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Настоящий номер журнала содержит статьи, представленные на второй международной конференции "Математика в Армении. Достижения и перспективы", посвященной 70-летнему юбилею основания Национальной Академии Наук Армении, который состоялся 24-31 августа 2013 года в городе Цахкадзор (Армения).

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ОБ АБСОЛЮТНОЙ СХОДИМОСТИ РЯДОВ ПО ОБЩЕЙ СИСТЕМЕ ФРАНКЛИНА

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Аннотация. Рассматривается общая система Франклина, порожденная сильно регулярным разбиением отрезка $[0;1]$. Для таких систем доказываются: 1) абсолютная сходимость ряда Фурье-Франклина в точке является локальным свойством; 2) ряд Фурье-Франклина функции ограниченной вариации почти всюду абсолютно сходится; 3) любая почти всюду конечная измеримая функция представляется почти всюду абсолютно сходящимся рядом по этой системе.

MSC2010 numbers: 42C15; 42C25.

Ключевые слова: Абсолютная сходимость; общая система Франклина.

1. Введение

Определение 1.1. *Последовательность (разбиение) $\mathcal{T} = \{t_n : n \geq 0\}$ называется допустимой, если $t_0 = 0$, $t_1 = 1$, $t_n \in (0;1)$, $n \geq 2$, $\mathcal{T}$ всюду плотно в $[0;1]$ и каждая точка $t \in (0;1)$ встречается в $\mathcal{T}$ не более чем два раза.*

Пусть $\mathcal{T} = \{t_n : n \geq 0\}$ допустимая последовательность. Для $n \geq 2$ обозначим $\mathcal{T}_n = \{t_i : 0 \leq i \leq n\}$. Допустим π_n получается из $\mathcal{T}_n$ неубывающей перестановкой: $\pi_n = \{\tau_i^n : \tau_i^n \leq \tau_{i+1}^n, 0 \leq i \leq n-1\}$, $\pi_n = \mathcal{T}_n$. Тогда через S_n обозначим пространство функций определенных на $[0;1]$, которые непрерывны слева, линейны на $(\tau_i^n; \tau_{i+1}^n)$ и непрерывны в τ_i^n, если $\tau_{i-1}^n < \tau_i^n < \tau_{i+1}^n$. Ясно, что $dim S_n = n+1$ и $S_{n-1} \subset S_n$. Следовательно, существует (с точностью до знака) единственная функция $f \in S_n$, которая ортогональна S_{n-1} и $\|f\|_2 = 1$. Эту функцию называют n-ой функцией Франклина, соответствующей разбиению $\mathcal{T}$. Известно, что $f(t_n) \neq 0$. Поэтому полагается $f(t_n) > 0$.

Определение 1.2. *Общая система Франклина $\{f_n(x) : n \geq 0\}$ соответствующая разбиению $\mathcal{T}$ определяется по правилу $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x-1)$ и*

[1] Исследование выполнено при финансовой поддержке ГКН МОН РА в рамках научного проекта 13-11A006.

для $n \geq 2$ функция $f_n(x)$ есть n-ая функция Франклина, соответствующая разбиению $\mathcal{T}$.

Общая система Франклина исследуется начиная с работы [1], где доказано, что если ряд $\sum_{n=0}^{\infty} a_n f_n(x)$ является рядом Фурье интегрируемой функции f, то

$$(1.1) \qquad |S_n(x)| \leq C \cdot \mathcal{M}(f, x),$$

где $S_n(x) = \sum_{k=0}^{n} a_k f_k(x)$ частичная сумма ряда Фурье-Франклина, а $\mathcal{M}(f,x)$ максимальная функция Харди-Литлвуда функции f.

Далее в работах [2], [3] для исследования свойств общей системы Франклина введены понятия сильной и слабой регулярности разбиения $\mathcal{T}$. Приведем необходимые определения.

Определение 1.3. *Последовательность $\mathcal{T}$ называется квазидиадической, если $t_{2^k+1} < t_{2^k+2} < \dots < t_{2^{k+1}}$ и между двумя соседними точками из $\{t_n : 0 \leq n \leq 2^k\}$ находится по одной точке из $\{t_n : 2^k < n \leq 2^{k+1}\}$.*

Определение 1.4. *Последовательность $\mathcal{T}$ называется сильно регулярной, если существует постоянная $\gamma > 1$, такая что*

$$\gamma^{-1} < \frac{\tau_{i+1}^n - \tau_i^n}{\tau_i^n - \tau_{i-1}^n} < \gamma, \qquad для \quad n = 2, 3, \dots, \qquad i = 1, 2, \dots, n-1.$$

Если в последовательности π_n точкой t_n является точка τ_i^n, то обозначаются $t_n^- = \tau_{i-1}^n$ и $t_n^+ = \tau_{i+1}^n$. Иными словами для фиксированного n через t_n^- и t_n^+ обозначаются ближайшие к t_n точки из $\mathcal{T}_{n-1}$, соответственно, слева и справа. Интервалы $(t_n^-; t_n)$ и $(t_n; t_n^+)$ назовем смежными.

Определение 1.5. *Последовательность $\mathcal{T}$ называется слабо регулярной, если существует постоянная $\gamma > 1$, такая что*

$$\gamma^{-1} < \frac{t_n^+ - t_n}{t_n - t_n^-} < \gamma, \qquad для \quad n = 2, 3, \dots.$$

Иними словами последовательность $\mathcal{T}$ слабо регулярная, если отношение длин смежных интервалов снизу и сверху ограничены числами $1/\gamma$, γ, соответственно.

Ясно, что если $\mathcal{T}$ слабо (сильно) регулярная, то в ней каждая точка встречается один раз, т.е. двойных точек нет.

Отметим, что если последовательность $\mathcal{T}$ диадическая, т.е. $t_n = \frac{2m-1}{2^{k+1}}$, где $n = 2^k + m$, $1 \leq m \leq 2^k$, $k = 0, 1, 2, \dots$, то соответствующая система Франклина есть классическая система Франклина [7].

Условимся о некоторых обозначениях.

Для $f \in L_1$ через $a_n(f)$, $n = 0, 1, 2, ...$, обозначим коэффициенты Фурье-Франклина функции f, т. е. $a_n(f) = \int_0^1 f(t) f_n(t) dt$.

Через $c, C, C_1, C_\gamma, ...$, обозначаются постоянные зависящие только от своих индексов. Значения этих постоянных в разных формулах могут быть разными.

Длину отрезка I обозначим через $|I|$.

$\rho(t, I)$-это расстояние между точкой t и интервалом I, и вообще $\rho(A, B)$-расстояние между множествами A и B, т.е. $\rho(A, B) = \inf_{x \in A, y \in B} |x - y|$.

Через $d_n(t, x)$ обозначим количество точек из $\mathcal{T}_n$, которые находятся между точками t и x. В случае $x = t_n$, вместо $d_n(t, t_n)$ обозначим $d_n(t)$, т.е. $d_n(t)$-количество точек из $\mathcal{T}_n$, которые находятся между точками t_n и t. Для множества A через $d_n(A)$ обозначим $d_n(A) = \min_{t \in A} d_n(t)$.

Запись $a \sim b$ означает, что существуют положительные постоянные c и C, такие что $c \cdot a \leq b \leq C \cdot a$, а запись $a \sim_\gamma b$ означает, что эти постоянные могут зависеть от γ.

Через $\chi_A(x)$ обозначим характеристическую функцию множества A, а через $\mu(A)$-лебегову меру множества A, A^c - дополнение множества A.

$V_I(f)$-вариация функции f на отрезке I, а $BV(I)$-множество функций ограниченной вариации на I. Если $I = [0; 1]$, то просто обозначим $V(f)$ и BV.

2. Некоторые свойства общей системы Франклина

Семейство интервалов линейности первых $n+1$ функций $\{f_k(x)\}_{k=0}^n$ обозначим через $\mathcal{I}_n$. Ясно, что интервалы семейства $\mathcal{I}_n$ получаются разбиением отрезка $[0; 1]$ точками $\mathcal{T}_n$. Обозначим также $\mathcal{I} = \bigcup_{n \geq 1} \mathcal{I}_n$. Следующая лемма доказана в работе [4], (см. лемму 2.2).

Лемма 2.1. *Пусть последовательность $\mathcal{T}$ сильно регулярная с параметром γ и интервалы I, I_1 с общим концом α принадлежат некоторому $\mathcal{I}_{n_1}$. Тогда, если $I_k \in \mathcal{I}_{n_k}$, $k = 1, ..., m$, различные интервалы, с общим концом α, $I_k \subset I_{k-1}$ и $I \in \mathcal{I}_{n_m}$, то $m \leq C_\gamma$.*

В исследованиях общей системы Франклина важную роль играют интервалы J_n и Δ_n связанные с функцией f_n. Для определения этих интервалов напомним, что при определении слабой регулярности мы определили точки t_n^- и t_n^+ как ближайшие к t_n точки из $\mathcal{T}_{n-1}$, соответственно, слева и справа. Тогда обозначим через Δ_n интервал $(t_n^-; t_n^+)$. Известно, что функция $|f_n(x)|$ достигает наибольшего значения на $[t_n^-; t_n^+]$. Поэтому интервал Δ_n называют интервалом пика функции f_n. Определим еще точки t_n^{--} и t_n^{++} как точки из $\mathfrak{T}_{n-1}$, соответственно, ближайшие к t_n^- слева и к t_n^+ справа. Тогда, если $t_n = \tau_i^n$, то $t_n^{--} = \tau_{i-2}^n$, $t_n^- = \tau_{i-1}^n$, $t_n^+ = \tau_{i+1}^n$ и $t_n^{++} = \tau_{i+2}^n$. Из интервалов $[\tau_{i-2}^n; \tau_i^n]$, $[\tau_{i-1}^n; \tau_{i+1}^n]$, $[\tau_i^n; \tau_{i+2}^n]$ тот, который имеет наименьшую длину обозначим $I^* = [\tau_{i^*}^n; \tau_{i^*+2}^n]$. Интервал J_n один из интервалов $[\tau_{i^*}^n; \tau_{i^*+1}^n]$ и $[\tau_{i^*+1}^n; \tau_{i^*+2}^n]$ с условием, что $|J_n| = \max(|[\tau_{i^*}^n; \tau_{i^*+1}^n]|, |[\tau_{i^*+1}^n; \tau_{i^*+2}^n]|)$.

Очевидно, что если последовательность слабо регулярная, то $|J_n| \sim_\gamma |\Delta_n|$. Поэтому, ради удобства, в оценках полученных в работах [2], [5] для функций f_n мы можем J_n заменить на Δ_n. Приведем некоторые оценки для функций Франклина, доказанные в работах [2] и [5]. Во первых, существует такая постоянная C_γ, что для всех n и x выполняются

$$|f_n(x)| \le C_\gamma q^{d_n(\delta)} \frac{|\Delta_n|^{\frac{1}{2}}}{|\Delta_n| + \rho(\delta, \Delta_n) + |\delta|} \quad \text{когда} \quad x \in \delta, \tag{2.1}$$

где q здесь и далее равно $\frac{2}{3}$, а δ-любой интервал линейности функции f_n, т.е. $\delta \in \mathfrak{T}_n$.

Далее, для любого $p \in [1; \infty)$ и некоторого $\epsilon \in (0; 1)$ имеют место

$$\int_{\tau_{i-2}^n}^{\tau_{i-1}^n} |f_n(x)|^p dx \le \epsilon^p \int_{\tau_{i-1}^n}^{\tau_i^n} |f_n(x)|^p dx \quad \text{если} \quad \tau_i^n < t_n, \tag{2.2}$$

$$\int_{\tau_{i+1}^n}^{\tau_{i+2}^n} |f_n(x)|^p dx \le \epsilon^p \int_{\tau_i^n}^{\tau_{i+1}^n} |f_n(x)|^p dx \quad \text{если} \quad \tau_i^n > t_n, \tag{2.3}$$

и

$$\|f_n\|_{L_p(\Delta_n)} \sim_\gamma \|f_n\|_p \sim_\gamma |\Delta_n|^{\frac{1}{p}-\frac{1}{2}} \quad \text{когда} \quad 1 \le p \le \infty. \tag{2.4}$$

Лемма 2.2. *Для*

$$f_n^{(-1)}(x) := \int_0^x f_n(t)dt, \quad x \in [0; 1], \quad n = 0, 1, 2, ...,$$

верна оценка

$$\left|f_n^{(-1)}(x)\right| \le C_\gamma q^{d_n(\delta)} \frac{|\Delta_n|^{\frac{1}{2}}|\delta|}{|\Delta_n| + \rho(\delta, \Delta_n) + |\delta|} \quad \textit{когда} \quad x \in \delta, \tag{2.5}$$

где δ любой интервал линейности функции f_n.

Доказательство. Когда $x \in \delta$ и δ левее Δ_n оценка (2.5) следует из (2.1) и (2.2). В случае когда δ правее Δ_n, учитывая, что $\int_0^1 f_n(t)dt = 0$, $f_n^{(-1)}(x)$ можно заменить на $-\int_x^1 f_n(t)dt$ и применить (2.1) и (2.3). Лемма доказана.

Для $I \in \mathfrak{I}$ обозначим

(2.6) $$D_I = \{n : f_n \quad \text{линейна на} \quad I\} \quad \text{и} \quad D_{I,k} = \{n \in D_I : d_n(I) = k\}.$$

В работе [3] доказана следующая (см. [3] лемма 3.4.)

Лемма 2.3. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда для любого интервала $I \in \mathfrak{I}$ и любого числа k имеет место*

$$\sum_{n:n\in D_{I,k}} \frac{|\Delta_n|}{|\Delta_n| + \rho(I,\Delta_n) + |I|} \le C_\gamma(k+1).$$

3. Свойство локализации абсолютной сходимости рядов Фурье-Франклина

В этом разделе будет доказано, что абсолютная сходимость ряда Фурье-Франклина в точке x локальное свойство, т.е. если $\sum_{n=0}^{\infty} a_n f_n(t)$ ряд Фурье-Франклина функции f, то сходимость ряда $\sum_{n=0}^{\infty} |a_n f_n(x)|$ зависит от поведения функции f в некоторой окрестности точки x. Для классической системы Франклина аналогичные вопросы рассмотрены в работе [8].

Лемма 3.1. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда если $f \in L_1[0;1]$ и $f(t) = 0$, когда $t \in [0;\beta]$, то*

1) $$\sum_{n=0}^{\infty} |a_n(f) f_n(x)| \le C_{\gamma,f,x} \qquad \text{для} \quad x < \beta.$$

2) ряд $\sum_{n=0}^{\infty} |a_n(f) f_n(x)|$ равномерно сходится на $[0;\beta']$, при любом $\beta' < \beta$.

Эта лемма вытекает из следующей леммы.

Лемма 3.2. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ, $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина и $k_0 > 2$. Допустим n_0 такое, что в интервале $(x;\beta)$ есть по крайней мере k_0 точек из $\mathfrak{T}_{n_0}$. Тогда если $f \in L_1[0;1]$ и $f(t) = 0$, когда $t \in [0;\beta]$, то*

$$\sum_{n=n_0}^{\infty} |a_n f_n(x)| \le C_\gamma \frac{\|f\|_1 q^{k_0}}{(\beta - x)}, \quad \text{когда} \quad x < \beta,$$

где $a_n = a_n(f)$.

Доказательство. Рассмотрим несколько случаев.

Первый случай. $\Delta_n \subset (x;\beta)$. Применяя (2.1), получаем

$$|a_n| \le C_\gamma \frac{|\Delta_n|^{1/2}}{|\Delta_n| + \rho(\Delta_n,\beta)} q^{d_n(\beta)} \|f\|_1. \tag{3.1}$$

Поэтому в силу (2.1), имеем

$$|a_n f_n(x)| \le C_\gamma \frac{|\Delta_n| \|f\|_1 q^{d_n(\beta)+d_n(x)}}{(|\Delta_n| + \rho(\Delta_n,\beta))(|\Delta_n| + \rho(\Delta_n,x))}$$

Нетрудно заметить, что

$$\frac{|\Delta_n|}{(|\Delta_n| + \rho(\Delta_n,\beta))(|\Delta_n| + \rho(\Delta_n,x))} \le \frac{1}{\beta - x}.$$

Поэтому для $\Delta_n \subset (x;\beta)$ имеем

$$|a_n f_n(x)| \le C_\gamma \frac{\|f\|_1}{\beta - x} q^{d_n(\beta)+d_n(x)}.$$

Отсюда, учитывая, что выражение $d_n(\beta) + d_n(x)$, с условием $\Delta_n \subset (x;\beta)$, растет вместе с n, принимая только натуральные значения не меньше k_0, получаем

$$\sum_{n:\Delta_n \subset (x;\beta)} |a_n f_n(x)| \le C_\gamma \frac{\|f\|_1 q^{k_0}}{\beta - x}. \tag{3.2}$$

Второй случай. $x \in \Delta_n$ и $n \ge n_0$. Применяя (2.1) получаем (3.1). Тогда в силу (2.4), получаем

$$|a_n f_n(x)| \le C_\gamma q^{d_n(\beta)} \frac{\|f\|_1}{|\Delta_n| + \rho(\Delta_n,\beta)}.$$

Из леммы 2.1 следует, что величина $d_n(\beta)$ каждое натуральное значение может принимать не более чем C_γ раз. Поэтому из неравенства $d_n(\beta) \ge k_0$ следует

$$\sum_{n \ge n_0: x \in \Delta_n} |a_n f_n(x)| \le C_\gamma \frac{\|f\|_1 q^{k_0}}{\beta - x}. \tag{3.3}$$

Третий случай. $\Delta_n \subset [0;x)$. Используя (2.1), получаем (3.1) и вновь используя (2.1) получаем

$$|a_n f_n(x)| \le C_\gamma \frac{|\Delta_n| \|f\|_1 \cdot q^{d_n(\beta)+d_n(x)}}{(|\Delta_n| + \rho(\Delta_n,\beta))(|\Delta_n| + \rho(\Delta_n, I_{n,x}) + |I_{n,x}|)}$$

$$\le C_\gamma \frac{\|f\|_1}{(\beta - x)} \cdot \frac{|\Delta_n| q^{d_n(\beta)+d_n(x)}}{|\Delta_n| + \rho(\Delta_n, I_{n,x}) + |I_{n,x}|},$$

где $I_{n,x}$ тот интервал линейности функции f_n, которому принадлежит точка x. Интервалы $I_{n,x}$ вложены. Через I_x^j обозначим те из интервалов $I_{n,x}$, которые не повторяются, т.е. $I_x^{j+1} \subset I_x^j$ и для любого n найдется такое j, что $I_{n,x} = I_x^j$. Тогда

$$\sum_{n:\Delta_n \subset [0;x)} |a_n f_n(x)| \le \tag{3.4}$$

$$C_\gamma \frac{\|f\|_1}{\beta - x} \sum_j \sum_{m\geq 1} q^m \sum_{\substack{n\,:\,\Delta_n \subset [0,x) \\ I_{n,x}=I^j, d_n(x)=m}} \frac{|\Delta_n| q^{d_n(\beta)}}{|\Delta_n| + \rho(\Delta_n, I_x^j) + |I_x^j|}.$$

Применяя лемму 2.3 и обозначив $\xi_{j,\beta} = \min_{n:I_{n,x}=I_x^j} d_n(\beta)$, из (3.4) получим

$$\sum_{n:\Delta_n\subset[0;x)} |a_n f_n(x)| \leq C_\gamma \frac{\|f\|_1}{\beta - x} \sum_j q^{\xi_{j,\beta}} \sum_{m\geq 1} q^m (m+1) \leq C_\gamma \frac{\|f\|_1}{\beta - x} \sum_j q^{\xi_{j,\beta}}. \tag{3.5}$$

Заметим, что числа $\xi_{j,\beta}$ принимают натуральные значения не меньше k_0. Из леммы 2.1, следует, что числа $\xi_{j,\beta}$ каждое натуральное значение могут принимать не более C_γ раз. Поэтому из (3.5) получим

$$\sum_{n\geq n_0, \Delta_n\subset[0;x)} |a_n f_n(x)| \leq \frac{C_\gamma \|f\|_1 q^{k_0}}{\beta - x}. \tag{3.6}$$

Четвертый случай. $\beta \in \Delta_n$. В этом случае, с учетом (2.4) и (2.1), получим

$$|a_n f_n(x)| \leq C_\gamma \frac{\|f\|_1 \cdot q^{d_n(x)}}{|\Delta_n| + \rho(\Delta_n, x)}. \tag{3.7}$$

Из леммы 2.1 следует, что $d_n(x)$ каждое натуральное значение может принимать не более C_γ раз. Поэтому из (3.7), с учетом того, что $d_n(x) \geq k_0$, следует

$$\sum_{\beta\in\Delta_n} |a_n f_n(x)| \leq C_\gamma \frac{\|f\|_1 q^{k_0}}{\beta - x}. \tag{3.8}$$

Пятый случай. $\Delta_n \subset (\beta; 1)$. Напомним, что $\Delta_n = (t_n^-; t_n^+)$. Обозначим $V_n^- = (\beta; t_n^-)$, $V_n^+ = (t_n^+; 1)$. Тогда

$$a_n = \int_{V_n^-} f(t) f_n(t)dt + \int_{\Delta_n} f(t) f_n(t)dt + \int_{V_n^+} f(t) f_n(t)dt =: a_{n,1} + a_{n,2} + a_{n,3}$$

и

$$\sum_{\Delta_n\subset(\beta;1)} |a_n f_n(x)| \leq \sum_{i=1}^{3} \sum_{\Delta_n\subset(\beta;1)} |a_{n,i} f_n(x)|.$$

A) Оценим $\sum_{\Delta_n\subset(\beta;1)} |a_{n,2} f_n(x)|$. Из неравенства (см. (2.4)) $|a_{n,2}| \leq C_\gamma |\Delta_n|^{-1/2} \int_{\Delta_n} |f(t)|dt$ и с учетом (2.1), получаем

$$|a_{n,2} f_n(x)| \leq C_\gamma \frac{q^{d_n(x)}}{|\Delta_n| + \rho(\Delta_n, x)} \int_{\Delta_n} |f_n(t)|dt \leq C_\gamma \frac{q^{d_n(x)}}{\beta - x} \int_{\Delta_n} |f_n(t)|dt.$$

Следовательно

$$\sum_{\Delta_n\subset(\beta;1)} |a_{n2} f_n(x)| \leq \frac{C_\gamma}{\beta - x} \sum_{k\geq k_0} q^k \sum_{n:d_n(x)=k} \int_{\Delta_n} |f_n(t)|dt. \tag{3.9}$$

Из леммы 2.1 следует, что количество вложенных интервалов Δ_n с условием $d_n(x) = k$ не превосходит C_γ. Поэтому из (3.9), учитывая, что $d_n(x) \geq k_0$, имеем

$$\sum_{\Delta_n \subset (\beta;1)} |a_{n2} f_n(x)| \leq \frac{C_\gamma}{\beta - x} \sum_{k \geq k_0} q^k \|f\|_1 \leq \frac{C_\gamma \|f\|_1 q^{k_0}}{\beta - x}. \tag{3.10}$$

B) Оценим $\sum_{\Delta_n \subset (\beta;1)} |a_{n3} f_n(x)|$. Напомним, что t_n^+ правая соседная с t_n точка из π_n. Нетрудно заметить, что если $n_1 < n_2 < \ldots$ такие числа, что $d_{n_m}(x) = k$, то $t_{n_{m+1}}^+ < t_{n_m}^+$ и $d_{n_m}(t_{n_l}^+) \geq m - l$, когда $m > l$. Положим также $t_{n_0} = 1$. Тогда в силу (2.1) получим

$$|a_{n_m,3}| \leq \sum_{l:l \leq m} \int_{t_{n_l}^+}^{t_{n_{l-1}}^+} |f(x)||f_{n_m}(x)|dx \leq C_\gamma \sum_{l:l \leq m} |\Delta_{n_m}|^{-1/2} q^{m-l} \int_{t_{n_l}^+}^{t_{n_{l-1}}^+} |f(x)|dx.$$

С учетом (2.1), для n_m с условиями $\Delta_{n_m} \subset (\beta;1)$ и $d_{n_m}(x) = k$ имеем

$$|a_{n_m,3} f_{n_m}(x)| \leq C_\gamma \frac{q^k}{\beta - x} \sum_{l:l \leq m} q^{m-l} \int_{t_{n_l}^+}^{t_{n_{l-1}}^+} |f(x)|dx.$$

Следовательно

$$\sum_{m:d_{n_m}(x)=k} |a_{n_m,3} f_{n_m}(x)| \leq C_\gamma \frac{q^k}{\beta - x} \sum_{m:d_{n_m}(x)=k} \sum_{l:l \leq m} q^{m-l} \int_{t_{n_l}^+}^{t_{n_{l-1}}^+} |f(x)|dx$$

$$\leq C_\gamma \frac{q^k}{\beta - x} \sum_l \int_{t_{n_l}^+}^{t_{n_{l-1}}^+} |f(x)|dx \leq C_\gamma \frac{q^k \|f\|_1}{\beta - x}.$$

Последнее неравенство суммируя по k и учитывая, что $k \geq k_0$, получим

$$\sum_{n:\Delta_n \subset (\beta;1)} |a_{n_m,3} f_{n_m}(x)| \leq C_\gamma \frac{\|f\|_1 q^{k_0}}{\beta - x}. \tag{3.11}$$

C) Оценим $\sum_{\Delta_n \subset (\beta;1)} |a_{n,1} f_n(x)|$. Для фиксированного натурального i обозначим

$$\mathcal{N}_i = \{n : \Delta_n \subset (\beta;1), \text{в } (x;\beta) \text{ есть } i \text{ интервалов линейности функции } f_n\},$$

и пусть Λ_i тот из этих интервалов линейности, который ближе β. При фиксированных i, k и для n, с условием $d_n(\Lambda_i) = k$ обозначим через $L_{n,j}$, $j = 1, 2, \ldots, k$, интервалы линейности функции f_n, пересекающиеся с V_n^-, причем $\max L_{n,j} = \min L_{n,j+1}$. Тогда

$$|a_{n,1}| \leq \sum_{j=1}^{k} \int_{L_{n,j}} |f_n(t)||f(t)|dt \leq C_\gamma \sum_{j=1}^{k} \frac{|\Delta_n|^{1/2} q^{k-j}}{|\Delta_n| + \rho(\Delta_n, L_{n,j}) + |L_{n,j}|} \int_{L_{n,j}} |f(t)|dt.$$

Следовательно (см. (2.1))

$$|a_{n,1} f_n(x)| \leq C_\gamma \frac{q^{k+i}}{|\Delta_n| + \rho(\Delta_n, x)} \sum_{j=1}^{k} \frac{|\Delta_n| q^{k-j}}{|\Delta_n| + \rho(\Delta_n, L_{n,j}) + |L_{n,j}|} \int_{L_{n,j}} |f(x)|dx.$$

Поэтому

$$\sum_{n:d_n(\Lambda_i)=k} |a_{n,1} f_n(x)|$$

$$\leq C_\gamma \frac{q^{k+i}}{|\Delta_n| + \rho(\Delta_n, x)} \sum_{n:d_n(\Lambda_i)=k} \sum_{j=1}^{k} \frac{|\Delta_n| q^{k-j}}{|\Delta_n| + \rho(\Delta_n, L_{n,j}) + |L_{n,j}|} \int_{L_{n,j}} |f(x)| dx.$$

Заметим, что при фиксированном j интервалы $L_{n,j}$ либо не пересекаются, либо имеют общий левый конец. Через $L^*_{j,p}$, $p = 1, 2, ...$, обозначим максимальные из них. Пусть J такой интервал, что для некоторых n и j имеет место $J = L_{n,j}$. Тогда $d_n(J) = k - j$. Из леммы 2.3 имеем

$$\sum_{n:d_n(J)=k-j} \frac{|\Delta_n|}{|\Delta_n| + \rho(\Delta_n, J) + |J|} \leq C_\gamma(k - j + 1).$$

Заметим также, что если $L_{n_1,j} \supset L_{n_2,j} \supset ... \supset L_{n_{m_j},j}$, то интервалы линейности $L_{n_l,j-1}$, $l = 1, 2, ...m_j$ неизменны. Поэтому в силу леммы 2.1 имеем $m_j \leq C_\gamma$. Поэтому

(3.12)
$$\sum_{k} \sum_{n:d_n(\Lambda_i)=k} |a_n^1 f_n(x)|$$

$$\leq \frac{C_\gamma}{\beta - x} \sum_{k} q^{k+i} \sum_{j=1}^{k} q^{k-j}(k - j + 1) \sum_{p} \int_{L^*_{j,p}} |f(x)| dx \leq \frac{C_\gamma}{\beta - x} q^i \|f\|_1.$$

Суммируя (3.12) по i и учитывая, что $i \geq k_0$ получим

(3.13)
$$\sum_{n:\Delta_n \subset (\beta;1)} |a_n^1 f_n(x)| \leq \frac{C_\gamma q^{k_0} \|f\|_1}{\beta - x}.$$

Из (3.10), (3.11), (3.13) следует

(3.14)
$$\sum_{n:\Delta_n \subset (\beta;1)} |a_n f_n(x)| \leq \frac{C_\gamma \|f\|_1}{\beta - x}.$$

Из (3.2), (3.3), (3.6), (3.8) и (3.14) следует утверждение леммы.

Аналогично доказываются следующие утверждения.

Лемма 3.3. *Пусть последовательность $\mathcal{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда если $f \in L_1[0;1]$ и $f(t) = 0$, когда $t \in [\alpha; 1]$, то*

1) $\sum_{n=0}^{\infty} |a_n(f) f_n(x)| \leq C_{\gamma,f,x}$, когда $x > \alpha$.

2) ряд $\sum_{n=0}^{\infty} |a_n f_n(x)|$ равномерно сходится на $[\alpha'; 1]$, при любом $\alpha' > \alpha$.

Лемма 3.4. *Пусть последовательность $\mathcal{T}$ сильно регулярная с параметром γ, $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина и $k_0 > 2$. Допустим n_0 такое, что в интервалс $(\alpha; x)$ есть по крайней мере k_0 точек из $\mathcal{T}_{n_0}$. Тогда если $f \in L_1[0;1]$ и $f(t) = 0$, когда $t \in [\alpha; 1]$, то*

$$\sum_{n=n_0}^{\infty} |a_n f_n(x)| \leq C_\gamma \frac{\|f\|_1 q^{k_0}}{(\alpha - x)}, \quad \text{когда} \quad x > \alpha,$$

где $a_n = a_n(f)$.

Из лемм 3.1, 3.3 получаются теоремы 3.1 и 3.2.

Теорема 3.1. *Пусть последовательность $\mathcal{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда если $f \in L_1[0;1]$ и $f(x) = 0$, когда $x \in (\alpha; \beta)$, то*

1)$\sum_{n=0}^{\infty} |a_n(f) f_n(x)| \leq C_{\gamma.f,x}$, когда $x \in (\alpha; \beta)$,

2) ряд $\sum_{n=0}^{\infty} |a_n(f) f_n(x)|$ равномерно сходится на $[\alpha'; \beta'] \subset (\alpha; \beta)$.

Теорема 3.2. *Пусть последовательность $\mathcal{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда если $f, g \in L_1[0;1]$ и $f(x) = g(x)$, когда $x \in (\alpha; \beta)$, то*

1)$\sum_{n=0}^{\infty} |a_n(f) - a_n(g)||f_n(x)| \leq C_{\gamma.f,x}$, когда $x \in (\alpha; \beta)$,

2) ряд $\sum_{n=0}^{\infty} |a_n(f) - a_n(g)||f_n(x)|$ равномерно сходится на $[\alpha'; \beta'] \subset (\alpha; \beta)$, т.е. ряды Фурье функций f и g равномерно абсолютно равносходятся на $[\alpha'; \beta']$.

Теорема 3.1 указывает на то, что абсолютная сходимость ряда Фурье-Франклина интегрируемой функции f в точке x зависит от поведения функции f в некоторой окрестности точки x. Следующая теорема указывает на то, что абсолютная сходимость ряда Фурье-Франклина интегрируемой функции f в точке x не зависит от тех функций f_n, интервалы пика которых не находятся в некоторой окрестности точки x.

Теорема 3.3. *Пусть последовательность $\mathcal{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда если f интегрируемая функция и $x \in (\alpha; \beta)$, то*

$$\sum_{\Delta_n \not\subset (\alpha;\beta)} |a_n f_n(x)| < \infty,$$

где $a_n = a_n(f)$.

Доказательство. В силу теоремы 3.2, без ограничения общности можно предположить, что $f(t) = 0$, когда $t \notin (\alpha; \beta)$. Количество тех n, для которых $x \in \Delta_n$

и $\Delta_n \not\subset (\alpha;\beta)$ ограничено. Поэтому достаточно доказать неравенства

$$\sum_{\alpha\in\Delta_n, x\notin\Delta_n} |a_n f_n(x)| < \infty, \qquad \sum_{\beta\in\Delta_n, x\notin\Delta_n} |a_n f_n(x)| < \infty, \tag{3.15}$$

$$\sum_{\Delta_n\subset(0;\alpha)} |a_n f_n(x)| < \infty, \qquad \sum_{\Delta_n\subset(\beta;1)} |a_n f_n(x)| < \infty. \tag{3.16}$$

Неравенства (3.15) доказываются как четвертый случай леммы 3.2. Неравенства (3.16) доказываются как третий случай леммы 3.2. Теорема 3.3 доказана.

4. Абсолютная сходимость рядов Фурье-Франклина функций ограниченной вариации

Теорема 4.1. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ, $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина и $f \in BV$. Тогда*

$$\sum_{n=0}^{\infty} |a_n| \|f_n\|_1 \le C_\gamma V(f).$$

Доказательство. Через $L_{n,i}$, $i = \pm1, \pm2, \ldots$, обозначим тот интервал линейности функции f_n, для которого $d_n(L_{n,i}) = |i|$ и интервал $L_{n,i}$ находится правее Δ_n, если $i > 0$ и левее Δ_n, если $i < 0$. Обозначим также $L_{n,0} = \Delta_n$. Тогда $[0;1] = \cup_i L_{n,i}$ и интервалы $L_{n,i}$ взаимно не пересекаются. Поэтому

$$|a_n| = \left|\int_0^1 f(t) f_n(t) dt\right| = \left|\int_0^1 f_n^{(-1)}(t) df(t)\right| = \left|\sum_i \int_{L_{n,i}} f_n^{(-1)}(t) df(t)\right|. \tag{4.1}$$

С учетом леммы 2.2, из (4.1) получим

$$\begin{aligned} |a_n| &\le \sum_i \max_{t\in L_{n,i}} |f^{(-1)}(t)| V_{L_{n,i}}(f) \\ &\le C_\gamma \sum_i \frac{|\Delta_n|^{1/2} |L_{n,i}| q^{d_n(L_{n,i})}}{|\Delta_n| + \rho(\Delta_n, L_{n,i}) + |L_{n,i}|} V_{L_{n,i}}(f). \end{aligned} \tag{4.2}$$

Поскольку $\|f_n\|_1 \le C_\gamma |\Delta_n|^{1/2}$, то из (4.2) имеем

$$\sum_{n=0}^{\infty} |a_n| \|f_n\|_1 \le C_\gamma \sum_n \sum_i \frac{|\Delta_n| |L_{n,i}| q^{d_n(L_{n,i})}}{|\Delta_n| + \rho(\Delta_n, L_{n,i}) + |L_{n,i}|} V_{L_{n,i}}(f). \tag{4.3}$$

Из обозначений (2.6) и из (4.3) следует

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| \|f_n\|_1 &\le C_\gamma \sum_{I\in\mathfrak{I}} V_I(f) |I| \sum_{n\in D_I} \frac{|\Delta_n| q^{d_n(I)}}{|\Delta_n| + \rho(\Delta_n, I) + |I|} \\ &= C_\gamma \sum_{I\in\mathfrak{I}} V_I(f) |I| \sum_k q^k \sum_{n\in D_{I,k}} \frac{|\Delta_n|}{|\Delta_n| + \rho(\Delta_n, I) + |I|}. \end{aligned} \tag{4.4}$$

Из леммы 2.3 имеем

$$\sum_{n\in D_{I,k}} \frac{|\Delta_n|}{|\Delta_n| + \rho(\Delta_n, I) + |I|} \le C_\gamma(k+1).$$

Поэтому из (4.4) получим

$$\sum_{n=0}^{\infty} |a_n| \|f_n\|_1 \le C_\gamma \sum_{I\in\mathfrak{I}} V_I(f)|I|. \tag{4.5}$$

Рангом интервала $I \in \mathfrak{I}$ назовем количество интервалов из $\mathfrak{I}$, которые содержат интервал I и различны между собой. Ясно, что ранг интервала $[0;1]$ равен 1. Нетрудно заметить, что любой интервал ранга k есть объединение двух непересекающихся интервалов ранга $k+1$. Следовательно, если обозначить

$$R_k = \{I \in \mathfrak{I} : \text{ранг интервала} \quad I \quad \text{равен} \quad k\},$$

то имеем

$$[0;1] = \bigcup_{I\in R_k} I, \quad \text{для любого} \quad k.$$

С учетом сильной регулярности $\mathfrak{T}$ имеем также

$$|I| \le \left(\frac{\gamma}{\gamma+1}\right)^k \quad \text{когда} \quad I \in R_k. \tag{4.6}$$

Из (4.5), (4.6) получим

$$\sum_{n=0}^{\infty} |a_n| \|f_n\|_1 \le C_\gamma \sum_k \sum_{I\in R_k} |I| V_I(f)$$

$$\le C_\gamma \sum_k \left(\frac{\gamma}{\gamma+1}\right)^k \sum_{I\in R_k} V_I(f) \le C_\gamma V(f).$$

Теорема 4.1 доказана.

В работе [10] для классической системы Франклина доказано более сильное утверждение. А именно, доказано что если $f \in BV$ и a_n-коэффициенты Фурье функции f по классической системе Франклина, тогда $\sum_{n=0}^{\infty} |a_n| \le C \cdot V(f)$.

Следствие 4.1. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ, $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина и $f \in BV$. Тогда ряд Фурье-Франклина функции f почти всюду абсолютно сходится.*

Отсюда и из теоремы 3.2 получается

Следствие 4.2. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ, $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина и интегрируемая на $[0;1]$ функция f на отрезке $[a;b]$ имеет ограниченную вариацию. Тогда ряд Фурье-Франклина функции f на $(a;b)$ почти всюду абсолютно сходится.*

Замечание 4.1. *В следствии 4.1 нельзя утверждать не только всюду абсолютную сходимость, но и сходимость всюду.*

В работе [8] доказано, что если интегрируемая функция в точке x_0 имеет разрыв первого рода, то ее ряд Фурье по классической системе Франклина в точке x_0 расходится. В общем случае аналогичное утверждение нам не удалось доказать. Однако, верно следующее утверждение.

Теорема 4.2. *Пусть последовательность $\mathcal{T}$ допустимая и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда для почти всех x, если $f \in L_1[0;1]$ и функция f в точке x имеет разрыв первого рода, то ряд Фурье-Франклина функции f в точке x ограничено расходится.*

Для доказательства этой теоремы нам нужна следующая

Лемма 4.1. *Пусть последовательность $\mathcal{T}$ допустимая и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда для почти всех $x \in [0;1]$ выполняется*

$$\limsup_{n\to\infty} \left| \int_0^x f_n(t)dt \right| |f_n(x)| > c_1. \tag{4.7}$$

Доказательство. В работе [5] доказано, что существуют постоянные c и C, такие, что для любого p, $1 \le p \le \infty$, выполняется следующая цепочка неравенств:

$$c|J_n|^{1/p-1/2} \le \|f_n\|_{L_p(J_n)} \le \|f_n\|_p \le C|J_n|^{1/p-1/2}. \tag{4.8}$$

Определение интервала J_n приведено перед (2.1). Однако, здесь важно то, что J_n один из интервалов $(t_n^{--};t_n^-)$, $(t_n^-;t_n)$, $(t_n;t_n^+)$, $(t_n^+;t_n^{++})$, на которых функция f_n линейная, и в концах интервала J_n функция f_n принимает значения разных знаков. Допустим $J_n = (\alpha;\beta)$ и $f_n(t) = a_n(t - z_n)$ когда $t \in [\alpha;\beta]$, где $z_n \in J_n$. Тогда из первого неравенства в (4.8) получим

$$|a_n| \ge c|J_n|^{-3/2}. \tag{4.9}$$

Очевидно, что функция

$$\phi_n(x) = \int_{z_n}^x f_n(t)dt, \quad x \in J_n \tag{4.10}$$

не меняет знак на J_n и

$$|\phi_n(x)| = \frac{1}{2}|a_n|(x - z_n)^2.$$

Следовательно

$$\mu\left\{x \in J_n : |\phi_n(x)| \le \frac{|a_n|}{72}|J_n|^2\right\} \ge \frac{|J_n|}{6} \tag{4.11}$$

и

$$\mu\left\{x \in J_n : |\phi_n(x)| \ge \frac{|a_n|}{18}|J_n|^2\right\} \ge \frac{|J_n|}{6} \tag{4.12}$$

Из (4.9)—(4.12) следует, что для функции

$$\psi_n(x) = \int_0^x f_n(t)dt = \int_0^{x_n} f_n(t)dt + \phi_n(x)$$

выполняется

$$\mu\left\{x \in J_n : |\psi_n(x)| > \frac{c}{50}|J_n|^{1/2}\right\} \ge \frac{|J_n|}{6}. \tag{4.13}$$

Из линейности функции f_n на J_n и первого неравенства в (4.8) следует

$$\mu\left\{t \in J_n : |f_n(t)| \ge \frac{c}{10}|J_n|^{-1/2}\right\} \ge \frac{9}{10}|J_n|. \tag{4.14}$$

Из (4.13) и (4.14) получим

$$\left\{x \in J_n : |\psi_n(x) f_n(x)| \ge \frac{c^2}{500}\right\} \ge \frac{|J_n|}{15}. \tag{4.15}$$

Нетрудно заметить, что

$$\mu\left(\limsup J_n\right) = \mu\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} J_n\right) = 1.$$

Поэтому из (4.15) следует утверждение леммы.

Доказательство теоремы 4.2. Пусть для точки x выполняется (4.7) и интегрируемая функция f в точке x имеет разрыв первого рода. Обозначим

$$\chi(t) = \begin{cases} 0, & \text{когда} \quad t < x, \\ f(x) - f(x-0), & \text{когда} \quad t = x, \\ f(x+0) - f(x-0), & \text{когда} \quad t > x. \end{cases}$$

Ясно, что интегрируемая функция $\varphi(t) = f(t) - \chi(t)$ непрерывна в точке x и ограничена в некоторой окрестности точки x. Тогда в силу (1.1) частичные суммы ряда Фурье функции φ ограничены в некоторой окрестности точки x и сходятся в точке x. Учитывая, что $\int_0^1 f_n(t)dt = 0$, когда $n > 0$, для коэффициентов $a_n(\chi)$ получим

$$a_n(\chi) = d\int_0^x f_n(t)dt \quad \text{для} \quad n > 0,$$

где $d = f(x+0) - f(x-0)$. Применяя лемму 4.2, получим

$$\limsup |a_n(\chi) f_n(x)| > d \cdot c_1.$$

Следовательно, ряд Фурье-Франклина функции f в точке x расходится. Теорема 4.2 доказана.

5. Представление функций абсолютно сходящимися рядами

В этом разделе доказываются теоремы о представлении измеримых функций п.в. абсолютно сходящимися рядами по общей системе Франклина, соответствующей сильно регулярному разбиению.

Лемма 5.1. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Допустим $I = [\alpha;\beta] \in \mathfrak{I}_{n_0}$, и $supp\varphi \subset I$. Тогда, если $I \subset \Delta \in \mathfrak{I}$, то*

$$\sum_{n:n\in D_\Delta} |a_n| \|f_n\|_1 \le C_\gamma \|\varphi\|_1,$$

где $D_\Delta = \{n : \Delta$ – интервал линейности функции $f_n\}$ и $a_n = a_n(\varphi)$.

Доказательство. Если $I \subset \Delta \in \mathfrak{I}$ и f_n линейна на Δ, то (см. (2.1))

$$|a_n| \le C_\gamma \|\varphi\|_1 \cdot \frac{|\Delta_n|^{1/2}}{|\Delta_n| + \rho(\Delta_n, \Delta) + |\Delta|} \cdot q^{d_n(\Delta)},$$

и поэтому

$$|a_n| \|f_n\|_1 \le C_\gamma \|\varphi\|_1 \cdot \frac{|\Delta_n|}{|\Delta_n| + \rho(\Delta_n, \Delta) + |\Delta|} \cdot q^{d_n(\Delta)}.$$

Обозначим $D_{\Delta,k} = \{n \in D_\Delta : d_n(\Delta) = k\}$. С применением леммы 2.3, суммируя последнее неравенство, получим

$$\sum_{n:n\in D_\Delta} |a_n| \|f_n\|_1 \le C_\gamma \|\varphi\|_1 \sum_{n\in D_\Delta} \frac{|\Delta_n|}{|\Delta_n| + \rho(\Delta_n, \Delta) + |\Delta|} q^{d_n(\Delta)}$$

$$= C_\gamma \|\varphi\|_1 \sum_k q^k \sum_{n\in D_{\Delta,k}} \frac{|\Delta_n|}{|\Delta_n| + \rho(\Delta_n, \Delta) + |\Delta|} \le C_\gamma \|\varphi\|_1 \sum_k q^k (k+1) \le C_\gamma \|\varphi\|_1.$$

Лемма 5.1 доказана.

Лемма 5.2. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда для произвольных $[a;b] \in \mathfrak{I}_{n_0}$ и $\varepsilon \in (0;0,1)$ существует полином $\phi(t) = \sum_{n=n_0}^{M} a_n f_n(t)$, такой что*

1. $\phi(t) = 0$, *когда* $t \notin (a;b)$,
2. $\phi(t) = 1$, *когда* $t \in E \subset [a;b]$ *и* $\mu(E) > (1-\varepsilon)(b-a)$,
3. $\int_0^1 \sum_{n=n_0}^{M} |a_n f_n(t)| dt \le C_\gamma (b-a) \ln \varepsilon^{-1}$,
4. $|\phi(t)| \le \frac{C_\gamma}{\varepsilon}$.

Доказательство. Через $I_{a,i}$, $i = 0,1,2,...$, обозначим интервалы из $\mathfrak{I}$ с левым концом a, причем $[a;b] = I_{a,0}$, $I_{a,i+1} \subset I_{a,i}$. Аналогично, через $I_{b,j}$, $j = 0,1,2,...$, обозначим интервалы из $\mathcal{T}$ с правым концом b. Учитывая сильную регулярность последовательности $\mathcal{T}$, имеем

$$(5.1)\qquad \frac{1}{\gamma+1}|I_{a,i}| \le |I_{a,i+1}| \le \frac{\gamma}{\gamma+1}|I_{a,i}| \quad \text{и} \quad \frac{1}{\gamma+1}|I_{b,i}| \le |I_{b,i+1}| \le \frac{\gamma}{\gamma+1}|I_{b,i}|.$$

Пусть i_0, j_0 такие, что

$$(5.2)\qquad |I_{a,i_0}| < \frac{\varepsilon}{2}(b-a), \quad |I_{b,j_0}| < \frac{\varepsilon}{2}(b-a)$$

и

$$(5.3)\qquad |I_{a,i_0-1}| \ge \frac{\varepsilon}{2}(b-a), \quad |I_{b,j_0-1}| \ge \frac{\varepsilon}{2}(b-a).$$

Из (5.1) и (5.3) следует, что

$$\left(\frac{\gamma}{\gamma+1}\right)^{i_0-1} \ge \frac{\varepsilon}{2}.$$

Поэтому

$$(5.4)\qquad i_0 \le C_\gamma \ln \varepsilon^{-1}.$$

Аналогично

$$(5.5)\qquad j_0 \le C_\gamma \ln \varepsilon^{-1}.$$

Пусть a' правый конец интервала I_{a,i_0}, а b'-левый конец интервала I_{b,j_0}, т.е. $I_{a,i_0} = [a; a']$, $I_{b,j_0} = [b', b]$. Обозначим также через a'', b'', соответственно, правый и левый концы интервалов I_{a,i_0+1} и I_{b,j_0+1}, т.е. a''-точка из $\mathcal{T} \cap I_{a,i_0}$ с наименьшим индексом и b''-точка из $\mathcal{T} \cap I_{b,j_0}$ с наименьшим индексом.

Обозначим

$$(5.6)\qquad \varphi_a(t) = \begin{cases} 1, & \text{когда} \quad t = a'', \\ 0, & \text{когда} \quad t \notin (a; a'), \\ \text{линейная на интервалах} & [a; a''], [a''; a']; \end{cases}$$

$$(5.7)\qquad \varphi_b(t) = \begin{cases} 1, & \text{когда} \quad t = b'', \\ 0, & \text{когда} \quad t \notin (b'; b), \\ \text{линейная на интервалах} & [b'; b''], [b''; b]; \end{cases}$$

$$(5.8)\qquad \varphi(t) = \begin{cases} 0, & \text{когда} \quad t \in [0; a] \bigcup [b; 1] \\ 1 & \text{когда} \quad t \in [a'; b'] \\ \text{линейная на интервалах} & [a; a'], [b'; b]. \end{cases}$$

Положим

$$(5.9)\qquad \phi(t) = \varphi(t) - \alpha\varphi_a(t) - \beta\varphi_b(t),$$

где числа α и β определяются из условий

$$(5.10)\qquad \int_0^1 \phi(t)(ct+d)dt = 0 \quad \text{для любых} \quad c, d.$$

Из (5.10) следует, что $a_n(\phi) = 0$, когда $n < n_0$, а из (5.6)-(5.9) следует, что $a_n(\phi) = 0$, когда $a'', b'' \in \mathfrak{T}_n$. Поэтому $\phi(t)$ имеет вид $\sum_{n=n_0}^{M} a_n f_n(t)$, где

$$M = \min\{n : a'' \in \mathfrak{T}_n, b'' \in \mathfrak{T}_n\}$$

Из (5.6)-(5.9) и (5.2) следуют пункты 1 и 2 леммы.

Для доказательства пунктов 3 и 4, сначала оценим α и β. Для выполнения (5.10) достаточно выполнение соотношений

$$\int_0^1 \phi(t)dt = 0 \quad \text{и} \quad \int_0^1 \phi(t)\omega(t)dt, \tag{5.11}$$

где $\omega(t) = t - \frac{a+b}{2}$. Равенства (5.11) равносильны системе

$$\begin{cases} h_{11}\alpha + h_{12}\beta = h_1, \\ h_{21}\alpha + h_{22}\beta = h_2, \end{cases}$$

где

$$h_{11} = \int_0^1 \varphi_a(t)dt, \quad h_{12} = \int_0^1 \varphi_b(t)dt, \quad h_1 = \int_0^1 \varphi(t)dt$$

и

$$h_{21} = \int_0^1 \varphi_a(t)\omega(t)dt, \quad h_{22} = \int_0^1 \varphi_b(t)\omega(t)dt, \quad h_2 = \int_0^1 \varphi(t)\omega(t)dt.$$

Нетрудно заметить, что

$$h_{11}, h_{12} \sim_\gamma \varepsilon(b-a), \quad h_1 \sim b-a,$$

$$-h_{21}, h_{22} \sim_\gamma \varepsilon(b-a)^2, \quad h_2 \sim (b-a)^2.$$

Поэтому, учитывая, что

$$\alpha = \frac{h_1 h_{22} - h_2 h_{12}}{h_{11}h_{22} - h_{21}h_{12}} \quad \text{и} \quad \beta = \frac{h_2 h_{11} - h_1 h_{21}}{h_{11}h_{22} - h_{21}h_{12}},$$

получаем

$$\alpha \sim_\gamma \frac{1}{\varepsilon}, \quad \beta \sim_\gamma \frac{1}{\varepsilon}. \tag{5.12}$$

Из (5.12), (5.6)-(5.9) следует пункт 4 леммы. Оценим $\sum_{n=n_0}^{M} \int_0^1 |a_n f_n(t)|dt$. Ясно, что $a_n = a_n(\phi) = a_n(\varphi) - \alpha a_n(\varphi_a) - \beta a_n(\varphi_b)$.

Применяя лемму 5.1, с учетом (5.2), получим

$$\sum_{n=n_0}^{M} \int_0^1 |a_n(\varphi_a) f_n(t)|dt = \sum_{i=0}^{i_0} \sum_{n: n \in D_{I_{a,i}}} \int_0^1 |a_n(\varphi_a) f_n(t)|dt \le (i_0+1)\|\varphi_a\|_1 \tag{5.13}$$

Из (5.13), с учетом (5.6), (5.4) и (5.12), следует

$$\alpha \sum_{n=n_0}^{M} \int_0^1 |a_n(\varphi_a) f_n(t)|dt \le C_\gamma (b-a) \ln \varepsilon^{-1}. \tag{5.14}$$

Аналогично, применяя (5.7), (5.5) и (5.12), получим

$$\beta \sum_{n=n_0}^{M} \int_0^1 |a_n(\varphi_b) f_n(t)| dt \le C_\gamma (b-a) \ln \varepsilon^{-1}. \tag{5.15}$$

Для оценки $\sum_{n=n_0}^{M} \int_0^1 |a_n f_n(t)| dt$ нам остается оценить $\sum_{n=n_0}^{M} \int_0^1 |a_n(\varphi) f_n(t)| dt$. Пусть $I_{a,i} = [a; b_i]$, $i = 0, 1, ..., i_0 + 1$, и $I_{b,j} = [a_j; b]$, $j = 0, 1, ..., j_0 + 1$. Ясно, что $a' = b_{i_0}$, $a'' = b_{i_0+1}$, $b' = a_{j_0}$ и $b'' = a_{j_0+1}$. Обозначим

$$n_i = \min\{n : b_i \in \mathcal{T}_n\}, \quad m_j = \min\{n : a_j \in \mathcal{T}_n\},$$

т.е. $b_i = t_{n_i}$, $a_j = t_{m_j}$. Далее, положим

$$L(i) = \{n : n_i < n \le n_{i+1}\}, \quad R(j) = \{n : m_j < n \le m_{j+1}\}, \quad S(i,j) = L(i) \cap R(j)$$

и

$$\phi_{i,j}(t) = \begin{cases} 0, \text{когда} & t \notin (a;b), \\ 1, \text{когда} & t \in [b_i; a_j], \\ \text{линейная на отрезках} & [a; b_i], [a_j; b]. \end{cases}$$

Из определения функций f_n следует, что

$$\int_0^1 f_n(t) \phi_{i,j}(t) dt = 0, \quad \text{если} \quad n \in S(i,j).$$

Поэтому, для $n \in S(i,j)$

$$a_n(\varphi) = \int_0^1 \varphi(t) f_n(t) dt = \int_0^1 (\varphi(t) - \phi_{i,j}(t)) f_n(t) dt = A_i(f_n) + B_j(f_n), \tag{5.16}$$

где

$$A_i(f_n) = \int_a^{b_i} (\varphi(t) - \phi_{i,j}(t)) f_n(t) dt, \tag{5.17}$$

$$B_j(f_n) = \int_{a_j}^{b} (\varphi(t) - \phi_{i,j}(t)) f_n(t) dt. \tag{5.18}$$

Применяя лемму 5.1, получим

$$\sum_{n \in S(i,j)} |A_i(f_n)| \|f_n\|_1 \le C_\gamma (b_i - a) \tag{5.19}$$

$$\sum_{n \in S(i,j)} |B_j(f_n)| \|f_n\|_1 \le C_\gamma (b - a_j). \tag{5.20}$$

Очевидно, что если $n \in [n_0; M]$, то $n \in S(i,j)$, для некоторых i, j и количество непустых $S(i,j)$ не превосходит $i_0 + j_0 + 2$. Поэтому из (5.16)-(5.20), (5.4), (5.5) имеем

$$\sum_{n=n_0}^{M} |a_n(\varphi)| \|f_n\|_1 \le C_\gamma (b-a) \ln \varepsilon^{-1}. \tag{5.21}$$

Из (5.14), (5.15), (5.21), (5.9) следует пункт 3 леммы. Лемма доказана.

Лемма 5.3. *Пусть последовательность $\mathcal{T}$ сильно регулярная с параметром γ, $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина и $a, b \in \mathcal{T}$, $a < b$. Тогда для любых $\varepsilon > 0$, $\delta > 0$ и натурального N существует полином $\phi(t) = \sum_{n=N+1}^{M} a_n f_n(t)$ по системе $\{f_n(x)\}_{n=0}^{\infty}$, такой что*

1. *$\phi(t) = 1$, когда $x \in E$. где E-объединение интервалов из $\mathcal{T}$ и $\mu(E) > (1-\varepsilon)(b-a)$,*
2. *$\phi(t) = 0$ когда $x \notin [a;b]$,*
3. *$|\phi(t)| \le \frac{C_\gamma}{\varepsilon}$, для любого t,*
4. *$\sum_n \int_0^1 |a_n f_n(t)|dt < C_\gamma(b-a)\ln\varepsilon^{-1}$,*
5. *$\sum_n |a_n f_n(t)| < \delta$, когда $t \notin [a-\delta; b+\delta]$.*

Доказательство. Напомним, что $\pi_n = \{\tau_i^n : \tau_i^n \le \tau_{i+1}^n, 0 \le i \le n-1\}$ неубывающая перестановка чисел из $\mathcal{T}_n$. Пусть $n_0 > N$, такое, что $\max(\tau_{i+1}^n - \tau_i^n) < \delta_1$, где число δ_1 будет выбрано позже и $a, b \in \mathcal{T}_{n_0}$. Для отрезков $[\tau_i^n; \tau_{i+1}^n] \subset [a;b]$ применяя лемму 5.2, получаем функции $\phi_i(t) = \sum_{n=n_0}^{M} a_{n,i} f_n(t)$, обладающие свойствами

А) $\phi_i(t) = 0$, когда $t \notin (\tau_i^n; \tau_{i+1}^n)$,

Б) $\phi_i(t) = 1$, когда $t \in E_i \subset (\tau_i^n; \tau_{i+1}^n)$ и $\mu(E_i) > (1-\varepsilon)(\tau_{i+1}^n - \tau_i^n)$,

В) $\int_0^1 \sum_{n=n_0}^{M} |a_n(\phi_i) f_n(t)|dt \le C_\gamma(\tau_{i+1}^n - \tau_i^n)\ln\varepsilon^{-1} \le C_\gamma \delta_1 \ln\varepsilon^{-1}$,

Г) $|\phi_i(t)| \le \frac{C_\gamma}{\varepsilon}$.

Положим $\phi(t) = \sum_i \phi_i(t) = \sum_{n=n_0}^{M} a_n f_n(t)$, где $a_n = \sum_i a_{n,i}$. Очевидно, что из А), Б), В) следуют пункты 1, 2, 4 леммы.

Из лемм 3.2, 3.4 имеем, что

$$\sum_{n=n_0}^{M} |a_{n,i} f_n(t)| \le \frac{C_\gamma q^{|i-j|}\|\phi_i\|_1}{\rho(t, (\tau_i^n; \tau_{i+1}^n))^2} \quad \text{когда} \quad t \in [\tau_j^n; \tau_{j+1}^n], \quad i \ne j. \tag{5.22}$$

При подходящем выборе δ_1, из (5.22) и В) получим пункт 5 леммы. Пункт 3 следует из Г) и (5.22). Лемма 5.3 доказана.

Лемма 5.4. *Пусть последовательность $\mathcal{T}$ сильно регулярная с параметром γ и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина. Тогда для любой почти всюду конечной измеримой функции $f(t)$, любого измеримого множества $A \subset [0;1]$, любых положительных чисел ε, δ и любого натурального числа N существуют измеримые множества B, D и полином $\phi(t) = \sum_{n=N+1}^{M} a_n f_n(t)$ такие, что $B \subset A \subset D$*

1. *$|\phi(t) - f(t)| < \delta$, когда $t \in B$ и $\mu(B) > (1-\varepsilon)\mu(A)$,*
2. *$\int_0^1 \sum_{n=N+1}^{M} |a_n f_n(t)|dt < \frac{C_\gamma}{\varepsilon}\int_B |f(t)|dt$,*
3. *$\sum_{n=N+1}^{M} |a_n f_n(t)| < \delta$, когда $t \in D^c$ и $\mu(D \setminus A) < \varepsilon\mu(A)$.*

Доказательство. Без ограничения общности, можем считать, что функция f ограниченная и неотрицательная, т.е. $0 \le f(t) < K$. Пусть числа α_i такие, что

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_p = K \quad \text{и} \quad \alpha_i - \alpha_{i-1} < \delta.$$

Обозначим $G_i = \{t \in [0;1] : \alpha_{i-1} \le f(t) < \alpha_i\}$. Существуют интервалы $I_m \in \mathfrak{I}$ и числа ν_i, такие что $\nu_i < \nu_{i+1}$, $I_m \cap I_{m'} = \emptyset$, когда $m \neq m'$ и

$$\mu\left(\left(\bigcup_{m=\nu_{i-1}}^{\nu_i - 1} I_m\right) \bigcap G_i\right) > \left(1 - \frac{\varepsilon}{2}\right)\mu(G_i).$$

Применяя лемму 5.3, получим полиномы $\phi_m(t) = \sum_{n=N_m+1}^{N_{m+1}} a_n f_n(t)$ и множества E_m, такие, что если $\nu_{i-1} \le m < \nu_i$, то

$$\phi_m(t) = \alpha_i, \quad \text{когда} \quad t \in E_m \subset I_m, \quad \mu(E_m) > \left(1 - \frac{\varepsilon}{2}\right)\mu(I_m),$$

$$\int_0^1 \sum_{N_m+1}^{N_{m+1}} |a_n f_n(t)| dt < \frac{C_\gamma}{\varepsilon} \cdot \alpha_i \cdot \mu(I_m),$$

$$\sum_{N_m+1}^{N_{m+1}} |a_n f_n(t)| < \eta_m, \quad \text{когда} \quad t \notin [a_m - \delta_m; b_m + \delta_m],$$

где $[a_m; b_m] = I_m$, а η_m и δ_m некоторые положительные числа. При подходящем выборе чисел η_m и δ_m, полином $\phi(t) = \sum_{m=1}^{\nu_p} \phi_m(t)$ будет удовлетворять условиям леммы. Лемма 5.4 доказана.

Теорема 5.1. *Пусть последовательность $\mathfrak{T}$ сильно регулярная с параметром γ, $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина и $a, b \in \mathfrak{T}$, $a < b$. Тогда для любой почти всюду конечной измеримой функции f существует ряд $\sum_{n=0}^{\infty} a_n f_n(t)$, который почти всюду абсолютно сходится к $f(t)$, т.е.*

1) $\sum_{n=0}^{\infty} a_n f_n(t) = f(t)$ *п.в. на* $[0;1]$
2) $\sum_{n=0}^{\infty} |a_n f_n(t)| < +\infty$ *п.в. на* $[0;1]$.

Доказательство. Пусть $\varepsilon_k = 2^{-k}$, $\delta_k = \varepsilon_{k+1}^2$, $k = 1, 2, \ldots$. Применяя лемму 5.4 найдем множество $B_1 \subset [0;1]$ и полином $\phi_1(t) = \sum_{n=1}^{N_1} a_n f_n(t)$ такие, что

$$|f(t) - \phi_1(t)| < \delta_1, \quad x \in B_1, \quad \mu(B_1) > 1 - \varepsilon_1$$

и

$$\int_0^1 \sum_{n=1}^{N_1} |a_n f_n(t)| < \frac{C_\gamma}{\varepsilon_1} \int_{B_1} |f(t)| dt.$$

Теперь, применив лемму 5.4 к функции $f(t) - \phi_1(t)$ и множеству B_1, получим полином $\phi_2'(t) = \sum_{n=N_1+1}^{M_1} a_n f_n(t)$ и $E_2 \subset B_1$ такие, что

$$\mu(E_2) > (1 - \varepsilon_2)\mu(B_1), \tag{5.23}$$

(5.24) $$|\phi_2'(t) - (f(t) - \phi_1(t))| < \delta_2, \quad \text{когда} \quad t \in E_2,$$

(5.25) $$\int_0^1 \sum_{n=N_1+1}^{M_1} |a_n f_n(t)| dt < \frac{C_\gamma}{\varepsilon_2} \int_{E_2} |f(t) - \phi_1(t)| dt < \frac{C_\gamma}{\varepsilon_2}\delta_1 = C_\gamma \varepsilon_2.$$

Пусть $F_2 \supset B_1$ и $\mu(F_2) > 1 - \varepsilon_2$. Опять, применив лемму 5.4, найдем полином $\phi_2'' = \sum_{n=M_1+1}^{N_2} a_n f_n(t)$ и множества G_2, R_2, такие что

(5.26) $$G_2 \subset F_2 \backslash E_2 \subset R_2, \quad \mu((F_2 \backslash E_2) \backslash G_2) < \varepsilon_2,$$

(5.27) $$|\phi_2''(t) - (f(t) - \phi_1(t) - \phi_2'(t))| < \delta_2, \quad \text{когда} \quad t \in G_2,$$

(5.28) $$\sum_{M_1+1}^{N_2} |a_n f_n(t)| < \delta_2, \quad t \in R_2^c \quad \text{и} \quad \mu(R_2 \backslash (F_2 \backslash E_2)) < \varepsilon_2.$$

Обозначим $A_2 = E_2 \cap R_2^c$ и $\phi_2(t) = \phi_2'(t) + \phi_2''(t) = \sum_{n=N_1+1}^{N_2} a_n f_n(t)$. Из (5.23)-(5.28) следуют

$$\int_{A_2} \sum_{n=N_1}^{N_2} |a_n f_n(t)| dt < 2C_\gamma \varepsilon_2 \quad \text{и} \quad \mu(A_2) > 1 - \varepsilon_1 - 2\varepsilon_2,$$

$$|f(t) - \phi_1(t) - \phi_2(t)| < 2\delta_2, \quad \text{когда} \quad t \in B_2,$$

где $B_2 \subset F_2$ и $\mu(F_2 \backslash B_2) < \varepsilon_2$.

Следовательно $\mu(B_2) > 1 - 2\varepsilon_2$. Продолжая этот процесс, получим множества B_m, A_m и полиномы $\phi_m(t) = \sum_{n=N_m+1}^{N_{m+1}} a_n f_n(t)$ такие, что

(5.29) $$\left| f(t) - \sum_{n=1}^{N_m} \phi_n(t) \right| < 2\delta_m, \quad t \in B_m, \quad \mu(B_m) > 1 - \varepsilon_m,$$

(5.30) $$\int_{A_m} \sum_{n=N_m+1}^{N_{m+1}} |a_n f_n(t)| dt < 2C_\gamma \varepsilon_m, \quad \mu(A_m) > 1 - \varepsilon_{m-1} - 2\varepsilon_m.$$

Из последнего вытекает

$$\int_{\bigcap_{n \geq m} A_n} \sum_{n=N_m+1}^{\infty} |a_n f_n(t)| dt < 2C_\gamma \sum_{n \geq m} \varepsilon_n < +\infty$$

и

$$\mu(\bigcap_{n \geq m} A_m) > 1 - 4\varepsilon_{m-1} - 4\varepsilon_m.$$

Отсюда следует, что ряд $\sum_{n=1}^{\infty} a_n f_n(t)$ почти всюду на $[0;1]$ абсолютно сходится. Из (5.29) следует, что почти всюду сходится к $f(t)$. Теорема доказана.

Когда последовательность $\mathcal{T}$ слабо регулярная и квазидиадическая и $\{f_n(x)\}_{n=0}^{\infty}$ соответствующая ей общая система Франклина, возможность представления измеримых функций почти всюду абсолютно сходящимися рядами по системе

$\{f_n(x)\}_{n=0}^{\infty}$ доказана в работе [6]. Для классической системы Франклина аналог теоремы 5.1 была доказана в работе [11]. Вопросы представления измеримых функций абсолютно сходящимися рядами по другим системам были рассмотрены в работах [12]–[15].

Abstract. The paper considers general Franklin systems generated by strong regular partitions of the segment [0; 1]. For such systems we prove the following assertions: 1) the absolute convergence of a Fourier-Franklin series at a point is a local property; 2) the Fourier-Franklin series of a function of bounded variation absolute converges almost everywhere; 3) any almost everywhere finite measurable function can be represented by an almost everywhere absolutely convergent series by a general Franklin system.

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CONVERGENCE IN MEASURE OF LOGARITHMIC MEANS OF MULTIPLE FOURIER SERIES

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Abstract. The maximal Orlicz space such that the mixed logarithmic means of rectangular partial sums of multiple Fourier series for the functions from this space converge in measure is found.

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1. Introduction and main results

Let $\mathbf{T}^d := [-\pi,\pi)^d$ denote a cube in the d-dimensional Euclidean space $\mathbf{R}^d$. The elements of $\mathbf{R}^d$ are denoted by $\mathbf{x} := (x_1, ..., x_d)$.
Let $D = \{1, 2, ..., d\}$, $B = \{l_1, l_2, ..., l_r\}$, $1 \le r \le d$, $B \subset D$, $l_k < l_{k+1}$, $k = 1, 2, ..., r-1$, $B' = D \backslash B$. For any $\mathbf{x} = (x_1, ..., x_d) \in \mathbf{R}^d$ and any $B \subset D$, denote $\mathbf{x}_B = (x_{l_1}, x_{l_2}, ..., x_{l_r}) \in \mathbf{R}^r$. The number of elements of a set B we denote by $|B|$. If $B \neq \varnothing$, then for any natural number n we suppose that $n_B := (n, n, ..., n) \in \mathbf{R}^{|B|}$. The notation $a \lesssim b$ stands for $a \le cb$, where c is a constant depending on the dimension d. Below we will identify the symbols

$$\sum_{i_B=0_B}^{n_B} \text{ and } \sum_{i_{l_1}=0}^{n_{l_1}} \cdots \sum_{i_{l_r}=0}^{n_{l_r}}; \quad dt_B \text{ and } dt_{l_1} \cdots dt_{l_r}, \text{ respectively.}$$

We denote by $L_0(\mathbf{T}^d)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbf{T}^d$. The Lebesgue measure of a set $A \subset \mathbf{T}^d$ we denote by $\mathrm{mes}(A)$. Also, we denote by $L_p(\mathbf{T}^d)$ the class of all measurable functions f that are

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2π-periodic with respect to all variables and satisfy

$$\|f\|_p := \left(\int_{\mathbb{T}^d} |f|^p \right)^{1/p} < \infty.$$

The $weak - L_1 \left(\mathbb{T}^d\right)$ space consists of all measurable, 2π-periodic with respect to each variable functions f, satisfying

$$\|f\|_{weak-L_1(\mathbb{T}^d)} := \sup_{\lambda} \lambda \operatorname{mes} \left\{ \mathbf{x} \in \mathbb{T}^d : |f(\mathbf{x})| > \lambda \right\} < \infty.$$

The Fourier series of a function $f \in L_1\left(\mathbb{T}^d\right)$ with respect to the trigonometric system is the series

$$S[f] := \sum_{n_1,\dots,n_d=-\infty}^{+\infty} \widehat{f}(n_1,\dots,n_d)\, e^{i(n_1x_1+\dots+n_dx_d)},$$

where

$$\widehat{f}(n_1,\dots,n_d) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x_1,\dots,x_d) e^{-i(n_1x_1+\dots+n_dx_d)} dx_1 \cdots dx_d$$

are the Fourier coefficients of f. The rectangular partial sums are defined as follows:

$$S_{N_D}(f;\mathbf{x}) := \sum_{n_D=-N_D}^{N_D} \widehat{f}(n_1,\dots,n_d)\, e^{i(n_1x_1+\dots+n_dx_d)}.$$

In the literature, it is known the notion of Riesz logarithmic means of a Fourier series. Given a natural n, the n-th Riesz logarithmic mean of the Fourier series of an integrable function f is defined by

$$\frac{1}{l_n} \sum_{k=0}^{n} \frac{S_k(f)}{k+1}, \qquad l_n := \sum_{k=0}^{n} \frac{1}{k+1},$$

where $S_k(f)$ is the partial sum of the Fourier series of f. The Riesz logarithmic means of Fourier series with respect to trigonometric system has been studied by a number of authors. Here we mention the papers by Szasz [13] and Yabuta [16], where additional references can be found. The Riesz logarithmic means of Fourier series with respect to Walsh and Vilenkin systems were discussed by Gát [2] and Simon [12].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{\sum_{k=0}^{n} q_k} \sum_{k=0}^{n} q_k S_{n-k}(f).$$

In the special case where $q_k = \frac{1}{k+1}$, we have the Nurlund logarithmic means:

$$L_n(f;x) := \frac{1}{l_n}\sum_{k=0}^{n}\frac{S_{n-k}(f)}{k+1}, \tag{1.1}$$

which represent kind of "reverse" Riesz logarithmic means. In [G] we have proved some convergence and divergence properties for the logarithmic means of Walsh-Fourier series of functions in the class of continuous functions and in the Lebesgue space L.

The Reisz and Nörlund logarithmic means of multiple Fourier series are defined by the following formulas:

$$R_{n_D}(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_i}\sum_{i_D=0_D}^{n_D}\frac{S_{i_D}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)},$$

$$L_{n_D}(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_i}\sum_{i_D=0_D}^{n_D}\frac{S_{n_D-i_D}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)}.$$

It is easy to see that

$$L_{n_D}(f;\mathbf{x}) = \frac{1}{\pi^d}\int\limits_{\mathbf{T}^d} f(\mathbf{t})\,F_{n_D}(\mathbf{x}-\mathbf{t})\,d\mathbf{t}$$

and

$$R_{n_D}(f;\mathbf{x}) = \frac{1}{\pi^d}\int\limits_{\mathbf{T}^d} f(\mathbf{t})\,G_{n_D}(\mathbf{x}-\mathbf{t})\,d\mathbf{t},$$

where

$$F_{n_D}(\mathbf{x}) := \prod_{j\in D} F_{n_j}(x_j), \qquad G_{n_D}(\mathbf{x}) := \prod_{j\in D} G_{n_j}(x_j),$$

$$F_n(u) := \frac{1}{l_n}\sum_{i=0}^{n}\frac{D_{n-i}(u)}{i+1}, \qquad G_n(u) := \frac{1}{l_n}\sum_{i=0}^{n}\frac{D_i(u)}{i+1}.$$

Let $B \subset D$. Then the mixed logarithmic means of multiple Fourier series are defined by

$$\left(L_{n_B}\circ R_{n_{B'}}\right)(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_i}\sum_{i_D=0_D}^{n_D}\frac{S_{n_B-i_B,i_{B'}}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)}.$$

It is easy to show that

$$\left(L_{n_B}\circ R_{n_{B'}}\right)(f;\mathbf{x}) = \frac{1}{\pi^d}\int\limits_{\mathbf{T}^d} f(\mathbf{t})\,F_{n_B}(\mathbf{x}_B-\mathbf{t}_B)\,G_{n_{B'}}(\mathbf{x}_{B'}-\mathbf{t}_{B'})\,d\mathbf{t}.$$

Let $L_Q = L_Q(\mathbf{T}^d)$ be the Orlicz space generated by Young function Q, that is, Q is a convex continuous even function such that $Q(0)=0$ and (see [10], Ch. 2)

$$\lim_{u\to+\infty}\frac{Q(u)}{u} = +\infty, \quad \lim_{u\to 0}\frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(\mathbf{T}^d)} = \inf\{k > 0 : \int_{\mathbf{T}^d} Q(|f|/k) \leq 1\}.$$

In particular, if $Q(u) = u\log^\beta(1+u)$ $(u, \beta > 0)$, then the corresponding space will be denoted by $L\log^\beta L(\mathbf{T}^d)$.

The rectangular partial sums of double Fourier series $S_{n,m}(f; x, y)$ of a function $f \in L_p(\mathbf{T}^2)$, $1 < p < \infty$, converge in L_p norm to the function f as $n \to \infty$ (see [17]). In the space $L_1(\mathbf{T}^2)$ this result does not hold. But for $f \in L_1(\mathbf{T})$, the operator $S_n(f; x)$ is of weak type $(1,1)$ (see [18]). This fact implies convergence of $S_n(f; x)$ in measure on $\mathbf{T}$ to the function $f \in L_1(\mathbf{T})$. However, for double Fourier series this result does not hold (see [8, 11]). Moreover, it is proved that quadratic partial sums $S_{n,n}(f; x, y)$ of double Fourier series do not converge in two-dimensional measure on $\mathbf{T}^2$ even for functions from Orlicz spaces wider than the Orlicz space $L\log L(\mathbf{T}^2)$. On the other hand, it is well-known that the rectangular partial sums $S_{n,m}(f; x, y)$ of a function $f \in L\log L(\mathbf{T}^2)$ converge in measure on $\mathbf{T}^2$.

Notice that the classical regular summation methods often improve the convergence of Fourier series. For instance, the Fejẽr means of the double Fourier series of a function $f \in L_1(\mathbf{T}^2)$ converge in $L_1(\mathbf{T}^2)$ norm to the function f (see [17]). These means represent the particular case of the Nϥrlund means.

It is well known that the method of Nϥrlund logarithmic means of double Fourier series is weaker than the Cesáro method of any positive order. In [14] Tkebuchava proved that these means of double Fourier series in general do not converge in two-dimensional measure on $\mathbf{T}^d$ even for functions from Orlicz spaces wider than the Orlicz space $L\log^{d-1} L(\mathbf{T}^d)$. Thus, not all classical regular summation methods can improve the convergence in measure of double Fourier series.

For the results on summability of logarithmic means of Walsh-Fourier series we refer the papers [4]–[6], [13, 16].

In this paper we consider the mixed logarithmic means $(L_{n_B} \circ R_{n_{B'}})(f)$ of rectangular partial sums of multiple Fourier series and prove that these means are acting from the space $L\log^{|B|-1} L(\mathbf{T}^d)$ into the space $weak - L_1(\mathbf{T}^d)$ (Theorem 1.1). This fact implies the convergence in measure of mixed logarithmic means of rectangular partial sums of multiple Fourier series (Theorem 1.2). We also prove the sharpness of this result (Theorem 1.3).

Theorem 1.1. *Let $B \subset D$ and $f \in L\log^{|B|-1} L\left(\mathbf{T}^d\right)$. Then*

$$\left\|\left(L_{n_B} \circ R_{n_{B'}}\right)(f)\right\|_{weak-L_1(\mathbf{T}^d)} \lesssim 1 + \left\||f|\log^{|B|-1}|f|\right\|_{L_1(\mathbf{T}^d)}.$$

Theorem 1.2. *Let $B \subset D$ and $f \in L\log^{|B|-1} L\left(\mathbf{T}^d\right)$. Then*

$$\left(L_{n_B} \circ R_{n_{B'}}\right)(f) \to f \text{ in measure on } \mathbf{T}^d \quad \text{as} \quad n_i \to \infty,\ i \in D.$$

Theorem 1.3. *Let $B \subset D$, $|B| > 1$ and $L_Q\left(\mathbf{T}^d\right)$ be an Orlicz space, such that*

$$L_Q\left(\mathbf{T}^d\right) \nsubseteq L\log^{|B|-1} L\left(\mathbf{T}^d\right).$$

Then the set of functions from the Orlicz space $L_Q\left(\mathbf{T}^d\right)$ with logarithmic means $\left(L_{n_B} \circ R_{n_{B'}}\right)(f)$ of rectangular partial sums of multiple Fourier series convergent in measure on $\mathbf{T}^d$ is of first Baire category in $L_Q\left(\mathbf{T}^d\right)$.

Corollary 1.1. *Let $B \subset D$, $|B| > 1$ and $\varphi : [0,\infty[\to [0,\infty[$ be a nondecreasing function satisfying the condition*

$$\varphi(x) = o(x\log^{|B|-1} x) \quad \text{as} \quad x \to +\infty.$$

Then there exists a function $f \in L_1(\mathbf{T}^d)$ such that

a)

$$\int_{\mathbf{T}^d} \varphi(|f|) < \infty;$$

b) the logarithmic means $\left(L_{n_B} \circ R_{n_{B'}}\right)(f)$ of rectangular partial sums of multiple Fourier series of f diverge in measure on $\mathbf{T}^d$.

2. Auxiliary results

In this section we state some auxiliary results that will be used in the proofs of our main results. For the next result we refer to ([1], Ch. 1).

Theorem 2.1. *Let $H : L_1(\mathbf{T}^d) \to L_0(\mathbf{T}^d)$ be a linear continuous operator, which commutes with a family of translations $\mathcal{E}$, that is, $HEf = EHf$ for all $E \in \mathcal{E}$ and all $f \in L_1(\mathbf{T}^d)$. Let $\|f\|_{L_1(\mathbf{T}^d)} = 1$ and $\lambda > 1$. Then for any $1 \leq r \in \mathbf{N}$ under the condition mes$\{\mathbf{x} \in \mathbf{T}^d : |Hf| > \lambda\} \geq \frac{1}{r}$ there exist $E_1, ..., E_r \in \mathcal{E}$, $j = 1, 2, ..., d$, and $\varepsilon_i = \pm 1$, $i = 1, ..., r$, such that*

$$\operatorname{mes}\left\{\mathbf{x} \in \mathbf{T}^d : \left|H\left(\sum_{i=1}^{r} \varepsilon_i f(E_i\mathbf{x})\right)\right| > \lambda\right\} \geq \frac{1}{8}.$$

The proof of the next result can be found in [3].

Lemma 2.1. *Let $\{H_m\}_{m=1}^{\infty}$ be a sequence of linear continuous operators, acting from Orlicz space $L_Q(\mathbb{T}^d)$ into the space $L_0(\mathbb{T}^d)$. Suppose that there exists a sequence of functions $\{\xi_k\}_{k=1}^{\infty}$ from the unit ball $S_Q(0,1)$ of the space $L_Q(\mathbb{T}^d)$, sequences of integers $\{m_k\}_{k=1}^{\infty}$ and $\{\nu_k\}_{k=1}^{\infty}$ increasing to infinity such that*

$$\varepsilon_0 = \inf_k \operatorname{mes}\{\mathbf{x} \in \mathbb{T}^d : |H_{m_k}\xi_k(x,y)| > \nu_k\} > 0.$$

Then the set K of functions f from the space $L_Q(\mathbb{T}^d)$, for which the sequence $\{H_m f\}$ converges in measure to an a. e. finite function, is of first Baire category in the space $L_Q(\mathbb{T}^d)$.

For the next lemma we refer to [4].

Lemma 2.2. *Let $L_\Phi(\mathbb{T}^d)$ be an Orlicz space and let $\varphi : [0,\infty) \to [0,\infty)$ be a measurable function satisfying the condition $\varphi(x) = o(\Phi(x))$ as $x \to \infty$. Then there exists an Orlicz space $L_\omega(\mathbb{T}^d)$, such that $\omega(x) = o(\Phi(x))$ as $x \to \infty$, and $\omega(x) \geq \varphi(x)$ for $x \geq c \geq 0$.*

To state the next lemma the proof of which can be found in [7], we first introduce the following notation:

$$\alpha_{mn} := \frac{\pi(12m+1)}{6(2^{2n}+1/2)},\ \beta_{mn} := \frac{\pi(12m+5)}{6(2^{2n}+1/2)},\ \gamma_n := \frac{\pi}{6(2^{2n}+1/2)},$$

and set

$$J_n := \bigcup_{m=1}^{2^n-1} [\alpha_{mn}+\gamma_n, \beta_{mn}-\gamma_n].$$

Lemma 2.3. *Let $0 \leq z \leq \gamma_n$ and $x \in J_n$. Then*

$$F_{2^{2n}}(x-z) \gtrsim \frac{1}{x}.$$

3. Proof of main results

Proof of Theorem 1.1. First, we prove that the one dimensional operator $L_n(f)$ (see (1.1)), has weak type $(1,1)$, that is, for $f \in L_1(\mathbb{T}^1)$ we have

$$\|L_n(f)\|_{weak-L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)}. \tag{3.1}$$

Setting $\alpha_n(t) := \sin((n+1)t)$ and $\beta_n(t) := \cos((n+1)t)$, we can write

$$S_{n-k}(f;x) = \frac{1}{\pi}\int_{\mathbf{T}} f(t)\frac{\sin((n-k+1/2)(x-t))}{2\sin((x-t)/2)}dt \tag{3.2}$$

$$= \frac{1}{\pi}\int_{\mathbf{T}} f(t)\sin((n+1)(x-t))\frac{\cos((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$-\frac{1}{\pi}\int_{\mathbf{T}} f(t)\cos((n+1)(x-t))\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$= \frac{1}{\pi}\int_{\mathbf{T}} f(t)\sin((n+1)(x-t))\left(\frac{\cos((k+1/2)(x-t))}{2\sin((x-t)/2)}-\frac{\cos((x-t)/2)}{2\sin((x-t)/2)}\right)dt$$

$$+\frac{1}{\pi}\int_{\mathbf{T}} f(t)\frac{\sin((n+1)(x-t))}{2\tan((x-t)/2)}dt$$

$$-\frac{1}{\pi}\int_{\mathbf{T}} f(t)\cos((n+1)(x-t))\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$= -\frac{\alpha_n(x)}{\pi}\int_{\mathbf{T}} f(t)\beta_n(t)\widetilde{D}_k(x-t)\,dt+\frac{\beta_n(x)}{\pi}\int_{\mathbf{T}} f(t)\alpha_n(t)\widetilde{D}_k(x-t)\,dt$$

$$+\frac{1}{\pi}\int_{\mathbf{T}} f(t)\frac{\sin((n+1)(x-t))}{2\tan((x-t)/2)}dt$$

$$-\frac{\beta_n(x)}{\pi}\int_{\mathbf{T}} f(t)\beta_n(t)\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$-\frac{\alpha_n(x)}{\pi}\int_{\mathbf{T}} f(t)\alpha_n(t)\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$= -\alpha_n(x)\widetilde{S}_k(f\beta_n;x)+\beta_n(x)\widetilde{S}_k(f\alpha_n;x)$$

$$-\beta_n(x)S_k(f\beta_n;x)-\alpha_n(x)S_k(f\alpha_n;x)+S^*_{n+1}(f;x),$$

where $\widetilde{S}_n(f;x)$, $S^*_n(f;x)$ and $\widetilde{D}_k(x)$ stand for the conjugate partial sums, the modified partial sums and the conjugate kernel, respectively. Taking into account that for $f\in L_1(\mathbf{T}^1)$ (see [18], Ch. 7):

$$\|S^*_n(f)\|_{weak-L_1(\mathbf{T}^1)} \lesssim \|f\|_{L_1(\mathbf{T}^1)},$$

$$\left\|\sup\frac{1}{n+1}\left|\sum_{k=0}^{n}S_k(f)\right|\right\|_{weak-L_1(\mathbf{T}^1)} \lesssim \|f\|_{L_1(\mathbf{T}^1)}$$

and

$$\left\|\sup \frac{1}{n+1}\left|\sum_{k=0}^{n} \widetilde{S}_k(f)\right|\right\|_{weak-L_1(\mathbf{T}^1)} \lesssim \|f\|_{L_1(\mathbf{T}^1)},$$

we can apply Abel's transform to infer (3.1) from (3.2).

We apply the following special case of the Marcinkiewicz interpolation theorem ([9], p. 173). Let $T : L_1(\mathbf{T}^1) \to L_0(\mathbf{T}^1)$ be a quasilinear operator of weak type (1, 1) and of type (α, α) for some $1 < \alpha < \infty$, that is, T satisfies the following conditions:

a) for all $y > 0$

$$\text{(3.3)} \qquad \operatorname{mes}\left\{x \in \mathbf{T}^1 : |T(f,x)| > y\right\} \lesssim \frac{1}{y}\int_{\mathbf{T}^1} |f(x)|\,dx; \quad \forall f \in L^1(\mathbf{T}^1);$$

b) for all $f \in L_\alpha(\mathbf{T}^1)$

$$\text{(3.4)} \qquad \|Tf\|_{L_\alpha(\mathbf{T}^1)} \lesssim \|f\|_{L_\alpha(\mathbf{T}^1)}.$$

Then for all $\beta \geq 0$

$$\text{(3.5)} \qquad \int_{\mathbf{T}^1} |T(f,x)| \ln^{\beta} |T(f,x)|\,dx \lesssim \int_{\mathbf{T}^1} |f(x)| \ln^{\beta+1} |f(x)|\,dx + 1.$$

On the other hand, it is easy to show that the operator $f * G_n$ has typo (1,1). Indeed, applying Abel's transform we get

$$\begin{aligned} l_n G_n(x) &= \sum_{i=0}^{n-1}\left(\frac{1}{i+1} - \frac{1}{i+2}\right)\sum_{j=0}^{i} D_j(x) + \frac{1}{n+1}\sum_{j=0}^{n} D_j(x) \\ &= \sum_{i=0}^{n-1} \frac{K_i(x)}{i+2} + K_n(x), \end{aligned}$$

where

$$K_i(x) := \frac{1}{i+1}\sum_{j=0}^{i} D_j(x).$$

Hence

$$\text{(3.6)} \qquad f * G_n = \frac{1}{l_n}\sum_{i=0}^{n-1} \frac{f * K_i}{i+2} + \frac{f * K_n}{l_n}.$$

Since $\|f * K_n\|_1 \lesssim \|f\|_1$ from (3.6) we conclude that

$$\text{(3.7)} \qquad \|f * G_n\|_{L_1(\mathbf{T}^1)} \lesssim \|f\|_{L_1(\mathbf{T}^1)}.$$

Next, setting

$$\Omega := \left\{\mathbf{x} \in \mathbf{T}^d : \left|\left(L_{n_B} \circ R_{n_{B'}}\right)(f, \mathbf{x})\right| > \lambda\right\}.$$

where $B' = \{s_1, s_2, ..., s_{r'}\}$, in view of (3.1)-(3.7) we can write

$$\begin{aligned}
& \lambda \text{mes} \left\{ \mathbf{x} \in \mathbf{T}^d : \left| \left(L_{n_B} \circ R_{n_{B'}} \right)(f, \mathbf{x}) \right| > \lambda \right\} \\
= \; & \lambda \int_{\mathbf{T}^d} 1_\Omega (\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{T}^{d-1}} \left(\int_{\mathbf{T}} 1_\Omega (\mathbf{x})\, dx_{l_1} \right) d\mathbf{x}_{D\setminus\{l_1\}} \\
\lesssim \; & \left\| \left(L_{n_{B\setminus\{l_1\}}} \circ R_{B'} \right)(f) \right\|_{L_1(\mathbf{T}^d)} \\
= \; & \left\| \left(R_{s_1} \circ \cdots \circ R_{s_{r'}} \circ L_{n_{l_2}} \circ \cdots \circ L_{n_{l_r}} \right)(f) \right\|_{L_1(\mathbf{T}^d)} \\
\lesssim \; & \cdots \lesssim \left\| \left(L_{n_{l_2}} \circ \cdots \circ L_{n_{l_r}} \right)(f) \right\|_{L_1(\mathbf{T}^d)} \\
\lesssim \; & 1 + \left\| \left| L_{n_{l_3}} \circ \cdots \circ L_{n_{l_r}} (f) \right| \log \left| L_{n_3} \circ \cdots \circ L_{n_{l_r}} (f) \right| \right\|_{L_1(\mathbf{T}^d)} \\
\lesssim \; & \cdots \lesssim 1 + \left\| \left| L_{n_{l_r}} (f) \right| \log^{r-2} \left| L_{n_{l_r}} (f) \right| \right\|_{L_1(\mathbf{T}^d)} \\
\lesssim \; & 1 + \left\| |f| \log^{r-1} |f| \right\|_{L_1(\mathbf{T}^d)},
\end{aligned}$$

and the result follows. Theorem 1.1 is proved. □

Proof of Theorem 1.2. By virtue of standard arguments (see, e.g., [18]), the result can easily be deduced from Theorem 1.1. So we omit the details. □

Proof of Theorem 1.3. By Lemma 2.1 the proof will be completed if we show that there exist sequences of integers $\{n_k : k \geq 1\}$ and $\{\nu_k : k \geq 1\}$ increasing to infinity, and a sequence of functions $\{\xi_k : k \geq 1\}$ from the unit ball $S_Q(0,1)$ of Orlicz space $L_Q(\mathbf{T}^d)$, such that for all k

$$\text{mes}\{\mathbf{x} \in \mathbf{T}^d : \left| L_{2^{2n_k}(B)} \circ R_{2^{2n_k}(B')} (\xi_k; \mathbf{x}) \right| > \nu_k\} \geq \frac{1}{8}. \tag{3.8}$$

First, we prove that

$$\begin{aligned}
& \text{mes} \left\{ \mathbf{x} \in \mathbf{T}^d : \left| L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \left(\frac{1_{[0,\gamma_n]^{|B|}}}{\gamma_n^{|B|}}; \mathbf{x} \right) \right| \gtrsim 2^{n(2|B|-1)} \right\} \\
\gtrsim \; & \frac{n^{|B|-1}}{2^{n(2|B|-1)}}, |B| > 1.
\end{aligned} \tag{3.9}$$

By Lemma 2.3 we have

$$\begin{aligned}
& L_{2^{2n}(B)} \circ R_{2^{2n}(B')} \left(\frac{1_{[0,\gamma_n]^{|B|}}}{\gamma_n^{|B|}}; \mathbf{x} \right) = \frac{1}{\gamma_n^{|B|}} \frac{1}{\pi^d} \int_{[0,\gamma_n]^{|B|}} \prod_{j \in B} F_{2^{2n}} (x_j - z_j)\, dz_B \\
& \times \int_{\mathbf{T}^{|B'|}} \prod_{i \in B'} G_{2^{2n}} (x_i - z_i)\, dz_{B'} = \frac{1}{\gamma_n^{|B|}} \frac{1}{\pi^{|B|}} \int_{[0,\gamma_n]^{|B|}} \prod_{j \in B} F_{2^{2n}} (x_j - z_j)\, dz_B \\
\gtrsim \; & \prod_{j \in B} \frac{1}{x_j}, x_j \in J_n, j \in B.
\end{aligned}$$

Consequently

$$
\begin{aligned}
& \operatorname{mes}\left\{\mathbf{x} \in \mathbf{T}^d : \left| L_{2^{2n}(B)} \circ R_{2^{2n}(B')}\left(\frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{\gamma_n^{|B|}};\mathbf{x}\right)\right| \gtrsim 2^{n(2|B|-1)}\right\} \\
\geq\ & \operatorname{mes}\left\{\mathbf{x} \in J_n^{|B|} \times \mathbf{T}^{|B'|} : \prod_{j\in B} \frac{1}{x_j} \gtrsim 2^{n(2|B|-1)}\right\} \\
=\ & (2\pi)^{|B'|} \operatorname{mes}\left\{\mathbf{x}_B \in J_n^{|B|} : x_{l_1} \lesssim \frac{1}{2^{n(2|B|-1)} \prod\limits_{j\in B\setminus\{l_1\}} x_j}\right\}.
\end{aligned}
$$

Setting

$$
r_{n,m_{l_2},\dots,m_{l_r}} := \max\left\{ l : \beta_{ln} \leq \frac{1}{2^{n(2|B|-1)} \prod\limits_{j\in B\setminus\{l_1\}} (\beta_{m_j n} - \gamma_n)} + \gamma_n \right\},
$$

after some algebra we obtain $r_{n,m_{l_2},\dots,m_{l_r}} \sim \frac{2^n}{\prod_{j\in B\setminus\{l_1\}} m_j}$. Then we have

$$
\begin{aligned}
& \operatorname{mes}\left\{\mathbf{x} \in \mathbf{T}^d : \left| L_{2^{2n}(B)} \circ R_{2^{2n}(B')}\left(\frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{\gamma_n^{|B|}};\mathbf{x}\right)\right| \gtrsim 2^{n(2|B|-1)}\right\} \\
\gtrsim\ & \frac{1}{2^{2n|B|}} \sum_{m_{B\setminus\{l_1\}}=1_{B\setminus\{l_1\}}}^{2^n(B\setminus\{l_1\})} \sum_{l=1}^{r_{n,m_{l_2},\dots,m_{l_r}}} 1 \\
\gtrsim\ & \frac{1}{2^{2n|B|}} \sum_{m_{B\setminus\{l_1\}}=1_{B\setminus\{l_1\}}}^{2^n(B\setminus\{l_1\})} \frac{2^n}{\prod\limits_{j\in B\setminus\{l_1\}} m_j} \gtrsim \frac{n^{|B|-1}}{2^{n(2|B|-1)}}, |B| > 1,
\end{aligned}
$$

yielding (3.9).

Next, under the conditions of the theorem we have

$$
\liminf_{u\to\infty} \frac{Q(u)}{u \log^{|B|-1} u} = 0.
$$

Therefore there exists a sequence of integers $\{n_k : k \geq 1\}$ increasing to infinity, such that for all k

$$
\lim_{k\to\infty} \frac{Q(2^{2n_k|B|})}{2^{2n_k|B|} n_k^{|B|-1}} = 0 \quad \text{and} \quad \frac{Q(2^{2n_k|B|})}{2^{2n_k|B|+4|B|}} \geq 1. \tag{3.10}
$$

From (3.9) we have

$$
\begin{aligned}
& \operatorname{mes}\left\{\mathbf{x} \in \mathbf{T}^d : \left| L_{2^{2n_k}(B)} \circ R_{2^{2n_k}(B')}\left(\frac{\mathbf{1}_{[0,\gamma_{n_k}]^{|B|}}}{\gamma_{n_k}^{|B|}};\mathbf{x}\right)\right| \gtrsim 2^{n_k(2|B|-1)}\right\} \\
\gtrsim\ & \frac{n_k^{|B|-1}}{2^{n_k(2|B|-1)}}.
\end{aligned}
$$

Then, by Theorem 2.1, there exist $E_1, ..., E_{r_k} \in \mathcal{E}$ and $\varepsilon_1, ..., \varepsilon_{r_k}$ $(\varepsilon_i = \pm 1)$ such that

$$\text{(3.11)} \qquad \begin{aligned} \operatorname{mes}\{\mathbf{x} \in \mathbf{T}^d : & \left| \sum_{i=1}^{r_k} \varepsilon_i L_{2^{2n_k}(B)} \circ R_{2^{2n_k}(B')} \left(\frac{1_{[0,\gamma_{n_k}]^{|B|}}}{\gamma_{n_k}^{|B|}}; E_i \mathbf{x} \right) \right| \\ & > 2^{n_k(2|B|-1)}\} > \frac{1}{8}, \end{aligned}$$

where $r_k \sim \frac{2^{n_k(2|B|-1)}}{n_k^{|B|-1}}$.

Denoting

$$\nu_k = \frac{2^{n_k(4|B|-1)-1}}{r_k Q\left(2^{2n_k|B|}\right)}, \quad \xi_k(\mathbf{x}) = \frac{2^{2|B|n_k-1}}{Q\left(2^{2n_k|B|}\right)} M_k(\mathbf{x}),$$

where

$$M_k(\mathbf{x}) = \frac{1}{r_k} \sum_{i=1}^{r_k} \varepsilon_i \frac{1_{[0,\gamma_{n_k}]^{|B|}}(E_i \mathbf{x})}{\gamma_{n_k}^{|B|}},$$

from (3.11) we obtain (3.8).

Finally, we prove that $\xi_k \in S_Q(0,1)$. To this end, observe that since

$$\|M_k\|_\infty \le 2^{2|B|(n_k+2)},$$

$$\|M_k\|_{L_1(\mathbf{T}^d)} \le 1,$$

$$\|\xi_k\|_{L_Q(\mathbf{T}^d)} \le \frac{1}{2} \left[\int_{\mathbf{T}^d} Q(2|\xi_k|) + 1 \right],$$

and $\frac{Q(u)}{u} < \frac{Q(u')}{u'}$ for $0 < u < u'$, in view of (3.10) we can write

$$\begin{aligned} \|\xi_k\|_{L_Q(\mathbf{T}^d)} & \le \frac{1}{2} \left[1 + \int_{\mathbf{T}^d} Q\left(\frac{2^{2|B|n_k} |M_k(x)|}{Q(2^{2|B|n_k})} \right) dx \right] \\ & \le \frac{1}{2} \left[1 + \int_{\mathbf{T}^d} \frac{Q\left(\frac{2^{2|B|n_k} 2^{2|B|(n_k+2)}}{Q(2^{2|B|n_k})} \right)}{\frac{2^{2|B|n_k} 2^{2|B|(n_k+2)}}{Q(2^{2|B|n_k})}} \frac{2^{2|B|n_k} |M_k(\mathbf{x})|}{Q(2^{2|B|n_k})} dx \right] \\ & \le \frac{1}{2} \left[1 + \int_{\mathbf{T}^d} \frac{Q\left(2^{2|B|n_k}\right)}{2^{2|B|n_k}} \frac{2^{2|B|n_k} |M_k(\mathbf{x})|}{Q(2^{2|B|n_k})} dx \right] \le 1, \end{aligned}$$

implying that $\xi_k \in S_Q(0,1)$. This completes the proof of Theorem 1.3. □

Proof of Corollary 1.1. The result follows from Theorem 1.3 and Lemma 2.2. □

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POINTWISE ESTIMATES FOR B-SPLINE GRAM MATRIX INVERSES

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Abstract. We present a new method to prove a certain geometric-decay inequality for entries of inverses of B-spline Gram matrices, which is given in [1].

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1. Introduction

Let $k, m \in \mathbb{N}$ be fixed. We define a knot-sequence $\Delta = (t_i)_{i=1}^{m+k}$ such that

$$t_i < t_{i+1}, \quad 0 = t_1 = \cdots = t_k, \quad t_{m+1} = \cdots = t_{m+k} = 1.$$

For convenience, we set $t_i = 0$ for $i \leq 0$ and $t_i = 1$ for $i \geq m+k+1$. Let $(N_{i,k})_{i=1}^{m} = (N_i)_{i=1}^{m}$ be the sequence of L_∞-normalized B-splines of order k on Δ. We use the notations

$$|\Delta| = \max_i (t_{i+1} - t_i), \quad J_{ij} = [t_{\min(i,j)}, t_{\max(i,j)+k}], \quad \eta_{ij} = |J_{ij}|.$$

Define the B-spline Gram matrix $A = (\langle N_i, N_j \rangle)_{i,j=1}^{m}$ and its inverse $B := (b_{i,j})_{i,j=1}^{m} := A^{-1}$. In this note, we will present a new method of proof of an inequality of the following type:

Theorem 1.1. *The entries of B satisfy the estimate*

$$|b_{i,j}| \leq K \gamma^{|i-j|} \eta_{ij}^{-1}, \quad 1 \leq i, j \leq n,$$

where $K > 0$ and $\gamma \in (0,1)$ are constants that depend only on k and not on the partition Δ.

This result was proved in [1] for general spline orders k and has many consequences, for instance

(1) a.e. convergence of the orthogonal projection $P_\Delta f$ as $|\Delta| \to 0$ arbitrarily, where P_Δ is the orthogonal projection operator onto $\operatorname{span}\{N_i : 1 \le i \le m\}$ [1],
(2) unconditionality of orthonormal spline series in reflexive L^p spaces [2].

In order to compare the methods used in [1] and those used here, we note that the proof in [1] strongly uses Shadrin's theorem [8] that $\|P_\Delta\|_\infty \le c$ for some constant c that depends only on k and not on Δ. Our proof does not use Shadrin's theorem, but we are able to prove Theorem 1.1 only for the spline orders $k = 2$ and $k = 3$. We note that for $k = 2$, i.e. in the piecewise linear case, there is a very direct argument by Z.Ciesielski to obtain Theorem 1.1, which is presented in [7] (cf. the results in [3, 4]). However, this proof is strictly limited to $k = 2$. One additional advantage of our approach is, that for $k = 3$, we get sharper constants K and γ than the ones obtained in [1].

This article is structured as follows. In Section 2, we collect a few preliminaries about matrices and B-splines. In Section 3 we derive a simple iteration formula for inverse matrices, which is the basis of our method of proving Theorem 1.1. Next, we use this iteration formula in Section 4 to give a new proof of Theorem 1.1 for $k = 2$. Finally, in Section 5, we show how our iterative method can be used to prove Theorem 1.1 for $k = 3$.

2. Preliminaries

2.1. The Sherman-Morrison formula. We have the following formula for the inverse of a rank j perturbation of a given matrix A.

Theorem 2.1. [9, 11] *Let A be an invertible $m \times m$ matrix and U, V $m \times j$ matrices. If the $j \times j$ matrix $1 + V^T A^{-1} U$ is invertible, then we have*

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(1 + V^T A^{-1} U)^{-1} V^T A^{-1}.$$

In applications of this formula, j is typically much smaller than m. We apply it for the choice $j = 2$.

2.2. B-splines. The B-spline functions N_i have the properties

$$\operatorname{supp} N_i = [t_i, t_{i+k}], \qquad N_i \ge 0, \qquad \sum_i N_i \equiv 1.$$

Moreover, we have the recursion formula for B-splines:

$$N_{i,k+1}(x) = \frac{x - t_i}{t_{i+k} - t_i} N_{i,k} + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} N_{i+1,k},$$

and the L^1-norm of $N_{i,k}$ is given by

$$\|N_{i,k}\|_1 = \int_0^1 N_{i,k}(x)\, dx = \frac{t_{i+k} - t_i}{k}.$$

For each Δ, we define then the space $\mathcal{S}_k(\Delta_n)$ of splines of order k with knots Δ as a linear span of (N_i), namely

$$s \in \mathcal{S}_k(\Delta) \quad \Leftrightarrow \quad s = \sum_{i=1}^{n} c_i N_i, \quad c_i \in \mathbb{R},$$

so that $\mathcal{S}_k(\Delta)$ is the space of piecewise polynomial functions of degree $k-1$, with $k-2$ continuous derivatives at t_i.

Observe that $\eta_{ij} = t_{\max(i,j)+k} - t_{\min(i,j)}$ satisfies the inequality

$$\eta_{j,n+1} \le \eta_{jn} \frac{t_{n+k+1} - t_{n+1}}{t_{n+k} - t_{n+1}}, \quad j \le n. \tag{2.1}$$

This follows from the elementary inequality $a + b + c \le \frac{(a+b)(a+c)}{a}$ for positive real numbers a, b, c with the choices $a = t_{n+k} - t_{n+1}$, $b = t_{n+1} - t_j$ and $c = t_{n+k+1} - t_{n+k}$.

In order to keep future expressions involving distances of points in Δ simple, we introduce the following notation for integer parameters ℓ, n, j:

$$(\ell, n)_j := (\ell n)_j := t_{j+\ell} - t_{j+n}. \tag{2.2}$$

2.3. Total positivity. We denote by $Q_{\ell,n}$ the set of strictly increasing sequences of ℓ integers from the set $\{1, \dots, n\}$. Let A be an $n \times n$-matrix. For $\alpha, \beta \in Q_{\ell,n}$, we denote by $A[\alpha; \beta]$ the submatrix of A consisting of the rows indexed by α and the columns indexed by β. Furthermore we let α' (the complement of α) be the uniquely determined element of $Q_{n-\ell,n}$ that consists of all integers in $\{1, \dots, n\}$ not occurring in α. In addition, we use the notation $A(\alpha; \beta) := A[\alpha'; \beta']$.

Definition 2.1. *Let A be an $n \times n$-matrix. A is called* totally positive, *if*

$$\det A[\alpha; \beta] \ge 0, \quad \textit{for } \alpha, \beta \in Q_{\ell,n},\ 1 \le \ell \le n. \tag{2.3}$$

The cofactor formula $b_{i,j} = (-1)^{i+j} \det A(j; i) / \det A$ for the inverse $B = (b_{i,j})_{i,j=1}^{n}$ of the matrix A leads to

Proposition 2.1. *Inverses $B = (b_{i,j})$ of totally positive matrices $A = (a_{i,j})$ have the checkerboard property. This means that $(-1)^{i+j} b_{i,j} \ge 0$ for all i, j.*

The theory of totally positive matrices can be applied to our Gram matrices of B-splines, since we have

Theorem 2.2 ([5]). *The Gram matrix of B-splines $A = (\langle N_{i,k}, N_{j,k}\rangle)_{i,j=1}^m$ of arbitrary order k and arbitrary partition Δ is totally positive.*

This theorem is a consequence of the so called basic composition formula (Equation (2.5), Chapter 1 in [6]) and the fact that the kernel $N_{i,k}(x)$, depending on the variables i and x, is totally positive (Theorem 4.1, Chapter 10 in [6]). Thus the inverse of A possesses the checkerboard property by the above proposition.

3. An iteration formula for inverses of matrices

In this section, we use the Sherman-Morrison formula to obtain an iterative expression for inverse matrices. Let $A = (a_{i,j})_{i,j=1}^m$ be an invertible $m \times m$-matrix and define $A_n = (a_{i,j})_{i,j=1}^n$ for $1 \le n \le m$ to be the $n \times n$ matrix consisting of the first n rows and the first n columns of A. Furthermore set $B_n := A_n^{-1}$ if A_n is invertible and let $(b_{i,j}^n)_{i,j=1}^n := B_n$. Then A_{n+1} may be written as the sum of two matrices as follows:

$$A_{n+1} = \begin{pmatrix} A_n & 0_{n\times 1} \\ 0_{1\times n} & a_{n+1,n+1} \end{pmatrix} + \begin{pmatrix} 0_{n\times n} & u^n \\ (v^n)^T & 0 \end{pmatrix} =: S_n + T_n, \tag{3.1}$$

where $v^n \in \mathbb{R}^n$ is the column vector $(a_{n+1,1}, \dots, a_{n+1,n})^T$ and u^n is the column vector $(a_{1,n+1}, \dots, a_{n,n+1})^T$. T_n is a rank 2 matrix that can be written as the product $T_n = U_n V_n^T$ where U_n and V_n are $(n+1) \times 2$ matrices and defined as

$$U_n = \begin{pmatrix} u^n & 0_{n\times 1} \\ 0 & 1 \end{pmatrix}, \quad V_n = \begin{pmatrix} 0_{n\times 1} & v^n \\ 1 & 0 \end{pmatrix}.$$

If we apply Theorem 2.1 to the above decomposition (3.1) of A_{n+1}, we get by some easy computations:

Corollary 3.1. Let $1 \le n \le m$. Additionally, suppose that A_n, B_n and v^n, u^n are defined as above and set $v = v^n, u = u^n$. If A_n is invertible, $a_{n+1,n+1} \ne 0$ and $a_{n+1,n+1} - v^T B_n u \ne 0$ we have the following formula for $B_{n+1} = A_{n+1}^{-1}$:

$$B_{n+1} = \begin{pmatrix} B_n & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{pmatrix} + \frac{1}{a_{n+1,n+1} - v^T B_n u} \begin{pmatrix} B_n u v^T B_n & -B_n u \\ -v^T B_n & 1 \end{pmatrix},$$

We employ this corollary in the cases where the matrix A is a symmetric tridiagonal or a symmetric 5-banded matrix.

3.1. Tridiagonal matrices. Let A be a symmetric tridiagonal $m \times m$-matrix. Using Corollary 3.1 on A, we get for $1 \le n \le m-1$

$$B_{n+1} = \begin{pmatrix} B_n & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{pmatrix} + (a_{n+1,n+1} - b^n_{n,n} a^2_{n,n+1})^{-1} C_{n+1}, \tag{3.2}$$

where C_{n+1} is given by

$$(C_{n+1})_{i,j} = \begin{cases} b^n_{i,n} b^n_{j,n} a^2_{n,n+1} & \text{if } 1 \le i,j \le n, \\ -b^n_{i,n} a_{n,n+1} & \text{if } 1 \le i \le n, j = n+1, \\ -b^n_{j,n} a_{n,n+1} & \text{if } 1 \le j \le n, i = n+1, \\ 1 & \text{if } i = j = n+1. \end{cases}$$

3.2. 5-banded matrices. Let $A = (a_{i,j})$ be a symmetric 5-banded matrix. This means that $a_{i,j} = 0$ for $|i-j| > 2$. Using Corollary 3.1 on A, we get for $2 \le n \le m-1$

$$B_{n+1} = \begin{pmatrix} B_n & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{pmatrix} + b^{n+1}_{n+1,n+1} C_{n+1}, \tag{3.3}$$

where

(3.4)

$$b^{n+1}_{n+1,n+1} = \left(a_{n+1,n+1} - b^n_{n,n} a^2_{n,n+1} - 2 b^n_{n-1,n} a_{n-1,n+1} a_{n,n+1} - b^n_{n-1,n-1} a^2_{n-1,n+1}\right)^{-1}$$

and C_{n+1} is given by

$$(C_{n+1})_{i,j} = \begin{cases} -b^n_{i,n} a_{n,n+1} - b^n_{i,n-1} a_{n-1,n+1} & \text{if } 1 \le i \le n, j = n+1, \\ -b^n_{j,n} a_{n,n+1} - b^n_{j,n-1} a_{n-1,n+1} & \text{if } 1 \le j \le n, i = n+1, \\ (C_{n+1})_{i,n+1} (C_{n+1})_{j,n+1} & \text{if } 1 \le i,j \le n, \\ 1 & \text{if } i = j = n+1. \end{cases} \tag{3.5}$$

The following two lemmas are independent of the special form of matrices considered in the next sections. We will use them in Section 5 to estimate inverses of Gram matrices corresponding to splines of order 3. The crucial fact about the following lemma is that inequality (3.6) depends only on the matrix A and on entries $b^n_{n,n}, b^{n-1}_{n-1,n-1}$ of the matrices B_n, B_{n-1} respectively.

Lemma 3.1. *Let A be a symmetric 5-banded $m \times m$-matrix such that $B_n = A_n^{-1}$ is checkerboard for $1 \le n \le m$. Then, the inequality*

(3.6)

$$b^{n+1}_{n+1,n+1} \le \Big(a_{n+1,n+1} - b^n_{n,n} a_{n,n+1} \Big(a_{n,n+1} - \frac{2 a_{n,n} a_{n-1,n+1}}{a_{n-1,n}}\Big) - 2 \frac{a_{n,n+1} a_{n-1,n+1}}{a_{n-1,n}}$$
$$- a^2_{n-1,n+1} b^{n-1}_{n-1,n-1} \big(1 + b^n_{n,n} b^{n-1}_{n-1,n-1} a^2_{n-1,n}\big)\Big)^{-1}$$

holds for $2 \le n \le m-1$.

Proof. By (3.4) and the checkerboard property of B_n, $b^{n+1}_{n+1,n+1}$ is given by

(3.7)
$$b^{n+1}_{n+1,n+1} = (a_{n+1,n+1} - b^n_{n,n}a^2_{n,n+1} + 2|b^n_{n-1,n}|a_{n,n+1}a_{n-1,n+1} - b^n_{n-1,n-1}a^2_{n-1,n+1})^{-1}.$$

Another consequence of the identities (3.3)–(3.5) is

$$b^n_{i,j} = b^{n-1}_{i,j} + b^n_{n,n}(b^{n-1}_{i,n-1}a_{n-1,n} + b^{n-1}_{i,n-2}a_{n-2,n})(b^{n-1}_{j,n-1}a_{n-1,n} + b^{n-1}_{j,n-2}a_{n-2,n})$$

for $1 \le i, j \le n-1$. The choice $i = j = n-1$ in these equations together with the checkerboard property of B_n yield the estimate

$$b^n_{n-1,n-1} \le b^{n-1}_{n-1,n-1}(1 + b^n_{n,n}b^{n-1}_{n-1,n-1}a^2_{n-1,n}). \tag{3.8}$$

The defining property of the inverse matrix $B_n = A_n^{-1}$, the fact that A_n is 2-banded and the checkerboard property of B_n again imply

$$|b^n_{n-1,n}| = a^{-1}_{n-1,n}(b^n_{n,n}a_{n,n} + b^n_{n-2,n}a_{n-2,n} - 1) \ge a^{-1}_{n-1,n}(b^n_{n,n}a_{n,n} - 1). \tag{3.9}$$

Finally, inserting (3.8) and (3.9) in (3.7) yields the conclusion of the lemma.

Lemma 3.2. *Let A be as in Lemma 3.1 and $2 \le n \le m$. Then, if $1 \le j \le n-1$, the estimate*

$$|b^n_{jn}| \le |b^{n-1}_{j,n-1}|b^n_{n,n}a_{n-1,n} \tag{3.10}$$

holds. Additionally for $3 \le n \le m$ and $1 \le j \le n-2$ we have

$$|b^n_{j,n}| \le |b^{n-1}_{j,n-1}|b^n_{n,n}\Big(a_{n-1,n} - \frac{a_{n-2,n}a_{n-1,n-1}}{a_{n-2,n-1}}\Big). \tag{3.11}$$

Proof. First, we note that (3.3) - (3.5) yield

$$b^n_{j,n} = -b^n_{n,n}(b^{n-1}_{j,n-2}a_{n-2,n} + b^{n-1}_{j,n-1}a_{n-1,n}) \quad \text{if } 1 \le j \le n-1. \tag{3.12}$$

Since B_n is checkerboard, (3.10) follows for $1 \le j \le n-1$. By the defining property of $B_n = A_n^{-1}$ we obtain

$$b^{n-1}_{j,n-3}a_{n-3,n-1} + b^{n-1}_{j,n-2}a_{n-2,n-1} + b^{n-1}_{j,n-1}a_{n-1,n-1} = \delta_{j,n-1}. \tag{3.13}$$

Thus we get, under the assumption $1 \le j \le n-2$,

(3.14)
$$|b^{n-1}_{j,n-2}| = a^{-1}_{n-2,n-1}(a_{n-3,n-1}|b^{n-1}_{j,n-3}| + a_{n-1,n-1}|b^{n-1}_{j,n-1}|) \ge a^{-1}_{n-2,n-1}a_{n-1,n-1}|b^{n-1}_{j,n-1}|,$$

since B_{n-1} is checkerboard. Now use the checkerboard property of B_n in (3.12) and insert estimate (3.14) in (3.12) to conclude the assertion of the lemma.

4. Picewise linear splines

We now apply Corollary 3.1 to the case of piecewise linear continuous splines to get geometric estimates for the entries of their Gram matrix inverse. The purpose of the following presentation is twofold: first, the simpler case $k = 2$ illustrates the basic proof steps that are essentially the same as for $k = 3$. Secondly, we give a new proof of the result in [7].

In this section, N_i is the unique piecewise linear continuous function on the unit interval $[0, 1]$ such that $N_i(t_j) = \delta_{i,j}$ for $1 \leq j \leq m$. We consider the Gram matrix $A = (a_{i,j})_{i,j=1}^m = (\langle N_i, N_j\rangle)_{i,j=1}^m$ obtained from these functions. Using the special form of the piecewise linear B-splines N_i, we get

$$a_{i,i} = (20)_i/3 \qquad \text{if } 1 \leq i \leq m, \tag{4.1}$$

$$a_{i,i+1} = a_{i+1,i} = (21)_i/6 \qquad \text{if } 1 \leq i \leq m-1, \tag{4.2}$$

$$a_{i,j} = 0 \qquad \text{if } |i-j| > 1. \tag{4.3}$$

Note that A is symmetric and tridiagonal, therefore we may apply the formulas of Section 3.1 to A. If we do and insert the expressions (4.1) – (4.3) for the Gram matrix A, we obtain

$$B_{n+1} = \begin{pmatrix} B_n & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{pmatrix} + 3\Big((20)_{n+1} - \frac{1}{12} b_{n,n}^n (21)_n^2\Big)^{-1} C_{n+1}, \tag{4.4}$$

where

$$(C_{n+1})_{i,j} = \begin{cases} b_{i,n}^n b_{j,n}^n (21)_n^2/36 & \text{if } 1 \leq i,j \leq n, \\ -b_{i,n}^n (21)_n/6 & \text{if } 1 \leq i \leq n, j = n+1, \\ -b_{j,n}^n (21)_n/6 & \text{if } 1 \leq j \leq n, i = n+1, \\ 1 & \text{if } i = j = n+1. \end{cases} \tag{4.5}$$

Lemma 4.1. *Let the matrix A be as above and $1 \leq n \leq m$. Then we have the estimates*

$$\frac{3}{(20)_n} \leq b_{n,n}^n \leq \frac{3}{\frac{3}{4}(10)_n + (21)_n} \leq \frac{4}{(20)_n}. \tag{4.6}$$

Proof. We proceed by induction on n. For $n = 1$, we have by definition of B_1 and (4.1)

$$\frac{3}{(20)_1} = b_{1,1}^1 = 3\Big(\frac{3}{4}(10)_1 + (21)_1\Big)^{-1},$$

since $(10)_1 = 0$ and thus equality on both sides of (4.6). We know from Theorem 2.2 that A is totally positive, so a fortiori A_n is totally positive for every $1 \leq n \leq m$.

Therefore, Proposition 2.1 yields $b^{n+1}_{n+1,n+1} \geq 0$. So we see the lower estimate in (4.6) immediately by glancing at formula (4.4) for $b^{n+1}_{n+1,n+1}$. For the upper estimate, we obtain as a consequence of the inductive hypothesis

$$\begin{aligned} b^{n+1}_{n+1,n+1} &= 3\Big((20)_{n+1} - \frac{1}{12} b^n_{n,n} (21)^2_n\Big)^{-1} \\ &\leq 3\Big((20)_{n+1} - \frac{3}{12}\big(\frac{3}{4}(10)_n + (21)_n\big)^{-1}(21)^2_n\Big)^{-1} \\ &\leq 3\Big((20)_{n+1} - \frac{1}{4}(21)_n\Big)^{-1} \\ &= 3\Big(\frac{3}{4}(10)_{n+1} + (21)_{n+1}\Big)^{-1}, \end{aligned}$$

thus the conclusion of the lemma.

Lemma 4.2. *Let the matrix A be as above and $1 \leq n \leq m$. Then, the elements $b^n_{j,n}$ of B_n satisfy the geometric estimate*

$$|b^n_{j,n}| \leq \frac{4q^{n-j}}{\eta_{jn}} \quad \text{for all } 1 \leq j \leq n, \tag{4.7}$$

where $q = 2/3$.

Proof. We infer from (4.4) and (4.5) that

$$b^{n+1}_{j,n+1} = -\frac{1}{6} b^{n+1}_{n+1,n+1} b^n_{j,n} (10)_{n+1}.$$

Invoking Lemma 4.1 for $b^{n+1}_{n+1,n+1}$ we get further

$$|b^{n+1}_{j,n+1}| \leq \frac{2}{3}\frac{(10)_{n+1}}{(20)_{n+1}} |b^n_{j,n}|. \tag{4.8}$$

Now we note that by Lemma 4.1, inequality (4.7) is true for $n = j$. For general $n > j$, inequality (4.7) follows from (4.8) by induction using inequality (2.1).

Theorem 4.1. *Let the matrix A be as above and $1 \leq n \leq m$. Then, the elements of $B_n = A_n^{-1}$ satisfy the geometric estimate*

$$|b^n_{i,j}| \leq \frac{36}{5} \frac{q^{|i-j|}}{\eta_{ij}} \quad \text{for all } 1 \leq i, j \leq n,$$

where $q = 2/3$.

Proof. We first observe that it is sufficient to prove the theorem for the parameter choices $i \leq j \leq n-1$, since B_n is symmetric and the case $j = n$ is already covered by Lemma 4.2. Equations (4.4) and (4.5) yield

$$b^n_{i,j} = b^{n-1}_{i,j} + b^n_{n,n} b^{n-1}_{i,n-1} b^{n-1}_{j,n-1} \frac{(10)^2_n}{36},$$

so by induction

$$b_{i,j}^n = b_{i,j}^j + \sum_{\ell=j}^{n-1} b_{\ell+1,\ell+1}^{\ell+1} b_{i,\ell}^{\ell} b_{j,\ell}^{\ell} \frac{(10)_{\ell+1}^2}{36}.$$

Using now Lemmas 4.1 and 4.2 we get

$$|b_{i,j}^n| \le |b_{i,j}^j| + \frac{16}{9} \sum_{\ell=j}^{n-1} \frac{q^{2\ell-i-j}(10)_{\ell+1}^2}{\eta_{i\ell}\eta_{j\ell}(20)_{\ell+1}} \le \frac{4q^{j-i}}{\eta_{ij}} \left(1 + \frac{4}{9} \sum_{\ell=j}^{n-1} \frac{q^{2(\ell-j)}\eta_{ij}(10)_{\ell+1}^2}{\eta_{i\ell}\eta_{j\ell}(20)_{\ell+1}}\right).$$

Since $(10)_{\ell+1} \le (20)_{\ell+1}$, $(10)_{\ell+1} \le (20)_\ell \le \eta_{j\ell}$ and $\eta_{ij} \le \eta_{i\ell}$ for $j \le \ell \le n-1$, we estimate this from above by

$$\frac{4q^{j-i}}{\eta_{ij}} \left(1 + \frac{4}{9} \sum_{\ell=j}^{\infty} q^{2(\ell-j)}\right) = \frac{36}{5} \frac{q^{j-i}}{\eta_{ij}},$$

proving the theorem.

This proves Theorem 1.1 for piecewise linear splines, i.e. $k = 2$.

5. Picewise quadratic splines

In this section, we apply Corollary 3.1 to the case of piecewise quadratic splines to get geometric estimates for the entries of their Gram matrix inverse.

5.1. The Gram matrix. We now calculate the Gram matrix of B-splines of order three. There is a standard formula for the inner product of B-splines of arbitrary order k involving divided differences (see for instance [10, Theorem 4.25]), but in this section we prefer to calculate the Gram matrix for $k = 3$ directly. The reason is a simplification of the general formula in our special case $k = 3$. As an easy consequence of the recursion formula for B-splines, we get the following

Corollary 5.1. The B-spline functions $N_i \equiv N_{i,3}$, $1 \le i \le m$ *of order* 3 *corresponding to the knot sequence* $\overline{\Delta} = (t_i)_{i=1}^{|\mu|}$ *are given by*

$$N_i(x) = \begin{cases} \dfrac{(x-t_i)^2}{(20)_i(10)_i} & \text{if } x \in [t_i, t_{i+1}), \\ \dfrac{(x-t_i)(t_{i+2}-x)}{(20)_i(21)_i} + \dfrac{(x-t_{i+1})(t_{i+3}-x)}{(31)_i(21)_i} & \text{if } x \in [t_{i+1}, t_{i+2}), \\ \dfrac{(t_{i+3}-x)^2}{(31)_i(32)_i} & \text{if } x \in [t_{i+2}, t_{i+3}), \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Lemma 5.1. *Let $1 \le i \le m-1$. Then we have*

$$\int_{t_{i+1}}^{t_{i+2}} N_i(x)N_{i+1}(x)\,\mathrm{d}x = \frac{(21)_i^2}{(31)_i}\left[\frac{1}{10} + \frac{(10)_i}{30(20)_i} + \frac{(32)_i}{5(31)_i}\right] \tag{5.2}$$

and due to symmetry

$$\int_{t_{i+2}}^{t_{i+3}} N_i(x)N_{i+1}(x)\,\mathrm{d}x = \frac{(32)_i^2}{(31)_i}\left[\frac{1}{10} + \frac{(43)_i}{30(42)_i} + \frac{(21)_i}{5(31)_i}\right] \tag{5.3}$$

Proof. A straightforward calculation using Corollary 5.1 yields

$$\int_{t_{i+1}}^{t_{i+2}} N_i(x)N_{i+1}(x)\,\mathrm{d}x = \frac{(21)_i^2}{(31)_i}\left[\frac{(21)_i}{20(20)_i} + \frac{(10)_i}{12(20)_i} + \frac{1}{4} - \frac{(21)_i}{5(31)_i}\right].$$

An easy reformulation of this identity gives us (5.2). Equation (5.3) now follows by symmetry.

Proposition 5.1. *The entries of the Gram matrix $A = (a_{i,j})_{i,j=1}^m = (\langle N_i, N_j\rangle)_{i,j=1}^m$ of B-splines of order 3 admit the following representation.*

$$a_{i,i+2} = a_{i+2,i} = \frac{(32)_i^3}{30(31)_i(42)_i} \qquad \text{if } 1 \le i \le m-2, \tag{5.4}$$

$$a_{i,i+1} = a_{i+1,i} = \frac{(31)_i}{10} + \frac{(21)_i^3(10)_i}{30(20)_i(31)_i} + \frac{(32)_i^3(43)_i}{30(31)_i(42)_i} \qquad \text{if } 1 \le i \le m-1, \tag{5.5}$$

$$a_{i,i} = \frac{(30)_i}{5} - \frac{(30)_i(21)_i^2}{15(20)_i(31)_i} \qquad \text{if } 1 \le i \le m. \tag{5.6}$$

Furthermore, $a_{i,j} = 0$ if $|i-j| > 2$.

Proof. Equation (5.4) is a simple consequence of Corollary 5.1. For equation (5.5), we observe that $\langle N_i, N_{i+1}\rangle$ is the sum of (5.2) and (5.3). Thus, (5.5) is a consequence of the identity

$$\frac{(21)^2}{(31)}\left[\frac{1}{10} + \frac{(32)}{5(31)}\right] + \frac{(32)^2}{(31)}\left[\frac{1}{10} + \frac{(21)}{5(31)}\right] = \frac{(21)^2 + (32)^2}{10(31)} + \frac{(21)(32)}{5(31)} = \frac{(31)}{10},$$

where every bracket (mn) is taken with respect to the subindex i. Since N_i has compact support and the B-splines form a partition of unity, we obtain

$$\left\langle N_i, \sum_{l=i-2}^{i+2} N_l \right\rangle = \langle N_i, 1\rangle = \frac{(30)_i}{3}. \tag{5.7}$$

The only unknown part in this equation is $\langle N_i, N_i\rangle$, so we use (5.7) and simple calculations to complete the proof of the proposition.

Observe that the Gram matrix A is symmetric, 5-banded and totally positive (cf. Theorem 2.2), so in particular, we can apply Lemmas 3.1 and 3.2 to A.

$$\mathtt{a[i\_,\ j\_]} := \frac{\sum_{n=i}^{i+2} \mathtt{x[n]}}{15}\left(3 - \frac{\mathtt{x[i+1]}^2}{(\mathtt{x[i]}+\mathtt{x[i+1]})\,(\mathtt{x[i+1]}+\mathtt{x[i+2]})}\right)\ /;\ \mathtt{j} == \mathtt{i};$$

$$\mathtt{a[i\_,\ j\_]} := \frac{\mathtt{x[i+1]}+\mathtt{x[i+2]}}{10} + \frac{\mathtt{x[i+1]}^2\,\mathtt{x[i]}}{30\,(\mathtt{x[i]}+\mathtt{x[i+1]})\,(\mathtt{x[i+1]}+\mathtt{x[i+2]})} + \frac{\mathtt{x[i+2]}^2\,\mathtt{x[i+3]}}{30\,(\mathtt{x[i+1]}+\mathtt{x[i+2]})\,(\mathtt{x[i+2]}+\mathtt{x[i+3]})}\ /;\ \mathtt{j} == \mathtt{i}+1;$$

$$\mathtt{a[i\_,\ j\_]} := \frac{\mathtt{x[i+2]}^3}{30\,(\mathtt{x[i+1]}+\mathtt{x[i+2]})\,(\mathtt{x[i+2]}+\mathtt{x[i+3]})}\ /;\ \mathtt{j} == \mathtt{i}+2;$$

FIG. 1. Definition of the Gram matrix A in *Mathematica*.

$$\varphi\mathtt{[i\_]} := \left(\frac{1}{9}\mathtt{x[i]} + \frac{1}{12}\mathtt{x[i+1]} + \frac{1}{5}\mathtt{x[i+2]} - \frac{\mathtt{x[i+1]}\,\mathtt{x[i+2]}}{30\,(\mathtt{x[i+1]}+\mathtt{x[i+2]})}\left(1 + \frac{\mathtt{x[i+2]}}{6\,(\mathtt{x[i+1]}+\mathtt{x[i+2]})} - \frac{20\,\mathtt{x[i]}}{9\,(\mathtt{x[i]}+\mathtt{x[i+1]})}\right) + \frac{5\,\mathtt{x[i-1]}\,\mathtt{x[i]}}{108\,(\mathtt{x[i-1]}+\mathtt{x[i]})} + \frac{2\,\mathtt{x[i-1]}^2\,\mathtt{x[i]}}{73\,(\mathtt{x[i-1]}+\mathtt{x[i]})^2}\right)^{-1};$$

FIG. 2. Definition of the function φ in *Mathematica*

In the following, we use a computer algebra system, in our case *Mathematica* 8.0, to show that certain given polynomials have only nonnegative coefficients. This allows us to deduce that the given polynomial itself is nonnegative for positive arguments. In the following results, we need the matrix A to be defined in *Mathematica*. This is done in Figure 1. Furthermore we need a *Mathematica* expression for φ_n to be defined in Proposition 5.2. For this, we refer to Figure 2.

The first thing to show is, as in Section 4, a suitable upper bound for b^n_{nn}. This is the content of the following Proposition 5.2.

Proposition 5.2. *Let A be as in Proposition 5.1 and $1 \leq n \leq m$. Then the element $b^n_{n,n}$ of $B_n = A_n^{-1}$ satisfies the estimate*

$$\begin{aligned} b^n_{n,n} \leq \Bigg(&\frac{(10)}{9} + \frac{(21)}{12} + \frac{(32)}{5} - \frac{(21)(32)}{30(31)}\left(1 + \frac{(32)}{6(31)} - \frac{20(10)}{9(20)}\right) \\ &+ \frac{5(0,-1)(10)}{108(1,-1)} + \frac{2(0,-1)^2(10)}{73(1,-1)^2}\Bigg)^{-1} =: \varphi_n, \end{aligned} \tag{5.8}$$

where every bracket (ij) in this formula is taken with respect to the subindex n.

Proof. We use Lemma 3.1 and induction to prove the claimed estimate. Recall that Lemma 3.1 stated that

(5.9)
$$b^{n+1}_{n+1,n+1} \leq \Big(a_{n+1,n+1} - b^{n}_{n,n}a_{n,n+1}\big(a_{n,n+1} - \frac{2a_{n,n}a_{n-1,n+1}}{a_{n-1,n}}\big) - 2\frac{a_{n,n+1}a_{n-1,n+1}}{a_{n-1,n}} - a^{2}_{n-1,n+1}b^{n-1}_{n-1,n-1}(1 + b^{n}_{n,n}b^{n-1}_{n-1,n-1}a^{2}_{n-1,n})\Big)^{-1}$$

for $2 \leq n \leq m-1$. Thus to apply induction, we need to verify (5.8) for $n = 1$ and $n = 2$. First, we suppose that $n = 1$; here we have $b^{1}_{1,1} = a^{-1}_{1,1} = 5/(30)_1$ by (5.6) and the fact that $(20)_1 = 0$. For $n = 2$, the formula $b^{2}_{2,2} = (a_{2,2} - b^{1}_{1,1}a^{2}_{1,2})^{-1}$ holds in view of Corollary 3.1. We thus obtain by (5.5), (5.6), $(20)_1 = 0$ and some elementary calculations

$$b^{2}_{2,2} = \Big(\frac{(21)_2}{12} + \frac{(32)_2}{5} - \frac{(21)_2(32)_2}{30(31)_2}\Big(1 + \frac{(32)_2}{(6(31)}\Big)\Big)^{-1}. \tag{5.10}$$

The expression for $b^{1}_{1,1}$ and (5.10) are special cases of (5.8) since $(20)_1 = 0$. Thus, the lemma is proved for $n = 1$ and $n = 2$.

Before we proceed with the actual proof of (5.8), we show that the term $a_{n,n+1} - \frac{2a_{n,n}a_{n-1,n+1}}{a_{n-1,n}}$, appearing in (5.9), is nonnegative. It is equivalent to show that the expression $a_{n,n+1}a_{n-1,n} - 2a_{n,n}a_{n-1,n+1}$ is nonnegative. This is done using the *mathematica*-code of Figure 3. The method of proof is as follows.

(1) Define the rational function $p = a_{n,n+1}a_{n-1,n} - 2a_{n,n}a_{n-1,n+1}$ depending on the variables $x[1] = (10)_{n-1}$, $x[2] = (21)_{n-1}, \dots, x[5] = (54)_{n-1}$.
(2) Determine the denominator d of the rational function p as $900(x[1]+x[2])(x[2]+x[3])^2(x[3]+x[4])^2(x[4]+x[5])$ and observe that d is positive for positive arguments.
(3) Calculate the coefficients of the polynomial $q := d \cdot p$ and verify that no coefficient of q is negative.

Now we continue with the proof of (5.8). Since every entry a_{ij} of the Gram matrix A is nonnegative and the matrices B_{n-1}, B_n are checkerboard, the argument of the last paragraph shows that a sufficient condition for (5.8) to be true for all $1 \leq n \leq m$ is the following recursive inequality for φ_n, $2 \leq n \leq m-1$.

$$\Big(a_{n+1,n+1} - \varphi_n a_{n,n+1}\big(a_{n,n+1} - \frac{2a_{n,n}a_{n-1,n+1}}{a_{n-1,n}}\big) - 2\frac{a_{n,n+1}a_{n-1,n+1}}{a_{n-1,n}} - a^{2}_{n-1,n+1}\varphi_{n-1}(1 + \varphi_n\varphi_{n-1}a^{2}_{n-1,n})\Big)^{-1} \leq \varphi_{n+1}. \tag{5.11}$$

```
p[z___] := Factor[a[1, 2] a[2, 3] - 2 a[2, 2] a[1, 3] /.
    Table[x[n] -> List[z][[n]], {n, 1, Length[List[z]]}]];
d[z___] := Denominator[p[z]];
q[z___] := d[z] p[z];
d[Sequence @@ Table[x[n], {n, 1, 5}]]
Select[Flatten[CoefficientList[
   q[Sequence @@ Table[x[n], {n, 1, 5}]], Table[x[n], {n, 1, 5}]]], # < 0 &]

900 (x[1] + x[2]) (x[2] + x[3])^2 (x[3] + x[4])^2 (x[4] + x[5])

{}
```

FIG. 3. Proof of the inequality $a_{n,n+1} - 2a_{nn}a_{n-1,n+1}/a_{n-1,n} \geq 0$

The proof that this inequality is true for all choices of $(0,-1)_{n-1}$, $(10)_{n-1}, \ldots, (54)_{n-1}$ is equivalent to prove that the rational function p defined as

(5.12)
$$\varphi_n^{-1}\varphi_{n-1}^{-1}a_{n-1,n}\Bigg(\Big(a_{n+1,n+1} - \varphi_n a_{n,n+1}\big(a_{n,n+1} - \frac{2a_{n,n}a_{n-1,n+1}}{a_{n-1,n}}\big) - 2\frac{a_{n,n+1}a_{n-1,n+1}}{a_{n-1,n}} - a_{n-1,n+1}^2\varphi_{n-1}(1+\varphi_n\varphi_{n-1}a_{n-1,n}^2)\Big) - \varphi_{n+1}^{-1}\Bigg)$$

and depending on the six variables $(0,-1)_{n-1}$, $(10)_{n-1}, \ldots, (54)_{n-1}$ is nonnegative for nonnegative arguments. This is done in the *mathematica*-code of Figure 4 using the following steps.

(1) Define the rational function p to be the expression in (5.12) depending on the variables $x[1] = (10)_{n-2}$, $x[2] = (21)_{n-2}, \ldots, x[5] = (54)_{n-2}$, $x[6] = (65)_{n-2}$.
(2) Determine the denominator d of the rational function p as an integer multiple of $(x[1]+x[2])^4(x[2]+x[3])^5(x[3]+x[4])^8(x[4]+x[5])^5(x[5]+x[6])^2$ and observe that d is positive for positive arguments.
(3) Calculate the coefficients of the polynomial $q := d \cdot p$ and verify that no coefficient of q is negative.

This proves the assertion of the proposition

Remark 1. *It is an easy consequence of the above lemma and the inequality*

$$\frac{(21)(32)}{(31)}\Big(1+\frac{(32)}{6(31)}\Big) \leq (32),$$

that we also have the estimate

$$b_{n,n}^n \leq \Big(\frac{(10)}{9} + \frac{(21)}{12} + \frac{(32)}{6}\Big)^{-1} =: \psi_n \tag{5.13}$$

```
p[z___] :=
  Factor[φ[3]^-1 φ[2]^-2 a[2, 3] (a[4, 4] - φ[3] a[3, 4] (a[3, 4] - (2 a[3, 3] a[2, 4])/a[2, 3]) -
       (2 a[3, 4] a[2, 4])/a[2, 3] - a[2, 4]^2 φ[2] (1 + φ[3] φ[2] a[2, 3]^2) - φ[4]^-1) /.
     Table[x[n] → List[z][[n]], {n, 1, Length[List[z]]}]];
d[z___] := Denominator[p[z]];
q[z___] := d[z] p[z];
d[Sequence @@ Table[x[n], {n, 1, 6}]]
Select[Flatten[CoefficientList[
    q[Sequence @@ Table[x[n], {n, 1, 6}]], Table[x[n], {n, 1, 6}]]], # < 0 &]

72 441 550 057 348 800 000 (x[1] + x[2])^4
 (x[2] + x[3])^5 (x[3] + x[4])^8 (x[4] + x[5])^5 (x[5] + x[6])^2

{}
```

FIG. 4. Proof of the estimate for b^n_{nn}

```
ψ[i_] := (1/9 x[i] + 1/12 x[i + 1] + 1/6 x[i + 2])^-1 ;
```

FIG. 5. Definition of the function ψ

for all $1 \le n \le m$. *The mathematica expression of* ψ_n *is given by Figure 5.*

Lemma 5.2. *Let A be as in Proposition 5.1 and $2 \le n \le m$. Then we have the estimate*

$$b^n_{n,n} a_{n-1,n} \le \frac{6}{5} \frac{(20)_n}{(30)_n}.$$

Proof. By Remark 1, it suffices to show that

$$\psi_n a_{n-1,n} \le \frac{6}{5} \frac{(20)_n}{(30)_n}.$$

We apply the same method of proof as in the above proposition and proceed with the *mathematica*-code of Figure 6 using the following steps.

(1) Define the rational function $p = \frac{6}{5}\frac{(20)_n}{(30)_n} - \psi_n a_{n-1,n}$ depending on the variables $x[1] = (10)_{n-1}, x[2] = (21)_{n-1}, x[3] = (32)_{n-1}, x[4] = (43)_{n-1}$.

(2) Determine the denominator d of the rational function p as $5(x[1] + x[2])(x[2] + x[3])(x[3] + x[4])(x[2] + x[3] + x[4])(4x[2] + 3x[3] + 6x[4])$ and observe that d is positive for positive arguments.

```
p[z___] := Factor[6/5 (x[2] + x[3])/(x[2] + x[3] + x[4]) - ψ[2] a[1, 2] /.
    Table[x[j] → List[z][[n]], {n, 1, Length[List[z]]}]];
d[z___] := Denominator[p[z]];
q[z___] := d[z] p[z];
d[Sequence @@ Table[x[n], {n, 1, 4}]]
Select[Flatten[CoefficientList[q[x, y, z, w], {x, y, z, w}]], # < 0 &]

5 (x[1] + x[2]) (x[2] + x[3]) (x[3] + x[4])
 (x[2] + x[3] + x[4]) (4 x[2] + 3 x[3] + 6 x[4])

{}
```

FIG. 6. Proof of the estimate for $b^n_{n,n}a_{n-1,n}$

(3) Calculate the coefficients of the polynomial $q := d \cdot p$ and verify that no coefficient of q is negative.

Let $\theta_n := b^n_{n,n}\left(a_{n-1,n} - \frac{a_{n-2,n}a_{n-1,n-1}}{a_{n-2,n-1}}\right)$. This expression is important, since it is used in Lemma 3.2 to estimate $|b^n_{j,n}|$ for $1 \leq j \leq n-1$. In the following lemma, we estimate the product of two consecutive values of θ_n. We will use this result in the proof of Proposition 5.3 to obtain explicit estimates for $|b^n_{j,n}|$.

Lemma 5.3. *Let A be as in Proposition 5.1 and $3 \leq n \leq m-1$. Then we have that*

$$\theta_n\theta_{n+1} \leq \frac{87}{100}\frac{(20)_n}{(30)_n}\frac{(20)_{n+1}}{(30)_{n+1}}. \tag{5.14}$$

Proof. We show that the rational function p, defined as

$$\frac{87}{100} - \frac{(30)_n}{(20)_n}\frac{(30)_{n+1}}{(20)_{n+1}}\theta_n\theta_{n+1}, \tag{5.15}$$

depending on the variables $x[1] = (10)_{n-2}, x[2] = (21)_{n-2}, \ldots, x[6] = (65)_{n-2}$ is nonnegative for nonnegative arguments. This is done with the *mathematica*-code in Figure 7 using the steps

(1) Define the rational function p as in (5.15).

(2) Determine the denominator d of the rational function p and verify that no coefficient of the polynomial d is negative.

(3) Determine the coefficients of the polynomial $q := d \cdot p$ and verify that no coefficient of q is negative.

```
p[x___] := Factor[87/100 - φ[3] φ[4] (x[3] + x[4] + x[5])/(x[3] + x[4]) (x[4] + x[5] + x[6])/(x[4] + x[5])
    (a[2, 3] - (a[1, 3] a[2, 2])/a[1, 2]) (a[3, 4] - (a[2, 4] a[3, 3])/a[2, 3]) /.
   Table[x[n] → List[x][[n]], {n, 1, Length[List[x]]}]];
d[x___] := Denominator[p[x]];
q[x___] := d[x] p[x];
Select[Flatten[CoefficientList[
   d[Sequence @@ Table[x[n], {n, 1, 6}]], Table[x[n], {n, 1, 6}]]], # < 0 &]
Select[Flatten[CoefficientList[q[Sequence @@ Table[x[n], {n, 1, 6}]],
   Table[x[n], {n, 1, 6}]]], # < 0 &]

{}

{}
```

FIG. 7. Proof of the estimate for the product of two consecutive values of θ_n

Proposition 5.3. *Let A be as in Proposition 5.1, $1 \le n \le m$ and $1 \le j \le n$. Then we have the estimate*

$$|b^n_{j,n}| \le C\frac{q^{n-j}}{\eta_{jn}} \quad \textit{with } q = (87/100)^{1/2},\ C = 12q^{-2}\Big(\frac{6}{5}\Big)^2.$$

Proof. We see in the first place that Remark 1 yields the estimate $b^n_{n,n} \le 12/(30)_n$ and thus the assertion of the theorem in the case $j = n$. If $j \le n-1$, we get from Lemma 3.2

$$|b^n_{j,n}| \le b^j_{j,j} b^{j+1}_{j+1,j+1} a_{j,j+1} \prod_{\ell=j+2}^{n} \theta_\ell. \tag{5.16}$$

Now assume that $n-(j+2)+1 = n-j-1$ is odd. By Lemmas 5.2 and 5.3 we conclude from (5.16) that

$$|b^n_{j,n}| \le \frac{12}{(30)_j}\Big(\frac{6}{5}\Big)^2 q^{n-j-2} \prod_{\ell=j+1}^{n} \frac{(20)_\ell}{(30)_\ell}. \tag{5.17}$$

We use induction and inequality (2.1) for η to obtain the final estimate

$$|b^n_{j,n}| \le 12q^{-2}\Big(\frac{6}{5}\Big)^2 \frac{q^{n-j}}{\eta_{jn}}. \tag{5.18}$$

If we assume that $n-(j+2)+1$ is even, the same reasoning yields the estimate

$$|b^n_{j,n}| \le 12q^{-1}\frac{6}{5}\frac{q^{n-j}}{\eta_{jn}}. \tag{5.19}$$

Inequalities (5.18) and (5.19) together now prove the proposition.

The passage to estimates of expressions $|b^n_{i,j}|$ for $1 \leq i, j \leq n$ is now an analogous calculation as in Theorem 4.1 and carried out in the following

Theorem 5.1. *Let A be as in Proposition 5.1 and $1 \leq n \leq m$. Then the entries $b^n_{i,j}$ of the matrix B_n satisfy the estimate*

$$|b^n_{i,j}| \leq C_1 \frac{q^{|i-j|}}{\eta_{ij}} \quad \text{for all } 1 \leq i, j \leq n, \tag{5.20}$$

where $C_1 = C\left(1 + \frac{12}{75}\frac{C}{1-q^2}\right)$ and C, q are as in Proposition 5.3.

Proof. Since B_n is symmetric and the case $j = n$ was treated in Proposition 5.3, it suffices to consider the cases $i \leq j \leq n-1$. Equations (3.3) – (3.5) yield the formula $b^n_{i,j} = b^{n-1}_{i,j} + b^n_{i,n} b^n_{j,n} / b^n_{n,n}$. By Lemma 3.2, inequality (3.10), we obtain

$$|b^n_{i,j}| \leq |b^{n-1}_{i,j}| + b^n_{n,n} a^2_{n-1,n} |b^{n-1}_{i,n-1} b^{n-1}_{j,n-1}|.$$

Applying Lemma 5.2 and Proposition 5.3 we estimate further

$$|b^n_{i,j}| \leq |b^{n-1}_{i,j}| + C^2 \frac{6}{5} a_{n-1,n} \frac{(20)_n}{(30)_n} \frac{q^{2n-2-i-j}}{\eta_{i,n-1}\eta_{j,n-1}}.$$

By induction, we get

$$\begin{aligned} |b^n_{i,j}| &\leq |b^j_{i,j}| + \frac{6}{5} C^2 \sum_{\ell=j}^{n-1} a_{\ell,\ell+1} \frac{(20)_{\ell+1}}{(30)_{\ell+1}} \frac{q^{2\ell-i-j}}{\eta_{i\ell}\eta_{j\ell}} \\ &\leq C \frac{q^{j-i}}{\eta_{ij}} \left(1 + \frac{6}{5} C \sum_{\ell=j}^{n-1} \frac{(20)_{\ell+1}}{(30)_{\ell+1}} a_{\ell,\ell+1} \frac{q^{2(\ell-j)}\eta_{ij}}{\eta_{i\ell}\eta_{j\ell}}\right). \end{aligned}$$

An easy consequence of formula (5.5) for $a_{\ell,\ell+1}$ is that $a_{\ell,\ell+1} \leq 2(31)_\ell/15 \leq 2\eta_{j\ell}/15$. This and the obvious inequalities $\eta_{ij} \leq \eta_{i\ell}$, $(20)_{\ell+1} \leq (30)_{\ell+1}$ for ℓ as in the above sum give the final estimate

$$|b^n_{i,j}| \leq C \frac{q^{j-i}}{\eta_{ij}} \left(1 + \frac{12}{75} C \sum_{\ell=j}^{\infty} q^{2(\ell-j)}\right) = C \frac{q^{j-i}}{\eta_{ij}} \left(1 + \frac{12}{75} \frac{C}{1-q^2}\right).$$

This proves Theorem 1.1 for piecewise quadratic splines, i.e. $k = 3$.

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SOLVING DECONVOLUTION TYPE PROBLEMS BY WAVELET DECOMPOSITION METHODS

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Abstract. In this paper we consider an inverse problem associated with equations of the form $\mathcal{K}f = g$, where $\mathcal{K}$ is a convolution-type operator. The aim is to find a solution f for given function g. We construct approximate solutions by applying a wavelet basis that is well adapted to this problem. For this basis we calculate the elementary solutions that are the approximate preimages of the wavelets. The solution for the inverse problem is then constructed as an appropriate finite linear combination of the elementary solutions. Under certain assumptions we estimate the approximation error and discuss the advantages of the proposed scheme.

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1. Introduction

We consider convolution type operators of the form

$$\mathcal{K}f(x) = 2\pi \int_{\mathbb{R}} \kappa(w) f(w) e^{iwx} dx, \tag{1.1}$$

where $\kappa \in \mathcal{C}^1(\mathbb{R})$ and f is a measurable function. Defining

$$\mathcal{D}_\kappa = \{\text{measurable } f : (\kappa f) \in L^2(\mathbb{R})\} \tag{1.2}$$

it results $\mathcal{K} : \mathcal{D}_\kappa \to L^2(\mathbb{R})$. If $g = \mathcal{K}f$, then its Fourier transform is $\widehat{g}(\omega) = (\kappa f)(\omega)$.

Although, in the general case, the integral operators $\mathcal{K}$ defined by (1.1) do not represent a convolution, by analogy, we call them *convolution-type operators*. The inverse problem (IP), associated with these operators consist in finding $f \in \mathcal{D}_\kappa$ such that

$$\mathcal{K}f = g \tag{1.3}$$

for a given $g \in L^2(\mathbb{R})$.

Note that in general we cannot assure that a solution f of the IP is the Fourier transform of a distribution $\check{f}$. In the ideal case we could construct f as : $f(\omega) = \widehat{g}(\omega)(\kappa(\omega))^{-1}$, but it is generally non viable because of its numerical instability if κ approaches to zero on a set of positive measure, or due to irregularities on $\widehat{g}$.

In order to find an approximate solution for the IP, we construct a sequence $f_J \in \mathcal{D}_\kappa \cap L^2(\mathbf{R})$ such that

$$\lim_{J\to+\infty} \|\mathcal{K}f_J - g\| = 0, \tag{1.4}$$

and the approximate solution will be the limiting function f: $f_J \to f$.

If we choose band-limited functions $f_J(\omega)$, $|\omega| \le \omega_J$, we have $f_J \in \mathcal{D}_\kappa \cap L^2(\mathbf{R})$. In that case if we set $\kappa_J(\omega)$, the restriction of $\kappa(\omega)$ to the band of frequencies, then $\mathcal{K}f_J$ is actually a convolution and we can write

$$\lim_{J\to+\infty} \|\check{\kappa}_J * \check{f}_J - g\| = 0. \tag{1.5}$$

Based on this idea, we construct f_J, the solution of the IP restricted to a compact set of frequencies, and consequently, the approximate solution f as the limit of f_J.

In this paper we choose an orthonormal wavelet basis $\psi_{jk}(x)$ associated with a hierarchical structure of the space, the multiresolution analysis (MRA) (see [15]). The scale function and the wavelets belong to the Schwartz class $\mathcal{S}$. They are smooth and infinitely oscillating functions with fast decay and compact support in two-sided bands Ω_j. They are well localized in both, time and frequency domains, and are well adapted to this problem (see [20]). Under suitable hypothesis on κ, it is possible to construct the *elementary solutions*, smooth functions μ_{jk} that are *nearly* the preimages of the wavelet basis $\psi_{jk}(x)$:

$$\begin{aligned} \mathcal{K}\mu_{jk}(x) &= (\check{\kappa} * \check{\mu}_{jk})(x) = 2\pi \int_{\Omega_j} \kappa(\omega)\mu_{jk}(\omega)e^{i\omega x}dw \\ &\cong \psi_{jk}(x). \end{aligned}$$

In this way, from the coefficients of the decomposition of g in the wavelet basis, we can estimate the components of the solution f on the subspaces generated by the elementary solutions (band limited) and the f_J are obtained. We also estimate the error of the approximation and discuss the advantages of the proposed scheme.

Integral operators, and in particular deconvolution problems, have been extensively studied, since they appear in various applications (electromagnetic measurements, design of digital filters , etc.). For instance, in [18] an efficient method for solving one dimensional deconvolution problems with noisy discrete data is presented and a regularizer is introduced, provided that the underlying operator can be decomposed into a sum of a compact and an invertible operators. Numerical aspects of this problem were analyzed in [19] and an optimal regularizer is constructed. The case of sampled independent identically distributed variables with random measurement error with deconvolution kernel density estimator is revisited in [7]. The relationship between deconvolution and correct sampling is explored in [13].

The Galerkin method for the case of integral operators with Hilbert kernels was proposed in [1]. In [11], [12] and [24] solvability and properties of the solutions of some integral and integro-differential equations were studied.
Techniques based on Wavelet Galerkin Methods and Wavelet Vaguelet Decomposition were studied in [5], [8] and [9] to find approximate solutions to IP associated to Pseudodifferential Operators (see [23]). Inverse problems associated with this kind of operators have been studied in [5], [8] and [9], where techniques based on wavelet Galerkin methods and wavelet Vaguelet decompositions were developed. In [22], using deconvolution techniques, we obtained approximate solutions for the equation $\mathcal{K}f = \check{\kappa} * \check{f} = g$ with $\kappa(\omega) = (1+\omega^2)^{-\alpha}$, $\alpha > 1$ for $n = 1$. In [22] a wavelet Vaguelet decomposition (WVD) was used to approximate f for a more general case. In [21] the inverse problem associated with a pseudodifferential operator whose symbol has separated variables is analyzed.

The present paper is organized as follows. In Section 2 we introduce the wavelet basis. In Section 3 we construct an approximate solution for IP, and analyze the approximation error. Two examples are discussed in Section 4. Finally, we state our conclusions in Section 5.

2. The Wavelet Basis

We choose a mother wavelet ψ, which is well localized in both time and frequency domains and possesses the following properties (see [14]):

- the family $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbf{Z}$, forms an orthonormal basis in $L^2(\mathbf{R})$ associated to MRA;
- $\psi \in \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space, is a smooth, infinitely oscillating mother wavelet with fast decay;
- the spectrum $|\widehat{\psi}(2^{-j}\omega)|$ is supported on the two-sided band
  $\Omega_j = \{\omega : 2^j(\pi-\alpha) \le |\omega| \le 2^{j+1}(\pi+\alpha)\}$, for some $0 < \alpha \le \pi/3$.
- the numerical implementation of the associated scheme is efficient.

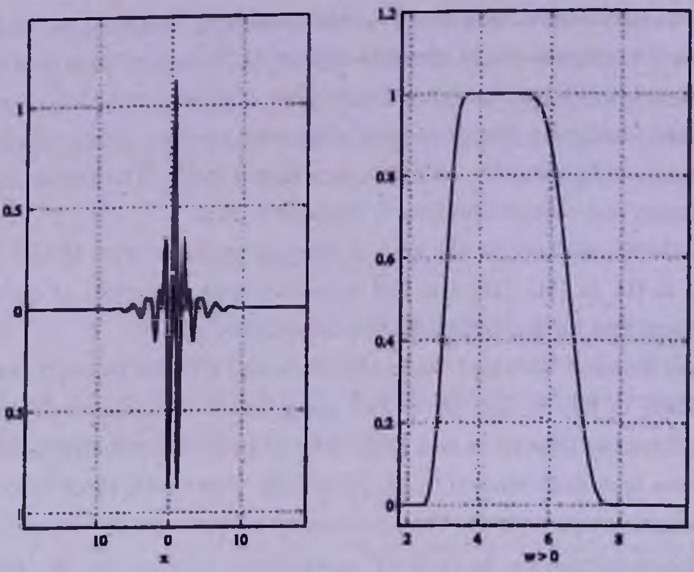

FIG. 1. (a) Mother wavelet with $\alpha = \pi/4$, (b) $|\widehat{\psi}|$ for $\omega \geq 0$

There also exists $\phi \in V_0$ such that $\{\phi(x-k),\ k \in Z\}$ is an orthonormal basis for V_0. The design of this basis and the implementation algorithm based on the Fourier fast transform (FFT) have been developed by authors in [22].

Let $W_j = \text{span}\{\psi_{jk}, k \in \mathbf{Z}\}$ and $V_J = \bigoplus_{j<J} W_j$ be the wavelet and the scales subspaces, respectively. Observe that each family $\phi_{jn}(x) = 2^{j/2}\phi(2^j x - n)$, $n \in \mathbf{Z}$, is an orthonormal basis of V_J.

For any signal $s \in L^2(\mathbf{R})$ we denote by $\mathcal{Q}_j(s)$ and $_j(s)$ the orthogonal projections of s onto the subspaces W_j and V_j, respectively. Then for any index J, the following representation holds:

$$
\begin{aligned}
s(x) &= \sum_{j\in\mathbf{Z}} \mathcal{Q}_j s(x) = \mathcal{P}_J s(x) + \sum_{j\geq J} \mathcal{Q}_j s(x) = \\
&= \sum_{n\in\mathbf{Z}} \langle s, \phi_{Jn}\rangle\, \phi_{Jn}(x) + \sum_{j\geq J} \sum_{k\in\mathbf{Z}} \langle f, \psi_{jk}\rangle\, \psi_{jk}(x). \qquad (2.1)
\end{aligned}
$$

We remark that $\widehat{\mathcal{Q}}_j s(\omega)$ is supported on Ω_j, while $\widehat{\mathcal{P}}_J s(\omega)$ is supported on $\cup_{j\leq J}\Omega_j$.

Notice also that the properties of ψ ensure uniform convergence on each W_j. In addition, since ψ is infinitely oscillating, it has vanishing moments: $\int_{\mathbf{R}} x^n \psi(x)\, dx = 0$ for all $n \in \mathbf{N}_0$, and the same occurs to its polynomials' components.

3. An approximate solution to the IP

3.1. The alternative problem. Without loss of generality we can assume that g is a real function and that the real and imaginary parts of the kernel $\kappa(w)$ are even and odd functions, respectively. If it is not the case, we can split the corresponding problems into two subproblems associated to the kernels $\kappa_1(w) = \frac{\kappa(w)+\kappa^*(-w)}{2}$ and $\kappa_2(w) = \frac{\kappa(w)-\kappa^*(-w)}{2}$, respectively.

Regarding the given function g, called also the data function, we suppose that there exist an integer J such that an appropriate approximation $g_J \in V_J$ of g is available, given by

$$g_J(x) = \mathcal{P}_J g(x) = 2\pi \int_{|w|\leq w_{max}} \kappa(\omega) f(\omega) e^{ix\omega}\, d\omega,$$

where $\omega_{max} = 2^{J+1}(\pi + \alpha)$. Then we consider the following alternative problem:

$$\mathcal{K} f = g_J. \tag{3.1}$$

Taking into account that

$$g_J(x) = \sum_{j<J} \mathcal{Q}_j g(x), \tag{3.2}$$

it seems natural to solve the IP on each subspace W_j:

$$g_j(x) = \mathcal{Q}_j g(x) = 2\pi \int_{\Omega_j} \kappa(\omega) f(\omega) e^{ix\omega}\, d\omega,$$

$$g_j(x) \triangleq \mathcal{K}_j f(x). \tag{3.3}$$

Observe that, in general, the subspaces W_j are not in the rank of the operator and we cannot assure that the problem (3.3) has a solution in the strict sense. In order to solve it, we propose to construct functions $f_j \in \mathcal{D}_\kappa \cap L^2(\mathbf{R})$ to satisfy

$$\mathcal{K}_j f_j(x) = g_j(x) + e_j$$

with $\|e_j\| \leq \epsilon \parallel g_j \parallel$ for some small ϵ.

In that case we will have

$$g_J = \sum_{j<J} \mathcal{K}_j f_j + e_J = \mathcal{K}\left(\sum_{j<J} f_j\right) + e_J$$

with $\|e_J\| \leq \epsilon \parallel g_J \parallel$. We remark that each f_j is a band limited function supported on Ω_j, and hence the approximate solution $f_J = \sum_{j<J} f_j$ is a band limited function in $\mathcal{D}_\kappa \cap L^2(\mathbf{R})$. We hope that under suitable assumptions, the sequence f_J will converge in some sense, to be explained later, to an appropriate approximate solution of

(1.3). In order to construct such solutions f_J, in the next subsection we consider the decomposition (3.2) and define the *preimages* of the wavelet basis.

3.2. The elementary solutions. In this subsection we construct the *elementary solutions* μ_{jk} to be the preimages of the wavelet basis ψ_{jk}. To this end, we will distinguish the following two cases:

- the functions μ_{jk} are the preimages of the wavelet basis ψ_{jk}: $\mathcal{K}_j\mu_{jk} = \psi_{jk}$.
- the wavelet basis ψ_{jk} is not in the rank of $\mathcal{K}_j$.

In the first case, it is worth noting that if $|\kappa(w)| \geq c > 0$ on Ω_j, then we can compute $\mu_{jk} = \frac{\widehat{\psi}_{jk}}{\kappa}$.

In the second case, where the wavelets ψ_{jk} are not in the rank of the operator $\mathcal{K}_j$, we can construct approximate preimages μ_{jk} that satisfy $\mathcal{K}_j\mu_{jk} = \widetilde{\psi}_{jk} \cong \psi_{jk}$.

We observe that if $|\kappa(\omega)| \equiv 0$ on some interval $\omega_1 \leq |\omega| \leq \omega_2$ in Ω_j, then the same occurs to $|kf_j(\omega)|$. In this case the problem $\mathcal{K}_j f_j = g_j$ either has no unique solution or it is incompatible. For these reasons, in what follows we assume:

Hypothesis 3.1. *The set* $\emptyset_j = \{\omega \in \Omega_j : |\kappa(\omega)| = 0\}$ *has null measure for all* $j \in \mathbb{Z}$. Then the eventual roots of κ must be isolated.

Definition 3.1. Under the Hypothesis 3.1 the *elementary solutions* μ_{jk}, $j, k \in \mathbb{Z}$, for the operator $\mathcal{K}_j$ are defined by

$$\mu_{jk}(\omega) = \frac{\widehat{\psi}_{jk}(\omega)\,\kappa^*(\omega)}{|\kappa(\omega)|^2 + \rho_j^2(\omega)}\,,\ \omega \in \Omega_j, \tag{3.4}$$

where ρ_j are even smooth functions satisfying $\rho_j(\omega) \geq \rho > 0$ in some neighborhood of the roots of $\kappa(\omega)$ in Ω_j.

Definition 3.2. For $j, k \in \mathbb{Z}$ we define $\widetilde{\psi}_{jk} \triangleq \mathcal{K}_j\mu_{jk}$. For the following proposition we refer to [21].

Proposition 3.1. *Under the Hypothesis 3.1 the following properties hold:*

- $\mu_{jk}(\omega) = \mu_{j0}(\omega)\, e^{-i\omega k/2^j}$, *that is,* $\check{\mu}_{jk}(x) = \check{\mu}_{j0}(x - 2^{-j}k)$.
- *If* μ_{j0} *is a band limited function on* Ω_j, *then* $\check{\mu}_{j0}$ *inherits the smoothness properties of* ψ.
- *If* $|k(w)| \geq c > 0$, *then* $\rho_j \equiv 0$ *on* Ω_j. *On the other hand,* ρ_j *can be chosen based on* $k(\omega)$ *for each level* j, *such that* $|support(\rho_j)| \leq \epsilon$.
- *In such cases,* $\| \psi_{jk} - \widetilde{\psi}_{jk} \| \leq \epsilon$.

- *For each j there exist constants m_j and M_j such that for all $w \in \Omega_j$*

$$\frac{|k(w)|^2}{(|k(w)|^2 + |\rho_j(w)|^2)^2} \leq m_j, \ \ \|\tilde{\mu}_{jk}\|^2 \leq M_j, \ k \in K_j.$$

- *The functions ρ_j can be chosen such that μ_{j0} preserves the smoothness properties of $\widehat{\psi}$.*
- *The family $\{\tilde{\mu}_{jk}, \ k \in \mathbf{Z}\}$ consists of linearly independent elements. Moreover, it is a basis of the subspace $\mathcal{U}_j = span\,\{\tilde{\mu}_{jk}, \ k \in \mathbf{Z}\} \subset L^2(\mathbf{R})$ and it is a Bessel sequence (see [4]).*

3.3. The proposed solution. Observe first that by Proposition 3.1, the families $\{\widetilde{\psi}_{jk}\}$ and $\{\psi_{jk}\}$ consist of exponentially decaying functions and are nearly biorthogonal, that is, $\left\langle \widetilde{\psi}_{jk}, \psi_{jk} \right\rangle \simeq 1$ and $\left\langle \widetilde{\psi}_{jn}, \psi_{jk} \right\rangle \simeq 0$ for $n \neq k$. Also, the family $\{\widetilde{\psi}_{jk}\}$ consist of linearly independent functions and, in addition, it is a Bessel sequence.

In order to calculate a solution for the alternative problem (3.1), for each $j < J$ we choose appropriate finite subsets $\mathbf{K}_j$, and construct an approximate solution on $\widetilde{W}_j = \text{span}\{\widetilde{\psi}_{jk}, k \in \mathbf{Z}\} \subset L_2(\mathbf{R})$ as follows: $\widetilde{\mathfrak{Q}}_j g = \sum_{k \in \mathbf{K}_j} d_{jk} \widetilde{\psi}_{jk}$. We define $\widetilde{\mathfrak{Q}}_j f = \sum_{k \in \mathbf{K}_j} d_{jk} \mu_{jk}$, and observe that

$$\mathcal{K} \widetilde{\mathfrak{Q}}_j f = \widetilde{\mathfrak{Q}}_j g = \mathfrak{Q}_j g + \Delta_j g.$$

We propose to select the coefficients d_{jk} to satisfy the linear system:

$$\sum_{k \in \mathbf{K}_j} \langle g, \psi_{jk} \rangle \, \psi_{jk} = \sum_{k \in \mathbf{K}_j} d_{jk} \, \widetilde{\psi}_{jk},$$

that is,

$$\langle g, \psi_{jn} \rangle = \sum_{k \in \mathbf{K}_j} d_{jk} \left\langle \psi_{jn}, \widetilde{\psi}_{jk} \right\rangle, \ \ n \in \mathbf{K}_j,$$

yielding

$$\Delta_j g = \sum_{k \notin \mathbf{K}_j} \langle g, \psi_{jk} \rangle \, .$$

We observe that $\Delta_j g$ is a projection that is nearly orthogonal to W_j on $\widetilde{W}_j$.

Since the error of the approximation will depend on the wavelet coefficients of the data function g, it can be controlled by choosing for each $J_0 \leq j < J$, the finite sets $\mathbf{K}_j$ to satisfy

$$\|\Delta_j g\|^2 = \sum_{k \notin \mathbf{K}_j} |\langle g, \psi_{jk} \rangle|^2 < \frac{\epsilon}{2} \, \| \, \mathfrak{Q}_j g \, \|^2 \tag{3.5}$$

for some small $\epsilon > 0$.

To carry out the numerical implementation of the scheme, we choose a minimum level J_0, such that

$$\| g_J - \sum_{j<J_0} \Omega_j g \|^2 < \frac{\epsilon}{2} \| g_J \|^2 . \tag{3.6}$$

Finally,

$$\tilde{f}_J = \sum_{J_0 \le j < J} \tilde{\Omega}_j f = \sum_{J_0 \le j < J} \sum_{k \in \mathbf{K}_j} d_{jk} \mu_{jk}$$

is an approximate solution for the IP associated with the alternative problem (3.1). We have

$$\mathcal{K}\tilde{f}_J = \sum_{J_0 \le j < J} \tilde{\Omega}_j g = \sum_{J_0 \le j < J} \sum_{k \in \mathbf{K}_j} \langle g, \psi_{jk} \rangle \psi_{jk} = g_J + \Delta_J g.$$

3.4. The Error. Under the hypothesis and definitions described above, from (3.5) - (3.6) we have

$$\begin{aligned} \| \Delta_J g \|^2 &= \| g_J - \mathcal{K}\tilde{f}_J \|^2 \\ &= \| \sum_{j<J_0} \Omega_j g \|^2 + \sum_{J_0 \le j < J} \sum_{k \notin \mathbf{K}_j} | \langle g, \psi_{jk} \rangle |^2 \\ &< \frac{\epsilon}{2} \| g_J \|^2 + \frac{\epsilon}{2} \sum_{J_0 \le j < J} \| \Omega_j g \|^2 = \epsilon \| g_J \|^2 \end{aligned}$$

Assuming that the initial approximate data function g_J satisfies

$$\| g - g_J \|^2 = \| \sum_{j \ge J} \langle g, \psi_{jk} \rangle \psi_{jk} \|^2 \| g \|^2 < \epsilon,$$

we obtain $\| g - \mathcal{K}\tilde{f}_J \|^2 < 2\epsilon$.

Remark 3.1. We note that it is not always convenient or even possible to disregard the low frequency components around $\omega = 0$. In particular, this is the case if g is not an oscillating function or if it is a distribution. In such cases it is important to include the component in the scale space V_{J_0}, and consequently the low frequencies of elementary solutions must be defined and included in the approximation scheme.

4. Examples

Example 4.1. We consider integro-differential operators with kernels:

$$\kappa(\omega) = (1 + |\omega|^2)^\alpha,$$

where $0 < |\alpha| < 1$. In this case the kernel $\kappa(\omega)$ is real and even.

Note that if $\alpha \le -\frac{1}{2}$ we have $\kappa \in L^2(\mathbf{R})$ and $\mathcal{D}_\kappa \supseteq L^2(\mathbf{R})$. On the other hand, if $\alpha > -\frac{1}{2}$, then $\mathcal{D}_\kappa \subset L^2(\mathbf{R})$. Since $\kappa(\omega) > 0$, we have $\tilde{\psi}_{jk}(\omega) = \psi_{jk}$ for all j and k.

Example 4.2. We consider transfer functions of the form:

$$H(s) = \frac{P(s)}{Q(s)},$$

where P and Q are polynomials of degree m and n ($m < n$), respectively. Observe that H describes the relationship between the input U and the output Y of a linear time-invariant system:

$$Y(s) = H(s)U(s) = \frac{P(s)}{Q(s)}U(s).$$

When the poles and zeros of H lie in the half-plane $\mathbf{C}^-$, the system is stable and we have $\kappa(\omega) = H(\imath\omega) \in L^2(\mathbb{R})$. In this case, for a given output Y it is possible to identify the input U based on the relationship between the Fourier transform and the convolution operator. In this case the assertion $\widetilde{\psi}_{jk}(\omega) = \psi_{jk}$ remains true.

When the output Y is a distribution, it is necessary to consider the low frequency elementary solutions as explained in the Remark 3.1.

In this frame, we could also propose an identification scheme, that is, a scheme that identifies H, considering the kernels $\kappa_n = U_n$ as test-inputs.

5. Conclusions

In this paper approximate solutions for inverse problems, associated with the equation $\mathcal{K}f = g$, where $\mathcal{K}$ is a convolution-type operator, are constructed, and the corresponding approximation error is analyzed. A perturbed data function g is considered. The data function g is decomposed in a suitable orthonormal wavelet basis ψ_{jk} and the elementary solutions $\tilde{\mu}_{jk}$ are calculated, which are the preimages of the wavelet basis trough $\mathcal{K}$. The case when ψ_{jk} are not in the rank of $\mathcal{K}$ is also studied. In both cases the data function g can be expressed as a linear combination of the images of the elementary solutions via the operator $\mathcal{K}$. The approximate solution of the IP is constructed as an appropriate finite linear combination of the elementary solutions. In this way the proposed scheme takes into account both, the characteristics of the data we want to invert and the properties of the underlying operator.

The proposed scheme can be adapted to make it applicable for the cases of noisy data and for convolution-type operators with more general kernels.

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VARIATION FORMULAS OF SOLUTION FOR A FUNCTIONAL DIFFERENTIAL EQUATION WITH DELAY FUNCTION PERTURBATION

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Abstract. Variation formulas of solution for a nonlinear functional differential equation with variable delay and continuous initial condition are proved. The effects of delay function perturbation and continuous initial condition are detected in the variation formulas. The continuity of the initial condition means that the values of the initial function and the trajectory always coincide at the initial moment.

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1. Introduction and main results

Let $I = [a, b]$ be a finite interval and let $\mathbb{R}^n_x$ be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T stands for transposition. Suppose that $O \subset \mathbb{R}^n_x$ is an open set, and E_f is the set of functions $f : I \times O^2 \to \mathbb{R}^n_x$ satisfying the following conditions:

- for almost all $t \in I$, the function $f(t, \cdot) : O^2 \to \mathbb{R}^n$ is continuously differentiable;
- for every $(x, y) \in O^2$, the functions $f(t, x, y), f_x(t, x, y)$ and $f_y(t, x, y)$ are measurable on I;
- for each $f \in E_f$ and for a compact set $K \subset O$ there exists a function $m_{f,K}(t) \in L(I, [0, \infty))$, such that

$$| f(t, x, y) | + | f_x(t, x, y) | + | f_y(t, x, y) | \leq m_{f,K}(t)$$

for all $(x, y) \in K^2$ and almost all $t \in I$.

Let D be the set of continuously differentiable scalar functions $\tau(t), t \in I$, called delay functions, satisfying the conditions: $\tau(t) < t$, $\dot{\tau}(t) > 0$ and $\hat{\tau} := \inf\{\tau(a) : \tau \in D\}$

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is finite. Let E_φ be the space of continuous functions $\varphi(t) \in \mathbb{R}^n_x$, $t \in I_1 := [\hat{\tau}, b]$. By $\Phi = \{\varphi \in E_\varphi : \varphi(t) \in O, t \in I_1\}$ we denote the set of initial functions.

To each element $\mu = (t_0, \tau, \varphi, f) \in \Lambda = [a, b) \times D \times \Phi \times E_f$ we assign the delay functional differential equation

$$\dot{x}(t) = f(t, x(t), x(\tau(t))) \tag{1.1}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\hat{\tau}, t_0]. \tag{1.2}$$

The condition (1.2) is said to be continuous initial condition since always we have $x(t_0) = \varphi(t_0)$.

Definition 1.1. *Let $\mu = (t_0, \tau, \varphi, f) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\hat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of equation* (1.1) *with the initial condition* (1.2), *or equivalently, a solution corresponding to the element μ and defined on the interval $[\hat{\tau}, t_1]$, if $x(t)$ satisfies the condition* (1.2), *is absolutely continuous on the interval $[t_0, t_1]$, and satisfies the equation* (1.1) *almost everywhere on $[t_0, t_1]$.*

Let $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0) \in \Lambda$ be a fixed element, with $a < t_{00}$. In the space $E_\mu = \mathbb{R}^1_t \times D \times E_\varphi \times E_f$ we introduce the set of variations:

$$V = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta f) \in E_\mu - \mu_0 : |\, \delta t_0 \,| \le \alpha, \|\, \delta\tau \,\| \le \alpha, \delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i,$$

$$\delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, |\, \lambda_i \,| \le \alpha, i = 1, \dots, k\Big\}, \tag{1.3}$$

where $\delta\varphi_i \in E_\varphi - \varphi_0$, $\delta f_i \in E_f - f_0, i = 1, \dots, k$, are fixed functions, $\alpha > 0$ is a fixed number, and $\|\tau\| := \sup\{|\tau(t)| : t \in I\}$.

Let $x_0(t)$ be the solution corresponding to the element μ_0 and defined on the interval $[\hat{\tau}, t_{10}]$, with $t_{00} < t_{10} < b$. Then there exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$ we have $\mu_0 + \varepsilon\delta\mu = (t_0, \tau, \varphi, f_0 + \varepsilon\delta f) \in \Lambda$, where $t_0 = t_{00} + \varepsilon\delta t_0$, $\tau = \tau_0 + \varepsilon\delta\tau$, $\varphi = \varphi_0 + \varepsilon\delta\varphi$, and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to the element $\mu_0 + \varepsilon\delta\mu$ (see Theorem 2.1). It is clear that $x(t; \mu_0 + \varepsilon\delta\mu)$ satisfies the perturbed equation $\dot{x}(t) = f_0(t, x(t), x(\tau(t))) + \varepsilon\delta f(t, x(t), x(\tau(t)))$ with the perturbed initial condition $x(t) = \varphi(t), t \in [\hat{\tau}, t_0]$.

Due to the uniqueness, the solution $x(t;\mu_0)$ is an extension of the solution $x_0(t)$ to the interval $[\hat{\tau}, t_{10}+\delta_1]$. Therefore, in what follows, the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10}+\delta_1]$.

Next, we define the increment of the solution $x_0(t) = x(t;\mu_0)$ as follows:

$$\Delta x(t) = \Delta x(t;\varepsilon\delta\mu) = x(t;\mu_0+\varepsilon\delta\mu) - x_0(t)$$

for all $(t,\varepsilon,\delta\mu) \in [\hat{\tau}, t_{10}+\delta_1] \times [0,\varepsilon_1] \times V$.

Now we are in position to state our main results.

Theorem 1.1. *Let the following conditions hold:*

1.1) *the function $\varphi_0(t), t \in I_1$ is absolutely continuous and the function $\dot{\varphi}_0(t)$ is bounded;*

1.2) *the function $f_0(t,x,y), (t,x,y) \in I \times O^2$ is bounded;*

1.3) *there exist the finite limits $\dot{\varphi}_0^- = \dot{\varphi}_0(t_{00}-)$, $\lim_{w\to w_0} f_0(w) = f_0^-$, $w = (t,x,y) \in (a, t_{00}] \times O^2$, where $w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$.*

Then there exist numbers $\varepsilon_2 \in (0,\varepsilon_1)$ and $\delta_2 \in (0,\delta_1)$ such that

$$\Delta x(t;\varepsilon\delta\mu) = \varepsilon\delta x(t;\delta\mu) + o(t;\varepsilon\delta\mu),^2 \tag{1.4}$$

for any $(t,\varepsilon,\delta\mu) \in [t_{00}, t_{10}+\delta_2] \times (0,\varepsilon_2) \times V^-$, where $V^- = \{\delta\mu \in V : \delta t_0 \le 0\}$, and

$$\delta x(t;\delta\mu) = Y(t_{00};t)(\dot{\varphi}_0^- - f_0^-)\delta t_0 + \beta(t;\delta\mu), \tag{1.5}$$

$$\beta(t;\delta\mu) = Y(t_{00};t)\delta\varphi(t_{00}) + \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds$$

$$- \int_{t_{00}}^{t} Y(s;t) f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)ds + \int_{t_{00}}^{t} Y(s;t)\delta f[s]ds. \tag{1.6}$$

Here $Y(s;t)$ stands for a $n \times n$-matrix function satisfying the equation

$$Y_s(s;t) = -Y(s;t)f_{0x}[s] - Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s), \quad s \in [t_{00}, t] \tag{1.7}$$

and the condition

$$Y(s;t) = \begin{cases} H & \text{for } s = t \\ \Theta & \text{for } s > t, \end{cases} \tag{1.8}$$

where

$$f_{0y}[s] = f_{0y}(s, x_0(s), x_0(\tau_0(s))), \quad \delta f[s] = \delta f(s, x_0(s), x_0(\tau_0(s))),$$

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[2] Here and below, the symbols $O(t;\varepsilon\delta\mu)$, and $o(t;\varepsilon\delta\mu)$ stand for quantities (scalar or vector) that have the corresponding order of smallness (with respect to ε) uniformly with respect to $(t,\delta\mu)$.

$\gamma_0(s)$ is the inverse of the function $\tau_0(t)$, H is the identity matrix and Θ is the zero matrix.

Some comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t), t \in [t_{00}, t_{10}+\delta_2]$, and the expression (1.5) is called the variation formula. On the basis of Cauchy formula on reprèsentation of solutions of linear delay functional differential equations (see [2], p.21), we conclude that the function

$$\delta x(t) = \begin{cases} \delta\varphi(t), & t \in [\hat{\tau}, t_{00}) \\ \delta x(t;\delta\mu), & t \in [t_{00}, t_{10}+\delta_2] \end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + f_{0y}[t]\delta x(\tau_0(t)) - f_{0y}[t]\dot{x}_0(\tau_0(t))\delta\tau(t) + \delta f[t]$$

with the initial condition

$$\delta x(t) = \delta\varphi(t),\ t \in [\hat{\tau}, t_{00}),\ \delta x(t_{00}) = (\dot{\varphi}_0^- - f_0^-)\delta t_0 + \delta\varphi(t_{00}).$$

The term

$$-\int_{t_{00}}^{t} Y(s;t) f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)ds$$

in formula (1.6) is the effect of perturbation of the delay function $\tau_0(t)$.

The expression $Y(t_{00};t)(\dot{\varphi}_0^- - f_0^-)\delta t_0$ is the effect of the initial condition (1.2) and perturbation of the initial moment t_{00}. The expression

$$Y(t_{00};t)\delta\varphi(t_{00}) + \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds + \int_{t_{00}}^{t} Y(s;t)\delta f[s]ds$$

in formula (1.6) is the effect of perturbations of the initial function $\varphi_0(t)$ and the right-hand side of equation (1.1).

Notice also that the variation formula allows to obtain an approximate solution of the perturbed functional differential equation $\dot{x}(t) = f(t, x(t), x(\tau(t))) + \varepsilon\delta f(t, x(t), x(\tau(t)))$ with the perturbed initial condition $x(t) = \varphi_0(t) + \varepsilon\delta\varphi(t)$, $t \in [\hat{\tau}, t_{00}+\varepsilon\delta t_0]$. In fact, for a sufficiently small $\varepsilon \in (0, \varepsilon_2]$ it follows from (1.4) that $x(t;\mu_0+\varepsilon\delta\mu) \approx x_0(t) + \varepsilon\delta x(t;\delta\mu)$.

Variation formulas of solution play an important role in proving optimality necessary conditions for optimization problems (see [1-3]). Variation formulas for various classes of functional differential equations without perturbation of delay function are given in [2,3].

Theorem 1.2. *Let the function $\varphi_0(t)$, $t \in I_1$ be absolutely continuous and let the functions $\dot{\varphi}_0(t)$ and $f_0(t,x,y)$, $(t,x,y) \in I \times O^2$ be bounded. Moreover, there exist the finite limits $\dot{\varphi}_0^+ = \dot{\varphi}_0(t_{00}+)$, $\lim_{w \to w_0} f_0(w) = f_0^+$, $w \in [t_{00}, b) \times O^2$. Then for each $\hat{t}_0 \in (t_{00}, t_{10})$ there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for any $(t, \varepsilon, \delta\mu) \in [\hat{t}_0, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^+$, where $V^+ = \{\delta\mu \in V : \delta t_0 \geq 0\}$, the formula* (1.4) *holds with*

$$(1.9) \qquad \delta x(t; \delta\mu) = Y(t_{00}; t)(\dot{\varphi}_0^+ - f_0^+)\delta t_0 + \beta(t; \delta\mu).$$

As an immediate consequence of Theorems 1.1 and 1.2 we can state the following result.

Theorem 1.3. *Let the assumptions of Theorems* 2.1 *and* 2.2 *be fulfilled. Moreover, assume that $\hat{f}_0 := \dot{\varphi}_0^- - f_0^- = \dot{\varphi}_0^+ - f_0^+$. Then for each $\hat{t}_0 \in (t_{00}, t_{10})$ there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for any $(t, \varepsilon, \delta\mu) \in [\hat{t}_0, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V$, the formula* (1.4) *holds with*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)\hat{f}_0 \delta t_0 + \beta(t; \delta\mu).$$

Notice that all the assumptions of Theorem 1.3 are satisfied if the function $f_0(t,x,y)$ is continuous and bounded, and the function $\varphi_0(t)$ is continuously differentiable. It is clear that in this case we have

$$\hat{f}_0 = \dot{\varphi}_0(t_{00}) - f_0(t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00}))).$$

2. Auxiliary assertions

In this section we state and prove some auxiliary results that will be used in the proofs of main results.

Theorem 2.1. *([4], Theorem 2). Let $x_0(t)$ be the solution corresponding to the element $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0) \in \Lambda$ and defined on $[\hat{\tau}, t_{10}]$ (see Definition 1.1), let $t_{i0} \in (a,b), i = 0,1$ and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following assertions hold:*

2.1) *there exist numbers $\varepsilon_1 > 0$ and$\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to this element. Moreover, $x(t; \mu_0 + \varepsilon\delta\mu) \in K_1, t \in [\hat{\tau}, t_{10} + \delta_1]$;*

2.2) $\lim_{\varepsilon\to 0}\sup\left\{|x(t;\mu_0+\varepsilon\delta\mu)-x(t;\mu_0)| : t\in[\hat{t},t_{10}+\delta_1]\right\}=0$, *uniformly in* $\delta\mu\in V$, *where* $\hat{t}=\max\{t_{00},t_0\}$.

Theorem 2.2. *Let the conditions of Theorem 1.1 hold. Then there exist numbers* $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that*

$$\max_{t\in[\hat{\tau},t_{10}+\delta_2]}|\Delta x(t)|\le O(\varepsilon\delta\mu) \tag{2.1}$$

for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2)\times V^-$. *Moreover,*

$$\Delta x(t_{00})=\varepsilon[\delta\varphi(t_{00})+(\dot{\varphi}_0^- - f_0^-)\delta t_0]+o(\varepsilon\delta\mu). \tag{2.2}$$

Proof. We first note that $\delta\mu\in V^-$, that is, $\delta t_0\le 0$. Therefore $t_0\le t_{00}$ and

$$\Delta x(t)=\begin{cases}\varphi(t)-\varphi_0(t)=\varepsilon\delta\varphi(t), & t\in[\hat{\tau},t_0),\\ x(t)-\varphi_0(t), & t\in[t_0,t_{00}),\\ x(t)-x_0(t), & t\in[t_{00},t_{10}+\delta_1].\end{cases}$$

Here, and in what follows, we assume that $t_0=t_{00}+\varepsilon\delta t_0$, $x(t)=x(t;\mu_0+\varepsilon\delta\mu)$ and $\varphi(t)=\varphi_0(t)+\varepsilon\delta\varphi(t)$.

Let $t\in[\hat{\tau},t_0]$, then by (1.3) we have

$$|\Delta x(t)|=\varepsilon|\delta\varphi(t)|\le\varepsilon\alpha\sum_{i=1}^{k}|\delta\varphi_i(t)|\le O(\varepsilon\delta\mu). \tag{2.3}$$

Let $t\in[t_0,t_{00}]$, then in view of (1.3), and the conditions 1.1) and 1.2) of Theorems 1.1 and 2.1, we can write

$$|\Delta x(t)|=|x(t)-\varphi_0(t)|=|\varphi(t_0)+\int_{t_0}^{t}\Big[f_0(s,x(s),x(\tau(s)))+\varepsilon\delta f(s,x(s),x(\tau(s)))\Big]ds$$

$$-\varphi_0(t)|\le\varepsilon|\delta\varphi(t_0)|+|\varphi_0(t_0)-\varphi_0(t)|+\int_{t_0}^{t_{00}}\Big[|f_0(s,x(s),x(\tau(s)))|$$

$$+\varepsilon\alpha\sum_{i=1}^{k}m_{\delta f_i,K_1}(s)\Big]ds\le O(\varepsilon\delta\mu)+\int_{t_0}^{t_{00}}|\dot{\varphi}_0(s)|\,ds\le O(\varepsilon\delta\mu). \tag{2.4}$$

Observe that on the interval $[t_{00},t_{10}+\delta_1]$ the function $\Delta x(t)=x(t)-x_0(t)$ satisfies the equation

$$\dot{\Delta x}(t)=a(t;\varepsilon\delta\mu)+b(t;\varepsilon\delta\mu), \tag{2.5}$$

where

(2.6)

$$a(t;\varepsilon\delta\mu)=f_0(t,x_0(t)+\Delta x(t),x_0(\tau(t))+\Delta x(\tau(t)))-f_0[t],\ f_0[t]=f_0(t,x_0(t),x_0(\tau_0(t)))$$

and

$$b(t;\varepsilon\delta\mu)=\varepsilon\delta f(t,x_0(t)+\Delta x(t),x_0(\tau(t))+\Delta x(\tau(t))). \tag{2.7}$$

We rewrite the equation (2.5) in the integral form

$$\Delta x(t)=\Delta x(t_{00})+\int_{t_{00}}^{t}[a(s;\varepsilon\delta\mu)+b(s;\varepsilon\delta\mu)]ds.$$

Hence it follows that

$$|\,\Delta x(t)\,|\leq|\,\Delta x(t_{00})\,|+a_1(t;\varepsilon\delta\mu)+b(\varepsilon\delta\mu), \tag{2.8}$$

where

$$a_1(t;\varepsilon\delta\mu)=\int_{t_{00}}^{t}|\,a(s;\varepsilon\delta\mu)\,|\,ds,\; b(\varepsilon\delta\mu)=\int_{t_{00}}^{t_{10}+\delta_1}|\,b(s;\varepsilon\delta\mu)\,|\,ds.$$

Let a number $\varepsilon_2' \in (0,\varepsilon_1)$ be chosen so small that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_0,t_{00}]\times(0,\varepsilon_2')\times V^-$ the following inequality holds:

$$\tau(t)=\tau_0(t)+\varepsilon\delta\tau(t)<t_0. \tag{2.9}$$

We first prove the formula (2.2). To this end observe that by (1.2) and (2.9) we have

$$\Delta x(t_{00})=x(t_{00})-x_0(t_{00})$$

$$=\varphi(t_0)+\int_{t_0}^{t_{00}}[f_0(t,x_0(t)+\Delta x(t),\varphi(\tau(t)))+b(t;\varepsilon\delta\mu)]dt-\varphi_0(t_{00}), \tag{2.10}$$

Next, since

$$\int_{t_{00}}^{t_0}\dot\varphi_0(t)dt=\varepsilon\dot\varphi_0^-\delta t_0+o(\varepsilon\delta\mu)$$

and

$$\lim_{\varepsilon\to0}\delta\varphi(t_0)=\delta\varphi(t_{00})\ \text{ uniformly with respect to } \delta\mu\in V^-,$$

we can write

$$\varphi(t_0)-\varphi_0(t_{00})=\varphi_0(t_0)+\varepsilon\delta\varphi(t_0)-\varphi_0(t_{00})=\int_{t_{00}}^{t_0}\dot\varphi_0(t)dt+\varepsilon\delta\varphi(t_{00})$$

$$+\varepsilon[\delta\varphi(t_0)-\delta\varphi(t_{00})]=\varepsilon[\dot\varphi_0^-\delta t_0+\delta\varphi(t_{00})]+o(\varepsilon\delta\mu). \tag{2.11}$$

In view of (2.4), it is clear that if $t\in[t_0,t_{00}]$, then

$$\lim_{\varepsilon\to0}(t,x_0(t)+\Delta x(t),\varphi(\tau(t)))=\lim_{t\to t_{00}-}(t,x_0(t),\varphi_0(\tau_0(t)))=w_0.$$

Consequently

$$\lim_{\varepsilon\to0}\sup_{t\in[t_0,t_{00}]}|\,f_0(t,x_0(t)+\Delta x(t),\varphi(\tau(t)))-f_0^-\,|=0,$$

implying that

$$\int_{t_0}^{t_{00}} f_0(t, x_0(t) + \Delta x(t), \varphi(\tau(t)))dt = -\varepsilon f_0^- \delta t_0$$

$$+\int_{t_0}^{t_{00}} [f_0(t, x_0(t) + \Delta x(t), \varphi(\tau(t))) - f_0^-]dt = -\varepsilon f_0^- \delta t_0 + o(\varepsilon\delta\mu). \tag{2.12}$$

Further, by (1.3) and (2.7), we have

$$\int_{t_0}^{t_{00}} | b(t; \varepsilon\delta\mu) | dt \le \varepsilon \int_{t_0}^{t_{00}} \sum_{i=1}^{k} | \lambda_i || \delta f_i(t, x(t), x(\tau(t))) | dt$$

$$\le \varepsilon\alpha \sum_{i=1}^{k} \int_{t_0}^{t_{00}} m_{\delta f_i, K_1}(t)dt = o(\varepsilon\delta\mu), \tag{2.13}$$

From (2.10), by virtue of (2.11) - (2.13) we obtain (2.2).

To prove the inequality (2.1), we first note that for any compact set $K \subset O$ and $f \in E_f$, there exists a function $L_{f,K}(t) \in L(I, [0, \infty))$ such that

$$| f(t, x_1, y_1) - f(t, x_2, y_2) |\le L_{f,K}(t)(| x_1 - x_2 | + | y_1 - y_2 |)$$

for almost all $t \in I$ and for any $x_i, y_i \in K, i = 1, 2$ (see [2], Lemma 3.1).

Now we assume $\gamma_0(t_{00}) \le t_{10}$, and estimate $a_1(t; \varepsilon\delta\mu), t \in [t_{00}, t_{10} + \delta_1]$. Obviously, we have (see (2.6)):

$$a_1(t; \varepsilon\delta\mu) \le \int_{t_{00}}^{t} L_{f_0, K_1}(s) | \Delta x(s) | ds + a_2(t; \varepsilon\delta\mu), \tag{2.14}$$

where

$$a_2(t; \varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{f_0, K_1}(s) | x_0(\tau(s)) + \Delta x(\tau(s)) - x_0(\tau_0(s)) | ds.$$

It follows from (2.9) that $\gamma(t) > t_{00}$ for $t \in [t_0, t_{00}]$, where $\gamma(t)$ is the inverse function of $\tau(t)$.

Introduce the quantities $\rho_\varepsilon = \min\{\gamma(t_0), \gamma_0(t_{00})\}$, $\theta_\varepsilon = \max\{\gamma(t_0), \gamma_0(t_{00})\}$, and observe that $\rho_\varepsilon > t_{00}$. Let $t \in [t_{00}, \rho_\varepsilon]$, then $\tau(t) < t_0$ and $\tau_0(t) < t_{00}$, and therefore we have

$$a_2(t; \varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{f_0, K_1}(s) | \varphi_0(\tau(s)) + \varepsilon\delta\varphi(\tau(s)) - \varphi_0(\tau_0(s)) | ds$$

$$\le O(\varepsilon\delta\mu) + \int_{t_{00}}^{b} L_{f_0, K_1}(s) | \varphi_0(\tau(s)) - \varphi_0(\tau_0(s)) | ds.$$

It follows from the boundedness of function $\dot\varphi_0(t), t \in I_1$, that

$$| \varphi_0(\tau(s)) - \varphi_0(\tau_0(s)) |= \left| \int_{\tau_0(s)}^{\tau(s)} \dot\varphi_0(\xi)d\xi \right| \le O(\varepsilon\delta\mu). \tag{2.15}$$

Thus, for $t \in [t_{00}, \rho_\varepsilon]$ we have

(2.16)
$$a_2(t;\varepsilon\delta\mu) \le O(\varepsilon\delta\mu).$$

Let $t \in [\rho_\varepsilon, \theta_\varepsilon]$. Here we consider two cases.

a) Let $\rho_\varepsilon = \gamma(t_0)$ and $\theta_\varepsilon = \gamma_0(t_{00})$. then for $t \in [\gamma(t_0), \gamma_0(t_{00})]$ we have $\tau(t) \ge t_0$ and $\tau_0(t) \le t_{00}$. Thus, in view of (2.15) we can write

$$a_2(t;\varepsilon\delta\mu) = a_2(\gamma(t_0);\varepsilon\delta\mu) + \int_{\gamma(t_0)}^{t} L_{f_0,K_1}(s) \mid x(\tau(s))) - \varphi_0(\tau_0(s)) \mid ds \le O(\varepsilon\delta\mu)$$

$$+ \int_{\gamma(t_0)}^{\gamma_0(t_{00})} L_{f_0,K_1}(s) \mid x(\tau(s)) - \varphi_0(\tau(s)) \mid ds + \int_{\gamma(t_0)}^{\gamma_0(t_{00})} L_{f_0,K_1}(s) \mid \varphi_0(\tau(s)) - \varphi_0(\tau_0(s)) \mid ds$$

$$\le O(\varepsilon\delta\mu) + \int_{t_0}^{\tau(\gamma_0(t_{00}))} L_{f_0,K_1}(\gamma(s)) \mid x(s) - \varphi_0(s) \mid \dot\gamma(s)ds.$$

Taking into account that

$$\mid \tau(\gamma_0(t_{00})) - t_0 \mid = \mid \tau_0(\gamma_0(t_{00})) + \varepsilon\delta\tau(\gamma_0(t_{00})) - t_0 \mid = \mid t_{00} - t_0 + \varepsilon\delta\tau(\gamma_0(t_{00})) \mid \le O(\varepsilon\delta\mu),$$

the arguments similar to that of used in (2.4) yield

$$\mid x(s) - \varphi_0(s) \mid \le O(\varepsilon\delta\mu), s \in [t_0, \tau(\gamma_0(t_{00}))].$$

Consequently, we have

(2.17)
$$a_2(t;\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + O(\varepsilon\delta\mu) \int_{\gamma(t_0)}^{\gamma_0(t_{00})} L_{f_0,K_1}(s)ds \le O(\varepsilon\delta\mu), t \in [\gamma(t_0), \gamma_0(t_{00})].$$

b) Let $\rho_\varepsilon = \gamma_0(t_{00})$ and $\theta_\varepsilon = \gamma(t_0)$, then for $t \in [\gamma_0(t_{00}), \gamma(t_0)]$ we have $\tau_0(t) \ge t_{00}$ and $\tau(t) \le t_0$. Thus, in view of (2.15) we can write

$$a_2(t;\varepsilon\delta\mu) = a_2(\gamma_0(t_{00});\varepsilon\delta\mu) + \int_{\gamma_0(t_{00})}^{t} L_{f_0,K_1}(s) \mid \varphi_0(\tau(s)) + \varepsilon\delta\varphi(\tau(s)) - x_0(\tau_0(s)) \mid ds$$

$$\le O(\varepsilon\delta\mu) + \int_{\gamma_0(t_{00})}^{\gamma(t_0)} L_{f_0,K_1}(s) \mid \varphi_0(\tau(s)) - \varphi_0(\tau_0(s)) \mid ds$$

$$+ \int_{\gamma_0(t_{00})}^{\gamma(t_0)} L_{f_0,K_1}(s) \mid \varphi_0(\tau_0(s)) - x_0(\tau_0(s)) \mid ds \le O(\varepsilon\delta\mu)$$

$$+ \int_{t_{00}}^{\tau_0(\gamma(t_0))} L_{f_0,K_1}(\gamma_0()) \mid \varphi_0(s) - x_0(s) \mid \dot\gamma_0(s)ds.$$

It is clear that

$$\mid \tau_0(\gamma(t_0)) - t_{00} \mid = \mid \tau_0(\gamma(t_0)) + \varepsilon\delta\tau(\gamma(t_0)) - \varepsilon\delta\tau(\gamma(t_0)) - t_{00} \mid$$

$$= \mid \tau(\gamma(t_0)) - t_{00} - \varepsilon\delta\tau(\gamma(t_0)) \mid \le \mid t_0 - t_{00} - \varepsilon\delta\tau(\gamma(t_0)) \mid \le O(\varepsilon\delta\mu).$$

For $s \in [t_{00}, \tau_0(\gamma(t_0))]$ we have

$$| \varphi_0(s) - x_0(s) |= \left|\varphi_0(s) - \varphi_0(t_{00}) - \int_{t_{00}}^{s} f_0(\xi, x_0(\xi), x_0(\tau_0(\xi))d\xi\right| \leq O(\varepsilon\delta\mu).$$

Therefore

$$a_2(t;\varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + O(\varepsilon\delta\mu) \int_{t_{00}}^{\tau_0(\gamma(t_0))} L_{f_0,K_1}(\gamma_0(s))\dot{\gamma}_0(s)ds, t \in [\gamma_0(t_{00}), \gamma(t_0)].$$

Thus, for $t \in [t_{00}, \theta_\varepsilon]$ the condition (2.16) is satisfied.

Let $t \in [\theta_\varepsilon, t_{10} + \delta_1]$, then $\tau(t) \geq t_0$ and $\tau_0(t) \geq t_{00}$. Therefore

$$a_2(t;\varepsilon\delta\mu) = a_2(\theta_\varepsilon;\varepsilon\delta\mu) + \int_{\theta_\varepsilon}^{t} L_{f_0,K_1}(s) \mid x_0(\tau(s)) + \Delta x(\tau(s)) - x_0(\tau_0(s)) \mid ds$$

$$\leq O(\varepsilon\delta\mu) + \int_{\tau(\theta_\varepsilon)}^{\tau(t)} L_{f_0,K_1}(\gamma(s)) \mid \Delta x(s) \mid \dot{\gamma}(s)ds$$

$$+ \int_{\theta_\varepsilon}^{t} L_{f_0,K_1}(s) \mid x_0(\tau(s)) - x_0(\tau_0(s)) \mid ds \leq O(\varepsilon\delta\mu)$$

$$+ \int_{t_0}^{t} \chi(\gamma(s))L_{f_0,K_1}(\gamma(s)) \mid \Delta x(s) \mid \dot{\gamma}(s)ds + \int_{\gamma_0(t_{00})}^{t_{10}+\delta_1} L_{f_0,K_1}(s) \mid x_0(\tau(s)) - x_0(\tau_0(s)) \mid ds,$$

where $\chi(s)$ is the characteristic function of the interval I.

Further, in view of (2.4) we have

$$| x_0(\tau(s)) - x_0(\tau_0(s)) |\leq \left| \int_{\tau_0(s)}^{\tau(s)} | f_0(\xi, x_0(\xi), x_0(\tau_0(\xi))) | d\xi\right| \leq O(\varepsilon\delta\mu)$$

and

$$\int_{t_0}^{t_{00}} \chi(\gamma(s))L_{f_0,K_1}(\gamma(s)) \mid \Delta x(s) \mid \dot{\gamma}(s)ds = O(\varepsilon\delta\mu)$$

For $t \in [\theta_\varepsilon, t_{10} + \delta_1]$ we get

$$a_2(t;\varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} \chi(\gamma(s))L_{f_0,K_1}(\gamma(s)) \mid \Delta x(s) \mid \dot{\gamma}(s)ds.$$

Thus, taking into account (2.14), in the case $\gamma_0(t_{00}) \leq t_{10}$ for $t \in [t_{00}, t_{10} + \delta_1]$, we have

$$\text{(2.17)}\quad a_1(t;\varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} \Big[L_{f,K_1}(s) + \chi(\gamma(s))L_{f,K_1}(\gamma(s))\dot{\gamma}(s)\Big] \mid \Delta x(s) \mid ds.$$

Now we consider the case where $\gamma_0(t_{00}) > t_{10}$. Let the numbers $\delta_2 \in (0, \delta_1)$ and $\varepsilon_2'' \in (0, \varepsilon_1)$ be chosen so small to satisfy $\tau_0(t_{10} + \delta_2) < t_{00}$ and $\tau(t_{10} + \delta_2) < t_0$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2'') \times V^-$.

It is easy to see that

$$a_1(t;\varepsilon\delta\mu) \leq \int_{t_{00}}^{t} L_{f_0,K_1}(s)\Big[\mid \Delta x(s) \mid + \mid \varphi(\tau(s)) - \varphi(\tau_0(s)) \mid \Big]ds$$

$$\leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} L_{f_0,K_1}(s) \mid \Delta x(s) \mid ds.$$

Thus, there exist numbers $\varepsilon_2 = \min\{\varepsilon_2', \varepsilon_2''\} \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^-$ the inequality (2.17) is fulfilled.

Obviously, we have

$$b(\varepsilon\delta\mu) \leq \varepsilon\alpha \int_{t_{00}}^{t_{10}+\delta_2} \sum_{i=1}^{k} m_{\delta f_i,K_1}(t)dt \leq O(\varepsilon\delta\mu). \tag{2.18}$$

According to (2.2), (2.17) and (2.18) from the inequality (2.8) for $t \in [t_{00}, t_{10} + \delta_2]$ we have

$$\mid \Delta x(t) \mid \leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} \Big[L_{f_0,K_1}(s) + \chi(\gamma(s))L_{f_0,K_1}(\gamma(s))\dot{\gamma}(s)\Big] \mid \Delta x(s) \mid ds.$$

An application of the Gronwall lemma yields

$$\mid \Delta x(t) \mid \leq O(\varepsilon\delta\mu), t \in [t_{00}, t_{10} + \delta_2]. \tag{2.19}$$

Finally, the inequalities (2.3), (2.4) and (2.19) imply (2.1).
Theorem 2.2 is proved.

Theorem 2.3. *Let the conditions of Theorem 1.2 be fulfilled. Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that (2.1) is valid for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times V^-$. Moreover, we have*

$$\Delta x(t_{00}) = \varepsilon[\delta\varphi(t_{00}) + (\dot{\varphi}_0^+ - f_0^+)\delta t_0] + o(\varepsilon\delta\mu).$$

Proof is similar to that of Theorem 2.2, and so is omitted.

3. Proof of Theorem 1.1

Observe first that on the interval $[t_{00}, t_{10} + \delta_2]$ the function $\Delta x(t)$ satisfies the equation

$$\dot{\Delta x}(t) = f_{0x}[t]\Delta x(t) + f_{0y}[t]\Delta x(\tau_0(t)) + \varepsilon\delta f[t] + \sum_{i=1}^{2} r_i[t], \tag{3.1}$$

where

$$r_1[t] = f_0(t, x_0(t) + \Delta x(t), x_0(\tau(t)) + \Delta x(\tau(t))) - f_0[t] - f_{0x}[t]\Delta x(t) - f_{0y}[t]\Delta x(\tau_0(t)),$$

$$r_2[t] = \varepsilon[\delta f(t, x_0(t) + \Delta x(t), x_0(\tau(t)) + \Delta x(\tau(t))) - \delta f[t]].$$

By using the Cauchy formula one can represent the solution of equation (3.1) in the form

$$\text{(3.2)}\quad \Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon \int_{t_{00}}^{t} Y(s;t)\delta f[s]ds + \sum_{i=0}^{2} R_i[t;t_{00}],\ t \in [t_{00}, t_{10}+\delta_2],$$

where

$$\text{(3.3)}\qquad R_0[t;t_{00}] = \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\Delta x(s)ds,$$

$$\text{(3.4)}\qquad R_i[t;t_{00}] = \int_{t_{00}}^{t} Y(s;t) r_i[s]ds,\ i = 1,2,$$

and $Y(s;t)$ is the matrix function satisfying the equation (1.7) and the condition (1.8).

Observe that the function $Y(s;t)$ is continuous on the set $\Pi = \{(s,t):\ t_{00} \le s \le t,\ t \in [t_{00}, t_{10}+\delta_2]\}$ (see [2], Lemma 2.1.7). Therefore, in view of (2.2) we have

$$\text{(3.5)}\qquad Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)[\delta\varphi(t_{00}) + (\dot{\varphi}_0^- - f_0^-)\delta t_0] + o(t;\varepsilon\delta\mu).$$

Next, one can readily see that

$$R_0[t;t_{00}] = \varepsilon \int_{\tau_0(t_{00})}^{t_0} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds$$

$$+ \int_{t_0}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\Delta x(s)d\xi = \varepsilon \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds$$

$$-\varepsilon \int_{t_0}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)d\xi + O(\varepsilon\delta\mu)\int_{t_0}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)ds$$

$$\text{(3.6)}\qquad = \varepsilon \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)d\xi + o(t;\varepsilon\delta\mu).$$

Introducing the notations:

$$f_0[t;s] = f_0(t, x_0(t) + s\Delta x(t), x_0(\tau_0(t)) + s(x_0(\tau(t)) - x_0(\tau_0(t)) + \Delta x(\tau(t)))),$$

$$\sigma[t;s] = f_{0x}[t;s] - f_{0x}[t],\ \rho[t;s] = f_{0y}[t;s] - f_{0y}[t],$$

it is easy to see that

$$f_0(t, x_0(t) + \Delta x(t), x_0(\tau(t)) + \Delta x(\tau(t))) - f_0[t]$$

$$= \int_0^1 \frac{d}{ds} f_0[t;s]ds = \int_0^1 \{f_{0x}[t;s]\Delta x(t) + f_{0y}[t;s](x_0(\tau(t)) - x_0(\tau_0(t)) + \Delta x(\tau(t)))\}ds$$

$$= \Big[\int_0^1 \sigma[t;s]ds\Big]\Delta x(t) + \Big[\int_0^1 \rho[t;s]ds\Big](x_0(t-\tau) - x_0(\tau_0(t)) + \Delta x(\tau(t)))$$

$$+ f_{0x}[t]\Delta x(t) + f_{0y}[t](x_0(\tau(t)) - x_0(\tau_0(t)) + \Delta x(\tau(t))).$$

Therefore for $t \in [t_{00}, t_{10} + \delta_2]$ we can write

$$R_1[t; t_{00}] = \sum_{i=1}^{4} R_{1i}[t; t_{00}],$$

where

$$R_{11}[t; t_{00}] = \int_{t_{00}}^{t} Y(\xi; t)\sigma_1[\xi]\Delta x(\xi)d\xi, \sigma_1[\xi] = \int_0^1 \sigma[\xi; s]ds; R_{12}[t; t_{00}] = \int_{t_{00}}^{t} Y(\xi; t)\rho_1[\xi](x_0(\tau(\xi))$$

$$-x_0(\tau_0(\xi)) + \Delta x(\tau(\xi)))d\xi, \rho_1[\xi] = \int_0^1 \rho[\xi; s]ds; R_{13}[t; t_{00}] = \int_{t_{00}}^{t} Y(\xi; t)f_{0y}[\xi](x_0(\tau(\xi))$$

$$-x_0(\tau_0(\xi)))d\xi; R_{14}[t; t_{00}] = \int_{t_{00}}^{t} Y(\xi; t)f_{0y}[\xi](\Delta x(\tau(\xi)) - \Delta x(\tau_0(\xi)))d\xi.$$

The function $\dot{x}_0(t), t \in [\hat{\tau}, t_{10} + \delta_2]$ is measurable. Hence for each fixed Lebesgue point $\tau_0(\xi) \in (\hat{\tau}, t_{10} + \delta_2)$ of function $\dot{x}_0(t)$ we have

$$x_0(\tau(\xi)) - x_0(\tau_0(\xi)) = \int_{\tau_0(\xi)}^{\tau(\xi)} \dot{x}_0(s)ds = \varepsilon\dot{x}_0(\tau_0(\xi))\delta\tau(\xi) + \omega(\xi; \varepsilon\delta\mu), \tag{3.7}$$

where

$$\lim_{\varepsilon \to 0} \frac{\omega(\xi; \varepsilon\delta\mu)}{\varepsilon} = 0 \text{ uniformly for } \delta\mu \in V^-. \tag{3.8}$$

Thus, (3.7) is valid for almost all points of the interval $(t_{00}, t_{10} + \delta_2)$.

Taking into account boundedness of the function

$$\dot{x}_0(t) = \begin{cases} \dot{\varphi}_0(t), t \in [\hat{\tau}, t_{00}] \\ f_0(t, x_0(t), x_0(\tau_0(t))), t \in (t_{00}, t_{10} + \delta_2], \end{cases}$$

from (3.7) we obtain

$$| x_0(\tau(\xi)) - x_0(\tau_0(\xi)) | \le O(\varepsilon\delta\mu) \text{ and } \left| \frac{\omega(\xi; \varepsilon\delta\mu)}{\varepsilon} \right| \le const. \tag{3.9}$$

It is clear that (see (2.1))

(3.10)

$$| \Delta x(\tau(\xi)) - \Delta x(\tau_0(\xi)) | = \varepsilon \, | \, \delta\varphi(\tau(\xi)) - \delta\varphi(\tau_0(\xi))) | = o(\xi; \varepsilon\delta\mu) \text{ for } \xi \in [t_{00}, \varrho_\varepsilon]$$

and

$$| \Delta x(\tau(\xi)) - \Delta x(\tau_0(\xi)) | \le O(\xi; \varepsilon\delta\mu) \text{ for } \xi \in [\varrho_\varepsilon, \theta_\varepsilon]. \tag{3.11}$$

Let $\xi \in [\theta_\varepsilon, t_{10} + \delta_1]$, then $\tau(\xi) \ge t_0$ and $\tau_0(\xi) \ge t_{00}$. Therefore, in view of (2.5) – (2.7), we have

$$| \Delta x(\tau(\xi)) - \Delta x(\tau_0(\xi)) | \le \int_{\tau_0(\xi)}^{\tau(\xi)} | \dot{\Delta x}(s) | ds \le \int_{\tau_0(\xi)}^{\tau(\xi)} L_{f_0, K_1}(s)(| \Delta x(s) |$$

(3.12)

$$+ \mid x_0(\tau(s)) - x_0(\tau_0(s)) \mid + \mid \Delta x(\tau(s)) \mid)ds + \varepsilon\alpha \int_{\tau_0(\xi)}^{\tau(\xi)} \sum_{i=1}^{k} m_{\delta f_0, K_1}(s)ds = o(\xi;\varepsilon\delta\mu).$$

According to (2.1) and (3.9) - (3.12) for $R_{1i}[t;t_{00}], i = 1,2,3$, we have

$$\mid R_{11}[t;t_{00}] \mid \leq \parallel Y \parallel O(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu), \mid R_{12}[t;t_{00}] \mid \leq \parallel Y \parallel O(\varepsilon\delta\mu)\rho_2(\varepsilon\delta\mu),$$

$$R_{13}[t;t_{00}] = \gamma_1(t;\varepsilon\delta\mu) - \varepsilon\Big[\int_{t_{00}}^{t} Y(\xi;t) f_{0y}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)d\xi\Big],$$

where

$$\parallel Y \parallel = \sup\{\mid Y(\xi;t) \mid : (\xi,t) \in \Pi\}, \sigma_2(\varepsilon\delta\mu) = \int_{t_{00}}^{t_{10}+\delta_1} \mid \sigma_1[\xi] \mid d\xi,$$

$$\rho_2(\varepsilon\delta\mu) = \int_{t_{00}}^{t_{10}+\delta_1} \mid \rho_1[\xi] \mid d\xi, \gamma_1(t;\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t) f_{0y}[\xi]\gamma(\xi;\varepsilon\delta\mu)d\xi.$$

Obviously,

$$\Big|\frac{\omega_1(t;\varepsilon\delta\mu)}{\varepsilon}\Big| \leq \parallel Y \parallel \int_{t_{00}}^{t_{10}+\delta_1} \mid f_{0y}[\xi] \mid \Big|\frac{\omega(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big| d\xi.$$

By the Lebesgue theorem on passage to the limit under the integral sign and formula (3.8), we can write

$$\lim_{\varepsilon\to 0} \sigma_2(\varepsilon\delta\mu) = 0, \ \lim_{\varepsilon\to 0} \rho_2(\varepsilon\delta\mu) = 0, \ \lim_{\varepsilon\to 0} \Big|\frac{\omega_1(t;\varepsilon\delta\mu)}{\varepsilon}\Big| = 0$$

uniformly for $(t,\delta\mu) \in [t_{00}, t_{10}+\delta_1] \times V^-$.

Therefore

$$R_{1i}[t;t_{00}] = o(t;\varepsilon\delta\mu), i = 1,2; \ R_{13}[t;t_{00}] = -\varepsilon \int_{t_{00}}^{t} Y(\xi;t) f_{0y}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\delta\xi$$

(3.13) $$+o(t;\varepsilon\delta\mu).$$

Further, we have

(3.14) $$\mid R_{14}[t;t_{00}] \mid \leq \parallel Y \parallel \int_{t_{00}}^{t_{10}+\delta_1} \mid f_{0y}[\xi] \mid\mid \Delta x(\tau(\xi)) - \Delta x(\tau_0(\xi)) \mid d\xi = o(\varepsilon\delta\mu)$$

By (3.13) and (3.14) we obtain

(3.15) $$R_1[t;t_{00}] = -\varepsilon \int_{t_{00}}^{t} Y(\xi;t) f_{0y}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)d\xi + o(t;\varepsilon\delta\mu).$$

Next, we have

$$\mid R_2[t;t_{00}] \mid \leq \varepsilon\alpha \int_{t_{00}}^{t_{10}+\delta_1} \Big[\sum_{i=1}^{k} L_{\delta f_i, K_1}(\xi)\Big(\mid \Delta x(\xi) \mid + \mid x_0(\tau(\xi))$$

(3.16) $$-x_0(\tau_0(\xi)) \mid + \mid \Delta x(\tau(\xi)) \mid \Big)\Big] d\xi = o(\varepsilon\delta\mu).$$

Finally, by virtue of (3.5), (3.6), (3.15) and (3.16), from (3.2) we obtain (1.4), with $\delta x(t;\delta\mu)$ given by (1.5) and (1.6).

Theorem 1.1 is proved.

4. Proof of Theorem 1.2

Let $\hat{t} \in (t_{00}, t_{10})$ be a fixed point, and let $\varepsilon_2 \in (0, \varepsilon_1)$ be chosen so small to satisfy $t_0 < \hat{t}$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times V^+$. The function $\Delta x(t)$ satisfies equation (3.1) on the interval $[t_0, t_{10} + \delta_2]$. Therefore, using Cauchy formula and (3.3), (3.4), we can represent $\Delta x(t)$ in the form

$$\Delta x(t) = Y(t_0;t)\Delta x(t_0) + \varepsilon \int_{t_0}^{t} Y(s;t)\delta f[s]ds + \sum_{i=0}^{2} R_i[t;t_0]. \tag{4.1}$$

The matrix function $Y(\xi;t)$ is continuous on $[t_{00}, \hat{t}] \times [\hat{t}, t_{10} + \delta_2]$, therefore

$$Y(t_0;t)\Delta x(t_0) = \varepsilon Y(t_{00};t)[\delta\varphi(t_{00}) + (\varphi_0^+ - f_0^+)]\delta t_0 + o(\varepsilon\delta\mu). \tag{4.2}$$

It is not difficult to check that $R_0[t;t_0]$ can be written in the following form

$$R_0[t;t_0] = \varepsilon \int_{\tau_0(t_0)}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds + \int_{t_{00}}^{t_0} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\Delta x(s)ds$$

$$= \varepsilon \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)ds + o(t;\varepsilon\delta\mu). \tag{4.3}$$

Using the above arguments, with some minor changes, one can show that for $t \in [\hat{t}, t_{10} + \delta_2]$

(4.4)

$$R_1[t;t_0] = -\varepsilon \int_{t_{00}}^{t} Y(s;t) f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)ds + o(t;\varepsilon\delta\mu),\ R_2[t;t_0] = o(t;\varepsilon\delta\mu).$$

Finally, note that for $t \in [\hat{t}, t_{10} + \delta_2]$

$$\varepsilon \int_{t_0}^{t} Y(s;t)\delta f[s]ds = \varepsilon \int_{t_{00}}^{t} Y(s;t)\delta f[s]ds + o(t;\varepsilon\delta\mu). \tag{4.5}$$

Taking into account (4.2) - (4.5), from (4.1) we obtain (1.4), with $\delta x(t;\varepsilon\delta\mu)$ given by (1.9).

Theorem 1.2 is proved.

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ТОЧНОСТЬ ГАУССОВСКОЙ АППРОКСИМАЦИИ ДЛЯ МАРТИНГАЛ–РАЗНОСТНЫХ СЛУЧАЙНЫХ ПОЛЕЙ

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Аннотация. Получена оценка скорости сходимости в центральной предельной теореме для однородных мартингал–разностных случайных полей на d-мерной целочисленной решетке.

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1. Введение

Предельные теоремы играют важнейшую роль как в прикладных, так и в теоретических задачах теории вероятностей. Особую значимость при этом имеют оценки скорости сходимости, без наличия которых использование указанных теорем в прикладных исследованиях практически невозможно.

Вопрос точности гауссовской аппроксимации для сумм случайных слагаемых имеет долгую историю и досконально изучен только для последовательностей независимых случайных величин (см., например, [1] – [4]). В определенной степени данная тема изучена и для слабо зависимых стационарных случайных процессов (отметим работы [5] – [7]) и мартингал-разностей (см. [8, 9] и приведенную в них литературу). Для многомерных же объектов (случайные поля) изучение этого вопроса находится в начальной стадии. В основном рассматриваются совокупности m-зависимых случайных величин ([10, 11]) и случайные поля с экспоненциальным убыванием корреляций ([12] – [14]). Касательно же мартингал–разностных случайных полей, отметим работу [15], в которой в рамках стандартных мартингальных условий (на моменты и на скорость сближения условных дисперсий компонент случайного поля с их безусловными дисперсиями) получены степенные оценки скорости сходимости в центральной предельной теореме (ЦПТ).

В настоящей работе применяется другой подход (восходящий к методу малых блоков, предложенному в работе [16]) к получению степенной оценки скорости сходимости в ЦПТ для однородных мартингал–разностных случайных полей. Он не предполагает близости условных и безусловных дисперсий, но требует наличия слабой зависимости между компонентами случайного поля. Такой подход открывает возможность применения полученных результатов к гиббсовским случайным полям.

2. Предварительные сведения

Приведем основные понятия и обозначения, используемые в работе. Пусть $\mathbf{Z}^d$, $d > 1$ — целочисленная решетка, $W = \left\{V \subset \mathbf{Z}^d, |V| < \infty\right\}$ – множество всех конечных подмножеств $\mathbf{Z}^d$ и $X \subset \mathbb{R}$ – некоторое конечное множество, $|X| > 1$. Здесь и далее символ $|V|$ используется для обозначения мощности конечного множества V.

Случайным полем, заданным на $\mathbf{Z}^d$, с фазовым пространством X, будем называть совокупность $(\xi_t) = \left(\xi_t, t \in \mathbf{Z}^d\right)$ случайных величин, каждая из которых принимает значение в X.

Пусть $\mathcal{B}$ — σ-алгебра всех подмножеств X и пусть $\mathcal{B}^{\mathbf{Z}^d}$ — σ-алгебра, порожденная цилиндрическими подмножествами $X^{\mathbf{Z}^d}$. Распределением случайного поля (ξ_t) называется вероятностная мера P на $\left(X^{\mathbf{Z}^d}, \mathcal{B}^{\mathbf{Z}^d}\right)$, такая, что

$$\Pr\left\{\left(\xi_t, t \in \mathbf{Z}^d\right) \in B\right\} = P(B), \qquad B \in \mathcal{B}^{\mathbf{Z}^d}.$$

Определим на $X^{\mathbf{Z}^d}$ группу преобразований τ_a, $a \in \mathbf{Z}^d$, такую, что $(\tau_a x)_t = x_{t+a}$ для всех $x \in X^{\mathbf{Z}^d}$. Случайное поле (ξ_t) с распределением P называется однородным, если $P(\tau_a A) = P(A)$ для любых $A \in \mathcal{B}^{\mathbf{Z}^d}$ и $a \in \mathbf{Z}^d$.

Для заданного случайного поля (ξ_t) обозначим через $\Im_S$ σ-алгебру, порожденную случайными величинами ξ_s, $s \in S$, $S \subset \mathbf{Z}^d$.

Случайное поле (ξ_t) называется мартингал–разностным случайным полем (см. [17]), если для всех $t \in \mathbf{Z}^d$, $M|\xi_t| < \infty$ и

$$M\left(\xi_t / \Im_{\mathbf{Z}^d \setminus \{t\}}\right) = 0 \text{ (п.н.)}, \tag{2.1}$$

где M – знак математического ожидания.

Нетрудно проверить, что условие (2.1) эквивалентно условию

$$M\left(\xi_t / \Im_V\right) = 0 \text{ (п.н.)} \tag{2.2}$$

для всех $t \in \mathbf{Z}^d$ и $V \in W$, $t \notin V$. Понятно также, что если (ξ_t) — мартингал–разностное случайное поле, то

$$M\xi_t = 0 \qquad \text{для всех } t \in \mathbf{Z}^d. \tag{2.3}$$

Будем говорить, что однородное случайное поле (ξ_t) удовлетворяет условию равномерного сильного перемешивания с коэффициентом φ_I, если для каждого фиксированного $I \in W$

$$\sup\left\{\left|P\left(A/B\right) - P\left(A\right)\right|, A \in \Im_I, B \in \Im_\Lambda, P\left(B\right) > 0\right\} \le \varphi_I\left(\rho\left(I, \Lambda\right)\right),$$

где функция $\varphi_I(\rho)$, $\rho \in \mathbb{R}$, такова, что $\varphi_I(\rho) \to 0$ при $\rho \to \infty$ и фиксированном $I \in W$. Здесь $\rho(I, \Lambda) = \inf\left\{|t - s|,\ t \in I,\ s \in \Lambda\right\}$, $|t - s| = \max\limits_{1\le i\le d}\left|t^{(i)} - s^{(i)}\right|$, $t = \left(t^{(1)}, ..., t^{(d)}\right)$, $s = \left(s^{(1)}, ..., s^{(d)}\right)$.

Справедливо следующее утверждение (см., например, [2, 7]).

Утверждение 1. *Пусть U, Y — случайные величины, измеримые относительно σ-алгебр $\Im_I$ и $\Im_\Lambda$ соответственно, $\rho(I, \Lambda) \ge r$, причем $|U| < B_1$, $|Y| < B_2$. Тогда*

$$|M(UY) - MU \cdot MY| \le 2B_1B_2\varphi_I(r),$$

где B_1, B_2 — положительные постоянные.

Для заданного случайного поля (ξ_t) обозначим

$$S(V) = \sum_{t\in V}\xi_t, \qquad V \in W.$$

При доказательстве предельных теорем для случайных полей с перемешиванием одним из основных требований является правильное поведение дисперсий сумм компонент случайного поля $S(V)$, $V \in W$, а именно, $DS(V) = \sigma^2|V|(1 + O(1))$, $0 < \sigma < \infty$. Для однородных мартингал-разностных случайных полей это требование выполняется автоматически, так как для таких полей $DS(V) = M\xi_0^2 \cdot |V|$.

3. Оценка скорости сходимости в ЦПТ

Основной результат настоящей работы — оценка скорости сходимости в ЦПТ для однородных мартингал–разностных случайных полей, удовлетворяющих условию равномерного сильного перемешивания — приводится в следующей теореме.

Теорема 3.1. *Пусть (ξ_t) — однородное мартингал-разностное случайное поле с конечным фазовым пространством X, удовлетворяющее условию равномерного сильного перемешивания с коэффициентом φ_I, $I \in W$, таким, что*

$$\varphi_I(j) \le |I| \cdot \varphi(j) \qquad и \qquad \sum_{j=1}^{\infty} j^{d-1}\varphi(j) < \infty.$$

Тогда

$$\sup_x\left|P\left(\frac{S(V_n)}{\sigma\sqrt{|V_n|}} < x\right) - \Phi(x)\right| \le C \cdot n^{-d/8},$$

где положительная постоянная C не зависит от n, $V_n = [-n;n]^d$ — d-мерный куб с центром в начале координат и стороной $2n+1$, $\sigma^2 = M\xi_0^2 > 0$ и $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-u^2/2}du$ — функция распределения стандартного нормального закона.

Доказательство. Опираясь на неравенство Бери-Эссеена (см., например, [18], стр. 317), мы будем оценивать близость функций распределения посредством оценки близости их характеристических функций. Пусть элементы множества V_n пронумерованы некоторым образом: $V_n = \{s_1, s_2, ..., s_N\}$, $N = (2n+1)^d$. Для удобства записи обозначим $\eta_j = \frac{\xi_{s_j}}{\sigma\sqrt{|V_n|}}$, $j = \overline{1,N}$. Пусть, далее, $f_n(t)$ — характеристическая функция случайной величины $\frac{S(V_n)}{\sqrt{DS(V_n)}} = \sum\limits_{j=1}^{N}\eta_j$. Оценим разность между $f_n(t)$ и $e^{-t^2/2}$. Имеем

$$
\begin{aligned}
(3.1)\qquad & \left|f_n(t) - e^{-t^2/2}\right| = \left|M\exp\left\{it\frac{S(V_n)}{\sqrt{DS(V_n)}}\right\} - e^{-t^2/2}\right| \le \\
& \le \left|M\prod_{j=1}^{N} e^{it\eta_j} - \prod_{j=1}^{N} Me^{it\eta_j}\right| + \left|\prod_{j=1}^{N} Me^{it\eta_j} - e^{-t^2/2}\right|.
\end{aligned}
$$

Рассмотрим первое слагаемое в (3.1). Можем написать

$$
\begin{aligned}
& \left|M\prod_{j=1}^{N} e^{it\eta_j} - \prod_{j=1}^{N} Me^{it\eta_j}\right| \le \sum_{k=1}^{N}\left|Me^{it\eta_k}\prod_{j=k+1}^{N} e^{it\eta_j} - Me^{it\eta_k}M\prod_{j=k+1}^{N} e^{it\eta_j}\right| \le \\
& \le \sum_{k=1}^{N}\left|M\left(e^{it\eta_k} - 1 - it\eta_k - \frac{(it)^2}{2}\eta_k^2\right)\prod_{j=k+1}^{N} e^{it\eta_j} - \right. \\
& \left. -M\left(e^{it\eta_k} - 1 - it\eta_k - \frac{(it)^2}{2}\eta_k^2\right)M\prod_{j=k+1}^{N} e^{it\eta_j}\right| + \\
& + |t|\sum_{k=1}^{N}\left|M\eta_k\prod_{j=k+1}^{N} e^{it\eta_j} - M\eta_k M\prod_{j=k+1}^{N} e^{it\eta_j}\right| + \\
& + \frac{t^2}{2}\sum_{k=1}^{N}\left|M\eta_k^2\prod_{j=k+1}^{N} e^{it\eta_j} - M\eta_k^2 M\prod_{j=k+1}^{N} e^{it\eta_j}\right| = T_1 + T_2 + T_3.
\end{aligned}
$$

Оценим слагаемые в правой части полученного выражения. Для первого слагаемого можем записать

$$
\begin{aligned}
T_1 \le & \sum_{k=1}^{N}\left(M\left|e^{it\eta_k} - 1 - it\eta_k - \frac{(it)^2}{2}\eta_k^2\right|\cdot\left|\prod_{j=k+1}^{N} e^{it\eta_j}\right| + \right. \\
& \left. + M\left|e^{it\eta_k} - 1 - it\eta_k - \frac{(it)^2}{2}\eta_k^2\right|\cdot M\left|\prod_{j=k+1}^{N} e^{it\eta_j}\right|\right) \le \\
\le & \, 2\sum_{k=1}^{N} M\left|e^{it\eta_k} - 1 - it\eta_k - \frac{(it)^2}{2}\eta_k^2\right|.
\end{aligned}
$$

Обозначим $M|\xi_0|^3 = \beta_3$. Воспользовавшись неравенством $\left|e^{ix} - 1 - ix - \frac{(ix)^2}{2!}\right| \le \frac{|x|^3}{3!}$, получаем

$$T_1 \le 2\frac{|t|^3}{3!}\sum_{k=1}^{N} M|\eta_k|^3 = \frac{|t|^3}{3}N\frac{\beta_3}{\sigma^3(2n+1)^{3d/2}} < |t|^3\frac{\beta_3}{3\sigma^3 2^{d/2}}\cdot\frac{1}{n^{d/2}}.$$

Покажем, что $T_2 = 0$. Учитывая свойство (2.3) мартингал–разностного случайного поля, можем написать

$$T_2 = |t|\sum_{k=1}^{N}\left|M\eta_k\prod_{j=k+1}^{N}e^{it\eta_j} - M\eta_k M\prod_{j=k+1}^{N}e^{it\eta_j}\right| =$$
$$= \frac{|t|}{\sigma\sqrt{|V_n|}}\sum_{k=1}^{N}\left|M\xi_{s_k}\prod_{j=k+1}^{N}\exp\left(it\frac{\xi_{s_j}}{\sigma\sqrt{|V_n|}}\right)\right|.$$

Далее, в силу свойства (2.2), $M\left(\xi_{s_k}/\xi_{s_j}, j=\overline{k+1,N}\right) = 0$ вне зависимости от способа нумерации элементов V_n, и, следовательно,

$$M\xi_{s_k}\prod_{j=k+1}^{N}\exp\left(it\frac{\xi_{s_j}}{\sigma\sqrt{|V_n|}}\right) =$$
$$= M\left(\left(\prod_{j=k+1}^{N}\exp\left(it\frac{\xi_{s_j}}{\sigma\sqrt{|V_n|}}\right)\right)\cdot M\left(\xi_{s_k}/\xi_{s_j}, j=\overline{k+1,N}\right)\right) = 0.$$

Рассмотрим T_3. Имеем

$$T_3 = \frac{t^2}{2\sigma^2|V_n|}\sum_{k=1}^{N}\left|M\xi_{s_k}^2\prod_{j=k+1}^{N}e^{it\eta_j} - M\xi_{s_k}^2 M\prod_{j=k+1}^{N}e^{it\eta_j}\right|.$$

Воспользовавшись соотношением

$$M\xi_{s_k}^2\prod_{j=k+1}^{N}e^{it\eta_j} - M\xi_{s_k}^2 M\prod_{j=k+1}^{N}e^{it\eta_j} =$$
$$= \sum_{m=k+1}^{N-1}\left(M\xi_{s_k}^2\left(e^{it\eta_m}-1\right)\prod_{j=m+1}^{N}e^{it\eta_j} - M\xi_{s_k}^2 M\left(e^{it\eta_m}-1\right)\prod_{j=m+1}^{N}e^{it\eta_j}\right),$$

получаем

$$T_3 \le \frac{t^2}{2\sigma^2|V_n|}\sum_{k=1}^{N}\sum_{m=k+1}^{N-1}\left|M\xi_{s_k}^2\left(e^{it\eta_m}-1\right)\prod_{j=m+2}^{N}e^{it\eta_j} - M\xi_{s_k}^2 M\left(e^{it\eta_m}-1\right)\prod_{j=m+2}^{N}e^{it\eta_j}\right|.$$

Обозначим через $x_0 = \max_{x\in X}|x|$. Поскольку $\xi_{s_k}^2 < x_0^2$ и

$$\left|\left(e^{it\eta_m}-1\right)\prod_{j=m+2}^{N}e^{it\eta_j}\right| \le \left|e^{it\eta_m}-1\right| \le |t|\frac{|\xi_{s_m}|}{\sigma\sqrt{|V_n|}} \le |t|\frac{x_0}{\sigma(2n+1)^{d/2}},$$

то, в силу Утверждения 1,

$$\left| M\xi_{s_k}^2 \left(e^{it\eta_m}-1\right) \prod_{j=m+2}^{N} e^{it\eta_j} - M\xi_{s_k}^2 M\left(e^{it\eta_m}-1\right) \prod_{j=m+2}^{N} e^{it\eta_j} \right| \le$$
$$\le 2\left|t\right| \frac{x_0^3}{\sigma\left(2n+1\right)^{d/2}} \varphi_{\{s_k\}}\left(\rho\left(\{s_k\},\{s_m\}\right)\right) \le 2\left|t\right| \frac{x_0^3}{\sigma\left(2n+1\right)^{d/2}} \varphi\left(\left|s_k-s_m\right|\right).$$

Следовательно, получаем

$$T_3 \le |t|^3 \frac{x_0^3}{\sigma^3} \cdot \frac{1}{(2n+1)^{3d/2}} \sum_{k=1}^{N} \sum_{m=k+1}^{N-1} \varphi\left(\left|s_k-s_m\right|\right).$$

Обозначим $V_n^{(k)}(j) = \{r \in V_n : |s_k - r| = j\}$, $j = 1, 2, ..., 2n+1$. Нетрудно проверить, что $\left|V_n^{(k)}(j)\right| < 2^{2d} j^{d-1}$. Тогда

$$\sum_{m=k+1}^{N-1} \varphi\left(\left|s_k-s_m\right|\right) < \sum_{j=1}^{2n+1} \sum_{r\in V_n^{(k)}(j)} \varphi\left(\left|s_k-r\right|\right) = \sum_{j=1}^{2n+1} \left|V_n^{(k)}(j)\right| \varphi(j) <$$
$$< 2^{2d} \sum_{j=1}^{2n+1} j^{d-1}\varphi(j) < 2^{2d} \sum_{j=1}^{\infty} j^{d-1}\varphi(j).$$

Окончательно получаем

$$T_3 < |t|^3 \frac{1}{(2n+1)^{d/2}} \cdot \frac{2^{2d}x_0^3}{\sigma^3} \sum_{j=1}^{\infty} j^{d-1}\varphi(j) < |t|^3 \frac{1}{n^{d/2}} \cdot \frac{2^{3d/2}x_0^3}{\sigma^3} \sum_{j=1}^{\infty} j^{d-1}\varphi(j)$$

Таким образом, для первого слагаемого в (3.1) получили следующую оценку

$$\left| M \prod_{j=1}^{N} e^{it\eta_j} - \prod_{j=1}^{N} M e^{it\eta_j} \right| < C_1 |t|^3 \cdot \frac{1}{n^{d/2}},$$

где $C_1 = \dfrac{\beta_3}{3\sigma^3 2^{d/2}} + \dfrac{2^{3d/2}x_0^3}{\sigma^3} \sum_{j=1}^{\infty} j^{d-1}\varphi(j)$.

Рассмотрим второе слагаемое в (3.1). Из теоремы Берри-Эссеена (см., например, [18, 4]) следует, что

$$\left| \prod_{j=1}^{N} M e^{it\eta_j} - e^{-t^2/2} \right| \le C_0 \frac{M\left|\eta_j\right|^3}{\sqrt{N}} |t|^3 e^{-t^2/4}, \qquad \text{при } |t| < \frac{\sqrt{N}}{5M\left|\eta_j\right|^3}.$$

Поскольку $M\left|\eta_j\right|^3 = \dfrac{\beta_3}{\sigma^3\left|V_n\right|^{3/2}}$, то при $|t| < \dfrac{\sigma^3}{5\beta_3}(2n+1)^{2d}$ имеем

$$\left| \prod_{j=1}^{N} M e^{it\eta_j} - e^{-t^2/2} \right| \le C_0 \frac{\beta_3}{\sigma^3 (2n+1)^{2d}} |t|^3 e^{-t^2/4} < C_2 \frac{1}{n^{2d}} |t|^3 e^{-t^2/4},$$

где $C_2 = \dfrac{C_0\beta_3}{\sigma^3 2^{2d}}$. Перейдем к оценке разности функций распределения

$$\Delta_n = \sup_x \left| P\left(\frac{S(V_n)}{\sigma\sqrt{|V_n|}} < x\right) - \Phi(x)\right|.$$

Согласно неравенству Берри-Эссеена,

$$\Delta_n \le \frac{2}{\pi}\int_0^T \left|\frac{f_n(t) - e^{-t^2/2}}{t}\right| dt + \frac{24}{\pi T}\cdot \max_x |\Phi'(x)| =$$
$$= \frac{2}{\pi}\int_0^T \left|\frac{f_n(t) - e^{-t^2/2}}{t}\right| dt + \frac{24}{\pi\sqrt{2\pi}}\cdot\frac{1}{T}.$$

Следовательно,

$$\Delta_n < \frac{2C_1}{\pi}\cdot\frac{1}{n^{d/2}}\int_0^T t^2 dt + \frac{2C_2}{\pi}\cdot\frac{1}{n^{2d}}\int_0^\infty t^2 e^{-t^2/4}dt + \frac{24}{\pi\sqrt{2\pi}}\cdot\frac{1}{T} <$$
$$< \frac{2C_1}{3\pi}\cdot\frac{T^3}{n^{d/2}} + \frac{4C_2}{\sqrt{\pi}}\cdot\frac{1}{n^{2d}} + \frac{24}{\pi\sqrt{2\pi}}\cdot\frac{1}{T}.$$

Положив $T = n^{d/8}$, окончательно получаем

$$\Delta_n < \left(\frac{2C_1}{3\pi} + \frac{24}{\pi\sqrt{2\pi}}\right)\cdot\frac{1}{n^{d/8}} + \frac{4C_2}{\sqrt{\pi}}\cdot\frac{1}{n^{2d}} < Cn^{-d/8}.$$

Теорема доказана.

Возможность применения полученных результатов к гиббсовским случайным полям основана на следующем замечании. Известно (см., например, [19], [7]), что при определенных условиях на потенциал гиббсовское случайное поле удовлетворяет условию равномерного сильного перемешивания. Кроме того, известны также условия на потенциал, при которых гиббсовское случайное поле является мартингал–разностным случайным полем ([17, 20]). Указанные условия не противоречат друг другу, что позволяет выделить класс гиббсовских случайных полей, удовлетворяющих условиям Теоремы 3.1. Таковыми будут, например, гиббсовские случайные поля с достаточно быстро убывающими четными потенциалами.

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Abstract. The estimations for the rate of convergence in the central limit theorem for homogenous martingale–difference random fields on the d-dimensional integer lattice are obtained.

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