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О ЯВЛЕНИИ ГИББСА ДЛЯ РАЗЛОЖЕНИЙ ПО СОБСТВЕННЫМ ФУНКЦИЯМ КРАЕВОЙ ЗАДАЧИ ДЛЯ СИСТЕМЫ ДИРАКА

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Аннотация. Рассмотрены разложения по собственным функциям системы Дирака. Выявлено явление Гиббса для таких разложений.

MSC2010 number: 34L10, 34L40.

Ключевые слова: Разложение по собственным функциям; система Дирака; явление Гиббса.

1. Введение

Как известно, разложения по собственным функциям регулярных краевых задач для обыкновенных дифференциальных уравнений с гладкими коэффициентами на конечном отрезке равномерно сходятся, если разлагаемая функция принадлежит области определения соответствующего оператора. В противном же случае может наблюдаться явление, аналогичное явлению Гиббса для классических рядов Фурье. Подобные явления, в частных случаях, были изучены в ряде работ (см. Л.Мишо [1, 2], Л.Брандолини и Л. Колзани [3] а также [4, 5, 6]).

В настоящей работе выявлено явление Гиббса для компонент вектор-функции краевой задачи для системы Дирака

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dy}{dx} - \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} y = \lambda y, \tag{1.1}$$

$$y_2(-1)\cos\alpha + y_1(-1)\sin\alpha = 0, \tag{1.2}$$

$$y_2(1)\cos\beta + y_1(1)\sin\beta = 0. \tag{1.3}$$

где p и r действительные на отрезке $[-1,1]$ функции.

Исходя из физических соображений, Р. Шмитковски показал (см. [7, 8]), что разложения по собственным функциям системы Дирака не сходится к разлагаемой функции в концах интервала, даже если каждая компонента этой функции из класса $C^\infty[-1,1]$, но не удовлетворяет краевым условиям.

Следуя [9] (см. стр. 71), приведем некоторые, необходимые нам, известные факты и формулы, связанные с задачей (1.1)-(1.3). Обозначим через $\{\lambda_n\}_{n=-\infty}^{\infty}$, $\{v_n = (v_{n,1}, v_{n,2})^T\}_{n=-\infty}^{\infty}$ множество собственных значений и нормированных собственных вектор-функций этой задачи. Не умаляя общности можем предположить, что число $\lambda = 0$ не является собственным значением. Для краткости ряды по собственным функциям $\{v_n\}$ будем также называть рядами Фурье, а соответствующие коэффициенты - коэффициентами Фурье.

Для вектор функции $f(x) = (f_1(x), f_2(x))^T \in L_2^2[-1,1] = L_2[-1,1] \times L_2[-1,1]$ введем обозначения,

$$S_N(f) = \sum_{n=-N}^{N} c_n v_n(x), \tag{1.4}$$

$$c_n = \int_{-1}^{1} v_n^T(x) f(x)\,dx, \tag{1.5}$$

$$R_N(f) = f(x) - S_N(f).$$

Известно [9] (см. стр. 82), что собственные вектор-функции задачи Дирака образуют полную ортогональную систему в гильбертовом пространстве $L_2^2[-1,1]$, т.е. $S_N(f)$ сходится к f по норме L_2^2

$$\|f\|_2 = \left(\int_{-1}^{1} (f_1^2(x) + f_2^2(x))\,dx\right)^{1/2}.$$

Имеют место следующие асимптотические формулы (см. [9] стр. 75, где при выводе формул (1.7),(1.8) была допущена опечатка. Здесь приведена уточненная формула):

$$\lambda_n = n - \frac{\theta}{\pi} + O\left(n^{-1}\right), \quad n \to \infty, \tag{1.6}$$

$$v_{n,1}(x) = \cos(\xi_n - \alpha) + O\left(n^{-1}\right), \quad n \to \infty, \tag{1.7}$$

$$v_{n,2}(x) = \sin(\xi_n - \alpha) + O\left(n^{-1}\right), \quad n \to \infty, \tag{1.8}$$

где

$$\theta = \beta - \alpha - \frac{1}{2}\int_{-1}^{1}(p(t) + r(t))dt, \tag{1.9}$$

$$\xi_n = \xi(x, \lambda_n) = \lambda_n(x+1) - \frac{1}{2}\int\limits_{-1}^{x}(p(\tau) + r(\tau))d\tau.$$

Основным результатом настоящей работы является

Теорема . *Пусть $p, r \in C^1[-1,1]$, $f \in C_2^{\frac{1}{2}}[-1,1]$ и функция f не удовлетворяет краевым условиям, тогда, если $B(f,-1,\alpha) \neq 0$ в точке -1 имеют место соотношения*

$$\limsup_{\substack{N\to\infty \\ x\to -1}} \frac{|B(S_N(f), x, \alpha)|}{|B(f,-1,\alpha)|} = \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt,$$

а если $B(f,1,\beta) \neq 0$ то

$$\limsup_{\substack{N\to\infty \\ x\to 1}} \frac{|B(S_N(f), x, \beta)|}{|B(f,1,\beta)|} = \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt,$$

где

$$B(f, x, \gamma) = f_1(x)\sin\gamma + f_2(x)\cos\gamma.$$

Замечание 1.1. *Так как все собственные функции удовлетворяют граничным условиям то если рассмотреть предел усеченного ряда то и он должен удовлетворять этим условиям (если сходимость равномерная). В случае когда разлагаемая функция не удовлетворяет граничным условиям, т.е. $B(f,-1,\alpha) \neq 0$ или $B(f,1,\beta) \neq 0$, то имеет место аналог явления Гиббса в терминах нарушения граничных условий.*

Замечание 1.2. *Если рассмотреть разложения* (1.4) *покомпонентно, то легко можно выяснить (следует из доказательства теоремы), что если* $\alpha = 0$ *или* $\alpha = \pi/2$ *то одна компонента ряда* $S_N(f)$ *сходиться равномерно а для другой имеет место явления Гиббса в точке* -1 *(см. §3). Вышесказанное имеет место также в точке* 1 *если* $\beta = 0$ *или* $\beta = \pi/2$.

Замечание 1.3. *Величина* $\frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}dt \approx 1.17898$ *это константа классического явления Гиббса для рядов Фурье.*

Замечание 1.4. *Для преодоления этого явления в работе [10] был предложен метод ускорения сходимости разложений по собственным вектор-функциям задачи* (1.1)-(1.3), *аналогичный методу Крылова-Экгофа ускорения сходимости классического ряда Фурье (см [11, 12])*

2. Доказательство теоремы

Обозначим

$$Lf = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{df}{dx} - \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} f,$$

$$Bf = B\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} -f_2(x) \\ f_1(x) \end{pmatrix},$$

$$\bar{f}_k(x) = BL^k f(x), \quad k \geqslant 0,$$

где L^0 -тождественный оператор.

Лемма 2.1. *Пусть* $p, r \in C^{q-1}[-1,1]$, $p^{(q-1)}, r^{(q-1)} \in AC[-1,1]$, $f \in C_2^q[-1,1]$ *и* $f^{(q)} \in AC_2[-1,1]$, *причем* $q \geqslant 1$. *Тогда для коэффициентов* c_n, *определенных в* (1.4), *имеет место представление* $c_n = P_n + F_n$, *где*

$$P_n = v_n^T(1)\sum_{k=0}^{q} \lambda_n^{-k-1}\bar{f}_k(1) - v_n^T(-1)\sum_{k=0}^{q} \lambda_n^{-k-1}\bar{f}_k(-1), \tag{2.1}$$

$$F_n = \lambda_n^{-q-1}\int_{-1}^{1} v_n^T(x)L^{q+1}(f(x))dx.$$

Доказательство. Имеем

$$c_n = \int_{-1}^{1} v_n^T(x) f(x) dx = \lambda_n^{-1} \int_{-1}^{1} (f_1(x)(v'_{n,2}(x) - p(x)v_{n,1}(x))$$

$$f_2(x)(v'_{n,1}(x) + r(x)v_{n,2}(x)))dx.$$

Интегрируя последнее равенство по частям, получим:

$$c_n = \lambda_n^{-1} \left(f_1(x)v_{n,2}(x) - f_2(x)v_{n,1}(x)\right)\Big|_{-1}^{1}$$

$$+ \lambda_n^{-1} \int_{-1}^{1} v_{n,1}(x)(f_2'(x) - p(x)f_1(x)) + v_{n,2}(x)(-f_1'(x) - r(x)f_2(x))dx$$

$$= \lambda_n^{-1} v_n^T(x)\tilde{f}_0(x)\Big|_{-1}^{1} + \lambda_n^{-1} \int_{-1}^{1} v_n^T(x) L(f(x)) dx$$

Повторяя интегрирование по частям q раз, получим требуемое. □

Рассмотрим теперь функцию

$$\kappa(x) = \begin{pmatrix} x - 1 \\ -x + 1 \end{pmatrix}, \tag{2.2}$$

которая не удовлетворяет краевым условиям в точке $x = 0$ при $\alpha \neq \frac{\pi}{4}$. Рассмотрим ее разложение по системе $\{v_n(x)\}$:

$$S_N(\kappa) = \sum_{n=-N}^{N} c_n(\kappa) v_n, \tag{2.3}$$

где $c_n(\kappa) = \int_{-1}^{1} v_n^T(x)\kappa(x)dx$.

Лемма 2.2. *Если $p, r \in C^1[-1,1]$, то коэффициенты разложения функции (2.2) имеют следующую асимптотику*

$$c_n(\kappa) = \sqrt{2}\lambda_n^{-1}(\cos\alpha - \sin\alpha) + \alpha_n,$$

где $\sum_{n=1}^{\infty} |\alpha_n| < +\infty$.

Доказательство. Используя лемму 2.1 для функции (2.2) получим:

$$c_n = v_n^T(1)\lambda_n^{-1}\tilde{\kappa}_0(1) - v_n^T(-1)\lambda_n^{-1}\tilde{\kappa}_0(-1) + \lambda_n^{-1}\int_{-1}^{1} v_n^T(x)L^1(\kappa(x))dx =$$

$$\binom{2}{2} v_n^T(-1)\lambda_n^{-1}\tilde{\kappa}_0(-1) + \lambda_n^{-1}\int_{-1}^{1} v_n^T(x)L^1(\kappa(x))dx =$$

$$\sqrt{2}\lambda_n^{-1}(\cos\alpha - \sin\alpha) + O\left(\frac{1}{n^2}\right) + \lambda_n^{-1}\int_{-1}^{1} v_n^T(x)L^1(\kappa(x))dx.$$

Учитывая тот факт что $L^1(\kappa(x)) \in L_2^2[-1,1]$, получим желаемый результат. □

Покажем теперь, что если $p, r \in C^1[-1,1]$ и функция κ определена в (2.2), тогда, если $\alpha \neq \frac{\pi}{4}$, для функции κ имеет место явление Гиббса. Рассмотрим ошибку при приближении функции (2.2) урезанным рядом

$$R_N(\kappa) = \kappa(x) - S_N(\kappa) = \sum_{\|n\|>N} c_n v_n(x) = \sum_{\|n\|>N} c_n \begin{pmatrix} v_{n,1}(x) \\ v_{n,2}(x) \end{pmatrix}.$$

Используя лемму 2.2 и асимптотику собственных функций κ, получим

$$\sum_{|n|>N} c_n v_{n,1}(x) = (\cos\alpha - \sin\alpha)\sum_{|n|>N}\left(\frac{\cos(\xi_n - \alpha)}{\lambda_n} + \alpha_n\right) = \tag{2.4}$$

$$(\cos\alpha - \sin\alpha)\sum_{|n|>N}\left(\frac{\cos(\frac{\pi}{2}n(x+1) + \varphi(x))}{\pi n/2} + \alpha_n\right),$$

где

$$\varphi(x) = -\alpha - \frac{\theta}{2}(x+1) - \frac{1}{2}\int_{-1}^{x}(p(\tau) + r(\tau))d\tau.$$

Из (2.4) следует, что

$$S_N(\kappa) = \kappa(x) - \sum_{|n|>N}\frac{\cos\alpha - \sin\alpha}{\pi n/2}\begin{pmatrix}\cos\left(\frac{\pi}{2}n(x+1) + \varphi(x)\right) \\ \sin\left(\frac{\pi}{2}n(x+1) + \varphi(x)\right)\end{pmatrix} + o(1) =$$

$$\kappa(x) - \frac{2(\cos\alpha - \sin\alpha)}{\pi}\begin{pmatrix}-\sin(\varphi(x)) \\ \cos(\varphi(x))\end{pmatrix}\sum_{|n|>N}\frac{\sin\left(\frac{\pi}{2}n(x+1)\right)}{n} + o(1).$$

Функция φ непрерывна, а сумма $\sum_{|n|>N} \frac{\sin(\frac{\pi}{2}n(x+1))}{n}$ является остаточной суммой ряда Фурье для функции $\frac{\pi}{2} - \frac{\pi}{2}x$, т.е.

$$\sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{\pi}{2}n(x+1)\right)}{n} = -\frac{\pi}{2}x + \frac{\pi}{2}, \quad x \in (-1,1]. \tag{2.5}$$

Разделим интервал $(-1,1]$ на две части $(-1,\xi]$, $(\xi,1]$ так, что бы на интервале $(-1,\xi]$ выполнялась оценка $|\varphi(x) - \varphi(-1)| < \epsilon$, а на интервале $(\xi,1]$ - оценка $|\kappa(x) - S_N(\kappa)| < \epsilon$, для достаточно больших N. Тогда

$$\limsup_{\substack{N\to\infty \\ x\to -1}} |B(S_N(\kappa), x, \alpha)| = |B(\kappa, -1, \alpha)| - \frac{|B(\kappa, -1, \alpha)|}{\pi} \limsup_{\substack{N\to\infty \\ x\to -1}} \sum_{|n|>N} \frac{\sin\left(\frac{\pi}{2}n(x+1)\right)}{n}. \tag{2.6}$$

Из формул (2.6) и (2.5) получим желаемый результат.

Аналогичный результат можно получить для функции

$$\varrho(x) = \begin{pmatrix} x+1 \\ -x-1 \end{pmatrix}, \tag{2.7}$$

которая имеет сингулярность в точке 1.

Если $p, r \in C^1[-1,1]$ и функция ϱ определена в (2.7), тогда при $\beta \neq \frac{\pi}{4}$ имеет место явление Гиббса, а именно

$$\limsup_{\substack{N\to\infty \\ x\to 1}} \frac{|B(S_N(\varrho), x, \beta)|}{|B(\varrho, 1, \beta)|} = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt.$$

Отсюда получен следующий общий результат

Предположим $\alpha, \beta \neq \frac{\pi}{4}$, аналогичным образом, можно доказать в случае когда α или β равны $\frac{\pi}{4}$.

Рассмотрим функцию $g(x) = f(x) - h\,\kappa(x) - H\varrho(x)$, где

$$h = \frac{f_2(-1)\cos\alpha + f_1(-1)\sin\alpha}{\kappa_2(-1)\cos\alpha + \kappa_1(-1)\sin\alpha}, \quad H = \frac{f_2(1)\cos\alpha + f_1(1)\sin\alpha}{\varrho_2(1)\cos\alpha + \varrho_1(1)\sin\alpha}.$$

Функция $g(x)$ будет удовлетворять краевым условиям и имеет непрерывную первую производную, следовательно ряд $S_N(f(x) - h\kappa(x) - H\varrho(x))$ сходится равномерно, что и доказывает утверждение теоремы. □

3. Численные иллюстрации

Проиллюстрируем сказанное на примере функции κ. Заметим также, что если $\alpha = 0$, то одна компонента в разложении (2.3) сходится равномерно, а другая ведет к явлению Гиббса. Для иллюстрации этого явления рассмотрим систему с нулевым потенциалом, и пусть $\alpha = 0$, $\beta = -\frac{\pi}{4}$. Для этой задачи легко можно вычислить собственные значения и собственные функции, а именно

$$\lambda_n = \frac{\pi k}{2} + \frac{\pi}{8}, \quad n = 0, \pm 1, \pm 2, \ldots,$$

а нормированные собственные вектор функции имеют вид

$$u_n(x) = \frac{1}{2}\sqrt{1 - \frac{(-1)^k}{\sqrt{2}}} \begin{pmatrix} \cos\left(\left(\frac{\pi k}{2} + \frac{\pi}{8}\right)x\right)\cot\left(\frac{\pi k}{2} + \frac{\pi}{8}\right) - \sin\left(\left(\frac{\pi k}{2} + \frac{\pi}{8}\right)x\right) \\ \cos\left(\left(\frac{\pi k}{2} + \frac{\pi}{8}\right)x\right) + \cot\left(\frac{\pi k}{2} + \frac{\pi}{8}\right)\sin\left(\left(\frac{\pi k}{2} + \frac{\pi}{8}\right)x\right) \end{pmatrix}.$$

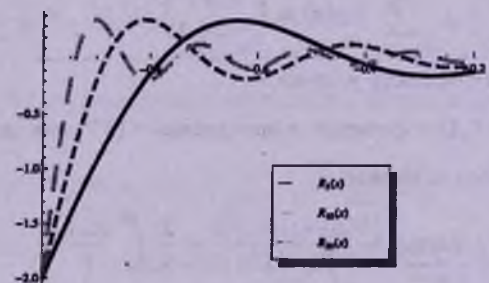

Рис. 1. Ошибка приближения второй компоненты функции $\kappa(x)$ конечной суммой ряда Фурье в окрестности точки $x = -1$ с использованием 5, 10, 20 коэффициентов Фурье

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Abstract. The paper considers expansions by eigenfunctions of the boundary problem for Dirac system. The Gibbs phenomenon for such expansions is revealed.

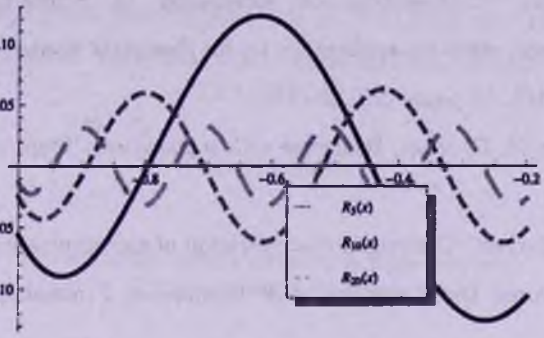

РИС. 2. Ошибка приближения первой компоненты функции $\kappa(x)$ конечной суммой ряда Фурье в окрестности точки $x = -1$, при $N = 5, 10, 20$ коэффициентов Фурье (равномерная ошибка уменьшается с ростом N).

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ОБ ОБРАТИМЫХ АЛГЕБРАХ ЛИНЕЙНЫХ НАД АБЕЛЕВОЙ ГРУППОЙ

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Аннотация. В работе с помощью формул второго порядка, а именно $\forall\exists(\forall)$–тождеств, характеризуются некоторые классы обратимых алгебр линейных над абелевой группой, имеющие ограничения на используемые автоморфизмы соответствующей группы.

MSC2010 number: 20N05.

Ключевые слова: квазигруппа; обратимая алгебра; линейная алгебра; обратимая Т-алгебра; формула второго порядка; сверхтождество.

1. Введение

При рассмотрении вопросов, связанных с многообразиями, квазигруппы представляют как алгебры с тремя операциями [1], добавляя к основной операции еще две дополнительные операции. Если основная операция является умножением $(\cdot)$, то остальные две операции называют правым и левым делением. Если операция квазигруппы обозначена через A, то правая и левая обратные операции обозначаются соответственно через A^{-1} и ${}^{-1}A$.

Квазигруппа $(Q;\cdot)$ называется изотопной квазигруппе $(Q;\circ)$, если существует такая тройка подстановок $T=(\alpha,\beta,\gamma)$ на множестве Q, что выполняется соотношение $\gamma(x\circ y)=\alpha x\cdot\beta y$. В классе квазигрупп, изотопных группам, представляют интерес линейные квазигруппы введенные В. Д. Белоусовым [2] в связи с исследованием уравновешенных тождеств в квазигруппах. Квазигруппа $(Q;\cdot)$ называется линейной над группой $(Q;+)$, если она имеет вид

$$x\cdot y=\varphi x+c+\psi y, \tag{1.1}$$

где $\varphi,\psi\in Aut(Q;+), c-$ фиксированный элемент из Q.

Важный класс линейных квазигрупп составляют T-квазигруппы. Согласно [3] T-квазигруппа- это квазигруппа с соотношением (1.1), где $(Q;+)$– абелева группа. Г. Б. Белявской и А. Х. Табаровым доказано, что (примитивные) T-квазигруппы составляют многообразие [4,5].

Бинарная алгебра $(Q;\Sigma)$ называется обратимой алгеброй, если каждая ее операция $A \in \Sigma$ является квазигруппой (обратимой). В работе [6], по аналогии с линейными квазигруппами введено понятие линейной обратимой алгебры и обратимой T-алгебры, а также дана их характеризация с помощью формул второго порядка.

Разными авторами изучались подклассы линейных квазигрупп с ограничениями на изотопные им группы и на используемые автоморфизмы и антиавтоморфизмы. Например, T-квазигруппы, медиальные, парамедиальные квазигруппы и т.д. рассматривались многими авторами(см. [7-13]).

В настоящей работе с помощью формул второго порядка, а именно $\forall\exists(\forall)$– тождеств, характеризуются некоторые классы обратимых T-алгебр имеющие ограничения на используемые автоморфизмы соответствующей группы. Полученные результаты являются обобщением результатов статьи [13] для некоторых классов обратимых T-алгебр и при их доказательстве используются некоторые методы данной работы.

2. Характеризация T-алгебр

Напомним [1], что квазиавтоморфизм (антиквазиавтоморфизм) квазигруппы $(Q;\cdot)$ это главная компонента γ автотопии (антиавтотопии) $T = (\alpha,\beta,\gamma)$ квазигруппы $(Q;\cdot)$, т.е. $\gamma(x\cdot y) = \alpha x\cdot\beta y$ ($\gamma(x\cdot y) = \alpha y\cdot\beta x$). Согласно лемме 2.5 [1], любой квазиавтоморфизм группы $(Q;+)$ имеет вид

$$\gamma x = R_s\gamma_0 x = L_s\gamma_0' x, \tag{2.1}$$

где γ_0, γ_0'– автоморфизмы группы $(Q;+)$, $R_s x = x+s$, $L_s x = s+x$. Как отмечено в [2], утверждение, аналогичное лемме 2.5 [1], справедливо и для антиквазиавтоморфизма γ. В этом случае γ_0 и γ_0' из (2.1) являются антиавтоморфизмами группы $(Q;+)$.

Хорошо известно [1], что с каждой квазигруппой A связаны следующие пять квазигрупп:

$$A^{-1},\ {}^{-1}A,\ {}^{-1}(A^{-1}),\ ({}^{-1}A)^{-1}\ A^{*},$$

где $A^{*}(x,y) = A(y,x)$. Других обратных операций для A не существует. Таким образом, с каждой обратимой алгеброй $(Q;\Sigma)$ связаны следующие пять обратимых алгебр:

$$(Q;\Sigma^{-1}),\quad (Q;{}^{-1}\Sigma),\quad (Q;{}^{-1}(\Sigma^{-1})),\quad (Q;({}^{-1}\Sigma)^{-1}),\quad (Q;\Sigma^{*}),$$

где $\Sigma^{-1} = \{A^{-1}|A \in \Sigma\}$, ${}^{-1}\Sigma = \{{}^{-1}A|A \in \Sigma\}$, ${}^{-1}(\Sigma^{-1}) = \{{}^{-1}(A^{-1})|A \in \Sigma\}$, $({}^{-1}\Sigma)^{-1} = \{({}^{-1}A)^{-1}|A \in \Sigma\}$, $\Sigma^{*} = \{A^{*}|A \in \Sigma\}$. Каждая из этих алгебр называется парастрофом исходной алгебры.

Для $A \in \Sigma$ и $a \in Q$ обозначим через $L_{A,a}$ ($R_{A,a}$) левую (правую) трансляцию алгебры $(Q;\Sigma)$, т.е. отображение $L_{A.a} : x \to A(a,x)$ ($R_{A,a} : x \to A(x,a)$).

Определение 2.1. [6]. *Обратимая алгебра $(Q;\Sigma)$ называется T-алгеброй, если каждая ее операция $A \in \Sigma$ изотопна одной и той же абелевой группе $(Q;+)$, причем изотопия имеет вид*

$$A(x,y) = \varphi_A x + c_A + \psi_A y, \tag{2.2}$$

где φ_A, ψ_A– автоморфизмы группы $(Q;+)$, c_A– фиксированный элемент Q.

Теорема 2.1. *Для обратимой алгебры $(Q;\Sigma)$ следующие условия эквивалентны:*

1) $(Q;\Sigma)$– обратимая T-алгебра, причем для всех $X,Y \in \Sigma$,
$\varphi_X\psi_Y\varphi_Y^{-1} = \psi_X\varphi_Y^{-1}\psi_Y$;

2) Для всех $X,Y \in \Sigma$, в алгебре $(Q;\Sigma\cup {}^{-1}\Sigma)$, выполняется следующая формула второго порядка:

$$X(Y(x,\ {}^{-1}Y(y,u)),z) = X(Y(x,\ {}^{-1}Y(u,u)),\ {}^{-1}Y(Y(z,y),u)). \tag{2.3}$$

Доказательство. 1) $\Rightarrow$ 2). Пусть $(Q;\Sigma)$– обратимая T-алгебра с условием $\varphi_X\psi_Y\varphi_Y^{-1} = \psi_X\varphi_Y^{-1}\psi_Y$ для всех $X,Y \in \Sigma$. Прежде всего заметим, что из (2.2) следует, что

$$X^{-1}(x,y) = -\psi_X^{-1}\varphi_X x - \psi_X^{-1}c_X + \psi_X^{-1}y,$$

$${}^{-1}X(x,y) = \varphi_X^{-1}x - \varphi_X^{-1}c_X - \varphi_X^{-1}\psi_X y.$$

Тогда, $X(Y(x,\ {}^{-1}Y(y,u)),z) =$

$$= \varphi_X\varphi_Y x + \varphi_X c_Y + \varphi_X\psi_Y(\varphi_Y^{-1}y - \varphi_Y^{-1}c_Y - \varphi_Y^{-1}\psi_Y u) + c_X + \psi_X z =$$

$$= \varphi_X\varphi_Y x + \varphi_X c_Y + \varphi_X\psi_Y\varphi_Y^{-1}y - \varphi_X\psi_Y\varphi_Y^{-1}c_Y - \varphi_X\psi_Y\varphi_Y^{-1}\psi_Y u + c_X + \psi_X z;$$

$$X(Y(x,{}^{-1}Y(u,u)),{}^{-1}Y(Y(z,y),u)) = \varphi_X Y(x,{}^{-1}Y(u,u)) + c_X + \psi_X{}^{-1}Y(Y(z,y),u) =$$

$$= \varphi_X(\varphi_Y x + c_Y + \psi_Y(\varphi_Y^{-1}u - \varphi_Y^{-1}c_Y - \varphi_Y^{-1}\psi_Y u)) + c_X + \psi_X(\varphi_Y^{-1}(\varphi_Y z + c_Y +$$

$$\psi_Y y) - \varphi_Y^{-1}c_Y - \varphi_Y^{-1}\psi_Y u) = \varphi_X\varphi_Y x + \varphi_X c_Y - \varphi_X\psi_Y\varphi_Y^{-1}c_Y - \varphi_X\psi_Y\varphi_Y^{-1}\psi_Y u +$$

$$+ c_X + \psi_X z + \psi_X\varphi_Y^{-1}\psi_Y y.$$

Следовательно, формула (2.3) выполняется в алгебре $(Q; \Sigma\cup\ {}^{-1}\Sigma)$.

2) $\Rightarrow$ 1). Пусть в обратимой алгебре $(Q; \Sigma\cup\ {}^{-1}\Sigma)$ выполняется формула (2.3). Фиксируем в (2.3) элемент u и операции $X = A$, $Y = B$, где $A, B \in \Sigma$, тогда получим:

$$A_1(A_2(x,y),z) = A_3(x, A_4(y,z)),$$

где $A_1(x,y) = A(x,y)$, $A_2(x,y) = B(x,\ {}^{-1}B(y,u))$, $A_3(x,y) = A(B(x,\ {}^{-1}B(u,u)),y)$, $A_4(x,y) = {}^{-1}B(B(y,x),u)$.

Из последнего равенства, по теореме Белоусова о четырех квазигруппах, связанных ассоциативным законом [2], каждая операция $A_i (i = 1,2,3,4)$ изотопна одной и той же групре. Следовательно, операции A и B изотопны одной и той же групре и поскольку эти операции произвольны, то любая операция из Σ изотопна одной и той же групре $(Q;*)$.

Для каждого $X \in \Sigma$ определим операции:

$$x \underset{X}{+} y = X(R_{X,a}^{-1}x, L_{X,b}^{-1}y), \tag{2.4}$$

где a и b некоторые элементы из Q. Эти операции- лупы с единичным элементом $0_X = X(b,a)$ [1,теорема 1.3], и они изотопны групре $(Q;*)$, поэтому согласно теореме Алберта [1, теорема 1.4], операции $\underset{X}{+}$ являются группами для всех $X \in \Sigma$.

Перейдем в (2.3) к операциям $\underset{X}{+}$:

$$R_{X,a}(R_{Y,a}x \underset{Y}{+} L_{Y,b}^{-1}Y(y,u)) \underset{X}{+} L_{X,b}z =$$

$$= R_{X,a}(R_{Y,a}x \underset{Y}{+} L_{Y,b}^{-1}Y(u,u)) \underset{X}{+} L_{X,b}({}^{-1}Y(R_{Y,a}z \underset{Y}{+} L_{Y,b}y, u)),$$

$$R_{X,a}(x \underset{Y}{+} L_{Y,b}^{-1}Y(Y(y,u),u)) \underset{X}{+} z =$$
$$= R_{X,a}(x \underset{Y}{+} L_{Y,b}^{-1}Y(u,u)) \underset{X}{+} L_{X,b}R_{Y,u}^{-1}(R_{Y,a}L_{X,b}^{-1}z \underset{Y}{+} L_{Y,b}R_{Y,u}y).$$

Взяв в последнем равенстве $z = 0_X$ и фиксируя элемент u, получаем

$$R_{X,a}(x \underset{Y}{+} L_{Y,b}y) = R_{X,a}(x \underset{Y}{+} L_{Y,b}^{-1}Y(u,u)) \underset{X}{+} L_{X,b}R_{Y,u}^{-1}(R_{Y,a}a \underset{Y}{+} L_{Y,b}R_{Y,u}y),$$

или

(2.5) $$R_{X,a}(x \underset{Y}{+} y) = \alpha_{X,Y}x \underset{X}{+} \beta_{X,Y}y,$$

где $\alpha_{X,Y}$ и $\beta_{X,Y}$ подстановки множества Q. Так как операции X и Y произвольны, мы можем в (2.5) взять $X = Y$, получим:

(2.6) $$R_{X,a}(x \underset{X}{+} y) = \alpha_{X,X}x \underset{X}{+} \beta_{X,X}y.$$

Из (2.5) и (2.6) имеем:

$$x \underset{X}{+} y = R_{X,a}(\alpha_{X,Y}^{-1}x \underset{Y}{+} \beta_{X,Y}^{-1}y),$$
$$x \underset{X}{+} y = R_{X,a}(\alpha_{X,X}^{-1}x \underset{X}{+} \beta_{X,X}^{-1}y),$$
$$\alpha_{X,X}^{-1}x \underset{X}{+} \beta_{X,X}y = \alpha_{X,Y}^{-1}x \underset{Y}{+} \beta_{X,Y}^{-1}y,$$

таким образом получаем

(2.7) $$x \underset{X}{+} y = \gamma_{X,Y}x \underset{Y}{+} \delta_{X,Y}y,$$

где $\gamma_{X,Y} = \alpha_{X,Y}^{-1}\alpha_{X,X}$ и $\delta_{X,Y} = \beta_{X,Y}^{-1}\beta_{X,X}$ подстановки множества Q. Следовательно, из (2.5) и (2.6) имеем

$$R_{X,a}(x \underset{Y}{+} y) = \gamma_{X,Y}\alpha_{X,Y}x \underset{Y}{+} \delta_{X,Y}\beta_{X,Y}y,$$

т.е. $R_{X,a}$– квазиавтоморфизм группы $(Q; \underset{Y}{+})$. Поскольку операции X, Y– произвольны, то для любой операции $X \in \Sigma$, $R_{X,a}$ будет квазиавтоморфизмом каждой из групп $(Q; \underset{Y}{+})$, где $Y \in \Sigma$.

Зафиксируем операцию $\underset{B}{+}$ для некоторого $B \in \Sigma$ и в дальнейшем будем обозначать ее через $+$. Согласно (2.4), для операций $A \in \Sigma$, имеем:

$$A(x,y) = R_{A,a}x \underset{A}{+} L_{A,b}y.$$

Из последнего равенства, согласно (2.7), получим:

(2.8) $$A(x,y) = \theta_1^{A,B}x + \theta_2^{A,B}y,$$

где $\theta_1^{A,B} = \gamma_{A,B} R_{A,a}$ и $\theta_2^{A,B} = \delta_{A,B} L_{A,b}$ подстановки множества Q.

Покажем, что $\theta_1^{A,B}$ – квазиавтоморфизм группы $(Q;+)$. Для этого, представим формулу (2.3) в следующем виде:

$$A(B(x,y),z) = A(B(x, {}^{-1}B(u,u)), {}^{-1}B(B(z,B(y,u)),u)). \tag{2.9}$$

Зафиксируем в (2.9) переменные $z = c,\ u = d$ и перепишем его с использованием операции $+$:

$$\theta_1^{A,B}(R_{B,a}x + L_{B,b}y) + \theta_2^{A,B}c = \theta_1^{A,B}B(x, {}^{-1}B(d,d)) + \theta_2^{A,B}\ {}^{-1}B(B(c,B(y,d)),d),$$

$$\theta_1^{A,B}(x+y) = \theta_1^{A,B}B(x,\ {}^{-1}B(d,d)) + \theta_2^{A,B}\ {}^{-1}B(B(c,B(L_{B,b}^{-1}y,d)),d) - \theta_2^{A,B}c.$$

Из последнего равенства получаем:

$$\theta_1^{A,B}(x+y) = \sigma_{A,B}x + \mu_{A,B}y,$$

где $\sigma_{A,B}$ и $\mu_{A,B}$ подстановки множества Q, следовательно $\theta_1^{A,B}$– квазиавтоморфизм группы $(Q;+)$.

Докажем теперь, что $\theta_2^{A,B}$– антиквазиавтоморфизм группы $(Q;+)$. Для этого, вновь перепишем формулу (2.9) в терминах операции $+$, получим:

$$\theta_1^{A,B}(R_{B,a}x + L_{B,b}y) + \theta_2^{A,B}z = \theta_1^{A,B}B(x,\ {}^{-1}B(u,u)) + \theta_2^{A,B}\ {}^{-1}B(B(z,B(y,u)),u),$$

$$\theta_1^{A,B}(R_{B,a}x + L_{B,b}y) + z = \theta_1^{A,B}(R_{B,a}x + L_{B,b}^{-1}B(u,u)) + \theta_2^{A,B}R_{B,u}^{-1}(R_{B,a}(\theta_2^{A,B})^{-1}z +$$

$$L_{B,b}R_{B,u}y).$$

В последнем равенстве возьмем $u = a$ и выберем x таким образом, чтобы $\theta_1^{A,B}(R_{B,a}x + L_{B,b}^{-1}B(a,a)) = 0$, тогда получим:

$$\alpha_{A,B}y + z = \theta_2^{A,B}R_{B,a}^{-1}(R_{B,a}(\theta_2^{A,B})^{-1}z + L_{B,b}R_{B,a}y),$$

где $\alpha_{A,B}$– подстановка множества Q, поэтому $\theta_2^{A,B}R_{B,a}^{-1}$– антиквазиавтоморфизм группы $(Q;+)$ и следовательно $\theta_2^{A,B}$ будет антиквазиавтоморфизмом, поскольку $R_{B,a}$– квазиавтоморфизм. Таким образом имеем:

$$\theta_1^{A,B}x = \varphi_A x + k_A,$$

$$\theta_2^{A,B}x = t_A + \overline{\psi}_A x,$$

где φ_A- автоморфизм, а $\overline{\psi}_A-$ антиавтоморфизм группы $(Q;+)$ и k_A, $t_A \in Q$. Поэтому из (2.8) получаем

$$A(x,y) = \varphi_A x + c_A + \overline{\psi}_A y, \tag{2.10}$$

где $c_A = k_A + t_A$. Поскольку операция A произвольна, получаем, что все операции из Σ могут быть представлены в виде (2.10) с помощью операции $+$. Перепишем формулу (2.9) в терминах операции $+$ с использованием равенства (2.10):

$$A(B(x,y),z) = \varphi_A\varphi_B x + \varphi_A c_B + \varphi_A\overline{\psi}_B y + c_A + \overline{\psi}_A z,$$

$$A(B(x,\ {}^{-1}B(u,u)),\ {}^{-1}B(B(z,B(y,u)),u)) = \varphi_A\varphi_B x + \varphi_A c_B - \varphi_A\overline{\psi}_B\varphi_B^{-1}c_B -$$

$$-\varphi_A\overline{\psi}_B\varphi_B^{-1}\overline{\psi}_B u + \varphi_A\overline{\psi}_B\varphi_B^{-1}u + c_A - \overline{\psi}_A\varphi_B^{-1}c_B - \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B u + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B\varphi_B y +$$

$$+\overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B c_B + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B\overline{\psi}_B u + \overline{\psi}_A\varphi_B^{-1}c_B + \overline{\psi}_A z,$$

таким образом

$$\varphi_A\overline{\psi}_B y + c_A = -\varphi_A\overline{\psi}_B\varphi_B^{-1}c_B - \varphi_A\overline{\psi}_B\varphi_B^{-1}\overline{\psi}_B u + \varphi_A\overline{\psi}_B\varphi_B^{-1}u + c_A - \overline{\psi}_A\varphi_B^{-1}c_B -$$

$$-\overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B u + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B\varphi_B y + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B c_B + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B\overline{\psi}_B u + \overline{\psi}_A\varphi_B^{-1}c_B.$$

Возьмем в последнем равенстве $u = 0$, получим:

$$\varphi_A\overline{\psi}_B y + c_A = -\varphi_A\overline{\psi}_B\varphi_B^{-1}c_B + c_A - \overline{\psi}_A\varphi_B^{-1}c_B + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B\varphi_B y + \tag{2.11}$$

$$+\overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B c_B + \overline{\psi}_A\varphi_B^{-1}c_B.$$

Положим в (2.11) $y = 0$, получим

$$c_A = -\varphi_A\overline{\psi}_B\varphi_B^{-1}c_B + c_A - \overline{\psi}_A\varphi_B^{-1}c_B + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B c_B + \overline{\psi}_A\varphi_B^{-1}c_B. \tag{2.12}$$

Подставляя (2.12) в (2.11) получаем:

$$\varphi_A\overline{\psi}_B y - \varphi_A\overline{\psi}_B\varphi_B^{-1}c_B + c_A - \overline{\psi}_A\varphi_B^{-1}c_B =$$

$$= -\varphi_A\overline{\psi}_B\varphi_B^{-1}c_B + c_A - \overline{\psi}_A\varphi_B^{-1}c_B + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B\varphi_B y.$$

Пусть, $-\varphi_A\overline{\psi}_B\varphi_B^{-1}c_B + c_A - \overline{\psi}_A\varphi_B^{-1}c_B = p$, тогда

$$\varphi_A\overline{\psi}_B y + p = p + \overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B\varphi_B y,$$

или

$$\widetilde{R}_p\varphi_A\overline{\psi}_B y = \widetilde{L}_p\overline{\psi}_A\varphi_B^{-1}\overline{\psi}_B\varphi_B y, \tag{2.13}$$

где $\widetilde{R}_p x = x + p$, $\widetilde{L}_p x = p + x$. Из (2.13) имеем

$$\widetilde{L}_p^{-1}\widetilde{R}_p y = \overline{\psi}_A \varphi_B^{-1}\overline{\psi}_B \varphi_B \overline{\psi}_B^{-1} \varphi_A^{-1} y,$$

поэтому $\widetilde{L}_p^{-1}\widetilde{R}_p$– антиавтоморфизм группы $(Q;+)$. Следовательно,

$$\widetilde{L}_p^{-1}\widetilde{R}_p(x+y) = \widetilde{L}_p^{-1}\widetilde{R}_p y + \widetilde{L}_p^{-1}\widetilde{R}_p x,$$

или

$$-p + x + y + p = -p + y + p - p + x + p,$$

поэтому $x + y = y + x$, т.е. группа $(Q;+)$– абелева группа. Поэтому $\overline{\psi}_A = \psi_A \in Aut(Q;+)$ и $\widetilde{L}_x = \widetilde{R}_x$ для всех $x \in Q$. Из (2.13) получаем:

$$\varphi_A \psi_B \varphi_B^{-1} = \psi_A \varphi_B^{-1} \psi_B.$$

□

Следующие утверждения доказываются аналогично.

Предложение 2.1. *Для обратимой алгебры $(Q;\Sigma)$ следующие условия эквивалентны:*

1) $(Q;\Sigma)$– обратимая T-алгебра, причем для всех $X, Y \in \Sigma$ имеем $\varphi_X \psi_Y \varphi_X^{-1} = \psi_X \varphi_X^{-1} \psi_X$;

2) Для всех $X, Y \in \Sigma$ в обратимой алгебре $(Q;\Sigma \cup {}^{-1}\Sigma)$ выполняется следующая формула второго порядка:

$$X(Y(x, {}^{-1}X(y,u)), z) = X(Y(x, {}^{-1}X(u,u)), {}^{-1}X(X(z,y),u)).$$

Предложение 2.2. *Для обратимой алгебры $(Q;\Sigma)$ следующие условия эквивалентны:*

1) $(Q;\Sigma)$ обратимая T-алгебра, причем для всех $X, Y \in \Sigma$ имеем $\varphi_X \psi_Y \varphi_X^{-1} = \psi_X \varphi_Y^{-1} \psi_Y$;

2) Для всех $X, Y \in \Sigma$ в обратимой алгебре $(Q;\Sigma \cup {}^{-1}\Sigma)$ выполняется следующая формула второго порядка:

$$X(Y(x, {}^{-1}X(y,u)), z) = X(Y(x, {}^{-1}X(u,u)), {}^{-1}Y(Y(z,y),u)).$$

Предложение 2.3. *Для обратимой алгебры $(Q;\Sigma)$ следующие условия эквивалентны:*

1) $(Q;\Sigma)$ *обратимая T-алгебра, причем для всех* $X,Y\in\Sigma$, $\varphi_X\psi_X^{-1}\varphi_X=\psi_X\varphi_Y\psi_X^{-1}$;

2) Для всех $X,Y\in\Sigma$ *в обратимой алгебре* $(Q;\Sigma\cup\Sigma^{-1})$ *выполняется следующая формула второго порядка:*

$$X(x,Y(X^{-1}(u,y),z))=X(X^{-1}(u,X(y,x)),Y(X^{-1}(u,u),z)).$$

Предложение 2.4. *Для обратимой алгебры* $(Q;\Sigma)$ *следующие условия эквивалентны:*

1) $(Q;\Sigma)$ *обратимая T-алгебра, причем для всех* $X,Y\in\Sigma$ *имеем* $\psi_X\varphi_Y\psi_X^{-1}=\varphi_X\psi_Y^{-1}\varphi_Y$;

2) Для всех $X,Y\in\Sigma$ *в обратимой алгебре* $(Q;\Sigma\cup\Sigma^{-1})$ *выполняется следующая формула второго порядка:*

$$X(x,Y(X^{-1}(u,y),z))=X(Y^{-1}(u,Y(y,x)),Y(X^{-1}(u,u),z)).$$

Предложение 2.5. *Для обратимой алгебры* $(Q;\Sigma)$ *следующие условия эквивалентны:*

1) $(Q;\Sigma)$ *обратимая T-алгебра, причем для всех* $X,Y\in\Sigma$ *имеем* $\psi_X\varphi_Y\psi_Y^{-1}=\varphi_X\psi_Y^{-1}\varphi_Y$;

2) Для всех $X,Y\in\Sigma$ *в обратимой алгебре* $(Q;\Sigma\cup\Sigma^{-1})$ *выполняется следующая формула второго порядка:*

$$X(x,Y(Y^{-1}(u,y),z))=X(Y^{-1}(u,Y(y,x)),Y(Y^{-1}(u,u),z)).$$

Приведем примеры обратимых T-алгебр с ограничениями на автоморфизмы соответствующей группы для каждого случая (Теорема 2.1 и Предложения 2.2–2.6). Рассмотрим четверную группу Клейна $K_4=\{0,1,2,3\}$. Как известно, ее группа автоморфизмов изоморфна групе S_3. Обозначим автоморфизмы группы $K_4(\cdot)$: $\varphi_1=\epsilon$, $\varphi_2=(12)$, $\varphi_3=(23)$, $\varphi_4=(13)$, $\varphi_5=(132)$, $\varphi_6=(123)$. Пусть $A_{i,j}(x,y)=\varphi_i x\cdot\varphi_j y$, $i,j=1,2,...,6$. Тогда, T-алгебры $(K_4;\{A_{4,3},A_{3,2}\})$ и $(K_4;\{A_{1,1},A_{5,5},A_{6,6}\})$ - удовлетворяют теореме 2.1; $(K_4;\{A_{2,4},A_{3,4}\})$ - предложению 2.2; $(K_4;\{A_{2,4},A_{3,3}\})$ - предложению 2.3; $(K_4;\{A_{2,3},A_{2,4}\})$ - предложению 2.4; $(K_4;\{A_{2,3},A_{4,4}\})$ - предложению 2.5; $(K_4;\{A_{3,4},A_{2,3}\})$ - предложению 2.6.

3. T-АЛГЕБРЫ И СВЕРХТОЖДЕСТВА

В данном параграфе приведем другие условия на автоморфизмы, приводящие к некоторым классам обратимых Т-алгебр, связанных с известными сверхтождествами:

$$X(x, Y(y,x)) = X(Y(y,x), y), \tag{3.1}$$

$$X(x, Y(y,x)) = Y(X(y,x), y), \tag{3.2}$$

$$X(Y(x,y), Y(y,x)) = y, \tag{3.3}$$

$$X(x, Y(x,y)) = X(Y(x,y), y), \tag{3.4}$$

$$X(x, Y(x,y)) = Y(X(x,y), y), \tag{3.5}$$

$$X(Y(x,y), Y(y,x)) = x. \tag{3.6}$$

Первые три сверхтождества называются сверхтождествами Стейна, другие три- сверхтождествами Шредера. Отметим, что в работе [14] В. Д. Белоусовым подробно исследованы квазигруппы с соответствующими тождествами Стейна и Шредера. Следующие две леммы очевидны.

Лемма 3.1. *Если в обратимой алгебре $(Q;\Sigma)$ выполняется одно из сверхтождеств (3.1) – (3.5), то алгебра $(Q;\Sigma)$– идемпотентна.*

Лемма 3.2. *Пусть $(Q;\Sigma)$ идемпотентная обратимая T-алгебра. Тогда, для любого $X \in \Sigma$, $c_X = 0$ и $\varphi_X + \psi_X = \varepsilon$, где ε– тождественная подстановка множества Q.*

Предложение 3.1. *Для обратимой T-алгебры $(Q;\Sigma)$ следующие условия эквивалентны:*

1) Обратимая алгебра $(Q;\Sigma)$ имеет вид

$$X(x,y) = \varphi_X x + \psi_X y,$$

где $\varphi_X + \psi_X\psi_Y = \varphi_X\psi_Y$, $\varphi_X\varphi_Y + \psi_X = \psi_X\varphi_Y$ для всех $X, Y \in \Sigma$;

2) В обратимой алгебре $(Q;\Sigma)$ выполняется сверхтождество Стейна (3.1).

Доказательство. 2) $\Rightarrow$ 1). Пусть в обратимой T-алгебре верно сверхтождество (3.1). Тогда, согласно лемме 3.1, $(Q;\Sigma)$ идемпотентная, а из леммы 3.2 следует, что $c_X = 0$ для всех $X \in \Sigma$. Учитывая это, переходим в сверхтождестве (3.1) к операции $+$, получим:

$$X(x,Y(y,x)) = \varphi_X x + \psi_X(\varphi_Y y + \psi_Y x) = \varphi_X x + \psi_X\varphi_Y y + \psi_X\psi_Y x,$$

$$X(Y(y,x),y) = \varphi_X(\varphi_Y y + \psi_Y x) + \psi_X y = \varphi_X\varphi_Y y + \varphi_X\psi_Y x + \psi_X y,$$

или

(3.7) $$\varphi_X x + \psi_X\varphi_Y y + \psi_X\psi_Y x = \varphi_X\varphi_Y y + \varphi_X\psi_Y x + \psi_x y.$$

Пусть $x = 0$, тогда

$$\psi_X\varphi_Y y = \varphi_X\varphi_Y y + \psi_X y, \ \varphi_X\varphi_Y + \psi_X = \psi_X\varphi_Y.$$

Положим в (3.7) $y = 0$:

$$\varphi_X x + \psi_X\psi_Y x = \varphi_X\psi_Y x, \ \varphi_X + \psi_X\psi_Y = \varphi_X\psi_Y.$$

Импликация 1) $\Rightarrow$ 2) легко проверяется. □

Предложение 3.2. *Для обратимой T-алгебры $(Q;\Sigma)$ следующие условия эквивалентны:*

1) Обратимая алгебра $(Q;\Sigma)$ имеет вид

$$X(x,y) = \varphi_X x + \psi_X y,$$

где $\varphi_X + \psi_X\psi_Y = \varphi_Y\psi_X$, $\psi_Y + \varphi_Y\varphi_X = \psi_X\varphi_Y$ для всех $X, Y \in \Sigma$;

2) В обратимой алгебре $(Q;\Sigma)$ выполняется сверхтождество Стейна (3.2).

Доказательство. Аналогично предложению 2.3. □

Предложение 3.3. *Для обратимой T-алгебры $(Q;\Sigma)$ следующие условия эквивалентны:*

1) Обратимая алгебра $(Q;\Sigma)$ имеет вид

$$X(x,y) = \varphi_X x + \psi_X y + c_X,$$

где $\varphi_X c_Y + \psi_X c_Y + c_X = 0$, $\varphi_X \psi_Y + \psi_X \varphi_Y = \varepsilon$, $\varphi_X \varphi_Y = J\psi_X \psi_Y$ $(Jx = -x)$ *для всех* $X, Y \in \Sigma$;

2) В обратимой алгебре $(Q; \Sigma)$ *выполняется сверхтождество Стейна (3.3).*

Доказательство. 2) $\Rightarrow$ 1). Пусть $(Q; \Sigma)$ обратимая Т-алгебра со сверхтождеством Стейна (3.3). Следовательно,

$$X(Y(x,y), Y(y,x)) = \varphi_X(\varphi_Y x + \psi_Y y + c_Y) + \psi_X(\varphi_Y y + \psi_Y x + c_Y) + c_X =$$

$$= \varphi_X \varphi_Y x + \varphi_X \psi_Y y + \varphi_X c_Y + \psi_X \varphi_Y y + \psi_X \psi_Y x + \psi_X c_Y + c_X = y.$$

Положим в последнем равенстве $x = y = 0$. Тогда

$$\varphi_X c_Y + \psi_X c_Y + c_X = 0.$$

Поэтому $\varphi_X \varphi_Y x + \varphi_X \psi_Y y + \psi_X \varphi_Y y + \psi_X \psi_Y x = y$. Положим в последнем равенстве $y = 0$:

$$\varphi_X \varphi_Y x + \psi_X \psi_Y x = 0, \ \varphi_X \varphi_Y = J\psi_X \psi_Y.$$

Тогда $\varphi_X \psi_Y y + \psi_X \varphi_Y y = y$, т.е. $\varphi_X \psi_Y + \psi_X \varphi_Y = \varepsilon$.

Импликация 1) $\Rightarrow$ 2). легко проверяется. □

Доказательство следующих предложений аналогичны доказательствам предыдущих.

Предложение 3.4. *Для обратимой Т-алгебры* $(Q; \Sigma)$ *следующие условия эквивалентны:*

1) Обратимая алгебра $(Q; \Sigma)$ *имеет вид*

$$X(x,y) = \varphi_X x + \psi_X y,$$

где $\psi_X \psi_Y = \varphi_X \psi_Y + \psi_X$, $\varphi_X + \psi_X \varphi_Y = \varphi_X \varphi_Y$ *для всех* $X, Y \in \Sigma$, *а* $(Q; +)$*– группа экспоненты два;*

2) В обратимой алгебре $(Q; \Sigma)$ *выполняется сверхтождество Шредера (3.4).*

Предложение 3.5. *Для обратимой Т-алгебры* $(Q; \Sigma)$ *следующие условия эквивалентны:*

1) Обратимая алгебра $(Q; \Sigma)$ *имеет вид*

$$X(x,y) = \varphi_X x + \psi_X y,$$

где $\psi_X\psi_Y = \varphi_Y\psi_X + \psi_Y$, $\varphi_Y\varphi_X = \psi_X\varphi_Y + \varphi_X$ *для всех* $X, Y \in \Sigma$, *а* $(Q;+)$– *группа экспоненты два;*

2) В обратимой алгебре $(Q;\Sigma)$ *выполняется сверхтождество Шредера (3.5).*

Предложение 3.6. *Для обратимой T-алгебры* $(Q;\Sigma)$ *следующие условия эквивалентны:*

1) Обратимая алгебра $(Q;\Sigma)$ *имеет вид*

$$X(x,y) = \varphi_X x + \psi_X y + c_X,$$

где $\varphi_X c_Y + \psi_X c_Y + c_X = 0$, $\varphi_X\psi_Y = J\psi_X\varphi_Y$ $(Jx = -x)$, $\varphi_X\varphi_Y + \psi_X\psi_Y = \varepsilon$ *для всех* $X, Y \in \Sigma$;

2) В обратимой алгебре $(Q;\Sigma)$ *выполняется сверхтождество Шредера (3.6).*

Abstract. In this paper using the second order formula, namely the $\forall\exists(\forall)-$ identities, we characterize some subclasses of the invertible algebras that are linear over an Abelian group and have restrictions on the use of the automorphisms of the corresponding group.

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PROOFS OF THE CONJECTURES BY MECKE FOR MIXED LINE-GENERATED TESSELLATIONS

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Abstract. For a compact and convex window, Mecke described a process of tessellations which arise from cell divisions in discrete time. At each time step, one of the existing cells is selected according to an equally-likely law. Independently, a line is thrown onto the window. If the line hits the selected cell the cell is divided. If the line does not hit the selected cell nothing happens in that time step. With a geometric distribution whose parameter depends on the time, Mecke transformed his construction into a continuous-time model. He put forward two conjectures in which he assumed this continuous-time model to have certain properties with respect to their iteration. These conjectures lead to a third conjecture which states the equivalence of the construction of STIT tessellations and Mecke's construction under some homogeneity conditions. In the present paper, all three conjectures are proven. A key tool to do that is a property of a continuous-time version of the *equally-likely* model classified by Cowan.

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Keywords: Random tessellation; stability by iterations; equally-likely model; Mecke's conjecture.

1. Introduction

The topic of this article are random tessellation processes in the plane. In general, random tessellations are constructed by lines or line segments that are thrown onto the plane under a certain probability law. In our context, line segments are always intersections of lines and a so called cell within a compact and convex window. Both the throwing of the lines and the selection of the cell to be divided are governed by specific probability laws. The timing of the cell division may depend on the selection rule for the cell to be divided.

In [4], Mecke developed a new process in discrete time in a convex and compact window W: At the first time step, a line is thrown onto the window according to a law Q dividing the window in two cells almost surely. At the second time step, one of the cells is selected for division according to an equally-likely law. Independently, a line is thrown onto the window. If the line intersects the selected cell, that cell is divided into another two cells. If, however, the line does not intersect the selected cell, although *no* new cell is created, another so-called quasi-cell arises. In any case, the number of quasi-cells (*real* cells plus empty quasi-cells) is always the number of time steps passed plus one.

At each time step one of the quasi-cells (which if they are empty cannot actually be divided) is selected with a probability equal to any other cell. If the line thrown independently hits a *real* cell, that cell is divided into two *real* cells. If a *real* cell is selected but the line does not hit it, the *real* cell remains; one new empty quasi-cell is added. If an empty quasi-cell is selected, there automatically arise two new empty quasi-cells.

Mecke proposed a way to transfer this process from discrete to continuous time by assigning to an arbitrary time t a geometrically-distributed random number of steps (dependent on t) in the discrete process. After formulating two conjectures, he examined the special case of a homogeneous line measure to be used for the (potential) cell division for which he stated another conjecture that his model in continuous time has the same distribution as the STIT tessellation process introduced and examined by Mecke, Nagel and Weiß in e.g. [5]—[7].

While his last conjecture, Conjecture 3 regarding the homogeneous case, was proven in [2] in rather lenghty terms, a by-product of that paper was a way to actually understand Mecke's construction as a *process* in continuous time. By this new access however, which is related to the *equally-likely* model Cowan examined in [3], the proofs of Mecke's remaining conjectures could be undertaken.

In this paper, after a short introduction into Mecke's construction (section 2), the distribution of the lifetime beyond an arbitrary point in time of a convex set within a cell of a fixed tessellation in Mecke's continuous-time model is calculated (section 3). This allows the proofs in section 4.

2. The Mecke process

Throughout this paper, we will consider, in the Euclidian plane, a compact and convex polygon $W \subset \mathbf{R}^2$ with non-empty interior. Let $[\mathcal{H}, \mathfrak{H}]$ be the measurable space of all lines in $\mathbf{R}^2$ where the σ-algebra is induced by the Borel σ-algebra on a parameter space of $\mathcal{H}$. For a set $A \subset \mathbf{R}^2$ we define

$$[A] = \{g \in \mathcal{H} : g \cap A \neq \emptyset\}.$$

Let Q be a non-zero locally finite measure on $[\mathcal{H}, \mathfrak{H}]$ which is not concentrated on one direction but which is bundle-free, i.e. there is no point $x \in \mathbf{R}^2$ such that $Q([\{x\}]) = Q(\{g \in \mathcal{H} : g \cap \{x\} \neq \emptyset\}) > 0$. Let additionally $Q([W]) > 0$ hold.

2.1. Mecke's process in discrete time. Let there be lines $\gamma_j, j = 1, 2, \ldots$, that are i.i.d. according to the law $Q([W])^{-1}Q(\cdot \cap [W])$. Further let us use, independently of γ_j, independent $\alpha_j, j = 1, 2, \ldots$ where α_j is uniformly distributed on the set $\{1, \ldots, j\}$. If a line γ_j does not contain the origin o then γ_j^+ shall be the open halfplane bounded by γ_j which contains the origin. Correspondingly, γ_j^- is the open halfplane bounded by γ_j which does not contain the origin. As the distribution of γ_j is bundle-free, we can neglect the possibility of γ_j going through the origin as the probability of this is zero.

Let be $\tilde{C}_{0,1} = W$, $\tilde{C}_{1,1} = W \cap \gamma_1^+$ and $\tilde{C}_{1,2} = W \cap \gamma_1^-$. For $n = 2, 3, \ldots$ we define

$$\tilde{C}_{n,j} = \begin{cases} \tilde{C}_{n-1,j} & \text{if } j \in \{1, \ldots, n\}, j \neq \alpha_n \\ \tilde{C}_{n-1,\alpha_n} \cap \gamma_n^- & \text{if } j = \alpha_n \\ \tilde{C}_{n-1,\alpha_n} \cap \gamma_n^+ & \text{if } j = n+1 \end{cases}$$

These entities $\tilde{C}_{n,j}$ are called **quasi-cells**. Some of these quasi-cells are empty. Those quasi-cells that are not empty will be called **cells**.

From this, we can deduce a random process: After each **decision time** n, $n = 1, 2, \ldots$, we consider the tessellation $\mathcal{T}_n$ consisting of the quasi-cells $\tilde{C}_{n,1}, \ldots, \tilde{C}_{n,n+1}$. This decision time is called the n-th decision time accordingly. If, at that decision time, the number of cells (i.e. non-empty quasi-cells) actually changes, that decision time is called a **jump time**. Obviously, the k-th jump time is that decision time at which the number of cells reaches $k + 1$. Let us denote the random closed set of the closure of the union of cell boundaries that are not part of the window's boundary at a step n for the tessellation $\mathcal{T}_n$ as

$$Y_d^M(n,W) = \overline{\bigcup_{j=1}^{n+1} \partial\tilde{C}_{n,j} \setminus \partial W}.$$

Then $(Y_d^M(n,W) : n \in \mathbf{N})$ is called the Mecke process in discrete time. Here, $\mathbf{N} = \{0,1,2,...\}$ is the set of the natural numbers.

2.2. The Mecke model in continuous time. In [4, Section 4], Mecke introduces a mixed line-generated tessellation model such that the tessellation $\mathcal{T}^t$ at the continuous time $t \in [0,\infty)$ corresponds to the tessellation $\mathcal{T}_{\nu(t)}$ at the discrete random time $\nu(t)$ where for the distribution of $\nu(t)$

$$\mathbf{P}(\nu(t) = k) = e^{-t}\left(1 - e^{-t}\right)^k, \; k = 0,1,...$$

holds. Mecke used $Q([W]) = 1$ for his considerations. For general Q, i.e. where $Q([W]) = 1$ is not necessarily true any more, the distribution is

$$\mathbf{P}(\nu(t) = k) = e^{-Q([W])t}\left(1 - e^{-Q([W])t}\right)^k, \; k = 0,1,... \tag{2.1}$$

This is the geometric distribution with parameter $e^{-Q([W])t}$; the model (which yields a random tessellation for any fixed time t but cannot yet be described as a process) thus has the characteristics Q and $tQ([W])$. (In Mecke's paper, these characteristics were Q and t due to $Q([W]) = 1$. Here, to have a connection between the characteristic and the exponential function's exponent, the characteristic is called $tQ([W])$.) A possible interpretation is that the decision times are no longer at equidistant discrete times $n = 1,2,...$ Instead, the law describes how many decisions take place until the time t. The $\nu(t)$ are assumed independent of all other random variables that are used in the construction of the Mecke process.

2.3. The sum of exponentially-distributed random variables. While there are more general results for the distribution of a sum of exponentially-distributed random variables with unequal parameters (e.g. see [1]), for the special case needed here the following calculations allow a quick understanding. If a random variable X is exponentially distributed with parameter λ, we will write $X \sim \mathcal{E}(\lambda)$.

Lemma 2.1. *Let $n \in \mathbb{N} \setminus \{0\}$ be fixed. Let further $S_n = \sum_{j=1}^{n} T_j$ be the sum of independent exponentially distributed random variables $T_1, ..., T_n$ with $T_j \sim \mathcal{E}(jR)$ for $j = 1, 2, ..., n$ and a fixed $R > 0$. Then*

$$\mathbb{P}(S_n \le t) = \int_0^t nRe^{-nxR}(e^{xR} - 1)^{n-1}dx = e^{-ntR}(e^{tR} - 1)^n = (1 - e^{-tR})^n.$$

Proof. The proof is by induction. For $n = 1$, obviously

$$\mathbb{P}(S_1 \le t) = \mathbb{P}(T_1 \le t) = \int_0^t Re^{-xR}dx = \int_0^t 1R \cdot e^{-1 \cdot xR} \cdot (e^{xR} - 1)^0 dx = 1 - e^{-tR}$$

holds which is true according to the condition $T_1 \sim \mathcal{E}(R)$.
Let the lemma be true for n. Then, because of $S_{n+1} = S_n + T_{n+1}$ with $T_{n+1} \sim \mathcal{E}((n+1)R)$ and the independence of S_n and T_{n+1}, for the density of S_{n+1}

$$\begin{aligned}
f_{S_{n+1}}(x) = \;& f_{S_n + T_{n+1}}(x) \\
= \;& \textstyle\int_0^x f_{S_n}(u) f_{T_{n+1}}(x - u)du \\
= \;& \textstyle\int_0^x nRe^{-nuR}(e^{uR} - 1)^{n-1}(n+1)Re^{-(n+1)(x-u)R}du \\
= \;& (n+1)Re^{-(n+1)xR} \textstyle\int_0^x nRe^{uR}(e^{uR} - 1)^{n-1}du \\
= \;& (n+1)Re^{-(n+1)xR}[(e^{uR} - 1)^n]_{u=0}^{u=x} \\
= \;& (n+1)Re^{-(n+1)xR}(e^{xR} - 1)^n
\end{aligned}$$

holds. Integration yields the second equation, straightforward calculation the third equation in the lemma. □

Lemma 2.2. *Let $N_t = \max\{n : \sum_{j=1}^{n} T_j \le t\}$ denote the number of $T_j \sim \mathcal{E}(jR)$, $j = 1, 2, ...$, which have consecutively expired until the time t. Then for $k = 0, 1, 2, ...$*

$$\mathbb{P}(N_t = k) = e^{-tR}\left(1 - e^{-tR}\right)^k. \tag{2.2}$$

Proof. From the distribution of the $S_k, k = 1, 2, \ldots$, one gets

$$\begin{aligned} & \mathbf{P}(N_t = k) \\ = & \ \mathbf{P}(S_k \leq t < S_{k+1}) \\ = & \ \mathbf{P}(S_k \leq t) - \mathbf{P}(S_{k+1} \leq t) \\ = & \ \left(1 - e^{-tR}\right)^k - \left(1 - e^{-tR}\right)^{k+1} \\ = & \ \left(1 - e^{-tR}\right)^k \left(1 - \left(1 - e^{-tR}\right)\right) \\ = & \ e^{-tR}\left(1 - e^{-tR}\right)^k \end{aligned}$$

For $N_t = 0$, the result follows from Lemma 2.1 immediately. □

2.4. The Mecke process in continuous time. Comparing the equations (2.1) and (2.2), we see that with $N_t = \nu(t)$ and $R = Q([W])$ both yield the same result. Therefore, the T_j from Lemma 2.2 with $T_j \sim \mathcal{E}(jQ([W]))$ can be interpreted as the (continuous-time) waiting times for the quasi-state of the tessellation to change from a quasi-state with j quasi-cells to a quasi-state with $j+1$ quasi-cells.

Definition 2.1. *Let us have a window $W \subset \mathbb{R}^2$. Let $(Y_d^M(n, W) : n \in \mathbf{N})$ be the Mecke process in discrete time as described in section 2.1. Let $(N_t : t \geq 0)$ be the process of the number of expired random variables $T_j \sim \mathcal{E}(jQ([W]))$ as in Lemma 2.2. Then for every $t \in [0, \infty)$ we define*

$$Y_c^M(t, W) = Y_d^M(N_t, W)$$

and the Mecke process in continuous time as $(Y_c^M(t, W) : t \geq 0)$.

3. The waiting time until a convex set is hit in the Mecke process in continuous time

We now give a formula for the waiting time of a convex set within a cell in the Mecke process in continuous time to be hit by a line.

Let us have a fixed time s. We work on the condition that, at this time, the tessellation has n quasi-cells, thus $\mathcal{T}^s = \mathcal{T}_{n-1}$. For the waiting time T_n^M in this state, $T_n^M \sim$

$\mathcal{E}(nQ([W]))$ holds. In this fixed tessellation with n quasi-cells, let us have κ ('real') cells.

Let these cells be called $C_1, ..., C_\kappa$. Let $S_j \subset C_j \subset W, j = 1, ..., \kappa$, be a convex set within a cell C_j, created Let these cells be called $C_1, ..., C_\kappa$. For each $j = 1, ..., \kappa$, we define

$$S_j = C_j \cap K$$

with K being a fixed convex set. Thus, S_j is a convex set within the cell C_j, created deterministically from C_j. It is possible for some (or all) of the S_j to be equal to C_j. We will only examine non-empty S_j.

Let us denote by X_{S_j} the waiting time for such a set S_j to be hit by a line for the first time after the time s. It may be possible that the cell C_j is hit by a line but the set S_j is not. In this case, the waiting time for S_j to be hit shall not begin anew but rather be extended until it is actually hit. The waiting time until the set S_j is hit is the waiting time T_n^M if and only if the cell C_j that contains S_j is selected for division in this step (i.e. $\alpha_n = j$ in Mecke's construction) *and* the set $S_j \subset C_j$ is hit by the line. The probability for this to happen is

$$\mathbf{P}(j = \alpha_n, S_j \cap \gamma_n \neq \emptyset | S_j \subset C_j \in \mathcal{T}_{n-1}) = \frac{1}{n} \frac{Q([S_j])}{Q([W])}.$$

If the set is not hit (which happens with probability $1 - \frac{1}{n}\frac{Q([S_j])}{Q([W])}$) the waiting time for the set to be hit is the sum of the waiting times $T_n^M + T_{n+1}^M$ if and only if the waiting times T_n^M and T_{n+1}^M have passed and the set is hit in the $(n+1)$-th division step the probability of which is

$$\mathbf{P}(j = \alpha_{n+1}, S_j \cap \gamma_{n+1} \neq \emptyset | S_j \subset C_j \in \mathcal{T}_n) = \frac{1}{n+1} \frac{Q([S_j])}{Q([W])}$$

and so on. The waiting times are independent of each other.

In general, one gets

$$\mathbf{P}\left(X_{S_j} \leq t | S_j \subset C_j \in \mathcal{T}_{n-1} = \mathcal{T}^s\right)$$

$$= \sum_{k=n}^{\infty} \mathbf{P}\left(\sum_{i=n}^{k} T_i^M \leq t\right) \frac{1}{k} \frac{Q([S_j])}{Q([W])} \prod_{i=n}^{k-1} \left(1 - \frac{1}{i} \frac{Q([S_j])}{Q([W])}\right). \tag{3.1}$$

Let us first calculate what one gets for $\mathbf{P}\left(\sum_{i=n}^{k} T_i^M \leq t\right)$ or the density of this respectively:

Lemma 3.1. *The following equation holds:*

$$\mathbf{P}\left(\sum_{i=n}^{k} T_i^M \leq t\right) =$$

$$\frac{1}{(k-n)!}\frac{k!}{(n-1)!}Q([W])\int_0^t \left(e^{Q([W])x}-1\right)^{k-n} e^{-kQ([W])x}dx. \tag{3.2}$$

Proof. We use the abbreviation $S_n^k = \sum_{i=n}^k T_i^M$. It is sufficient to show that for the density $f_{S_n^k}(x)$ of the probability distribution

$$f_{S_n^k}(x) = \frac{1}{(k-n)!}\frac{k!}{(n-1)!}Q([W])\left(e^{Q([W])x}-1\right)^{k-n} e^{-kQ([W])x} \tag{3.3}$$

holds.

The proof is by induction over k.

For the base case $k = n$, because of $T_n^M \sim \mathcal{E}(nQ([W]))$ the equation $f_{S_n^n}(x) = nQ([W])e^{-nQ([W])x}$ should hold. Indeed,

$$f_{S_n^n}(x) = \frac{1}{(n-n)!}\frac{n!}{(n-1)!}Q([W])\left(e^{Q([W])x}-1\right)^{n-n} e^{-nQ([W])x} = nQ([W])e^{-nQ([W])x}.$$

Let now equation (3.3) be true for any k. Then for $k+1$, due to the convolution formula (the waiting times are independent of each other)

$$\begin{aligned}
&f_{S_n^{k+1}}(x)\\
=\;& \int_0^x \tfrac{1}{(k-n)!}\tfrac{k!}{(n-1)!}\\
&\times Q([W])\left(e^{Q([W])u}-1\right)^{k-n} e^{-kQ([W])u}(k+1)Q([W])e^{-(k+1)Q([W])(x-u)}du\\
=\;& \tfrac{1}{(k-n)!}\tfrac{(k+1)!}{(n-1)!}(Q([W]))^2\int_0^x \left(e^{Q([W])u}-1\right)^{k-n}\\
&\times e^{-kQ([W])u}e^{-(k+1)Q([W])x+kQ([W])u+Q([W])u}du\\
=\;& \tfrac{1}{(k-n)!}\tfrac{(k+1)!}{(n-1)!}e^{-(k+1)Q([W])x}(Q([W]))^2\int_0^x \left(e^{Q([W])u}-1\right)^{k-n} e^{Q([W])u}du\\
=\;& \tfrac{1}{(k+1-n)!}\tfrac{(k+1)!}{(n-1)!}e^{-(k+1)Q([W])x}\tfrac{1}{Q([W])}(Q([W]))^2\\
&\times\int_0^x (k+1-n)Q([W])\left(e^{Q([W])u}-1\right)^{k-n} e^{Q([W])u}du\\
=\;& \tfrac{1}{(k+1-n)!}\tfrac{(k+1)!}{(n-1)!}e^{-(k+1)Q([W])x}Q([W])\left[\left(e^{Q([W])u}-1\right)^{k+1-n}\right]_{u=0}^{u=x}\\
=\;& \tfrac{1}{(k+1-n)!}\tfrac{(k+1)!}{(n-1)!}e^{-(k+1)Q([W])x}Q([W])\left(e^{Q([W])x}-1\right)^{k+1-n}
\end{aligned}$$

holds what is exactly what equation (3.3) yields for $k+1$. □

Theorem 3.1. *Let a time s be fixed. At this time s, let us have a tessellation $\mathcal{T}^s$ with an arbitrary number of cells. Let a convex set S_j be contained in the cell C_j ($S_j \subset C_j \subset W$). For the waiting time X_{S_j} for this convex set S_j to be hit from the time s on,*

$$\mathbf{P}\left(X_{S_j} \leq t | S_j \subset C_j \in \mathcal{T}^s\right) = 1 - e^{-tQ([S_j])} \tag{3.4}$$

holds.

Proof. Let us first keep the number n of quasi-cells fixed. For equation (3.1) we get (at some point we will abbreviate $A = 1 - e^{-Q([W])x}$)

$$\begin{aligned}
&\mathbf{P}\left(X_{S_j} \leq t | S_j \subset C_j \in \mathcal{T}_{n-1} = \mathcal{T}^s\right)\\
= \;& \textstyle\sum_{k=n}^{\infty} \mathbf{P}\left(\sum_{i=n}^{k} T_i^M \leq t\right) \frac{1}{k}\frac{Q([S_j])}{Q([W])} \prod_{i=n}^{k-1}\left(1 - \frac{1}{i}\frac{Q([S_j])}{Q([W])}\right)\\
= \;& \textstyle\sum_{k=n}^{\infty} \frac{1}{(k-n)!}\frac{k!}{(n-1)!} Q([W]) \int_0^t \left(e^{Q([W])x} - 1\right)^{k-n} e^{-kQ([W])x} dx\\
\times \;& \textstyle\frac{1}{k}\frac{Q([S_j])}{Q([W])}\left[\prod_{i=n}^{k-1} \frac{1}{i}\left(i - \frac{Q([S_j])}{Q([W])}\right)\right]\\
= \;& \textstyle\sum_{k=n}^{\infty} Q([S_j]) \frac{1}{(k-n)!}\frac{k!}{(n-1)!} \int_0^t \left(e^{Q([W])x} - 1\right)^{k-n} e^{-kQ([W])x} dx\\
\times \;& \textstyle\frac{(n-1)!}{k!}\left[\prod_{i=n}^{k-1}\left(i - \frac{Q([S_j])}{Q([W])}\right)\right]\\
= \;& \textstyle Q([S_j]) \int_0^t \sum_{k=0}^{\infty} \frac{1}{k!}\left(e^{Q([W])x} - 1\right)^k e^{-(n+k)Q([W])x} \frac{\Gamma\left(n+k-\frac{Q([S_j])}{Q([W])}\right)}{\Gamma\left(n-\frac{Q([S_j])}{Q([W])}\right)} dx\\
= \;& \textstyle\frac{Q([S_j])}{\Gamma\left(n-\frac{Q([S_j])}{Q([W])}\right)} \int_0^t e^{-nQ([W])x} \sum_{k=0}^{\infty} \frac{1}{k!}\left(1 - e^{-Q([W])x}\right)^k \Gamma\left(n + k - \frac{Q([S_j])}{Q([W])}\right) dx\\
= \;& \textstyle\frac{Q([S_j])}{\Gamma\left(n-\frac{Q([S_j])}{Q([W])}\right)} \int_0^t e^{-nQ([W])x} \sum_{k=0}^{\infty} \frac{1}{k!} A^k \int_0^{\infty} u^{n+k-\frac{Q([S_j])}{Q([W])}-1} e^{-u} du dx\\
= \;& \frac{Q([S_j])}{\Gamma\left(n - \frac{Q([S_j])}{Q([W])}\right)} \int_0^t e^{-nQ([W])x} \int_0^{\infty} e^{-u(1-A)} u^{n-\frac{Q([S_j])}{Q([W])}-1} du dx\\
\overset{(a)}{=} \;& \frac{Q([S_j])}{\Gamma\left(n - \frac{Q([S_j])}{Q([W])}\right)} \int_0^t e^{-nQ([W])x} \int_0^{\infty} e^{-v}\left(\frac{v}{1-A}\right)^{n-\frac{Q([S_j])}{Q([W])}-1} \frac{1}{1-A} dv dx
\end{aligned}$$

$$= \frac{Q([S_j])}{\Gamma\left(n - \frac{Q([S_j])}{Q([W])}\right)} \int_0^t e^{-nQ([W])x}(1-A)^{\frac{Q([S_j])}{Q([W])}-n} \int_0^\infty e^{-v} v^{n-\frac{Q([S_j])}{Q([W])}-1} dv dx$$

$$= \frac{Q([S_j])}{\Gamma\left(n - \frac{Q([S_j])}{Q([W])}\right)} \int_0^t e^{-nQ([W])x}(1-A)^{\frac{Q([S_j])}{Q([W])}-n} \Gamma\left(n - \frac{Q([S_j])}{Q([W])}\right) dx$$

$$= Q([S_j]) \int_0^t e^{-xQ([S_j])} dx = 1 - e^{-tQ([S_j])}.$$

Equation (a) follows from the substitution $v = u(1-A)$.

Let us now have a time s. The probability that there are exactly n quasi-cells in the tessellation $\mathcal{T}^s$ (or that $\mathcal{T}^s = \mathcal{T}_{n-1}$) is just

$$\mathbf{P}(\mathcal{T}^s = \mathcal{T}_{n-1}) = e^{-Q([W])t}\left(1 - e^{-Q([W])t}\right)^{n-1}$$

Thus, we get

$$\begin{aligned}
\mathbf{P}\left(X_{S_j} \le t | S_j \subset C_j \in \mathcal{T}^s\right) = & \textstyle\sum_{n=1}^\infty \mathbf{P}(\mathcal{T}^s = \mathcal{T}_{n-1})\left(1 - e^{-tQ([S_j])}\right) \\
= & \textstyle\sum_{n=1}^\infty e^{-Q([W])t}\left(1 - e^{-Q([W])t}\right)^{n-1}\left(1 - e^{-tQ([S_j])}\right) \\
= & \left(1 - e^{-tQ([S_j])}\right)\textstyle\sum_{n=1}^\infty e^{-Q([W])t}\left(1 - e^{-Q([W])t}\right)^{n-1} \\
= & 1 - e^{-tQ([S_j])}.
\end{aligned}$$

Thus the theorem is proven. □

It is worth to mention that (as shown by the last equation) the result does *not* depend on the number of quasi-cells n at the time s. For translation-invariant Q, this result is the same result one has for the STIT process. Obviously, the lifetime of a cell C_j (which is a convex set contained within a fixed cell, namely C_j) can be described in this manner as well.

4. Proofs of Mecke's Conjectures

4.1. Conjecture 1. Lemma 2.2 makes clear the relation between those properties Mecke calls 'characteristics' and the waiting time in a state with n quasi-cells. Let us have a tessellation in a window $\hat{W}$ with characteristics $\hat{Q}$ and $\hat{t}$; then we get

$$\mathbf{P}(\hat{N}_t = k) = e^{-\hat{t}}\left(1 - e^{-\hat{t}}\right)^k.$$

Theorem 4.1. *(Mecke's Conjecture 1) Let $\mathcal{T}_W$ be a mixed line-generated tessellation in W with characteristics Q and $tQ([W])$, and let $\hat{W}$ be a window with $\hat{W} \subset W$ and*

$Q([\hat{W}]) > 0$. *Then the cutout of* $\mathcal{T}_W$ *in* $\hat{W}$ *can be interpreted as a mixed line-generated tessellation in* $\hat{W}$ *with characteristics*

$$\hat{Q} = \frac{1}{Q([\hat{W}])} Q(\cdot \cap [\hat{W}]) \text{ and } \hat{t} = tQ([\hat{W}]).$$

Proof. We will first examine the tessellation $\mathcal{T}_W$ in W with characteristics Q and $tQ([W])$. For the probability that until a time t the convex subset $\hat{W} \subset W$ was hit, according to equation (3.4)

$$\mathbf{P}(X_{\hat{W}} \leq t) = 1 - e^{-Q([\hat{W}])t}.$$

If we condition on $\hat{W}$ being hit the hitting line has distribution $\frac{1}{Q([\hat{W}])} Q(\cdot \cap [\hat{W}]) = \hat{Q}$. Let us now examine the tessellation $\mathcal{T}_{\hat{W}}$ in $\hat{W}$ with characteristics $\hat{Q} = \frac{1}{Q([\hat{W}])} Q(\cdot \cap [\hat{W}])$ and $\hat{t} = tQ([\hat{W}])$. The distribution of the number of decisions $\hat{\nu}(t)$ until time t is

$$\mathbf{P}(\hat{\nu}(t) = k) = e^{-\hat{t}} \left(1 - e^{-\hat{t}}\right)^k = e^{-tQ([\hat{W}])} \left(1 - e^{-tQ([\hat{W}])}\right)^k.$$

From this, we can deduce the lifetime of the first cell $\hat{W}$ to be

$$\mathbf{P}(X_{\hat{W}} \leq t) = 1 - e^{-tQ([\hat{W}])}.$$

Thus, the distribution of the lifetimes of $\hat{W}$ is the same in $\mathcal{T}_W$ and $\mathcal{T}_{\hat{W}}$; additionally, the distribution of the segment dividing $\hat{W}$ is identical as well.

Let us now have $\mathcal{T}_W \cap \hat{W} = \mathcal{T}_{\hat{W}}$ at an arbitrary time. Then, in $\mathcal{T}_W$ there exist the cells $Cells(\mathcal{T}_W) = \{C_1^W, ..., C_n^W\}$ and accordingly in $\mathcal{T}_{\hat{W}}$ the cells $Cells(\mathcal{T}_{\hat{W}}) = \{C_1^W \cap \hat{W}, ..., C_n^W \cap \hat{W}\} \setminus \{\emptyset\}$. Note that some of the intersections $C_j^W \cap \hat{W}$ can be empty; therefore the empty set is taken out of the set in order to have only 'real' cells with non-empty interior in $Cells(\mathcal{T}_{\hat{W}})$.

We now examine a cell $C \in Cells(\mathcal{T}_W)$ with $C \cap \hat{W} \neq \emptyset$. This cell has, as calculated above, the lifetime $X_C \sim \mathcal{E}(Q([C]))$. If we take a look at this cell's intersection with the subwindow $\hat{W}$ we have a waiting time $X_{C \cap \hat{W}} \sim \mathcal{E}(Q([C \cap \hat{W}]))$ for this convex set to be hit. For the distribution of the dividing line we have, due to the conditioning on the division of the set, $\frac{1}{Q([\hat{W}])} Q([C \cap \hat{W}]) \frac{1}{Q([C \cap \hat{W}])} Q([\cdot \cap C \cap \hat{W}]) = \hat{Q}(\cdot \cap [C])$.

In the tessellation $\mathcal{T}_{\hat{W}}$, we have a cell $\hat{C} = C \cap \hat{W}$ with a lifetime $X_{\hat{C}} \sim \mathcal{E}(Q([\hat{C}])) = \mathcal{E}(Q([C \cap \hat{W}]))$. For the distribution of the line dividing $\hat{C}$ we have $\hat{Q}(\cdot \cap [\hat{C}])$.

So, the waiting time for the set $C \cap \hat{W}$ in $\mathcal{T}_W$ to be hit and the lifetime of the cell $C \cap \hat{W}$ in $\mathcal{T}_{\hat{W}}$ respectively are identically distributed. Because of $\hat{Q}(\cdot \cap [C]) =$

$\hat{Q}(\cdot \cap [C \cap \hat{W}]) = \hat{Q}(\cdot \cap [\hat{C}])$ the distributions of the dividing lines are identical as well. Under the condition of the equality $\mathcal{T}_W \cap \hat{W} = \mathcal{T}_{\hat{W}}$ the distributions of the time of the next segment falling in $\hat{W}$ are identical in both considered windows as are the distributions of that next segment. As we always start in the same configuration of an empty subset $\hat{W} \subset W$ and window $\hat{W}$ respectively, the theorem is proven. □

4.2. Conjecture 2. Let us first define the iteration and its symbol ⊞.

Definition 4.1. ([4, Subsection 4.3, Remark]) *Let $\mathcal{T}^t$ be a mixed line-generated tessellation with distribution law P^t and $\mathcal{T}^s$ such a tessellation with law P^s. Let further $\mathcal{Y}_1, \mathcal{Y}_2, \ldots$ be a sequence of i.i.d. copies of $\mathcal{T}^s$ which are independent of $\mathcal{T}^t$. Let $\{Z_1, \ldots, Z_\kappa\}$ be the set of cells of $\mathcal{T}^t$ and $\mathfrak{Y}_n$ be the set of cells of $\mathcal{Y}_n$ for $n = 1, 2, \ldots$. Then the set of cells*

$$\bigcup_{n=1}^{\kappa} (\mathfrak{Y}_n \cap Z_n)$$

is a new tessellation and its distribution law is denoted by $P^t \boxplus P^s$.

With this definition and Theorem 4.1, Mecke's Conjecture 2 can be proven in quite a straightforward manner:

Theorem 4.2. *(Mecke's Conjecture 2) The class of all mixed line-generated tessellations (related to Q) as a whole is stable under iteration in the following sense: Every operation of iteration maps the mentioned class into itself, i.e. an iterated mixed line-generated tessellation is again a mixed line-generated tessellation. If the mixed line-generated tessellation $\mathcal{T}^t$ is iterated according to the law P^s of $\mathcal{T}^s$, then the law $P^t \boxplus P^s$ of the outcome fulfils*

$$P^t \boxplus P^s = P^{t+s}.$$

Proof. Let $\mathcal{T}^t$ be a tessellation with the cells $Cells(\mathcal{T}^t) = \{Z_1, \ldots, Z_\kappa\}$. Then each of those cells Z_j has a lifetime $X_{Z_j} \sim \mathcal{E}(Q([Z_j]))$ which is (under the condition of the existence of these cells) independent of the other lifetimes which because of the memorylessness of the exponential distribution does non depend on the time the cell was created before the time t. After the lifetime has expired (provided it is smaller than s), a segment of a line with distribution $\frac{1}{Q([Z_j])} Q(\cdot \cap [Z_j])$ falls into the cell. Afterwards, the process goes on with its cells and their exponentially-distributed

lifetimes until time s. Thus, one gets the resulting tessellation $\mathcal{T}^{t+s}$.
According to Theorem 4.1, one can interpret the cutout $\mathcal{T}_W \cap Z_j$ of $\mathcal{T}_W$ with characteristics Q and $sQ([W])$ as a process $\mathcal{T}_{Z_j}$ with characteristics $\hat{Q} = \frac{1}{Q([Z_j])}Q(\cdot \cap [Z_j])$ and $\hat{s} = sQ([Z_j])$. If one considers a cell Z_j now, its lifetime is $\mathcal{E}(Q([Z_j]))$-distributed as verified in the proof of Theorem 4.1; after this lifetime's expiry, a segment falls with the corresponding line having a distribution $\hat{Q} = \frac{1}{Q([Z_j])}Q(\cdot \cap [Z_j])$.
This cutout process runs independently in all cells $Z_j, j = 1, ..., \kappa$, with the same lifetime and segment distribution as in the process $\mathcal{T}^{t+s}$. Thus, because the processes are identically distributed,

$$P^t \boxplus P^s = P^{t+s}$$

holds as claimed in Mecke's Conjecture 2. □

5. Conclusion

From Theorem 4.1 and Theorem 4.2 we can deduce

Theorem 5.1. *(Mecke's Conjecture 3) Let Λ be a non-zero locally-finite translation-invariant measure not concentrated on one direction and W a window with $0 < \Lambda([W]) < \infty$. Then, the STIT construction and the Mecke construction with $Q = \Lambda$ are equivalent in the sense that they yield identically-distributed tessellations within the window W.*

Proof. The equivalence of the STIT and the Mecke construction follows from Theorem 4.2 for the given translation-invariant measure Λ as the property $P^t \boxplus P^s = P^{t+s}$ is the defining property of the STIT tessellation, namely to be stable with respect to iteration, and from the fact that the STIT tessellation is unique in this property, as per [6, Corollary 2]. □

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A WEIGHTED TRANSPLANTATION THEOREM FOR LAGUERRE FUNCTION EXPANSIONS

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Abstract. We establish a generalized weighted transplantation theorem for Laguerre function expansions, which extends the corresponding result by G. Garrigós et al. "A sharp weighted transplantation theorem for Laguerre function expansions" (J. Funct Anal. 244 (2007), pp. 247-276).

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1. Introduction

We consider the system of Laguerre functions defined by

$$\mathcal{L}_k^\alpha(y) = c_{k,\alpha} y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} L_k^{(\alpha)}(y), y \in \mathbb{R}_+ = (0,\infty),\ k \in \mathbb{N},$$

where $L_k^{(\alpha)}(y) = (y^{\alpha+k}e^{-y})^{(k)}/(k!y^\alpha e^{-y})$ is the usual Laguerre polynomial of degree k. For $\alpha > -1$ this system forms an orthonormal basis in $L^2(\mathbb{R}_+)$ when we choose the normalizing constants

$$c_{k,\alpha} = \sqrt{\Gamma(k+1)/\Gamma(\alpha+k+1)},\ k \in \mathbb{N}.$$

This produces a formal expansion $f = \sum_{k=0}^\infty < f, \mathcal{L}_k^\alpha > \mathcal{L}_k^\alpha$, which is convergent in norm at least for $f \in L^2(\mathbb{R}_+)$.

The main object in the theory of Laguerre function expansions is the set of transplantation operators, defined for $\alpha, \beta > -1$ and $f \in L^2(\mathbb{R}_+)$ by

$$T_\beta^\alpha f = \sum_{k=0}^\infty < f, \mathcal{L}_k^\alpha > \mathcal{L}_k^\beta.$$

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The L^p boundedness of such operators was first established by Kanjin [7]. Recently, G. Garrigós et al. [5] extended the Kanjin's result to the power weighted spaces (see also [16]).

The purpose of this paper is to establish a generalized weighted transplantation theorem for Laguerre function expansions, which extends the corresponding result by G. Garrigós et al. [5]. The main result of the paper is the following theorem.

Theorem 1.1. *Let $-1 < \alpha < \beta$ and $1 < p < \infty$. Then the operators T_α^β and T_β^α admit bounded extensions to the weighted space $L^p(\omega)$ whenever $\omega(x) = (1+x)^{p\gamma}x^{p\delta}$ with $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$ and $\gamma \in \mathbf{R}$.*

We remark that in the special case $\gamma = 0$, Theorem 1.1 has been proved by G. Garrigós et al. (see [5], Theorem 1.4). So, our result extends essentially the main result of [5]. Also, the proof of Theorem 1.1 is curried out by using arguments similar to one used in [5].

To prove Theorem 1.1, we need to establish new weighted multiplier theorems for Hermite function expansions in $\mathbf{R}^d$ and Laguerre function expansions in $\mathbf{R}_+$, respectively. Recall that the Hermite functions in $\mathbf{R}^d$ are defined by

$$\eta_{\mathbf{k}}(x) = d_{\mathbf{k},d}e^{-|x|^2/2}\prod_{i=1}^{d} H_{k_i}(x_i), \ \mathbf{k} = (k_1, \cdots, k_n), k_i \in \mathbf{N},$$

where $H_k(t) = (-1)^k e^{t^2} D^{(k)}(e^{-t^2})$ is the usual Hermite polynomial in $\mathbf{R}$ and $\mathbf{N} = \{0, 1, 2, \cdots\}$. Normalizing by $d_{\mathbf{k},d} = \prod_{i=1}^{d}(2^{k_i}k_i!\sqrt{\pi})^{-1/2}$, the system $\{\eta_{\mathbf{k}}\}_{\mathbf{k}}$ becomes an orthonormal basis in $L^2(\mathbf{R}^d)$ and a complete system of eigenvectors for the Hermite operator $-\Delta + |x|^2$.

Theorem 1.2. *Let $1 < p < \infty$ and $m \in l^\infty(\mathbf{N}^d)$ be such that*

$$|\Delta^\alpha m(\mathbf{k})| \le C_\alpha(1 + |\mathbf{k}|)^{-|\alpha|}, \ \mathbf{k} \in \mathbf{N}^d, \ \forall\, \alpha \in \mathbf{N}^d, \tag{1.1}$$

where Δ^α is a difference operator. Consider the operator $T_m f = \sum_{\mathbf{k}} m(\mathbf{k}) < f, \eta_{\mathbf{k}} > \eta_{\mathbf{k}}$, defined at least for $f \in L^2(\mathbf{R}^d)$. Then T_m admits a bounded extension to the weighted space $L^p(\omega)$ whenever $\omega(x) = (1+|x|)^\gamma \mu(x)$ with $\mu \in A_p(\mathbf{R}^d)$ and $\gamma \in \mathbf{R}$, where $\mu \in A_p(\mathbf{R}^d)$ stands for the Muckenhoupt class.

We remark that in the special case $\gamma = 0$, Theorem 1.2 has been proved in [5] under weaker conditions (see [5], Theorem 1.6). But, our Theorem 1.2 cannot be deduced from the conditions imposed in [5].

Theorem 1.3. *Let $\alpha > -1$, $1 < p < \infty$ and $m \in C^\infty[0,\infty)$ be such that*

$$|D^l m(\xi)| \leq C_l(1+\xi)^{-l}, \quad \xi \geq 0, l \in \mathbf{N}. \tag{1.2}$$

Consider the operator $T_m f = \sum_{k\geq 0} m(k) < f, L_k^\alpha > L_k^\alpha$, defined at least for $f \in L^2(\mathbf{R}_+)$. Then T_m admits a bounded extension to the weighted space $L^p(\omega)$ whenever $\omega(x) = (1+x)^{p\gamma}x^{p\delta}$ with $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$ and $\gamma \in \mathbf{R}$.

We remark that in the special case $\gamma = 0$, Theorem 1.3 has been proved in [5], Theorem 1.8.

The paper is organized as follows. In Section 2 we prove Theorem 1.2 by using a new class of weights $A_p(\varphi)$. In Section 3 we establish Theorem 1.3. The main result of the paper - Theorem 1.1 is proved in Section 4. Finally, Section 5 is devoted to the applications of Theorems 1.1-1.3 to the boundedness property of the Littlewood-Paley g-functions associated with the Laguerre expansions.

2. Multipliers for Hermite expansions

In this section we prove Theorem 1.2. First we introduce some notation and properties of the new weight function class $A_p(\varphi)$.

Throughout the paper, $Q(x,t)$ denotes a cube centered at x and of the side length t. Given a cube $Q = Q(x,t)$ and a number $\lambda > 0$, we will write λQ for the λ-dilate cube, which is the cube with the same center x and with side length λt. Given a Lebesgue measurable set E and a weight ω, $|E|$ will denote the Lebesgue measure of E and $\omega(E) = \int_E \omega dx$. The symbol $\|f\|_{L^p(\omega)}$ denotes $(\int_{\mathbf{R}^d} |f(y)|^p\omega(y)dy)^{1/p}$ for $0 < p < \infty$, and $\|f\|_{L^{1,\infty}(\omega)}$ denotes $\sup_{\lambda>0} \lambda^{-1}\omega(\{x \in \mathbf{R}^d : |f(x)| > \lambda\})$. The letter C denotes constants that are independent of the main parameters involved, but whose value may vary from line to line. For a measurable set E, by χ_E we denote the characteristic function of E. By $A \sim B$ we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

In this section, we let $\varphi(t) = (1+t)^{\beta_0}$ for $\beta_0 > 0$ and $t \geq 0$.

A weight always means a positive function which is locally integrable. We say that a weight ω belongs to the class $A_p(\varphi)$ for $1 < p < \infty$, if there is a constant C such that for all cubes $Q = Q(x,r)$ with center x and side length r

$$\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega(y)\,dy\right)\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \leq C.$$

Also, we say that a nonnegative function ω belongs to the class $A_1(\varphi)$ (or satisfies the $A_1(\varphi)$ condition), if there exists a constant C such that for all cubes Q

$$M_\varphi(\omega)(x) \leq C\omega(x), \ \ a.e.\ x \in \mathbf{R}^d.$$

where

$$M_\varphi f(x) = \sup_{x\in Q} \frac{1}{\varphi(|Q|)|Q|}\int_Q |f(y)|\,dy.$$

Since $\varphi(|Q|) \geq 1$, we have $A_p(\mathbf{R}^d) \subset A_p(\varphi)$ for $1 \leq p < \infty$, where $A_p(\mathbf{R}^d)$ denotes the class of classical Muckenhoupt weights (see [4]). It is well known that if $\omega \in A_\infty(\mathbf{R}^d) = \bigcup_{p\geq 1} A_p(\mathbf{R}^d)$, then $\omega(x)dx$ is a doubling measure, that is, there exist a constant $C > 0$ such that for any cube Q

$$\omega(2Q) \leq C\omega(Q).$$

Now we list some properties of weights $\omega \in A_\infty(\varphi) = \bigcup_{p\geq 1} A_p(\varphi)$, similar to that of classical Muckenhoupt weights.

Lemma 2.1. *For any cube $Q \subset \mathbf{R}^d$ the following assertions hold:*

(i) *If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}(\varphi) \subset A_{p_2}(\varphi)$.*

(ii) *$\omega \in A_p(\varphi)$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}(\varphi)$, where $1/p + 1/p' = 1$.*

(iii) *If $\omega_1,\ \omega_2 \in A_1(\varphi)$, $p > 1$, then $\omega_1\omega_2^{1-p} \in A_p(\varphi)$.*

(iv) *If $\omega \in A_p$ for $1 \leq p < \infty$, then*

$$\frac{1}{\varphi(|Q|)|Q|}\int_Q |f(y)|dy \leq C\left(\frac{1}{\omega(Q)}\int_Q |f|^p\omega(y)dy\right)^{1/p}.$$

In particular, if $f = \chi_E$ for any measurable set $E \subset Q$, then

$$\frac{|E|}{\varphi(|Q|)|Q|} \leq C\left(\frac{\omega(E)}{\omega(Q)}\right)^{1/p}.$$

Remark 2.1. *It follows from the definition of $A_p(\varphi)$ and Lemma 2.1 (iii), that if $\omega \in A_p(\varphi)$, then $\omega(x)dx$ generally is not a doubling measure. Indeed, let $0 \leq \gamma \leq \beta_0/d$,* *it is easy to check that* $\omega(x) = (1+|x|)^{-(d+\gamma)} \notin A_\infty(\mathbf{R}^d)$ *and* $\omega(x)dx$ *is not a doubling measure, but* $\omega(x) = (1+|x|)^{-(d+\gamma)} \in A_1(\varphi)$.

It is easy to see that the set of all Schwartz functions, denoted by $\mathcal{S}$, is dense in $L^p(\omega)$ for $\omega \in A_\infty(\varphi)$ and $1 \le p < \infty$. Hence, we always can assume that $f \in \mathcal{S}$ if $f \in L^p(\omega)$ for $1 \le p < \infty$.

Lemma 2.2. *Let* $1 \le p_1 < \infty$ *and* $\omega \in A_{p_1}(\varphi)$. *Then for* $p_1 < p < \infty$ *the inequality holds:*

$$\int_{\mathbf{R}^d} |M_\varphi f(x)|^p \omega(x)dx \le C_p \int_{\mathbf{R}^d} |f(x)|^p \omega(x)dx.$$

Further, let $1 \le p < \infty$, *then* $\omega \in A_p(\varphi)$ *if and only if*

$$\omega(\{x \in \mathbf{R}^d :\ M_\varphi f(x) > \lambda\}) \le \frac{C_p}{\lambda^p} \int_{\mathbf{R}^d} |f(x)|^p \omega(x)dx, \quad \lambda > 0.$$

The dyadic sharp maximal operator $M_\varphi^{\sharp,\triangle} f(x)$ is defined by

$$\begin{aligned} M_\varphi^{\sharp,\triangle} f(x) \ &:= \sup_{x\in Q, r<1} \frac{1}{|Q|}\int_{Q(x_0,r)} |f(x) - f_Q|\, dx + \sup_{x\in Q, r\ge 1} \frac{1}{\varphi(|Q|)|Q|}\int_{Q(x_0,r)} |f|\, dx \\ &\simeq \sup_{x\in Q, r<1} \inf_C \frac{1}{|Q|}\int_{Q(x_0,r)} |f(y) - C|\, dy + \sup_{x\in Q, r\ge 1} \frac{1}{\varphi(|Q|)|Q|}\int_{Q(x_0,r)} |f|\, dx, \end{aligned}$$

where Q denotes a dyadic cube and $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$. Similarly, we define the sharp maximal operator $M_\varphi^\sharp f(x)$ for an arbitrary cube with sides parallel to the coordinate axes.

Lemma 2.3. *Let* $1 < p < \infty$, $\omega \in A_\infty(\varphi)$ *and* $f \in L^p(\omega)$, *then*

$$\|M_\varphi^\triangle f\|_{L^p(\omega)} \le C\|M_\varphi^{\sharp,\triangle} f\|_{L^p(\omega)}.$$

Here $M_\varphi^\triangle f(x)$ denotes the dyadic maximal operator. Lemmas 2.2 and 2.3 follow from [19].

Note that $|f(x)| \le M_\varphi^\triangle f(x)$ *a.e.* $x \in \mathbf{R}^d$ and $M_\varphi^{\sharp,\triangle} f(x) \le M_\varphi^\sharp f(x)$ for $x \in \mathbf{R}^d$. By Lemma 2.3, we have

Proposition 2.1. *Let* $1 < p < \infty$, $\omega \in A_\infty(\varphi)$ *and* $f \in L^p(\omega)$, *then*

$$\|f\|_{L^p(\omega)} \le \|M_\varphi^\triangle f\|_{L^p(\omega)} \le C\|M_\varphi^\sharp f\|_{L^p(\omega)}.$$

In order to prove Theorem 1.1, we need to introduce some vector-valued spaces. Let X be a Hilbert space with norm $|\cdot|_X$, and let $\|f\|_{L^p_X(\omega)}$ denote $(\int_{\mathbf{R}^d} |f(y)|_X^p \omega(y)dy)^{1/p}$ for $0 < p < \infty$.

Consider the Bochner integral operator T, defined by

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy,$$

where the X-valued kernel K satisfies the following conditions (for $N \geq n\beta_0 + 1$):

(i) $|K(x,z)|_X \leq C_N|x-z|^{-d}(1+|x-z|)^{-N}$,

(ii) $|K(x,z)-K(x_0,z)|_X \leq C_N \frac{|x-x_0|}{(1+|x-z|)^N|x_0-z|^{d+1}}$, if $2|x-x_0| < |x-z|$.

The next result can be deduced from Lemmas 2.2 and 2.3, and Proposition 2.1.

Proposition 2.2. *If the Bochner operator T is bounded from $L^p(\mathbb{R}^d)$ into $L^p_X(\mathbb{R}^d)$, then for any $r > 1$,*

$$M_\varphi^\sharp(|Tf|_X)(x) \leq CM_{\varphi,r}f(x),$$

where $M_{\varphi,r}f(x) = [M_\varphi(|f|^r)(x)]^{1/r}$, and, as a consequence, the inequality

$$\|Tf\|_{L^p_X(\omega)} \leq C\|f\|_{L^p(\omega)}$$

holds for $1 \leq p_1 < p < \infty$ and $\omega \in A_{p_1}(\varphi)$.

For the proofs of the above stated results we refer to [19].

Now we proceed to prove Theorem 1.2. We define the Hermite g-function and g^*-function, respectively, by the following formulas:

$$g_l(f)(x) = \left[\int_0^\infty |s^l\partial_s^l T_s f(x)|^2 \frac{ds}{s}\right]^{1/2}, \ l = 1, 2, \cdots,$$

$$g_\lambda^*(f)(x) = \left[\int_{\mathbb{R}^d}\int_0^\infty |s^l \frac{s^{-\frac{d}{2}}}{(1+\frac{|x-y|}{\sqrt{s}})^{d\lambda}} \partial_s^l T_s f(x)|^2 \frac{ds}{s}\right]^{1/2}, \ \lambda > 1,$$

where $T_s = e^{-s(-\Delta+|x|^2)}$ denotes the Hermite heat semigroup.

Denoting by $T_s(y,z)$ the kernel of T_s, we can write

$$s^l\partial_s^l T_s f(y) = \int_{\mathbb{R}^d} s^l \left[\frac{\partial^l T_s(y,z)}{\partial s^l}\right] f(z)dz.$$

For convenience, we change the variable $s = t^2$ in the definition of g and g^*, and denote by $Q_t(y,z)$ the new(normalized) kernels $t^{2l}[\frac{\partial^l T_s(y,z)}{\partial s^l}]|_{s=t^2}$ for $l \geq 1$. It is easy to check that these kernels are symmetric and satisfy the inequalities (see [20], pp. 98-99):

(a) $|Q_t(y,z)| \leq Ct^{-d}e^{-\frac{a}{t^2}|x-y|^2}$, $0 < t < 1$,

(b) $|Q_t(y,z)| \leq C2^{-dt}e^{-b|x-y|^2}$, $t \geq 1$,

(c) $|Q_t(y+h,z)-Q_t(y,z)|+|Q_t(y,z+h)-Q_t(y,z)| \leq Cht^{-d-1}e^{-\frac{a}{2}t|x-y|^2}$, for $0<t<1$, $\forall|h|\leq t$,

(d) $|Q_t(y+h,z)-Q_t(y,z)|+|Q_t(y,z+h)-Q_t(y,z)| \leq Ch2^{-dt}e^{-b|x-y|^2}$, for $t\geq 1, \forall|h|\leq t$, where C, a and b are positive constants, independent of x, y, t.

To prove Theorem 1.2 it is convenient to look at the functions g and g^* as vector-valued singular integrals. Let A denote the Hilbert space $L^2(\mathbb{R}_+, dt/t)$, and B denote the Hilbert space $L^2(\mathbb{R}_+\times\mathbb{R}^d, dtdy/t^{n+1})$.

Consider the operator $G_1: L^2(\mathbb{R}^d)\to L^2_A(\mathbb{R}^d)$ defined by

$$G_1f(x)=\int_{\mathbb{R}^d}K_1(x,z)f(z)dz,$$

where $K_1(x,z)$ is the A-valued kernel: $K_1(x,z) := \{Q_t(x,z)\}_t$, and the operator $G_2: L^2(\mathbb{R}^d)\to L^2_B(\mathbb{R}^d)$ defined by

$$G_2f(x)=\int_{\mathbb{R}^d}K_2(x,z)f(z)dz,$$

where $K_2(x,z)$ is the B-valued kernel:

$$K_2(x,z) := \left\{\left(1+\frac{|x-y|}{t}\right)^{-\frac{d\lambda}{2}} Q_t(y,z)\right\}_{(t,y)}.$$

Observe that $|G_1f(x)|_A = g_l(x)$ and $|G_2f(x)|_B = g^*_\lambda(x)$. Therefore, the boundedness of g_l and g^*_λ in $L^p(\omega)$ are equivalent to the boundedness of G_1 from $L^p(\omega)$ into $L^p_A(\omega)$ and G_2 from $L^p(\omega)$ into $L^p_B(\omega)$, respectively. Moreover, boundedness holds for the Muckenhoupt weights for $1<p<\infty$ (see [5]). Hence, in order to apply Proposition 2.2, we need to establish the following lemmas.

Lemma 2.4. *There exist positive constants c_1 and c_2 such that*

(i) $|K_1(x,y)|_A \leq c_1|x-y|^{-d}e^{-c_2|x-y|^2}$,

(ii) $|K_1(x,y)-K_1(x_0,y)|_A \leq c_1\dfrac{|x-x_0|}{|x-y|^{d+1}}e^{-c_2|x-y|^2}$, *if* $2|x-x_0|<|x-y|$.

Proof. We use the above stated inequalities (a)–(d), and fist prove the assertion (i). Note that

$$\begin{aligned}|K_1(x,y)|^2_A &= \int_0^\infty |Q_t(x,y)|^2\frac{dt}{t}\\ &= \int_0^1 |Q_t(x,y)|^2\frac{dt}{t}+\int_1^\infty |Q_t(x,y)|^2\frac{dt}{t}\\ &:= I_1+I_2.\end{aligned}$$

Using the inequality (a), for I_1 we have

$$\begin{aligned} I_1 &\leq C\int_0^1 t^{-2d}e^{-a\frac{|x-y|^2}{t^2}}\frac{dt}{t} \\ &= C\int_0^1 t^{-2d}e^{-a\frac{|x-y|^2}{2t^2}}e^{-a\frac{|x-y|^2}{2t^2}}\frac{dt}{t} \\ &\leq Ce^{-a\frac{|x-y|^2}{2}}\int_0^1 \frac{1}{(t+|x-y|)^{2d+1}}dt \\ &\leq C\frac{1}{|x-y|^{2d}}e^{-a\frac{|x-y|^2}{2}}. \end{aligned}$$

Now using the inequality (b), for I_2 we obtain

$$I_2 \leq C\int_1^\infty e^{-t}e^{-a|x-y|^2}\frac{dt}{t} \leq Ce^{-b|x-y|^2} \leq C\frac{1}{|x-y|^{2d}}e^{-b\frac{|x-y|^2}{2}}.$$

To prove the assertion (ii), first note that

$$\begin{aligned} |K_1(x,y) &- K_1(x_0,y)|_A^2 \\ &\leq \int_0^\infty |Q_t(x,y) - Q_t(x_0,y)|^2\frac{dt}{t} \\ &= \int_0^1 |Q_t(x,y) - Q_t(x_0,y)|^2\frac{dt}{t} + \int_1^\infty |Q_t(x,y) - Q_t(x_0,y)|^2\frac{dt}{t} \\ &:= I_3 + I_4. \end{aligned}$$

If $|x-y| \geq 1$, then by the inequality (c), for I_3 we have

$$\begin{aligned} I_3 &\leq C\int_0^1 \left(\frac{|x-x_0|}{t}\right)^2 t^{-2d}e^{-a\frac{|x-y|^2}{t^2}}\frac{dt}{t} \\ &\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+2)}}\int_0^1 e^{-a\frac{|x-y|^2}{2t^2}}t^2\frac{dt}{t} \\ &\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+2)}}e^{-a\frac{|x-y|^2}{2}}\int_0^1 tdt \\ &\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+1)}}e^{-a\frac{|x-y|^2}{2}}. \end{aligned}$$

If $|x-y| < 1$, an application of the inequality (c) yields

$$\begin{aligned} I_3 &\leq C\int_0^{|x-y|} \left(\frac{|x-x_0|}{t}\right)^2 t^{-2d}e^{-a\frac{|x-y|^2}{t^2}}\frac{dt}{t} \\ &+C\int_{|x-y|}^1 \left(\frac{|x-x_0|}{t}\right)^2 t^{-2d}e^{-a\frac{|x-y|^2}{t^2}}\frac{dt}{t} \end{aligned}$$

$$\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+2)}}\int_0^{|x-y|} e^{-a\frac{|x-y|^2}{2t^2}}t^2\frac{dt}{t}$$
$$+C\frac{|x-x_0|^2}{|x-y|^{2d}}\int_{|x-y|}^1 e^{-a\frac{|x-y|^2}{2t^2}}t^{-2}\frac{dt}{t}$$
$$\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+2)}}e^{-a\frac{|x-y|^2}{2}}\int_0^{|x-y|} tdt$$
$$+C\frac{|x-x_0|^2}{|x-y|^{2d}}e^{-a\frac{|x-y|^2}{2}}\int_{|x-y|}^1 t^{-3}dt$$
$$\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+1)}}e^{-a\frac{|x-y|^2}{2}}.$$

To estimate I_4, we apply the inequality (d) to obtain

$$I_4 \leq C\int_1^\infty |x-x_0|^2 e^{-t}e^{-b|x-y|^2}dt \leq C\frac{|x-x_0|^2}{|x-y|^{2(d+1)}}e^{-b\frac{|x-y|^2}{2}}.$$

This completes the proof of Lemma 2.4.

Lemma 2.5. *Let $\lambda > 4$, $N_1 = d(\frac{\lambda}{2}-1)$ and $N_2 = d(\frac{\lambda}{2}-1)-1$, then there exist positive constants C_{N_1}, C_{N_2} such that*

(i) $|K_2(x,z)|_B \leq C_{N_1}|x-z|^{-d}(1+|x-z|)^{-N_1}$,

(ii) $|K_2(x,z)-K_2(x,z_0)|_B \leq C_{N_2}\frac{|z-z_0|}{(1+|x-z|)^{N_2}|z_0-z|^{d+1}}$, *if* $2|z-z_0| < |x-z|$.

The proof is similar to that of Lemma 2.4, and hence, is omitted.

Theorem 2.1. *Let $1 \leq p_1 < p < \infty$ and $\omega \in A_{p_1}(\varphi)$. Then, for $l \geq 1$ there is a constant $C > 0$ so that*

$$\|g_l(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

Obviously, Theorem 2.1 is a consequence of Lemma 2.4 and Proposition 2.2.

As an immediate consequence of Theorem 2.1 we can state the following result.

Corollary 2.1. Let $1 \leq p_1 < p < \infty$ and $\omega \in A_{p_1}(\varphi)$. Then for $l \geq 1$ there is a constant $C > 0$ so that

$$C^{-1}\|f\|_{L^p(\omega)} \leq \|g_l(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

Theorem 2.2. *Let $1 \leq p_1 < p < \infty$ and $\omega \in A_{p_1}(\varphi)$. Then for each $\lambda > 2(\beta_0+4)$ there is a constant $C > 0$ so that*

$$\|g_\lambda^*(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

Proof. Adapting the arguments used in [15], pp. 43-44, and using a duality argument and Lemma 2.5 and Proposition 2.1, we obtain the desired result.

As a consequence of Theorem 2.2, we have the following result.

Corollary 2.2. *Let* $1 < p < \infty$ *and* $\omega(x) = (1 + |x|)^\gamma$ *with* $|\gamma| < \beta_0$. *Then there exists a constant* $C > 0$ *such that for each* $\lambda > 2(\beta_0 + 4)$

$$\|g_\lambda^* f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Proof. Note that $\omega(x) = (1 + |x|)^\gamma \in A_1(\varphi)$ if $-\beta_0 < \gamma < \beta_0$. Applying Theorem 2.2, we get

$$\|g_\lambda^* f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

This implies

$$\|g_\lambda^* f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Thus, Corollary 2.2 is proved.

Theorem 2.3. *Let* $\omega(x) = (1 + |x|)^\gamma \mu(x)$ *with* $\mu \in A_p(\mathbf{R}^d)$ *(Muckenhoupt class) and* $\gamma \in \mathbf{R}$. *Then there exists a positive constant* λ_0 *depending on* γ *and* μ *such that for each* $\lambda > \lambda_0$

$$\|g_l\|_{L^p(\omega)} + \|g_\lambda^*\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Proof. Using the results from [5], we obtain

$$\|g_l\|_{L^p(\mu)} + \|g_\lambda^*\|_{L^p(\mu)} \le C\|f\|_{L^p(\mu)},$$

where $\mu \in A_p(\mathbf{R}^d)$. By the properties of the Muckenhoupt class $A_p(\mathbf{R}^d)$ (see [4]), there exists $\epsilon > 0$ such that

$$\|g_l\|_{L^p(\mu^{1+\epsilon}))} + \|g_\lambda^*\|_{L^p(\mu^{1+\epsilon}))} \le C\|f\|_{L^p(\mu^{1+\epsilon}))}. \tag{2.1}$$

On the other hand, for $\omega_1(x) = (1 + |x|)^{\gamma(1+\epsilon)/\epsilon}$ and $\lambda_0 = 2(|\gamma| + 4)(1 + \epsilon)/\epsilon$, by Corollaries 2.1 and 2.2, we have

$$\|g_l\|_{L^p(\omega_1)} + \|g_\lambda^*\|_{L^p(\omega_1)} \le C\|f\|_{L^p(\omega_1)}. \tag{2.2}$$

Putting together (2.1) and (2.2), we obtain the desired result.

To prove Theorem 1.2, we also need the following result proved in [5].

Proposition 2.3. *Let $\lambda > 2$ and T_m be as in Theorem 1.2. Then for all $l \geq d\lambda/2+1$ we have*

$$g_l(T_m f)(x) \leq C g_\lambda^*(f)(x), \text{ a.e. } x \in \mathbf{R}^d.$$

Proof of Theorem 1.2. Combining Corollary 2.1, Theorem 2.3 and Proposition 2.3 we have, for $f \in C_c(\mathbf{R}^d)$

$$\|T_m f\|_{L^p(\omega)} \leq C\|g_l(T_m f)\|_{L^p(\omega)} \leq C\|g_\lambda^* f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}$$

provided that ω, l and λ satisfy the conditions of Theorem 2.3 and Proposition 2.3. The proof is complete.

Using the same transference principle as in Corollary 3.4 from [6], we obtain a counterpart of Theorem 1.2 for Laguerre expansions when $\alpha = \frac{n}{2} - 1$.

The next two lemmas were stated in [5].

Lemma 2.6. *Let $\alpha = \frac{n-2}{2}$ where $n \in \mathbf{N}_+$. Then for some constants $\alpha_{\mathbf{k}} \in \mathbf{R}$, $k = 1, 2\cdots$, the following equalities hold:*

$$L_k^\alpha(|z|^2) = \sum_{|\mathbf{k}|=k} \alpha_{\mathbf{k}} \eta_{2\mathbf{k}}(z)|z|^\alpha, \ \forall z \in \mathbf{R}^d, \ k = 1, 2\cdots.$$

We shall also need the following elementary fact.

Lemma 2.7. *For every $f \in L^1(0,\infty)$ we have*

$$\int_{\mathbf{R}^d} f(|z|^2)|z|^{-(d-2)}dz = c_d \int_0^\infty f(t)dt.$$

Corollary 2.3. *The assertion of Theorem 1.3 remains valid when $\alpha = \frac{n-2}{2}$ and n is a positive integer.*

Proof. Let $m(\xi)$ be as in Theorem 1.3. The function $M(\xi) = m((\xi_1 + \cdots + \xi_d)/2)$ restricted to the lattice $\mathbf{N}^d$ defines a multiplier $\{M(\mathbf{k})\}$ which satisfies the conditions (1.2).

By Lemma 2.6 we have

$$(T_m f)(|z|^2) = \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} m(\mathbf{k}) < f, L_{\mathbf{k}}^\alpha > \alpha_{\mathbf{k}} \eta_{2\mathbf{k}}(z)|z|^\alpha, \ z \in \mathbf{R}^d.$$

Let $\omega(|z|) = (1+|z|)^\gamma |z|^{p\delta}$ for $\gamma \in \mathbf{R}$ and $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$. Observing that $|z|^{(d-2)(\frac{p}{2}-1)}|z|^{p2\delta} \in A_p(\mathbf{R}^d)$, we can use Lemmas 2.4 and 2.5 to apply Theorem 1.1

to obtain

$$\begin{aligned}\|T_m f\|^p_{L^p(\omega)} &= \int_0^\infty |(T_m f)(t)|^p \omega(t)dt \\ &= c_n \int_{\mathbb{R}^d} \left| \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} M(2\mathbf{k}) < f, L^\alpha_{\mathbf{k}} > \alpha_{\mathbf{k}} \eta_{2\mathbf{k}}(z) \right|^p |z|^{\alpha p-(d-2)} \omega(|z|^2) dz \\ &= c_n \int_{\mathbb{R}^d} \left| \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} < f, L^\alpha_{\mathbf{k}} > \alpha_{\mathbf{k}} \eta_{2\mathbf{k}}(z) \right|^p |z|^{(d-2)(\frac{p}{2}-1)} \omega(|z|^2) dz \\ &\le C\|f\|_{L^p(\omega)},\end{aligned}$$

Thus, Corollary 2.3 is proved.

3. Multipliers for Laguerre expansions

In this section we prove Theorem 1.3. The strategy is to deduce the result from the special case discussed in Corollary 2.3, by using interpolation of the following analytic family of operators

$$T^z_m f = \sum_{k=0}^{\infty} m_k < f, L^z_k > L^z_k, \text{ where } z \in \mathbf{C} \text{ and } Rez > -1.$$

We first recall the definition of Kanjin's operators $T^{\alpha+i\theta}_\alpha$ and prove their boundedness for the range of $L^p_{\alpha,\gamma}(\mathbf{R}_+)$.

In this section we will use the following notation from [5, 7]. We denote $M(\theta) := (1+|\theta|)^N e^{c|\theta|}$ for suitably large constants N and c. The constants appearing in the section such as C, c or N may depend on α, p, δ and γ, but are independent of $\theta \in \mathbf{R}$. Finally, it is also convenient to denote the admissible range of indices by

(3.1)

$$\mathcal{A} = \left\{ \left(\frac{1}{p}, \alpha, \delta, \gamma\right) \in (0,1) \times (-1,\infty) \times \mathbf{R} \times \mathbf{R} : -\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2} \right\}.$$

We first state the boundedness of $T^{\alpha+i\theta}_\alpha$ in $L^p_{\delta,\gamma}(\mathbf{R}_+)$ for special values of α.

Observe that (see [7], p. 539), the Laguerre polynomials can be extended to complex parameters $z \in \mathbf{C}$ with $Rez > -1$ by the formula

$$L^{(z)}_k(y) = \frac{D^{(k)}_y[y^{z+k}e^{-y}]}{k! y^z e^{-y}} = \sum_{j=0}^{k} \frac{\Gamma(k+z+1)}{\Gamma(k-j+1)\Gamma(j+z+1)} \frac{(-y)^j}{j!}, \ y > 0,$$

and likewise for the corresponding Laguerre functions we have

$$L^z_k(y) = \left(\frac{\Gamma(k+1)}{\Gamma(z+k+1)}\right)^{1/2} y^{1/2} e^{-y/2} L^{(z)}_k(y), \ y > 0.$$

Moreover, the following lemma is true, which was proved in [7].

Lemma 3.1. *Let $\alpha > -1$ and $f \in C_c^\infty(0,\infty)$. Then for each $N \geq 1$ there exist a constant $C > 0$ and a number $k_0 \in \mathbf{N}$ such that for all $k \geq k_0$ and $\theta \in \mathbf{R}$*

$$(3.2) \qquad |< f, L_k^{\alpha+i\theta} >| \leq C(1+|\theta|)^{4N+\alpha} e^{\frac{\pi}{2}|\theta|}(1+k)^{-N}.$$

Using this lemma one can define the complex transplantation operators

$$T_\alpha^z f = \sum_{k=0}^{\infty} < f, L_k^z > L_k^\alpha, \; Rez > -1, \; \alpha > -1,$$

at least for functions $f \in C_0^\infty(\mathbf{R}_+)$.

For every $\alpha > -1$ and $\theta \in \mathbf{R}$ we define a multiplier by

$$(3.3) \qquad \lambda(\xi) = \lambda_{\alpha,\theta}(\xi) = \left(\frac{\Gamma(\xi+\alpha+1+i\theta)}{\Gamma(\xi+\alpha+1)}\right)^{1/2}, \; \xi \geq 0.$$

Observe that λ is an analytic function of ξ when $Re\xi > -1-\alpha$. The following result has been proved in [5].

Lemma 3.2. *Let $\alpha > -1$. Then the function $\lambda(\xi)$ defined by (3.3) belongs to $C^\infty(0,\infty)$ and satisfies*

$$\sup_{\xi\in[0,\infty)} (1+|\xi|)^l |D^l \lambda(\xi)| \leq C_l(1+|\theta|)^l, \; \forall\, \theta \in \mathbf{R}, \; l = 0,1,2,\cdots,$$

where the constants C_l are independent of θ.

We prove Theorem 1.2 under the following assumption on the indices $(\frac{1}{p},\alpha,\delta,\gamma)$.

Assumption (A). The point $(\frac{1}{p},\alpha,\delta,\gamma)$ is so that the multiplier operator $T_\lambda f = \sum_{k=0}^{\infty} \lambda(k) < f, L_k^\alpha > L_k^\alpha$, with $\lambda = \lambda_{\alpha,\theta}$ as in (3.3), is bounded on $L^p_{\delta,\gamma}(\mathbf{R}_+)$ and satisfies

$$\|T_\lambda f\|_{L^p_{\delta,\gamma}} \leq C(1+|\theta|)^N e^{c|\theta|}\|f\|_{L^p_{\delta,\gamma}}, \; \forall\, \theta \in \mathbf{R},$$

for some constants $C, c, N > 0$, where

$$\|f\|_{L^p_{\delta,\gamma}} = \left(\int_{\mathbf{R}_+} (|f(x)|(1+x)^\gamma x^\delta)^p dx\right)^{1/p}.$$

Remark 3.1. *It follows from Corollary 2.3 and Lemma 3.2 that the Assumption (A) is fulfilled for parameters from the set A (see (3.1)) of the form $(\frac{1}{p},\frac{n-2}{2},\delta,\gamma)$, whenever $n \in \mathbf{Z}_+$. Moreover, the Assumption (A) also holds for $(\frac{1}{2},\alpha,0,0)$ and for* *all $\alpha > -1$, and, by the duality, it holds for a fixed $(\frac{1}{p}, \alpha, \delta, \gamma)$ if and only if it is true for $(\frac{1}{p'}, \alpha, -\delta, -\gamma)$.*

In order to prove Theorem 1.3, we also need the following complex interpolation result.

Lemma 3.3. *Let $P_0 = (\frac{1}{p_0}, \alpha_0, \delta_0, \gamma_0)$ and $P_1 = (\frac{1}{p_1}, \alpha_1, \delta_1, \gamma_1)$ be two fixed points from A for which the assertion of Theorem 1.2 holds. Then the assertion of the theorem must also hold at the points $P = (\frac{1}{p}, \alpha, \delta, \gamma)$ of the form*

$$P = (1-t)P_0 + tP_1, \ t \in (0,1). \tag{3.4}$$

Proof. As in Lemma 3.20 from [5], we define

$$\alpha(z) = (1-z)\alpha_0 + z\alpha_1, \ \delta(z) = (1-z)\delta_0 + z\delta_1, \ \text{and } \gamma(z) = (1-z)\gamma_0 + z\gamma_1,$$

for complex $z = s + i\theta$ and $0 \le s \le 1$. Recall that $M(\theta) = (1+|\theta|)^N e^{c|\theta|}$ for suitably large constants N and c. By Lemma 3.1, the operator

$$T_m^{\sigma+i\tau} f = \sum_{k=0}^{\infty} m(k) < f, L_k^{\sigma-i\tau} > L_k^{\sigma+i\tau} = (T_\sigma^{\sigma+i\tau})^* T_m^\sigma T_\sigma^{\sigma-i\tau} f$$

is well defined and bounded at least when $f \in L^2(\mathbb{R}_+)$. We define an analytic family of operators by letting

$$S_z F(y) = y^{\delta(z)}(1+y)^{\gamma(z)} T_m^{\alpha(z)}(F(x)x^{-\delta(z)}(1+x)^{-\gamma(z)})(y)$$

at least for $F \in L_c^2(0,\infty)$.

Now we are going to show that $\{S_z\}$ satisfies the conditions of Stein's interpolation theorem (see [2]). To this end, observe first that, given any two subsets E_1, E_2 compactly contained in $(0,\infty)$, the function

$$z \mapsto \Phi(z) =< S_z(\chi_{E_1}), \chi_{E_2} >$$

is well defined whenever $0 \le Rez \le 1$, and satisfies

(3.5)
$$\begin{aligned} |\Phi(z)| &\le \|T_m^{\sigma(z)}(x^{-\delta(z)}(1+x)^{-\gamma(z)}\chi_{E_1})\|_2 \|(y^{-\delta(z)}(1+y)^{-\gamma(z)}\chi_{E_2})\|_2 \\ &\le C_{E_2} \|(T_{\alpha(s)}^{\alpha(s)+i(\alpha_1-\alpha_0)\theta})^* T^{\alpha(s)}{}_m T^{\alpha(s)-i(\alpha_1-\alpha_0)\theta}(x^{-\delta(z)}(1+x)^{-\gamma(z)}\chi_{E_1})\|_2 \\ &\le C_{E_2} M(\theta) \|(x^{-\delta(z)}(1+x)^{-\gamma(z)}\chi_{E_1})\|_2 \\ &\le C_{E_1} C_{E_2} M(\theta), \end{aligned}$$

by the L^2 boundedness of $T_\sigma^{\sigma+i\tau}$, $\forall\, \sigma > -1$.

Next, we show that the function Φ is holomorphic in a neighborhood of the strip $\bar{S} := \{0 \leq Re z \leq 1\}$. Indeed, since $\|T_m^{\sigma(z)}\|_{L^2 \to L^2}$ is uniformly bounded in the compact sets of $\bar{S}$, similar to (3.4), it is enough to show the holomorphy of $z \mapsto < S_z F, G >$ for all $F, G \in C_c^\infty(0,\infty)$. Denoting $f(x) = x^{-\delta(z)}(1+x)^{\gamma(z)}F(x)$, $g(y) = y^{-\delta(\bar{z})}(1+y)^{\gamma(\bar{z})}G(y)$ and $\alpha(z) = \sigma + i\tau$, we can write

$$\begin{aligned} < S_z F, G > &= < T_m^{\alpha(z)}(f), g > = < T_m^{(\sigma)} T_\sigma^{\sigma - i\tau}(f), \overline{T_\sigma^{\sigma+i\tau}(g)} > \\ &= \sum_k m_k < f, L_k^{\sigma - i\tau} > \overline{< g, L_k^{\sigma+i\tau} >} \\ &= \sum_k m_k < x^{-\delta(z)}(1+x)^{\gamma(z)} F, L_k^{\alpha(z)} >< y^{-\delta(\bar{z})}(1+y)^{\gamma(\bar{z})} F, L_k^{\alpha(\bar{z})} > . \end{aligned}$$

Since the series converges uniformly when z belongs to a compact set of $\bar{S}$, it is easy to show the holomorphy of the map

$$z \in \bar{S} \mapsto < x^{\pm\delta(z)}(1+x)^{\pm\gamma(z)} F, L_k^{\alpha(\bar{z})} > = \int_0^\infty x^{\pm\delta(z)}(1+x)^{\pm\gamma(z)} F(x) L_k^{\alpha(\bar{z})}(x) dx,$$

for all $F \in C_c^\infty(0,\infty)$.

Combining this with (3.4) we conclude that Φ is holomorphic in the strip $\{0 < Re\, z < 1\}$, continuous in the closure and has admissible growth for complex interpolation. To verify the conditions of Stein's interpolation theorem (see [2]), we only need to show the boundedness of the operator S_z at the limiting bands

$$S_{i\theta} : L^{p_0}(\mathbb{R}_+) \to L^{p_0}(\mathbb{R}_+) \text{ and } S_{1+i\theta} : L^{p_1}(\mathbb{R}_+) \to L^{p_1}(\mathbb{R}_+).$$

When $Re z = 0$ we use the assumption that Theorem 1.2 (and hence Assumption (A)) holds for the point p_0. Then, both $T_m^{\alpha_0}$ and $T_{\alpha_0}^{\alpha_0 - i(\alpha_1 - \alpha_0)\theta}$ are bounded in $L_{\delta_0,\gamma_0}^{p_0}$ and in $L_{-\delta_0,-\gamma_0}^{p_0'}$, which implies

$$\begin{aligned} \|S_{i\theta} F\|_{p_0} &= \|(T_{\alpha_0}^{\alpha_0 + i(\alpha_1 - \alpha_0)})^* T_m^{\alpha_0} T_{\alpha_0 + i\tau}^{\alpha_0 - i(\alpha_1 - \alpha_0)}(x^{-\delta(i\theta)}(1+x)^{\gamma(i\theta)} F)\|L_{\delta_0,\gamma_0}^{p_0} \\ &\leq M(\theta)\|x^{-\delta_0 - i(\delta_1 - \delta_0)\theta}(1+x)^{\gamma_0 - i(\gamma_1 - \gamma_0)\theta} F(x)\|L_{\delta_0,\gamma_0}^{p_0} \\ &= M(\theta)\|F\|_{p_0}. \end{aligned}$$

When $Re\ z = 1$, we have a similar result. Thus, by Stein's theorem S_s must be bounded in $L^{p_s}(\mathbb{R}_+)$ for $\frac{1}{p_s} = \frac{1-s}{p_0} + \frac{s}{p_1}$ and all $s \in (0,1)$. Letting $s = t$ and using (3.4), we have $p_t = p$, $\alpha(t) = \alpha$ and $\delta(t) = \delta$.

Moreover, such boundedness translates into

$$\begin{aligned} \|T_m^\alpha f\|_{L_{\delta,\gamma}^p} &= \|y^{\delta(t)}(1+y)^{\gamma(t)} T_m^{\alpha(t)}(x^{\delta(t)}(1+x)^{\gamma(t)} f(x) x^{-\delta(t)}(1+x)^{-\gamma(t)})\|_{L^p} \\ &= \|S_t(x^{\delta(t)}(1+x)^{\gamma(t)} f(x))\|_{L^p} \\ &\leq M\|x^{\delta(t)}(1+x)^{\gamma(t)} f(x)\|_{L^p} = M\|f\|_{L_{\delta,\gamma}^p}, \end{aligned}$$

showing that the assertion of Theorem 1.2 holds for the point $P = (\frac{1}{p}, \alpha, \delta, \gamma)$. This completes the proof of Lemma 3.3.

Proof of Theorem 1.3. We need to show that the operator T_m^α is bounded in $L^p_{\delta,\gamma}$ for every fixed $P = (\frac{1}{p}, \alpha, \delta, \gamma) \in \mathcal{A}$. When $\alpha > 0$, $\alpha = \alpha_n := \frac{n-2}{2}$, and n is an integer so that $\alpha_{n-1} < \alpha < \alpha_n$, then we easily find two points from $\mathcal{A}$ of the form $P_0 = (\frac{1}{p}, \alpha_{n-1}, \delta_0, \gamma_0)$, $P_1 = (\frac{1}{p}, \alpha_n, \delta_1, \gamma_1)$ and some $t \in (0,1)$ to satisfy $P = (1-t)P_0 + tP_1$. When $-1 < \alpha < 0$, one can choose a number α_0 close enough to -1, and interpolate between the points $P_0 = (\frac{1}{2}, \alpha_0, 0, 0)$, $P_1 = (\frac{1}{p}, 0, \delta_1, \gamma_1)$. By Corollary 2.3, the assertion of Theorem 1.3 holds for points P_0 and P_1, and therefore, by Lemma 3.3, it must also hold for the point P. Theorem 1.3 is proved.

4. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following result.

Theorem 4.1. *Let $\alpha > -1$, $\gamma \in \mathbf{R}$ and $\theta \in \mathbf{R}$. Then the operator $T_\alpha^{\alpha+i\theta}$ can be boundedly extended to $L^p_{\delta,\gamma}(\mathbf{R}_+)$ for all $1 < p < \infty$ and $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$. Moreover, there exist constants $C, c > 0$ and a number $N \in \mathbf{N}$(depending only on $\alpha, p, \delta, \gamma$) such that*

$$\|T_\alpha^{\alpha+i\theta} f\|_{L^p_{\delta,\gamma}} \le C(1+|\theta|)^N e^{c|\theta|} \|f\|_{L^p_{\delta,\gamma}}, \ \forall\, \theta \in \mathbf{R}. \tag{4.1}$$

The proof of the theorem follows the scheme proposed by Garrigós et al. in [5] and Kanjin in [7]. Obviously, under Assumption (A), it is enough to show (4.1) for the operator

$$\widetilde{T}_\alpha^{\alpha+i\theta} f = \sum_{k=0}^{\infty} \left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+1+i\theta)} \right)^{1/2} < f, L_k^{\alpha+i\theta} > L_k^\alpha$$

instead of $T_\alpha^{\alpha+i\theta}$.

Following [5] and [7], we can define for $\epsilon > 0$ the operators

$$G_{\theta,\epsilon}(f) = \sum_{k=0}^{\infty} \left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+1+\epsilon+i\theta)} \right)^{1/2} < f, L_k^{\alpha+\epsilon+i\theta} > L_k^\alpha$$

so that $\widetilde{T}_\alpha^{\alpha+i\theta} f(x) = \lim_{\epsilon\to 0} G_{\theta,\epsilon} f(x)$ for all $x > 0$, at least $f \in C_c^\infty(0,\infty)$ by Lemma 3.1. Moreover, the following remarkable formula holds (see [7]):

$$G_{\theta,\epsilon} f(x) = \frac{1}{\Gamma(\epsilon+i\theta)} \int_x^\infty f(t) e^{-\frac{t-x}{2}} \left(1 - \frac{x}{t}\right)^{\epsilon-1+i\theta} \left(\frac{x}{t}\right)^{\frac{\alpha}{2}} t^{\frac{\epsilon+i\theta}{2}} \frac{dt}{t}.$$

Adapting the arguments used in [5], pp. 260-263, we can obtain the following result.

Proposition 4.1. *Let $\alpha > -1$, $\gamma \in \mathbf{R}$ and let p, δ be such that $\delta > -\frac{1}{p} - \frac{\alpha}{2}$. Then, there exist constants $C, c > 0$ and a number $N \in \mathbf{N}$ (depending only on $\alpha, p, \delta, \gamma$) such that*

$$\|G_{\theta,\epsilon}\|_{L^p_{\delta,\gamma}} \le C(1+|\theta|)^N e^{c|\theta|}\left(\|f(x)x^{\frac{1}{2}}\|_{L^p_{\delta,\gamma}} + \|f(x)x^{-\frac{1}{2}}\|_{L^p_{\delta,\gamma}}\right),$$

for all $\theta \in \mathbf{R}$ and all $0 < \epsilon \le 1$.

Proof of Theorem 4.1. By Lemma 3.2 all the multipliers $\lambda = \lambda_{\alpha,\theta}$ in (3.3) satisfy the conditions of Theorem 1.3. Hence, Assumption (A) is satisfied for all $(\frac{1}{p}, \alpha, \delta, \gamma) \in \mathcal{A}$, and we can infer Theorem 4.1 immediately from Proposition 4.1 and Fatou's lemma. Indeed, using these facts, for $f \in C_c^\infty(0,\infty)$ and with some constant C (independent of ϵ) we obtain

$$\begin{aligned}\|T_\alpha^{\alpha+i\theta} f\|_{L^p_{\delta,\gamma}} &= \|T_\lambda \widetilde{T}_\alpha^{\alpha+i\theta} f\|_{L^p_{\delta,\gamma}} \\ &\le CM(\theta)\|\widetilde{T}_\alpha^{\alpha+i\theta} f\|_{L^p_{\delta,\gamma}} \\ &\le CM(\theta)\lim_{\epsilon\to 0}\|G_{\theta,\epsilon} f\|_{L^p_{\delta,\gamma}} \\ &\le CM(\theta)\lim_{\epsilon\to 0}\left(\|f(x)x^{\frac{1}{2}}\|_{L^p_{\delta,\gamma}} + \|f(x)x^{-\frac{1}{2}}\|_{L^p_{\delta,\gamma}}\right) \\ &\le CM(\theta)\|f\|_{L^p_{\delta,\gamma}},\end{aligned}$$

The proof is complete.

We also need the following lemma proved in [5].

Lemma 4.1. *Let $\alpha > -1$ and $z = \sigma + i\tau$ with $\sigma > -1$. Then the operator T_α^z is bounded in $L^2(\mathbf{R}_+)$.*

Proof of Theorem 1.1. We fix $\beta > \alpha_0 > -1$ so that $-\frac{\alpha_0}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha_0}{2}$. Hence, we need only to show that $T_{\alpha_0}^\beta$ and $T_\beta^{\alpha_0}$ are bounded operators in $L^p_{\alpha,\gamma}(\mathbf{R}_+)$. We let $P := (\frac{1}{p}, \alpha, \delta, \gamma)$, which clearly belongs to $\mathcal{A}$. It is easy to see that there exist two other points in $\mathcal{A}$ of the form $P_0 = (\frac{1}{p_0}, \alpha_0, \delta_0, \gamma_0)$ and $P_1 = (\frac{1}{2}, \alpha_1, 0, 0)$, and some $t \in (0,1)$ such that $P = (1-t)P_0 + tP_1$. This can be done explicitly if α_1 is chosen sufficiently large, by taking $\delta_0 = \delta/(1-t)$ and $t = \frac{\beta-\alpha_0}{\alpha_1-\alpha_0}$. As in Section 3, we use the notation $\alpha(z) = (1-z)\alpha_0 + z\alpha_1$, $\delta(z) = (1-z)\delta_0$ and $\gamma(z) = (1-z)\gamma_0$ for $z \in \mathbf{C}$.

By Lemma 4.1, we can define the analytic family of operators

$$S_z = y^{\alpha(z)}(1+y)^{\gamma(z)} T_{\alpha_0}^{\alpha(z)}(F(x)x^{-\alpha(z)}(1+x)^{-\gamma(z)})(y),\ 0 \le Re z \le 1,$$

at least for $F \in L_c^\infty(0,\infty)$. Then, arguing as in Section 3, we conclude that S_z satisfies the conditions of Stein's theorem, where the boundedness of the operators

$$S_{i\theta} : L^{p_0}(\mathbb{R}_+) \to L^{p_0}(\mathbb{R}_+) \text{ and } S_{1+i\theta} : L^{p_0}(\mathbb{R}_+) \to L^{p_0}(\mathbb{R}_+)$$

follows from Theorem 4.1 and Lemma 4.1, respectively. Thus, S_t must be bounded in $L^{p_t} = L^p$, which translates into

$$\begin{aligned} \|T^\beta_{\alpha_0} f\|_{L^p_{\delta,\gamma}} &= \|S_t(x^{(1-t)\delta_0}(1+x)^{(1-t)\gamma_0} f(x))\|_{L^p} \\ &\le M\|x^{(1-t)\delta_0}(1+x)^{(1-t)\gamma_0} f(x)\|_{L^p} \\ &= M\|f\|_{L^p_{\delta,\gamma}}. \end{aligned}$$

This proves the required $L^p_{\delta,\gamma}$ boundedness for the operators $T^\beta_{\alpha_0}$ for any $\beta > \alpha_0 > -1$. The boundedness of $T^{\alpha_0}_\beta$ follows by duality. Indeed, if $(\frac{1}{p}, \alpha_0, \delta, \gamma) \in \mathcal{A}$, then an elementary algebraic manipulation shows that $(\frac{1}{p'}, \alpha_0, -\delta, -\gamma) \in \mathcal{A}$ as well, where $\frac{1}{p'} = 1 - \frac{1}{p}$. Then, for all $f \in C_c^\infty(0,\infty)$ we have

$$\begin{aligned} \|T^{\alpha_0}_\beta f\|_{L^p_{\delta,\gamma}} &= \sup_{\|g\|_{p'}=1} \left| \int_0^\infty T^{\alpha_0}_\beta f(x) x^\delta (1+x)^\gamma g(x)dx \right| \\ &= \sup_{\|g\|_{p'}=1} \left| \int_0^\infty f(y) T^\beta_{\alpha_0} f(x^\delta(1+x)^\gamma g)dx \right| \\ &\le \|y^\delta(1+y)^\gamma f(y)\|_{L^p} \sup_{\|g\|_{p'}=1} \|T^\beta_{\alpha_0}(x^\delta(1+x)^\gamma g))\|_{L^p_{-\delta,-\gamma}} \\ &\le \|f\|_{L^p_{\delta,\gamma}} M \sup_{\|g\|_{p'}=1} \|x^\delta(1+x)^\gamma g)\|_{L^p_{-\delta,-\gamma}} \\ &= M\|f\|_{L^p_{-\delta,-\gamma}}. \end{aligned}$$

Theorem 1.1 is proved.

5. Applications

In this section, we study the Littlewood-Paley g-functions for the Laguerre semigroup. Consider the heat diffusion semigroup e^{-tL} associated with the Laguerre operator $L = L^{(\alpha)}$. Similar to the classical case, treated in [16], g-functions of order $l = 1, 2, \cdots$ can be defined by

$$g_l^{(\alpha)}(f) = \left\{ \int_0^\infty \left| t^l \frac{\partial^l}{\partial t^l}(e^{-tL^{(\alpha)}} f) \right|^2 \frac{dt}{t} \right\}^{1/2}.$$

The main purpose of this section is to extend Theorem 5.4 from [5] to our case. More precisely, we are going to prove the following result.

Theorem 5.1. *Let $\alpha > -1$, $\gamma \in \mathbb{R}$, $1 < p < \infty$, and δ be such that $-1/p - \alpha/2 < \delta < 1 - 1/p + \alpha/2$. Then for every $l = 1, 2, \cdots$, there is a positive constant C such that*

$$\frac{1}{C}\|f\|_{L^p_{\delta,\gamma}} \le \|g_l^{(\alpha)} f\|_{L^p_{\delta,\gamma}} \le C\|f\|_{L^p_{\delta,\gamma}}, \ f \in C_c^\infty(0,\infty).$$

Proof. The proof is similar to that of Theorem 5.4 from [5]. So, we only give a sketch of the proof. Since the first inequality follows from the usual polarization argument, we need only to prove the second inequality. We first consider the case $l = 1$. For simplicity we write $g(f) = g_1^{(\alpha)}(f)$, and drop the superscript (α) when reference to such index is clear. Recall that the kernel $h_t(x, y)$ of e^{-tL} is given explicitly by

$$h_t(y,z) = \sum_{k=0}^{\infty} e^{-t(k+\frac{\alpha+1}{2})} L_k^\alpha(y) L_k^\alpha(z) = \frac{r^{1/2}}{1-r} exp\left\{-\frac{1}{2}\frac{1+r}{1-r}(y+z)\right\} I_\alpha\left(\frac{2(ryz)^{1/2}}{1-r}\right),$$

where $r = e^{-t}$, $I_\alpha = i^{-\alpha} J_\alpha(is)$, and J_α is the usual Bessel function of order α (see [10]).

We first claim that the assertion of the theorem is true when $\alpha = \frac{n-2}{2}$. Indeed, denoting $\Phi(x) = |x|^2$, it is easy to see that for $x \in \mathbb{R}^d$ (see [10]):

$$e^{-tL}(f)(|x|^2) = e^{-\frac{t}{4}(-\Delta+|x|^2)}\left(\frac{f \circ \Phi}{|\cdot|^\alpha}\right).$$

Hence $g(f)(|x|^2) = 4g_1(\frac{f\circ\Phi}{|\cdot|^\alpha})|x|^\alpha$, where g_1 was defined in Section 2. Following the same lines as in the proof of Corollary 2.1, the claim can be obtained from Proposition 2.1.

To prove the assertion for any index $\alpha > -1$, we split the operator into two parts as follows:

$$g^*(f) = \left\{\int_{t_0}^{\infty} \left|t^l \frac{\partial^l}{\partial t^l}(e^{-tL^{(\alpha)}} f)\right|^2 \frac{dt}{t}\right\}^{1/2} \text{ and } g_*(f) = \left\{\int_0^{t_0} \left|t^l \frac{\partial^l}{\partial t^l}(e^{-tL^{(\alpha)}} f)\right|^2 \frac{dt}{t}\right\}^{1/2},$$

where t_0 is a sufficiently large number to be chosen later. In the remaining part of the proof, we will need the following result, proved in [5]: there exist a small number $r_0 \in (0, r_0)$ and a constant $C = C(\alpha, r_0) > 0$ such that

$$\sup_{0<r\le r_0} \left|\frac{\partial}{\partial r}[h_{\ln\frac{1}{r}}(y,z)]\right| \le C r_0^{\frac{\alpha-1}{2}} y^{\alpha/2} z^{\alpha/2} e^{-(y+z)/8}, \ \forall\, y,\ z > 0. \tag{5.1}$$

We begin with the operator $g^*(f)$, and choose t_0 such that $e^{-t_0} = r_0$. By (5.1) we have

$$\begin{aligned} g^*(f)(y) &\leq \left\{\int_{t_0}^{\infty}\left[\int_{\mathbb{R}_+}\left|\frac{\partial}{\partial t}[h_t(y,z)]\right||f(z)|dz\right]^2 tdt\right\}^{1/2} \\ &\leq \int_{\mathbb{R}_+}\left\{\int_{t_0}^{\infty}\left|\frac{\partial}{\partial t}[h_t(y,z)]\right|^2 tdt\right\}^{1/2}|f(z)|dz \\ &= \int_{\mathbb{R}_+}\left\{\int_0^{r_0}\left|\frac{\partial}{\partial r}[h_{\ln\frac{1}{r}}(y,z)]\right|^2 r\ln r dr\right\}^{1/2}|f(z)|dz \\ &\leq C\int_{\mathbb{R}_+} y^{\alpha/2}z^{\alpha/2}e^{-(y+z)/8}|f(z)|dz. \end{aligned}$$

Hence,

$$\begin{aligned} \|g^*(f)\|_{L^p_{\delta,\gamma}} &\leq C\left[\int_{\mathbb{R}_+} e^{\frac{-py}{8}}y^{(\frac{\alpha}{2}+\delta)p}(1+y)^{\gamma p}dy\right]^{1/p} \\ &\quad\times\left[\int_{\mathbb{R}_+} e^{\frac{-p'z}{8}}z^{(\frac{\alpha}{2}-\delta)p'}(1+z)^{-\gamma p'}dz\right]^{1/p'}\|f\|_{L^p_{\delta,\gamma}}, \end{aligned}$$

and both integrals are finite since $-1/p-\alpha/2<\delta<1-1/p+\alpha/2$ and $\gamma\in\mathbb{R}$.

Now we turn to the operator g_*, which we need to write as a linear vector-valued operator in order to use transplantation. Let H denote the Hilbert space $L^2((0,\infty),\frac{dt}{t})$. Consider the mapping $G: L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+;H)$ defined by

$$G(f) = G^{(\alpha)}(f) = \left\{t\frac{\partial}{\partial t}(e^{-tL^{(\alpha)}}f)\right\}_{t>0}, \quad f\in L^2(\mathbb{R}_+).$$

Since $g(f) = |G(f)|_H$, the $L^p_{\delta,\gamma}$ boundedness of g is equivalent to the boundedness of G from $L^p_{\delta,\gamma}$ into $L^p_{\delta,\gamma}(\mathbb{R}_+;H)$. Likewise we define

$$G_*(f) = G_*^{(\alpha)} = \left\{t\frac{\partial}{\partial t}(e^{-tL^{(\alpha)}}f)\chi_{(0,t_0]}(t)\right\}_{t>0}.$$

Finally, we denote by $\overline{T^\alpha_\beta}$ the vector-valued extension of the transplantation operator to $L^2(\mathbb{R}_+;H)$, defined as follows

$$\overline{T^\alpha_\beta}(\{f_t\}_{t>0}) = \{T^\alpha_\beta(f_t)\}_{t>0}, \quad \{f_t\}_{t>0}\in L^2(\mathbb{R}_+;H).$$

By Krivine's theorem (see, e.g., [8]), the vector-valued operator $\overline{T^\alpha_\beta}$ is bounded in $L^p_{\delta,\gamma}(\mathbb{R}_+;H)$ if and only if T^α_β is bounded in $L^p_{\alpha,\gamma}(\mathbb{R}_+)$. Denote by $\bar{M}$ the vector-valued extension of the multiplier operator $Mf = \sum_{k\geq 0} m(k) < f, L^\beta_k > L^\beta_k$, where $m(s) = \frac{2s+\alpha+1}{2s+\beta+1}$. It is easy to see that this multiplier satisfies the conditions of Theorem 1.3..

Given $\alpha > -1$, we choose $\beta = \frac{n}{2} - 1$, for some positive integer n such that $\beta \geq \alpha$. It is known that (see [5], p. 272):

$$G_*^{(\alpha)} = \overline{T_\alpha^\beta} \circ \bar{N}_{\beta-\alpha} \circ \bar{M} \circ G^{(\beta)} \circ T_\beta^\alpha,$$

where

$$\overline{N}_{\beta-\alpha}(\{f_t\}_{t>0}) = \left\{e^{\frac{\beta-\alpha}{2}t}\chi_{(0,t_0]}(t)f_t\right\}_{t>0}, \ \{f_t\}_{t>0} \in L^2(\mathbb{R}_+; H).$$

Applying Theorems 1.1 and 1.3, we can obtain the boundedness of these operators in $L^p_{\delta,\gamma}$ or $L^p_{\delta,\gamma}(\mathbb{R}_+; H)$ when $-\frac{1}{p} - \frac{\alpha}{2} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$ and $\gamma \in \mathbb{R}$. Thus, Theorem 5.1 is proved for $l = 1$.

Now we proceed to prove the $L^p_{\delta,\gamma}$-boundedness of g_l when $l \geq 2$. Observe first that by the previous result we know the boundedness of $G : L^p_{\delta,\gamma} \to L^p_{\delta,\gamma}(\mathbb{R}_+; H)$, which by Krivine's theorem implies the boundedness of the vector-valued extension $\bar{G} : L^p_{\delta,\gamma}(H) \to L^p_{\delta,\gamma}(H \times H)$ given by

$$\{f_s\}_{s>0} \mapsto \{Gf_s\}_{s>0} = \left\{t\frac{\partial}{\partial t}[e^{-tL}f_s]\right\}_{(t,s)}.$$

Thus, we obtain the boundedness for the composition operator $\bar{G} \circ G : L^p_{\delta,\gamma}(H) \to L^p_{\delta,\gamma}(H \times H)$. Note that $|\bar{G} \circ Gf|^2_{H\times H} = \frac{1}{6}g_2(f)^2$ (see [5], p. 273). Combining all the above facts we obtain the desired estimate $\|g_2(f)\|_{\delta,\gamma} \leq C\|f\|_{\delta,\gamma}$. Similar arguments and induction yield the same conclusion for g_l for all $l \geq 1$. This completes the proof Theorem 5.1.

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POSITIVE SOLUTIONS FOR MULTI-POINT BOUNDARY VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the problem of existence of positive solution to the following boundary value problem: $D^{\sigma}_{0+}u''(t) - g(t)f(u(t)) = 0$, $t \in (0,1)$, $u''(0) = u''(1) = 0$, $au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, $cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, where D^{σ}_{0+} is the Riemann-Liouville fractional derivative of order $1 < \sigma \leq 2$ and f is a lower semi-continuous function. Using Krasnoselskii's fixed point theorems in a cone, the existence of one positive solution and multiple positive solutions for nonlinear singular boundary value problems is established.

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1. Introduction

The purpose of this paper is to study the problem of existence of positive solutions for the following m-point boundary value problem for fractional differential equation

$$\begin{cases} D^{\sigma}_{0+}u''(t) - g(t)f(u(t)) = 0, & t \in (0,1), \\ u''(0) = u''(1) = 0, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1.1}$$

where D^{σ}_{0+} is the Riemann-Liouville fractional derivative of order $1 < \sigma \leq 2$, $m > 2$ $(m \in \mathbb{N})$, $a, b, c, d \geq 0$, $\rho = ac + bc + ad > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0, +\infty)$ $(i = 1, 2, \ldots, m-2)$, $g \in C((0,1); [0, +\infty))$ and $0 < \int_0^1 g(r)dr < \infty$, and f is a nonnegative, lower semi-continuous function defined on $[0, +\infty)$.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and

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the applications of such constructions in various scientific fields, such as physics, mechanics, chemistry, engineering, etc. For details we refer to [5, 8, 9] and references therein.

The solution of differential equations of fractional order is much involved. Some analytical methods have been developed, such as the popular Laplace transform method [21, 22], the Fourier transform method [16], the iteration method [23], and Green function method [15, 24]. Numerical schemes for solving fractional differential equations also were introduced (see, e.g. [3, 4, 18]). A great deal of effort has been expended over the last years in attempting to find robust and stable numerical as well as analytical methods for solving fractional differential equations of physical interest. The Adomian decomposition method [20], homotopy perturbation method [19], homotopy analysis method [2], differential transformation method [17] and variational method [6] are relatively new approaches to provide analytical approximate solutions to linear and nonlinear fractional differential equations.

The problem of existence of solutions of initial value problems for fractional order differential equations have been studied in the literature (see [1, 11, 21, 23, 27] and the references therein).

In [13], Liu and Jia have investigated existence of multiple solutions for the problem:

$$\begin{cases} {}^cD^{\sigma}_{0+}(p(t)u'(t)) + q(t)f(t,u(t)) = 0, & t>0, \ \ 0<\sigma<1, \\ p(0)u'(0) = 0, \\ \lim_{t\to\infty} u(t) = \int_0^{+\infty} g(t)u(t)dt, \end{cases}$$

where ${}^cD^{\sigma}_{0+}$ stands for the standard Caputo's derivative of order σ. Some existence results for the problem (1.1) with $\sigma = 2$ were obtained by Yanga et al. [25] and Zhao et al. [28].

In [12], Liu has considered existence of positive solutions for the following generalized Sturm-Liouville four-point boundary value problem:

$$\begin{cases} u''(t) + g(t)f(u(t)) = \theta, & t\in(0,1), \\ au(0) - bu'(0) = a_1u(\xi_1), \\ cu(1) + du'(1) = b_1u(\xi_2), \end{cases}$$

by using the fixed points of strict-set-contractions.

In [26], Zhou and Chua have studied the following fractional differential equation with multi-point boundary conditions

$$\begin{cases} {}^cD^{\sigma}_{0+}u(t) = f(t, u(t), (Ku)(t), (Hu)(t)), \quad t \in (0,1), \\ au(0) - bu'(0) = d_1u(\xi_1), \\ cu(1) + du'(1) = d_2u(\xi_2), \end{cases}$$

where D^{σ}_{0+} is the Caputo's fractional derivative of order $1 < \sigma \leq 2$. By using the contraction mapping principle and the Krasnoselskii's fixed point theorem, the existence of solutions was established.

In this paper, motivated by the above-mentioned works, and using Krasnoselskii's fixed point theorems in a cone, we show that the problem (1.1) has positive solutions.

The remainder of the paper is organized as follows. In Section 2 we state some preliminary facts needed in the proofs of the main results. We also state a version of the Krasnoselakii's fixed point theorem. In Section 3, we state the main results of the paper, that establish existence of at least one or multiple positive solutions for the problem (1.1). Finally, in Section 4 we discuss an example that illustrates the main results of the paper.

2. Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in the proofs of the main results.

We work in the space $C([0,1])$ with the norm $||u|| = \max_{0\leq t\leq 1}|u(t)|$. We make the following assumptions:

(H1) $f \in C([0,+\infty);[0,+\infty))$;

(H1*) f is a nonnegative, lower semi-continuous function defined on $[0,+\infty)$, i.e., there exist $I \subset [0,+\infty)$ such that for all $x_n \in I$, $x_n \to x_0$ as $n \to \infty$, one has $f(x_0) \leq \underline{\lim}_{n\to\infty} f(x_n)$. Moreover, f has only a finite number of discontinuity points in each compact subinterval of $[0,+\infty)$.

(H2) $g \in C((0,1);[0,+\infty))$ and $0 < \int_0^1 g(r)dr < +\infty$. Moreover, $g(t)$ does not vanish identically on any subinterval of $[0,1]$;

(H3) $a,b,c,d \geq 0$, $\rho = ac + bc + ad > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0,+\infty)$ $(i = 1,2,\ldots,m-2)$, $\rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) > 0$ and $\Delta < 0$, where

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix},$$

and for $t \in [0,1]$

$$\psi(t) = b + at \quad \text{and} \quad \varphi(t) = c + d - ct \tag{2.1}$$

are linearly independent solutions of the equation $x''(t) = 0$, $t \in [0,1]$. Observe that ψ is non-decreasing on $[0,1]$ while φ is non-increasing on $[0,1]$.

Definition 2.1. *Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:*
(1) $x \in P, \mu \geq 0$ implies $\mu x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in L^1(\mathbb{R}^+)$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. *The Riemann-Liouville fractional derivative of order α ($n-1 < \alpha < n$, $n \in \mathbb{N}$) is defined as*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.1. *([7]). The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L^1(0,1)$.*

Lemma 2.2. *([7]). Let $\alpha > 0$. Then the differential equation*

$$D_{0+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $n-1 < \alpha \leq n$.

Lemma 2.3. *([7]). Let $\alpha > 0$. Then the following equality holds for $u \in L^1(0,1)$, $D_{0+}^{\alpha} u \in L^1(0,1)$;*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $n-1 < \alpha \leq n$.

Now we present the Green function for a boundary value problem involving fractional differential equation.

Observe first that for $y(t) = u''(t)$ the problem

$$\begin{cases} D^{\sigma}_{0+}u''(t) - g(t)f(u(t)) = 0, & t \in (0,1), \\ u''(0) = u''(1) = 0, \end{cases}$$

becomes into the problem

$$\begin{cases} D^{\sigma}_{0+}y(t) - g(t)f(u(t)) = 0, & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2.2}$$

Lemma 2.4. *If (H1) and (H2) are satisfied, then the boundary value problem (2.2) has a unique solution given by*

$$y(t) = -\int_0^1 H(t,s)g(s)f(u(s))ds, \tag{2.3}$$

where

$$H(t,s) = \begin{cases} \frac{t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \le s \le t \le 1, \\ \frac{t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Proof. According to Lemma 2.3 we can write

$$\begin{aligned} y(t) &= I^{\sigma}_{0+}\Big(g(t)f(u(t))\Big) - c_1 t^{\sigma-1} - c_2 t^{\sigma-2} \\ &= \tfrac{1}{\Gamma(\sigma)}\textstyle\int_0^t (t-s)^{\sigma-1} g(s)f(u(s))ds - c_1 t^{\sigma-1} - c_2 t^{\sigma-2}. \end{aligned}$$

Since $\sigma - 2 \le 0$, in view of the boundary condition $y(0) = 0$, we must set $c_2 = 0$ if $\sigma = 2$, and if $\sigma < 2$ then in order to have $c_2 t^{\sigma-2}$ well defined we must choose $c_2 = 0$. Also, using the boundary condition $y(1) = 0$ we must set $c_1 = \frac{1}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1} g(s)f(u(s))ds$.

Thus, the unique solution of problem (2.2) is given by

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}g(s)f(u(s))ds - \frac{t^{\sigma-1}}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1}g(s)f(u(s))ds \\ &= -\int_0^t \frac{t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1}}{\Gamma(\sigma)} g(s)f(u(s))ds - \int_t^1 \frac{t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)} g(s)f(u(s))ds \\ &= -\int_0^1 H(t,s)g(s)f(u(s))ds. \end{aligned}$$

This completes the proof.

Lemma 2.5. *If (H3) holds, then for $y \in C[0,1]$ the boundary value problem*

$$(2.5)\qquad \begin{cases} u''(t) = y(t), & t \in (0,1), \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

has a unique solution given by

$$(2.6)\qquad u(t) = -\Big[\int_0^1 G(t,s)y(s)ds + A(y(s))\psi(t) + B(y(s))\varphi(t)\Big],$$

where

$$(2.7)\qquad G(t,s) = \frac{1}{\rho}\begin{cases} \varphi(t)\psi(s), & 0 \le s \le t \le 1, \\ \varphi(s)\psi(t), & 0 \le t \le s \le 1, \end{cases}$$

$$(2.8)\qquad A(y(s)) = \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)y(s)ds & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)y(s)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix},$$

$$(2.9)\qquad B(y(s)) = \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)y(s)ds \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)y(s)ds \end{vmatrix}.$$

Proof. The proof is similar to that of Lemma 5.5.1 in [14], and it is omitted. □

We assume that $\theta \in (0, \frac{1}{2})$, and for convenience, we set

$$\Lambda_1 = \min\Big\{\frac{\varphi(1-\theta)}{\varphi(0)}, \frac{\psi(\theta)}{\psi(1)}\Big\}, \qquad \Gamma = \min\{\Lambda_1, \frac{\Lambda_2}{\Lambda_3}\},$$
$$\Lambda_2 = \min\{\min_{\theta \le t \le 1-\theta} \varphi(t), \min_{\theta \le t \le 1-\theta} \psi(t), 1\}, \qquad \Lambda_3 = \max\{1, ||\varphi||, ||\psi||\}.$$

Lemma 2.6. *Let $\rho, \Delta \neq 0$ and $\theta \in (0, \frac{1}{2})$, then the following inequalities hold:*

$$(2.10)\qquad 0 \le G(t,s) \le G(s,s), \quad \text{for } t,s \in [0,1],$$

and

$$(2.11)\qquad G(t,s) \ge \Lambda_1 G(s,s), \quad \text{for } t \in \Big[\theta, 1-\theta\Big] \text{ and } s \in [0,1].$$

Proof. The inequality (2.10) is obvious. So, we have to verify only the inequality (2.11). To this end, observe that for $t \in [\theta, 1-\theta]$ and $s \in [0,1]$ we have

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \begin{cases} \frac{\varphi(t)}{\varphi(s)}, & 0 \le s \le t \le 1-\theta, \\ \frac{\psi(t)}{\psi(s)}, & \theta \le t \le s \le 1, \end{cases} \\ &\ge \begin{cases} \frac{\varphi(1-\theta)}{\varphi(0)}, & 0 \le s \le t \le 1-\theta, \\ \frac{\psi(\theta)}{\psi(1)}, & \theta \le t \le s \le 1, \end{cases} \ge \Lambda_1. \end{aligned}$$

This completes the proof.

Proposition 2.1. *For $t, s \in [0,1]$ we have*

$$0 \le H(t,s) \le H(s,s) \le \frac{1}{\Gamma(\sigma)}\left(\frac{1}{4}\right)^{\sigma-1}.$$

Proposition 2.2. *Let $\theta \in (0, \frac{1}{2})$, then there exists a positive function $\varrho \in C(0,1)$ such that*

$$\min_{\theta \le t \le 1-\theta} H(t,s) \ge \varrho(s)H(s,s), \quad s \in (0,1).$$

Proof. For $\theta \in (0, \frac{1}{2})$ we define

$$\begin{aligned} g_1(t,s) &= t^{\sigma-1}(1-s)^{\sigma-1} - (t-s)^{\sigma-1}, \quad && 0 \le s \le t \le 1, \\ g_2(t,s) &= t^{\sigma-1}(1-s)^{\sigma-1}, && 0 \le t \le s \le 1. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d}{dt} g_1(t,s) &= (\sigma-1)\Big(t^{\sigma-2}(1-s)^{\sigma-1} - (t-s)^{\sigma-2}\Big) \\ &= (\sigma-1)t^{\sigma-2}\Big((1-s)^{\sigma-1} - (1-\frac{s}{t})^{\sigma-2}\Big) \\ &\le (\sigma-1)t^{\sigma-2}\Big((1-s)^{\sigma-1} - (1-s)^{\sigma-2}\Big), \end{aligned}$$

implying that $g_1(\cdot,s)$ is non-increasing for all $s \in (0,1]$. Also, taking into account that $g_2(\cdot,s)$ is non-decreasing for all $s \in (0,1)$, we can write

$$\begin{aligned} \min_{\theta \le t \le 1-\theta} H(t,s) &= \begin{cases} \frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, & s \in (0,\theta], \\ \min\{\frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, \frac{g_2(\theta,s)}{\Gamma(\sigma)}\}, & s \in [\theta, 1-\theta], \\ \frac{g_2(\theta,s)}{\Gamma(\sigma)}, & s \in [1-\theta, 1). \end{cases} \\ &= \begin{cases} \frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, & s \in (0,\mu], \\ \frac{g_2(\theta,s)}{\Gamma(\sigma)}, & s \in [\mu, 1). \end{cases} \\ &= \begin{cases} \frac{(1-\theta)^{\sigma-1}(1-s)^{\sigma-1} - (1-\theta-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in (0,\mu], \\ \frac{\theta^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in [\mu, 1), \end{cases} \end{aligned}$$

where $\theta < \mu < 1-\theta$ is a solution of the equation

$$(1-\theta)^{\sigma-1}(1-\mu)^{\sigma-1} - (1-\theta-\mu)^{\sigma-1} = \theta^{\sigma-1}(1-\mu)^{\sigma-1}.$$

It follows from the monotonicity of g_1 and g_2 that

$$\max_{0 \le t \le 1} H(t,s) = H(s,s) = \frac{s^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, \quad s \in (0,1).$$

Therefore, setting

$$\varrho(s) = \begin{cases} \frac{(1-\theta)^{\sigma-1}(1-s)^{\sigma-1} - (1-\theta-s)^{\sigma-1}}{s^{\sigma-1}(1-s)^{\sigma-1}}, & s \in (0,\mu], \\ \left(\frac{\theta}{s}\right)^{\sigma-1} & s \in [\mu, 1), \end{cases}$$

we complete the proof.

Remark 2.1. *It follows from Lemmas 2.4 and 2.5 that $u(t)$ is a solution of the problem (1.1) if and only if*

$$u(t) = \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t), \tag{2.12}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Lemma 2.7. *Let (H1), (H2) and (H3) be fulfilled. Then the solution u of the problem (1.1) satisfies the following conditions:*

(i) $u(t) \geq 0$ for $t \in [0,1]$,

(ii) $\min_{\theta \leq t \leq 1-\theta} u(t) \geq \Gamma\|u\|$.

Proof. (i) By Lemma 2.6, Proposition 2.1, formulas (2.3) and (2.6)-(2.9), we have

$$G(t,s) \geq 0, \quad W(s) \geq 0, \quad A(W(s)) \geq 0, \quad B(W(s)) \geq 0,$$

implying that $u(t) \geq 0$ for $t \in [0,1]$.

(ii) By Lemma 2.6 and formula (2.12) for $t \in [\theta, 1-\theta]$ we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)W(s)ds + \frac{\Lambda_2}{\Lambda_3} \cdot \Lambda_3[A(W(s)) + B(W(s))] \\ &\geq \Gamma\Big[\int_0^1 G(s,s)W(s)ds + \Lambda_3[A(W(s)) + B(W(s))]\Big] \\ &\geq \Gamma\|u\|. \end{aligned}$$

This implies $\min_{\theta \leq t \leq 1-\theta} u(t) \geq \Gamma\|u\|$. Lemma 2.7 is proved.

Next, for $\theta \in (0, \frac{1}{2})$ we choose a cone $K = K_\theta$ in $C^1([0,1])$ by setting

$$K = K_\theta = \{u \in C[0,1] \mid u(t) \geq 0, \min_{\theta \leq t \leq 1-\theta} u(t) \geq \Gamma\|u\|\},$$

and define an operator T by

$$(Tu)(t) = \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t), \tag{2.13}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

It is clear that the existence of a positive solution for the system (1.1) is equivalent to the existence of nontrivial fixed point of T in K.

Lemma 2.8. *Suppose that the conditions (H1) and (A1) hold, then $T(K) \subseteq K$ and $T : K \to K$ is completely continuous.*

Proof. By (2.13), for any $u \in K$ we have $(Tu)(t) \geq 0$, and for $t \in [0,1]$ we can write

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\leq \int_0^1 G(s,s)W(s)ds + \Lambda_3[A(W(s)) + B(W(s))].\end{aligned}$$

Thus,

$$||Tu|| \leq \int_0^1 G(s,s)W(s)ds + \Lambda_3[A(W(s)) + B(W(s))].$$

On the other hand for $t \in [\theta, 1-\theta]$ we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)W(s)ds + \frac{\Lambda_2}{\Lambda_3} \cdot \Lambda_3[A(W(s)) + B(W(s))] \\ &\geq \Gamma\Big[\int_0^1 G(s,s)W(s)ds + \Lambda_3[A(W(s)) + B(W(s))]\Big] \\ &\geq \Gamma||Tu||.\end{aligned}$$

This implies $TK \subseteq K$. Using standard arguments and Arzela-Ascoli theorem it can be easily verified that $T : K \to K$ is completely continuous, so we omit the details. Thus, Lemma 2.8 is proved.

As it was mentioned above, our approach to the existence of positive solutions for boundary value problems for fractional differential equations is based on the Krasnoselskii's fixed point theorems in a cone. For completeness of the presentation here we state the following Guo-Krasnoselskii fixed point theorem in a cone (see [10]).

Theorem 2.1. *Let E be a Banach space and $K \subseteqq E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \bigcap(\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator. Then under each of the following conditions the operator T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$:*

(A) $||Tu|| \leq ||u||$, $\forall u \in K \cap \partial\Omega_1$ *and* $||Tu|| \geq ||u||$, $\forall u \in K \cap \partial\Omega_2$;

(B) $||Tu|| \geq ||u||$, $\forall u \in K \cap \partial\Omega_1$ *and* $||Tu|| \leq ||u||$, $\forall u \in K \cap \partial\Omega_2$.

3. Main results

We define $\Omega_l = \{u \in K : \|u\| < l\}$ and $\partial\Omega_l = \{u \in K : \|u\| = l\}$, where $l > 0$. Observe that if $u \in \partial\Omega_l$ for $t \in [\theta, 1-\theta]$, then we have $\Gamma l \leq u \leq l$. Also, for convenience, we introduce the following notation:

$$\begin{aligned}
f_l &= \inf\left\{\frac{f(u)}{l}\Big| u \in [\Gamma l, l]\right\}, \qquad f^l = \sup\left\{\frac{f(u)}{l}\Big| u \in [0, l]\right\},\\
f_\varrho &= \liminf_{u\to\varrho}\frac{f(u)}{u}, \quad f^\varrho = \limsup_{u\to\varrho}\frac{f(u)}{u}; \quad (\varrho := 0^+ \text{ or } +\infty),\\
\eta &= \min_{\theta\leq s\leq 1-\theta}\varrho(s),\\
\frac{1}{\omega} &= \frac{1}{\Gamma(\sigma)}\left(\frac{1}{4}\right)^{(\sigma-1)}\left[\left(\int_0^1 G(s,s)ds\right)\left(\int_0^1 g(\tau)d\tau\right) + \Lambda_3\widetilde{A} + \Lambda_3\widetilde{B}\right],\\
\frac{1}{M} &= \frac{\eta}{\Gamma(\sigma)}\theta^{2(\sigma-1)}\left[\frac{\Lambda_1}{\rho}\varphi(1-\theta)\psi(\theta)\left(\int_\theta^{1-\theta} g(\tau)d\tau\right) + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}\right].
\end{aligned}$$

In the theorems that follow, we always assume that the assumption (H1) is fulfilled.

Theorem 3.1. *Suppose that there exist constants $r, R > 0$ with $r < \Gamma R$ for $r < R$, such that the following two conditions are satisfied:*

(H4) $f^r \leq \omega$,

(H5) $f_R \geq M$.

Then the problem (1.1) has at least one positive solution $u \in K$, such that

$$0 < r \leq \|u\| \leq R.$$

Proof. Case 1. We prove the result assuming that (H1) is satisfied. Also, without loss of generality, we can assume that $r < \Gamma R$ for $r < R$.

By (H4), Proposition 2.1, and formulas (2.8) and (2.9), for $u \in \Omega_r$ we have

$$\begin{aligned}
A(W) &\leq \frac{\left(\frac{1}{\Gamma(\sigma)}\right)\left(\frac{1}{4}\right)^{(\sigma-1)}\omega r}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)(\int_0^1 g(\tau)d\tau)ds & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)(\int_0^1 g(\tau)d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix},\\
&= \frac{1}{\Gamma(\sigma)}\left(\frac{1}{4}\right)^{(\sigma-1)}\omega r\widetilde{A}, \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
B(W) &\leq \frac{\left(\frac{1}{\Gamma(\sigma)}\right)\left(\frac{1}{4}\right)^{(\sigma-1)}\omega r}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)(\int_0^1 g(\tau)d\tau)ds \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)(\int_0^1 g(\tau)d\tau)ds\end{vmatrix}\\
&= \left(\frac{1}{\Gamma(\sigma)}\right)\left(\frac{1}{4}\right)^{(\sigma-1)}\omega r\widetilde{B}. \qquad (3.2)
\end{aligned}$$

Therefore, by (H4), Lemma 2.6, and formulas (2.13) – (3.2), for $t \in [0,1]$ and $u \in \Omega_r$ we can write

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t)\\
&\le \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\Big(\int_0^1 G(s,s)ds\Big)\Big(\int_0^1 g(\tau)d\tau\Big)\\
&\quad + \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\widetilde{A}\psi(t) + \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\widetilde{B}\varphi(t)\\
&\le \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\Big[\Big(\int_0^1 G(s,s)ds\Big)\Big(\int_0^1 g(\tau)d\tau\Big) + \Lambda_3\widetilde{A} + \Lambda_3\widetilde{B}\Big]\\
&= r = ||u||.
\end{aligned}$$

This implies that $||Tu|| \le ||u||$ for $u \in \Omega_r$.

On the other hand, by (H5), Proposition 2.2 and formulas (2.8), (2.9) and (2.13), for $u \in \Omega_R$ we have

$$\begin{aligned}
A(W) &\ge \frac{(\frac{\eta}{\Gamma(\sigma)})\theta^{2(\sigma-1)}MR}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_\theta^{1-\theta} G(\xi_i,s)(\int_\theta^{1-\theta} g(\tau)d\tau)ds & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2} b_i\int_\theta^{1-\theta} G(\xi_i,s)(\int_\theta^{1-\theta} g(\tau)d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix},\\
&= \Big(\frac{\eta}{\Gamma(\sigma)}\Big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR\widehat{A}, \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
B(W) &\ge \frac{(\frac{\eta}{\Gamma(\sigma)})\theta^{2(\sigma-1)}MR}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_\theta^{1-\theta} G(\xi_i,s)(\int_\theta^{1-\theta} g(\tau)d\tau)ds\\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_\theta^{1-\theta} G(\xi_i,s)(\int_\theta^{1-\theta} g(\tau)d\tau)ds\end{vmatrix}\\
&= \Big(\frac{\eta}{\Gamma(\sigma)}\Big)\theta^{2(\sigma-1)}MR\widehat{B}. \qquad (3.4)
\end{aligned}$$

Therefore, by (H5), Lemma 2.6 and formulas (2.13), (3.3) and (3.4), for $t \in [0,1]$ and $u \in \Omega_R$ we have

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t)\\
&\ge \frac{\eta}{\Gamma(\sigma)}\theta^{2(\sigma-1)}MR\Big[\frac{\Lambda_1}{\rho}\varphi(1-\theta)\psi(\theta)\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big) + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}\Big]\\
&= R = ||u||.
\end{aligned}$$

This implies that $||Tu|| \ge ||u||$ for $u \in \Omega_R$.

Therefore, by Theorem 2.1, it follows that T has a fixed point u in $K \cap (\overline{\Omega_R} \setminus \Omega_r)$. This means that the problem (1.1) has at least one positive solution $u \in K$ satisfying $0 < r \le ||u|| \le R$.

Case 2. When (H1*) holds, by applying the linear approaching method on the domain of discontinuous points of f we can construct a sequence $\{f_j\}_{j=1}^{\infty}$ satisfying the following two conditions

(i) $f_j \in C[0,\infty)$ and $0 \le f_j \le f_{j+1}$ on $[0,\infty)$,

(ii) $\lim_{j\to\infty} f_j = f$, $j = 1,2,\dots$, is pointwisely convergent on $[0,\infty)$.

According to the Case 1, for $f = f_j$ the problem (1.1) has a positive solution $u_j(t)$ given by

$$\begin{aligned} u_j(t) &= \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds \\ &+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix} \\ &+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds \end{vmatrix} \\ &= \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds + \psi(t)A_j + \varphi(t)B_j, \end{aligned}$$

for all $t \in [0,1]$ and $r \le ||u_j|| \le R$, where r and R are independent of j.

By uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$, and $\varphi(t)$, $\psi(t)$ on $[0,1]$, for any small enough $\epsilon > 0$ there exists $\delta > 0$ such that for $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$, one has $|G(t_1,s) - G(t_2,s)| < \epsilon$, $|\varphi(t_1) - \varphi(t_2)| < \epsilon$ and $|\psi(t_1) - \psi(t_2)| < \epsilon$. Thus, for $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$ we can write

$$\begin{aligned} |u_j(t_1) - u_j(t_2)| &\le \int_0^1 |G(t_1,s) - G(t_2,s)|\cdot\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds \\ &\quad +A_j|\psi(t_1)-\psi(t_2)| + B_j|\varphi(t_1)-\varphi(t_2)| \\ &\le \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\cdot\max_{||u_j||\le R} f_j(u_j)\cdot\Big(\int_0^1 g(\tau)d\tau\Big)\cdot\epsilon + A_j\cdot\epsilon + B_j\cdot\epsilon. \end{aligned}$$

Thus, $\{u_j\}_{j=1}^{\infty}$ are equicontinuous on $[0,1]$, and hence by Arzela-Ascoli theorem there exists a convergent subsequence of $\{u_j\}_{j=1}^{\infty}$. For convenience, we denote this convergent subsequence by $\{u_j\}_{j=1}^{\infty}$, and without loss of generality, we assume that $\lim_{j\to\infty} u_j(t) = u(t)$, $\forall t \in [0,1]$, and $r \le ||u|| \le R$. By Fatou's Lemma and Lebesgue dominated convergence theorem we have

$$\lim_{j\to\infty} u_j(t) \geq \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau\Big)ds$$
$$+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}$$
$$+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds\end{vmatrix},$$

implying

$$u(t) \geq \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t), \tag{3.5}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

On the other hand, by the conditions (i) and (ii) we have

$$u_j(t) \leq \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau\Big)ds$$
$$+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}$$
$$+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u(\tau))d\tau)ds\end{vmatrix}.$$

By the lower semi-continuity of f, we can pass to the limit in the above inequality as $j\to\infty$ to obtain

$$u(t) \leq \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds$$
$$+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}$$
$$+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds\end{vmatrix}.$$

Therefore

$$u(t) \leq \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t), \tag{3.6}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Finally, by (3.5) and (3.6) we obtain

$$u(t) = \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t),$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Therefore $u(t)$ is a positive solution of the problem (1.1). This completes the proof of Theorem 3.1.

Similarly, we can prove the following theorem.

Theorem 3.2. *Assume that there exist constants $r, R > 0$ with $r < \Gamma R$ for $r < R$, such that the following two conditions are satisfied:*

(H4)* $f^r < \omega$,

(H5)* $f_R > M$.

Then the problem (1.1) has at least one positive solution $u \in K$ such that

$$0 < r < \|u\| < R.$$

Theorem 3.3. *Assume that one of the following two conditions is satisfied:*

(H6) $f^0 \leq \omega$, $\quad f_\infty \geq \frac{M}{\Gamma}$,

(H7) $f_0 \geq \frac{M}{\Gamma}$, $\quad f^\infty \leq \omega$

Then the problem (1.1) has at least one positive solution.

Proof. It is enough to prove the assertion of the theorem for nonnegative and continuous on $[0,\infty)$ functions. Then using the arguments of the proof of Theorem 3.1 we can extend the result to the case of nonnegative and lower semi-continuous on $[0,\infty)$ functions.

We show that (H6) implies (H4) and (H5). Suppose that (H6) holds, then there exist r and R with $0 < r < \gamma R$, such that

$$\frac{f(u)}{u} \leq \omega, \quad 0 < u \leq r \qquad \text{and} \qquad \frac{f(u)}{u} \geq \frac{M}{\Gamma}, \quad u \geq \Gamma R.$$

Hence

$$f(u) \leq \omega u \leq \omega r, \quad 0 < u \leq r$$

and

$$f(u) \geq \frac{M}{\Gamma}u \geq \frac{M}{\Gamma}\Gamma R = MR, \quad u \geq \Gamma R,$$

implying (H4) and (H5). Therefore, by Theorem 3.1 the problem (1.1) has at least one positive solution.

Now suppose that (H7) holds, then there exist $0 < r < R$ with $Mr < \omega R$ such that

$$\frac{f(u)}{u} \geq \frac{M}{\Gamma}, \quad 0 < u \leq r. \tag{3.7}$$

and

$$\frac{f(u)}{u} \leq \omega, \quad u \geq R. \tag{3.8}$$

By (3.7), it follows that

$$f(u) \geq \frac{M}{\Gamma}u \geq \frac{M}{\Gamma}\Gamma r = Mr, \quad \Gamma r \leq u \leq r.$$

So, the condition (H5) holds for $r > 0$. As for (H4), we consider two cases.

(i) If $f(u)$ is bounded, then there exists a constant $D > 0$ such that $f(u) \leq D$ for $0 \leq u < \infty$. By (3.8) there exists a constant $\lambda \geq R$ with $Mr < \omega R \leq \lambda\omega$ satisfying $\lambda \geq \max\{R, \frac{D}{\omega}\}$, such that $f(u) \leq D \leq \lambda\omega$ for $0 \leq u \leq \lambda$, implying (H4).

(ii) If $f(u)$ is unbounded, then there exist $\lambda_1 \geq R$ with $Mr < \omega R \leq \lambda_1\omega$ such that $f(u) \leq f(\lambda_1)$ for $0 \leq u \leq \lambda_1$. This yields $f(u) \leq f(\lambda_1) \leq \lambda_1\omega$ for $0 \leq u \leq \lambda_1$. Thus, condition (H4) holds for λ_1.

Therefore, by Theorem 3.1, the problem (1.1) has at least one positive solution. Theorem 3.3 is proved.

Remark 3.1. *It is easy to see that the assertion of Theorem 3.3 remains valid under each of the following conditions: either $f^0 = 0$ and $f_\infty = +\infty$ or $f_0 = +\infty$ and $f^\infty = 0$.*

Now we are going to give some conclusions about the existence of multiple positive solutions. In the theorems that follow we assume that the assumptions (H1*), (H2) and (H3) are fulfilled.

Theorem 3.4. *Assume that one of the following conditions is satisfied:*

(H8) $f^r < \omega$,

(H9) $f_0 \geq \frac{M}{\Gamma}$ *and* $f_\infty \geq \frac{M}{\Gamma}$.

Then the problem (1.1) has at least two positive solutions satisfying

$$0 < ||u_1|| < r < ||u_2||.$$

Proof. By the proof of Theorem 3.3, we can take $0 < r_1 < r < \Gamma r_2$ such that $f(u) \geq r_1 M$ for $\Gamma r_1 \leq u \leq r_1$ and $f(u) \geq r_2 M$ for $\Gamma r_2 \leq u \leq r_2$. Therefore, by Theorems 3.2 and 3.3, it follows that problem (1.1) has at least two positive solutions satisfying $0 < ||u_1|| < r < ||u_2||$. □

Theorem 3.5. *Assume that one of the following conditions is satisfied:*

(H10) $f_R > M$,

(H11) $f^0 \leq \omega$ *and* $f^\infty \leq \omega$.

Then the problem (1.1) has at least two positive solutions satisfying

$$0 < \|u_1\| < R < \|u_2\|.$$

Theorem 3.6. *Assume that (H6) (or (H7)) holds, and there exist constants* $r_1, r_2 > 0$ *with* $r_1 M < r_2 \omega$ *(or* $r_1 < \Gamma r_2$*) such that (H8) holds for* $r = r_2$ *(or* $r = r_1$*) and (H10) holds for* $R = r_1$ *(or* $R = r_2$*). Then the problem (1.1) has at least three positive solutions satisfying*

$$0 < \|u_1\| < r_1 < \|u_2\| < r_2 < \|u_3\|.$$

Theorem 3.7. *Let* $n = 2k+1$, $k \in \mathbb{N}$. *Assume (H6) (or (H7)) holds. If there exist constants* $r_1, r_2, \dots, r_{n-1} > 0$ *with* $r_{2i} < \Gamma r_{2i+1}$, *for* $1 \leq i \leq k-1$ *and* $r_{2i-1} M < r_{2i}\omega$ *for* $1 \leq i \leq k$ *(or with* $r_{2i-1} < \Gamma r_{2i}$, *for* $1 \leq i \leq k$ *and* $r_{2i} M < r_{2i+1}\omega$ *for* $1 \leq i \leq k-1$*) such that (H10) (or (H8)) holds for* r_{2i-1}, $1 \leq i \leq k$ *and (H8) (or (H10)) holds for* r_{2i}, $1 \leq i \leq k$. *Then the problem (1.1) has at least* n *positive solutions* $u_1, \dots, u_n$ *satisfying*

$$0 < \|u_1\| < r_1 < \|u_2\| < r_2 < \cdots < \|u_{n-1}\| < r_{n-1} < \|u_n\|.$$

The proofs of Theorems 3.5 - 3.7 are similar to that of Theorem 3.4, and so are omitted.

4. An Example.

In this section we discuss an example that illustrates the main results of the paper.

Example. Consider the following singular boundary value problem

$$\begin{cases} D_{0+}^{\frac{3}{2}}(u''(t)) - t^{-\frac{1}{2}} f(u(t)) = 0, & t \in (0,1), \\ u''(0) = u''(1) = 0, \\ u(0) - u'(0) = \frac{1}{2}u(\frac{1}{2}), \\ u(1) + u'(1) = \frac{1}{2}u(\frac{1}{2}), \end{cases} \tag{4.1}$$

where

$$f(u) = \begin{cases} \frac{e^{-u}}{3}, & 0 \leq u \leq 10, \\ (n+1)e^{-u}, & n < u \leq n+1, \quad n = 10, 11, \dots, 20, \\ e^{\sqrt{u}}, & u > 21. \end{cases}$$

We note that

$$a=b=c=d=1,\quad \rho=3,\quad m=3,\quad \xi_1=\frac{1}{2},\quad \sigma=\frac{3}{2},$$
$$a_1=b_1=\frac{1}{2},\quad f_0=+\infty,\quad f_\infty=+\infty,\quad \Delta=-\frac{9}{2},\quad g(t)=t^{-\frac{1}{2}}.$$

Let $\theta=\frac{1}{3}$, then we have

$$\Lambda_1=\frac{2}{3}=1,\quad \Lambda_2=1,\quad \Lambda_3=2,\quad \Gamma=\frac{1}{2},$$
$$\omega=\frac{9\pi}{131},\quad M=\frac{729\pi}{944(3-2\sqrt{2})\eta^2},$$

where $\eta=\min_{\frac{1}{3}\le s\le\frac{2}{3}} H(s,s)$.

By calculating, we can let $\mu=\frac{2\sqrt{2}-1}{2\sqrt{2}}$. So, $f_\infty>\frac{M}{\Gamma}$ and $f_0>\frac{M}{\Gamma}$. Choosing $r=10$, we get

$$f^r=\sup\left\{\frac{f(u)}{r}\Big|u\in[0,r]\right\}=0.105409<0.2157519=\omega,$$

showing that (H8) and (H9) are fulfilled. It is easy to see that (H1*), (H2) and (H3) are satisfied as well. So, we can apply Theorem 3.4 to conclude that the problem (4.1) has at least two positive solutions $u_1,u_2\in K$ satisfying $0<\|u_1\|<4<\|u_2\|$.

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