the given differential equation.

An alternative approach is to integrate the differential equation q times, where q is the order of the equation. The advantage of this approach is that the underlying equation resulted an algebraic system that contains a finite number of terms. Doha et al. [12] have followed this approach, to solve the integrated forms of third- and fifth-order elliptic differential equations using general parameters of the generalized Jacobi polynomials. Some other papers were concerned with obtaining analytical formulae for the q times repeated integration of some orthogonal polynomials (see, e.g. Doha [7, 8], and Phillips and Karageorghis [21]).

In this paper, we derive two new formulae that express explicitly the repeated integrals of Chebyshev polynomials of third and fourth kinds in terms of the same polynomials. Then using these formulae, we develop two Galerkin-type algorithms, (C3GM) and (C4GM), for solving the integrated forms of some high even-order differential equations with polynomial coefficients.

The paper is organized as follows. In Section 2, some properties of Chebyshev polynomials of third and fourth kinds are given, and some new relations of these polynomials are stated and proved. In Section 3, we derive two new formulae which express explicitly the repeated integrals of Chebyshev polynomials of third and fourth kinds in terms of the same polynomials. In Section 4, we present two Galerkin-type algorithms for solving the integrated forms of some high-order boundary value problems with polynomial coefficients. In Section 5, a numerical example is discussed to demonstrate the accuracy and efficiency of the algorithms proposed in Section 4.

2. Some properties of Chebyshev polynomials of third and fourth kinds

Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of third and fourth kinds are polynomials in x, which can be defined by one of the following two equivalent forms (see Mason and Handscomb [20]):

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}} = \frac{2^{2n}}{\binom{2n}{n}}\, P_n^{(-\frac{1}{2},\frac{1}{2})}(x),$$

and

$$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} = \frac{2^{2n}}{\binom{2n}{n}}\, P_n^{(\frac{1}{2},-\frac{1}{2})}(x),$$

where $x = \cos\theta$, and $P_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomial of degree n.

It is clear that

$$W_n(x) = (-1)^n V_n(-x). \tag{2.1}$$

The polynoamils $V_n(x)$ and $W_n(x)$ are orthogonal on $(-1,1)$ with respect to the weight functions $\sqrt{\frac{1+x}{1-x}}$ and $\sqrt{\frac{1-x}{1+x}}$, respectively, that is, we have

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\, V_n(x)\, V_m(x)\, dx = \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\, W_n(x)\, W_m(x)\, dx = \begin{cases} 0, & \text{if } m\neq n,\\ \pi, & \text{if } m=n, \end{cases} \tag{2.2}$$

and can be generated by using the following two recurrence relations:

$$V_n(x) = 2x\, V_{n-1}(x) - V_{n-2}(x), \quad n = 2,3,\ldots, \tag{2.3}$$

with $V_0(x) = 1$, $V_1(x) = 2x-1$, and

$$W_n(x) = 2x\, W_{n-1}(x) - W_{n-2}(x), \quad n = 2,3,\ldots,$$

with $W_0(x) = 1$, $W_1(x) = 2x+1$. Below the following special values will be of importance:

$$V_n(1) = (-1)^n W_n(-1) = 1, \tag{2.4}$$

$$W_n(1) = (-1)^n V_n(-1) = 2n+1, \tag{2.5}$$

$$D^q V_n(1) = (-1)^{n+q} D^q W_n(-1) = \prod_{k=0}^{q-1} \frac{(n-k)(n+k+1)}{2k+1}, \ q \geq 1, \tag{2.6}$$

$$D^q W_n(1) = (-1)^{n+q} D^q V_n(-1) = (2n+1) \prod_{k=0}^{q-1} \frac{(n-k)(n+k+1)}{2k+3}, \ q \geq 1. \tag{2.7}$$

The following two theorems and lemma are needed in the sequel.

Theorem 2.1. *The explicit power form of the polynomial $V_n(x)$, $n \geq 1$ is given by the formula*

$$V_n(x) = \sum_{k=0}^{[\frac{n}{2}]} a_{n,k}\, x^{n-2k} + \sum_{k=0}^{[\frac{n}{2}]} b_{n,k}\, x^{n-2k-1}, \tag{2.8}$$

where

$$a_{n,k} = \frac{(-1)^k (n-k)!\, 2^{n-2k}}{k!\,(n-2k)!}, \quad b_{n,k} = \frac{(-1)^{k+1} (n-k-1)!\, 2^{n-2k-1}}{k!\,(n-2k-1)!}. \tag{2.9}$$

Proof. We proceed by induction on n. Assume that the relation (2.8) holds for $(n-1)$ and $(n-2)$. Then starting with the recurrence relation (2.3) and applying the induction hypothesis twice, we obtain

$$V_n(x) = 2 \sum_{k=0}^{[\frac{n-1}{2}]} a_{n-1,k}\, x^{n-2k} - \sum_{k=0}^{[\frac{n}{2}]-1} a_{n-2,k}\, x^{n-2k-2} + \tag{2.10}$$

$$+2 \sum_{k=0}^{[\frac{n-1}{2}]} b_{n-1,k}\, x^{n-2k-1} - \sum_{k=0}^{[\frac{n}{2}]-1} b_{n-2,k}\, x^{n-2k-3},$$

which can be written in the form

$$V_n(x) = \sum_1 + \sum_2,$$

where

$$\sum_1 = -a_{n-2,\frac{n}{2}-1}\, \delta_n + 2\, a_{n-1,0}\, x^n + \sum_{k=1}^{[\frac{n-1}{2}]} \{2\, a_{n-1,k} - a_{n-2,k-1}\}\, x^{n-2k},$$

$$\sum_2 = 2\, b_{n-1,0}\, x^{n-1} + \sum_{k=1}^{[\frac{n-1}{2}]} \{2\, b_{n-1,k} - b_{n-2,k-1}\}\, x^{n-2k-1},$$

and $\delta_n = \begin{cases} 1, & \text{n even,} \\ 0, & \text{n odd.} \end{cases}$

It is not difficult to show that

$2\,a_{n-1,0} = a_{n,0},\ -a_{n-2,\frac{n}{2}-1} = a_{n,\frac{n}{2}},\ 2\,a_{n-1,k} - a_{n-2,k-1} = a_{n,k},\ 1 \le k \le \left[\frac{n-1}{2}\right],$

$2\,b_{n-1,0} = b_{n,0},\ 2\,b_{n-1,k} - b_{n-2,k-1} = b_{n,k},\ 1 \le k \le \left[\frac{n-1}{2}\right],$

and therefore we can write

$$\sum_1 = \sum_{k=0}^{[\frac{n}{2}]} a_{n,k}\, x^{n-2k},$$

and

$$\sum_2 = \sum_{k=0}^{[\frac{n}{2}]} b_{n,k}\, x^{n-2k-1},$$

where $a_{n,k}$ and $b_{n,k}$ are given in (2.9). This completes the proof of Theorem 2.1. □

The next theorem was proved in Doha et al. [13].

Theorem 2.2. *For all $k, m \in \mathbf{Z}^+$, we have*

$$x^m\, V_k(x) = \frac{1}{2^m} \sum_{s=0}^{m} \binom{m}{s} V_{k+m-2s}(x). \tag{2.11}$$

In particular, the following inversion formula holds

$$x^m = \frac{1}{2^m} \sum_{s=0}^{m} \binom{m}{s} V_{m-2s}(x). \tag{2.12}$$

Lemma 2.1. *For every nonnegative integer r and a natural $n > r$, we have*

$$\sum_{j=0}^{r} \frac{(-1)^j\,(n-j-1)!}{j!\,(n-j+q-r)!\,(r-j)!} = \frac{(-1)^r\, q!\,(n-r-1)!}{r!\,(q-r)!\,(n+q-r)!}. \tag{2.13}$$

Proof. Setting

$$M_{n,q,r} = \sum_{j=0}^{r} \frac{(-1)^j\,(n-j-1)!}{j!\,(n-j+q-r)!\,(r-j)!},$$

and using Zeilberger's algorithm (see, e.g., Koepf [18]), we conclude that $M_{n,q,r}$ satisfies the following difference equation of order one:

$$(r+1)(n-r-1)\,M_{n,q,r+1} + (q-r)(n+q-r)\,M_{n,q,r} = 0, \quad M_{n,q,0} = \frac{(n-1)!}{(n+q)!},$$

which can be solved to obtain

$$M_{n,q,r} = \frac{(-1)^r\, q!\,(n-r-1)!}{r!\,(q-r)!\,(n+q-r)!}.$$

Lemma 2.1 is proved. □

Remark 2.1. *The counterparts of Theorems 2.1 and 2.2 for the polynomials $W_n(x)$ can easily be deduced with the aid of relation* (2.1).

3. Formulas for repeated integrals of Chebyshev polynomials $V_n(x)$ and $W_n(x)$

The objective of this section is to state and prove two theorems, which express explicitly the repeated integrals of Chebyshev polynomials $V_n(x)$ and $W_n(x)$ in terms of the same polynomials.

Given a natural number q, the q-times repeated integral of the third kind Chebyshev polynomial $V_n(x)$ is denoted by

$$I_n^{(q)}(x) = \int^{(q)} V_n(x)\,(dx)^q = \overbrace{\int\int\dots\int}^{q\ times} V_n(x)\,\overbrace{dx\,dx\dots dx}^{q\ times}.$$

Theorem 3.1. *The following formula holds.*

$$I_n^{(q)}(x) = \sum_{r=0}^{q} A_{n,r,q}\, V_{n+q-2r}(x) + \sum_{r=0}^{q} B_{n,r,q}\, V_{n+q-2r-1}(x) + \pi_{q-1}(x), \tag{3.1}$$

where

$$A_{n,r,q} = \frac{(-1)^r\,(n-r)!\,q!}{2^q\,r!\,(q-r)!\,(n+q-r)!}, \quad B_{n,r,q} = \frac{(-1)^{r+1}\,(n-r-1)!\,q!}{2^q\,r!\,(q-r-1)!\,(n+q-r)!},$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Proof. Integrating the relation (2.8) q-times, and using the equality

$$\int^{(q)} x^i\,(dx)^q = \frac{x^{i+q}}{(i+1)_q} + \pi_{q-1}(x),$$

we get

$$I_n^{(q)}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} e_{n,k,q}\, x^{n-2k-q} + \sum_{k=0}^{\left[\frac{n}{2}\right]} f_{n,k,q}\, x^{n-2k+q-1} + \pi_{q-1}(x),$$

where

$$e_{n,k,q} = \frac{(-1)^k\,2^{n-2k}(n-k)!}{k!\,(n-2k+q)!}, \quad f_{n,k,q} = \frac{(-1)^{k+1}\,2^{n-2k-1}(n-k-1)!}{k!\,(n-2k+q-1)!}. \tag{3.2}$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Taking into account the relation (2.12), we can write

$$I_n^{(q)}(x) = \sum_1 + \sum_2 + \pi_{q-1}(x),$$

where

$$\sum_1 = \sum_{k=0}^{[\frac{q}{2}]} e_{n,k,q} \sum_{i=0}^{[\frac{n+q-1}{2}]-k} c_{i,n-2k+q}\, V_{n+q-2k-2i}(x) +$$

$$\sum_{k=0}^{[\frac{q}{2}]} f_{n,k,q} \sum_{i=0}^{[\frac{n+q}{2}]-k-1} c_{i,n-2k+q-1}\, V_{n+q-2k-2i-2}(x),$$

$$\sum_2 = \sum_{k=0}^{[\frac{q}{2}]} e_{n,k,q} \sum_{i=0}^{[\frac{n+q-1}{2}]-k} c_{i,n-2k+q}\, V_{n+q-2k-2i-1}(x) +$$

$$\sum_{k=0}^{[\frac{q}{2}]} f_{n,k,q} \sum_{i=0}^{[\frac{n+q-1}{2}]-k} c_{i,n-2k+q-1}\, V_{n-q-2k-2i-3}(x),$$

the coefficients $e_{n,k,q}$ and $f_{n,k,q}$ are given in (3.2) and

$$c_{i,m} = \frac{\binom{m}{i}}{2^m}.$$

Expanding $\sum_1$ and $\sum_2$, and collecting similar terms, after some algebra we get

$$I_n^{(q)}(x) = \sum_{r=0}^{q} A_{n,r,q}\, V_{n+q-2r}(x) + \sum_{r=0}^{q} B_{n,r,q}\, V_{n+q-2r-1}(x) + \pi_{q-1}(x),$$

where

$$A_{n,r,q} = \sum_{j=0}^{r} \left\{ e_{n,j,q}\, c_{r-j,n+q-2j} + f_{n,j,q}\, c_{r-j-1,n+q-2j-1} \right\}, \tag{3.3}$$

$$B_{n,r,q} = \sum_{j=0}^{r} \left\{ e_{n,j,q}\, c_{r-j,n+q-2j} + f_{n,j,q}\, c_{r-j,n+q-2j-1} \right\}. \tag{3.4}$$

Next, it is not difficult to show that

$$e_{n,j,q}\, c_{r-j,n+q-2j} + f_{n,j,q}\, c_{r-j-1,n+q-2j-1} = \frac{(-1)^j\, 2^{-q}\, (n-r)\, (n-j-1)!}{j!\, (n+q-j-r)!\, (r-j)!}, \tag{3.5}$$

and

$$e_{n,j,q}\, c_{r-j,n+q-2j} + f_{n,j,q}\, c_{r-j,n+q-2j-1} = \frac{(-1)^{j+1}\, 2^{-q}\, (q-r)\, (n-j-1)!}{j!\, (n+q-j-r)!\, (r-j)!}. \tag{3.6}$$

Finally, substituting the relations (3.5) and (3.6) into (3.3) and (3.4), and using (2.13), for $A_{n,r,q}$ and $B_{n,r,q}$ we obtain

$$A_{n,r,q} = \frac{(-1)^r\, 2^{-q}\, (n-r)!\, q!}{r!\, (q-r)!\, (n+q-r)!},$$

$$B_{n,r,q} = \frac{(-1)^{r+1}\, 2^{-q}\, (n-r-1)!\, q!}{r!\, (q-r-1)!\, (n+q-r)!},$$

and the result follows. Theorem 3.1 is proved. □

Remark 3.1. Note that the relation (3.1) may be written in the following equivalent form:

$$I_n^{(q)}(x) = \sum_{i=0}^{2q} E_{n,i,q}\, V_{n+q-i}(x) + \pi_{q-1}(x), \quad n \ge q \ge 1, \tag{3.7}$$

where

$$E_{n,i,q} = \frac{q!}{2^q} \begin{cases} \dfrac{(-1)^{\frac{i}{2}}\ (n-\frac{i}{2})!}{(\frac{i}{2})!\ (q-\frac{i}{2})!\ (n+q-\frac{i}{2})!}, & \text{if } i \text{ is even,} \\ \dfrac{(-1)^{\frac{i+1}{2}}\ (n-(\frac{i+1}{2}))!}{(\frac{i-1}{2})!\ (q-(\frac{i+1}{2}))!\ (n+q-(\frac{i-1}{2}))!}, & \text{if } i \text{ is odd,} \end{cases} \tag{3.8}$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Using the arguments of the proof of Theorem 3.1 and formula (2.1), we can obtain a formula that express explicitly the repeated integral of Chebyshev fourth kind polynomial $W_n(x)$ in terms of the same polynomial. The corresponding result is stated in the following theorem.

Theorem 3.2. *Let $J_n^{(q)}(x)$ be the q-times repeated integral of the polynomial $W_n(x)$:*

$$J_n^{(q)}(x) = \int^{(q)} W_n(x)(dx)^q,$$

then

$$J_n^{(q)}(x) = \sum_{i=0}^{2q} S_{n,i,q}\, W_{n+q-i}(x) + \bar{\pi}_{q-1},$$

where

$$S_{n,i,q} = (-1)^i\, E_{n,i,q}, \tag{3.9}$$

and $\bar{\pi}_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

4. An application to a high-order two point boundary value problem

In this section, we are interested in applying the formulas, obtained in Section 3, to solve the following high-order boundary value problem:

$$(-1)^n\, u^{(2n)}(x) + \gamma\, p(x)\, u(x) = f(x), \quad x \in (-1,1),\ n \ge 1, \tag{4.1}$$

subject to the nonhomogeneous Dirichlet boundary conditions

$$u^{(j)}(\pm 1) = \pm\alpha_j, \quad 0 \le j \le n-1, \tag{4.2}$$

where $p(x)$ is a given polynomial and γ is a real constant.

It is worth to note that if we use the transformation:

$$y(x) = u(x) + \sum_{i=0}^{2n-1} \xi_i x^i,$$

where ξ_i, $0 \le i \le 2n-1$, are coefficients to be determined such that $y(x)$ satisfies the homogeneous boundary conditions

$$y^{(j)}(\pm 1) = 0, \quad 0 \le j \le n-1, \tag{4.3}$$

then the equation (4.1) takes the form

$$(-1)^n y^{(2n)}(x) + \gamma p(x) y(x) = g(x), \quad x \in (-1,1),\ n \ge 1, \tag{4.4}$$

where

$$g(x) = f(x) + \sum_{i=0}^{2n-1} \eta_i x^i,$$

and η_i, $0 \le i \le 2n-1$ are some constants that are determined in terms of ξ_i. For details we refer to Doha et al. [13].

In what follows, we take $p(x) = x^{\mu}$, $\mu \in \mathbb{Z}^{\ge 0}$, and instead of the problem (4.4) subject to (4.3), consider its integrated form:

(4.5)

$$\left.\begin{aligned} &(-1)^n y(x) + \gamma \int^{(2n)} x^{\mu} y(x) (dx)^{(2n)} = h(x) + \sum_{i=0}^{2n-1} \alpha_i x^i, \quad x \in (-1,1), \\ &y^{(j)}(\pm 1) = 0, \quad 0 \le j \le n-1, \quad h(x) = \int^{(2n)} g(x) (dx)^{(2n)}, \end{aligned}\right\}$$

where α_i are arbitrary constants, and

$$\int^{(q)} y(x) (dx)^q = \overbrace{\int\int \dots \int}^{q \text{ times}} y(x) \overbrace{dx\, dx \dots dx}^{q \text{ times}}.$$

Define the following spaces

$$\begin{aligned} S_N &= \text{span}\{V_0(x), V_1(x), V_2(x), \dots, V_N(x)\}, \\ \bar{S}_N &= \text{span}\{W_0(x), W_1(x), W_2(x), \dots, W_N(x)\}, \\ X_N &= \{v(x) \in S_N \ : D^j v(\pm 1) = 0,\ 0 \le j \le n-1\}, \\ \bar{X}_N &= \{\bar{v}(x) \in \bar{S}_N \ : D^j \bar{v}(\pm 1) = 0,\ 0 \le j \le n-1\}. \end{aligned}$$

Then the Chebyshev third and fourth kinds Galerkin procedures for solving (4.5) consist of finding $y_N^n(x) \in X_N$ and $\bar{y}_N^n(x) \in \bar{X}_N$ to satisfy

$$
\begin{aligned}
&\left((-1)^n y_N^n(x), v(x)\right)_{w_1(x)} + \gamma \left(\int^{(2n)} x^\mu y_N^n(x)(dx)^{(2n)}, v(x)\right)_{w_1(x)} \\
&= \left(h(x) + \sum_{i=0}^{2n-1} b_i V_i(x), v(x)\right)_{w_1(x)}, \quad 0 \le k \le N-2n, \quad \forall\, v(x) \in X_N,
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
&\left((-1)^n \bar{y}_N^n(x), \bar{v}(x)\right)_{w_2(x)} + \gamma \left(\int^{(2n)} x^\mu \bar{y}_N^n(x)(dx)^{(2n)}, \bar{v}(x)\right)_{w_2(x)} \\
&= \left(h(x) + \sum_{i=0}^{2n-1} \bar{b}_i W_i(x), \bar{v}(x)\right)_{w_2(x)}, \quad 0 \le k \le N-2n, \quad \forall\, \bar{v}(x) \in \bar{X}_N,
\end{aligned} \tag{4.7}
$$

where $w_1(x) = \sqrt{\frac{1+x}{1-x}}$, $w_2(x) = \sqrt{\frac{1-x}{1+x}}$, $(u(x), v(x))_{w_i(x)} = \int_{-1}^{1} w_i(x)\, u(x)\, v(x)\, dx$ is the inner product in the weighted space $L^2_{w_i(x)}(-1,1)$, and b_i, $\bar{b}_i$, $i = 1, 2$ are some constants.

We can construct two kinds of bases functions as compact combinations of the Chebyshev polynomials of third and fourth kinds by setting

$$
\phi_{k,n}(x) = V_k(x) + \sum_{m=1}^{2n} d_{m,k}\, V_{k+m}(x), \quad 0 \le k \le N-2n, \; n \ge 1, \tag{4.8}
$$

$$
\psi_{k,n}(x) = W_k(x) + \sum_{m=1}^{2n} \bar{d}_{m,k}\, W_{k+m}(x), \quad 0 \le k \le N-2n, \; n \ge 1, \tag{4.9}
$$

where the coefficients $\{d_{m,k}\}$ and $\{\bar{d}_{m,k}\}$ are chosen such that $\phi_{k,n}(x) \in X_{k+2n}$ and $\psi_{k,n}(x) \in \bar{X}_{k+2n}$. In view of relations (2.4)-(2.7), the boundary conditions (4.3) lead to the following linear system to determine the constants $\{d_{m,k}\}$:

$$\begin{cases} 1+\sum_{m=1}^{2n+1} d_{m,k}=0, \\ \sum_{m=1}^{2n+1}(-1)^m\,(2\,k+2\,m+1)\,d_{m,k}=-(2\,k+1), \\ \prod_{s=0}^{q-1}(k-s)\,(k+s+1)+\sum_{m=1}^{2\,n+1} d_{m,k}\prod_{s=0}^{q-1}(k+m-s)(k+m+s+1)=0, \\ (2\,k+1)\prod_{s=0}^{q-1}(k-s)\,(k+s+1)+\sum_{m=1}^{2\,n+1}(-1)^m\,d_{m,k}\,(2\,k+2m+1)\times \\ \prod_{s=0}^{q-1}(k+m-s)(k+m+s+1)=0, \\ \qquad 0\le k\le N-2n, \quad \text{and} \quad 1\le q\le n-1. \end{cases}$$

The determinant of the above system is different from zero, hence $\{d_{m,k}\}$ can be uniquely determined to obtain

$$d_{m,k}=\begin{cases} \dfrac{(-1)^{\frac{m}{2}}\binom{n}{\frac{m}{2}}\,(k+1)_{\frac{m}{2}}}{(k+n+2)_{\frac{m}{2}}}, & \text{if } m \text{ is even,} \\[2ex] \dfrac{(-1)^{\frac{m+1}{2}}\binom{n}{\frac{m-1}{2}}\,(k+1)_{\frac{m-1}{2}}}{(k+n+2)_{\frac{m-1}{2}}}, & \text{if } m \text{ is odd,} \end{cases} \tag{4.10}$$

and hence the basis functions $\phi_{k,n}(x)$ take the form:

$$\phi_{k,n}(x)=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}(k+1)_m}{(k+n+1)_m}V_{k+2m}(x)+ \sum_{m=0}^{n-1}\frac{(-1)^m\binom{n}{m}(m-n)\,(k+1)_m}{(k+n+1)_{m+1}}V_{k+2m+1}(x). \tag{4.11}$$

Similarly, the constants $\bar{d}_{m,k}$ can be uniquely determined to obtain

$$\bar{d}_{m,k}=(-1)^m\,d_{m,k}, \tag{4.12}$$

and hence

$$\psi_{k,n}(x)=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}(k+1)_m}{(k+n+1)_m}W_{k+2m}(x)+ \sum_{m=0}^{n-1}\frac{(-1)^{m+1}\binom{n}{m}(m-n)\,(k+1)_m}{(k+n+1)_{m+1}}W_{k+2m+1}(x). \tag{4.13}$$

It is obvious that $\{\phi_{k,n}(x)\}$ and $\{\psi_{k,n}(x)\}$ are linearly independent. Therefore we have

$$X_N = \text{span}\{\phi_{k,n}(x) : 0 \le k \le N-2n\}, \quad \bar{X}_N = \text{span}\{\psi_{k,n}(x) : 0 \le k \le N-2n\}.$$

Thus, the variational relations (4.6) and (4.7) are respectively equivalent to the following:

$$\begin{aligned}(4.14)\quad &\left((-1)^n y_N^n(x), \phi_{k,n}(x)\right)_{w_1(x)} + \left(\int^{(2n)} x^\mu y_N^n(x)(dx)^{(2n)}, \phi_{k,n}(x)\right)_{w_1(x)} \\ &= \left(h(x) + \sum_{i=0}^{2n-1} b_i V_i(x), \phi_{k,n}(x)\right)_{w_1(x)}, \quad 0 \le k \le N-2n,\end{aligned}$$

and

$$\begin{aligned}(4.15)\quad &\left((-1)^n \bar{y}_N^n(x), \psi_{k,n}(x)\right)_{w_2(x)} + \left(\int^{(2n)} x^\mu \bar{y}_N^n(x)(dx)^{(2n)}, \psi_{k,n}(x)\right)_{w_2(x)} \\ &= \left(h(x) + \sum_{i=0}^{2n-1} \bar{b}_i W_i(x), \psi_{k,n}(x)\right)_{w_2(x)}, \quad 0 \le k \le N-2n.\end{aligned}$$

Noting that the constants b_i, $\bar{b}_i$, $0 \le i \le 2n-1$, should not appear if we take $k \ge 2n$ in (4.14) and (4.15), we can write

$$\begin{aligned}(4.16)\quad &\left((-1)^n y_N^n(x), \phi_{k,n}(x)\right)_{w_1(x)} + \left(\int^{(2n)} x^\mu y_N^n(x)(dx)^{(2n)}, \phi_{k,n}(x)\right)_{w_1(x)} \\ &= \left(h(x), \phi_{k,n}(x)\right)_{w_1(x)}, \quad 2n \le k \le N,\end{aligned}$$

and

$$\begin{aligned}(4.17)\quad &\left((-1)^n \bar{y}_N^n(x), \psi_{k,n}(x)\right)_{w_2(x)} + \left(\int^{(2n)} x^\mu \bar{y}_N^n(x)(dx)^{(2n)}, \psi_{k,n}(x)\right)_{w_2(x)} \\ &= \left(h(x), \psi_{k,n}(x)\right)_{w_2(x)}, \quad 2n \le k \le N.\end{aligned}$$

Denoting

$$h_k^n = (h(x), \phi_{k,n}(x))_{w_1(x)}, \qquad \mathbf{h}^n = (h_{2n}^n, h_{2n+1}^n, \dots, h_N^n)^T,$$

$$\bar{h}_k^n = (h(x), \psi_{k,n}(x))_{w_2(x)}, \qquad \bar{\mathbf{h}}^n = (\bar{h}_{2n}^n, \bar{h}_{2n+1}^n, \dots, \bar{h}_N^n)^T,$$

$$y_N^n(x) = \sum_{m=0}^{N-2n} c_m^n\, \phi_{m,n}(x), \qquad \mathbf{c}^n = (c_0^n, c_1^n, \dots, c_{N-2n}^n)^T,$$

$$\bar{y}_N^n(x) = \sum_{m=0}^{N-2n} \bar{c}_m^n\, \psi_{m,n}(x), \qquad \bar{\mathbf{c}}^n = (\bar{c}_0^n, \bar{c}_1^n, \dots, \bar{c}_{N-2n}^n)^T,$$

$$A_n = (a_{kj}^n)_{2n \le k,j \le N}, \qquad B_n = (b_{kj}^n)_{2n \le k,j \le N},$$

$$R_n = (r_{kj}^n)_{2n \le k,j \le N}, \qquad S_n = (s_{kj}^n)_{2n \le k,j \le N},$$

the equations (4.16) and (4.17) can be written in the following equivalent matrix forms:

$$(A_n + \gamma\, B_n)\, \mathbf{c}^n = \mathbf{h}^n,$$

and

$$(R_n + \gamma\, S_n)\, \bar{\mathbf{c}}^n = \bar{\mathbf{h}}^n,$$

where the nonzero elements of the matrices A_n, B_n, R_n and S_n are given explicitly in the following two theorems.

Theorem 4.1. *Let the basis functions $\phi_{k,n}(x)$ be defined as in (4.11), and let*

$$a_{kj}^n = (-1)^n\, (\phi_{j-2n,n}(x), \phi_{k,n}(x))_{w_1(x)}$$

and

$$b_{kj}^n = \left(\int^{(2n)} x^\mu\, \phi_{j-2n,n}(x) (dx)^{(2n)}, \phi_{k,n}(x) \right)_{w_1(x)}.$$

Then

$$X_{N+2n} = span\{\phi_{0,n}(x), \phi_{1,n}(x), \dots, \phi_{N,n}(x)\},$$

and the nonzero elements of the matrices A_n and B_n are given by

$$\text{(4.18)} \qquad a_{kj}^n = (-1)^n \pi \sum_{m=0}^{2n} d_{m,j-2n}\, d_{j-k+m-2n,k},$$

$$\text{(4.19)} \qquad b_{kj}^n = \frac{\pi}{2^\mu} \sum_{m=0}^{2n} \sum_{s=0}^{\mu} \sum_{i=0}^{4n} \binom{\mu}{s} d_{m,j-2n}\, d_{\mu+j+m-i-k-2s,k}\, E_{\mu+j+m-2n-2s,i,2n},$$

where $d_{m,k}$ and $E_{n,i,q}$ are as in (4.10) and (3.8), respectively.

Theorem 4.2. *Let the basis functions $\psi_{k,n}(x)$ be defined as in (4.13), and let*

$$r_{kj}^n = (-1)^n \left(\psi_{j-2n,n}(x), \psi_{k,n}(x)\right)_{w_2(x)},$$

and

$$s_{kj}^n = \left(\int^{(2n)} x^\mu \psi_{j-2n,n}(x)(dx)^{(2n)}, \psi_{k,n}(x)\right)_{w_2(x)}.$$

Then

$$\bar{X}_{N+2n} = span\{\psi_{0,n}(x), \psi_{1,n}(x), \ldots, \psi_{N,n}(x)\},$$

and the nonzero elements of the matrices R_n and S_n are given by

$$r_{kj}^n = (-1)^n \pi \sum_{m=0}^{2n} \bar{d}_{m,j-2n}\, \bar{d}_{j-k+m-2n,k},$$

$$s_{kj}^n = \frac{\pi}{2^\mu} \sum_{m=0}^{2n} \sum_{s=0}^{\mu} \sum_{i=0}^{4n} \binom{\mu}{s} \bar{d}_{m,j-2n}\, \bar{d}_{\mu+j+m-i-k-2s,k}\, S_{\mu+j+m-2n-2s,i,2n},$$

where $\bar{d}_{m,k}$ and $S_{k,n,q}$ are as in (4.12) and (3.9), respectively.

The proofs of Theorems 4.1 and 4.2 are similar, so it suffices to prove only Theorem 4.1.

Proof of Theorem 4.1. The basis functions $\phi_{k,n}(x)$ we choose such that $\phi_{k,n}(x) \in X_{N+2n}$ for $k = 0, 1, \ldots, N$. On the other hand, it is clear that $\{\phi_{k,n}(x)\}_{0\le k\le N}$ are linearly independent and the dimension of X_{N+2n} is equal to $N+1$. Hence, we have

$$X_{N+2n} = \text{span}\{\phi_{0,n}(x), \phi_{1,n}(x), \ldots, \phi_{N,n}(x)\}.$$

To obtain the nonzero elements (a_{kj}^n) for $2n \le k, j \le N$, we use formula (4.8) to get

$$a_{kj}^n = (-1)^n \sum_{m=0}^{2n} \sum_{i=0}^{2n} d_{m,j-2n}\, d_{i,k} \left(V_{j-2n+m}(x), V_{k+i}(x)\right)_{w_1(x)},$$

which in turn, with the aid of the orthogonality relation (2.2), yields

$$a_{kj}^n = (-1)^n \pi \sum_{m=0}^{2n} d_{m,j-2n}\, d_{j-2n+m-k,k}, \quad j = k+s, s \ge 0.$$

Thus, the formula (4.18) is proved. To prove (4.19), observe that since

$$b_{kj}^n = \left(\int^{(2n)} x^\mu \phi_{j-2n,n}(x)(dx)^{(2n)}, \phi_{k,n}(x)\right)_{w_1(x)},$$

we can use the formulas (2.11) and (4.8) to obtain

$$x^{\mu}\,\phi_{j-2n}(x)=\frac{1}{2^{\mu}}\sum_{m=0}^{2n}\sum_{s=0}^{\mu}\binom{\mu}{s}d_{m,j-2n}\;V_{\mu+j+m-2n-2s}(x),$$

and therefore relation (3.7) gives

$$\int^{(2n)} x^{\mu}\,\phi_{j-2n,n}(x)(dx)^{(2n)}$$

$$=\frac{1}{2^{\mu}}\sum_{m=0}^{2n}\sum_{s=0}^{\mu}\sum_{i=0}^{4n}\binom{\mu}{s}\,d_{m,j-2n}\,E_{\mu+j+m-2n-2s,i,2n}\,V_{\mu+j+m-i-2s}(x).$$

Finally, using the orthogonality relation (2.2), we obtain

$$b_{kj}^{n}=\frac{\pi}{2^{\mu}}\sum_{m=0}^{2n}\sum_{s=0}^{\mu}\sum_{i=0}^{4n}\binom{\mu}{s}\,d_{m,j-2n}\,d_{\mu+j+m-i-k-2s,k}\,E_{\mu+j+m-2n-2s,i,2n}.$$

This completes the proof of Theorem 4.1. □

5. NUMERICAL RESULTS

In this section we give a numerical example to show the accuracy and the efficiency of the proposed algorithms.

Example 1. *Consider the following linear sixth-order boundary value problem (see, Siddiqi and Akram [23]):*

(5.1)

$$y^{(6)}(x)+(5x+1)\,y(x)=(185\,x-25\,x^{2}+10\,x^{4})\cos(x)+(270-36\,x^{2})\sin(x),\quad x\in[-1,1],$$

subject to the boundary conditions:

$$y(-1)=4\cos(1),\qquad y(1)=-2\cos(1),$$

$$y^{(1)}(-1)=\cos(1)+4\sin(1),\qquad y^{(1)}(1)=\cos(1)+2\sin(1),$$

$$y^{(2)}(-1)=-16\cos(1)+2\sin(1),\qquad y^{(2)}(1)=14\cos(1)-2\sin(1).$$

The analytical solution of this problem is given by

$$y(x)=(2x^{3}-5x+1)\cos(x).$$

Table 1 below contains the maximum pointwise error E of $|u-u_N|$ using our algorithms C3GM and C4GM for various values of N, while Table 2 contains the

TABLE 1. Maximum pointwise error for Example 1, $N = 14, 16, 18, 20, 22$.

N	C3GM	C4GM
14	2.16553×10^{-9}	2.16799×10^{-9}
16	7.20557×10^{-12}	7.21269×10^{-12}
18	1.65128×10^{-14}	1.70084×10^{-14}
20	1.3765×10^{-15}	1.38995×10^{-15}

TABLE 2. Comparison between best error for Example 1 by different methods

Best error	C3GM	C4GM	Siddiqi and Akram [23]
E	1.3765×10^{-15}	1.38995×10^{-15}	8.68×10^{-7}

best errors obtained by our methods (C3GM and C4GM) and by the septic spline method developed in [23].

Comparing the errors given in Table 2, we conclude that our two methods, C3GM and C4GM, are more accurate than the method developed in [23].

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О ХАРАКТЕРИЗАЦИИ ЭКСТРЕМАЛЬНЫХ МНОЖЕСТВ ДИФФЕРЕНЦИРОВАНИЯ ИНТЕГРАЛОВ В $\mathbb{R}^2$

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Аннотация. В настоящей работе рассматривается вопрос характеризации множеств точек недифференцируемости интегралов по базисам прямоугольников и квадратов. В частности, дана полная характеризация множеств неопределенностей для интегралов функций из $L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$, по базису квадратов.

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1. Введение

Вопросы характеризации экстремальных множеств актуальны во многих направлениях математического анализа. Экстремальные множества часто встречаются в теории ортогональных рядов и в исследованиях граничных поведений аналитических и гармонических функций. Этим вопросам посвящены много работ. Задачи характеризации множеств точек расходимости тригонометрических рядов рассматривались в работах [4, 5, 19, 10, 26, 27, 34]. Аналогичные вопросы для рядов по другим классическим ортогональным систем были рассмотрены в [2, 3, 6, 12, 14, 15, 21, 23, 30, 31]. Подробный обзор некоторых из этих результатов можно найти в статьях П. Л. Ульянова [28, 29]. Характеризациям предельных множеств аналитических функций посвящена монография Э. Коллингвуда и А. Ловатера [17].

Первоисточником для многих этих исследований является следующее утверждение.

Теорема А. [Хан-Серпинский, [8, 25]] *Для того, чтобы множество $E \subset \mathbb{R}$ было множеством расходимости (неограниченной расходимости) некоторой последовательности непрерывных функций на $\mathbb{R}$, необходимо и достаточно, чтобы оно было множеством $G_{\delta\sigma}$ (G_δ).*

Одними из редких результатов, дающих полные характеризации экстремальных множеств являются следующие теоремы.

Теорема B. [Колесников [18], 1994] *Для того, чтобы множество $E \subset \mathbf{T}$ было множеством радиальной расходимости некоторой аналитической в единичном круге функции, необходимо и достаточно, чтобы E было бы объединением двух множеств, одно из которых G_δ, а другое - нуль множество типа $G_{\delta\sigma}$.*

Теорема C. [Загорский [33], 1946] *Для того чтобы множество $E \subset \mathbb{R}$ было множеством недифференцируемости некоторой непрерывной на $\mathbb{R}$ функции, необходимо и достаточно, что E было объединением множества типа G_δ и множества типа $G_{\delta\sigma}$ меры нуль.*

В работах [11, 13] установлены общие теоремы характеризации экстремальных множеств последовательностей операторов со свойством локализации, из которых, в частности, следуют некоторые результаты, упомянутых выше работ. С помощью этих теорем, получены также полные характеризации множеств точек расходимости (C, α) средних рядов Фурье по классическим ортонормированным системам, а также обычных частичных сумм рядов Хаара и Франклина.

В настоящей работе рассматривается вопрос характеризации множеств точек недифференцируемости интегралов функций из $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$. Пусть $\mathfrak{R}$ есть множество всех полуоткрытых прямоугольников (декартовы произведения двух интервалов вида $[a, b)$) в $\mathbb{R}^2$, а через Ω обозначим множество полуоткрытых квадратов на $\mathbb{R}^2$. Очевидно имеем $\Omega \subset \mathfrak{R}$. Длину большей стороны прямоугольника $R \in \mathfrak{R}$ обозначим через $\operatorname{diam}(R)$. Пусть $\mathfrak{M}$ есть один из базисов $\mathfrak{R}$ или Ω. Для произвольной функции $f \in L^1(\mathbb{R}^2)$ определим

$$\delta_{\mathfrak{M}}(x, f) = \limsup_{\operatorname{diam}(R)\to 0:\, x\in R\in\mathfrak{M}} \left|\frac{1}{|R|}\int_R f(t)dt - f(x)\right|.$$

Говорят, что интеграл функции $f \in L^1(\mathbb{R}^2)$ в точке $x \in \mathbb{R}^2$ дифференцируем по базису $\mathfrak{M}$, если $\delta_{\mathfrak{M}}(x, f) = 0$. Известны следующие классические теоремы о дифференцировании интегралов (см. [7]).

Теорема D. [Лебег, [20]] *Если $f \in L^1(\mathbb{R}^2)$, то $\delta_\Omega(x, f) = 0$ почти всюду на $\mathbb{R}^2$.*

Теорема E. [Йессен-Марцинкевич-Зигмунд, [9]] *Если $f \in L(1 + \log L)(\mathbb{R}^2)$, то $\delta_{\mathfrak{R}}(x, f) = 0$ почти всюду на $\mathbb{R}^2$.*

Отметим, что класс $L(1 + \log L)(\mathbb{R}^2)$ представляет собою множество функций $f(x)$, удовлетворяющие условию

$$\int_{\mathbb{R}^2} |f(t)| \left(1 + \log^+ |f(t)|\right) dt < \infty,$$

и имеем

$$L^1(\mathbf{R}^2) \supset L(1+\log L)(\mathbf{R}^2) \supset L^p(\mathbf{R}^2), \quad 1 < p < \infty.$$

Согласно теореме E получаем, что интегралы функций класса $L^p(\mathbf{R}^2)$, $1 < p \leq \infty$, дифференцируемы почти всюду по базису $\mathfrak{R}$.

Теорема F. [Безикович, [1]] *Если $f \in L^1(\mathbf{R}^2)$ и $\delta_{\mathfrak{R}}(x, f) < \infty$ на некотором множестве $E \subset \mathbf{R}^2$, то $\delta_{\mathfrak{R}}(x, f) = 0$ почти всюду на E.*

Теорема G. [Сакс, [24]] *Существует функция $f \in L^1(\mathbf{R}^2)$, для которой $\delta_{\mathfrak{R}}(x, f) = \infty$ всюду на $\mathbf{R}^2$.*

Определение 1.1. *Множества*

$$C_{\mathfrak{M}}(f) = \{x \in \mathbf{R}^2 : \delta_{\mathfrak{M}}(x, f) = 0\},$$
$$B_{\mathfrak{M}}(f) = \{x \in \mathbf{R}^2 : 0 < \delta_{\mathfrak{M}}(x, f) < \infty\},$$
$$U_{\mathfrak{M}}(f) = \{x \in \mathbf{R}^2 : \delta_{\mathfrak{M}}(x, f) = \infty\}$$

назовем, соответственно, C, B и U множествами функции $f \in L^1(\mathbf{R}^2)$ относительно базиса $\mathfrak{M}$ (равного $\mathfrak{R}$ или Ω).

Следующие теоремы дают характеризации U, B и C множеств в некоторых пространствах $L^p(\mathbf{R}^2)$ по базисам $\mathfrak{R}$ и Ω.

Теорема 1.1. *Для того чтобы $E \subset \mathbf{R}^2$ было U-множеством некоторой функции $f \in L^1(\mathbf{R}^2)$ относительно базиса $\mathfrak{R}$, необходимо и достаточно, чтобы оно было множеством типа G_δ.*

Теорема 1.2. *Для того чтобы $E \subset \mathbf{R}^2$ было B-множеством некоторой функции $f \in L^p(\mathbf{R}^2)$, $1 \leq p \leq \infty$, относительно базиса Ω, необходимо и достаточно, чтобы оно было $G_{\delta\sigma}$-множеством меры нуль.*

Теорема 1.3. *Для того чтобы $E \subset \mathbf{R}^2$ было U-множеством некоторой функции $f \in L^p(\mathbf{R}^2)$, $1 \leq p < \infty$, относительно базиса Ω, необходимо и достаточно, чтобы оно было G_δ-множеством меры нуль.*

В теореме 1.3 случай $p = \infty$ не рассматривается, так как из $f \in L^\infty$ следует $U_\Omega(f) = \varnothing$. Из теорем 1.2 и 1.3 получаем следующий результат.

Следствие 1.1. *Для того, чтобы попарно непересекающиеся множества E_1, E_2, $E_3 \subset \mathbf{R}^2$ с $E_1 \cup E_2 \cup E_3 = \mathbf{R}^2$ являлись бы соответственно U, B и C множествами некоторой функции $f \in L^p(\mathbf{R}^2)$, $1 \leq p < \infty$, относительно базиса Ω,* *необходимо и достаточно, чтобы* $|E_1| = |E_2| = 0$, E_1 *имело тип* G_δ, *а* E_2 *было множеством* $G_{\delta\sigma}$.

Следствие 1.2. *Для того, чтобы попарно непересекающиеся множества* E_1, E_2, $E_3 \subset \mathbb{R}^2$ *являлись соответственно* U, B *и* C *множествами некоторой функции* $f \in L^\infty(\mathbb{R}^2)$ *относительно базиса* Ω, *необходимо и достаточно, чтобы* $E_1 = \varnothing$, E_2 *было* $G_{\delta\sigma}$ *множеством меры нуль.*

2. Развиения и ключевые функции

Через $\mathfrak{R}_d$ и Ω_d обозначим соответственно семейства двоичных прямоугольников и квадратов. Имеем $\mathfrak{R}_d \subset \mathfrak{R}$ и $\Omega_d \subset \Omega$. Через $\overline{E}$ и $\dot{E}$ обозначим соответственно замыкание и внутренность множества $E \subset \mathbb{R}^2$. Множество $E \subset \mathbb{R}^2$ называется множеством типа G_δ если оно представимо в виде счетного пересечения открытых множеств, а счетные объединения множеств типа G_δ называются $G_{\delta\sigma}$ множествами.

Определение 2.1. *Семейства* A *и* B *двоичных прямоугольников из* $\mathfrak{R}_d$ *обладают соотношением* $A \preceq B$, *если для любых элементов* $a \in A$ *и* $b \in B$ *выполняется одно из условий* $a \subset b$ *или* $a \cap b = \varnothing$. *При этом если* A *состоит из единственного элемента* a, *то это соотношение можно записывать* $a \preceq B$.

Определение 2.2. *Семейство двоичных прямоугольников* $A \subset \mathfrak{R}_d$ *локально-конечно относительно некоторого открытого множества* $G \subset \mathbb{R}^2$, *если любой компакт* $K \subset G$ *пересекается лишь с конечным числом элементов из* A.

Определение 2.3. *Семейство* $\Omega \subset \Omega_d$ *попарно непересекающихся двоичных квадратов назовем* δ*-разбиением открытого множества* $G \subset \mathbb{R}^2$, *если оно локально-конечно относительно* G *и выполняются соотношения*

$$G = \bigcup_{\omega \in \Omega} \omega,$$

$$\delta > \frac{\operatorname{diam}(\omega)}{\operatorname{dist}(\omega, G^c)} \to 0, \text{ при } \operatorname{diam}(\omega) \to 0, \quad \omega \in \Omega. \tag{2.1}$$

Лемма 2.1. *Если* $\delta > 0$, *а* $B \subset \mathfrak{R}_d$ *есть любое семейство, локально-конечное относительно некоторого ограниченного открытого множества* $G \subset \mathbb{R}^2$, *то существует* δ*-разбиение* Ω *множества* G, *такое, что* $\Omega \preceq B$.

Доказательство. Пусть A_k, $k = 1, 2\ldots$, есть семейство всевозможных двоичных квадратов, длины сторон которых равны 2^{-k} и выполняются соотношения

$$\omega \preceq B, \quad \operatorname{diam}(\omega) < \frac{\delta \cdot \operatorname{dist}(\omega, G^c)}{k}, \quad \omega \in A_k.$$

Заметим, что A_k будет не пустым при $k > k_0$. Обозначим

$$A'_k = \left\{ \omega \in A_k : \omega \not\subset \bigcup_{\omega \in A_{k-1}} \omega \right\}, \quad k = 2, 3, \ldots.$$

Легко проверить, что семейство квадратов

$$\Omega = A_1 \cup \left(\bigcup_{k=2}^{\infty} A'_k \right)$$

удовлетворяет условиям леммы. □

Для данного квадрата $\omega \in \Omega$ определим пирамидообразную функцию

$$\lambda_\omega(x) = \operatorname{dist}(x, \omega^c), \quad x \in \mathbb{R}^2,$$

которая очевидно является непрерывной. Рассматриваются функции

$$u(x, n) = 3 \cdot (n+1) 2^{n-2} \big(\lambda_{[0,2^{-n})\times[0,2^{-n})}(x) + \lambda_{[1/2,1/2+2^{-n})\times[1/2,1/2+2^{-n})}(x)$$
$$- \lambda_{[0,2^{-n})\times[1/2,1/2+2^{-n})}(x) - \lambda_{[1/2,1/2+2^{-n})\times[0,2^{-n})}(x) \big), \quad n \in \mathbf{N},$$

$$v(x) = \lambda_{[0,1/2)\times[0,1/2)}(x) + \lambda_{[1/2,1)\times[1/2,1)}(x) - \lambda_{[0,1/2)\times[1/2,1)}(x) - \lambda_{[1/2,1)\times[0,1/2)}(x).$$

Определим четыре множества

$$E_{ij}(n) = \bigcup_{k=0}^{n-1} \left[\frac{i}{2}, \frac{i}{2} + \frac{1}{2^{k+1}} \right] \times \left[\frac{j}{2}, \frac{j}{2} + \frac{1}{2^{n-k}} \right], \quad i, j = 0, 1, \tag{2.2}$$

и обозначим

$$E(n) = E_{00}(n) \cup E_{01}(n) \cup E_{10}(n) \cup E_{11}(n). \tag{2.3}$$

Пусть $\omega \in \Omega$ есть произвольный квадрат, а ϕ_ω линейное преобразование в $\mathbf{R}^2$, отображающее ω на единичный квадрат $[0,1)^2 \subset \mathbf{R}^2$. Обозначим

$$u_\omega(x, n) = u(\phi_\omega(x), n), \quad v_\omega(x) = v(\phi_\omega(x)), \quad E_\omega(n) = (\phi_\omega)^{-1}(E(n)).$$

Простые вычисления показывают, что

$$\|u_\omega(x, n)\|_1 = |E_\omega(n)| = |E(n)||\omega| = \frac{n+1}{2^n}|\omega|, \quad \|v_\omega(x)\|_1 = |\omega|/3. \tag{2.4}$$

Далее, заметим, что если $\omega \in \Omega_d$ есть двоичный прямоугольник, то для любой точки $x \in E_\omega(n)$ существует двоичный прямоугольник $R(x)$ для которого имеем

$$\frac{1}{|R(x)|}\left|\int_{R(x)} u_\omega(x,n)dx\right| = \frac{n+1}{2}, \quad x \in R(x) \subset E_\omega(n), \tag{2.5}$$

и этот прямоугольник совпадает с одним из прямоугольников, участвующих в объединениях (2.2). Аналогично, если вновь $\omega \in \Omega_d$, то

$$\frac{1}{|R(x)|}\left|\int_{R(x)} v_\omega(x)dx\right| = \frac{1}{3}, \quad x \in R(x) \subset \omega \tag{2.6}$$

для некоторого квадрата $R(x)$ с $|R(x)| = |\omega|/4$. На этот раз $R(x)$ просто совпадает с одним из четырех квадратов, составляющих ω. При этом отметим, что в обоих случаях $R(x)$ выбирается из конечного набора прямоугольников.

3. Доказательство теоремы 1.1

Для данного прямоугольника $R \in \mathfrak{R}$ обозначим через $\operatorname{vert}(R)$ множество четырех вершин прямоугольника R.

Лемма 3.1. *Если $Q \in \Omega$ и функция $f(x) = f(x_1,x_2) \in L(\mathbb{R}^2)$ удовлетворяет условиям $\operatorname{supp} f(x) \subset Q$ и*

$$\int_{\mathbb{R}} f(x_1,t)dt = \int_{\mathbb{R}} f(t,x_2)dt = 0, \quad x_1,x_2 \in \mathbb{R}, \tag{3.1}$$

то для любого прямоугольника $R \in \mathfrak{R}$ с условием $Q \cap \operatorname{vert}(R) = \varnothing$ имеем

$$\int_R f(x)dx = 0. \tag{3.2}$$

Доказательство. Если $R \cap Q = \varnothing$, то (3.2) тривиально. Предположим, что

$$Q = [\alpha_1,\beta_1) \times [\alpha_2,\beta_2), \quad R = [a_1,b_1) \times [a_2,b_2).$$

С учетом условия $Q \cap \operatorname{vert}(R) = \varnothing$, остается лишь рассмотреть случаи $[\alpha_1,\beta_1) \subset [a_1,b_1)$ или $[\alpha_2,\beta_2) \subset [a_2,b_2)$. Если имеем первое соотношение, то с учетом (3.1), легко усмотреть, что

$$\int_R f(x)dx = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1,x_2)dx_1dx_2 = \int_{a_2}^{b_2}\int_{\alpha_1}^{\beta_1} f(x_1,x_2)dx_1dx_2$$
$$= \int_{a_2}^{b_2}\left(\int_{\mathbb{R}} f(x_1,x_2)dx_1\right)dx_2 = 0.$$

Второй случай рассматривается аналогичным образом. □

Лемма 3.2. *Если $Q \in \Omega$ и функция $f \in L^\infty(\mathbb{R}^2)$ удовлетворяет условию*

$$\operatorname{supp} f \subset Q, \tag{3.3}$$

то для любого прямоугольника $R \in \mathfrak{R}$ имеет место неравенство

$$\frac{1}{|R|}\int_R |f(x)|dx \le \frac{\|f\|_\infty \cdot \operatorname{diam}(Q)}{\operatorname{diam}(R)}.$$

Доказательство. Покажем неравенство

$$\frac{|Q\cap R|}{|R|} \le \frac{\operatorname{diam}(Q)}{\operatorname{diam}(R)}. \tag{3.4}$$

Параллельным переносом квадрат Q можно переставить так, чтобы его вершина совпадала бы с одной из вершин прямоугольника. Заметим, что тогда величина $|Q\cap R|$ принимает наибольшее значение, а все остальные величины в неравенстве (3.4) сохраняются. Поэтому без потери общности можно рассматривать только следующие случаи взаимного расположения Q и R (см. Рис. 1).

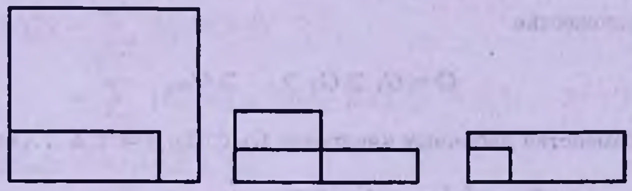

Рис. 1

Пусть сторона квадрата равна c, а стороны прямоугольника равны a, b ($a \le b$). В первом случае имеем

$$\frac{|Q\cap R|}{|R|} = 1 \le \frac{c}{b} = \frac{\operatorname{diam}(Q)}{\operatorname{diam}(R)},$$

во втором имеем

$$\frac{|Q\cap R|}{|R|} = \frac{a\cdot c}{a\cdot b} = \frac{c}{b} = \frac{\operatorname{diam}(Q)}{\operatorname{diam}(R)},$$

а в третьем случае

$$\frac{|Q\cap R|}{|R|} = \frac{c^2}{a\cdot b} \le \frac{c}{b} = \frac{\operatorname{diam}(Q)}{\operatorname{diam}(R)}.$$

Из (3.3) и (3.4) вытекает

$$\frac{1}{|R|}\int_R |f(x)|dx \le \frac{\|f\|_\infty |Q\cap R|}{|R|} \le \frac{\|f\|_\infty \cdot \operatorname{diam}(Q)}{\operatorname{diam}(R)}.$$

□

Лемма 3.3. *Если $L > 1$, а $Q \in \Omega_d$ есть произвольный двоичный квадрат, то существует функция $f \in C(\mathbb{R}^2)$ и число $c(L) > 0$ такие, что*

$$\operatorname{supp} f \subset Q, \tag{3.5}$$

$$\|f\|_\infty \le c(L), \tag{3.6}$$

$$\|f\|_1 \le 2|Q|, \tag{3.7}$$

$$\int_R f(x)dx = 0, \quad R \in \mathfrak{R}, \quad Q \cap \operatorname{vert}(R) = \varnothing, \tag{3.8}$$

и для любой точки $x \in Q$ существует двоичный прямоугольник $R(x) \subset Q$ такой, что

$$\frac{1}{|R(x)|}\left|\int_{R(x)} f(t)dt\right| \ge L, \quad x \in Q, \tag{3.9}$$

при этом семейство $\{R(x) : x \in Q\}$ состоит из конечного числа элементов.

Доказательство. Пусть $n = [3L] + 1$ и m есть некоторое натуральное число. Определим множества

$$Q = G_1 \supset G_2 \supset \ldots \supset G_m, \tag{3.10}$$

и конечные семейства двоичных квадратов $\Omega_k \subset \Omega_d$, $k = 1, 2, \ldots, m$, такие, что

$$G_k = \bigcup_{\omega \in \Omega_k} \omega, \quad k = 1, 2, \ldots, m, \tag{3.11}$$

$$G_k = G_{k-1} \setminus \bigcup_{\omega \in \Omega_{k-1}} E_\omega(n), \quad k = 2, \ldots, m, \tag{3.12}$$

где множества $E_\omega(n)$ определены в (2.3). Воспользуемся математической индукцией. В качестве первого шага возьмем просто $G_1 = Q$ и $\Omega_1 = \{Q\}$. Ясно, что условия (3.10) – (3.12) выполняются. Предположим, что уже определены множества G_k и семейства Ω_k при $k = 1, 2, \ldots, p$, удовлетворяющие условиям (3.10)-(3.12). Определим множество

$$G_{p+1} = G_p \setminus \bigcup_{\omega \in \Omega_p} E_\omega(n).$$

Очевидно, оно представимо в виде конечного объединения двоичных квадратов, составляющих Ω_{p+1}. В итоге получаем

$$G_{p+1} = \bigcup_{\omega \in \Omega_{p+1}} \omega.$$

Очевидно G_{p+1} и Ω_{p+1} удовлетворяют условиям (3.10)-(3.12) при $k = p+1$. Учитывая (2.4), (3.11) и (3.12), имеем

$$|G_k| = |G_{k-1}| - \left|\bigcup_{\omega\in\Omega_{k-1}} E_\omega(n)\right| = |G_{k-1}| - \frac{n+1}{2^n}|G_{k-1}| = \left(1 - \frac{n+1}{2^n}\right)|G_{k-1}|$$

и, следовательно, получаем

$$|G_m| = \left(1 - \frac{n+1}{2^n}\right)^m |Q| < \frac{|Q|}{n}, \tag{3.13}$$

при достаточно большом $m = m(n)$. Обозначим

$$f_k(x) = \sum_{\omega\in\Omega_k} u_\omega(x,n), \quad k = 1,2,\ldots,m-1, \tag{3.14}$$

$$f_m(x) = n\sum_{\omega\in\Omega_m} v_\omega(x). \tag{3.15}$$

Учитывая (2.4), (3.11) – (3.15), легко проверить, что

$$\operatorname{supp} f_k \subset G_k \setminus G_{k+1} = \bigcup_{\omega\in\Omega_k} E_\omega(n), \quad k = 1,2,\ldots,m, \tag{3.16}$$

$$\begin{aligned} \|f_k\|_1 &= \sum_{\omega\in\Omega_k} \|a_\omega(x,n)\|_1 \\ &= \sum_{\omega\in\Omega_k} |E_\omega(n)| = |G_k| - |G_{k+1}|, \quad k = 1,\ldots,m-1, \end{aligned} \tag{3.17}$$

$$\|f_m\|_1 = n\cdot|G_m| \le |Q|, \tag{3.18}$$

$$\|f_k\|_\infty = 3(n+1)2^{n-2} = c(L). \tag{3.19}$$

Определим

$$f(x) = \sum_{k=1}^m f_k(x).$$

Очевидно $f \in C(\mathbb{R}^2)$ и имеют место (3.5) и (3.6). Далее, с учетом (3.10), (3.17) и (3.18), получаем

$$\int_{\mathbb{R}} |f(t)|dt = \sum_{k=1}^{m-1}(|G_k| - |G_{k+1}|) + |Q| = 2|Q| - |G_m| \le 2|Q|,$$

откуда следует (3.7). Условие (3.8) немедленно следует из леммы 3.2. Чтобы показать (3.9), возьмем любую точку $x \in Q$. Имеем $x \in G_k \setminus G_{k+1}$ при некотором $k = 1,2,\ldots,m$, где предполагается $G_{m+1} = \varnothing$. Из (3.16) получаем, что $x \in E_\omega(n)$ для некоторого квадрата $\omega \in \Omega_k$. Если $k < m$, то согласно (2.5) существует прямоугольник $R = R(x)$, $x \in R \subset E_\omega(n)$, такой что

$$\frac{1}{|R|}\left|\int_R f(t)dt\right| = \frac{1}{|R|}\left|\int_R a_\omega(t,n)dt\right| = \frac{n+1}{2} > L.$$

Если же $k = m$, то из (2.6) вытекает

$$\frac{1}{|R|}\left|\int_R f(t)dt\right| = \frac{n}{|R|}\left|\int_R b_\omega(t)dt\right| = \frac{n}{3} > L$$

для некоторого квадрата $R = R(x)$, $x \in R \subset \omega$. При этом семейство $\{R(x),\ x \in Q\}$ составляет конечный набор. □

Лемма 3.4. *Если $G \subset \mathbb{R}^2$–ограниченное открытое множество, а семейство двоичных прямоугольников $A \subset \mathfrak{R}_d$ локально-конечно относительно G, то для любых чисел $L > 1$, $\varepsilon > 0$ существует функция $f \in L^\infty(\mathbb{R})$ такая, что*

(3.20) $$\operatorname{supp} f \subset G,$$

(3.21) $$\|f\|_1 \le 1,$$

(3.22) $$\int_R f(x)dx = 0, \quad R \in A,$$

(3.23) $$\delta_{\mathfrak{R}}(f,x) = 0, \quad x \in \mathbb{R}^2,$$

(3.24) $$\frac{1}{|R|}\left|\int_R f(x)dx\right| < \varepsilon, \quad R \in \mathfrak{R},\ R \not\subset G,$$

и для любой точки $x \in G$ существует двоичный прямоугольник $R = R(x) \subset G$, такой что

(3.25) $$\frac{1}{|R(x)|}\left|\int_{R(x)} f(t)dt\right| \ge L, \quad x \in R(x) \subset G,$$

при этом семейство $\{R(x) : x \in G\}$ локально-конечно относительно G.

Доказательство. Пусть $\delta > 0$ есть некоторое число. Согласно лемме 2.1 существует δ-разбиение Ω для множества G, такое что

(3.26) $$\Omega \preceq A.$$

Применив лемму 3.3 над каждым квадратом $Q \in \Omega$, мы получим функции $f_Q(x)$, $Q \in \Omega$, удовлетворяющие условиям (3.5)-(3.9). Обозначим

(3.27) $$f(x) = \frac{1}{2|G|}\sum_{Q\in\Omega} f_Q(x).$$

Ясно, что имеет место (3.20) и

(3.28) $$\|f\|_\infty \le \frac{c(L)}{2|G|}.$$

Далее, в силу (3.7) (для функций f_Q), имеем

$$\|f\|_1 \le \frac{1}{2|G|}\sum_{Q\in\Omega}\|f_Q\|_1 \le \frac{1}{2|G|}\sum_{Q\in\Omega} 2|Q| = 1,$$

и получим (3.21). Из (3.8), в частности, имеем

$$\int_Q f_Q(t)dt = 0. \tag{3.29}$$

Из условия (3.26) следует, что для любых $R \in A$ и $Q \in \Omega$ имеем $Q \subset R$ или $Q \cap R = \varnothing$, и поэтому, с учетом (3.29), получаем

$$\int_R f_Q(x)dx = 0, \quad Q \in \Omega,$$

откуда следует (3.22). Если $x \in G$, то $x \in Q$ при некотором $Q \in \Omega$. Согласно (3.9), существует прямоугольник $R = R(x) \subset Q$, удовлетворяющий условию

$$\frac{1}{|R|}\left|\int_R f_Q(t)dt\right| \geq 2L|G|.$$

Отсюда получаем

$$\frac{1}{|R|}\left|\int_R f(t)dt\right| = \frac{1}{2|G||R|}\left|\int_R f_Q(t)dt\right| \geq L,$$

которое дает (3.25). Согласно лемме 3.3 семейство $\{R(x) : x \in Q\}$ состоит из конечного числа прямоугольников. С учетом (3.26), легко заметить, что $\{R(x) : x \in G\}$ будет локально-конечным относительно G. Теперь рассмотрим любые $Q \in \Omega$ и $R \in \mathfrak{R}$. Если имеет место $Q \cap \operatorname{vert}(R) = \varnothing$, то согласно лемме 3.1, получим

$$\int_R f_Q(x)dx = 0.$$

Если же имеем

$$R \not\subset G, \quad Q \cap \operatorname{vert}(R) \neq \varnothing, \tag{3.30}$$

то легко усмотреть, что $\operatorname{diam}(R) \geq \operatorname{dist}(Q, G^c)/2$. Отсюда, применив лемму 3.2 и соотношения (2.1), (3.28), заключаем

$$\frac{1}{|R|}\left|\int_R f_Q(x)dx\right| \leq \frac{\|f\|_\infty \cdot \operatorname{diam}(Q)}{\operatorname{diam}(R)} \leq \frac{c(L)}{|G|} \cdot \frac{\operatorname{diam}(Q)}{\operatorname{dist}(Q, G^c)} \to 0, \tag{3.31}$$

при $\operatorname{diam}(R) \to 0$. Отметим, что при фиксированном R количество таких квадратов $Q \in \Omega$, удовлетворяющих (3.30) не превосходит 4. Отсюда и из (3.27) вытекает

$$\begin{aligned}\delta_{\mathfrak{R}}(x, f) &= \lim_{\operatorname{diam}(R)\to 0:\, x\in R\in\mathfrak{R}} \frac{1}{|R|}\left|\int_R f(t)dt - f(x)\right| \\ &= \lim_{\operatorname{diam}(R)\to 0:\, x\in R\in\mathfrak{R}} \frac{1}{|R|}\left|\int_R f(t)dt\right| = 0, \quad x \in G^c,\end{aligned}$$

т.е. имеет место (3.23) при $x \in G^c$. Если же $x \in G$, то любой прямоугольник R у которого величина $\operatorname{diam}(R)$ достаточно мала пересекается лишь конечным числом квадратов из Ω. С другой стороны, $\delta_{\mathfrak{R}}(x, f_Q) = 0$, так как каждая из f_Q является непрерывной функцией. Отсюда легко следует

$$\delta_{\mathfrak{R}}(x, f) = 0, \quad x \in G,$$

и мы получаем (3.23). Аналогичными рассуждениями, при условиях (3.30), из (2.1) и (3.31) заключаем

$$\frac{1}{|R|}\left|\int_R f(x)dx\right| \le 4 \cdot \frac{c(L)}{|G|} \cdot \frac{\operatorname{diam}(Q)}{\operatorname{dist}(Q, G^c)} \le \frac{4c(L)\delta}{|G|}, \tag{3.32}$$

которое дает (3.24) при достаточно малом δ. □

Доказательство теоремы 1.1. Необходимость: Для данной функции $f \in L^1(\mathbb{R}^2)$ рассмотрим множества

$$A_n(f) = \left\{x \in \mathbb{R}^2 : \exists R \in \mathfrak{R},\, x \in R,\, \operatorname{diam}(R) < 1/n, \left|\frac{1}{|R|}\int_R f(t)dt\right| > n\right\},$$

которые очевидно являются открытыми. Докажем, что

$$A(f) = \bigcap_{n\ge 1} A_n(f) = \{x \in \mathbb{R}^2 : \delta_{\mathfrak{R}}(x, f) = \infty\}. \tag{3.33}$$

Действительно,

$$\begin{aligned} x \in A(f), &\Leftrightarrow x \in A_n(f), \text{ при любом } n = 1, 2, \ldots \\ &\Leftrightarrow \exists R_k \in \mathfrak{R},\, x \in R_k,\, \operatorname{diam}(R_k) \to 0, \left|\frac{1}{|R_k|}\int_{R_k} f(t)dt\right| \to \infty, \\ &\Leftrightarrow \delta_{\mathfrak{R}}(x, f) = \infty. \end{aligned}$$

Из соотношения (3.33) вытекает, что $\{x \in \mathbb{R}^2 : \delta_{\mathfrak{R}}(x, f) = \infty\}$ является множеством типа G_δ.

Достаточность: Пусть E есть некоторое множество типа G_δ. Сначала предположим, что $E \subset Q$, где Q есть некоторый квадрат. Тогда имеем

$$E = \bigcap_{k=1}^{\infty} G_k$$

где, без ограничения общности, можно предполагать, что множества $G_k \subset 2Q$ и $G_{k+1} \subset G_k$, $k = 1, 2, \ldots$. Докажем, что существует последовательность функций $f_k \in L^\infty(\mathbb{R}^2)$, $k = 1, 2, \ldots$, для которых имеют место соотношения

$$\operatorname{supp} f_k \subset G_k, \quad \|f_k(t)\|_1 = 1, \tag{3.34}$$

$$\delta_{\mathfrak{R}}(x, f_k) = 0, \quad x \in \mathbb{R}^2, \tag{3.35}$$

$$\frac{1}{|R|}\left|\int_R f_k(x)dx\right| < 2^{-k}, \quad R \in \mathfrak{R},\ R \not\subset G_k, \tag{3.36}$$

и для любой точки $x \in G_k$ существует двоичный прямоугольник $R_k(x)$, такой что

$$\frac{1}{|R_k(x)|}\left|\int_{R_k(x)} f_k(t)dt\right| \geq 2^k + \sum_{j=1}^{k-1}\|f_j\|_\infty, \quad x \in R_k(x) \subset G_k, \quad k \geq 1, \tag{3.37}$$

$$\int_{R_j(x)} f_k(t)dt = 0, \quad x \in G_j, \quad j = 1, 2, \ldots, k-1, \tag{3.38}$$

при этом семейство $\{R_k(x) : x \in G_k\}$ является локально-конечным относительно G_k. Воспользуемся индукцией. Случай $k = 1$ вытекает из леммы 3.4 при $G = G_1$ и $A = \varnothing$. Предположим, что уже определены функции $f_k(x)$, $k = 1, 2, \ldots, p$ со свойствами (3.34)-(3.38). Обозначим

$$A = \bigcup_{k=1}^{p}\{R_k(x) : x \in G_k\}. \tag{3.39}$$

По предположению индукции, каждое из семейств $\{R_k(x) : x \in G_k\}$, $k = 1, 2, \ldots p$, является локально-конечным относительно G_k. Отсюда, очевидно, что множество прямоугольников A будет локально-конечным относительно G_{p+1}. Тогда применив лемму 3.4 для A определенного в (3.39) и

$$G = G_{p+1}, \quad \delta = 2^{-p-1}, \quad L = 2^{p+1} + \sum_{j=1}^{p}\|f_j\|_\infty,$$

можно определить функцию $f_{p+1}(x)$, удовлетворяющую условиям леммы 3.4. Обозначим

$$f(x) = \sum_{k=1}^{\infty} f_k(x).$$

Возьмем любую точку $x \in E$. Имеем $x \in G_k$, $k = 1, 2, \ldots$. Пусть $R_k(x)$ есть последовательность прямоугольников, удовлетворяющих условиям (3.37) и (3.38). Имеем

$$\frac{1}{|R_k(x)|}\left|\int_{R_k(x)} f(t)dt\right| = \frac{1}{|R_k(x)|}\left|\int_{R_k(x)} \sum_{j=1}^{k} f_j(t)dt\right|$$

$$\geq \frac{1}{|R_k(x)|}\left|\int_{R_k(x)} f_k(t)dt\right| - \sum_{j=1}^{k-1}\frac{1}{|R_k(x)|}\left|\int_{R_k(x)} f_j(t)dt\right|$$

$$\geq \frac{1}{|R_k(x)|}\left|\int_{R_k(x)} f_k(t)dt\right| - \sum_{j=1}^{k-1}\|f_j\|_\infty \geq 2^k,$$

и следовательно $\delta_{\mathfrak{R}}(x,f)=\infty$. Теперь предположим, что $x\in E^c$. Тогда имеем $x\in G_{k-1}\setminus G_k$ при некотором $k=1,2,\ldots$, где предполагается $G_0=\mathbb{R}$ и имеем

$$f_j(x)=0,\quad j\geq k. \tag{3.40}$$

Если $m\geq k$, то, с учетом (3.35), (3.36) и (3.40), получаем

$$\delta_{\mathfrak{R}}(x,f)=\delta_{\mathfrak{R}}\left(x,\sum_{j=m}^{\infty}f_j\right)\leq\limsup_{\operatorname{diam}(R)\to0,\,x\in R\in\mathfrak{R}}\sum_{j=m}^{\infty}\left|\frac{1}{|R|}\int_R f_j(t)dt-f_j(x)\right|$$

$$=\limsup_{\operatorname{diam}(R)\to0,\,x\in R\in\mathfrak{R}}\sum_{j=m}^{\infty}\left|\frac{1}{|R|}\int_R f_j(t)dt\right|\leq\sum_{j=m}^{\infty}2^{-j}.$$

Так как $m\in\mathbb{R}$–произвольное число, то отсюда получаем $\delta_{\mathfrak{R}}(x,f)=0$, что и доказывает достаточность в случае когда E содержится в некотором квадрате Q. Дополнительно отметим, что построенная функция f удовлетворяет условию $\operatorname{supp}f\subset 2Q$. Теперь предположим, что E есть произвольное множество типа G_δ. Пусть Q_k есть последовательность квадратов с

$$\operatorname{diam}(Q_k)=1,\quad \cup_k Q_k=\mathbb{R}^2.$$

Очевидно, что каждое множество $E_k=E\cap Q_k$ имеет тип G_δ. Применив доказанное для каждого множества E_k, найдем функции $F_k(x)$, с условиями

$$\operatorname{supp}F_k\subset 2Q_k,\quad \delta_{\mathfrak{R}}(x,F_k)=0,\quad x\in(E_k)^c,\quad \delta_{\mathfrak{R}}(x,F_k)=\infty,\quad x\in E_k.$$

Функция

$$F(x)=\sum_{k=1}^{\infty}F_k(x)$$

будет искомой. Действительно, если $x\in E^c$, то имеем

$$\lim_{\operatorname{diam}(R)\to0,\,x\in R\in\mathfrak{R}}\frac{1}{|R|}\int_R F_k(t)dt=F_k(x),\quad k=1,2,\ldots.$$

Так как каждый прямоугольник $R\in\mathfrak{R}$ пересекается лишь с конечным числом квадратов $2Q_k$, то получим

$$\lim_{\operatorname{diam}(R)\to0,\,x\in R\in\mathfrak{R}}\frac{1}{|R|}\int_R F(t)dt=\sum_k\lim_{\operatorname{diam}(R)\to0,\,x\in R\in\mathfrak{R}}\frac{1}{|R|}\int_R F_k(t)dt=F(x).$$

Если же $x\in E$, то имеем $x\in E_k$, при некотором k и $x\notin E_j$, $j\neq k$. Отсюда, аналогичным образом, легко получить $\delta_{\mathfrak{R}}(x,F)=\infty$. Теорема 1.1 доказана. □

4. Доказательства теорем 1.2 и 1.3

Для измеримого множества $A \subset \mathbb{R}^2$ обозначим

$$\lambda(A) = A \cup \left\{x \in \mathbb{R}^2 : \delta_\Omega(x, \mathbf{I}_A) > 0\right\}.$$

Так как по теореме D имеем $\delta_\Omega(x, \mathbf{I}_A) = 0$ п.в., то получим

$$|\lambda(A) \setminus A| = 0. \tag{4.1}$$

Легко проверить также следующие свойства

$$\lambda(A \cup B) \subset \lambda(A) \cup \lambda(B), \tag{4.2}$$

$$\lambda(A) \subset \lambda(B), \text{ при } A \subset B. \tag{4.3}$$

Лемма 4.1. *Если $\delta > 0$, G есть открытое множество, а $E \subset G$ имеет меру нуль, то существует открытое множество U такое, что*

$$E \subset U \subset G, \tag{4.4}$$

$$|Q \cap U| < \delta |Q|, \quad Q \in \Omega, \quad \overline{Q} \not\subset G, \tag{4.5}$$

$$\lambda(U) \subset G. \tag{4.6}$$

Доказательство. Согласно лемме 2.1, существует разбиение Ω множества G, такое, что

$$\frac{1}{5} > \frac{\operatorname{diam}(\omega)}{\operatorname{dist}(\omega, G^c)} \to 0, \text{ при } \operatorname{diam}(\omega) \to 0, \quad \omega \in \Omega. \tag{4.7}$$

Очевидно, можно выбрать открытое множество U такое, что

$$E \subset U, \quad |\omega \cap U| < \min\{\delta/4, \operatorname{dist}(\omega, G^c)\}|\omega|.$$

Пусть Q есть произвольный квадрат, такой что $\overline{Q} \not\subset G$. Тогда, если $\omega \in \Omega$ и $\omega \cap Q \neq \varnothing$, то согласно (4.7) имеем

$$\operatorname{diam}(\omega) < \frac{\operatorname{dist}(\omega, G^c)}{5} \le \frac{\sqrt{2} \cdot \operatorname{diam}(Q)}{5} < \frac{\operatorname{diam}(Q)}{2}.$$

Отсюда, с учетом соотношения $\omega \cap Q \neq \varnothing$, легко следует $\omega \subset 2Q$ и, следовательно, имеем

$$\operatorname{dist}(\omega, G^c) \le \sqrt{2} \cdot \operatorname{diam}(2Q) = 2\sqrt{2} \cdot \operatorname{diam}(Q) < 4 \cdot \operatorname{diam}(Q).$$

Отсюда получаем

$$
\begin{aligned}
|Q\cap U| &\le \sum_{\omega\in\Omega:\,\omega\cap Q\ne\varnothing} |\omega\cap U| \\
&\le \sum_{\omega\in\Omega:\,\omega\cap Q\ne\varnothing} \min\{\delta/4, \mathrm{dist}\,(\omega, G^c)\}|\omega| \\
&\le \min\{\delta/4, 4\cdot \mathrm{diam}\,(Q)\} \sum_{\omega\in\Omega:\,\omega\cap Q\ne\varnothing} |\omega| \qquad (4.8)\\
&\le \min\{\delta/4, 4\cdot \mathrm{diam}\,(Q)\} \sum_{\omega\in\Omega:\,\omega\subset 2Q} |\omega| \\
&\le \min\{\delta, 4\cdot \mathrm{diam}\,(Q)\}|Q|,
\end{aligned}
$$

откуда немедленно следует (4.5). Чтобы установить (4.6) возьмем любую точку $x\in G^c$. Из (4.8) следует

$$\lim_{\mathrm{diam}\,(Q)\to 0:\, x\in Q\in\Omega} \frac{1}{|Q|}\int_Q \mathrm{I}_U(t)dt = \lim_{\mathrm{diam}\,(Q)\to 0:\, x\in Q\in\mathfrak{R}} \frac{|Q\cap U|}{|Q|} = 0,$$

и следовательно получим $\delta_\Omega(x, \mathrm{I}_U) = 0$ при $x\in G^c$. Это значит, что $\lambda(U)\subset G$. Лемма 4.1 доказана. □

Лемма 4.2. *Если $A\subset B\subset \mathbb{R}^2$ есть открытые множества, то существует открытое множество G, такое, что*

$$A\subset G\subset B, \qquad (4.9)$$

$$|G| = \frac{|A|+|B|}{2}, \qquad (4.10)$$

$$\lambda(G\setminus A)\subset B. \qquad (4.11)$$

Доказательство. Если $|A| = |B|$, то возьмем $G = A$ и утверждение очевидно. Так что, предположим $|A| < |B|$. Пусть $\Omega = \{\omega_k\}$ есть произвольное разбиение множества B. Существует число $p\in\mathbb{N}$ такое, что

$$\sum_{k=1}^{p} |\omega_k| > \frac{|A|+|B|}{2}. \qquad (4.12)$$

Обозначим

$$G(t) = A\cup\left(\bigcup_{k=1}^{p}(t\mathring{\omega}_k)\right), \quad 0\le t\le 1, \qquad (4.13)$$

и рассмотрим функцию $f(t) = |G(t)|$. Из (4.12) и (4.13) следует

$$f(0) = |A| < \frac{|A|+|B|}{2}, \quad f(1)\ge \sum_{k=1}^{p}|\omega_k| > \frac{|A|+|B|}{2}.$$

Отсюда и из непрерывности $f(t)$ следует, что для некоторого $t_0 \in (01, 1)$ имеет место равенство

$$|G(t_0)| = \frac{|A| + |B|}{2}.$$

Легко видеть, что $G = G(t_0)$ удовлетворяет условиям (4.9) и (4.10). Для проверки (4.11), заметим, что согласно (4.13) имеем соотношение

$$G \setminus A \subset \bigcup_{k=1}^{p} (t_0 \hat{\omega}_k),$$

из которого легко вытекает (4.11). □

Лемма 4.3. *Если $B \subset \mathbb{R}^2$ есть открытое, а $A \subset B$ измеримое множества, с $\lambda(A) \subset B$, то существует открытое множество $G \subset B$ такое, что*

(4.14) $$\lambda(A) \subset G, \quad \lambda(G) \subset B, \quad |G| = \frac{|A| + |B|}{2}.$$

Доказательство. Так как $|\lambda(A) \setminus A| = 0$, то используя лемму 4.1, найдем открытое множество C такое, что $\lambda(A) \setminus A \subset C$, $\lambda(C) \subset B$. Далее, согласно лемме 4.2, существует открытое множество G такое, что

$$A \cup C \subset G \subset B, \quad \lambda(G \setminus (A \cup C)) \subset B, \quad |G| = \frac{|A| + |B|}{2}.$$

Отсюда и из (4.1)-(4.3) получаем

$$\begin{aligned} &\lambda(A) \subset A \cup C \subset G, \\ &\lambda(G) \subset \lambda(G \setminus (A \cup C)) \cup \lambda(A \cup C) \\ &\qquad \subset \lambda(G \setminus (A \cup C)) \cup \lambda(A) \cup \lambda(C) \subset B. \end{aligned}$$

□

Рассмотрим семейство открытых множеств $\{G_r : r \in E\}$, где $E \subset \mathbb{R}$-некоторое множество индексов. Будем говорить, что это семейство является цепью, если $\lambda(G_r) \subset G_{r'}$ при любых $r, r' \in E$, с $r < r'$.

Лемма 4.4. *Если A и B-открытые множества в $\mathbb{R}^2$, с условиями $\lambda(A) \subset B$ и $\alpha = |A| < |B| = \beta$, то существует цепь открытых множеств*

$$G_r, \quad r \in \mathcal{D} = \left\{ \alpha + \frac{i(\beta - \alpha)}{2^k}, 0 \le i \le 2^k, k = 0, 1, \dots \right\}$$

такая, что

(4.15) $$G_\alpha = A, G_\beta = B, \quad |G_r| = r, \quad r \in [\alpha, \beta].$$

Доказательство. Определим $G_\alpha = A$, $G_\beta = B$ и применим лемму 4.3 для пары открытых множеств G_α, G_β. Этим определяется открытое множество $G = G_{(\alpha+\beta)/2}$, с условиями (4.14), что означает множества G_α, $G_{(\alpha+\beta)/2}$, G_β образуют цепь. Далее, продолжим рассуждения по индукции. Обозначим

$$\mathcal{D}_k[\alpha,\beta] = \left\{\alpha + \frac{i(\beta-\alpha)}{2^k}, 0 \le i \le 2^k\right\},$$

и предположим, что уже выбраны множества G_r, для всех $r \in \mathcal{D}_k[\alpha,\beta]$, при этом они образуют цепь и $|G_r| = r$. Применив лемму 4.3 для каждой пары множеств $G_{i/2^k}$, $G_{(i+1)/2^k}$, получим множества $G_{(2i+1)/2^{k+1}}$, $0 \le i \le 2^k - 1$. Ясно, что полученное таким путем семейство $\{G_r,\ r \in \mathcal{D}_{k+1}[\alpha,\beta]\}$ тоже будет цепью. При этом сохраняется свойство $|G_r| = r$ теперь уже при $r \in \mathcal{D}_{k+1}[\alpha,\beta]$. В самом деле, имеем

$$|G_{(2i+1)/2^{k+1}}| = \frac{1}{2}(|G_{i/2^k}| + |G_{(i+1)/2^k}|) = \frac{1}{2}\left(\frac{i}{2^k} + \frac{i+1}{2^k}\right) = \frac{2i+1}{2^{k+1}}.$$

Продолжив этот процесс, получим семейство множеств G_r, определенных при всех $r \in \mathcal{D}$, которое будет цепью и $|G(r)| = r$. Лемма 4.4 доказана. □

Лемма 4.5. *Если $\varepsilon > 0$, $G \subset \mathbb{R}^2$ есть открытое множество, а $E \subset G$ имеет меру нуль, то существуют открытое множество A, с $E \subset A \subset G$, и функция $h(x)$, $x \in \mathbb{R}^2$, такие, что*

(4.16) $$\operatorname{supp} h \subset G, \quad h(x) = 1,\ x \in A,$$

(4.17) $$0 \le h(x) \le 1, \quad x \in \mathbb{R}^2,$$

(4.18) $$\frac{1}{|Q|}\left|\int_Q h(t)dt\right| \le \varepsilon, \quad Q \in \Omega, \quad \overline{Q} \not\subset G,$$

(4.19) $$\delta_\Omega(x,h) = 0, \quad x \in \mathbb{R}^2.$$

Доказательство. Применив лемму 4.1, получим открытое множество B с условиями

(4.20) $$E \subset B \subset G,$$

(4.21) $$|Q \cap B| < \varepsilon|Q|, \quad Q \in \Omega, \quad \overline{Q} \not\subset G,$$

(4.22) $$\lambda(B) \subset G.$$

Далее, применив лемму 4.3, получим открытое множество A, с условиями $E \subset A$, $|A| < |B|$ и $\lambda(A) \subset B$. Пусть $\alpha = |A|$, $\beta = |B|$. Согласно лемме 4.4, существует цепь открытых множеств $\{G(r) : r \in \mathcal{D}\}$, удовлетворяющая условиям

$$G_\alpha = A,\ G_\beta = B, \quad |G_r| = r.$$

Обозначим $\tau(x) = \inf\{r : x \in G_r\}$, $x \in B \setminus A$. Отметим, что $\tau(x)$ отображает множество $B \setminus A$ в $[\alpha, \beta]$. Определим непрерывную функцию

$$f(x) = \begin{cases} 1 & \text{при} \quad x \in [0, \alpha], \\ 0 & \text{при} \quad x \in [\beta, 1], \\ & \text{линейна на} \quad [\alpha, \beta], \end{cases}$$

и функцию

$$h(x) = \begin{cases} 1 & \text{при} \quad x \in A, \\ 0 & \text{при} \quad x \in \mathbb{R}^2 \setminus B, \\ f(\tau(x)) & \text{при} \quad x \in B \setminus A. \end{cases} \tag{4.23}$$

Очевидно $\operatorname{supp} h(x) \subset B \subset G$ и выполняются условия (4.16) и (4.17). Из (4.21) следует

$$\frac{1}{|Q|}\left|\int_Q h(t)dt\right| \le \frac{|Q \cap B|}{|Q|} < \varepsilon, \quad Q \in \Omega, \quad \overline{Q} \not\subset G$$

и получаем (4.18). Остается проверить условие (4.19). Рассмотрим функцию

$$p(x) = \mathbb{I}_{G_{r_0}}(x) + \sum_{k=0}^{m-1} f(r_k)\mathbb{I}_{G_{r_{k+1}} \setminus G_{r_k}}(x), \tag{4.24}$$

где числа $r_k \in \mathcal{D}$ удовлетворяют неравенству

$$\alpha = r_0 < r_1 < \ldots < r_m = \beta.$$

Докажем, что для любого $\varepsilon > 0$ можно выбрать r_k такими, что

$$|h(x) - p(x)| < \varepsilon, \quad x \in \mathbb{R}^2. \tag{4.25}$$

В самом деле, имеем

$$h(x) = p(x) = 1, \quad x \in G_\alpha, \tag{4.26}$$

$$h(x) = p(x) = 0, \quad x \in \mathbb{R}^2 \setminus G_\beta. \tag{4.27}$$

Если же $x \in G_\beta \setminus G_\alpha$, то имеем $x \in G_{r_{k+1}} \setminus G_{r_k}$ при некотором $k = 0, 1, \ldots, m-1$. Тогда из определения отображения τ следует, что $r_k \le \tau(x) \le r_{k+1}$. Учитывая (4.23), получаем

$$\inf_{t \in [r_i, r_{i+1}]} f(t) \le h(u) \le \sup_{t \in [r_i, r_{i+1}]} f(t), \quad u \in G_{r_{i+1}} \setminus G_{r_i}. \tag{4.28}$$

Имеем также

$$\inf_{t \in [r_i, r_{i+1}]} f(t) \le f(r_i) \le \sup_{t \in [r_i, r_{i+1}]} f(t). \tag{4.29}$$

Из непрерывности функции f следует, что при достаточно малом

$$\delta = \max_{0 \le i < m} (r_{i+1} - r_i)$$

имеем

$$\sup_{t,t'\in[r_i,r_{i+1}]} |f(t)-f(t')|<\varepsilon, \quad i=0,1,\dots,m-1. \tag{4.30}$$

Комбинируя (4.26)-(4.30), получаем (4.25). Из (4.25), (4.24) следует

$$\delta_\Omega(x,h)\le\delta_\Omega(x,h-p)+\delta_\Omega(x,p)\le\varepsilon+\delta_\Omega(x,p). \tag{4.31}$$

Для доказательства (4.19) рассмотрим три случая.

Случай 1: $x\in G_\alpha$. Имеем, что G_α-открытое множество и $h(t)=1$ при $t\in G_\alpha$. Отсюда следует (4.19) при таких x.

Случай 2: $x\in G_\beta\setminus G_\alpha$. В этом случае имеем

$$x\in G_{r_{k+1}}\setminus G_{r_k} \tag{4.32}$$

при некотором $k=0,1,\dots,m-1$. Тогда из открытости множеств G_{r_i} следует

$$\delta_\Omega(x,\mathbb{I}_{G_{r_i}})=0, \quad i\ge k+1.$$

С другой стороны, учитывая соотношения $\lambda(G_{r_i})\subset G_{r_{i+1}}$, имеем

$$\delta_\Omega(x,\mathbb{I}_{G_{r_i}})=0, \quad i\le k-1, \quad k>0.$$

В итоге получаем

$$\delta_\Omega(x,\mathbb{I}_{G_{r_{i+1}}\setminus G_{r_i}})\le\delta_\Omega(x,\mathbb{I}_{G_{r_{i+1}}})+\delta_\Omega(x,\mathbb{I}_{G_{r_i}})=0, \quad i\in\mathbb{N}, \quad i\ne k,k-1. \tag{4.33}$$

Из (4.24) и (4.32) вытекает $p(x)=f(r_k)$. Имеем также $\delta_\Omega(x,\mathbb{I}_B)=0$. Отсюда следует

$$\delta_\Omega(x,p)=\delta_\Omega\left(x,p-f(r_k)\mathbb{I}_B\right). \tag{4.34}$$

Имеем

$$p(x)-f(r_k)\mathbb{I}_B(x)=(1-f(r_k))\mathbb{I}_{G_{r_0}}(x)+\sum_{i=0}^{m}(f(r_i)-f(r_k))\mathbb{I}_{G_{r_{i+1}}\setminus G_{r_i}}(x).$$

Далее, имея в виду (4.30) и (4.33), получим

$$\delta_\Omega\left(x,p-f(r_k)\mathbb{I}_B\right)\le\sum_{i\in\mathbb{N}\cap\{k-1,k\}}(f(r_i)-f(r_k))\delta_\Omega\left(x,\mathbb{I}_{G_{r_{i+1}}\setminus G_{r_i}}(x)\right)\le2\varepsilon$$

Комбинируя это с (4.31) и (4.34), получим $\delta_\Omega(x,h)=0$.

Случай 3: $x\in\mathbb{R}^2\setminus B$. Из соотношений $\lambda(G_{r_i})\subset G_{r_m}=B$, $i=1,2,\dots,m-1$, следует

$$\lim_{n\to\infty}\delta_\Omega(x,\mathbb{I}_{G_{r_{i+1}}\setminus G_{r_i}})=0, \quad i=1,2,\dots,m-2,$$

и следовательно, с учетом (4.24), (4.30) и равенства $f(r_m)=0$, получим

$$\delta_\Omega(x,p)=|f(r_{m-1})|\delta_\Omega(x,\mathbb{I}_{G_{r_m}\setminus G_{r_{m-1}}})<\varepsilon.$$

Это завершает доказательство (4.19). Лемма 4.5 доказана. □

Лемма 4.6. *Для любого нуль-множества $E \subset \mathbb{R}^2$ типа G_δ, существует функция $g(x) \in L^\infty(\mathbb{R}^2)$, удовлетворяющая условиям*

a) $0 \le g(x) \le 1, \quad x \in \mathbb{R}^2$,

b) $\delta_\Omega(x, g) = 0$ *в каждой точке* $x \in E^c$,

c) $\delta_\Omega(x, g) = 1$ *в каждой точке* $x \in E$.

Доказательство. Имеем

$$E = \cap_{k=1}^{\infty} E_k,$$

где E_k–некоторые открытые множества. Построим функции $g_k \in L^\infty(\mathbb{R}^2)$, открытые множества G_k, $k = 1, 2, \ldots,$, удовлетворяющие условиям

1) $E \subset G_k \subset E_k$, $G_k \subset G_{k-1}$, $k \ge 1$ $(G_0 = \mathbb{R}^2)$,

2) $g_k(x) = 1$, $x \in G_k$, $k \ge 1$,

3) $g_k(x) = 0$, $x \in \mathbb{R}^2 \setminus G_{k-1}$, $k \ge 1$,

4) $0 \le g_k(x) \le 1$, $x \in \mathbb{R}^2$, $k \ge 1$,

5) $\frac{1}{|Q|} \int_Q g_k(t)dt < 2^{-k}$, $Q \in \Omega$, $\overline{Q} \not\subset G_{k-1}$,

6) $\delta_\Omega(x, g_k) = 0$, $x \in \mathbb{R}^2$,

Сделаем эти построения по индукции. Возьмем $G_0 = \mathbb{R}^2$. Применив лемму 4.5 при $G = G_0$ и $\varepsilon = 1/2$, найдем функцию $h(x)$, и открытое множество A, удовлетворяющие условиям леммы. Обозначим $g_1(x) = h(x)$ и $G_1 = A \cap E_1$. Легко проверить, что тогда будут выполнены условия 1)-6) при $k = 1$. Предположим, что уже выбрали множества G_k и функции $g_k(x)$, с условиями 1)-6) при $k = 1, 2, \ldots, p$. Далее, вновь применив лемму 4.5 при $G = G_p$ и $\varepsilon = 2^{-p-1}$, найдем функцию $h(x)$ и открытое множество A, удовлетворяющие условиям той же леммы. Обозначив $g_{p+1}(x) = h(x)$ и $G_{p+1} = A \cap E_{p+1}$ мы получим

$$\operatorname{supp} g_{p+1} \subset G_p,$$

$$g_{p+1}(x) = 1, \; x \in G_{p+1},$$

$$0 \le g_{p+1}(x) \le 1, \quad x \in \mathbb{R}^2,$$

$$\frac{1}{|Q|} \left| \int_Q g_{p+1}(t)dt \right| < 2^{-p-1}, \quad Q \in \Omega, \quad \overline{Q} \not\subset G_p,$$

$$\delta_\Omega(x, g_{p+1}) = 0, \quad x \in \mathbb{R}^2.$$

Легко проверить, что тогда будут выполнены также условия 1)-6) при $k = p+1$, что и завершает процесс индукции. Из 1) следует, что

$$E = \bigcap_{i=1}^{\infty} G_i.$$

Определим

$$g(x) = \begin{cases} \sum_{k=1}^{\infty}(-1)^{k+1}g_k(x), & \text{при} \quad x \in \mathbb{R}^2 \setminus E, \\ 0, & \text{при} \quad x \in E, \end{cases} \tag{4.35}$$

Отметим, что ряд в (4.35) сходится если $x \notin E$. Из соотношений 1), 2) и 3) легко следует условие a) леммы, Если $x \notin E$, то имеем

$$x \in G_{k-1} \setminus G_k, \tag{4.36}$$

для некоторого $k = 1, 2, \ldots$, а это значит, что

$$g_i(x) = 0, \quad i > k. \tag{4.37}$$

Возьмем любую точку x с условием (4.36) и пусть $Q \ni x$ есть произвольный квадрат. Из соотношения 5) и (4.37) следует

$$\frac{1}{|Q|}\left|\int_Q g_i(t)dt - g_i(x)\right| = \frac{1}{|Q|}\left|\int_Q g_i(t)dt\right| < 2^{-i}, \quad i > k,$$

откуда, с учетом 6), легко получить

$$\delta_\Omega(x, g) = \delta_\Omega\left(x, \sum_{i=m}^{\infty}(-1)^i g_i\right) < \sum_{i=m}^{\infty} 2^{-i}, \text{ при } m > k. \tag{4.38}$$

Так как $m \in \mathbb{N}$ может быть произвольным числом, то получим условие b) леммы.

Чтобы установить условие c), предположим $x \in E$. Тогда имеем $x \in G_k$, $k = 1, 2, \ldots$. Очевидно, что существует последовательность квадратов Q_k таких, что

$$\dot{Q}_k \subset G_k, \quad \overline{Q_k} \not\subset G_k.$$

Отсюда, с учетом 2) и 5), получим

$$\frac{1}{|Q_k|}\int_{Q_k} g_i(t)dt = 1, \quad i \le k,$$

$$\frac{1}{|Q_k|}\int_{Q_k} g_i(t)dt < 2^{-i}, \quad i > k.$$

Отсюда получим

$$\left|\frac{1}{|Q_k|}\int_{Q_k} g(t)dt - \sum_{i=1}^{k}(-1)^{i+1}\right|$$

$$\le \sum_{i=1}^{k}\left|\frac{1}{|Q_k|}\int_{Q_k} g_i(t)dt - 1\right| + \sum_{i=k+1}^{\infty}\left|\frac{1}{|Q_k|}\int_{Q_k} g_i(t)dt\right| \le k \cdot 2^{-k} + \sum_{i=k+1}^{\infty} 2^{-k}.$$

Так как сумма $\sum_{i=1}^{k}(-1)^{k+1}$ принимаст значения 0 и 1 по очереди, то получаем условие с) для любой точки $x \in E$. Лемма 4.6 доказана. □

Доказательство теоремы 1.2. Очевидно, что достаточно установить часть необходимости в случае $p = 1$, а часть достаточности в случае $p = \infty$.

Необходимость: Пусть $f \in L^1(\mathbb{R}^2)$ есть произвольная функция. Во первых отметим, что совершенно аналогично доказательству необходимой части теоремы 1.1 можно установить, что $U_\Omega(f)$ является множеством типа G_δ. Далее, рассмотрим множества

$$A_{n,m}(f) = \Big\{x \in \mathbb{R}^2 : \exists Q, Q' \in \Omega,\ x \in Q \cap Q',$$

$$\operatorname{diam}(Q) < 1/n, \operatorname{diam}(Q') < 1/n, \left|\frac{1}{|Q|}\int_Q f(t)dt - \frac{1}{|Q'|}\int_{Q'} f(t)dt\right| > \frac{1}{m}\Big\}.$$

Из соображений непрерывности, легко проверить, что они являются открытыми. Докажем, что

$$A(f) = \bigcup_{m\geq 1}\bigcap_{n\geq 1} A_{n,m}(f) = \{x \in \mathbb{R}^2 : \delta_\Omega(x, f) > 0\}. \tag{4.39}$$

Для этого проверим эквивалентность следующих соотношений:

$$\begin{aligned} x \in A(f), &\Leftrightarrow \exists m_0, \text{ так что } x \in A_{n,m_0}(f), \text{ при любом } n = 1, 2, \dots \\ &\Leftrightarrow \exists Q_k,\ Q'_k \in \Omega,\ \operatorname{diam}(Q_k) \to 0,\ \operatorname{diam}(Q_k) \to 0, \\ &\quad \left|\frac{1}{|Q_k|}\int_{Q_k} f(t)dt - \frac{1}{|Q'_k|}\int_{Q'_k} f(t)dt\right| > \frac{1}{m_0}, \\ &\Leftrightarrow \delta_\Omega(x, f) > 0. \end{aligned}$$

Отсюда и из(4.39) получим,что $\{x \in \mathbb{R}^2 : \delta_\Omega(x, f) > 0\}$ является множеством типа $G_{\delta\sigma}$. Имеем

$$B_\Omega(f) = \{x \in \mathbb{R}^2 : \delta_\Omega(x, f) > 0\} \setminus U_\Omega(f),$$

и $U_\Omega(f)$ является множеством типа G_δ. Отсюда следует, что $B_\Omega(f)$ есть множество типа $G_{\delta\sigma}$. То, что оно имеет меру нуль, следует из теоремы D.

Достаточность: Предположим, что E есть множество типа $G_{\delta\sigma}$ меры нуль и представим его в виде

$$E = \bigcup_{k=1}^{\infty} E_k,$$

где E_k-множества типа G_δ меры нуль. Очевидно можно предполагать, что $E_k \subset E_{k+1}$. В противном случае могли бы рассматривать множества

$$E'_k = \bigcup_{j=1}^{k} E_j$$

которые тоже являются множествами типа $G_{\delta\sigma}$ и их объединение равно E. Применив лемму 4.6, найдем функции $g_k(x)$, такие, что

a) $0 \le g_k(x) \le 1$,

b) $\delta_\Omega(x, g_k) = 0$ в каждой точке $x \notin E_k$,

c) $\delta_\Omega(x, g_k) = 1$ для любой точки $x \in E_k$.

Обозпачим

$$f(x) = \sum_{k=1}^{\infty} 4^{-k} g_k(x)$$

Из свойства a) следует, что $f \in L^\infty(\mathbb{R}^2)$. Пусть $x \in E$. Для некоторого k имеем $x \in E_k \setminus E_{k-1}$. Отсюда вытекает

$$\delta_\Omega\left(x, \sum_{i=1}^{k-1} 4^{-i} g_i\right) = 0, \quad \delta_\Omega(x, g_k) = 1.$$

Отсюда получаем

$$\delta_\Omega(x, f) = \delta_\Omega\left(x, \sum_{i=k}^{\infty} 4^{-i} g_i\right) \ge 4^{-k}\delta_\Omega(x, g_k) - \delta_\Omega\left(x, \sum_{i=k+1}^{\infty} 4^{-i} g_i\right)$$

$$\ge 4^{-k} - \sum_{i=k+1}^{\infty} 4^{-i} > 0.$$

Если же $x \notin E$, то имеем $x \notin E_i$, $i = 1, 2, \ldots$. При любом фиксированном $k \in \mathbb{N}$, с учетом свойств a)-c), следует

$$\delta_\Omega(x, f) = \delta\left(x, \sum_{i=k}^{\infty} 4^{-i} g_i\right) \le \sum_{i=k}^{\infty} 4^{-i}.$$

Так как последнее имеет место при любом k, то получим $\delta_\Omega(x, f) = 0$. Теорема доказапа. □

Доказательство теоремы 1.3. Необходимость утверждения аналогично доказательству необходимости предыдущей теоремы. Приступим к доказательству достаточности. Будем воспользоваться функциями $g_k(x)$, построенные в начале доказательства леммы 4.6. Не трудно выяснить, что в месте условиями 1)-6) можно также гарантировать условие $|G_k| < 4^{-k}$. Определим функцию

$$f(x) = \sum_{k=1}^{\infty} g_k(x).$$

Очевидно, что она принадлежит всем пространствам $L^p(\mathbb{R}^2)$, $1 \le p < \infty$. Из соотношения 2) сразу же следует, что $\delta_{\mathfrak{Q}}(x, f) = \infty$ при $x \in E$. Если же $x \notin E$, то имеем (4.36) и (4.37). Возьмем любую точку x с условием (4.36) и пусть $Q \ni x$ есть произвольный квадрат. Из соотношения 5) и (4.37) следует

$$\frac{1}{|Q|}\left|\int_Q g_i(t)dt - g_i(x)\right| = \frac{1}{|Q|}\left|\int_Q g_i(t)dt\right| < 2^{-i}, \quad i > k,$$

откуда, с учетом 6), легко получить

$$\delta_{\mathfrak{Q}}(x, f) = \delta_{\mathfrak{Q}}\left(x, \sum_{i=m}^{\infty} g_i\right) < \sum_{i=m}^{\infty} 2^{-i}, \text{ при } m > k,$$

которое устанавливает $\delta_{\mathfrak{Q}}(x, f) = 0$. Теорема 1.3 доказана. □

Abstract. The paper considers a question of characterization of the sets of points of differentiation of integrals by bases of rectangles and squares. In particular, a complete characterization of the sets of ambiguous points for integrals of functions from $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, by the basis of squares is obtained.

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FINSLER MANIFOLDS WITH A SPECIAL CLASS OF g-NATURAL METRICS

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Abstract. [1] In this paper we first study the horizontal and vertical distributions of a class of g-natural metrics on the tangent bundle of Finsler manifolds. Then we characterize the Riemannian manifolds among Finsler manifolds from the viewpoint of the geometry of slit tangent bundle and obtain some results on the Riemannian curvature of these metrics. Finally we prove that if the slit tangent bundle is locally symmetric, then the base manifold is locally Euclidean.

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Keywords: Horizontal and vertical distributions; Finsler manifold; g-natural metrics; symmetric space.

1. Introduction

A Riemannian metric g on a smooth manifold M gives rise to several Riemannian metrics on the tangent bundle TM of M, and maybe the best known example is the Sasaki metric g_S introduced in [18]. Although the Sasaki metric is naturally defined, it is very rigid. For example, Kowalski [10] showed that TM with the Sasaki metric is never locally symmetric unless the metric g on the base manifold M is flat. On the other hand, the Sasaki metric is not a "good" metric in the sense of [7], since its Ricci curvature is not constant, that is, the Sasaki metric is not generally Einstein. For this reason, a number of mathematicians tried to construct metrics that save more geometrical properties than Sasaki metric (see, e.g., [1] - [4, 11, 12], and references therein). Also, Oproiu and his collaborators (see [14], [15]) have constructed a family of Riemannian metrics on TM, for which the Ricci curvature is constant, and the tangent bundle TM endowed a metric from this family is locally symmetric.

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In this paper, by using Finsler metric F on a manifold M, we introduce a lift metric $\widetilde{G}$ on TM, called generalized Sasaki metric, or g-natural metric as follows:

$$\widetilde{G}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = g_{ij}(x,y), \quad \widetilde{G}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = a(F^2)g_{ij}(x,y), \quad \widetilde{G}(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0,$$

where $a : Im(F^2) \subset \mathbf{R}^+ \to \mathbf{R}^+$.

Let F be a positive-definite Finsler metric on the manifold M. We prove that F is a Landsberg metric if and only if the vertical distribution $V\widetilde{TM}$ is totally geodesic in $T\widetilde{TM}$, generalizing a result, previously known only in the case of Sasaki metrics (see [5]). Next, we prove that a necessary and sufficient condition for F be a weakly Landsberg metric is that the vertical distribution $V\widetilde{TM}$ is minimal in $T\widetilde{TM}$. This result is an extension of Shen's theorem, proved in [19] for Sasaki metrics. Then we show that the horizontal distribution $H\widetilde{TM}$ is integrable if and only if F has zero flag curvature, which generalizes Mo's result obtained in [13].

According to Deicke's theorem (see [9]), a Finsler metric F on a manifold M is Riemannian if and only if the Cartan tensor is zero. In this paper, we characterize the Riemannian manifolds among Finsler manifolds from the viewpoint of the geometry of tangent bundle, and prove that a Finsler metric F is a Riemannian metric if and only if the horizontal distribution $H\widetilde{TM}$ is minimal in $T\widetilde{TM}$. In [10], Kowalski proved that the tangent bundle of a Riemannian manifold with Sasaki metric is locally symmetric if and only if the base manifold is locally Euclidean. Then Wu [22] extended this result to the case of Finsler manifolds. In this paper, we show that under some conditions, this result remains true for g-natural metrics on Finsler manifolds.

2. Preliminaries

In this section we introduce the necessary notation and definitions, as well as the Chern-Rund connection ∇, associated with a Finsler manifold (M, F).

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ that has the following properties:

(i) F is C^∞ on $\widetilde{TM}$;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iii) for each $y \in T_xM$, the quadratic form $\mathbf{g}_y : T_xM \otimes T_xM \to \mathbf{R}$ defined by $\mathbf{g}_y(u, v) := g_{ij}(y)u^iv^j$ is positive definite, where

$$g_{ij}(y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad u = u^i\frac{\partial}{\partial x^i}|_x \quad \text{and} \quad v = v^i\frac{\partial}{\partial x^j}|_x.$$

Lemma 2.1 (Euler's Lemma). *Let H be a real-valued function on $\mathbb{R}$ of positively homogeneous of degree r. If H is differentiable away from the origin of $\mathbb{R}$, then*

$$y^i \frac{\partial}{\partial y^i} H(y) = rH(y).$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{C}_y(u, v, w) := C_{ijk}(y)u^i v^j w^k$, where

$$C_{ijk}(y) = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

The family $\mathbf{C} = \{\mathbf{C}_y\}_{y \in \widetilde{TM}}$ is called the Cartan torsion. By using the notion of Cartan torsion, define the mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i(y) := g^{jk}C_{ijk}(y)$ and $g^{jk} = (g_{jk})^{-1}$. It is well known that $\mathbf{I} = 0$ if and only if F is Riemannian (see [6], [9]).

Put $C^i{}_{jk} := g^{is}C_{sjk}$. The formal Christofell symbols of the second kind are

$$\gamma^k_{ij} = \frac{1}{2} g^{kl}\Big(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\Big),$$

which are functions on $\widetilde{TM}$. Also, we define the following quantities on $\widetilde{TM}$, called the nonlinear connection coefficients on $\widetilde{TM}$:

$$N^i_j(x, y) := \gamma^i{}_{jk}y^k - C^i{}_{jk}\gamma^k{}_{rs}y^r y^s, \quad y \in \widetilde{TM}.$$

There are a number of connections in Finsler geometry (see [6], [20]). In this paper, we use the Chern connection. According to [8], the pulled-back bundle π^*TM admits a unique linear connection, which is called the Chern connection. Its connection forms are characterized by the structure equations:

(1) Torsion freeness: $dx^j \wedge \omega^i_j = 0$.

(2) Almost g-compatibility: $dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = 2C_{ijk}(dy^k + N^k_l dx^l)$.

It is easy to see that the torsion freeness is equivalent to the absence of dy^k terms in ω^i_j, namely, $\omega^i_j = \Gamma^i_{jk}dx^k$, together with the symmetry property $\Gamma^i_{jk} = \Gamma^i_{kj}$. Let

$$\delta_i = \partial_i - N^k_i \partial_k,$$

where $\delta_i := \frac{\delta}{\delta x^i}$, $\partial_i := \frac{\partial}{\partial x^i}$ and $\partial_{\dot{k}} := \frac{\partial}{\partial y^k}$. The *Riemannian curvature tensor* $R_j{}^i{}_{kl}$ and the *Landsberg curvature tensor* L^i_{jk} can be expressed by the equations

$$R_j{}^i{}_{kl} = \delta_k\Gamma^i_{jl} - \delta_l\Gamma^i_{jk} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^i_{ls}\Gamma^s_{jl}, \tag{2.1}$$

$$L^i_{jk} = y^l \partial_{\dot{j}}\Gamma^i_{lk}, \tag{2.2}$$

respectively. Obviously, we have $R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}$. Let $L_{ijk} = g_{il}L^l_{jk}$, then both C_{ijk} and L_{ijk} are symmetric on all their indices, and by Euler's lemma we have $y^iC_{ijk} = y^iL_{ijk} = 0$. Denote $R^s{}_{kl} := y^jR_j{}^s{}_{kl}$, and observe that

$$R^s{}_{kl} = -R^s{}_{lk}, \quad R^s_{kl} = \delta_k(N^s_l) - \delta_l(N^s_k).$$

Setting

(2.3)

$$(i)\ R_{jikl} := g_{is}R_j{}^s{}_{kl}, \quad (ii)\ R_{ikl} := g_{is}R^s{}_{kl}, \quad (iii)\ R^i{}_k := R^i{}_{kl}y^l, \quad (iv)\ R_{ij} := y_{im}R^m{}_j,$$

we can write (see [8])

$$R_{klji} - R_{jikl} = C_{jis}R^s{}_{kl} - C_{kls}R^s{}_{ji} - C_{kis}R^s{}_{lj} - C_{ljs}R^s{}_{ki} - C_{ils}R^s{}_{jk} - C_{jks}R^s{}_{il},$$

(2.4) $$y^jR_{jikl} = R_{ikl}, \quad y^lR_{ikl} = -y^lR_{ilk} = R_{ik}, \quad y^kR^i{}_k = 0, \quad R_{ik} = R_{ki},$$

(2.5) $$y^iR_{ijk} = 0,$$

and

(2.6) $$R^i{}_{kl} = \frac{1}{3}\{\partial_l R^i{}_k - \partial_k R^i{}_l\}.$$

The flag curvature of the Chern-Rund connection ∇ associated with F is a geometrical invariant which generalizes the sectional curvature in Riemannian geometry. Let $x \in M$ and $0 \neq y \in T_xM$, then $V := V^i\frac{\partial}{\partial x^i}$ is called the transverse edge. The flag curvature is obtained by carrying out the following computation at the point $(x, y) \in \widetilde{TM}$, and viewing y and V as sections of π^*TM:

$$K(y, V) := \frac{R_{ik}V^iV^k}{g(y, y)g(V, V) - [g(y, V)]^2}.$$

If $K(y, V)$ is independent of the transverse edge V, that is, there is a scalar function $\lambda(x, y)$ on $\widetilde{TM}$ such that $K(y, V) = \lambda(x, y)$, then (M, F) is called of *scalar flag curvature*. If furthermore $\lambda(x, y)$ is constant on $\widetilde{TM}$, then the Finsler manifold (M, F) is called of *constant flag curvature*. By using (2.6) and part (iii) of formula (2.3), we conclude that $R^i{}_k = 0$ if and only if $R^i_{kl} = 0$, and this is just the condition for (M, F) to have zero flag curvature (cf. [8], [21], [22]).

2.1. g-natural Metrics. For a given Finsler manifold (M, F), we can endow its slit tangent bundle $\widetilde{TM}$ with a Riemannian metric, known as the g-natural metric or generalized Sasaki metric. It can be described in local coordinates as follows. Let $(x, y) = (x^i, y^i)$ be the local coordinates on $\widetilde{TM}$. It is well known that the tangent space to $\widetilde{TM}$ at (x, y) splits into the direct sum of the vertical subspace $V\widetilde{TM}_{(x,y)} = span\{\partial_i\}$ and the horizontal subspace $H\widetilde{TM}_{(x,y)} = span\{\delta_i\}$ as follows:

$$T_{(x,y)}\widetilde{TM} = V\widetilde{TM}_{(x,y)} \oplus H\widetilde{TM}_{(x,y)}.$$

The g-natural metric or generalized Sasaki metric $\widetilde{G}$ on $\widetilde{TM}$ is defined by

$$\widetilde{G}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = g_{ij}(x, y), \quad \widetilde{G}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = a(F^2)g_{ij}(x, y), \quad \widetilde{G}(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0, \tag{2.7}$$

where $a : Im(F^2) \subset \mathbf{R}^+ \to \mathbf{R}^+$. For $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$, its horizontal lift X^h and vertical lift X^v are defined by

$$X^h := (X^i \circ \pi)\frac{\delta}{\delta x^i}, \quad \text{and} \quad X^v := (X^i \circ \pi)\frac{\partial}{\partial y^i}.$$

The Levi-Civita connection $\widetilde{\nabla}$ on $\widetilde{TM}$ with respect to $\widetilde{G}$ is given by Koszul formula

$$\begin{aligned} 2\widetilde{G}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) &= \widetilde{X}\widetilde{G}(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y}\widetilde{G}(\widetilde{Z}, \widetilde{X}) - \widetilde{Z}\widetilde{G}(\widetilde{X}, \widetilde{Y}) \\ &+\widetilde{G}([\widetilde{X}, \widetilde{Y}], \widetilde{Z}) - \widetilde{G}([\widetilde{Y}, \widetilde{Z}], \widetilde{X}) + \widetilde{G}([\widetilde{Z}, \widetilde{X}], \widetilde{Y}), \end{aligned} \tag{2.8}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \chi(\widetilde{TM})$. We say that the vertical distribution $V\widetilde{TM}$ is totally geodesic (resp. minimal) in $T\widetilde{TM}$ if $\mathcal{H}\widetilde{\nabla}_{\partial_i}\partial_j = 0$ (resp. $g^{ij}\mathcal{H}\widetilde{\nabla}_{\partial_i}\partial_j = 0$), where $\mathcal{H}$ denotes the horizontal projection. Similarly, if we denote by $\mathcal{V}$ the vertical projection, then we say that the horizontal distribution $H\widetilde{TM}$ is totally geodesic (resp. minimal) in $T\widetilde{TM}$ if $\mathcal{V}\widetilde{\nabla}_{\delta_i}\delta_j = 0$ (resp. $g^{ij}\mathcal{V}\widetilde{\nabla}_{\delta_i}\delta_j = 0$). By a simple calculation, we get the following result.

Lemma 2.2. *(see [16]) Let (M, F) be a Finsler manifold. Then we have*

$$[\delta_i, \delta_j] = -R^k{}_{ij}\partial_k, \quad [\partial_i, \partial_j] = 0, \quad [\delta_i, \partial_j] = (\Gamma^k_{ij} + L^k{}_{ij})\partial_k.$$

Lemma 2.3. *(see [16]) Let (M, F) be a Finsler manifold. Then the Levi-Civita connection on the Riemannian manifold $(\widetilde{TM}, \widetilde{G})$ is locally expressed as follows:*

$$\widetilde{\nabla}_{\partial_i}\partial_j = a(F^2)L^k{}_{ij}\delta_k + [C^k{}_{ij} + \frac{a'(F^2)}{a(F^2)}(y_i\delta^k_j + y_j\delta^k_i - g_{ij}y^k)]\partial_k, \tag{2.9}$$

$$\widetilde{\nabla}_{\partial_i}\delta_j = [C^k{}_{ij} + \frac{1}{2}a(F^2)y^l R_{lij}{}^k]\delta_k - L^k{}_{ij}\partial_k, \tag{2.10}$$

$$\widetilde{\nabla}_{\delta_i}\partial_j = [C^k{}_{ij} + \frac{1}{2}a(F^2)y^l R_{lji}{}^k]\delta_k + \Gamma^k{}_{ij}\partial_k, \tag{2.11}$$

$$\widetilde{\nabla}_{\delta_i}\delta_j = \Gamma^k{}_{ij}\delta_k - [\frac{1}{a(F^2)}C^k{}_{ij} + \frac{1}{2}R^k{}_{ij}]\partial_k, \tag{2.12}$$

where

$$R_{lij}{}^k = g_{is}g^{kt}R_l{}^s{}_{jt}. \tag{2.13}$$

The following proposition contains some relationships between the Finsler manifold (M, F) and the vertical or horizontal distributions in $\widetilde{TTM}$.

Proposition 2.1. *Let (M, F) be a Finsler manifold. Then the following assertions hold.*

1) *F is a Landsberg metric if and only if $\widetilde{VTM}$ is totally geodesic in $\widetilde{TTM}$.*
2) *F is a weakly Landsberg metric if and only if $\widetilde{VTM}$ is minimal in $\widetilde{TTM}$.*
3) *F has zero flag curvature if and only if $\widetilde{HTM}$ is integrable.*

Proof. Taking into account that $a(F^2) \neq 0$, the assertions 1) and 2) we infer from (2.9), as for assertion 3), it follows from Lemma 2.2. □

3. Reduction of Finsler Manifolds to Riemannian Manifolds

Let $\widetilde{div}$ denotes the divergence operator of $(\widetilde{TM}, \widetilde{G})$. In this section, we characterize the Riemannian manifolds as the Finsler manifolds such that the vertical lift of any vector field is divergence-free, or equivalently, the horizontal distribution is minimal in the tangent bundle of the slit tangent bundle.

Theorem 3.1. *Let (M, F) be a Finsler manifold. Then the following statements are equivalent.*

1) *(M, F) is a Riemannian manifold;*
2) *$\widetilde{div}(X^v) = n\frac{a'(F^2)}{a(F^2)}X^i y_i$ for any $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$;*
3) *The horizontal distribution $\widetilde{HTM}$ is minimal in $\widetilde{TTM}$.*

Proof. By (2.12) we have $g^{ij}\mathcal{V}\widetilde{\nabla}_{\delta_i}\delta_j = -\frac{1}{a(F^2)}I^k\partial_{\bar{k}}$. Equivalence of statements 1) and 3) follows from the above equation and the Deicke's theorem. On the other hand, by the definition of divergence we have

$$\widetilde{div}(\partial_{\bar{i}}) = g^{jl}\widetilde{G}(\widetilde{\nabla}_{\delta_j}\partial_{\bar{i}}, \delta_l) + \frac{1}{a(F^2)}g^{jl}\widetilde{G}(\widetilde{\nabla}_{\partial_{\bar{j}}}\partial_{\bar{i}}, \partial_{\bar{l}})$$

$$= g^{jl}(C^k_{\ jl} + \frac{1}{2}a(F^2)y^m R_{m\ ij}^{\ k})g_{kl} + g^{jl}[C^k_{\ jl} + \frac{a'(F^2)}{a(F^2)}(y_j\delta^k_i + y_i\delta^k_j - g_{ji}y^k)]g_{kl}$$

$$= 2g^{jl}C_{ijl} + n\frac{a'(F^2)}{a(F^2)}y_i = 2I_i + n\frac{a'(F^2)}{a(F^2)}y_i,$$

implying that

$$\widetilde{div}(X^v) = 2X^iI_i + n\frac{a'(F^2)}{a(F^2)}X^iy_i \tag{3.1}$$

for $X = X^i\frac{\partial}{\partial x^i} \in \chi(M)$. Hence the equivalence of statements 1) and 2) follows from (3.1). Theorem 3.1. is proved. □

Corollary 3.1. *Let (M,F) be a Finsler manifold and $a'(F^2)=0$. Then F is a Riemannian metric if and only if $\widetilde{div}(X^v)=0$.*

Proof. Since $a'(F^2)=0$, then by (3.1) we get $\widetilde{div}(X^v)=2X^iI_i$. It follows that $\widetilde{div}(X^v)=0$ if and only if $I_i=0$. □

Lemma 3.1. *Let (M,F) be a Riemannian manifold. Then the coefficients of the Riemannian curvature tensor with respect to $\widetilde{G}$ are given by the following formulas:*

$$\widetilde{R}(\partial_i,\partial_j)\partial_k=[\frac{2aa''-3a'^2}{a^2}(y_iy_k\delta^s_j-y_jy_k\delta^s_i-g_{jk}y_iy^s+g_{ik}y_jy^s)$$

$$+\frac{a'(2a+F^2a')}{a^2}(g_{ik}\delta^s_j-g_{jk}\delta^s_i)]\partial_s, \tag{3.2}$$

$$\widetilde{R}(\partial_i,\partial_j)\delta_k=[\frac{a}{2}(R_{ijk}{}^s-R_{jik}{}^s)+\frac{a^2}{4}y^my^l(R_{mjk}{}^rR_{lir}{}^s-R_{mik}{}^rR_{ljr}{}^s)$$

$$+a'y^l(y_iR_{ljk}{}^s-y_jR_{lik}{}^s)]\delta_s, \tag{3.3}$$

$$\widetilde{R}(\delta_i,\delta_j)\delta_k=[R_k{}^s{}_{ij}-\frac{a}{4}y^lR^r{}_{jk}R_{lri}{}^s+\frac{a}{4}y^lR^r{}_{ik}R_{lrj}{}^s+\frac{a}{2}y^lR^r{}_{ij}R_{lrk}{}^s]\delta_s$$

$$+\frac{1}{2}[R^s{}_{ik|j}-R^s{}_{jk|i}]\partial_s, \tag{3.4}$$

$$\widetilde{R}(\delta_i,\delta_j)\partial_k=\frac{a}{2}y^l[R_{lkj}{}^s{}_{|i}-R_{lki}{}^s{}_{|j}]\delta_s+[\frac{a}{4}y^l(R_{lki}{}^rR^s{}_{jr}-R_{lkj}{}^rR^s{}_{ir})$$

$$+R_k{}^s{}_{ij}+\frac{a'}{a}R^r{}_{ij}(y_r\delta^s_k+y_k\delta^s_r-g_{rk}y^s)]\partial_s, \tag{3.5}$$

$$\widetilde{R}(\delta_i,\partial_j)\delta_k=\frac{a}{2}y^lR_{ljk}{}^s{}_{|i}\delta_s+[\frac{a'}{2a}(y_jR^s{}_{ik}+y_rR^r{}_{ik}\delta^s_j-R^r{}_{ik}g_{jr}y^s)$$

$$-\frac{a}{4}y^lR^s{}_{ir}R_{ljk}{}^r+\frac{1}{2}R_j{}^s{}_{ik}]\partial_s, \tag{3.6}$$

$$\widetilde{R}(\delta_i,\partial_j)\partial_k=[(\frac{a'}{a}-a')y_jy^lR_{lki}{}^s+\frac{a'}{a}y_ky^lR_{lji}{}^s-$$

$$-\ \frac{a}{2}R_{jki}{}^s-\frac{a^2}{4}y^ly^mR_{lki}{}^rR_{mjr}{}^s]\delta_s, \tag{3.7}$$

where $R_{lkj}{}^s{}_{|i}$ denote the horizontal covariant derivative of the tensor $R_{lkj}{}^s$ with respect to the Chern connection.

Proof. Recall that the curvature tensor in terms of the Levi-Civita connection $\widetilde{\nabla}$ is given by the following formula

$$\widetilde{R}(X,Y)Z=\widetilde{\nabla}_X\widetilde{\nabla}_YZ-\widetilde{\nabla}_Y\widetilde{\nabla}_XZ-\widetilde{\nabla}_{[X,Y]}Z, \tag{3.8}$$

where $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\chi(\widetilde{TM})$. Since (M,F) is a Riemannian manifold, we have

$$L^k_{ij}=C^k_{ij}=0. \tag{3.9}$$

Substituting (3.9) into (2.9)-(2.12) and using (3.8), we obtain (3.2)-(3.7). □

Theorem 3.2. *Let (M,F) be a Riemannian manifold. Then the tangent bundle $\widetilde{TM}$ with the lift metric $\widetilde{G}$ has constant sectional curvature k if and only if M is a flat manifold and either $a(F^2)=c$ or $a(F^2)=\frac{c}{F^2}$, where c is an arbitrary real constant. Furthermore, in this case $k=0$ and $\widetilde{TM}$ is a flat manifold.*

Proof. We first recall that the curvature tensor field of the Riemannian manifold $(\widetilde{TM},\widetilde{G})$ with constant sectional curvature k satisfies the following equation:

$$\widetilde{R}(X,Y)Z = k[\widetilde{G}(Y,Z)X - \widetilde{G}(X,Z)Y], \tag{3.10}$$

where $\widetilde{X},\widetilde{Y},\widetilde{Z}\chi(\widetilde{TM})$. Assuming that $(\widetilde{TM},\widetilde{G})$ has a constant sectional curvature k, we obtain

$$\widetilde{R}(\delta_i,\delta_j)\partial_{\bar{k}} = k[\widetilde{G}(\delta_j,\partial_{\bar{k}})\delta_i - \widetilde{G}(\delta_i,\partial_{\bar{k}})\delta_j] = 0. \tag{3.11}$$

Using (3.5) and (3.11), we get

$$\frac{a}{4}y^l(R_{lki}{}^rR^s{}_{jr} - R_{lkj}{}^rR^s{}_{ir}) + R_k{}^s{}_{ij} + \frac{a'}{a}R^r{}_{ij}(y_r\delta^s_k + y_k\delta^s_r - g_{rk}y^s) = 0,$$

or

$$R_k{}^s{}_{ij} = -\frac{a}{4}y^l(R_{lki}{}^rR^s{}_{jr} - R_{lkj}{}^rR^s{}_{ir}) - \frac{a'}{a}R^r{}_{ij}(y_r\delta^s_k + y_k\delta^s_r - g_{rk}y^s). \tag{3.12}$$

By (2.5) and (2.13) we conclude that

$$y^ly^iR_{lij}{}^s = y^ly^ig_{is}g^{kt}R_l{}^s{}_{jt} = y^ig_{is}g^{kt}R^s{}_{jt} = g^{kt}y^iR_{ijt} = 0. \tag{3.13}$$

Multiplying (3.12) by y^k and using (3.13), we can write

$$(1+F^2\frac{a'}{a})R^s{}_{ij} = 0. \tag{3.14}$$

The equation (3.14) implies that either $R^s{}_{ij}=0$ or $1+F^2\frac{a'}{a}=0$.

Now, we prove that in these cases (M,F) is a flat manifold.

Case 1. If $R^s{}_{ij}=0$, then by using (3.12) we obtain $R_k{}^s{}_{ij}=0$, implying that (M,F) is a flat manifold.

Case 2. If $1+F^2\frac{a'}{a}=0$, then we get $a+a'F^2=0$. For $t=F^2$, solving this equation we get

$$a(F^2) = \frac{c}{F^2}, \quad a'(F^2) = -\frac{c}{F^4}, \tag{3.15}$$

where c is a constant. Since $(\widetilde{TM},\widetilde{G})$ has a constant sectional curvature k, then we have $\widetilde{R}(\partial_{\bar{i}},\partial_{\bar{j}})\delta_k = k\Big(G(\partial_{\bar{j}},\delta_k)\partial_{\bar{i}} - G(\partial_{\bar{i}},\delta_k)\partial_{\bar{j}}\Big) = 0$. Plugging (3.15) into (3.3) we get

(3.16)

$$\frac{c}{2F^2}(R_{ijk}{}^s - R_{jik}{}^s) + \frac{c^2}{4F^4}y^m y^l (R_{mjk}{}^r R_{lir}{}^s - R_{mik}{}^r R_{ljr}{}^s) - \frac{c}{F^4}y^l(y_i R_{ljk}{}^s - y_j R_{lik}{}^s) = 0.$$

By contracting (3.16) with y^j and using (3.13), we obtain

$$\frac{c}{2F^2}y^l R_{ilk}{}^s + \frac{c}{2F^2}y^l R_{lik}{}^s = 0. \tag{3.17}$$

Also, from (3.10) we have

$$\widetilde{R}(\delta_i, \partial_{\bar{j}})\partial_{\bar{k}} = k(G(\partial_{\bar{j}}, \partial_{\bar{k}})\delta_i - G(\delta_i, \partial_{\bar{k}})\partial_{\bar{j}}) = ka(F^2)g_{jk}\delta_i = \frac{kc}{F^2}g_{jk}\delta_i. \tag{3.18}$$

The relations (3.7), (3.15) and (3.18) imply

(3.19)

$$[(-\frac{1}{F^2} + \frac{c}{F^4})y_j y^l R_{lki}{}^s - \frac{1}{F^2}y_k y^l R_{lji}{}^s - \frac{c}{2F^2}R_{jki}{}^s - \frac{c^2}{4F^4}y^l y^m R_{lki}{}^r R_{mjr}{}^s]\delta_s = \frac{kc}{F^2}g_{jk}\delta_i.$$

Multiplying both sides of (3.19) by $y^j y^k$ and using (3.13), we conclude that $kc = 0$. Since $c \neq 0$, then $k = 0$. In this case ($k = 0$), by contracting (3.19) with y^k, we get

$$-y^l R_{lji}{}^s - \frac{c}{2F^2}y^l R_{jli}{}^s = 0. \tag{3.20}$$

By replacing $j \to i$ and $i \to k$ in (3.20), we get

$$-y^l R_{lik}{}^s - \frac{c}{2F^2}y^l R_{ilk}{}^s = 0, \tag{3.21}$$

and in view of (3.17) and (3.21) we obtain

$$\left(\frac{c}{2F^2} - 1\right) y^l R_{lik}{}^s = 0. \tag{3.22}$$

Observe that if $\frac{c}{2F^2} - 1 = 0$, then $F^2 = \frac{c}{2} = constant$, which is contradiction. Thus, in view of $\frac{c}{2F^2} - 1 \neq 0$ and (3.22), we conclude that $y^l R_{lik}{}^s = 0$. Consequently, by using (2.13), we get

(3.23)

$$0 = g_{ms}g^{ir}y^l R_{lik}{}^s = g_{ms}g^{ir}y^l g_{ih}g^{st}R_l{}^h{}_{kt} = g_{ms}g^{ir}g_{ih}g^{st}R^h{}_{kt} = \delta^r_h\delta^t_m R^h{}_{kt} = R^r{}_{km}.$$

Substituting (3.23) into (3.12), we get $R_k{}^s{}_{ij} = 0$. Therefore in this case, and hence in both cases, (M, F) is a flat Finsler manifold.

Now, we show that $(\widetilde{TM}, \widetilde{G})$ is flat. To this end, we first substitute $R_k{}^s{}_{ij} = 0$ and $R^s_{ij} = 0$ into (3.4) to obtain $\widetilde{R}(\delta_i, \delta_j)\delta_k = 0$. Hence from (3.10), we get

$$k(g_{jr}\delta^s_i - g_{ir}\delta^s_j) = 0,$$

implying that $k = 0$, that is, $(\widetilde{TM}, \widetilde{G})$ is flat. Therefore

$$\widetilde{R}(\partial_{\bar{i}}, \partial_{\bar{j}})\partial_{\bar{k}} = 0. \tag{3.24}$$

Since (M,F) is a Riemannian manifold, then g_{ik} is a function of position. Therefore, in view of (3.2) and (3.24), we obtain

$$2aa'' - 3a'^2 = 0 \quad \text{and} \quad a'(2a + F^2a') = 0. \tag{3.25}$$

For $t = F^2$, solving the equations in (3.25), we get $a(F^2) = c$ and $a(F^2) = \frac{c}{F^4}$. Finally, using (3.2) - (3.7), it is easy to see that the converse of this theorem is true. Theorem 3.4 is proved. □

4. Locally Symmetric Finsler Metrics

In [10], Kowalski proved that the tangent bundle of a Riemannian manifold with Sasaki metric is locally symmetric if and only if the base manifold is locally Euclidean. Then Wu [22] extended this result to the case of Finsler manifolds. In this section, we show that under some conditions, this result remains true for Finsler manifolds with g-natural metrics. To this end, we first prove the following result.

Theorem 4.1. *Let (M,F) be a Finsler manifold and $a(F^2)$ be a positively homogeneous of degree $n \neq -2$. If $(\widetilde{TM},\widetilde{G})$ is locally symmetric, then (M,F) is locally Euclidean.*

Proof. Let $\eta \in \chi(\widetilde{TM})$ be such that $\eta(x,y) = y^v = y^i\partial_i$. Then by (2.9) we get

$$\widetilde{\nabla}_\eta\partial_j = y^i\Big\{aL^k{}_{ij}\delta_k + [C^k{}_{ij} + \frac{a'}{a}(y_i\delta^k_j + y_j\delta^k_i - g_{ij}y^k)]\partial_k\Big\} = \frac{F^2a'}{a}\partial_j. \tag{4.1}$$

Also, using (2.10) and (3.13), we obtain

$$\widetilde{\nabla}_\eta\delta_j = y^i[C^k{}_{ij} + \frac{1}{2}a(F^2)y^lR_{lij}{}^k]\delta_k - y^iL^k{}_{ij}\partial_k = \frac{1}{2}a(F^2)y^iy^lR_{lij}{}^k = 0. \tag{4.2}$$

Then we have

$$y^j(\widetilde{\nabla}_\eta\widetilde{R})(\partial_i,\partial_j)\partial_k = y^j\widetilde{\nabla}_\eta(\widetilde{R}(\partial_i,\partial_j)\partial_k) - 3\frac{F^2a'}{a}y^j\widetilde{R}(\partial_i,\partial_j)\partial_k. \tag{4.3}$$

Using (2.9) and (2.10), a direct computation shows that

$$\begin{aligned}
\widetilde{R}(\partial_i,\partial_j)\partial_k = &\Big\{a'L^s{}_{ir}(y_j\delta^r_k + y_k\delta^r_j - g_{jk}y^r) - a'L^s{}_{jr}(y_i\delta^r_k + y_k\delta^r_i - g_{ik}y^r) + \frac{1}{2}a^2y^lL^r{}_{jk}R_l{}^s{}_{ir}\\
&+\partial_i(aL^s{}_{jk}) - \partial_j(aL^s{}_{ik}) + a(L^r{}_{jk}C^s{}_{ir} - L^r{}_{ik}C^s_{jr} + C^r{}_{jk}L^s{}_{ir} - C^r{}_{ik}L^s{}_{jr})\\
&-\frac{1}{2}a^2y^lL^r{}_{ik}R_l{}^s{}_{jr}\Big\}\,\delta_s + \Big\{a(L^r{}_{ik}L^s{}_{jr} - L^r{}_{jk}L^s{}_{ir}) - (C^r{}_{jk}C^s{}_{ir} - C^r{}_{ik}C^s{}_{jr})\\
&+(\frac{a'}{a})^2(y_jy_k\delta^s_i - y_iy_k\delta^s_j - y_jg_{ik}y^s + y_ig_{jk}y^s - F^2g_{jk}\delta^s_i + F^2g_{ik}\delta^s_j)\\
&+\partial_i[\frac{a'}{a}(y_j\delta^s_k + y_k\delta^s_j - g_{jk}y^s)] - \partial_j[\frac{a'}{a}(y_i\delta^s_k + y_k\delta^s_i - g_{ik}y^s)]\Big\}\,\partial_s,
\end{aligned} \tag{4.4}$$

where $\widetilde{R}$ denotes the Riemannian curvature of $(\widetilde{TM}, \widetilde{G})$. Let $a(t)$ be positively homogeneous of degree $\frac{n}{2}$, then $a(F^2)$, $a'(F^2)$ and $a''(F^2)$ are positively homogeneous of degrees n, $n-2$ and $n-4$, respectively. Contracting (4.4) with y^j we get

$$y^j\widetilde{R}(\partial_i,\partial_j)\partial_k = [F^2a' - (n+1)a]L^s{}_{ik}\delta_s + 2\frac{(a''a - a'^2)F^2 + aa'}{a^2}y_i\delta^s_k\partial_s. \tag{4.5}$$

Taking into account the equation (4.4), we obtain

$$\begin{aligned} y^j\widetilde{\nabla}_\eta(\widetilde{R}(\partial_i,\partial_j)\partial_k) = &\Big\{a(1-n^2) + (n-1)F^2a'\Big\}L^s_{ik}\delta_s + \Big\{2F^4\frac{a'(aa''-a'^2)}{a^3} \\ &+2F^2\frac{a'^2}{a^2} - 4F^2\frac{aa''-a'^2}{a^2} - 4\frac{a'}{a}\Big\}y_i\delta^s_k\partial_s. \end{aligned} \tag{4.6}$$

Assuming that $\widetilde{\nabla}\widetilde{R} = 0$, in view of (4.5) and (4.6), we have

$$\begin{aligned} 0 = y^j(\widetilde{\nabla}_\eta\widetilde{R})(\partial_i,\partial_j)\partial_k = &\Big\{a(1-n^2) + F^2a'(4n+2-3F^2\frac{a'}{a})\Big\}L^s{}_{ik}\delta_s \\ &+\Big\{\frac{4(a+F^2a')[-F^2aa'' + a'(F^2a'-a)]}{a^3}\Big\}y_i\delta^s_k\partial_s. \end{aligned} \tag{4.7}$$

Let $t = F^2$, since $a(F^2)$ is positively homogeneous of degree n, then $a(t)$ is positively homogeneous of degree $\frac{n}{2}$. Hence using Lemma 2.1 we obtain $ta'(t) = \frac{n}{2}a(t)$. For the solution of the above equation we have $a(t) = ct^{\frac{n}{2}}$. Therefore

$$a(F^2) = cF^n, \quad a'(F^2) = c\frac{n}{2}F^{n-2}, \quad a''(F^2) = c\frac{n}{2}(\frac{n}{2}-1)F^{n-4}.$$

Using the above relations we get

$$[a(1-n^2) + F^2a'(4n+2-3F^2\frac{a'}{a})]L^s_{ik}\delta_s = \frac{c}{4}(n+2)^2L^s{}_{ik}\delta_s, \tag{4.8}$$

$$-F^2aa'' + a'(F^2a'-a) = 0. \tag{4.9}$$

In view of (4.7), (4.8) and (4.9) we have $\frac{c}{4}(n+2)^2L^s{}_{ik}\delta_s = 0$, implying that $L^i_{jk} = 0$ $(n \neq -2)$, which together with yields

$$\begin{aligned} \widetilde{R}(\partial_i,\partial_j)\partial_k = [&C^r{}_{ik}C^s{}_{jr} - C^r{}_{jk}C^s{}_{ir} + \frac{n}{2}\partial_i[\frac{1}{F^2}(y_j\delta^s_k + y_k\delta^s_j - g_{jk}y^s)] \\ &-\frac{n}{2}\partial_j[\frac{1}{F^2}(y_i\delta^s_k + y_k\delta^s_i - g_{ik}y^s)] + \frac{n^2}{4F^4}(y_jy_k\delta^s_i - y_iy_k\delta^s_j \\ &-y_jg_{ik}y^s + y_ig_{jk}y^s - g_{jk}\delta^s_iF^2 + g_{ik}\delta^s_jF^2)]\partial_s. \end{aligned} \tag{4.10}$$

Consequently we have

$$\begin{aligned} 0 = (\widetilde{\nabla}\widetilde{R})(\partial_i,\partial_j)\partial_k &= \widetilde{\nabla}_\eta(\widetilde{R}(\partial_i,\partial_j)\partial_k) - 3\frac{F^2a'}{a}\widetilde{R}(\partial_i,\partial_j)\partial_k \\ &= -(n+2)\Big\{\frac{n}{2}\partial_i[\frac{1}{F^2}(y_j\delta^s_k + y_k\delta^s_j - g_{jk}y^s)] - \frac{n}{2}\partial_j[\frac{1}{F^2}(y_i\delta^s_k + y_k\delta^s_i - g_{ik}y^s)] \\ &+\frac{n^2}{4F^4}\Big(y_jy_k\delta^s_i - y_iy_k\delta^s_j - y_jg_{ik}y^s + y_ig_{jk}y^s - g_{jk}\delta^s_iF^2 + g_{ik}\delta^s_jF^2\Big) + C^r{}_{ik}C^s{}_{jr} - C^r{}_{jk}C^s{}_{ir}\Big\}\partial_s. \end{aligned} \tag{4.11}$$

Taking into account that $n \neq -2$, we can use (4.10) and (4.11) to conclude that $\widetilde{R}(\partial_i, \partial_j)\partial_k = 0$.

Now let $\xi \in \chi(\widetilde{TM})$ be such that $\xi(x,y) = y^h = y^i\delta_i$, then from (2.11) we obtain

$$\widetilde{\nabla}_\xi \partial_i = N_i^j \partial_j - \frac{a}{2}R^j{}_i\delta_j = N_i^j\partial_j - c\frac{F^n}{2}R^j{}_i\delta_j.$$

This implies

$$\begin{aligned} 0 = (\widetilde{\nabla}_\xi \widetilde{R})(\eta, \partial_i)\partial_j &= -\widetilde{R}(\eta, \widetilde{\nabla}_\xi \partial_i)\partial_j - \widetilde{R}(\eta, \partial_i)\widetilde{\nabla}_\xi \partial_j \\ &= c\frac{F^n}{2}\Big(R^k{}_i\widetilde{R}(\eta, \delta_k)\partial_j + R^k{}_j\widetilde{R}(\eta, \partial_i)\delta_k\Big). \end{aligned} \tag{4.12}$$

It can directly be verified that

$$\widetilde{R}(\eta, \delta_k)\partial_j = (\frac{n+2}{2})(-C^s{}_{kj} + \frac{c}{2}y^l R_l{}^s{}_{jk}F^n)\delta_s, \tag{4.13}$$

$$\widetilde{R}(\eta, \partial_i)\delta_k = \frac{c}{2}(n+2)y^l R_l{}^s{}_{ik}F^n\delta_s. \tag{4.14}$$

Combining (4.12)-(4.14) we obtain

$$c(\frac{n+2}{2})\frac{F^n}{2}[R^k{}_i(-C^s_{kj} + \frac{c}{2}y^l R_l{}^s{}_{jk}F^n) + cR^k{}_j y^l R_l{}^s{}_{ik}F^n] = 0. \tag{4.15}$$

Multiplying (4.15) by $y_s = g_{si}y^i$ and then summing up, we get

$$cF^n(\frac{1}{2}R^k{}_i R_{jk} + R^k{}_j R_{ik}) = 0. \tag{4.16}$$

It is easy to deduce from (4.16) that $R_{ij} = 0$, or equivalently, $R^k{}_{ij} = 0$. To see that the last two equalities are equivalent, observe that if $R^k{}_{ij} = 0$, then by using parts (iii) and (iv) of (2.3) we have $R_{ij} = 0$. Conversely, if $R_{ij} = 0$, then by using (iv) of (2.3) we get $R^k{}_i = g^{kj}R_{ij} = 0$. Hence, in view of (2.6) we conclude that $R^k{}_{ij} = 0$.

Next, since $L^k{}_{ij} = R^k{}_{ij} = 0$, it follows that

$$0 = (\widetilde{\nabla}_{\delta_k}\widetilde{R})(\eta, \partial_i)\partial_j = -\widetilde{R}(\eta, \widetilde{\nabla}_{\delta_k}\partial_i)\partial_j = C^r{}_{ki}\widetilde{R}(\eta, \delta_r)\partial_j = -(\frac{n+2}{2})C^r{}_{ki}C^s{}_{rj}\delta_s.$$

Therefore $C^r{}_{ki}C^s{}_{rj} = 0$, or equivalently, $C_{ikr}C_{sjl}g^{rl} = 0$, and since F is positively definite, we get $C_{ijk} = 0$, implying that (M, F) is Riemannian. On the other hand, if $R^k{}_{ij} = 0$, then (M, F) is a flat Riemannian manifold, and thus, it is locally Euclidean. Theorem 4.1 is proved. □

Corollary 4.1. *Let (M, F) be a Finsler manifold. If either $a(F^2) = c$ or $a(F^2) = \frac{c}{F^4}$, then the Riemannian manifold $(\widetilde{TM}, \widetilde{G})$ is locally symmetric if and only if (M, F) is locally Euclidean.*

Proof. Observe that $a(F^2) = c$ and $a(F^2) = \frac{c}{F^4}$ are positively homogenous of degrees $n = 0$ and $n = -4$, respectively. If $(\widetilde{TM}, \widetilde{G})$ is locally symmetric, then in view of

Theorem 4.1 we conclude that (M, F) is locally Euclidean. Conversely, if (M, F) is locally Euclidean, then it follows from Theorem 3.2 that $(\widetilde{TM}, \widetilde{C})$ is flat, and consequently, it is locally symmetric. This completes the proof. □

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A CHARACTERIZATION OF TIGHT WAVELET FRAMES ON LOCAL FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. The objective of this paper is to establish a complete characterization of tight wavelet frames on local fields of positive characteristic by means of two basic equations in the Fourier domain.

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1. Introduction

Tight wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. Tight wavelet frames provide representations of signals and images in applications, where redundancy of the representation is preferred and the perfect reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets, is kept.

In recent years there has been a considerable interest in the problem of constructing wavelet bases on locally compact Abelian groups. For example, Dahlke [4] introduced multiresolution analysis and wavelets on locally compact Abelian groups, Lang [12], by following the procedure of Daubechies [5], has constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ via scaling filters, and these wavelets turn out to be certain lacunary Walsh series on the real line. Later on, Farkov [6] extended the results of Lang [12] on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact Abelian group G_p, which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. Concerning the construction of wavelets on the half-line $\mathbb{R}^+$, Farkov [7] has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ and obtained necessary and sufficient conditions for scaling filters with p^n many terms ($p, n \geq 2$) to generate a

p-MRA analysis in $L^2(\mathbb{R}^+)$. These studies were continued by Shah and Debnath in [15-17], where they have given some new algorithms for constructing the wavelet and Gabor frames on the positive half-line $\mathbb{R}^+$. More results in this direction can also be found in [8, 9] and in the references therein.

A field $\mathbb{K}$ equipped with a topology is called a *local field* if both the additive and multiplicative groups of $\mathbb{K}$ are locally compact Abelian groups. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. Local fields have attracted the attention of several mathematicians, and have found innumerable applications not only in the number theory, but also in the representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as a part of the standard repertoire of contemporary mathematics. For more details we referr to [14, 19].

Recently, R. L. Benedetto and J. J. Benedetto [3] developed a wavelet theory for local fields and related groups. Jiang *et al.*[11] pointed out a method for constructing orthogonal wavelets on a local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. Subsequently, the tight wavelet frames on the local fields were constructed by Li and Jiang in [13]. They have obtained a necessary condition and sufficient conditions for tight wavelet frame on local fields in the frequency domain. Behera and Jahan [1] have constructed wavelet packets and wavelet frame packets on a local field $\mathbb{K}$ of positive characteristic, and show how to construct an orthonormal basis from a Riesz basis. Further, Behera and Jahan [2] have given a characterization of scaling functions associated with given multiresolution analysis of positive characteristic on a local field $\mathbb{K}$. Recently, Shah and Debnath [18], by following the procedure of Daubechies [5], have constructed tight wavelet frames on a local field $\mathbb{K}$ via extension principles.

Finally, E. Hermandes and Weiss (see [10]) have given a general characterization of all tight wavelet frames in $L^2(\mathbb{R})$ by means of the Fourier transform. As for the corresponding counterpart for a local field $\mathbb{K}$, such a result is not yet reported. So in this paper, we give a complete characterization of tight wavelet frames on local fields of positive characteristic, using the Fourier transform with different machinery as that of used in [10].

The paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and also some results which are required in the subsequent section. A characterization of tight wavelet frames on local fields of positive characteristic is given in Section 3.

2. Preliminaries on local fields

Let $\mathbb{K}$ be both a field and a topological space. Then $\mathbb{K}$ is called a *local field* if both $\mathbb{K}^+$ and $\mathbb{K}^*$ are locally compact Abelian groups, where $\mathbb{K}^+$ and $\mathbb{K}^*$ denote the additive and multiplicative groups of $\mathbb{K}$, respectively. If $\mathbb{K}$ is any field and is endowed with the discrete topology, then $\mathbb{K}$ is a local field. Further, if $\mathbb{K}$ is connected, then $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{K}$ is not connected, then it is totally disconnected. Hence by a local field, we mean a field $\mathbb{K}$ which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. For more details we refer the monographs [14, 19]. In the rest of the paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let $\mathbb{K}$ be a fixed local field. Then there is an integer $q = \mathfrak{p}^r$, where $\mathfrak{p}$ is a fixed prime element of $\mathbb{K}$ and r is a positive integer, and a norm $|\cdot|$ on $\mathbb{K}$ such that for all $x \in \mathbb{K}$ we have $|x| \geq 0$ and for each $x \in \mathbb{K} \setminus \{0\}$ we can write $|x| = q^k$ for some integer k. This norm is non-Archimedean, that is, $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$ and $|x + y| = \max\{|x|, |y|\}$ whenever $|x| \neq |y|$. Let dx be the Haar measure on the locally compact topological group $(\mathbb{K}, +)$. This measure is normalized so that $\int_{\mathfrak{D}} dx = 1$, where $\mathfrak{D} = \{x \in \mathbb{K} : |x| \leq 1\}$ is the *ring of integers* in $\mathbb{K}$. Define $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in $\mathbb{K}$. The prime ideal in $\mathbb{K}$ is the unique maximal ideal in $\mathfrak{D}$, and hence as result $\mathfrak{B}$ is both principal and prime. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in \mathbb{K} : |x| = 1\}$. Then, it is easy to verify that $\mathfrak{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. Moreover, $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in \mathbb{K} : |x| < q^{-k}\}$ are compact subgroups of $\mathbb{K}^+$, and are known as the *fractional ideals* of $\mathbb{K}^+$ (see [14]). Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in \mathbb{K}$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$, implying that if

$y \in \mathcal{B}^k$, then $\chi_y(x) = \chi(yx)$ for $x \in \mathbf{K}$. Suppose that χ_u is any character on $\mathbf{K}^+$, then clearly the restriction $\chi_u|\mathcal{D}$ is also a character on $\mathcal{D}$. Therefore, if $\{u(n) : n \in \mathbf{N}_0\}$ is a complete list of distinct coset representatives of $\mathcal{D}$ in $\mathbf{K}^+$, then, as it was proved in [19], the set $\left\{\chi_{u(n)} : n \in \mathbf{N}_0\right\}$ of distinct characters on $\mathcal{D}$ is a complete orthonormal system on $\mathcal{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n\in\mathbf{N}_0}$. We have $\mathcal{D}/\mathcal{B} \cong GF(q) = \Gamma$, where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$ (see [19]). We choose a set $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, ..., \epsilon_{c-1}\} \subset \mathcal{D}^*$ such that span $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, ..., \epsilon_{c-1}\} \cong GF(q)$. For $n \in \mathbf{N}_0$ satisfying

$$0 \le n < q, \quad n = a_0 + a_1 p + ... + a_{c-1}p^{c-1}, \; 0 \le a_k < p \quad \text{and} \quad k = 0, 1, ..., c-1,$$

we define

$$u(n) = (a_0 + a_1\epsilon_1 + ... + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n = b_0 + b_1 q + ... + b_s q^s, n \ge 0, \; 0 \le b_k < q$, we set

$$u(n) = u(b_0) + \mathfrak{p}^{-1}u(b_1) + ... + \mathfrak{p}^{-s}u(b_s).$$

Then, it is easy to verify that for $\ell \in \mathbf{N}_0$

$$\{u(k) : k \in \mathbf{N}_0\} = \{-u(k) : k \in \mathbf{N}_0\} = \{u(k) + u(\ell) : k \in \mathbf{N}_0\},$$

and $u(n) = 0$ iff $n = 0$ (see [19]). Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$. Also, by Ω we denote the test function space on $\mathbf{K}$, that is, each function f in Ω is a finite linear combination of functions of the form $\mathbf{1}_k(x - h)$, $h \in \mathbf{K}$, $k \in \mathbf{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathcal{B}^k$. Then, it is clear that Ω is dense in $L^p(\mathbf{K})$, $1 \le p < \infty$, and each function in Ω is of compact support and so is its Fourier transform. The Fourier transform of a function $f \in L^1(\mathbf{K})$ is defined by

$$\hat{f}(\xi) = \int_{\mathbf{K}} f(x)\overline{\chi_\xi(x)}dx.$$

Note that

$$\hat{f}(\xi) = \int_{\mathbf{K}} f(x)\overline{\chi_\xi(x)}dx = \int_{\mathbf{K}} f(x)\chi(-\xi x)dx.$$

The properties of the Fourier transform on the local field $\mathbf{K}$ are quite similar to those of the Fourier analysis on the real line (see [14, 19]). In particular, if $f \in L^1(\mathbf{K}) \cap L^2(\mathbf{K})$, then $\hat{f} \in L^2(\mathbf{K})$ and $\|\hat{f}\|_2 = \|f\|_2$. For a given $\psi \in L^2(\mathbf{K})$, define the wavelet system

$$X(\Psi) = \{\psi_{j,k} : j \in \mathbf{Z}, k \in \mathbf{N}_0\}, \tag{2.2}$$

where $\psi_{j,k} = q^{j/2}\psi(\mathrm{p}^j. - u(k))$. The wavelet system (2.2) is called a *wavelet frame*, if there exist positive numbers $0 < A \le B < \infty$ such that for all $f \in L^2(\mathbf{K})$

$$A\|f\|^2 \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{j,k}\rangle|^2 \le B\|f\|^2. \tag{2.3}$$

The largest constant A and the smallest constant B satisfying (2.3) are called the *lower and upper wavelet frame bounds*, respectively. A wavelet frame is a *tight wavelet frame* if A and B are chosen so that $A = B$, and the wavelet frame is called a *Parseval's wavelet frame* if $A = B = 1$, that is, for all $f \in L^2(\mathbf{K})$

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{j,k}\rangle|^2 = \|f\|^2, \tag{2.4}$$

and in this case, every function $f \in L^2(\mathbf{K})$ can be written as

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} \langle f, \psi_{j,k}\rangle \psi_{j,k}(x).$$

Since Ω is dense in $L^2(\mathbf{K})$ and is closed under the Fourier transform, the set

$$\Omega^0 = \left\{ f \in \Omega : \operatorname{supp}\hat{f} \subset \mathbf{K}\backslash\{0\}\right\}$$

is also dense in $L^2(\mathbf{K})$. Therefore, it is enough to verify that the system $X(\Psi)$ given by (2.2) is a frame and tight frame for $L^2(\mathbf{K})$ if (2.3) and (2.4) hold for all $f \in \Omega^0$.

In order to prove the main result to be presented in next section, we need the following lemma whose proof can be found in [13].

Lemma 2.1. *Let $f \in \Omega^0$ and $\psi \in L^2(\mathbf{K})$. If ess sup$\{\sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathrm{p}^j\xi)|^2 : \xi \in \mathfrak{B}^{-1}\backslash\mathfrak{D}\} < \infty$, then*

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{j,k}\rangle|^2 = \int_{\mathbf{K}} \left|\hat{f}(\xi)\right|^2 \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(\mathrm{p}^j\xi)\right|^2 d\xi + R_\psi(f), \tag{2.5}$$

where

$$R_\psi(f) = \sum_{j\in\mathbb{Z}} \int_{\mathbf{K}} \overline{\hat{f}(\xi)}\hat{\psi}(\mathrm{p}^j\xi) \left[\sum_{l\in N} \hat{f}(\xi + \mathrm{p}^{-j}u(l))\, \overline{\hat{\psi}(\mathrm{p}^j\xi + u(l))}\right] d\xi$$

$$= \sum_{j\in\mathbb{Z}}\sum_{l\in N} \int_{\mathbf{K}} \overline{\hat{f}(\xi)}\hat{\psi}(\mathrm{p}^j\xi)\, \hat{f}(\xi + p^{-j}u(l))\, \overline{\hat{\psi}(\mathrm{p}^j\xi + u(l))} d\xi. \tag{2.6}$$

Furthermore, the iterated series in (2.6) is absolutely convergent.

Remark 2.1. The left hand side of (2.5) converges for all $f \in \Omega^0$ if and only if $\sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathrm{p}^j\xi)|^2$ is locally integrable in $\mathbf{K}\backslash \cup_{j\in\mathbb{Z}} E_j^c$, where E_j is the set of regular points of $\left|\psi(\mathrm{p}^j\xi)\right|^2$, which means that for each $x \in E_j$, we have

$$q^n \int_{\xi - x\in\mathfrak{B}^n} |\hat{\psi}(\mathrm{p}^j\xi)|^2 d\xi \to |\hat{\psi}(\mathrm{p}^j\xi)|^2 \quad \text{as } n \to \infty.$$

3. The main result

In this section, we establish our main result concerning the characterization of the wavelet system $X(\Psi)$ given by (2.2) to be a tight frame for $L^2(\mathbf{K})$.

Theorem 3.1. *The wavelet system $X(\Psi)$ given by (2.2) is a tight wavelet frame for $L^2(\mathbf{K})$ if and only if ψ satisfies*

$$\sum_{j\in\mathbf{Z}}\left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2 = 1 \quad \textit{for a.e.} \quad \xi\in\mathfrak{B}^{-1}\backslash\mathfrak{D} \tag{3.1}$$

and

$$\sum_{j\in\mathbf{N}_0}\hat{\psi}\left(\mathfrak{p}^{-j}\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^{-j}(\xi+u(m))\right)} = 0 \quad \textit{for a.e.} \quad \xi\in\mathfrak{B}^{-1}\backslash\mathfrak{D}, \quad m\in q\mathbf{N}_0+\bar{Q}, \tag{3.2}$$

where $q\mathbf{N}_0=\{qk:\ k=0,1,2,\dots\}$ and $\bar{Q}=\{1,2,\dots,q-1\}$.

Proof. Let

$$t_\psi(u(m),\xi)=\sum_{k\in\mathbf{N}_0}\hat{\psi}\left(\mathfrak{p}^{-k}\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^{-k}(\xi+u(m))\right)}.$$

Assume $f\in\Omega^0$, then for each $l\in\mathbf{N}$, there exists $k\in\mathbf{N}_0$ and a unique $m\in q\mathbf{N}_0+\bar{Q}$ such that $l=q^k m$. Thus, by virtue of (2.1) we have that $\{u(l)\}_{l\in\mathbf{N}}=\{\mathfrak{p}^{-k}u(m)\}_{(k,m)\in\mathbf{N}_0\times\{q\mathbf{N}_0+\bar{Q}\}}$. Since the series in (2.4) is absolutely convergent, we can estimate $R_\psi(f)$, defined by (2.6), as follows:

$$\begin{aligned}
R_\psi(f) &= \sum_{j\in\mathbf{Z}}\int_{\mathbf{K}}\overline{\hat{f}(\xi)}\hat{\psi}\left(\mathfrak{p}^j\xi\right)\left\{\sum_{l\in\mathbf{N}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(l)\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+u(l)\right)}\right\}d\xi\\
&=\sum_{j\in\mathbf{Z}}\int_{\mathbf{K}}\overline{\hat{f}(\xi)}\hat{\psi}\left(\mathfrak{p}^j\xi\right)\left\{\sum_{k\in\mathbf{N}_0}\sum_{m\in q\mathbf{N}_0+\bar{Q}}\hat{f}\left(\xi+\mathfrak{p}^{-j-k}u(m)\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+\mathfrak{p}^{-k}u(m)\right)}\right\}d\xi\\
&=\int_{\mathbf{K}}\overline{\hat{f}(\xi)}\left\{\sum_{k\in\mathbf{N}_0}\sum_{m\in q\mathbf{N}_0+\bar{Q}}\sum_{j\in\mathbf{Z}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right)\hat{\psi}\left(\mathfrak{p}^{j-k}\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^{j-k}\xi+\mathfrak{p}^{-k}u(m)\right)}\right\}d\xi\\
&=\int_{\mathbf{K}}\overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbf{Z}}\sum_{m\in q\mathbf{N}_0+\bar{Q}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right)\sum_{k\in\mathbf{N}_0}\hat{\psi}\left(\mathfrak{p}^{j-k}\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^{-k}\left(\mathfrak{p}^j\xi+u(m)\right)\right)}\right\}d\xi\\
&=\int_{\mathbf{K}}\overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbf{Z}}\sum_{m\in q\mathbf{N}_0+\bar{Q}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right)t_\psi\left(u(m),\mathfrak{p}^j\xi\right)\right\}d\xi.
\end{aligned}$$

Let us collect the results we have obtained: if $\psi\in L^2(\mathbf{K})$ and $f\in\Omega^0$, then

$$\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{N}_0}|\langle f,\psi_{j,k}\rangle|^2=\int_{\mathbf{K}}|\hat{f}(\xi)|^2\sum_{j\in\mathbf{Z}}|\hat{\psi}\left(\mathfrak{p}^j\xi\right)|^2d\xi$$

$$+\int_{\mathbb{K}}\overline{\hat{f}(\xi)}\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\bar{Q}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right)t_{\psi}(u(m),\mathfrak{p}^{j}\xi)d\xi. \tag{3.3}$$

The last integrand is integrable, and so is the first when $\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\mathfrak{p}^{j}\xi\right)\right|^{2}$ is locally integrable in $\mathbb{K}\setminus\cup_{j\in\mathbb{Z}}E_j^c$. Further, equation (3.2) implies that

$$t_{\psi}(u(m),\xi)=0 \text{ for all } m\in q\mathbb{N}_0+\bar{Q}.$$

Combining all together with (3.1) and (3.2) we get

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\langle f,\psi_{j,k}\rangle\right|^2=\|f\|_2^2,\quad \forall\, f\in\Omega^0.$$

Since Ω^0 is dense in $L^2(\mathbb{K})$, we conclude that the wavelet system $X(\Psi)$ given by (2.2) is a tight frame for $L^2(\mathbb{K})$. Conversely, suppose that the system $X(\Psi)$ given by (2.2) is a tight wavelet frame for $L^2(\mathbb{K})$, then we need to show that both equations (3.1) and (3.2) are satisfied. Since $\{\psi_{j,k}(x): j\in\mathbb{Z}, k\in\mathbb{N}_0\}$ is a tight wavelet frame for $L^2(\mathbb{K})$, then for all $f\in\Omega^0$ we have

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\langle f,\psi_{j,k}\rangle\right|^2=\|f\|_2^2. \tag{3.4}$$

By remark 2.1, $\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\mathfrak{p}^{j}\xi\right)\right|^{2}$ is locally integrable in $\mathbb{K}\setminus\cup_{j\in\mathbb{Z}}E_j^c$. Therefore, for each $\xi_0\in\mathbb{K}\setminus\cup_{j\in\mathbb{Z}}E_j^c$, we consider

$$\hat{f}_1(\xi)=q^{\frac{M}{2}}\Phi_M(\xi-\xi_0),$$

where $f=f_1$ and $\Phi_M(\xi-\xi_0)$ is the characteristic function of $\xi_0+\mathfrak{B}^M$. Then, it follows that $\overline{\hat{f}(\xi)}\hat{f}(\xi+\mathfrak{p}^{-j}u(l))\equiv 0$ for $l\in\mathbb{N}$, since ξ and $\xi+\mathfrak{p}^{-j}u(l)$ can not be in $\xi_0+\mathfrak{B}^M$ simultaneously, and hence $\|f_1\|_2^2=1$. Furthermore, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\langle f,\psi_{j,k}\rangle\right|^2&=\|f_1\|_2^2=\|\hat{f}_1\|_2^2=1\\&=\int_{\xi_0+\mathfrak{B}^M}\sum_{j\in\mathbb{Z}}q^M\left|\hat{\psi}\left(\mathfrak{p}^{j}\xi\right)\right|^2d\xi+R_{\psi}(f_1).\end{aligned}$$

By letting $M\to\infty$, we obtain

$$1=\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\mathfrak{p}^{j}\xi_0\right)\right|^2+\lim_{M\to\infty}R_{\psi}(f_1). \tag{3.5}$$

Now, we estimate $R_{\psi}(f_1)$ as follows:

$$R_{\psi}(f_1)=\sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\overline{\hat{f}_1(\xi)}\hat{\psi}\left(\mathfrak{p}^{j}\xi\right)\left\{\sum_{l\in\mathbb{N}}\hat{f}_1\left(\xi+\mathfrak{p}^{-j}u(l)\right)\overline{\hat{\psi}\left(\mathfrak{p}^{j}\xi+u(l)\right)}\right\}d\xi$$

$$|R_\psi(f_1)| \le \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \int_{\mathbb{K}} \left|\hat{f}_1(\xi)\hat{\psi}\left(\mathfrak{p}^j\xi\right)\hat{f}_1\left(\xi+\mathfrak{p}^{-j}u(l)\right)\hat{\psi}\left(\mathfrak{p}^j\xi+u(l)\right)\right| d\xi$$

$$= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} q^j \int_{\mathbb{K}} \left|\hat{f}_1(\mathfrak{p}^{-j}\xi)\hat{f}_1\left(\mathfrak{p}^{-j}(\xi+u(l))\right)\hat{\psi}(\xi)\hat{\psi}(\xi+u(l))\right| d\xi.$$

Note that

$$\left|\hat{\psi}(\xi)\hat{\psi}(\xi+u(l))\right| \le \frac{1}{2}\left(\left|\hat{\psi}(\xi)\right|^2 + \left|\hat{\psi}(\xi+u(l))\right|^2\right).$$

Therefore, we have

$$|R_\psi(f_1)| \le \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} q^j \int_{\mathbb{K}} |\hat{f}_1(\mathfrak{p}^{-j}\xi)\hat{f}_1\left(\mathfrak{p}^{-j}(\xi+u(l))\right)| \, |\hat{\psi}(\xi)|^2 d\xi. \tag{3.6}$$

Since $u(l) \ne 0$ $(l \in \mathbb{N})$ and $f_1 \in \Omega^0$, there exists a constant $J > 0$ such that

$$\hat{f}_1(\mathfrak{p}^{-j}t)\hat{f}_1\left(\mathfrak{p}^{-j}t+\mathfrak{p}^{-j}u(l)\right) = 0, \quad \forall\, |j| > J.$$

On the other hand, for each $|j| \le J$, there exists a constant L such that

$$\hat{f}_1\left(\mathfrak{p}^{-j}t+\mathfrak{p}^{-j}u(l)\right) = 0, \quad \forall\, l > L.$$

This means that only a finite number of terms of the series on the right hand side of (3.6) are non-zero. Consequently, there exits a constant C such that

$$|R_\psi(f_1)| \le C\|\hat{f}_1\|_\infty^2\|\hat{\psi}\|_2^2 = Cq^m\|\hat{\psi}\|_2^2,$$

implying

$$\lim_{M\to\infty} |R_\psi(f_1)| = 0.$$

Hence equation (3.5) becomes

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\mathfrak{p}^j\xi_0\right)\right|^2 = 1.$$

Finally, we must show that if (3.4) holds for all $f \in \Omega^0$, then equation (3.2) is true. From equalities (3.3), (3.4) and just established equality (3.1), for all $f \in \Omega^0$ we have

$$\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\bar{Q}} \int_{\mathbb{K}} \overline{\hat{f}(\xi)}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right) t_\psi(u(m),\mathfrak{p}^j\xi)d\xi = 0.$$

Also, by polarization, for all $f, g \in \Omega^0$ we have

$$\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\bar{Q}} \int_{\mathbb{K}} \overline{\hat{f}(\xi)}\hat{g}\left(\xi+\mathfrak{p}^{-j}u(m)\right) t_\psi(u(m),\mathfrak{p}^j\xi)\, d\xi = 0. \tag{3.7}$$

Let us fix $m_0 \in q\mathbb{N}_0+\bar{Q}$ and $\xi_0 \in \mathbb{K}\setminus\cup_{j\in\mathbb{Z}}E_j^c$ such that neither $\xi_0 \ne 0$ nor $\xi_0+u(m_0) \ne 0$. Setting $f = f_1$ and $g = g_1$ such that

$$\hat{f}_1(\xi) = q^{\frac{M}{2}}\Phi_M(\xi-\xi_0) \quad \text{and} \quad \hat{g}_1(\xi) = \hat{f}_1(\xi-u(m_0)),$$

we obtain $\hat{f}_1(\xi)\hat{g}_1(\xi+u(m_0)) = q^M \Phi_M(\xi-\xi_0)$. Now, equality (3.7) can be written as

$$0 = q^M \int_{\xi_0+\mathfrak{B}^M} t_\psi(u(m_0),\xi)d\xi + J_1, \tag{3.8}$$

where

$$J_1 = \sum_{\substack{j\in\mathbb{Z} \\ (j,m)\neq(0,m_0)}} \sum_{m\in q\mathbb{N}_0+\bar{Q}} \int_{\mathbb{K}} \overline{\hat{f}_1(\xi)}\hat{g}_1\left(\xi+\mathfrak{p}^{-j}u(m)\right) t_\psi(u(m),\mathfrak{p}^j\xi)d\xi.$$

Since the first summand in (3.8) tends to $t_\psi(u(m_0),\xi_0)$ as $M\to\infty$, we have to prove that

$$\lim_{M\to\infty} J_1 = 0.$$

Since $u(m)\neq 0,\ (m\in\mathbb{N})$ and $f_1, g_1\in\Omega^0$, there exists a constant $J_0>0$ such that

$$\overline{\hat{f}_1(\xi)}\,\hat{g}_1\left(\xi+\mathfrak{p}^{-j}u(m)\right) = 0 \quad \forall\, j > J_0.$$

Therefore, we have

$$J_1 = \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\bar{Q}} \int_{\mathbb{K}} \overline{\hat{f}_1(\xi)}\hat{g}_1\left(\xi+\mathfrak{p}^{-j}u(m)\right) t_\psi(u(m),\mathfrak{p}^j\xi)d\xi$$

$$|J_1| \le \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\bar{Q}} q^j \int_{\mathbb{K}} \left|\overline{\hat{f}_1(\mathfrak{p}^{-j}\xi)}\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right)\right| |t_\psi(u(m),\xi)|\,d\xi.$$

Since

$$2\,|t_\psi(u(m),\xi)| \le \sum_{k\in\mathbb{N}_0} |\hat{\psi}\left(\mathfrak{p}^{-k}\xi\right)|^2 + \sum_{k\in\mathbb{N}_0} |\hat{\psi}\left(\mathfrak{p}^{-k}(\xi+u(m))\right)|^2,$$

we have

$$|J_1| \le J_1^{(1)} + J_1^{(2)},$$

where

$$J_1^{(1)} = \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\bar{Q}} q^j \int_{\mathbb{K}} \left|\hat{f}_1(\mathfrak{p}^{-j}\xi)\right| \left|\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right)\right| [\tau(\xi)]^2 d\xi$$

with

$$\int_{\mathbb{K}} [\tau(\xi)]^2 d\xi = \frac{1}{2}\sum_{k\in\mathbb{N}_0} \int_{\mathbb{K}} \left|\hat{\psi}\left(\mathfrak{p}^{-k}\xi\right)\right|^2 d\xi = \|\hat{\psi}\|_2^2 < \infty,$$

and

$$J_1^{(2)} = \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\bar{Q}} q^j \int_{\mathbb{K}} \left|\hat{f}_1(\mathfrak{p}^{-j}\xi)\right| \left|\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right)\right| [\tau(\xi+u(m))]^2 d\xi$$

$$= \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\bar{Q}} q^j \int_{\mathbb{K}} \left|\hat{f}_1(\mathfrak{p}^{-j}(\eta-u(m)))\right| \left|\hat{g}_1\left(\mathfrak{p}^{-j}\eta\right)\right| [\tau(\eta)]^2 d\xi.$$

Thus $J_1^{(2)}$ has the same form as $J_1^{(1)}$ with the roles of $\hat{f}_1$ and $\hat{g}_1$ interchanged. Next, since $\hat{f}_1(\xi) = q^{\frac{M}{2}}\Phi_M(\xi - \xi_0)$, we can write

$$J_1^{(1)} = \sum_{j \le J_0} \sum_{m \in q\mathbb{N}_0 + \bar{Q}} q^j q^{\frac{M}{2}} \int_{\mathrm{p}^{-j}\xi_0 + \mathcal{B}^{-j+M}} \left|\hat{g}_1\left(\mathrm{p}^{-j}(\xi + u(m))\right)\right| [\tau(\xi)]^2 d\xi.$$

Now, if $\hat{g}_1\left(\mathrm{p}^{-j}(\xi + u(m))\right) \neq 0$, then we must have $\mathrm{p}^{-j}\xi + \mathrm{p}^{-j}u(m) \in \xi_0 + \mathcal{B}^M + u(m_0)$ and $|\mathrm{p}^{-j}u(m)| \le q^{-M}$, and hence $|u(m)| \le q^{-M-j}$. Thus, we have

(3.9)

$$J_1^{(1)} = \sum_{j \le J_0} q^j q^{\frac{M}{2}} \int_{\mathrm{p}^{-j}\xi_0 + \mathcal{B}^{-j+M}} [\tau(\xi)]^2 \sum_{m \in q\mathbb{N}_0 + \bar{Q}} \left|\hat{g}_1\left(\mathrm{p}^{-j}(\xi + u(m))\right)\right| d\xi$$

$$\le \sum_{j \le J_0} q^j q^{\frac{M}{2}} \int_{\mathrm{p}^{-j}\xi_0 + \mathcal{B}^{-j+M}} [\tau(\xi)]^2 q^{-M-j} q^{\frac{M}{2}} d\xi = \sum_{j \le J_0} \int_{\mathrm{p}^{-j}\xi_0 + \mathcal{B}^{-j+M}} [\tau(\xi)]^2 d\xi .$$

For given $\xi_0 \neq 0$, we choose $q^{J_0} < |\xi_0| = q^{-M}$ to obtain

$$\mathrm{p}^{-j}\xi_0 + \mathcal{B}^{-j+M} \subset \mathcal{B}^{-J_0+M} \quad \forall j \le J_0, \tag{3.10}$$

as $|\mathrm{p}^{-j}\xi_0| = q^j q^{-M} \le q^{J_0} q^{-M}$ and $\mathcal{B}^{-j+M} \subset \mathcal{B}^{-J_0+M}$. On the other hand, for any $j_1 < j_2 \le J_0$, we claim that

$$\{\mathrm{p}^{-j_1}\xi_0 + \mathcal{B}^{-j_1+M}\} \cap \{\mathrm{p}^{-j_2}\xi_0 + \mathcal{B}^{-j_2+M}\} = \emptyset. \tag{3.11}$$

Indeed, for any $x \in \mathrm{p}^{-j_1}\xi_0 + \mathcal{B}^{-j_1+M}$ and $y \in \mathrm{p}^{-j_2}\xi_0 + \mathcal{B}^{-j_2+M}$, write $x = \mathrm{p}^{-j_1}\xi_0 + x_1$ and $y = \mathrm{p}^{-j_2}\xi_0 + y_1$, then we have $|x - y| = \max\{|\mathrm{p}^{-j_1}\xi_0 - \mathrm{p}^{-j_2}\xi_0|,\ |x_1 - y_1|\} = q^{j_2 - M} \neq 0$, implying that (3.11) holds. Combining (3.9) - (3.11), we obtain

$$J_1^{(1)} \le \int_{\mathcal{B}^{-J_0+M}} [\tau(\xi)]^2 d\xi \to 0 \quad \text{as} \quad M \to \infty.$$

This completes the proof of Theorem 3.1.

Example 3.1. Consider the functions

$$\psi_1(x) = \begin{cases} 1 & \text{if } x \in \mathcal{D}, \\ 0 & \text{if } x \notin \mathcal{D}, \end{cases} \quad \text{and} \quad \psi_2(x) = \begin{cases} q^{-1} & \text{if } x \in \mathcal{B}^{-1}, \\ 0 & \text{if } x \notin \mathcal{B}^{-1}, \end{cases}$$

and define $\psi(x) = \psi_1(x) - \psi_2(x)$. Since $\hat{\psi}_1(\xi) = \psi_1(\xi)$ and

$$\hat{\psi}_2(\xi) = \begin{cases} 1 & x \in \mathcal{B}, \\ 0 & x \notin \mathcal{B}, \end{cases}$$

we have

$$\hat{\psi}(\xi) = \begin{cases} 1 & x \in \mathcal{B}^{-1} \backslash \mathcal{D}, \\ 0 & \text{otherwise}. \end{cases}$$

Next, for $\xi \neq 0$, we see that $\sum_{j \in \mathbb{Z}} \left|\hat{\psi}\left(\mathfrak{p}^j \xi\right)\right|^2 = 1$, and since $\mathfrak{p}^{-j}\xi$ and $\mathfrak{p}^{-j}(\xi + u(m))$ can not be in $\mathfrak{B}^{-1} \backslash \mathfrak{D}$ simultaneously, we conclude that

$$\sum_{j=0}^{\infty} \hat{\psi}\left(\mathfrak{p}^{-j}\xi\right) \overline{\hat{\psi}\left(\mathfrak{p}^{-j}(\xi + u(m))\right)} = 0.$$

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EXISTENCE AND NONEXISTENCE RESULTS FOR A $2n$-TH ORDER p-LAPLACIAN DISCRETE DIRICHLET BOUNDARY VALUE PROBLEM

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Abstract. [1] In this paper $2n$-th order p-Laplacian difference equations are considered. Using the critical point method, we establish various sufficient conditions for the existence and nonexistence of solutions for Dirichlet boundary value problem. Recent results in the literature are generalized and significantly complemented, as well as, some new results are obtained.

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1. Introduction

Throughout the paper the letters $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural, integer and real numbers, respectively. The letter k stands for a positive integer. For any a, $b \in \mathbf{Z}$ $(a < b)$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$ and $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$. Also, the symbol $*$ denotes the transpose of a vector.

Recently, the difference equations have widely occurred as the mathematical models describing real life situations in many fields, such as: probability theory, matrix theory, electrical circuit analysis, combinatorial analysis, queuing theory, number theory, psychology and sociology, etc.

For the general background of difference equations, one can refer the monograph [1, 8]. Since the last decade, there has been much progress on the study of qualitative

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properties of difference equations, which includes results on stability, attractivity, oscillation and other topics (see, [6, 11, 12, 16], and reference therein).

In this paper we consider the following $2n$-th order p-Laplacian difference equation

$$\Delta^n \left(\gamma_{i-n+1}\varphi_p\left(\Delta^n u_{i-1}\right)\right) = (-1)^n f(i, u_{i+1}, u_i, u_{i-1}), \ n \in \mathbf{Z}(1), \ i \in \mathbf{Z}(1, k), \tag{1.1}$$

with boundary value conditions:

$$u_{1-n} = u_{2-n} = \cdots = u_0 = 0, \ u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0, \tag{1.2}$$

where Δ is the forward difference operator: $\Delta u_i = u_{i+1} - u_i$, $\Delta^n u_i = \Delta^{n-1}(\Delta u_i)$, γ_i is nonzero and real-valued for each $i \in \mathbf{Z}(2-n, k+1)$, $\varphi_p(s)$ is the p-Laplacian operator: $\varphi_p(s) = |s|^{p-2}s (1 < p < \infty)$, and $f \in C(\mathbf{R}^4, \mathbf{R})$.

We may think of (1.1) as a discrete analogue of the following $2n$-th order p-Laplacian functional differential equation

$$\frac{d^n}{dt^n}\left[\gamma(t)\varphi_p\left(\frac{d^n u(t)}{dt^n}\right)\right] = (-1)^n f(t, u(t+1), u(t), u(t-1)), \ t \in [a, b], \tag{1.3}$$

with boundary value conditions:

$$u(a) = u'(a) = \cdots = u^{(n-1)}(a) = 0, \ u(b) = u'(b) = \cdots = u^{(n-1)}(b) = 0. \tag{1.4}$$

Note that equations of type (1.3) arise in the study of solitary waves [15], in lattice differential equations and periodic solutions [9], and in the study of functional differential equations.

In recent years, the boundary value problems for differential equations were in the focus of a number of researchers. By using various methods and techniques, such as the Schauder fixed point theory, the topological degree theory, the coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained (see, [4, 9]). Another important and powerful tool that was used to deal with problems on differential equations is critical point theory (see [7, 13]). However, only since 2003, the critical point theory has been employed to establish sufficient conditions for the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [10] and Shi et al. [14] have found sufficient conditions for the existence of periodic solutions of second-order nonlinear difference equations. We also refer to [16] for the discrete boundary value problems.

Compared to the first- or second-order difference equations, the study of higher-order equations, and in particular, $2n$-th order equations, has received considerably less attention (see, [2, 3, 5] and references therein).

The authors [2] studied the following $2n$-th order difference equation:

$$\sum_{j=0}^{n} \Delta^j \left(\gamma_j(i-j)\Delta^j u(i-j)\right) = 0 \tag{1.5}$$

in the context of the discrete calculus of variations, and Peil and Peterson [12] studied the asymptotic behavior of solutions of (1.5) with $\gamma_j(i) \equiv 0$ for $1 \leq j \leq n-1$. In 1998, Anderson [3] considered (1.5) for $i \in \mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (n,n)-disconjugacy for (1.5). In 2004, Migda [11] studied an m-th order linear difference equation. In 2007, Cai and Yu [5] have obtained some criteria for the existence of periodic solutions of the following $2n$-th order difference equation:

$$\Delta^n \left(\gamma_{i-n}\Delta^n u_{i-n}\right) + f(i,u_i) = 0,\ n \in \mathbf{Z}(3),\ i \in \mathbf{Z}, \tag{1.6}$$

in the case where f grows superlinearly both at 0 and at ∞. In 2007, Chen and Fang [6], using the critical point theory, have obtained a sufficient condition for the existence of periodic and subharmonic solutions of the following p-Laplacian difference equation:

$$\Delta\left(\varphi_p\left(\Delta u_{i-1}\right)\right) + f(i,u_{i+1},u_i,u_{i-1}) = 0,\ i \in \mathbf{Z}. \tag{1.7}$$

The study of boundary value problems (BVP) to determine the existence of solutions of difference equations has been a very active research area in the last twenty years. For the surveys of recent results in this area, we refer the reader to the monographs [1, 8]. However, to the best of our knowledge, results on solutions to boundary value problems of higher-order nonlinear difference equations are very scarce in the literature. Furthermore, since the equation (1.1) contains both advance and retardation, not surprisingly, there are only few papers dealing with this subject.

Motivated by the above results, we use the critical point theory to give some sufficient conditions for the existence and nonexistence of solutions for the BVP (1.1), (1.2). We study both the superlinear and sublinear cases. The main idea used in this paper is to transfer the existence of solutions of the BVP (1.1), (1.2) into the existence of the critical points of some functional. The proofs are based on the celebrated Mountain Pass Lemma in combination with variational technique. The purpose of this paper is two-folded. On one hand, we further demonstrate the powerfulness of critical point theory in the study of solutions for boundary value problems of difference equations. On the other hand, we complement the existing results. The motivation for the present work stems from the recent papers [7, 9].

For the basic concepts of variational methods, we refer the reader to monographs [8, 13]. Let

$$\overline{\gamma} = \max\{\gamma_i : i \in \mathbf{Z}(2-n, k+1)\}, \quad \underline{\gamma} = \min\{\gamma_i : i \in \mathbf{Z}(2-n, k+1)\}.$$

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*

(γ) $\gamma_i < 0$ *for any* $i \in Z(2-n, k+1)$;

(F_1) *there exists a functional* $F(i,\cdot) \in C^1(Z \times R^2, R)$ *with* $F(0,\cdot) = 0$, *such that*

$$\frac{\partial F(i-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(i, v_1, v_2)}{\partial v_2} = f(i, v_1, v_2, v_3), \ \forall i \in Z(1,k);$$

(F_2) *there exists a constant* $M_0 > 0$, *such that for all* $(i, v_1, v_2) \in Z(1,k) \times R^2$

$$\left|\frac{\partial F(i, v_1, v_2)}{\partial v_1}\right| \le M_0, \ \left|\frac{\partial F(i, v_1, v_2)}{\partial v_2}\right| \le M_0.$$

Then the BVP (1.1), (1.2) possesses at least one solution.

Remark 1.1. Assumption (F_2) implies that there exists a constant $M_1 > 0$, such that

(F_2') $|F(i, v_1, v_2)| \le M_1 + M_0(|v_1| + |v_2|)$, $\forall (i, v_1, v_2) \in \mathbf{Z}(1,k) \times \mathbf{R}^2$.

Theorem 1.2. *Suppose that* (F_1) *and the following hypotheses are satisfied:*

(γ') $\gamma_i > 0$ *for any* $i \in Z(2-n, k+1)$;

(F_3) *there exists a functional* $F(i,\cdot) \in C^1(Z \times R^2, R)$, *such that*

$$\lim_{r\to 0} \frac{F(i, v_1, v_2)}{r^p} = 0, \ r = \sqrt{v_1^2 + v_2^2}, \ \forall i \in Z(1,k);$$

(F_4) *there exists a constant* $\beta > p$, *such that for any* $i \in Z(1,k)$

$$0 < \frac{\partial F(i, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(i, v_1, v_2)}{\partial v_2} v_2 < \beta F(i, v_1, v_2), \ \forall (v_1, v_2) \ne 0.$$

Then the BVP (1.1), (1.2) possesses at least two nontrivial solutions.

Remark 1.2. Assumption (F_4) implies that there exist constants $a_1 > 0$ and $a_2 > 0$, such that

(F_4') $F(i, v_1, v_2) > a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^{\beta} - a_2$, $\forall i \in \mathbf{Z}(1,k)$.

Theorem 1.3. *Suppose that* (γ'), (F_1) *and the following assumption are satisfied:*

(F_5) *there exist constants* $R > 0$ *and* $1 < \alpha < 2$, *such that for* $i \in Z(1,k)$ *and* $\sqrt{v_1^2 + v_2^2} \ge R$,

$$0 < \frac{\partial F(i, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(i, v_1, v_2)}{\partial v_2} v_2 \le \frac{\alpha}{2} p F(i, v_1, v_2).$$

Then the BVP (1.1), (1.2) possesses at least one solution.

Remark 1.3. Assumption (F_5) implies that for each $i \in \mathbf{Z}(1,k)$ there exist constants $a_3 > 0$ and $a_4 > 0$, such that

(F_5') $\quad F(i, v_1, v_2) \leq a_3 \left(v_1^2 + v_2^2\right)^{\frac{q}{2}p} + a_4, \ \forall (i, v_1, v_2) \in \mathbf{Z}(1,k) \times \mathbf{R}^2.$

Theorem 1.4. *Suppose that* (γ), (F_1) *and the following assumption are satisfied:*

(F_6) $v_2 f(i, v_1, v_2, v_3) > 0$, *for* $v_2 \neq 0$, $\forall i \in Z(1,k)$.

Then the BVP (1.1), (1.2) has no nontrivial solutions.

Remark 1.4. As it was mentioned above, results on the nonexistence of solutions of problem (1.1), (1.2) are very scarce. Hence, Theorem 1.4 complements the existing results.

The rest of the paper is organized as follows. In Section 2 we establish the variational framework for the BVP (1.1), (1.2), and transfer the problem of existence of solutions of BVP (1.1), (1.2) into that of the existence of critical points of the corresponding functional. Some related fundamental results are also recalled. Finally, in Section 3 we prove our main results, by using the critical point method.

2. Variational structure and some lemmas

In order to apply the critical point theory, we first establish the corresponding variational framework for the BVP (1.1) with (1.2), and state some lemmas, which are used in the proofs of our main results. We start with some basic notation.

Let $\mathbf{R}^k$ be the real Euclidean space of dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v\rangle = \sum_{j=1}^{k} u_j v_j, \quad u, v \in \mathbf{R}^k, \tag{2.1}$$

which induced the norm $\|\cdot\|$:

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2\right)^{\frac{1}{2}}, \ u \in \mathbf{R}^k. \tag{2.2}$$

On the other hand, for all $u \in \mathbf{R}^k$ and $s > 1$, we define the norm $\|u\|_s$ on $\mathbf{R}^k$ as follows:

$$\|u\|_s = \left(\sum_{j=1}^{k} |u_j|^s\right)^{\frac{1}{s}}. \tag{2.3}$$

Since the norms $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants c_1, c_2 ($c_2 \geq c_1 > 0$), such that

$$c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2, \quad u \in \mathbf{R}^k. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For the BVP (1.1), (1.2), consider the functional J defined on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{p}\sum_{i=1-n}^{k} \gamma_{i+1}\,|\Delta^n u_i|^p - \sum_{i=1}^{k} F(i, u_{i+1}, u_i), \quad u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k, \tag{2.5}$$

where $u_{1-n} = u_{2-n} = \cdots = u_0 = 0$, $u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0$ and

$$\frac{\partial F(i-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(i, v_1, v_2)}{\partial v_2} = f(i, v_1, v_2, v_3).$$

It is easy to see that $J \in C^1(\mathbf{R}^k, \mathbf{R})$, and for any $u = \{u_i\}_{i=1}^{k} = (u_1, u_2, \ldots, u_k)^*$, by using $u_{1-n} = u_{2-n} = \cdots = u_0 = 0$, $u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0$, and

$$\Delta^n u_i = \sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{i+n-j},$$

we can compute the partial derivatives of J by formula:

$$\frac{\partial J}{\partial u_i} = (-1)^n \Delta^n\left(\gamma_{i-n+1}\varphi_p(\Delta^n u_{i-1})\right) - f(i, u_{i+1}, u_i, u_{i-1}), \ \forall i \in \mathbf{Z}(1, k).$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$\Delta^n\left(\gamma_{i-n+1}\varphi_p(\Delta^n u_{i-1})\right) = (-1)^n f(i, u_{i+1}, u_i, u_{i-1}), \ \ i \in \mathbf{Z}(1, k),$$

and so, we can reduce the existence of solutions of the BVP (1.1), (1.2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1.1), (1.2).

Let D be the $(k+n) \times (k+n)$ matrix defined by

$$D = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Clearly, D is positive definite. Let $\lambda_{1-n}, \lambda_{2-n}, \cdots, \lambda_k$ be the eigenvalues of D. Applying matrix theory, we see that $\lambda_j > 0$, $j = 1-n, 2-n, \cdots, k$, and, without loss of generality, we can assume that

$$0 < \lambda_{1-n} \leq \lambda_{2-n} \leq \cdots \leq \lambda_k. \tag{2.6}$$

Let E be a real Banach space. We assume that $J \in C^1(E, \mathbf{R})$, that is, J is a continuously Fréchet-differentiable functional defined on E. The functional J is said to satisfy the Palais-Smale condition, (PS)-condition, for short, if any sequence $\{u^{(l)}\} \subset E$ for which $\{J(u^{(l)})\}$ is bounded and $J'(u^{(l)}) \to 0$ as $l \to \infty$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E of radius ρ centered at 0, and let ∂B_ρ denote its boundary.

Lemma 2.1 *(Mountain Pass Lemma [13]). Let E be a real Banach space and let $J \in C^1(E, R)$ satisfy the (PS)-condition. If $J(0) = 0$ and*
(J_1) there exist constants ρ, $a > 0$ such that $J|_{\partial B_\rho} \geq a$,
(J_2) there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.
Then J possesses a critical value $c \geq a$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} J(g(s)), \tag{2.7}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}. \tag{2.8}$$

Lemma 2.2. *Suppose that the conditions (γ'), (F_1), (F_3) and (F_4) are satisfied. Then the functional J satisfies the (PS)-condition.*

Proof. Let $u^{(l)} \in \mathbf{R}^k$ and $l \in \mathbf{Z}(1)$ be such that $\{J(u^{(l)})\}$ is bounded. Then there exists a positive constant M_2, such that

$$-M_2 \leq J\left(u^{(l)}\right) \leq M_2,\ \forall l \in \mathbf{N}.$$

By (F_4'), we can write

$$-M_2 \leq J\left(u^{(l)}\right) = \frac{1}{p}\sum_{i=1-n}^{k} \gamma_{i+1}\left|\Delta^n u_i^{(l)}\right|^p - \sum_{i=1}^{k} F\left(i, u_{i+1}^{(l)}, u_i^{(l)}\right)$$

$$\leq \frac{\bar\gamma}{p}c_2^p\left[\sum_{i=1-n}^{k}\left(\Delta^{n-1}u_{i+1}^{(l)} - \Delta^{n-1}u_i^{(l)}\right)^2\right]^{\frac{p}{2}} - a_1\sum_{i=1}^{k}\left[\sqrt{\left(u_{i+1}^{(l)}\right)^2 + \left(u_i^{(l)}\right)^2}\right]^\beta + a_2 k$$

$$\leq \frac{\bar\gamma}{p}c_2^p\left[\left(x^{(l)}\right)^* Dx^{(l)}\right]^{\frac{p}{2}} - a_1 c_1^\beta\left\|u^{(l)}\right\|^\beta + a_2 k$$

$$\leq \frac{\bar\gamma}{p}c_2^p\lambda_k^{\frac{p}{2}}\left\|x^{(l)}\right\|^p - a_1 c_1^\beta\left\|u^{(l)}\right\|^\beta + a_2 k,$$

where $x^{(l)} = \left(\Delta^{n-1}u_{1-n}^{(l)}, \Delta^{n-1}u_{2-n}^{(l)}, \cdots, \Delta^{n-1}u_k^{(l)}\right)^*$. Taking into account that

$$\left\|x^{(l)}\right\|^p = \left[\sum_{i=1-n}^{k}\left(\Delta^{n-2}u_{i+1}^{(l)} - \Delta^{n-2}u_i^{(l)}\right)^2\right]^{\frac{p}{2}} \leq \left[\lambda_k\sum_{i=1-n}^{k}\left(\Delta^{n-2}u_i^{(l)}\right)^2\right]^{\frac{p}{2}} \leq \lambda_k^{\frac{(n-1)p}{2}}\left\|u^{(l)}\right\|^p,$$

we obtain

$$J\left(u^{(l)}\right) \le \frac{\bar{\gamma}}{p} c_2^p \lambda_k^{\frac{np}{2}} \left\|u^{(l)}\right\|^p - a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta + a_2 k.$$

That is,

$$a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta - \frac{\bar{\gamma}}{p} c_2^p \lambda_k^{\frac{np}{2}} \left\|u^{(l)}\right\|^p \le M_2 + a_2 k.$$

Since $\beta > p$, there exists a constant $M_3 > 0$ to satisfy $\|u^{(l)}\| \le M_3$, $l \in \mathbf{N}$. Therefore, $\{u^{(l)}\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\{u^{(l)}\}$ possesses a convergence subsequence in $\mathbf{R}^k$, implying the (PS)-condition. Lemma 2.2 is proved. □

3. Proofs of the main results

In this Section, we prove our main results by using the critical point theory.

Proof of Theorem 1.1. By (F_2'), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned} J(u) &= \frac{1}{p} \sum_{i=1-n}^{k} \gamma_{i+1} |\Delta^n u_i|^p - \sum_{i=1}^{k} F(i, u_{i+1}, u_i) \\ &\le \frac{\bar{\gamma}}{p} c_1^p \left[\sum_{i=1-n}^{k} \left(\Delta^{n-1} u_{i+1} - \Delta^{n-1} u_i\right)^2 \right]^{\frac{p}{2}} + M_0 \sum_{i=1}^{k} (|u_{i+1}| + |u_i|) + M_1 k \\ &\le \frac{\bar{\gamma}}{p} c_1^p (x^* D x)^{\frac{p}{2}} + 2M_0 \sum_{i=1}^{k} |u_i| + M_1 k \le \frac{\bar{\gamma}}{p} c_1^p \lambda_{1-n}^{\frac{p}{2}} \|x\|^p + 2M_0 \|u\| + M_1 k, \end{aligned}$$

where $x = (\Delta^{n-1} u_{1-n}, \Delta^{n-1} u_{2-n}, \cdots, \Delta^{n-1} u_k)^*$. Since

$$\|x\|^p = \left[\sum_{i=1-n}^{k} \left(\Delta^{n-2} u_{i+1} - \Delta^{n-2} u_i\right)^2 \right]^{\frac{p}{2}} \ge \left[\lambda_{1-n} \sum_{i=1-n}^{k} \left(\Delta^{n-2} u_i\right)^2 \right]^{\frac{p}{2}} \ge \lambda_{1-n}^{\frac{(n-1)p}{2}} \|u\|^p,$$

we have

$$J(u) \le \frac{\bar{\gamma}}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p + 2M_0 \sqrt{k} \|u\| + M_1 k \to -\infty \text{ as } \|u\| \to +\infty.$$

The above inequality means that $-J(u)$ is coercive. By the continuity of $J(u)$, J attains its maximum at some point, which we denote by $\check{u}$, that is,

$$J(\check{u}) = \max\left\{ J(u) | u \in \mathbf{R}^k \right\}.$$

Clearly, $\check{u}$ is a critical point of the functional J, and the result follows. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By (F_3), for any $\epsilon = \frac{\underline{\gamma}}{2p} c_1^p \lambda_{1-n}^{\frac{np}{2}}$, where λ_{1-n} is as in (2.6), there exists $\rho > 0$, such that

$$|F(i, v_1, v_2)| \le \frac{\underline{\gamma}}{2p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \left(v_1^2 + v_2^2\right)^{\frac{p}{2}}, \forall i \in \mathbf{Z}(1, k),$$

for $\sqrt{v_1^2+v_2^2} \le \sqrt{2}\rho$.

For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$ and $\|u\| \le \rho$, we have $|u_i| \le \rho,\ i \in \mathbf{Z}(1,k)$.

It follows from the proof of the Theorem 1.1 that for any $u \in \mathbf{R}^k$,

$$J(u) = \frac{1}{p}\sum_{i=1-n}^{k} \gamma_{i+1}\,|\Delta^n u_i|^p - \sum_{i=1}^{k} F(i, u_{i+1}, u_i)$$

$$\ge \frac{\underline{\gamma}}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - \frac{\underline{\gamma}}{2p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \sum_{i=1}^{k} (u_{i+1}^2 + u_i^2)^{\frac{p}{2}}$$

$$\ge \frac{\underline{\gamma}}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - \frac{\underline{\gamma}}{2p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p = \frac{\underline{\gamma}}{2p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p.$$

Taking $a = \frac{\underline{\gamma}}{2p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \rho^p > 0$, we obtain

$$J(u) \ge a > 0,\ \forall u \in \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho} \ge a$, implying that J satisfies the condition (J_1) of Lemma 2.1.

For our setting, clearly $J(0) = 0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify other conditions of this lemma. By Lemma 2.2, J satisfies the (PS)-condition. So; it remains to verify the condition (J_2).

It follows from the proof of Lemma 2.2 that

$$J(u) \le \frac{\bar{\gamma}}{p} c_2^p \lambda_k^{\frac{np}{2}} \|u\|^p - a_1 c_1^\beta \|u\|^\beta + a_2 k.$$

Since $\beta > p$, we can choose $\bar{u}$ large enough to ensure that $J(\bar{u}) < 0$.

Now we can apply the Mountain Pass Lemma to conclude that the functional J possesses a critical value $c \ge a > 0$, where

$$c = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s)),$$

and

$$\Gamma = \{h \in C([0,1], \mathbf{R}^k) \mid h(0) = 0,\ h(1) = \bar{u}\}.$$

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated with the critical value c of J, that is, $J(\tilde{u}) = c$. Similar to the proof of Lemma 2.2 ((PS)- condition), we can conclude that $J(u)$ is bounded on $\mathbf{R}^k$. As a consequence, there exists $\hat{u} \in \mathbf{R}^k$ such that

$$J(\hat{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s)).$$

Clearly, $\hat{u} \ne 0$. If $\tilde{u} \ne \hat{u}$, then the conclusion of Theorem 1.2 holds. Otherwise, $\tilde{u} = \hat{u}$, and we have $c = J(\tilde{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s))$. That is,

$$\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s)).$$

Therefore,

$$c_{\max} = \max_{s \in [0,1]} J(h(s)), \ \forall h \in \Gamma.$$

By the continuity of $J(h(s))$ with respect to s, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0,1)$, such that $J(h(s_0)) = c_{\max}$. Choose h_1, $h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_2(s) \mid s \in (0,1)\}$ is empty, then there exists s_1, $s_2 \in (0,1)$ to satisfy $J(h_1(s_1)) = J(h_2(s_2)) = c_{\max}$. Thus, we get two different critical points of J on $\mathbf{R}^k$ denoted by

$$u^1 = h_1(s_1), \ u^2 = h_2(s_2).$$

The above arguments can be applied to conclude that the BVP (1.1), (1.2) possesses at least two nontrivial solutions. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. In view of above arguments, we only need to find at least one critical point of the functional J defined by (2.5).

By (F_5'), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we can write

$$\begin{aligned} J(u) &= \frac{1}{p}\sum_{i=1-n}^{k} \gamma_{i+1}\left|\Delta^n u_i\right|^p - \sum_{i=1}^{k} F(i, u_{i+1}, u_i) \\ &\geq \frac{\gamma}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - a_3 \sum_{i=1}^{k} \left(\sqrt{u_{i+1}^2 + u_i^2}\right)^{\frac{\alpha}{2}p} - a_4 k \\ &\geq \frac{\gamma}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - a_3 \left\{\left[\sum_{i=1}^{k} \left(\sqrt{u_{i+1}^2 + u_i^2}\right)^{\frac{\alpha}{2}p}\right]^{\frac{2}{\alpha p}}\right\}^{\frac{\alpha}{2}p} - a_4 k \\ &\geq \frac{\gamma}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - a_3 c_2^{\frac{\alpha}{2}p} \left\{\left[\sum_{i=1}^{k} \left(u_{i+1}^2 + u_i^2\right)\right]^{\frac{1}{2}}\right\}^{\frac{\alpha}{2}p} - a_4 k \\ &\geq \frac{\gamma}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - 2^{\frac{\alpha}{2}p} a_3 c_2^{\frac{\alpha}{2}p} \|u\|^{\frac{\alpha}{2}p} - a_4 k \to +\infty \quad \text{as} \quad \|u\| \to +\infty. \end{aligned}$$

By the continuity of J, the above inequality implies that there exist lower bounds of the values of J. This means that J attains its minimal value at some point, which is just the critical point of J with the finite norm. Theorem 1.3 is proved. □

Proof of Theorem 1.4. Assume the opposite, that the BVP (1.1), (1.2) has a nontrivial solution. Then J has a nonzero critical point u^*. Since

$$\frac{\partial J}{\partial u_i} = (-1)^n \Delta^n \left(\gamma_{i-n+1} \varphi_p \left(\Delta^n u_{i-1}\right)\right) - f(i, u_{i+1}, u_i, u_{i-1}),$$

we get

$$\sum_{i=1}^{k} f(i,u_{i+1}^*,u_i^*,u_{i-1}^*)u_i^* = \sum_{i=1}^{k} \left[(-1)^n\Delta^n\left(\gamma_{i-n+1}\varphi_p\left(\Delta^n u_{i-1}^*\right)\right)\right]u_i^* \tag{3.1}$$

$$= \sum_{i=1-n}^{k} \gamma_{i+1}\left|\Delta^n u_i^*\right|^p \le 0.$$

On the other hand, it follows from (F_6) that

$$\sum_{i=1}^{k} f(i,u_{i+1}^*,u_i^*,u_{i-1}^*)u_i^* > 0. \tag{3.2}$$

This contradicts (3.1), and the result follows. Theorem 1.4 is proved. □

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CYCLIC REPRESENTATIONS OF C^*-ALGEBRAS ON FOCK SPACE

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Abstract. [1] The paper characterises C^*-algebras generated by semigroups of composition operators acting upon Fock space. With an approach of approximation from nonharmonic analysis, sufficient conditions for cyclic representations are obtained. We also characterize the cyclic representation of a C^*-algebra of measures generated by Heisenberg groups.

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1. Introduction

This paper is devoted to the study of properties of the C^*-algebras generated by semigroups of composition operators acting upon Fock space of $\mathbf{C}^n$, $n \geq 1$. Representations of a C^*-algebra of measures generated by Heisenberg groups are also considered.

The main novelty of the analysis carried out in this paper lies precisely in the fact that we analyze the cyclic representations of the C^*-algebras with the approach of uniqueness of analytic functions, which is crucial in solving approximation problems in nonharmonic analysis (see, [8], [16], [17]).

Semigroups appear in many areas of analysis (harmonic analysis, representation theory, operator theory, ergodic theory, etc.) The properties of semigroups of holomorphic flows have been extensively studied during the past several decades. Here we mention two known facts that are related to our work in this paper. In [4], Berkson, Kaufman and Porta proved the strong continuity of these flows on Hardy spaces. A complete description of semigroups of holomorphic flows on $\mathbf{C}$ was obtained in [11], by using an approach and techniques, which are quite different and independent of operator-theoretic considerations.

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It was Grothendieck (see [9]), who initiated the study of approximation properties of operator algebras associated with discrete groups, whose fundamental ideas have been applied to the study of groups. In this case one discovers that some important properties of groups can be expressed in terms of approximation properties of the associated operator algebras. Also, various important properties of the groups can be expressed in terms of analytic properties of these algebras. An illustration of nontrivial interaction between analytic and geometric properties of groups and a short survey of approximation properties of operators algebras associated with discrete groups can be found in [6].

Throughout the paper we use the following notation: the points of $\mathbf{C}^n$ are denoted by $z = (z_1, ..., z_n)$, where $z_k \in \mathbf{C}$. If $z_k = x_k + iy_k$, $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, then we write $z = x + iy$. The vectors $x = \Re z$ and $y = \Im z$ are the real and imaginary parts of z, respectively. $\mathbf{R}^n$ stands for the set of all $z \in \mathbf{C}^n$ with $\Im z = 0$. Also, we denote

$$|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2},\ |\Re z| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2},\ |\Im z| = (|y_1|^2 + \cdots + |y_n|^2)^{1/2},$$
$$z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}, \quad (z, t) = z_1 t_1 + \cdots + z_n t_n.$$

The Bargmann-Fock space $\mathcal{F}_n^2(\mathbf{C}^n)$ is defined to be the Hilbert space of entire functions on $\mathbf{C}^n$ equipped with the inner product:

$$\langle f, g \rangle = \frac{1}{(2\pi)^n} \int_{\mathbf{C}^n} f(z)\overline{g(z)} e^{-\frac{1}{2}|z|^2} dv(z),$$

where v denotes the n-dimensional Lebesgue measure on $\mathbf{C}^n$. The norm in $\mathcal{F}_n^2(\mathbf{C}^n)$ is defined by $\|f\| = \sqrt{\langle f, f \rangle}$ (see, [2], [3], [18]).

The reproducing kernel for the Fock space is given by $K_w(z) = e^{\langle z, w \rangle /2}$, where $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$. It is well known that $\|K_w\| = e^{|w|^2/4}$. The Bargmann-Fock spaces has been studied by many authors and it is rooted from mathematical problems of relativistic physics (see [15]) or from quantum optics (see [13]). In physics the Bargmann-Fock space contains the canonical coherent states, so it is the main tool for studying the bosonic coherent state theory of radiation field (see [14]).

The Bargmann-Fock spaces has also been proved invaluable in the theory of wavelets. In fact, the Bargmann transform is a unitary map from $L^2(\mathbf{R})$ onto the Bargmann-Fock space $\mathcal{F}_1^2(\mathbf{C})$, transforming the family of evaluation functionals at a point into canonical coherent states, which are nothing but the Gabor wavelets.

In the last years there was an increasing interest to the characterization of composition operators acting upon Fock space. For instance, bounded and compact composition operators acting upon Fock space $\mathcal{F}^2_n(\mathbf{C}^n)$ were described in [7]. In the recent paper [19], boundedness and compactness of densely defined operators on Fock space $\mathcal{F}^2_1(\mathbf{C})$ were characterized in terms of Berezin transform.

Motivated by [5], [6], [11], [18], [19], it is rather natural to study the approximation properties of C^*-algebras generated by semigroups of composition operators acting upon Fock space.

The paper is organized as follows. Section 2 is devoted to the study of composition operators on Fock space of $\mathbf{C}^n$ which induce holomorphic flows. In Section 3, we obtain sufficient conditions for representations of a C^*-algebra of composition flow to be cyclic in Fock space of $\mathbf{C}^n$. In Section 4, similar conditions are obtained for a C^*-algebra of measures generated by Heisenberg groups.

2. Holomorphic flows induced by a bounded composition operator on Fock space of $\mathbf{C}^n$

In this section we describe the holomorphic flows induced by a bounded composition operator on Fock space of $\mathbf{C}^n$. To this end, we first recall some basic definitions and results.

Let G be a domain in $\mathbf{C}^n$, and let $H(G)$ be the set of holomorphic functions on G. A one-parameter family $\varphi(t,z)$ of nonconstant functions from G to G satisfying $\varphi(0,z)=z$ and $\varphi(t+s,z)=\varphi(s,(t,z))$ for all $s,t\geq 0$ and $z\in G$ is called a semigroup flow (see [11]). The family $(C_{\varphi_t})_{t\geq 0}$ of composition operators on $H(G)$ is given by

$$(C_{\varphi_t}f)(z)=f(\varphi(t,z))$$

for every $t\geq 0$ and $f\in H(G)$. Notice that since $\varphi(t,z)$ is a flow, the semigroup property $C_{\varphi_{t+s}}=C_{\varphi_t}C_{\varphi_s}$ is satisfied.

The next result, which was established in [7], contains the boundedness and compactness of composition operator on Fock space of $\mathbf{C}^n$.

Lemma 2.1 ([7]). *Let $\varphi:\mathbf{C}^n\to\mathbf{C}^n$ be a holomorphic mapping. If for $f\in\mathcal{F}^2_n(\mathbf{C}^n)$, $C_\varphi(f):=f(\varphi(z))$ is bounded on $\mathcal{F}^2_n(\mathbf{C}^n)$, then $\varphi(z)=Az+B$, where A is an $n\times n$ matrix and B is an $n\times 1$ vector. Furthermore, $\|A\|\leq 1$, and if $|A\xi|=|\xi|$ for some $\xi\in\mathbf{C}^n$, then $\langle A\xi,B\rangle=0$; if C_φ is compact, then $\|A\|<1$.*

Conversely, let $\varphi(z)=Az+B$, where A is an $n\times n$ matrix and B is an $n\times 1$ vector. If $\langle A\xi,B\rangle=0$ whenever $|A\xi|=|\xi|$, then C_φ is bounded on $\mathcal{F}^2_n(\mathbf{C}^n)$; if $\|A\|<1$, then C_φ is compact on $\mathcal{F}^2_n(\mathbf{C}^n)$.

The main result of this section is the following theorem.

Theorem 2.1. *Suppose that $\varphi(t,z) : \mathbf{C}^n \to \mathbf{C}^n$ is an one-parameter family of holomorphic mapping satisfying $\varphi(0,z) = z$ and $\varphi(t+s,z) = \varphi(s,(t,z))$ for all $s,t \geq 0$ and $z \in \mathbf{C}^n$. If the family $(C_{\varphi_t})_{t\geq 0}$ is bounded on $\mathfrak{F}^2_n(\mathbf{C}^n)$, then $\varphi(t,z) = e^{Ft}z + (e^{Ft_0} - I)^{-1}(e^{Ft} - I)B$ or $\varphi(t,z) = z + Dt$, where F is an $n \times n$ matrix, and B and D are $n \times 1$ vectors.*

Proof. By Lemma 2.1 we have $\varphi(t,z) = A(t)z + B(t)$, where $A(t) = (a_{ij}(t))_{n\times n}$ is an $n \times n$ matrix satisfying $\|A(t)\| \leq 1$ and $B(t) = (b_{ij}(t))_{n\times 1}$ is an $n \times 1$ vector, $a_{ij}(t)$ and $b_{ij}(t)$ are differentiable functions of t.

From the semigroup property of the flow, we have

$$A(t+s)z + B(t+s) = A(t)(A(s)z + B(s)) + B(t) = A(t)A(s)z + A(t)B(s) + B(t).$$

Equating the coefficients of z, we get

$$A(t+s) = A(t)A(s) \tag{2.1}$$

and

$$B(t+s) = A(t)B(s) + B(t). \tag{2.2}$$

Since $\varphi(0,z) = z$, we have $A(0) = I$, where I is the unit matrix, and $B(0) = O$, where O is the zero vector. The differentiability of $A(t)$ and equality (2.1) imply $A(t) = e^{Ft}$, where F is an $n \times n$ matrix. Actually, we have

$$A'(t) = \lim_{\Delta t\to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t} = A(t) \lim_{\Delta t\to 0} \frac{A(\Delta t) - A(0)}{\Delta t} = A(t)A'(0).$$

Next, the equality

$$A(t+s) = A(s)A(t) = A(t)A(s)$$

implies $A'(t) = A(t)A'(0) = A'(0)A(t)$. If $e^{Ft_0} \neq I$ for some $t_0 \geq 0$, then from (2.2) we have

$$B(t+s) = e^{Ft}B(s) + B(t), \tag{2.3}$$

$$B(s+t) = e^{Fs}B(t) + B(s). \tag{2.4}$$

Taking $s = t_0$ in (2.3) and (2.4), we get $B(t) = (e^{Ft_0} - I)^{-1}(e^{Ft} - I)B(t_0)$.

If $e^{Ft} \equiv I$, then from (2.3) we obtain $B(s+t) = B(t) + B(t)$. Thus, $B(t)$ is a continuous linear vector function in t with $B(0) = 0!$. So, we can conclude that $B(t) = Dt$ for some $n \times 1$ vector D in $\mathbf{C}^n$. This completes the proof of Theorem 2.1. □

3. Cyclic representations of C^*-algebras of composition flow

In this section we deal with the C^*-algebra:

$$\mathfrak{C}_{T_t} := C^*(\{C_{\varphi_t} : \varphi_t \in \Gamma\}),$$

where $C_{\varphi_t}(f(z)) = f(\varphi(t,z))$ and Γ is a discrete semigroup flow of $\mathbf{C}^n$.

We first recall some definitions from the theory of C^*-algebras (see, [1]). A bounded linear map $\pi : X \to Y$ between C^*-algebras X and Y is called a *-homomorphism if it preserves the algebraic operations and satisfies $\pi(x^*) = \pi(x)^*$ for any $x \in X$. A representation of a C^*-algebra $\mathfrak{C}$ is a *-homomorphism of $\mathfrak{C}$ into the C^*-algebra $\mathfrak{L}(H)$ of all bounded operators on some Hilbert space H. It is customary to refer the map $\rho : \mathfrak{C} \to \mathfrak{L}(H)$ as a representation of $\mathfrak{C}$ on H. An invariant subspace $\mathfrak{M}$ of the C^*-algebra $\rho(\mathfrak{C})$ is called a cyclic subspace if it contains a vector ξ, such that $\{\rho(\mathfrak{C})\xi,\ \xi \in H\}$ is dense in $\mathfrak{M}$. A representation ρ is called a cyclic representation if H itself is a cyclic subspace for ρ. Let Λ be a complex set and $f(z)$ be some holomorphic function. If $f(\Lambda) = 0$ implies $f(z) \equiv 0$, then Λ is called a *uniqueness set* for f.

The main result of this section is the following theorem.

Theorem 3.1. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of nonnegative real numbers. Suppose that $\rho(\varphi(t,z))$ is analytic on t and z separately, where $\varphi(t,z)$ is as in Theorem 2.1. Furthermore, suppose that $0 \leq t \in \Lambda$ is the uniqueness set of some bounded holomorphic function in the right half plane. Then ρ is a cyclic representation of $\mathfrak{C}_{T_t}$ on $\mathfrak{F}_n^2(\mathbf{C})$.*

Proof. To prove that ρ is a cyclic representation of the C^*-algebra $\mathfrak{C}_{T_t}$, it is enough to show that $\operatorname{span}\{\rho(\mathfrak{C}_{T_t})f\}$ is dense in $\mathfrak{F}_n^2(\mathbf{C}^n)$ for some fixed $f \in \mathfrak{F}_n^2(\mathbf{C}^n)$. Without loss of generality, let g be a function in $\mathfrak{F}_n^2(\mathbf{C}^n)$ such that

$$|f(\rho(\varphi(t,z)))| = |g(\varphi(t,z))| \leq e^{\alpha|z|^2}, \tag{3.1}$$

where the function $\varphi(t,z)$ is as in Theorem 2.1 and $\alpha < 1/2$ is some fixed positive constant. If the conditions of the theorem are satisfied, but $\operatorname{span}\{\rho(\mathfrak{C}_{T_t})f\}$ is not dense in $\mathfrak{F}_n^2(\mathbf{C}^n)$, then by the Hahn-Banach Theorem, there exists a nontrivial bounded linear functional L which annihilates $\{\rho(\mathfrak{C}_{T_t})f\}$. Thus,

$$L(f(\rho(\varphi(t,z)))) = L(g(\varphi(t,z))) = 0.$$

Define

$$L(w) = \frac{1}{(2\pi)^n}\int_{\mathbf{C}^n} g(\varphi(w,z))\overline{q(z)}e^{-\frac{1}{2}|z|^2}dv(z),$$

and observe that $L(w)$ is an analytic function in the right half plane $\mathbf{C}_+ = \{w : \Re w > 0\}$. It follows from (3.1) that $L(w)$ is bounded in $\mathbf{C}_+$. Since by assumption $0 \leq t \in \Lambda$ is the uniqueness set, we can conclude that $L(w) \equiv 0$. This completes the proof of Theorem 3.1. □

The following example illustrates Theorem 3.1.

Example. The mapping ρ defined by $\rho_t f = e^{-t} f$ is a cyclic representation of $\mathcal{C}_{T_t}$ on $\mathcal{F}_1^2(\mathbf{C})$. Indeed, in this case we take $f(z) = e^z$,

$$L(w) := \frac{1}{(2\pi)^n} \int_{\mathbf{C}^n} e^{ae^{-w}z+b}\overline{q(z)}e^{-\frac{1}{2}|z|^2} dv(z),$$

and

$$\Lambda = \left\{ \lambda_k : \lambda_k > 0, \sum_{k=1}^{\infty} \frac{\lambda_k}{1+\lambda_k^2} = +\infty \right\}.$$

Then using the arguments of the proof of Theorem 3.1 and [10], we can conclude that $L(w) \equiv 0$, and the result follows.

4. Cyclicity of Segal-Bargmann representation.

In this section we study the cyclicity property of Segal-Bargmann representation. To this end, we first recall some basic facts on C^*-algebras generated by Heisenberg groups (see [5]). The Heisenberg group H_n is given by $\mathbf{C}^n \times \mathbf{R}$ with multiplication

$$(a,t)(b,s) = (a+b, s+t+\Im b.a/2),$$

where $\Im b.a/2 = (b.a - a.b)/2i$. It is well-known that the Lebesgue measure on $\mathbf{C}^n \times \mathbf{R}$ is bi-invariant Haar measure on H_n. In [5], Coburn focused on the Segal-Bargmann representation on Fock space. The representation is given by $\rho(a,t) = e^{it}W_a$, where

$$(W_a f)(z) = k_a(z) f(z-a)$$

and $k_a(z) = \exp\{\langle z, a\rangle - |a|^2/2\}$ is the normalized reproducing kernel. In representation theory, ρ is often extended to $M(H_n)$ and $L^1(H_n)$, the convolution algebra of bounded regular complex valued Borel measures on H_n and its closed two-sided ideal of measures that are absolutely continuous with respect to left Haar measure, respectively. It is represented as follows:

$$\rho(\sigma) = \int_{H_n} \rho(a,t) d\sigma(a,t). \tag{4.1}$$

The equation (4.1) determines an operator on $\mathcal{F}_n^2(\mathbf{C}^n)$ by

$$\langle \rho(\sigma) f, g\rangle = \int_{H_n} \langle \rho(a,t) f, g\rangle d\sigma(a,t),$$

where f and g are arbitrary functions in $\mathcal{F}_n^2(\mathbf{C}^n)$ and $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $L^2(\mathbf{C}^n)$.

The cut-down of ρ to $M(H_n)$, $L^1(H_n)$ is defined as follows:

$$\tilde{\rho}(\sigma) = \int_{\mathbf{C}^n} W_a d\sigma(a). \tag{4.2}$$

Observe that $M(H_n)$ and $M(\mathbf{C}^n)$ are involution Banach algebras with $d\sigma^* = \overline{d\sigma(-a)}$. The twisted convolution $\tau\natural\sigma$ on $M(\mathbf{C}^n)$ is defined for any continuous on $\mathbf{C}^n$ function ϕ which vanishes at infinity as follows:

$$\int_{\mathbf{C}^n} \phi(a)d(\tau\natural\sigma)(a) = \int_{\mathbf{C}^n}\int_{\mathbf{C}^n} \phi(a+b)\chi_a(b/2)d\tau(a)d\tau(b),$$

where $\chi_a(z) = \exp\{i\Im\langle z,a\rangle\}$. Denote by $B(\mathbf{C}^n)$ the linear span of all continuous positive-definite functions on $\mathbf{C}^n$.

For bounded and continuous φ and $f \in \mathcal{F}_n^2(\mathbf{C}^n)$, the Berezin-Toeplitz operator T_φ is defined by

$$(T_\varphi f)(z) = \frac{1}{(2\pi)^n}\int_{\mathbf{C}^n} \varphi(a)f(a)e^{\langle z,a\rangle/2}e^{-\frac{1}{2}|a|^2}dv(a).$$

In [5], it is proved that $\tilde{\rho}$, defined by (4.2), is a faithful representation of $M(\mathbf{C}^n)_\natural$ on $\mathcal{F}_n^2(\mathbf{C}^n)$. The following identities are also proved in [5]:

$$\mathfrak{C}^*\{\tilde{\rho}[M(\mathbf{C}^n)_\natural]\} = \mathfrak{C}^*\{\rho[M(H_n)]\} = \text{closure}\{T_\varphi : \varphi \in B(\mathbf{C}^n)\}.$$

Below we prove that both $\tilde{\rho}$ and ρ are cyclic representations on $\mathcal{F}_n^2(\mathbf{C}^n)$. To this end, we need the Jensen's formula for entire functions of several variables.

Let $\mathbf{e} = (e_1, e_2, ..., e_n)$ be a unit vector satisfying $e_j > 0$ $(j = 1,2,...,n)$. Denote by $N(\Lambda,\mathbf{e},t)$ the number of points of Λ lying on the segment $S(\mathbf{e},t) = \{(e_1\xi, e_2\xi, ..., e_n\xi) : |\xi| \le t\}$ of the complex affine straight line $S(\mathbf{e}) = \{(e_1\xi, e_2\xi, ..., e_n\xi) : \xi \in \mathbf{C}\}$. For an entire function $f(z)$, by $N(f,\mathbf{e},t)$ we denote the number of zeros of $f(z)$ in $S(\mathbf{e},t)$, counted according to their multiplicity. Also, we denote

$$S(r,\alpha) := \Big(\frac{r_1}{r}e^{i\alpha_1}, \frac{r_2}{r}e^{i\alpha_2}, ..., \frac{r_{n-1}}{r}e^{i\alpha_{n-1}}, \frac{r_n}{r}\Big),$$

where $r = (r_1, r_2, ..., r_n)$ satisfying $r_j > 0$ $(j = 1,2,...,n)$. For an entire function $f(z)$ $(f(0) \neq 0)$, the Jensen's formula is as follows (see [12], Chapter 4, Section 2.1-2.4):

$$\int_0^r \Big(\frac{1}{t}\int_0^{2\pi} \cdots \int_0^{2\pi} N(f, S(r,\alpha), t)d\alpha_1 ... d\alpha_{n-1}\Big)dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} \log|f(r_1e^{i\theta_1}, ..., r_ne^{i\theta_n})|d\theta_1 ... d\theta_n - \log|f(0)|.$$

Theorem 4.1. *The representations ρ and $\widetilde{\rho}$ defined by (4.1) and (7), respectively, are cyclic representations of $M(H_n)$ and $M(\mathbf{C}^n)_I$ on $\mathcal{F}_n^2(\mathbf{C}^n)$, respectively.*

Proof. Since both of the closures of $M(H_n)$ and $M(\mathbf{C}^n)_I$ under the discussed representations are equal to closure$\{T_\varphi : \varphi \in B(\mathbf{C}^n)\}$, it is enough to show that

$$\text{closure}\{T_\varphi : \varphi \in B(\mathbf{C}^n)\} = \mathcal{F}_n^2(\mathbf{C}^n). \tag{4.3}$$

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of nonnegative real numbers in $\mathbf{R}^n$, where $\lambda_k = (\lambda_k^1, \lambda_k^2, ..., \lambda_k^n)$. It is easy to see that the functions $\varphi_\lambda := e^{i\Im\langle\lambda,a\rangle/2} = e^{i\langle\lambda,\Im a\rangle/2}$, $\lambda \in \Lambda$ are in $B(\mathbf{C}^n)$. Thus, to prove (4.3), it is enough to show that the closure of linear span $\{T_{\varphi_\lambda} : \lambda \in \Lambda\}$ coincides with $\mathcal{F}_n^2(\mathbf{C}^n)$ for a suitable selected sequence Λ. Denote

$$J(r,f) = \int_0^r \left(\frac{1}{t}\int_0^{2\pi} ... \int_0^{2\pi} N(f, S(r,\alpha), t)d\alpha_1...d\alpha_{n-1}\right)dt,$$

and observe that if Λ satisfies the condition

$$\limsup_{r\to+\infty} \frac{J(r,f)}{r^2} = +\infty, \tag{4.4}$$

then the closure of linear span $\{T_{\varphi_\lambda} : \lambda \in \Lambda\}$ coincides with $\mathcal{F}_n^2(\mathbf{C}^n)$. Actually, if (4.4) is fulfilled, but the closure of linear span $\{T_{\varphi_\lambda} : \lambda \in \Lambda\} \neq \mathcal{F}_n^2(\mathbf{C}^n)$, then by the Hahn-Banach Theorem, there exists a nontrivial bounded linear functional L which annihilates T_{φ_λ}, that is $L(T_{\varphi_\lambda} f) = 0$. Without loss of generality, let $f(z)$ be a function in $\mathcal{F}_n^2(\mathbf{C}^n)$ satisfying

$$|f(z)| \leq e^{-\varepsilon_0|z|^2}, \tag{4.5}$$

where ε_0 is some fixed positive number. Define

$$L(w) = \frac{1}{(2\pi)^n}\int_{\mathbf{C}^n} (T_{\varphi_w} f)(z)\overline{q(z)}e^{-\frac{1}{2}|z|^2}dv(z)$$

$$= \frac{1}{(2\pi)^{2n}}\int_{\mathbf{C}^n}\int_{\mathbf{C}^n} e^{i\langle w,\Im a\rangle} f(a)e^{\langle z,a\rangle/2}e^{-\frac{1}{2}|a|^2}dv(a)\overline{q(z)}e^{-\frac{1}{2}|z|^2}dv(z),$$

and observe that $L(w)$ is an entire function.

By (4.5), with some positive constant A_1 we have

$$|(T_{\varphi_w} f)(z)| \leq A_1 e^{-\frac{1}{8\varepsilon_0}(|w|+|z|)^2}. \tag{4.6}$$

It follows from (4.6) that for sufficiently large r with some positive constant A_2 $\log|f(z)| \leq A_2 r^2$. Hence by the Jensen's formula, we should have

$$\limsup_{r\to+\infty} \frac{J(r,f)}{r^2} < +\infty,$$

which contradicts (4.4). Thus, we conclude that $L(w) \equiv 0$, and the result follows. Theorem 4.1 is proved. □

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UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING ONE VALUE OR FIXED POINTS

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Abstract. [1] In this paper we study the uniqueness problems on meromorphic functions sharing a nonzero finite value or fixed points. Our results improve or generalize those given by Fang and Hua [7], Yang and Hua [18], Fang and Qiu [9], Cao and Zhang [2].

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Keywords: Uniqueness; meromorphic function; sharing value; fixed point.

1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane and let $f(z)$ be a non-constant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard notions used in the Nevanlinna value distribution theory, such as $T(r,f)$, $m(r,f)$, $N(r,f)$ (see [10, 12, 19, 20]). Let $S(r,f)$ denote any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside possible an exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function of $f(z)$ if $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a(z)$ be a small function of $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities, and we say that $f(z)$ and $g(z)$ share $a(z)$ IM (ignoring multiplicities) if the multiplicities are ignored. We denote by $N_{k)}(r, \frac{1}{f-a})$ (or $\overline{N}_{k)}(r, \frac{1}{f-a})$) the counting function for zeros of $f - a$ with multiplicity $\leq k$ (ignoring multiplicities), and by $N_{(k}(r, \frac{1}{f-a})$ (or $\overline{N}_{(k}(r, \frac{1}{f-a})$) the counting function for zeros of $f-a$ with multiplicity $\geq k$ (ignoring multiplicities). Also, we set

$$N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + \overline{N}_{(3}(r, \frac{1}{f-a}) + \cdots + \overline{N}_{(k}(r, \frac{1}{f-a}).$$

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We say that a finite value z_0 is a fixed point of f if $f(z_0) = z_0$ or z_0 is a zero of $f(z) - z$.

The following theorem is well known in the value distribution theory (see [1, 3]).

Theorem A. *Let $f(z)$ be a transcendental meromorphic function, and let $n \geq 1$ be a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [7], and Yang and Hua [18], respectively have obtained a unicity theorem corresponding to Theorem A.

Theorem B. *Let f and g be two non-constant entire (resp., meromorphic) functions, and let $n \geq 6$ (resp., $n \geq 11$) be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Corresponding to the uniqueness of entire or meromorphic functions sharing fixed points, Fang and Qiu [9] obtained the following result.

Theorem C. *Let f and g be two non-constant meromorphic (resp., entire) functions, and let $n \geq 11$ (resp., $n \geq 6$) be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}$ and $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

For more results in this direction, we refer the reader to [4] — [9], [11],[13] – [16], [18], [21] – [24]. Cao and Zhang [2] extended Theorems B and C as follows.

Theorem D. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer, and let $n > \max\{2k-1, k+4/k+4\}$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds:*

(1) $f^n f^{(k)} = g^n g^{(k)}$;

(2) $f = c_1e^{cz^2}$, $g = c_2e^{-cz^2}$ where c_1, c_2, c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

Theorem E. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer, and let $n > \max\{2k-1, k+4/k+4\}$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, and f and g share ∞ IM, then one of the following two conclusions holds:*

(1) $f^n f^{(k)} = g^n g^{(k)}$;

(2) $f = c_3e^{dz}$, $g = c_4e^{-dz}$ where c_3, c_4, d are constants such that $(-1)^k(c_3c_4)^{n+1}d^{2k} = 1$.

In this paper we show that in Theorems D and E the condition "f and g share ∞ IM" can be removed. Specifically we prove the following results.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < +\infty$, whose zeros are of multiplicities at least k, where k is a positive integer, and let $n > \max\{2k+1, 2(\sigma(f)-1)k-3, k+4/k+8\}$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share z CM, then one of the following two conclusions holds:*
(1) $f^n f^{(k)} = g^n g^{(k)}$;
(2) $f = c_1 e^{cz^2}$, $g = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < +\infty$, whose zeros are of multiplicities at least k, where k is a positive integer, and let $n > \max\{2k-1, 2(\sigma(f)-1)k-1, k+4/k+5\}$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, then one of the following two conclusions holds:*
(1) $f^n f^{(k)} = g^n g^{(k)}$;
(2) $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$, where c_3, c_4, d are constants such that $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.

To prove Theorems 1.1 and 1.2, we need the following results.

Proposition 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < +\infty$, and let n and k be two positive integers such that $n > \max\{2k+1, 2(\sigma(f)-1)k-3\}$. If $f^n f^{(k)} g^n g^{(k)} = z^2$, then $f = c_1 e^{cz^2}$, $g = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.*

Proposition 1.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < +\infty$, and let n and k be two positive integers such that $n > \max\{2k-1, k+1, 2(\sigma(f)-1)k-1\}$. If $f^n f^{(k)} g^n g^{(k)} = 1$, then $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.*

2. Preliminary lemmas

Lemma 2.1 (see [19]). *Let $f(z)$ be a non-constant meromorphic function and let $a_0(z), a_1(z), \ldots, a_n(z) (\not\equiv 0)$ be small functions of f. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([19], p. 21). *Let $f(z)$ be a non-constant meromorphic function in the complex plane. If the order of $f(z)$ is finite, then*

$$m(r, \frac{f'}{f}) = O(\log r), \quad r \to \infty.$$

Lemma 2.3 ([19], p. 65). *Let $h(z)$ be a non-constant entire function and let $f(z) = e^{h(z)}$. Let λ and μ be the order and the lower order of $f(z)$, respectively. We have*
(i) *If $\mu < \infty$, then μ is a positive integer, $h(z)$ is a polynomial of degree μ, and $\lambda = \mu$.*
(ii) *If $\mu = \infty$, then $h(z)$ is transcendental and $\lambda = \mu$.*

Lemma 2.4. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + O(\log r).$$

Proof. Since f is of finite order, by Lemma 2.2, we have

$$m(r, \frac{f'}{f}) = O(\log r).$$

Now we use mathematical induction to prove that $m(r, \frac{f^{(k)}}{f}) = O(\log r)$. Suppose that the conclusion is true for $k = m$. For $k = m + 1$ we have

$$\frac{f^{(m+1)}}{f} = (\frac{f^{(m)}}{f})' + \frac{f^{(m)}}{f}\frac{f'}{f}.$$

Then we can write

$$\begin{aligned} & m(r, \frac{f^{(m+1)}}{f}) \leq m(r, (\frac{f^{(m)}}{f})') + m(r, \frac{f^{(m)}}{f}) + m(r, \frac{f'}{f}) + O(1) \\ = \ & m(r, \frac{(\frac{f^{(m)}}{f})'}{\frac{f^{(m)}}{f}}\frac{f^{(m)}}{f}) + O(\log r) \leq m(r, \frac{(\frac{f^{(m)}}{f})'}{\frac{f^{(m)}}{f}}) + m(r, \frac{f^{(m)}}{f}) + O(\log r) \\ = \ & O(\log r). \end{aligned}$$

Moreover, we have

$$m(r, \frac{1}{f}) \leq m(r, \frac{1}{f^{(k)}}) + m(r, \frac{f^{(k)}}{f}) = m(r, \frac{1}{f^{(k)}}) + O(\log r).$$

Hence

$$T(r, f) - N(r, \frac{1}{f}) \leq T(r, f^{(k)}) - N(r, \frac{1}{f^{(k)}}) + O(\log r).$$

Therefore

$$\begin{aligned} N(r,\frac{1}{f^{(k)}}) &\le T(r,f^{(k)}) - T(r,f) + N(r,\frac{1}{f}) + O(\log r) \\ &= m(r,f^{(k)}) + N(r,f^{(k)}) - T(r,f) + N(r,\frac{1}{f}) + O(\log r) \\ &\le m(r,f) + m(r,\frac{f^{(k)}}{f}) + N(r,f) + k\overline{N}(r,f) - T(r,f) + N(r,\frac{1}{f}) + O(\log r) \\ &= N(r,\frac{1}{f}) + k\overline{N}(r,f) + O(\log r). \end{aligned}$$

This completes the proof of Lemma 2.4.

Lemma 2.5 ([18]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, n and k be two positive integers, and a be a finite nonzero constant. If f and g share a CM, then one of the following conclusions holds:*
(i) $T(r,f) \le N_2(r,1/f) + N_2(r,1/g) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$, the same inequality holding for $T(r,g)$;
(ii) $fg \equiv a^2$;
(iii) $f \equiv g$.

3. Proofs of Propositions 1.1–1.2

Proof of Proposition 1.1. We first prove that

$$f \neq 0, \quad g \neq 0. \tag{3.1}$$

We have

$$f^n f^{(k)} g^n g^{(k)} = z^2. \tag{3.2}$$

Suppose that $z_0 \neq 0$ is a zero of f, say of multiplicity l, then z_0 is a pole of g, say of multiplicity s. Then we have $nl + l - k = ns + s + k$, implying $(n+1)(l-s) = 2k$, which is impossible since by assumption $n > 2k+1$.
Now suppose that $z = 0$ is a zero of f, say of multiplicity l_1. If $z = 0$ is not a pole of g, then $z = 0$ must be the zero of z^2 of multiplicity $nl_1 + l_1 - k > 2$, which is a contradiction. If $z = 0$ is a pole of g, say of multiplicity s_1, then we have

$$(n+1)(l_1 - s_1) = 2k + 2,$$

which is impossible since by assumption $n > 2k+1$. So f has no zeros. Similarly, it can be shown that g also has no zeros. Thus (3.1) is proved.
Next, we prove that

$$N(r,f) = O(\log r), \quad N(r,g) = O(\log r). \tag{3.3}$$

To this end, we rewrite (3.2) as follows

$$f^n f^{(k)} = \frac{z^2}{g^n g^{(k)}}. \tag{3.4}$$

From (3.4) we deduce that

$$N(r, f^n f^{(k)}) = N(r, \frac{1}{g^n g^{(k)}}). \tag{3.5}$$

Since $N(r, f^n f^{(k)}) = (n+1)N(r, f) + k\overline{N}(r, f)$, using (3.5) and Lemma 2.4, we obtain

$$(n+1)N(r, f) + k\overline{N}(r, f) \leq k\overline{N}(r, g) + O(\log r). \tag{3.6}$$

Similarly we get

$$(n+1)N(r, g) + k\overline{N}(r, g) \leq k\overline{N}(r, f) + O(\log r). \tag{3.7}$$

A combination of (3.6) and (3.7) yields

$$N(r, f) + N(r, g) = O(\log r). \tag{3.8}$$

Thus we obtain (3.3), which means that both f and g have at most finitely many poles. Now we prove that

$$\sigma(f) = \sigma(g). \tag{3.9}$$

It is easy to show that both f and g must be transcendental meromorphic functions. Note that $nT(r, f) = T(r, f^n) =$

$$= T(r, f^n f^{(k)}/f^{(k)}) \leq T(r, f^n f^{(k)}) + (k+1)T(r, f) + S(r, f), \tag{3.10}$$

and $T(r, f^n f^{(k)}) = T(r, \frac{z^2}{g^n g^{(k)}}) \leq$

$$\leq T(r, g^n g^{(k)}) + S(r, g) \leq (n+k+1)T(r, g) + S(r, g). \tag{3.11}$$

Combining (3.10) and (3.11) we get

$$(n-k-1)T(r, f) \leq (n+k+1)T(r, g) + S(r, f) + S(r, g).$$

Since $n > 2k+1$, we have $T(r, f) = O(T(r, g))$. Similarly we obtain $T(r, g) = O(T(r, f))$. Thus (3.9) is proved. Note that $\sigma(f) < +\infty$. Let

$$f = \frac{e^{h(z)}}{p(z)}, \quad g = \frac{e^{h_1(z)}}{q(z)},$$

where $p(z)$ and $q(z)$ are polynomials with $\deg(p(z)) = p$ and $\deg(q(z)) = q$, while $h(z)$ and $h_1(z)$ are non-constant entire functions. By Lemma 2.3, $h(z)$ and $h_1(z)$ are polynomials with $\deg(h(z)) = \deg(h_1(z)) = h = \sigma(f)$. Then we have

$$f^n = \frac{e^{nh(z)}}{p^n(z)}, \quad g^n = \frac{e^{nh_1(z)}}{q^n(z)}.$$

By mathematical induction we get

$$f^n f^{(k)} = \frac{e^{(n+1)h(z)}P_k(z)}{p^{n+k+1}(z)}, \quad g^n g^{(k)} = \frac{e^{(n+1)h_1(z)}Q_k(z)}{q^{n+k+1}(z)},$$

where $P_k(z)$ and $Q_k(z)$ are two polynomials with $\deg(P_k(z)) = k(h-1+p)$ and $\deg(Q_k(z)) = k(h-1+q)$. By (3.2), we get $h(z)+h_1(z) \equiv C$, where C is a constant. Furthermore, we have

$$\deg(P_k(z)) + \deg(Q_k(z)) = \deg(p^{n+k+1}(z)) + \deg(q^{n+k+1}(z)) + 2,$$

implying that

$$2k(h-1) = (n+1)(p+q)+2. \tag{3.12}$$

By (3.8), if $N(r,f)+N(r,g) \neq 0$, then $p+q \geq 1$, and from (3.12) we obtain $n \leq 2k(h-1)-3 = 2k(\sigma(f)-1)-3$, which contradicts the assumption that $n > 2k(\sigma(f)-1)-3$. Therefore $N(r,f)+N(r,g) = 0$, showing that both f and g are entire functions and $p = q = 0$. From (3.12) we obtain that $h = 2$ and $k = 1$, and from (3.2) we have $h'(z) = l_2 z$, $h_1'(z) = l_3 z$ and $h(z) = cz^2 + l_3$, $h_1(z) = -cz^2 + l_4$. So, we can rewrite f and g as $f = c_1 e^{cz^2}$ and $g = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

This completes the proof of Proposition 1.1.

Proof of Proposition 1.2. By the same reasoning as in the proof of Proposition 1.1, we get

$$2k(h-1) = (n+1)(p+q). \tag{3.13}$$

In view of $f^n f^{(k)} g^n g^{(k)} = 1$, if $N(r,f)+N(r,g) \neq 0$, then $p+q \geq 1$, and from (3.13) we obtain $n \leq 2k(h-1)-1 = 2k(\sigma(f)-1)-1$, which contradicts the assumption that $n > 2k(\sigma(f)-1)-1$. Therefore $N(r,f)+N(r,g) = 0$, showing that both f and g are entire functions and $p = q = 0$. From (3.13) we obtain that $h = 1$. Thus $h(z) = dz + l_5$ and $h_1(z) = -dz + l_6$. Finally, we rewrite f and g as $f = c_3 e^{dz}$ and $g = c_4 e^{-dz}$, where c_3, c_4 and d are nonzero constants, and deduce that $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$. This completes the proof of Proposition 1.2.

4. Proof of Theorem 1.1

Let $F = f^n f^{(k)}$, $G = g^n g^{(k)}$, $F^* = F/z$, and $G^* = G/z$. Then F^* and G^* share 1 CM. In view of Lemma 2.5, we consider three cases.

Case 1.

$$T(r,F^*) \leq N_2(r,1/F^*)$$

$$+N_2(r,1/G^*)+N_2(r,F^*)+N_2(r,G^*)+S(r,f)+S(r,g). \tag{4.1}$$

We deduce from (4.1) that

$$T(r,F)\le N_2(r,1/F)+N_2(r,1/G)+2\overline{N}(r,f)+2\overline{N}(r,g)+3\log r+S(r,f)+S(r,g).$$

Obviously,

$$N(r,F)=(n+1)N(r,f)+k\overline{N}(r,f)+S(r,f). \tag{4.2}$$

Also, we have

$$\begin{aligned} nm(r,f) &= m(r,F/f^{(k)})\le m(r,F)+m(r,1/f^{(k)})+S(r,f)\\ &= m(r,F)+T(r,f^{(k)})-N(r,1/f^{(k)})+S(r,f)\\ &\le m(r,F)+T(r,f)+k\overline{N}(r,f)-N(r,1/f^{(k)})+S(r,f). \end{aligned} \tag{4.3}$$

It follows from (4.2), (4.3) and Lemma 2.1 that

$$\begin{aligned} (n-1)T(r,f) &\le T(r,F)-N(r,f)-N(r,1/f^{(k)})+S(r,f)\\ &\le N_2(r,1/F)+N_2(r,1/G)+2\overline{N}(r,f)+2\overline{N}(r,g)\\ &\quad -N(r,f)-N(r,1/f^{(k)})+3\log r+S(r,f)+S(r,g)\\ &\le 2\overline{N}(r,1/f)+2\overline{N}(r,1/g)+N(r,1/g)+k\overline{N}(r,g)\\ &\quad +\overline{N}(r,f)+2\overline{N}(r,g)+3\log r+S(r,f)+S(r,g)\\ &\le \frac{2}{k}(T(r,f)+T(r,g))+(k+4)T(r,g)+3\log r+S(r,f)+S(r,g). \end{aligned} \tag{4.4}$$

Similarly we obtain

$$\begin{aligned} (n-1)T(r,g) &\le \frac{2}{k}(T(r,f)\\ &+T(r,g))+(k+4)T(r,f)+3\log r+S(r,f)+S(r,g). \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) we get

$$\begin{aligned} &(n-1)(T(r,f)+T(r,g))\\ &\le (\frac{4}{k}+k+4)(T(r,f)+T(r,g))+6\log r+S(r,f)+S(r,g). \end{aligned} \tag{4.6}$$

Noting that $T(r,f)\ge \log r+O(1)$, $T(r,g)\ge \log r+O(1)$ and $n>k+4/k+8$, we get a contradiction from (4.6).

Case 2. We have $f^nf^{(k)}g^ng^{(k)}=z^2$, and by Proposition 1.1 we get conclusion (2) of the theorem 1.1.

Case 3. We have $f^nf^{(k)}=g^ng^{(k)}$. This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 is similar to that of Theorem 1.1, the only difference is that instead of Proposition 1.1, we use Proposition 1.2.

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