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Настоящий номер журнала продолжает публикацию (начало в № № 4 – 6 за 2003) оригинальных статей, представленных на международной конференции “МАТЕМАТИКА в АРМЕНИИ: Достижения и перспективы” сентябрь 30 - октябрь 7, 2003, Цахкадзор, Армения. Основной целью конференции, посвящённой 60 летию Армянской Национальной Академии Наук, было обсуждение текущей работы и перспективы математики в Армении. Конференция привлекла внимание многих участников из-за границы, включая многих математиков из армянской диаспоры. Научная программа конференции охватила следующие направления современной математики: комплексный анализ, вещественный анализ, теория аппроксимации, теория вероятностей и математическая статистика, дифференциальные и интегральные уравнения, математическая физика, алгебра, геометрия, топология.

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THE CAUCHY INTEGRAL AND CERTAIN OF ITS APPLICATIONS

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§INTRODUCTION

On the occasion of the 60th anniversary of the National Academy of Sciences I am happy to recall a number of interests I have shared with Armenian mathematicians over many years. Among those whose work had a particular influence on me as a young mathematician are A. L. Shahinian, M. M. Djrbashian, S. N. Mergelian and S. O. Sinanian. A common feature of our individual efforts was the use of the Cauchy integral in problems of approximation theory. The various aspects of the subject to which I was especially drawn can be roughly classified as follows :

- Invariant subspaces and subnormal operators,
- The L^p -approximation by polynomials and rational functions,
- The uniqueness property of analytic functions on sets without interior points,
- The L^p -integrability of the derivative in conformal mapping.

It is my intention to summarize some of what has been achieved on the first three topics and to indicate the manner in which the Cauchy integral enters in different situations. In the process I shall also indicate the contributions of Armenian mathematicians in these areas of approximation theory ; and, I will draw on the work of Beurling and Vol'berg on **general quasianalyticity**, the work of Maz'ja and Havin on **nonlinear potential theory**, as well as more recent work of Mel'nikov and Tolsa on **analytic capacity**. As to the fourth and final topic, at least one very important question concerning the L^p -integrability of the derivative of a conformal mapping initially arose in connection with a problem in approximation theory studied by Shahinian and Djrbashian, and in this way can be linked to questions pertaining to the growth of the Cauchy integral.

The integrability question itself remains open to this day, and is currently being widely and intensively studied. As a result, it has now been shown to have significant

connections with open questions in such diverse areas as geometric function theory, hyperbolic geometry in $\mathbb{R}^3$, and mathematical physics. Here I will limit my remarks to a brief description of the history and current state of the problem.

§1. THE INVARIANT SUBSPACE PROBLEM

A bounded linear operator T on an infinite dimensional Hilbert space H is **subnormal**, if it has a normal extension; or equivalently, if T is the restriction of a normal operator to a closed invariant subspace. The study of such operators was begun by Halmos [40] in 1950, and quickly became a catalyst for increased activity at the interface between operator theory and the theory of analytic functions. The fundamental problem here is to determine whether or not an arbitrary subnormal operator T admits a nontrivial closed invariant subspace.

Evidently, it can be assumed from the outset that T has a cyclic vector; that is, a vector x for which the linear span of $x, Tx, T^2x, \dots$ is dense in the underlying space H . Otherwise, invariant subspaces are plentiful and there is nothing to prove. Under these circumstances the spectral theorem guarantees that there is a positive measure μ carried on the spectrum of T such that the given operator is unitarily equivalent to multiplication by the complex identity function z on $H^2(d\mu)$, where $H^2(d\mu)$ is the closure of the polynomials in $L^2(d\mu)$. Thus, the study of subnormal operators leads directly to questions concerning approximation by polynomials in $L^2(d\mu)$ (cf. Bram [15], pp. 83-86).

In this setting there are at least two ways in which invariant subspaces can arise: If $H^2(d\mu) = L^2(d\mu)$ and if X is any subset of the support of μ with $0 < \mu(X) < \|\mu\|$ then the collection

$$S = \{f \in H^2(d\mu) : f = 0 \text{ a.e. } - d\mu \text{ on } X\}$$

is a nontrivial closed subspace invariant under multiplication by z . If, on the other hand, $H^2(d\mu) \neq L^2(d\mu)$ it may happen that there is a point ξ somewhere in the complex plane $\mathbb{C}$ such that the map $P \rightarrow P(\xi)$ can be extended from the polynomials to a bounded linear functional on $H^2(d\mu)$, that is,

$$|P(\xi)| \leq C\|P\|_{L^2(d\mu)} \quad (1.1)$$

for every polynomial P and some absolute constant C . Such a point ξ is called a **bounded point evaluation** or BPE for $H^2(d\mu)$. If we let S be the closure in $H^2(d\mu)$ of the set of polynomials vanishing at ξ we again obtain a nontrivial invariant subspace, since $(z - \xi) \in S$ and $1 \notin S$.

In order, therefore, to settle the invariant subspace question for subnormal operators in the affirmative it is sufficient to prove that if μ is a positive Borel measure in $\mathbb{C}$, not concentrated at a single point, then either

(1) $H^2(d\mu)$ has a BPE, or

(2) $H^2(d\mu) = L^2(d\mu)$.

Moreover, initial confirmation that the suggested dichotomy is valid for a large class of measures came quickly. In 1955 Wermer [96] verified it for all measures μ whose closed support X has two-dimensional Lebesgue measure (i.e. dA measure) zero. In his reasoning he made use of a theorem of Hartogs and Rosenthal [41] on uniform rational approximation, but in essence it is this : Let g be any function in $L^2(d\mu)$ which is orthogonal to the polynomials in the sense that $\int P g d\mu = 0$ for every polynomial P , and form the Cauchy integral

$$\widehat{g\mu}(\xi) = \int \frac{g(z)}{z - \xi} d\mu(z)$$

which converges absolutely a.e. dA in $\mathbb{C}$. By an argument due essentially to Cauchy

$$P(\xi) = \frac{1}{\widehat{g\mu}(\xi)} \int \frac{P(z)g(z)}{z - \xi} d\mu(z) \quad (1.2)$$

at every point $\xi \in \mathbb{C}$ where $\widehat{g\mu}(\xi)$ is defined and $\widehat{g\mu}(\xi) \neq 0$. In particular, if $\xi \in \mathbb{C} \setminus X$ and $\widehat{g\mu}(\xi) \neq 0$ then (1.2) holds and, since the kernel $(z - \xi)^{-1}$ is bounded on X ,

$$|P(\xi)| \leq C \int |P| |g| d\mu$$

for all polynomials P and a suitable constant C . Hence, the inequality (1.1) is also satisfied and $H^2(d\mu)$ has a BPE at ξ . If, therefore, $H^2(d\mu)$ has no BPE's it follows that $\widehat{g\mu} \equiv 0$ in $\mathbb{C} \setminus X$; that is, $\widehat{g\mu} = 0$ a.e. $-dA$ in $\mathbb{C}$, since X has area zero. Because $\bar{\partial}(\widehat{g\mu}) = -\pi g\mu$ as a distribution, $g\mu = 0$ as a measure, and so $H^2(d\mu) = L^2(d\mu)$.

Subsequent efforts to extend this line of reasoning to absolutely continuous measures $w dA$ would yield clues eventually leading to a complete solution of the bounded point evaluation problem. In a very real sense the essential difficulties can all be found here. Building on ideas introduced in [21] in order to confirm the BPE conjecture for $H^2(w dA)$ when $w \in L^{1+\epsilon}(dA)$, Thomson [86] overcame substantial technical difficulties to achieve a final resolution. His result in this :

Theorem 1 (1991). *If μ is a positive measure of compact support in $\mathbb{C}$, not concentrated at a single point, then $H^2(d\mu) = L^2(d\mu)$ if and only if $H^2(d\mu)$ has no BPE's.*

Thomson's argument, however, is quite complicated incorporating ideas and techniques from a variety of disciplines. On the other hand, it is now possible to give a more direct proof which is less obscured by peripheral matters, and which can be more easily adapted to other problems as well (cf. [25]). Our goal here is to take a retrospective look at and to indicate the current disposition of some of these problems.

§2. APPROXIMATION THEORY

Questions concerning approximation by analytic functions have a long history stretching back over a century to the seminal works of Weierstrass [95] and Runge [75]. In its most general form the central issue is this : **Given a subset $X \subset \mathbb{C}$, a Banach space B of functions defined on X and a subfamily $\mathcal{F} \subset B$ of functions analytic in a neighborhood of X , is $\mathcal{F}$ dense in B ?** The difficulties can be quite varied depending on the choice of X , the space B and the family $\mathcal{F}$. Of particular interest here is the situation that arises when $\mathcal{F}$ is the set of all complex analytic polynomials, and either :

- (1) X is compact and $B = A(X)$ is the space of all functions continuous on X , analytic in its interior, and endowed with the uniform norm.
- (2) $X = \Omega$ is a bounded simply connected domain and $B = L^p_a(\Omega, dA)$, $1 \leq p < \infty$, is the set of all functions in $L^p(\Omega, dA)$ which are analytic in Ω .

In the case of uniform polynomial approximation there is an obvious obstacle. Suppose, for example, that X is compact and separates the plane, so that $\mathbb{C} \setminus X$ has at least one bounded component G . Then any sequence of polynomials which converges uniformly on X necessarily converges uniformly on $X \cup G$ as well. In particular, if $a \in G$ then $(z - a)^{-1} \in A(X)$, but $(z - a)^{-1}$ cannot be uniformly approximated on X by polynomials. Moreover, by a theorem of Mergelian (cf. [64] or [38], p. 48) this is the only obstacle to the density of the polynomials in $A(X)$.

While the polynomial approximation question for $A(X)$ can be settled in a purely topological way, the same cannot be said for approximation in the $L^p(\Omega, dA)$ -norm. In this setting the **completeness problem** was first examined by Carleman [26] in 1923, at which time he succeeded in proving that the polynomials are dense in $L^p_a(\Omega, dA)$ for all p , $1 \leq p < \infty$ whenever Ω is a Jordan domain. A decade later Markuševič [58] and Farrell [36] obtained, independently, the corresponding theorem for Carathéodory domains. A **Carathéodory domain** is by definition a domain Ω whose boundary coincides with the boundary of Ω_∞ , the unbounded complementary component of its closure; that is, $\partial\Omega = \partial\Omega_\infty$. And so, by the mid 1930's this is where the matter stood. There was no discernable evidence to suggest that there might be an intrinsic dissimilarity between completeness criteria for the uniform and integral

metrics, insofar as the polynomials were concerned.

To the contrary, experience seemed to indicate that L^p -completeness of the polynomials can never occur within the class of non-Carathéodory domains, a class of regions for which the following are typical examples :

(A) **the crescent** ; that is, a region topologically equivalent to one bounded by two internally tangent circles.

(B) **a domain with boundary cuts** ; that is, a domain with cuts or incisions in the form of simple arcs extending from the interior to the boundary.

In both instances, however, intuition would prove to be misleading. In 1939 Keldysh [52] made the initial discovery : for a crescent Ω the polynomials may or may not be dense in $L^p_a(\Omega, dA)$ depending on the **thickness, or metric properties**, of Ω near the multiple boundary point. Not until 1947-48, and with rather strong regularity restrictions placed on $\partial\Omega$, was a condition found that is both necessary and sufficient for completeness in this context. That was due to the combined efforts of M. M. Djrbashian [35], who established sufficiency, and A. L. Shahinian [76], who established necessity. No essentially new results were obtained for another twenty years until the subject was once again taken up in earnest, this time by Havin and Maz'ja [59] and the author [17]. Nevertheless, the initial achievements of Djrbashian and Shahinian can be viewed as genuine precursors to much subsequent work.

(A) **The Crescent**. By a **crescent** we shall mean a region Ω whose closure $\bar{\Omega}$ is a compact connected set having two complementary components : a bound component G and an unbounded component Ω_∞ , with $\partial G \cap \partial\Omega_\infty \neq \emptyset$. We do not require that $\partial G \cap \partial\Omega_\infty$ be a singleton as was the case in the early work of Keldysh, Djrbashian and Shahinian (cf. [65]). The theorem of the latter two authors is this :

Theorem 2 (1948). *Let Ω be a crescent with a single multiple boundary point at the origin such that Ω is situated between the two circles $|z - 1| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$. Denote by $l(r)$ the linear measure of $(|z| = r) \cap \Omega$, and assume that $r \frac{l'(r)}{l(r)} \uparrow +\infty$ as $r \downarrow 0$. Then, in order for the polynomials to be dense in $L^2_a(\Omega, dA)$ it is necessary and sufficient that*

$$\int_0 \log l(r) dr = -\infty. \quad (2.1)$$

By requiring Ω to lie between the two circles $|z - 1| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ one precludes the possibility of a cusp at the multiple boundary point. This fact plays a key role in the proof of the theorem, and cannot be omitted (cf. [19], p. 142). In order to address the completeness question for the most general regions of crescent type it is essential

that (2.1) be replaced with a more suitable criterion; by a criterion which, not only measures the thickness of a region in the neighborhood of a boundary point, but is also sensitive to the underlying geometry as well. Theorems 3 and 4 below offer one possible alternative.

Before preceeding in that direction, however, it should be noted that Theorem 2 is actually valid in a somewhat wider context. For each p , $1 \leq p < \infty$, and consistent with notation introduced earlier $H^p(\Omega, dA)$ will stand for the closure of the polynomials in $L^p_a(\Omega, dA)$. Under the various assumptions of Theorem 2 the argument of Djrbashian and Shahinian can be easily modified to yield the more general conclusion:

In order that $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ for any p it is necessary and sufficient that

$$\int_0^\infty \log l(r) dr = -\infty.$$

And, since the intervening integral does not depend on p , it follows, here at least, that $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ if and only if $H^1(\Omega, dA) = L^1_a(\Omega, dA)$. We shall not, therefore, limit our discussion solely to the case $p = 2$.

Let Ω be a crescent whose closure has two complementary components G and Ω_∞ as previously indicated; that is, $(\mathbb{C} \setminus \bar{\Omega}) = (G \cup \Omega_\infty)$. For each $z \in \mathbb{C}$ let $\delta(z)$ be the Euclidean distance from z to Ω_∞ . Here are two theorems which owe a considerable debt to the early efforts of Djrbashian and Shahinian, and which are indicative of what is presently known (cf. [17] and [19]).

Theorem 3 (1973). Suppose that ∂G is a C^1 curve whose exterior unit normal n satisfies a Lipschitz condition $|n(z_1) - n(z_2)| \leq C|z_1 - z_2|$ for all $z_1, z_2 \in \partial G$. Then in order that $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ for any p it is necessary and sufficient that

$$\int_{\partial G} \log \delta(z) |dz| = -\infty. \quad (2.2)$$

Interestingly, the pertinent density criterion (2.2) is once again independent of p , and therefore if $H^1(\Omega, dA) = L^1_a(\Omega, dA)$ then $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ for all p . But, it is still not known if this is also the case in the absence of all smoothness restrictions on ∂G . In full generality, however, we have the following:

Theorem 4 (1978). Let Ω be an arbitrary crescent, and let ω be harmonic measure on ∂G relative to some fixed point $x_0 \in G$. Then there exists a universal constant $\tau > 0$ such that if

$$\int_{\partial G} \log \delta(z) d\omega(z) = -\infty, \quad (2.3)$$

then $H^p(\Omega, dA) = L_a^p(\Omega, dA)$ whenever $1 \leq p < 3 + \tau$.

Between 1968 and 1973 results similar to these, but rather more restrictive, were also obtained by Havin and Maz'ja [59], [61]. A characteristic feature of their work was the reduction via duality of questions on approximation in $L_a^p(\Omega)$, $1 < p < \infty$, to corresponding questions in the Sobolev space $W_1^q(\Omega)$, where $q = p/(p-1)$ is the index conjugate to p and $W_1^q(\Omega)$ consists of those functions in $L^q(\Omega)$ whose first order partial derivatives (taken in the sense of distribution theory) also belong to $L^q(\Omega)$. Here we have suppressed explicit reference to the underlying measure which, unless otherwise stated, is assumed to be Lebesgue measure dA . The actual transition is achieved by means of the Cauchy integral.

Suppose, for example, that $1 < p < \infty$ and that we wish to prove $H^p(\Omega) = L_a^p(\Omega)$ for a given bounded open set Ω . One way to proceed in this : Let $k \in L^q(\Omega)$ be any function with the property that $\int Pk dA = 0$ for all polynomials P , and form the Cauchy integral

$$\hat{k}(z) = \int_{\Omega} \frac{k(\zeta)}{\zeta - z} dA_{\zeta}.$$

By assumption $\hat{k} \equiv 0$ in Ω_{∞} . Since $q > 1$ and $\hat{k}$ has compact support, it follows from a theorem of Calderón and Zygmund (cf. [84], p. 35) that

$$\|\text{grad } \hat{k}\|_q \leq C \|\bar{\partial} \hat{k}\|_q \leq C\pi \|k\|_q. \quad (2.4)$$

Thus, $\hat{k} \in W_1^q(\mathbb{R}^2)$ and as such $\hat{k}$ enjoys a certain degree of continuity, which for $1 < q \leq 2$ is best described in terms of capacity. If $q > 2$, then $\hat{k}$ is actually Hölder continuous. In either case, however, the key idea is to use the available information to show that $\hat{k} \equiv 0$ in the full complement $\mathbb{C} \setminus \bar{\Omega}$, and then to conclude by continuity that $\hat{k} = 0$ on a sufficiently large portion of $\partial\Omega$ so that $\hat{k} \in W_1^q(\Omega)$, that is, so that there exists a sequence of C^{∞} -functions η_j , $j = 1, 2, \dots$, with support in Ω such that $\eta_j \rightarrow \hat{k}$ in the norm of $W_1^q(\Omega)$. In particular, so that

$$\int_{\Omega} |\bar{\partial} \eta_j - k|^q dA \rightarrow 0, \quad j \rightarrow \infty.$$

If $F \in L_a^p(\Omega)$ we see, after an integration by parts, that

$$\int_{\Omega} F \bar{\partial} \eta_j dA = - \int_{\Omega} \bar{\partial} F \eta_j dA = 0, \quad j = 1, 2, \dots$$

And, because

$$\lim_{j \rightarrow \infty} \int_{\Omega} F \bar{\partial} \eta_j dA = \int_{\Omega} F k dA,$$

we have $\int F k dA = 0$, from which we can conclude that $F \in H^p(\Omega)$. Therefore, $H^p(\Omega) = L_a^p(\Omega)$.

This argument depends in an essential way on several ideas from **nonlinear potential theory**, a theory whose origins lie in the works of Havin and Maz'ja [60] and Meyers [69]. For each q , $1 < q \leq 2$, there is a set function γ_q , called q -**capacity**, which accurately measures the exceptional sets associated with W_1^q functions. A function $h \in W_1^q$ is said to be q -**finely continuous** at a point x_0 if there exists a set E which is **thin**, or **sparse**, in a potential theoretic sense at x_0 and

$$\lim_{x \rightarrow x_0, x \in \mathbb{C} \setminus E} h(x) = h(x_0).$$

The precise sense in which E is understood to be thin is expressed in terms of q -capacity (cf. [1], p. 176), but for our purposes it is sufficient to know that a set X is **thick** at x_0 if almost every circle $|z - x_0| = r$, $r \leq r_0$, intersects X . We have made implicit use above of the fact that if $k \in L^q(\Omega)$, $1 < q \leq 2$, then $\hat{k}$ is q -finely continuous almost everywhere with respect to q -capacity. By a theorem due to Deny [32] and Havin [42] for $q = 2$, and to Bagby [3] for $1 < q < \infty$ it follows that $\hat{k} \in \dot{W}_1^q(\Omega)$ if and only if $\hat{k} = 0$ almost everywhere on $\mathbb{C} \setminus \Omega$ with respect to q -capacity, that is q -quasieverywhere. In this way we are able to deal with questions concerning approximation in $L^p(\Omega)$ when $1 < p < \infty$, since then $\hat{k} \in W_1^q$ by virtue of (2.4). In case $p = 1$ the argument needs to be modified along the lines found in Bers [6], [7] and Hedberg [47], making use of the **Ahlfors mollifier** introduced in [2].

The following pointwise estimate for the Cauchy integral due to Hedberg (cf. [47], p. 164 and also [17], p. 176) plays a critical role in the proof of Theorem 4, and was initially employed by him, in lieu of the Sobolev space approach of Havin and Maz'ja, to investigate the completeness question for $L_p^p(\Omega)$, $1 \leq p < \infty$.

Lemma 1 (1971). *Let E be a compact set and let $k \in L^q(E)$, $1 < q \leq 2$. Assume that for each $\xi \in \partial E$ and each $r > 0$ the circle $|z - \xi| = r$ meets $\mathbb{C} \setminus E$. If $\hat{k} \equiv 0$ in $\mathbb{C} \setminus E$ and z is a point of E at a distance $\delta < 1/e$ from ∂E , then*

$$|\hat{k}(z)| < C \left\{ k^*(z) \delta \log \frac{1}{\delta} + \left(\Gamma_q(\delta) \int_{|\zeta - z| \leq 4\delta} |k(\zeta)|^q dA \right)^{1/q} \right\} \quad (2.5)$$

where

$$k^*(z) = \sup_r (\pi r^2)^{-1} \int_{|\zeta - z| < r} |k(\zeta)| dA$$

is the Hardy-Littlewood maximal function, $\Gamma_q(\delta)$ is equal to $\log 1/\delta$ or δ^{q-2} according to whether $q = 2$ or $q < 2$, and C is a constant depending only on q and the diameter of E .

We are now in a position to outline the arguments in support of the assertions that have been made. The proof of Theorem 4 will be presented first; the sufficiency portion of Theorem 3 then follows easily.

Proof of Theorem 4. Assume that

$$\int_{\partial G} \log \delta(z) d\omega(z) = -\infty,$$

where $\delta(z)$ is the distance from z to Ω_∞ . Fix p and let $k \in L^q(\Omega)$, $q = p/(p-1)$, have the property that $\int Pk dA = 0$ for every polynomial P . Thus, $\hat{k} \equiv 0$ in Ω_∞ . We seek to prove that $\hat{k} \equiv 0$ in G , and therefore that $H^p(\Omega) = L_a^p(\Omega)$.

To accomplish this choose a sequence of smoothly bounded domains G_j , $j = 1, 2, \dots$, so that

- (i) $x_0 \in G_j$ and $\bar{G}_j \subset G_{j+1}$ for every j
- (ii) $G = \bigcup_{j=1}^{\infty} G_j$

The key is to show that $\int_{\partial G_j} \log |\hat{k}| d\omega_j \rightarrow -\infty$ as $j \rightarrow +\infty$, where ω_j is harmonic measure for x_0 on ∂G_j . Since $\log |\hat{k}|$ is subharmonic in G ,

$$\log |\hat{k}(x_0)| \leq \int_{\partial G_j} \log |\hat{k}| d\omega_j \rightarrow -\infty,$$

and so $\hat{k}(x_0) = 0$. Hence $\hat{k} \equiv 0$ in G since x_0 is an arbitrary point of G .

In order to relate the integrals of the preceding paragraph to the one in the statement of the theorem we multiply and divide $|\hat{k}(z)|$ by $\delta(z)^\epsilon$, where $\epsilon > 0$ is to be determined later. This yields the identity

$$\int_{\partial G_j} \log |\hat{k}(z)| d\omega_j = \epsilon \int_{\partial G_j} \log \delta(z) d\omega_j + \int_{\partial G_j} \log \left(\frac{|\hat{k}(z)|}{\delta(z)^\epsilon} \right) d\omega_j$$

As $j \rightarrow \infty$ the first integral on the right approaches $-\infty$ by assumption. If we can prove that

$$\sup_j \int_{\partial G_j} \log \left(\frac{|\hat{k}(z)|}{\delta(z)^\epsilon} \right) d\omega_j < \infty, \quad (2.6)$$

it will follow that

$$\lim_{j \rightarrow \infty} \int_{\partial G_j} \log |\hat{k}| d\omega_j = -\infty$$

as claimed. The two cases $1 \leq p < 2$ and $2 \leq 3 + \tau$ will be considered separately. At the moment, of course, τ has yet to be specified.

If $1 \leq p < 2$ then $k \in L^q(\Omega)$, $q > 2$, and $\hat{k}$ satisfies a Hölder condition of order α where $0 < \alpha < 1$. For $q = \infty$ any $\alpha < 1$ will do; otherwise take $\alpha = (q - 2)/q$. Hence, $|\hat{k}(z)| \leq C\delta(z)^\alpha$ for all $z \in \mathbb{C}$ and the supremum in (2.6) will be finite as soon as $\varepsilon \leq \alpha$.

If $2 \leq p < 3 + \tau$ then $k \in L^q(\Omega)$, $q \leq 2$, and $\hat{k}$ need not satisfy a Hölder condition. To verify (2.6) in this case we begin by noting that

$$\int_{\partial G_j} \log \left(\frac{|\hat{k}(z)|}{\delta(z)^\varepsilon} \right) d\omega_j \leq \int_{\partial G_j} \frac{|\hat{k}(z)|}{\delta(z)^\varepsilon} d\omega_j,$$

and we can then direct our efforts to finding a bound for the larger integral. In that endeavor some additional notation will be employed: g_j will denote Green's function for G_j with pole at x_0 , ∇ will be the gradient operator and $\partial/\partial n$ will stand for differentiation with respect to the outward normal on appropriate boundary curves. Since ∂G_j is smooth, $d\omega_j = -\frac{\partial g_j}{\partial n} |dz|$.

By removing a small disk $|z - x_0| \leq \rho$ from G_j we obtain a domain G'_j on which $|\hat{k}(z)|\delta(z)^{-\varepsilon}$ is Lipschitz and on which g_j is uniformly well-behaved. We shall assume that ρ is fixed and that $|z - x_0| \leq \rho$ is contained in every G_j . According to the divergence theorem

$$\int_{\partial G'_j} \frac{|\hat{k}(z)|}{\delta(z)^\varepsilon} \frac{\partial g_j}{\partial n} |d(z)| = \int_{G'_j} \nabla \left(\frac{|\hat{k}(z)|}{\delta(z)^\varepsilon} \right) \cdot \nabla g_j dA$$

from which we obtain constants C_1 and C_2 such that

$$\begin{aligned} \int_{G_j} \frac{|\hat{k}(z)|}{\delta(z)^\varepsilon} d\omega_j &\leq C_1 \int_{G'_j} \frac{|\hat{k}(z)|}{\delta(z)^{1+\varepsilon}} |\nabla g_j| dA + C_2 \int_{G'_j} |\nabla(|\hat{k}|)| \frac{|\nabla g_j|}{\delta(z)^\varepsilon} dA = \\ &= C_1 I_1 + C_2 I_2. \end{aligned}$$

Our problem, therefore, is one of finding bounds for I_1 and I_2 which are independent of j . We shall give only a brief indication of how this can be done for I_2 , and then point out the necessary modifications for dealing with I_1 .

It is a consequence of Hölder's inequality and the Calderón-Zygmund theorem on the continuity of singular integral operators (cf. [84], p. 35) that

$$I_2 \leq C \|k\|_q \left(\int_{G'_j} \left| \frac{\nabla g_j}{\delta(z)^\varepsilon} \right|^p dA \right)^{1/p}.$$

If $\delta(z)$ is replaced by $\delta_j(z) = \text{dist}(z, \partial G_j)$ we retain an upper bound for I_2 and the resulting integral is estimated as follows: For each j let ϕ_j be a conformal map of G_j

onto $|w| < 1$ with $\phi_j(x_0) = 0$; put $\psi_j = \phi_j^{-1}$. The following inequalities are satisfied with all constants independent of j :

$$(a) \quad 1 - |\phi_j(z)| \leq C \sqrt{\delta_j(z)} \text{ for all } z \in G_j;$$

$$(b) \quad |\nabla g_j| \leq C|\phi'_j| \text{ on } G'_j;$$

$$(c) \quad |\psi'_j(w)| \geq C(1 - |w|).$$

(a) and (c) are consequences of the Koebe distortion theorem (cf. [71], pp. 21-22). The remaining inequality (b) can be derived from the relation $g_j = -\log |\phi_j|$. These inequalities and the fact that $\delta_j(z) \leq \delta(z)$ yield the estimate:

$$\begin{aligned} |I_2|^p &\leq C_1 \int_{G'_j} \left| \frac{\nabla g_j}{\delta_j(z)^\epsilon} \right|^p dA \leq C_2 \int_{G_j} \frac{|\phi'_j|^{p-2}}{(1 - |\phi_j|)^{2p\epsilon}} |\phi'_j|^2 dA = \\ &= C_2 \int_{|w|<1} \frac{1}{|\psi'_j|^{p-2}} \frac{1}{(1 - |w|)^{2p\epsilon}} dA. \end{aligned}$$

In order to complete the task of estimating I_2 it is essential to find a bound for the last integral which does not depend on j . As an aid in that process let ϕ be a conformal map of G onto $|w| < 1$ with $\phi(x_0) = 0$, and let $\psi = \phi^{-1}$. We may assume without loss of generality that $\psi_j(w) = \psi((1 - 1/j)w)$, $j = 2, 3, \dots$. For any pair of conjugate indices r and s it is easy to see by a change of variables and a short calculation that

$$\int_{|w|<1} |\psi'_j|^{2-p} (1 - |w|)^{-2p\epsilon} dA \leq C \|(\psi')^{2-p}\|_r \| (1 - |w|)^{-2p\epsilon} \|_s,$$

where C is an absolute constant. Given any $\lambda > 0$ (λ small) we can choose r sufficiently close to 1 so that $r(p - 2) = p - 2 + \lambda$, and we can then adjust ϵ to ensure that the second factor is finite. Thus, once $\lambda > 0$ has been chosen there exists a corresponding constant C for which

$$|I_2|^p \leq C \left(\int_{|w|<1} \frac{1}{|\psi'|^{p-2+\lambda}} dA \right)^{1/r} = C \left(\int_G |\phi'|^{p+\lambda} dA \right)^{1/r}.$$

On the other hand, there is another absolute constant $\tau > 0$ such that $\int |\phi'|^{p+\lambda} dA < \infty$, whenever $p + \lambda < 3 + \tau$ (cf. [20] and Section 4 of this article). Therefore, if $p < 3 + \tau$ the various constants, and in particular λ and ϵ , can be chosen so as to yield a finite upper bound for I_2 .

The estimation of I_1 can be carried out along similar lines. Rewriting I_1 in the form

$$I_1 = \int_{G'_j} \frac{|\hat{k}(z)|}{\delta(z)^{1-\epsilon}} \frac{|\nabla g_j|}{\delta(z)^{2p\epsilon}} dA$$

and then applying Hölder's inequality we find that

$$I_1 \leq \left(\int_{G'_j} \frac{|\hat{k}(z)|^q}{\delta(z)^{(1-\varepsilon)q}} dA \right)^{1/q} \left(\int_{G'_j} \frac{|\nabla g_j|^p}{\delta(z)^{2p\varepsilon}} dA \right)^{1/p}.$$

As before, for any $\lambda > 0$ we can choose $\varepsilon > 0$ so that the second factor is bounded by a constant times some power of $\int |\phi'|^{p+\lambda} dA$, and so that the latter is finite since $p < 3 + \tau$. With ε fixed in this way it follows from Lemma 1 that the first factor is majorized by a quantity of the form $C(\varepsilon) \|k\|_q$.

In conclusion : if $p < 3 + \tau$ it is always possible to choose $\varepsilon > 0$ so that the supremum in (2.6) is finite, from which the theorem follows.

Proof of Theorem 3. Here, we are assuming that Ω is a crescent and that ∂G , the boundary of its bounded complementary component, is a C^1 curve with a Lipschitz normal. Again, we fix a point $x_0 \in G$ and we let ϕ be a conformal map of G onto the open unit disk $|w| < 1$ with $\phi(x_0) = 0$; and $\psi = \phi^{-1}$. Because ∂G has a Lipschitz normal, ψ extends to a continuously differentiable function on $|w| \leq 1$, and there are positive constants C_1 and C_2 for which $0 < C_1 \leq |\psi'(w)| \leq C_2 < \infty$ everywhere on $|w| = 1$ (cf. [54], p. 123). Consequently, harmonic measure $d\omega = |\phi'| |dz|$ and arc length $|dz|$ are boundedly equivalent on ∂G . Hence, the integrals

$$\int_{\partial G} \log \delta(z) |dz|, \quad \int_{\partial G} \log \delta(z) d\omega$$

converge or diverge simultaneously.

If we assume that $\int_{\partial G} \log \delta(z) |dz| = -\infty$ we may argue exactly as in the proof of Theorem 4 to conclude that $H^p(\Omega) = L_a^p(\Omega)$ for all p , since under the current hypothesis $\int |\phi'|^p dA$ is evidently finite for all p . This establishes the sufficiency of the putative completeness criterion.

Apart from certain technical difficulties the key idea in establishing necessity is due to Shahinian (cf. [65], p. 121). Under the assumption that $\int_{\partial G} \log \delta(z) |dz| > -\infty$, we can construct a Jordan curve Γ lying in Ω and so that

(1) G lies entirely inside Γ ;

(2) Any sequence of polynomials bounded in the $L^p(\Omega, dA)$ norm forms a normal family inside Γ .

Thus, every function in $H^p(\Omega)$ admits an analytic continuation to G , and so $H^p(\Omega) \neq L_a^p(\Omega)$.

For our purpose it is important that the new curve Γ also enjoys the same degree of smoothness as ∂G ; that is, Γ must itself be a C^1 curve with a Lipschitz normal.

In order to carry out the required construction we first replace $\delta(z)$ by a regularized distance function $\Delta(z)$ which is defined at all points of the plane, has essentially the same profile as $\delta(z)$, but is smooth off $\partial\Omega_\infty$. In particular, $\Delta(z)$ has these properties :

$$(a) \quad C_1\delta(z) \leq \Delta(z) \leq C_2\delta(z) \text{ for all } z \in \mathbb{C};$$

$$(b) \quad \Delta(z) \text{ is } C^\infty \text{ in } \mathbb{C} \setminus \partial\Omega_\infty \text{ and } \left| \frac{\partial^\alpha \Delta}{\partial x^\alpha}(z) \right| \leq B_\alpha \delta(z)^{1-|\alpha|};$$

where C_1, C_2 and B_α are positive constants (cf. [84], p. 171). Here $\alpha = (\alpha_1, \alpha_2)$ is a pair of positive integers and $\frac{\partial^\alpha}{\partial x^\alpha}$ is the corresponding partial derivative of order $|\alpha| = \alpha_1 + \alpha_2$. Although $\Delta(z)$ does not appear to be smooth at $\partial\Omega_\infty$, it is easy to see that for any integer $m \geq 3$ the power $\Delta^m(z)$ is an $(m-2)$ -smooth function throughout the entire plane vanishing identically in Ω_∞ .

Now let $\rho(z) = \text{dist}(z, \partial G)$, and extend $n(z)$ off ∂G by setting

$$(i) \quad n(z) = -\text{grad } \rho(z) \text{ if } z \in G;$$

$$(ii) \quad n(z) = \text{grad } \rho(z) \text{ if } z \notin G \cup \partial G.$$

As extended, n is defined almost everywhere, has unit modulus, and since ∂G has positive reach (cf. [37], p. 432), it is Lipschitz and everywhere defined in a neighborhood of ∂G (cf. [17], p. 186). By convolving n with a suitable C^∞ function χ and renormalizing the result, we obtain a field of unit vectors $N(z)$ which is C^∞ in the vicinity of ∂G and satisfies $n(z) \cdot N(z) \geq \frac{1}{2}$. Thus, $n(z)$ and $N(z)$ make an angle of not more than $\pi/6$ radians. Since the vector field n is transverse along ∂G the same is true of N , and so the vectors $\varepsilon N(z)$ attached to ∂G at z fill out a tubular neighborhood T around ∂G in a one-to-one fashion provided ε is sufficiently small (cf. [97], p. 157). By choosing ε even smaller if necessary, we can arrange that for each $z \in \partial\Omega$

$$(c) \quad \text{the vectors } \varepsilon \Delta^4(z) N(z) \text{ lie entirely inside } T;$$

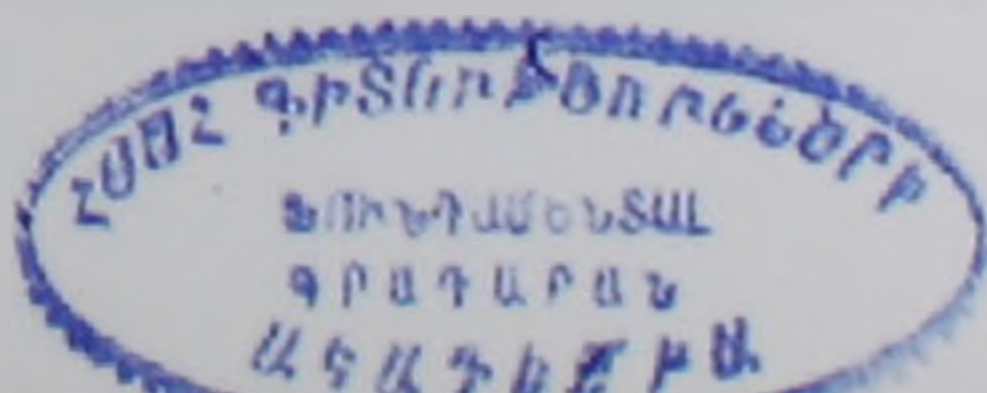
$$(d) \quad \varepsilon \Delta^4(z) \leq \delta(z)/2.$$

Therefore, if $\beta(t)$, $0 \leq t \leq 1$ is a parametric representation for ∂G with $\beta'(t)$ nonvanishing and Lipschitz in t , the curve Γ parameterized by

$$\gamma(t) = \beta(t) + \varepsilon \Delta^4(\beta(t)) N(\beta(t)), \quad 0 \leq t \leq 1$$

is a simple closed Jordan curve lying in Ω satisfying property (1), and enjoying the same degree of smoothness as ∂G .

To complete the proof of necessity it remains to verify that property (2) is also satisfied. Suppose in this regard that Q_j , $j = 1, 2, \dots$ is a sequence of polynomials which is bounded in the $L^p(\Omega, dA)$ -norm. By construction the disk with center at $\gamma(t)$ and radius $\varepsilon \Delta^4(\beta(t))/3$ is contained in Ω for all t , and so by the area mean value



theorem

$$|Q_j(\gamma(t))| \leq \frac{C}{\Delta^{8/p}(\beta(t))} \left(\int_{\Omega} |Q_j|^p dA \right)^{1/p} \leq \frac{C'}{\Delta^{8/p}(\beta(t))}, \quad (2.7)$$

for $j = 1, 2, \dots$ and constants C, C' which depend on p , but not on t . We shall presently see that this is sufficient to ensure that $\{Q_j, j = 1, 2, \dots\}$ is a normal family inside the region bounded by Γ .

To that end let λ be defined along Γ by setting $\lambda(\gamma(t)) = \Delta^{8/p}(\beta(t))$. Because $|\gamma'(t)| \leq K|\beta'(t)|$, there exists a constant K_1 such that

$$\begin{aligned} \int_{\Gamma} \log \lambda(z) |dz| &= \int_0^1 \log \lambda(\gamma(t)) |\gamma'(t)| dt \geq \\ &\geq K_1 \int_0^1 \log \delta(\beta(t)) |\beta'(t)| dt = K_1 \int_{\partial G} \log \delta(z) |dz| > -\infty. \end{aligned}$$

And, since harmonic measure $d\omega$ and arc length $|dz|$ are boundedly equivalent on Γ , it follows that

$$\int_{\Gamma} \log \lambda(z) d\omega > -\infty. \quad (2.8)$$

By a standard argument (cf. [50], p. 53) the convergence of the integral in (2.8) guarantees the existence of a function h analytic and nowhere zero inside Γ with the additional property that $|h|$ takes the boundary value $\lambda(\gamma(t)) = \Delta^{8/p}(\beta(t))$ almost everywhere with respect to harmonic measure on Γ . In view of (2.7) the inequality $|Q_j h| \leq C'$, $j = 1, 2, \dots$, obtains a.e. $-d\omega$ on Γ and everywhere inside. It follows that $\{Q_j h, j = 1, 2, \dots\}$ is a normal family there, and since $h \neq 0$ the same is true of $\{Q_j, j = 1, 2, \dots\}$.

In conclusion : Under the assumption that $\int_{\partial G} \log \delta(z) |dz| > -\infty$ every function in $H^p(\Omega)$ admits an analytic continuation to G . Therefore $H^p(\Omega) \neq L_a^p(\Omega)$.

Theorem 3 is illustrative of a far more general phenomenon first suggested by Mergelian ([66], p. 904) in 1955, and subsequently confirmed in a somewhat weaker form by Sinanian ([83], pp. 412 – 415) in 1970 (cf. also [63], pp. 204 – 206). Eventually established in full generality by the author ([21], pp. 418 – 419) in 1979, it is this : For an arbitrary bounded simply connected domain Ω and any finite $p \geq 1$ either $H^p(\Omega) = L_a^p(\Omega)$ or there exists a point $x_0 \in \partial\Omega$ and an open set U containing x_0 such that every function $f \in H^p(\Omega)$ admits an analytic continuation to U . In case Ω is of crescent type and $H^p(\Omega) \neq L_a^p(\Omega)$ then every $f \in H^p(\Omega)$ necessarily extends analytically to the entire bounded complementary component G . We shall have more to say later concerning the reciprocal relation between completeness and analytic continuation in various situations. Our focus for

the present will be directed at the following problem : Assuming that $H^p(\Omega) \neq L_a^p(\Omega)$ give a complete description of the functions in $L_a^p(\Omega)$ which admit approximation by polynomials.

Without additional restrictions on Ω the problem as stated may not have a simple answer. Consider, therefore, the simplest domain Ω_0 of crescent type ; that is, the region bounded by the two internally tangent circles $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$. By Theorem 3, $H^2(\Omega_0) \neq L_a^2(\Omega_0)$, and Keldysh (cf. [53], p. 4 and [65], p. 135) has shown that every $f \in H^2(\Omega_0)$ admits an extension $\bar{f}$ to the disk $|z - \frac{1}{2}| < \frac{1}{2}$ which is subject to the growth restriction

$$|\bar{f}(x)| \leq \frac{K}{(1-x)^{3/2}} \quad \text{for } \frac{1}{2} < x < 1.$$

Three decades later Shapiro ([78], p. 293) sharpened this by replacing growth along the x -axis with membership in a certain Smirnov class, and he raised at that time the question of determining exactly those functions in $L^p(\Omega_0)$ which admit approximation by polynomials. Shortly thereafter Havin [43] gave a complete solution to the problem for any crescent Ω where $\partial\Omega_\infty$ and ∂G are both Jordan curves of class C^3 .

For the remainder of this discussion Ω will be a crescent. We assume that ∂G is a Jordan curve of at least class C^2 so that harmonic measure ω_x on ∂G relative to an arbitrary point $x \in G$ is boundedly equivalent to arc length $|dz|$. It is further assumed that

$$\int_{\partial G} \log \delta(z) |dz| > -\infty, \quad (2.9)$$

where $\delta(z) = \text{dist}(z, \Omega_\infty)$. Thus, $H^p(\Omega) \neq L_a^p(\Omega)$ for any p . With additional assumptions on $\partial\Omega$ (principally on $\partial\Omega_\infty$) it is possible to characterize, by means of an auxiliary function Q , those functions in $L_a^p(\Omega)$ which also belong to $H^p(\Omega)$. The function Q is defined as follows : Let

$$u(z) = \int_{\partial G} \log \delta(t) d\omega_z(t),$$

let $v(z)$ be the harmonic conjugate of u and set $Q = e^{u+iv}$. As defined Q is analytic in G , continuous on $\bar{G}$ and $|Q(z)| = \delta(z)$ for each $z \in \partial G$. A function F analytic in G is said to belong to the Smirnov class $E^p(G)$ if there exists a sequence of rectifiable Jordan curves $C_1, C_2, \dots$ in G , tending to ∂G , so that

$$\sup_n \int_{C_n} |F(z)|^p |dz| < \infty.$$

Additional information concerning the Smirnov class can be found in [33].

The following represents a strengthening of a theorem of Havin [43] who obtained an analogous result, but with more stringent regularity restrictions placed on $\partial\Omega$ (cf. [19], pp. 149 – 155).

Theorem 5 (1977). Let Ω be a crescent whose interior and exterior boundaries (i.e. ∂G and $\partial\Omega_\infty$, respectively) are Jordan curves of class $C^{2+\alpha}$. Assume that (2.9) is satisfied. If $f \in L^p_\alpha(\Omega)$ the following are equivalent :

- (1) $f \in H^p(\Omega)$;
- (2) f can be extended to a function $\bar{f}$ which is analytic in G and so that $\bar{f}Q^{1/p} \in E^p(G)$.

The proof of the theorem depends in an essential way on the extent to which certain analytic functions are influenced by the moduli of their boundary values. To be more precise it is necessary to introduce some additional notation and terminology : D will denote the open unit disk and F will be a function analytic in D and continuous on $\bar{D}$. We further assume that F is an **outer function**, or equivalently, that

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{i\theta})| d\theta$$

(cf. [50], p. 62). A function k (defined on either $\bar{D}$ or ∂D) is said to belong to Λ_α (i.e. $\Lambda_\alpha(\bar{D})$ or $\Lambda_\alpha(\partial D)$, respectively) if

- i) $k \in Lip_\alpha$ in case $0 < \alpha \leq 1$;
- ii) $k^{[\alpha]} \in Lip_{\alpha-[\alpha]}$ in case $\alpha > 1$ and $[\alpha] < \alpha$. Here $[\alpha]$ denotes the greatest integer $\leq \alpha$, and $k^{[\alpha]}$ is the $[\alpha]$ -th derivative of k .

Lemma 2 (1977). If F is as above, and $|F| \in \Lambda_\alpha(\partial D)$ then

- (1) $F \in \Lambda_{\alpha/2}(\bar{D})$ if $0 < \alpha < 2$;
- (2) $F \in \Lambda_1(\bar{D})$ if $\alpha > 2$.

Moreover, the Lipschitz constant associated to F on $\bar{D}$ depends only on the Lipschitz constants and bounds for the derivatives of $|F|$ on ∂D .

If $0 < \alpha < 1$ and F does not vanish on ∂D it is a consequence of the Privalov-Zygmund theorem on the modulus of continuity of the conjugate function that $F \in \Lambda_\alpha(\bar{D})$. If, however, F is allowed to have zeros on ∂D then $F \in \Lambda_{\alpha/2}(\bar{D})$ and the exponent $\alpha/2$ is, in general, best possible. That result was first discovered by Jacobs for $0 < \alpha \leq 1$ in his thesis [51], unpublished. Later Havin [44] rediscovered and strengthened it to include functions which vanish in the interior of D provided those zeros do not accumulate tangentially at any point of the boundary. Between these two events Havin and Shamoyan [45] announced the same result for outer functions in the restricted range $0 < \alpha < 1$. As stated, Lemma 2 was obtained by the author [19] specifically in connection with the proof of Theorem 5. Subsequently, the corresponding result for all α , $0 < \alpha < \infty$ was obtained by Shirokov (cf. [79] and [80]).

Proof of Theorem 5. Here we include only a brief sketch of the implication (1) $\Rightarrow$ (2) which is sufficient to indicated a few of the more important ideas involved

(cf. [19] for a complete proof). Since $\int_{\partial G} \log |\delta(z)| |dz| > -\infty$, it follows that each $f \in H^p(\Omega)$ extends to a function $\bar{f}$ which is analytic in $\Omega \cup \bar{G}$. It must be shown that $\bar{f}Q^{1/p} \in E^p(G)$.

First let $\gamma(t)$, $0 \leq t \leq 1$, be a $C^{2+\alpha}$ parametric representation of ∂G with $\gamma'(t)$ nowhere zero. For each $z \in \partial G$ let $n(z)$ be the unit outward pointing normal. As in the proof of Theorem 3 let $N(z)$ be a smooth vector field such that $n(z)$ and $N(z)$ make an angle of not more than $\pi/6$ radians. Thus, the field N is transverse along ∂G and the vectors $\varepsilon N(z)$, attached to ∂G at z , sweep out in a one-to-one fashion a tubular neighborhood T_ε abutting ∂G . The family of curves γ_c , $0 \leq c \leq \varepsilon$, parameterized by

$$\gamma_c(t) = \gamma(t) + c\delta(\gamma(t))N(\gamma(t)), \quad 0 \leq t \leq 1,$$

are simple closed curves lying in Ω . Since $\partial\Omega_\infty \in C^{2+\alpha}$, it is a consequence of the implicit function theorem that $\delta(z)$ is a function of class $C^{2+\alpha}$ in $\Omega \cap W$ for some neighborhood W of $\partial\Omega_\infty$. We may assume without loss of generality that $\partial G \subset W$. Each γ_c is then a curve of class $C^{2+\alpha}$.

The γ_c 's can also be expressed as the level sets of a single Lipschitz function u defined as follows. Let $\xi : T_\varepsilon \rightarrow \partial G$ be the map which associates to each $z \in T_\varepsilon$ its projection $\xi(z)$ onto ∂G along the unique vector $N(\xi(z))$ passing through z . It is easily checked that ξ satisfies a Lipschitz condition of order 1 in a neighborhood of ∂G (cf. [37], p. 434 and [17], p. 186). Setting

$$u(z) = \frac{|z - \xi(z)|}{\delta(\xi(z))},$$

we obtain a function $u \in Lip_1$ away from the zeros of δ and $\gamma_c = \{z : u(z) = c\}$. Because $\delta(z)$ and $\delta(\xi(z))$ are comparable, for almost every c sufficiently small, $c \leq \varepsilon$ say, $|\nabla u(z)| \leq K/\delta(z)$ along γ_c with K independent of c .

Suppose now that $f \in H^p(\Omega)$. By definition there exists a sequence of polynomials f_j , $j = 1, 2, \dots$, with

$$\int_{\Omega} |f_j - f|^p dA \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Let Q_c be an outer function defined in the region bounded by γ_c with $|Q_c(z)| = \delta(z)$ for all $z \in \gamma_c$. If we let $T = \bigcup_{0 \leq c \leq \varepsilon} \gamma_c$, then by the co-area formula (cf. [37], pp. 426 - 427 and [39], p. 118)

$$\int_0^\varepsilon \left(\int_{\gamma_c} |f_j - f|^p |Q_c| |dz| \right) dc = \int_T |f_j - f|^p \delta(z) |\nabla u| dA \leq K \int_{\Omega} |f_j - f|^p dA$$

Since the latter integral tends to zero as $j \rightarrow \infty$, it follows that

$$\int_{\gamma_c} |f_j - f|^p |Q_c| |dz| \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

for almost every $c \leq \varepsilon$. On the other hand, it can be shown that for a sufficiently small fixed c

$$\int_{\partial G} |f_j - \tilde{f}|^p |Q| |dz| \leq^{(i)} K_1 \int_{\partial G} |f_j - \tilde{f}|^p |Q_c| |dz| \leq^{(ii)} K_2 \int_{\gamma_c} |f_j - \tilde{f}|^p |Q_c| |dz|,$$

where K_1 and K_2 are constants which do not depend on j or c . By a suitable choice of c we are forced to conclude that $f_j Q^{1/p} \rightarrow \tilde{f} Q^{1/p}$ in $L^p(\partial G, |dz|)$. But, G is a Smirnov domain and so $Q^{1/p}$ can also be approximated by a sequence of polynomials in the $L^p(\partial G, |dz|)$ -norm (cf. [33], p. 173). Consequently, $\tilde{f} Q^{1/p}$ is itself the limit of a sequence of polynomials in $L^p(\partial G, |dz|)$ and therefore $\tilde{f} Q^{1/p} \in E^p(G)$.

The first inequality (i) follows from the fact that $|Q_c| \in C^{2+\alpha}(\gamma_c)$ and so, by Lemma 2, $Q_c \in \text{Lip}_1$ on the closed region bounded by γ_c . Here we have made use of the fact that a conformal map of the open unit disk D onto $\text{int}(\gamma_c)$ necessarily has a $C^{2+\alpha}$ extension to $\bar{D}$, since $\gamma_c \in C^{2+\alpha}$ (cf. [94], p. 73). The result is that $Q_c \in \text{Lip}_1$ on the closure of $\text{int}(\gamma_c)$ and that $|Q_c(x)| \geq K|Q(x)|$ for all $x \in \partial G$, from which (i) follows. Inequality (ii) is a consequence of the subharmonicity of $|f_j - f|^p$ and the fact that $|Q_c| |dz|$ is boundedly equivalent to harmonic measure on each of the curves ∂G and γ_c .

(B) Domains with Boundary Cuts. In order to study polynomial approximation on domains with cuts one is led rather naturally to consider a weighted measure $w dA$. Roughly speaking, the polynomials will be dense in $L^p_\alpha(\Omega, w dA)$ if the weight $w \rightarrow 0$ sufficiently rapidly at the cuts; otherwise, they will not. In this setting, the completeness problem bears a strong resemblance to the well-known Bernstein problem concerning uniform weighted polynomial approximation on the real line, where the cuts or inner boundary of Ω now play the role of the point at infinity. Over the years the Bernstein problem has been studied by Akhiezer, Carleson, Pollard, and Mergelian while the corresponding problem for domains has been studied by Keldysh, Djrbashian, Mergelian, Beurling and the author.

The Bernstein approximation problem [5] was initially posed in 1924 and is now largely solved. Here $w(x) \geq 0$ is a bounded measurable function defined on the whole real line,

$$C_w = \left\{ f : \lim_{|x| \rightarrow \infty} f(x)w(x) = 0, \quad f \text{ continuous} \right\},$$

and the question is this : For which weights w are the polynomials dense in C_w in the sense that given any $f \in C_w$ there exists a sequence of polynomials Q_j , $j = 1, 2, \dots$, such that

$$\sup_x |Q_j(x) - f(x)|w(x) \rightarrow 0, \quad \text{as } j \rightarrow \infty?$$

If $w(x)$ vanishes outside a finite interval $[-R, R]$ the matter is settled by the Weierstrass approximation theorem ; the polynomials are dense. If, however, $w(x)$ has unbounded support then the question is more subtle and in order that the polynomials belong to C_w we must assume from the outset that

$$\lim_{|x| \rightarrow \infty} |x|^n w(x) = 0, \quad n = 0, 1, \dots$$

A second and completely analogous problem is this :

Can every $f \in L^p(wdx)$, $p \geq 1$ be approximated by a sequence of polynomials Q_j , $j = 1, 2, \dots$, in the sense that

$$\int_{-\infty}^{\infty} |Q_j(x) - f(x)|^p w(x) dx \rightarrow 0, \quad \text{as } j \rightarrow \infty?$$

And, correspondingly it must be assumed that

$$\int_{-\infty}^{\infty} |x|^n w(x) dx < \infty, \quad n = 0, 1, 2, \dots$$

Complete solutions to the approximation problem for C_w have been given by several authors, including Akhiezer, Pollard and Mergelian. Of particular interest here are certain results of Mergelian, each of which corresponds to a parallel result in connection with weighted polynomial approximation on plane domains. Given any $z \in \mathbb{C}$ define

$$w^*(z) = \sup_Q |Q(z)|,$$

the supremum being taken over all polynomials Q such that $|Q(x)|(1+|x|)^{-1}w(x) \leq 1$ for all real x . In other words, $w^*(z)$ is the norm of the mapping $Q \rightarrow Q(z)$ regarded as a linear functional on the polynomials in the $C_{\frac{w}{1+|x|}}$ norm. Mergelian [67] has shown that : each of the following is necessary and sufficient for the polynomials to be dense in C_w :

$$(1) \quad \int_{-\infty}^{\infty} \frac{\log w^*(x)}{1+x^2} dx = +\infty,$$

$$(2) \quad w^*(z) = \infty, \quad \text{Im } z \neq 0.$$

As a consequence it is easy to establish a reciprocal relation between completeness in C_w and a continuation phenomenon, first noted by Bernstein in 1924 and similar to one already encountered here :

(3) either the polynomials are dense in C_w , or the only functions which can be so approximated are restrictions of entire functions to the real axis. Much of what has just been said has a natural meaning and interpretation in the context of uniform weighted approximation on planar domains. Beurling [10] considered, for example, the following generalization of the classical Bernstein problem : Given a bounded simply connected domain Ω and $w(z) > 0$ a positive continuous function on Ω with $w(z) \rightarrow 0$ as $z \rightarrow \partial\Omega$, let $C_w(\Omega)$ be the Banach space of all complex-valued functions f for which the product $f(z)w(z)$ is continuous on $\bar{\Omega}$ and vanishes on $\partial\Omega$, the norm being defined by

$$\|f\|_w = \sup_{\bar{\Omega}} |f|w.$$

Evidently, the collection of functions $A_w(\Omega) = \{f \in C_w(\Omega) : f \text{ is analytic in } \Omega\}$ is a closed linear subspace of $C_w(\Omega)$. The problem is to determine whether or not the polynomials are dense in $A_w(\Omega)$.

Under the supposition that $w(z) = W(g(z))$ is a weight depending only on Green's function g for Ω and satisfying an additional mild regularity restriction, the following correspond to (1), (2) and (3) for the Bernstein problem :

(1') the polynomials are dense in $A_w(\Omega)$ if $\int_0^1 \log \log \frac{1}{W(t)} dt = +\infty$;

(2') the polynomials are dense in $A_w(\Omega)$ if and only if $A_w(\Omega)$ has no BPE's on $\partial\Omega$;

(3') either the polynomials are dense in $A_w(\Omega)$, or every $f \in A_w(\Omega)$ admits an analytic continuation across a fixed piece of $\partial\Omega$.

The log log -integral in (1') has been studied extensively for a variety of reasons. It is of fundamental importance, for example, in the Beurling-Levinson-Sjöberg theorem on the existence of a greatest subharmonic minorant of a given function (cf. [22] and [24] for additional references). It first appears in connection with the completeness problem for $L^p_\alpha(\Omega, wdA)$ in the work of Djrbashian (cf. [65], p. 143) initiated in the late 1940's. He showed that if Ω_0 is the open unit disk with a single radial cut and if $W(t)$ satisfies the regularity conditions alluded to above, then $H^p(\Omega_0, wdA) = L^p_\alpha(\Omega_0, wdA)$ whenever (1') is fulfilled. In general, if $\partial\Omega$ has an isolated smooth arc, as is the case for Ω_0 , then the divergence of the log log -integral is also necessary for completeness to occur (cf. [22], p. 46).

Since we are assuming that w depends only on Green's function, it can be expressed in the form

$$w(z) = e^{-h(\log 1/|\phi(z)|)},$$

where ϕ is a conformal map of Ω onto the open unit disk D . With the additional requirement that $yh(y) \uparrow +\infty$ as $y \downarrow 0$, we have

Theorem 6 (1994).

$$H^2(\Omega, w dA) = L^2_a, \quad \text{whenever} \quad \int_0^1 \log h(y) dy = +\infty.$$

Results on the Bernstein problem all depend in one way or another on the now standard theory of quasianalyticity as developed by Denjoy, Carleman and Bernstein. In order to address the difficulties associated with the corresponding problem for domains this must be replaced by a more recent theory of asymptotically holomorphic functions in a form due principally to Beurling [9], Dyn'kin [34] and Vol'berg [91]. Roughly speaking, a function $F(e^{i\theta})$ defined initially on ∂D is **asymptotically holomorphic** if it can be realized as the boundary values of a corresponding Sobolev function $F(z)$ defined on D and having the property that

$$|\bar{\partial}F(z)| \rightarrow 0, \quad \text{rapidly as } |z| \rightarrow 1.$$

The rate at which $|\bar{\partial}F|$ drops off at ∂D can be taken as a measure (or asymptotic estimate) of the extent to which $F(e^{i\theta})$ deviates from the boundary values of a truly analytic function. If the drop off is sufficiently rapid and the consequent deviation below a certain critical level, then $F(e^{i\theta})$ will continue to enjoy many of the properties usually associated with analyticity. A more extensive discussion of this notion can be found in Koosis' book [56].

Proof of Theorem 6 (outline). Let $g \in L^2(\Omega, w dA)$ be any function such that $\int_{\Omega} P g w dA = 0$ for all polynomials P and form the Cauchy integral

$$f(z) = \int_{\Omega} \frac{g w(\zeta)}{\zeta - z} dA_{\zeta},$$

which by (2.4) is a function belonging to the Sobolev space $W_1^2(\mathbb{R}^2)$. The use of g here to represent something other than Green's function will now be suppressed and no confusion will result. By assumption $f \equiv 0$ in Ω_{∞} , and the rate at which w decreases at the boundary guarantees that f is continuous when restricted to $\partial\Omega$. And so, it follows from the fine continuity associated with potentials that $f \equiv 0$ on

$\partial\Omega_\infty$. The proof can then be completed and the theorem established by verifying that $f \equiv 0$ on all of $\partial\Omega$ and arguing as in the paragraph following (2.4).

In order to carry this out it is convenient to transfer the problem to the unit disk by setting $F = f(\psi)$, where $\psi = \phi^{-1}$. In this way we obtain a function F which belongs to the Sobolev space $W_1^2(D)$, and therefore has radial limits $F(e^{i\theta})$ almost everywhere $d\theta$ on ∂D . Although F will not in general be analytic it is, nevertheless, almost analytic in the sense that $F(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ and the negative Fourier coefficients a_{-n} , $n = 1, 2, \dots$ are such that

$$(i) |a_{-n}| \leq e^{-k(n)}, \quad n = 1, 2, \dots$$

$$(ii) \sum_{n=1}^{\infty} \frac{k(n)}{n^2} = +\infty,$$

where $k(x) = \inf_{y>0} (h(y) + xy)$ is the Legendre transform of h . The divergence of the series in (ii) follows from the fact that the two integrals

$$\int_0^\delta \log h(y) dy \quad \text{and} \quad \int_1^\infty \frac{k(x)}{x^2} dx$$

converge or diverge simultaneously, and $\int_0^\delta \log h(y) dy = -\infty$ by assumption. Because $yh(y) \uparrow +\infty$ as $y \downarrow 0$ it follows that $x^{-1/2}k(x) \uparrow +\infty$ as $x \uparrow +\infty$. This, together with the fact that $k(x)$ is concave and (i) and (ii) are satisfied, implies that $F(e^{i\theta})$ is asymptotically holomorphic in the sense previously described. An extension to the full disk D is achieved by first introducing an auxiliary weight

$$W(z) = W(|z|) = e^{-h(\log 1/|z|)}$$

which is itself defined on D . Since the negative Fourier coefficients satisfy $|a_{-n}| \leq e^{-k(n)}$, and so tend to zero very fast we can find a function $\rho \in L^\infty(D)$ such that

$$\bar{F}(z) = \sum_{n=0}^{\infty} a_n z^n + \frac{1}{\pi} \int_D \frac{\rho W(\zeta)}{(\zeta - z)} dA_\zeta$$

has the same boundary values $F(e^{i\theta})$. This is known as the Dyn'kin extension theorem, and is central to our entire argument (cf. [56], pp. 339 – 343). In this way we obtain a new Sobolev function $\bar{F}$ such that $|\partial \bar{F}(z)| \leq CW(z)$ and $(F - \bar{F}) \in W_1^2(D)$. Our goal then is to prove that $\bar{F} = 0$ a.e. on ∂D . To this end let

$$E = \left\{ z \in D : |\bar{F}(z)| \leq W(z) \right\},$$

and set $U = D \setminus E$. Since $\bar{F}$ is continuous in the interior of D , the set U is open. We may assume with no loss of generality that E is the union of countably many

disjoint, smoothly bounded, Jordan regions only finitely many of which meet any compact subset of D (cf. [24], p. 771). There are two possibilities : either $\partial U \cap D$ is so sparse that harmonic measure for U , denoted $d\omega$, is boundedly equivalent to $d\theta$ along ∂D , or it is not. In the second instance an argument originating with Beurling [9], and based on the Fourier transform, shows that $F(e^{i\theta}) \equiv 0$ on ∂D . It is here that some additional restriction such as $y h(y) \uparrow +\infty$ as $y \downarrow 0$ is needed (cf. [24], pp. 772-773). In the case $d\omega \approx \frac{d\theta}{2\pi}$ on ∂D we present only a brief description of the argument, sufficient to indicate the manner in which the Cauchy integral enters once again ; this time in an unusual way.

The first step is to introduce an auxiliary function $\Phi = \bar{F}e^R$, where

$$R(z) = -\frac{1}{\pi} \int_U \frac{1}{\bar{F}(\zeta)} \frac{\rho W(\zeta)}{(\zeta - z)} dA_\zeta.$$

Because $\bar{\partial}\Phi = 0$ a.e. in U , and because the integrand is bounded

(iii) Φ is analytic in U , and

(iv) $C_1|\bar{F}(z)| \leq |\Phi(z)| \leq C_2|\bar{F}(z)|$ with $C_1, C_2 > 0$.

The idea of regularizing a function in this way was first employed by Theodorescu in 1931, rediscovered by Bers in 1951, and rediscovered again by Vol'berg [91] in 1982. It is now commonly referred to as the **similarity principle** (cf. [89], 122 – 123). To determine the behavior of Φ in U fix an arbitrary point $z_0 \in U$. For each $\varepsilon > 0$ and sufficiently small let $D_\varepsilon = \{z : |z| < 1 - \varepsilon\}$ and denote by U_ε the component of $D_\varepsilon \setminus E$ containing z_0 . Next, let $d\omega_\varepsilon$ be harmonic measure for z_0 relative to U_ε and integrate $\log |\Phi|$ over ∂U_ε to obtain the identity

$$\int_{\partial U_\varepsilon} \log |\Phi| d\omega_\varepsilon = \int_{\partial U_\varepsilon} \log |\bar{F}| d\omega_\varepsilon + \int_{\partial U_\varepsilon} \operatorname{Re} R d\omega_\varepsilon. \quad (2.10)$$

The proof then proceeds by using the divergence of the integral $\int_0^\infty \log h(y) dy$ to show that $\int \log |\bar{F}| d\omega_\varepsilon \rightarrow 0$ for some sequence of $\varepsilon \rightarrow 0$. Since the second term on the right side of (2.10) is bounded it follows that

$$\int_{\partial U_\varepsilon} \log |\Phi| d\omega_\varepsilon \rightarrow -\infty,$$

from which it follows that $\Phi \equiv 0$ in U . Therefore, $\bar{F}(e^{i\theta}) = F(e^{i\theta}) = 0$ on ∂D as claimed, since $\bar{F} \equiv 0$ in U and $\bar{F}(z) \rightarrow 0$ as $z \rightarrow \partial D$ through $D \setminus U$ by construction. This is sufficient to imply that the original Cauchy integral $f \equiv 0$ on $\partial\Omega$, and the theorem follows.

In order to better appreciate the difficulties involved in the proof of Theorem 6, consider for a moment the domain Ω obtained by removing from the open unit disk D a spiral $z = re^{i\theta}$ defined by

$$r = e^{-1/\log \theta}, \quad \theta > 2\pi + 1.$$

As given, the domain Ω was first studied by Keldysh in connection with weighted polynomial approximation in 1941, but not published until 1945 (cf. [53], p. 19 and [65], p. 140). The essential feature here consists in the fact that $\partial\Omega_\infty$ now represents a single prime end for Ω . Thus, even though a Sobolev function f vanishes identically on $\partial\Omega_\infty$ the corresponding function F on D , obtained by means of conformal mapping, may not take the boundary value zero at more than a single point. It then becomes a delicate matter to show that

$$\int_{\partial D} \log |F(e^{i\theta})| d\theta = -\infty,$$

and to argue from there that $F(e^{i\theta}) \equiv 0$ on ∂D , etc.

We have so far been concerned exclusively with approximation in the $L^2(\Omega, w dA)$ norm. The principal reason for doing so is that when transferring the problem from Ω to D by a conformal map the Sobolev space W_1^2 is preserved. It is clear, however, that our results (and Theorem 6, in particular) are also valid whenever $1 \leq p \leq 2$. In order to deal with the case $p > 2$ we shall invoke the following theorem (cf. [21], pp. 416 and 418) which gives a complete solution to the weighted approximation problem in terms of BPE's, confirming an earlier conjecture of Mergelian [66] in a more general setting than that in which it was originally posed.

Theorem 7 (1979). *Let Ω be a bounded simply connected domain, and let $w \in L^\infty$ be a weight which is continuous in Ω and depends only on Green's function. Then, for any p*

- (1) $H^p(\Omega, w dA) = L_a^p$, if and only if, H^p has no BPE's on $\partial\Omega$.
- (2) If $H^p(\Omega, w dA) \neq L_a^p$ there exists a point $\xi_0 \in \partial\Omega$ and an open set U containing ξ_0 such that every $f \in H^p$ admits an analytic continuation to U .

For the present we shall assume that Ω is a bounded simply connected domain, that $\phi : \Omega \rightarrow D$ is a conformal map, and that

$$w(z) = w(|z|) = e^{-h(\log 1/|\phi(z)|)}$$

is a weight satisfying the two conditions

- (i) $yh(y) \uparrow +\infty$ as $y \downarrow 0$,

$$(ii) \int_0 \log h(y) dy = +\infty.$$

Given any such weight w we can find an analogous weight $\bar{w}$ such that for any polynomial Q

$$\sup_{\Omega} |Q| \bar{w} \leq C \|Q\|_{L^1(\Omega, w dA)}. \quad (2.11)$$

The existence of such a weight is easily verified as follows : Fix a point $\xi \in \Omega$, set $y = 1 - |\phi(\xi)|$, and let $\delta = \text{dist}(\xi, \partial\Omega)$. By the Koebe distortion theorem ([71], pp. 21-22), $y \leq C\sqrt{\delta}$ for some absolute constant C . If B is the disk with center at ξ and radius $\delta/2$ the area mean value theorem gives

$$Q(\xi) = \frac{1}{|B|} \int_B Q dA = \frac{4}{\pi\delta^2} \int_B Q \frac{w}{w} dA.$$

Because w is monotone in y and $y \leq C\sqrt{\delta}$, we obtain the inequality

$$|Q(\xi)| y^4 w(y/2) \leq C \int_{\Omega} |Q| w dA,$$

valid for all polynomials Q . Finally, taking

$$\bar{w}(z) = e^{-2h(\frac{1}{2} \log \frac{1}{|\phi(z)|})}$$

and adjusting the constant C accordingly (2.11) is satisfied. In summary :

- (a) Given w we can find $\bar{w}$ and (2.11) holds ;
- (b) Conversely, given $\bar{w}$ we can find a corresponding w ;
- (c) w and $\bar{w}$ both enjoy properties (i) and (ii).

The Case $p > 2$. We are now in a position to establish Theorem 6 for all finite p . Suppose, for example, that $\bar{w}$ is a weight with properties (i) and (ii) and suppose

$$H^p(\Omega, \bar{w} dA) \neq L^p_{\alpha}.$$

Hence, $H^p(\Omega, \bar{w} dA)$ has a BPE at some point $\xi_0 \in \partial\Omega$. By our remarks above there exists another weight w also having properties (i) and (ii) such that

$$|Q(\xi_0)| \leq C_1 \|Q\|_{L^p(\bar{w} dA)} \leq C_2 \|Q\|_{L^1(w dA)}$$

for all polynomials Q . This contradicts the fact that $H^1(\Omega, w dA) = L^1_{\alpha}$ as we have already proved.

The Case $p = \infty$. Beurling's question which is the planar version of the Bernstein problem can now be answered in the affirmative :

Theorem 8 (1994). *Assuming properties (i) and (ii) the polynomials are dense in $A_w(\Omega)$.*

Proof : Let $f \in A_w(\Omega)$. For each $r < 1$ put $f_r = F(r\phi)$, where $F = f(\phi^{-1})$. Evidently, $f_r \in A_w(\Omega)$ and, moreover,

$$\|f_r - f\|_w = \sup_{\Omega} |f_r - f|_w \rightarrow 0, \quad \text{as } r \rightarrow 0.$$

Since f_r is bounded on Ω it can be approximated arbitrarily closely by polynomials in the $L^1(\Omega, \bar{w}dA)$ norm, where $\bar{w}$ dominates w in the sense of (2.11). Therefore, the same sequence of polynomials converges to f_r in the norm of $A_w(\Omega)$. Theorem 8 is proved.

We began our treatment of the completeness problem on domains with boundary cuts by suggesting that it was somehow necessary to consider only weighted approximation in this setting. That, however, is not, strictly speaking, entirely true. In 1964 Mergelian and Tamadian [68] made a rather surprising discovery. They showed that in certain special situations the cuts themselves can actually be made to play the role of a weight. Consider, for example, a perfect nowhere dense set of points E on the unit circle ∂D . Given ρ , $0 < \rho < 1$, let

$$S_x = \{z : \arg z = \arg x, 1 - \rho \leq |z| \leq 1\}$$

and put $S_E = \bigcup_{x \in E} S_x$. Thus, $\Omega_E = D \setminus S_E$ is a simply connected domain and $\partial\Omega_E$ consists almost entirely of cuts. Let $E^c = \partial D \setminus E$, set $\Delta_t(x) = \{e^{i\theta} : |\theta - \arg x| \leq t\}$, and denote 1-dimensional Lebesgue measure by m . The following is a strengthening of the result in [68], p. 80, and depends on the Denjoy quasianalyticity criterion :

Theorem 9 (1977). *The polynomials are complete in $L^p_a(\Omega_E, dA)$ for every p if there exists a countable set E' , everywhere dense in E , such that for each $x \in E'$*

- (1) $m(\Delta_t(x) \cap E^c) \leq V(t)$ for $t \leq t(x)$ and some regular majorant V ;
- (2) $\int_0 \log \log \frac{1}{V(t)} dt = +\infty$.

A weight $V(t)$ is **regular** if it is logarithmically convex ; that is, if $t \frac{V'(t)}{V(t)} \uparrow +\infty$ as $t \downarrow 0$. Recall that the same condition appears in the early work of Djrbashian and Shahinian as described here in Theorem 2.

§3. THE UNIQUENESS PROPERTY

We have encountered several instances of what now appears to be a general principle. It is this : Either $H^p = L^p_a$, or every $f \in H^p$ admits an analytic extension to a set larger than that on which f is presumably defined. Here we ask : **To what extent,**

and in what form, might such a principle be extended to cover rational approximation on closed sets without interior points?

In this discussion X will be a compact nowhere dense set in the plane and $\mathcal{R}(X)$ will stand for the set of all rational functions having no poles on X . As usual, $C(X)$ will be the Banach space of all continuous complex valued functions on X endowed with the uniform norm $\|f\| = \max_{x \in X} |f(x)|$. We will denote by $R(X)$ and $R^p(X)$ the closures of $\mathcal{R}(X)$ in $C(X)$ and in $L^p(X, dA)$, respectively. There is a fundamental distinction between approximation in the ranges $1 \leq p < 2$ and $p \geq 2$. The following disparity was first noticed by Sinanian [81] in 1965 and subsequently rediscovered by the author [16] :

(1) $R^p(X) = L^p(X)$, whenever $1 \leq p < 2$,

(2) It can happen that $R^p(X) \neq L^p(X)$, when $p \geq 2$, depending on the set X .

The arguments presented here in support of (1) and (2) are those of the author.

In order to verify the first assertion fix p with $1 \leq p < 2$ and let $g \in L^q(X)$, $q = p/(p-1)$ be any function in the dual space such that $\int Qg dA = 0$ for all rational functions Q . By assumption the Cauchy integral

$$f(z) = \int_X \frac{g(\zeta)}{\zeta - z} dA_\zeta$$

vanishes identically in the complement of X . It is also continuous since f is the convolution of an L^q function with a function in L^p for all $p < 2$. Because X has no interior, $f \equiv 0$ in the entire plane and so $g = 0$. Evidently, then, $R^p(X) = L^p(X)$.

To obtain a set X for which $R^2(X) \neq L^2(X)$ we can take advantage of the manner in which the Bergman kernel varies with certain deformations of the underlying region. Beginning with a countable dense set of points $S = \{\alpha_1, \alpha_2, \dots\}$ in $D \setminus \{0\}$, let $a_1 = \alpha_1$ and remove from D an open disk $D_1 = \{z : |z - a_1| < r_1\}$ so that $0 \in \Omega_1 = \bar{D} \setminus D_1$. For any $f \in \mathcal{R}(\Omega_1)$,

$$|f(0)|^2 \leq K_1(0, 0) \int_{\Omega_1} |f|^2 dA,$$

where $K_1(z, \zeta)$ is the Bergman kernel for Ω_1 . Next, let $a_2 = \alpha_{j_2}$ be the first point of S not contained in the closure of D_1 . Remove a second disk $D_2 = \{z : |z - a_2| < r_2\}$ from D in such a way that

- (i) D_1 and D_2 have disjoint closures;
- (ii) $0 \in \Omega_2 = \bar{D} \setminus (D_1 \cup D_2)$;
- (iii) $K_2(0, 0) - K_1(0, 0) < \frac{1}{2}$ with $K_j(z, \zeta)$ being the Bergman kernel for Ω_j , $j = 1, 2$.

To ensure that property (iii) is satisfied one has only to choose the radius of D_2 sufficiently small (cf. [16], p. 301). Continue in this way to obtain a sequence of disks

D_j , $j = 1, 2, \dots$, and a sequence of closed regions $\Omega_n = \bar{D} \setminus \bigcup_{j=1}^n D_j$, $n = 1, 2, \dots$, such that

(i) the D_j have mutually disjoint closures;

(ii) $\bigcap_{n=1}^{\infty} \Omega_n$ has no interior and contains the point $z = 0$;

(iii) $K_{n+1}(0, 0) - K_n(0, 0) \leq \frac{1}{2^n}$,

where $K_j(z, \zeta)$ is the Bergman kernel for Ω_j , $j = 1, 2, \dots$.

The set $X = \bar{D} \setminus \bigcup_{j=1}^{\infty} D_j = \bigcap_{j=1}^{\infty} \Omega_j$ is compact, and $0 \in X$. If $f \in \mathcal{R}(X)$ then there exists an integer n_0 such that $f \in \mathcal{R}(\Omega_n)$ whenever $n \geq n_0$, and therefore

$$|f(0)|^2 \leq K_n(0, 0) \int_{\Omega_n} |f|^2 dA$$

as soon as $n \geq n_0$. It follows from the monotone convergence theorem for integrals that

$$|f(0)|^2 \leq \left(K_1(0, 0) + \sum_{n=1}^{\infty} \frac{1}{2^n} \right) \int_X |f|^2 dA$$

for all $f \in \mathcal{R}(X)$. Thus, $R^2(X)$ has a BPE at 0, and so $R^2(X) \neq L^2(X)$.

The first steps toward answering the question which initiated this discussion were taken by Émile Borel during the last decade of the nineteenth century. By 1892 he had conceived that it must be possible to extend the theory of analytic functions to larger classes of functions defined on sets without interior points in such a way that the distinctive property of unique continuation is retained. To that end he introduced the notion of a monogenic function. By definition a function f defined on a set E is **monogenic** at a point $x_0 \in E$ if it has a derivative at x_0 in the sense that

$$\lim_{x \rightarrow x_0, x \in E} \frac{f(x) - f(x_0)}{x - x_0}$$

exists through points of E . Thus, if Ω is an open set then a function f is monogenic at each point of Ω if and only if it is analytic in Ω . Borel's main result [14], published twenty-five years later, was to construct a compact set X having no interior, but containing a large dense subset E such that every function monogenic on X is uniquely determined by its value and the values of all its derivatives at any individual point of E (cf. also [12]).

In the process he also obtained an integral representation of Cauchy type for monogenic functions. Although this aspect of Borel's many contributions to mathematics never gained wide attention, it did nevertheless provide a basis and inspiration for

the subsequent development of the theory of quasianalytic functions of a real variable by Denjoy, Carleman, Mandelbrojt and Bang. All of these ideas are, in one way or another, embedded in the following example.

Theorem 10 (1973). *There exists a compact set X_0 having no interior and positive dA measure such that whenever two functions in $R^p(X_0)$, $p \geq 2$ coincide on a set of positive measure in X_0 , they coincide almost everywhere.*

The first example of this kind in connection with L^p approximation was constructed by Sinanian [81] in 1965. At that time he was able to establish the existence of a compact nowhere dense set X with the property that whenever two functions in $R^p(X)$, $p \geq 2$, agree on a **relatively open** subset of X , they agree almost everywhere. Theorem 10, therefore, represents a strengthening of Sinanian's result. The history of the various attempts to extend the uniqueness property of the analytic functions to broader classes defined on sets without interior points is not, however, entirely clear. There is an oblique reference in Mergelian's famous paper [64], pp. 317-318, to the existence of a compact nowhere dense X such that the functions in $R(X)$ are monogenic in the sense described above, and so if two of them coincide on a certain portion of X , they are identical everywhere on X . Sinanian ([81], p. 1365 and [63], p. 745) seems to credit this result to Keldysh, but no reference is ever given. In 1975 Gonchar constructed an example in which any subset of X having positive 1-dimensional Hausdorff measure uniquely determines the functions in $R(X)$. His argument is similar in spirit to the proof of Theorem 10 outlined below in that it avoids lengthy and cumbersome computations.

Proof of Theorem 10. The idea is to construct a compact nowhere dense set X_0 of finite perimeter (a **Swiss cheese**) in such a way that it contains a sequence of subsets X_k , $k = 1, 2, \dots$, having the following properties :

- (1) each X_k is a Swiss cheese whose complementary components are bounded by polygonal arcs ;
- (2) $\text{meas } \{X_0 \setminus X_k\} \rightarrow 0$ as $k \rightarrow \infty$;
- (3) for each k there are positive constants A_n , $n = 0, 1, 2, \dots$, such that

$$|f^{(n)}(\zeta)| \leq A_n \|f\|_{L^2(X_0)}$$

for all $f \in \mathcal{R}(X_0)$ and all $\zeta \in X_k$;

$$(4) \sum_{n=0}^{\infty} \left(\frac{1}{A_n} \right)^{1/n} = \infty.$$

The crucial properties (3) and (4) are easily arranged by taking advantage of the fact that, not only the values of the Bergman kernel, but also those of all its derivatives

can be controlled under deformation of the underlying region in the manner described above.

Suppose now that X_0 has been constructed as indicated, and that a function $f \in \mathcal{R}^2(X_0)$ vanishes on a set of positive dA measure. By assumption f is the limit in norm of a sequence f_j , $j = 1, 2, \dots$, belonging to $\mathcal{R}(X_0)$, and by (3) the sequence of derivatives $f_j^{(n)}$, $j = 1, 2, \dots$, converges uniformly on X_k , $k = 1, 2, \dots$, for each n . Thus, f can be viewed as a C^∞ function on any line segment lying in X_k , $k = 1, 2, \dots$. Since X_k has finite perimeter almost all lines in a fixed direction meet only finitely many of its complementary components, and therefore intersect X_k in a collection of linear segments. By choosing k sufficiently large we may assume that $f = 0$ on a set of positive dA measure in X_k , and we can then infer from Fubini's theorem that $f = 0$ on a set of positive linear measure on some line segment l lying in X_k . Because $\sup |f^{(n)}| \leq A_n \|f\|_{L^2(X_0)}$ and property (4) is satisfied, f satisfies Denjoy's criterion (cf. [18], p. 314 and [12], p. 126) for membership in a quasianalytic class on l . If, on the other hand, x_0 is a point of linear density in l for the zero set of f one can easily verify that $f^{(n)}(x_0) = 0$, $n = 0, 1, \dots$. Hence, by quasianalyticity, f vanishes identically on l . Moreover, we can arrange that x_0 can be joined to almost any other point of X_k by a polygonal arc in X_k whose initial segment lies in l . Again, by quasianalyticity, f vanishes identically along any such arc, since at each vertex its derivatives coincide in the appropriate directions. Consequently, $f = 0$ almost everywhere on X_k , $k = 1, 2, \dots$, and therefore by property (2) almost everywhere on X_0 . Theorem 10 is proved.

§4. THE INTEGRABILITY PROBLEM

Let Ω be a simply connected domain having at least two boundary points in the extended complex plane, and let $\phi : \Omega \rightarrow D$ be a conformal map of Ω onto the open unit disk D . The following question arose in connection with the L^p -completeness problem for polynomials on domains of crescent type as presented here in Section 2.

Question : For which values of p , $1 \leq p < \infty$ is $\int_\Omega |\phi'|^p dA < \infty$?

For $p = 2$ the integral represents the area of the image disk D , and is therefore finite. By the early 1970's it was known to converge for $\frac{4}{3} < p < 3$, and if Ω is the plane slit along the negative real axis it clearly diverges for $p = \frac{4}{3}$ and $p = 4$, as the Koebe function shows. These facts were apparently first discovered by Gehring and Hayman (unpublished) for $\frac{4}{3} < p < 2$ and published by Metzger (cf. [20]) for $2 < p < 3$. It has been known for some time that the upper bound 3 can be increased (cf. [20]).

Theorem 11 (1978). *There exists a constant $\tau > 0$, not depending on the region*

Ω , such that

$$\int_{\Omega} |\phi'|^p dA < \infty,$$

whenever $\frac{4}{3} < p < 3 + \tau$.

For a wide class of regions, including **star-like** and **close-to-convex** domains, $p = 4$ is the correct upper bound. The same is true if either of these is satisfied locally along the boundary; for example, if $\partial\Omega$ is locally the graph of a function. All available evidence seems to indicate that $\int |\phi'|^p dA < \infty$ for $\frac{4}{3} < p < 4$ in all cases, but that is still unproven.

Here is a sketch of the author's original proof of Theorem 11 in [20]. Fix $x_0 \in \Omega$ with $\phi(x_0) = 0$ and let $\delta(z) = \text{dist}(z, \partial\Omega)$. It follows from the coarea formula, or in this case from polar coordinates, that

$$\int_{\Omega} |\phi'|^p dA = \int_0^1 2\pi r \int_{|\phi|=r} |\phi'|^{p-2} d\omega_r dr, \quad (4.1)$$

where $d\omega_r$ is harmonic measure on the curve $|\phi| = r$ relative to x_0 . Moreover, by the Koebe distortion theorem (cf. [71], p. 22) $|\phi'(z)| \approx \frac{1-|\phi(z)|}{\delta(z)}$ near $\partial\Omega$ and, consequently, the integral in (4.1) converges if, and only if,

$$\int_0^1 (1-r)^{p-2} \int_{|\phi|=r} \frac{d\omega_r}{\delta(z)^{p-2}} dr < \infty.$$

The proof of the theorem is now completed by showing that there exists a constant $\rho > 0$, such that if $\lambda > \frac{1}{2}$, then

$$\int_{|\phi|=r} \frac{d\omega_r}{\delta(z)^\lambda} = O \left[\frac{1}{(1-r)^{2\lambda-\rho}} \right], \quad r \rightarrow 1, \quad (4.2)$$

Theorem 11 is proved.

If we could establish (4.2) for all $\rho < 1$ it would follow that $\int |\phi'|^p dA < \infty$ for $\frac{4}{3} < p < 4$. However, the reasoning employed here, which is based on a combinatorial argument of Carleson [28], will not give that result. The principle focus in [28] was concentrated on the question : On a Jordan curve, is harmonic measure absolutely continuous with respect to α -dimensional Hausdorff measure for every $\alpha < 1$? This and the integrability problem are each questions about distortion under a conformal mapping, and that distortion can be manifested in two ways : By the

- (i) expansion and/or compression of boundary sets;
- (ii) growth and/or decay of $|\phi'|$ at the boundary.

In the case of harmonic measure Makarov has shown that the answer is yes, while Lavrent'ev, McMillan and Piranian, and Carleson have shown by means of counterexamples that absolute continuity does not always occur if $\alpha = 1$. Each of those examples depends on the existence of an abundance of twist points in $\partial\Omega$. By definition a point $\xi \in \partial\Omega$ is a **twist point** if

$$\limsup_{z \rightarrow \xi, z \in \Omega} \arg(z - \xi) = +\infty, \quad \liminf_{z \rightarrow \xi, z \in \Omega} \arg(z - \xi) = -\infty.$$

Additional information and references on this topic can be found in Pommerenke's two books [71] and [73]. It would be interesting to know to what extent the similarity between the two problems can be pursued. Is the extremal situation for the integrability of the derivative of a conformal map tied to the existence of twist points? If there are no twist points in $\partial\Omega$ does it follow more easily that $\int_{\Omega} |\phi'|^p dA < \infty$ if $\frac{4}{3} < p < 4$?

One can choose to work either on Ω or on D . Setting $f = \phi^{-1}$, it suffices to show that

$$(a) \quad \int_D |f'|^{2-p} dA < \infty, \quad \frac{4}{3} < p < 4,$$

or, by analogy with (4.2), to obtain a suitable estimate for the rate of growth of the integral means

$$(b) \quad I_t(r, f') = \int_0^{2\pi} |f'(re^{i\theta})|^t d\theta,$$

particularly in the range $-2 < t < 0$. The precise bound depends on the special nature of the function

$$B(t) = \inf \{ \beta : I_t(r, f') = O((1-r)^{-\beta}) \text{ for all univalent } f \},$$

which was introduced by Makarov and Pommerenke and is known as the **universal integral means spectrum** for conformal maps. The main integrability conjecture is more or less equivalent to the assertion that $B(-2) = 1$. In general, $B(t)$ is a convex function defined for $-\infty < t < \infty$, and Pommerenke [72] (cf. also [73] and [74]) has shown that

$$B(t) \leq -\frac{1}{2} + t + \sqrt{\frac{1}{4} - t + 4t^2}, \quad B(t) < |t| - 0.399 \text{ for } t \leq -1,$$

from which it follows that $B(-2) < 1.601$ and that the integral in (a) is finite whenever $\frac{4}{3} < p < 3.399$. His argument is carried out on the disk D , and is based on an estimate

for the Schwarzian derivative and certain differential inequalities. By considering higher or generalized Schwarzian derivatives in his thesis [8], Bertilsson has shown that the upper bound for the range of integrability can be increased to 3.421, and that is currently the best known estimate of its kind.

A more detailed analysis of the function $B(t)$ can be found in the article of Carleson and Makarov [29], where a kind of phase transition is identified (cf. also Bertilsson [8] for an excellent summary and description of those results). It is now known that there exist constants $t_- < 0$ and $t_+ > 0$ such that

$$B(t) = \begin{cases} 3t - 1 & \text{if } t > t_+, \\ -t - 1 & \text{if } t < t_-, \end{cases} \quad (4.3)$$

while examples constructed by Makarov and Rohde involving lacunary series demonstrate that

$$B(t) \sim c t^2 \text{ if } |t| \text{ is small.} \quad (4.4)$$

It follows from the work of Feng and MacGregor (cf. [73], p. 177) that $t_+ \leq \frac{1}{3}$ and from that of Carleson and Makarov [29] that $t_- \leq -2$. For a proof of (4.4) see Pommerenke [73], p. 192. Evidently, then, if $t < t_-$ the Koebe function gives nearly the maximal rate of growth for the integral means (b), but if $t_- < t < t_+$ the rate of growth is more rapid. And, as Bertilsson has suggested, for $|t|$ small the extremal rate is (due to the intervention of lacunary series) most likely achieved by a function f mapping D onto a region Ω with a fractal boundary. Should phase transition occur at $t_- = -2$ then the main integrability conjecture is true, and we obtain as a corollary a comparison between the Hausdorff dimension of an arbitrary set E in ∂D with that of its image $f(E)$ in $\partial\Omega$; in particular,

$$\dim f(E) > \frac{2 \dim E}{4 - \dim E} > \frac{\dim E}{2},$$

which is consistent with earlier results of Makarov (cf. [73], p. 232). This, of course, tends to support the conjecture.

In order to test the integrability conjecture on domains with fractal boundaries one is led to consider complex dynamics as a source of such domains. For a fixed $c \in \mathbb{C}$ let $F(z) = F_c(z) = z^2 + c$. By definition $F^n = F \circ \dots \circ F$ is the n -fold iterate of F with itself and

$$\Omega_c = \{z \in \mathbb{C} : F^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

Thus, $\infty \in \Omega_c$, and in some cases Ω_c is simply connected. Also by definition

$$\mathcal{M} = \{c \in \mathbb{C} : \Omega_c \text{ is simply connected}\} = \{c \in \mathbb{C} : J = \partial\Omega_c \text{ is connected}\},$$

and is known as the **Mandelbrot set**. Here, $J = \partial\Omega_c$ is the **Julia set** of F and is most often a fractal curve.

Given $c \in \mathcal{M}$ let $\psi : D \mapsto \Omega_c$ be a conformal map with $\psi(0) = \infty$. Recently, Baranski, Vol'berg and Zdunik [4] have verified :

Theorem 12 (1998). *For every $c \in \mathcal{M}$, the integral $\int_{\Omega_c} |\psi'|^{2-p} dA < \infty$ whenever $\frac{4}{3} < p < 4$.*

The main difficulty here, of course, is to establish 4 as the correct upper bound. The proof, on the other hand, is surprisingly elementary in comparison with other arguments which have been brought to bear on the problem, particularly given the extreme fractal nature of $\partial\Omega_c$.

Over the years the integrability question has gained wide attention and some interesting connections with other problems have been uncovered. As recently as 2002 Bishop [13] has exhibited a deep connection between some work of Sullivan and Thurston on hyperbolic geometry in three dimensions, and more specifically on work related to a long standing open problem concerning quasiconformal extension in $\mathbb{R}^3$. Carleson and Makarov ([30], [31]) have indicated some connections with certain problems in mathematical physics. Links to other areas of function theory can also be found in Hedenmalm [49].

§5. CAUCHY INTEGRAL AND ANALYTIC CAPACITY

The concept of analytic capacity was introduced by Ahlfors in 1947 in connection with the problem of characterizing sets of removable singularities for bounded analytic functions, otherwise known as the **Painlevé problem**. In the ensuing years others, and A. G. Vitushkin in particular, further developed the concept and used it to settle a number of questions concerning uniform approximation by rational functions on compact subsets of the plane. An extensive discussion of analytic capacity and its many applications can be found collectively in the articles of Vitushkin [90], Mel'nikov and Sinanian [63] and in Gamelin's book [38].

The **analytic capacity** of a compact set X , denoted $\gamma(X)$, is defined as follows :

$$\gamma(X) = \sup |f'(\infty)|,$$

where the supremum is extended over all functions f analytic in $\mathbb{C} \setminus X$ and normalized so that

$$\|f\|_\infty = \sup_{\mathbb{C} \setminus X} |f| \leq 1, \quad f(\infty) = 0.$$

In this case there exists a unique admissible f with $f'(\infty) = \gamma(X)$. For an arbitrary planar set E we let $\gamma(E) = \sup \gamma(X)$, the supremum being taken over all compact

sets $X \subseteq E$. Two properties of capacity which are of critical importance here are these :

- (i) $\gamma(B_r) = r$ for every disk B_r of radius r
- (ii) $\gamma(X) \approx \text{diam}(X)$ whenever X is compact and connected ; in particular, $\gamma(X) \leq \text{diam}(X) \leq 4\gamma(X)$.

Unfortunately, insofar as it is not known to be subadditive, analytic capacity does not appear to be a genuine capacity. In the past this has made it especially difficult to work with in certain situations. Tolsa [88], however, has recently shown that analytic capacity is **semiadditive** in the sense that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F))$$

for all compact sets $E, F \subseteq \mathbb{C}$ and some absolute constant C . The key consists in showing that γ is equivalent to another capacity γ^+ which is itself known to be semiadditive, and which for our purpose is more directly linked to the Cauchy integral. For a compact set X ,

$$\gamma^+(X) = \sup_{\nu} \nu(X),$$

where the supremum is taken over all positive measures ν supported on X such that the Cauchy integral $\hat{\nu} \in L^\infty(\mathbb{C})$ and $\|\hat{\nu}\|_\infty \leq 1$. Since $\hat{\nu}$ is analytic in $\mathbb{C} \setminus X$ and $\nu'(\infty) = \nu(X)$, the function $\hat{\nu}$ is admissible in the definition of γ and therefore $\gamma^+(X) \leq \gamma(X)$. As before, if E is an arbitrary planar set, then

$$\gamma^+(E) = \sup_X \gamma^+(X),$$

where X is compact and $X \subseteq E$.

The capacity γ^+ seems to have been effectively introduced by Havinson at an international conference on function theory held in Erevan during the summer of 1965 (cf. [46]). In defining $\gamma^+(X)$, however, he would allow all measures ν for which $\|\hat{\nu}\|_\infty \leq 1$ in $\mathbb{C} \setminus X$, which in retrospect is evidently an unnecessary requirement. Tolsa's theorem (cf. [88]) which represents a major advance in the theory of analytic capacity is this :

Theorem 13 (2003). *There exists an absolute constant $C > 0$ so that*

$$\gamma^+(E) \leq \gamma(E) \leq C\gamma^+(E)$$

for all sets $E \subseteq \mathbb{C}$; that is, $\gamma \approx \gamma^+$.

Since γ^+ is in fact countably semiadditive (cf. [87], [88]), the same is also true of γ :
 If E_n , $n = 1, 2, \dots$, are Borel sets, then

$$\gamma\left(\bigcup_n E_n\right) \leq C \sum_n \gamma(E_n), \quad (5.1)$$

where C is once again an absolute constant.

In order to employ analytic capacity in connection with certain problems it is convenient to have a substitute for the notion of fine continuity, a concept described earlier in Section 2. A substitute is provided by the following lemma. It can be viewed as a strengthening of a prior result of the author ([17], Lemma 2), and was suggested by an idea of Carleson ([27], Lemma 1) used to give a short proof of Mergelian's theorem on uniform polynomial approximation. As is the custom, given a positive measure ν the corresponding Newtonian potential is designated

$$U^\nu(z) = \int |\zeta - z|^{-1} d\nu(\zeta)$$

Lemma 3 (2003). *Let μ be a finite complex, compactly supported, Borel measure in $\mathbb{C}$, and let x_0 be any point where $U^{|\mu|}(x_0) < \infty$. For each $r > 0$ let $B_r = B(x_0, r)$ be the disk with center at x_0 and radius r , and let E be a set with the property that for every $r > 0$ there is a relatively large subset $E_r \subseteq (E \cap B_r)$ on which $U^{|\mu|}$ is bounded ; that is*

$$(1) \ U^{|\mu|} \leq M_r < \infty \text{ on } E_r$$

$$(2) \ \gamma(E_r) \geq \varepsilon \gamma(E \cap B_r) \text{ for some absolute constant } \varepsilon.$$

If, moreover, E is thick at x_0 in the sense that

$$\limsup_{r \rightarrow 0} \frac{\gamma(E \cap B_r)}{r} > 0, \quad (5.2)$$

then

$$|\hat{\mu}(x_0)| \leq \limsup_{z \rightarrow x_0, z \in E} |\hat{\mu}(z)|.$$

Proof : By Tolsa's theorem $\gamma \approx \gamma^+$, and so each of the hypotheses in the lemma remain valid for γ^+ with possibly different constants. In particular, according to (5.2),

$$\limsup_{r \rightarrow 0} \frac{\gamma^+(E \cap B_r)}{r} > 0.$$

Thus, there exists a constant $C > 0$ and a sequence of $r \rightarrow 0$ such that $\gamma^+(E_r) > Cr$ for each corresponding r . Consistent with the definition of γ^+ we can then select positive measures σ_r on E_r such that

$$(a) \|\sigma_r\| = \sigma_r(E_r) > Cr$$

$$(b) |\hat{\sigma}_r| \leq 1 \text{ a.e.} - dA.$$

Setting $\nu_r = \frac{\sigma_r}{\|\sigma_r\|}$ we obtain a sequence of probability measures, and it can be easily checked that

$$(c) \nu_r \text{ is carried by } E_r \subseteq (E \cap B_r)$$

$$(d) \lim_{r \rightarrow 0} \int \hat{\nu} d\nu_r = \hat{\nu}(x_0).$$

From (d) the desired conclusion is immediate :

$$|\hat{\nu}(x_0)| \leq \limsup_{z \rightarrow x_0, z \in E} |\hat{\nu}(z)|.$$

Lemma 3 is proved.

The following two well-known theorems of Lavrent'ev and Vitushkin are straightforward corollaries of Lemma 3. Given a compact set $X \subseteq \mathbb{C}$ let $P(X)$ and $R(X)$ denote the closure in $C(X)$ of the polynomials and rational functions, respectively.

Corollary 1 (Lavrent'ev 1936). $P(X) = C(X)$ if and only if X has no interior and the complement $\mathbb{C} \setminus X$ is connected.

Corollary 2 (Vitushkin 1958). $R(X) = C(X)$ if and only if,

$$\limsup_{r \rightarrow 0} \frac{\gamma(B(x, r) \setminus X)}{r} > 0$$

for dA almost every $x \in X$.

Proof of Corollary 2. Let ν be any measure on X such that $\int f d\nu = 0$ for every rational function f having no poles on X ; that is, for every $f \in \mathcal{R}(X)$. Clearly, then, $\hat{\nu} = 0$ identically in $\mathbb{C} \setminus X$. Fix a point $x_0 \in X$ where $U^{|\nu|}(x_0) < \infty$, and let $B_r = B(x_0, r)$ be the disk with center at x_0 and radius r . Since $U^{|\nu|}$ is bounded on any compact subset of $\mathbb{C} \setminus X$ and

$$\limsup_{r \rightarrow 0} \frac{\gamma(B_r \setminus X)}{r} > 0,$$

the requirements of Lemma 3 are satisfied and therefore

$$|\hat{\nu}(x_0)| \leq \limsup_{z \rightarrow x_0, z \in B_r \setminus X} |\hat{\nu}(z)| = 0.$$

Because $U^{|\nu|} < \infty$ a.e.- dA on X , it follows that $\hat{\nu} = 0$ a.e.- dA . Hence, $\nu = 0$ as a measure and $R(X) = C(X)$. For a proof in the other direction, see [38], p. 207.

Corollary 2 is proved.

Except for certain technical, but substantial, difficulties Theorem 1 can now be established along similar lines. In order to give a brief indication of the argument consider the following : For each positive integer n form a grid in the plane consisting of lines parallel to the coordinate axes, intersecting at those points whose coordinates are both integral multiples of 2^{-n} . The resulting collection of squares $\mathcal{G}_n = \{S_n\}_{j=1}^\infty$ of side length 2^{-n} is an edge-to-edge tiling of the plane ; its members will be referred to as squares of the n -th generation.

Theorem 1 (1991). *If μ is a positive measure of compact support in $\mathbb{C}$, not concentrated at a single point, then $H^2(d\mu) = L^2(d\mu)$ if and only if $H^2(d\mu)$ has no BPE's.*

Proof (outline 2003) : Suppose that $H^2(d\mu)$ has no BPE's. Let g be any function in $L^2(d\mu)$ with the property that $\int P g d\mu = 0$ for all polynomials P , and set $\nu = g\mu$. And, fix a point x_0 at which the potential $U^{|\nu|}(x_0) < \infty$.

For an arbitrary, but fixed, $\lambda > 0$ consider the set $E_\lambda = \{z : |\hat{\nu}(z)| < \lambda\}$. Because there are no BPE's the set E_λ cannot surround x_0 , and in a sense must escape to ∞ . More specifically, we can find a compact connected set X linking x_0 to ∞ such that X is the union of squares from the n -th generation and higher, and certain narrow rectangular channels R_j , $j > n$, where

- (1) $|E_\lambda \cap S| > \frac{1}{100} |S|$ for each square $S \subseteq X$;
- (2) $\text{diam}(R_j) \approx j^2 2^{-j}$.

Given $r > 0$, let $B_r = B(x_0, r)$. By discarding certain superfluous pieces we can assume that $X \cap B_r$ is connected and joins x_0 to ∂B_r . Hence, by property (ii)

$$\gamma(X \cap B_r) \geq \frac{1}{4} \text{diam}(X \cap B_r) \geq \frac{r}{8}.$$

On the other hand, it follows from the countable semiadditivity of analytic capacity that

$$\frac{r}{16} \leq \gamma(X \cap B_{r/2}) \leq C[\gamma(K) + \sum_{j=n}^{\infty} j^2 2^{-j}], \quad (5.3)$$

where K is the union of the squares in X for which (1) is satisfied, and where C is an absolute constant. Since construction of the set X can begin with squares from an arbitrary generation $\mathcal{G}_n$, we are free to choose n as large as we please and (5.3) remains valid with the same constant C . In particular, n can be chosen large enough so that the infinite sum on the right side of (5.3) is negligible, from which we conclude that $\gamma(K) > Cr$ for another absolute constant. It follows from (1) and (2) that

$$\gamma(E_\lambda \cap B_r) \geq C\epsilon\gamma(K) > C\epsilon r,$$

for some constant ε independent of r (cf. [25], Lemma 3 and [62]). Therefore, E_λ is thick at x_0 in the sense that

$$\limsup_{r \rightarrow 0} \frac{\gamma(E_\lambda \cap B_r)}{r} > 0.$$

The hypotheses of Lemma 3 are therefore satisfied, and so

$$|\hat{\nu}(x_0)| \leq \limsup_{z \rightarrow x_0, z \in E_\lambda} |\hat{\nu}(z)| \leq \lambda.$$

Since the latter inequality is valid for all $\lambda > 0$, we can infer that $\hat{\nu}(x_0) = 0$. Hence $\hat{\nu} = 0$ a.e.- dA and $\nu = g\mu = 0$ as a measure. Consequently, $H^2(d\mu) = L^2(d\mu)$. Theorem 1 is proved.

In case $H^2(d\mu) \neq L^2(d\mu)$ it follows from an argument based on the Cauchy integral that there exists a point ξ_0 and an open set U containing ξ_0 such that every function $f \in H^2(d\mu)$ admits an analytic continuation to U (cf. [25]). The same technique can be used in many different situations, including those considered here, to establish a reciprocal relation between completeness and analytic continuation. Several of these applications can be found in [23].

REFERENCES

1. D. R. Adams, L. I. Hedberg, *Function Spaces and Potential Theory*, Springer, Berlin, 1996.
2. L. V. Ahlfors, "Finitely generated Kleinian groups", *Amer. J. Math.*, vol. 86, pp. 413 – 429, 1964.
3. T. Bagby, "Quasi topologies and rational approximation", *J. Funct. Anal.*, vol. 10, pp. 259 – 268, 1972.
4. K. Baranski, A. Vol'berg, A. Zdunik, "Brennan's conjecture and the Mandelbrot Set", *Internat. Math. Res. Notices*, no. 12, pp. 589 – 600, 1998.
5. S. N. Bernstein, "Le probleme de l'approximation des fonctions continues sur tout l'axe réel et l'une de ses applications", *Bull. Soc. Math. France*, vol. 52, pp. 399 – 410, 1924.
6. L. Bers, "An approximation theorem", *J. Anal. Math.*, vol. 14, pp. 1 – 4, 1965.
7. L. Bers, " L_1 approximation of analytic functions", *J. Indian Math. Soc.*, vol. 34, pp. 193 – 201, 1970.
8. D. Bertilsson, "On Brennan's conjecture in conformal mapping", *Doctoral Thesis*, Royal Institute of Technology, Stockholm, Sweden, 1999.
9. A. Beurling, "On quasianalyticity and general distributions", *Multilithed Lecture Notes*, Summer School, Stanford University, Stanford, 1961.
10. A. Beurling, "Analytic continuation across a linear boundary", *Acta Math.*, vol. 128, pp. 153 – 182, 1972.
11. A. Beurling, *Collected Works*, vol. 1, Birkhäuser, Boston, 1989.
12. G. G. Bilodeau, "The origin and early development of non-analytic infinitely differentiable functions", *Arch. Hist. Exact Sci.*, vol. 27, pp. 115 – 135, 1982.

13. C. J. Bishop, "Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture", *Ark. Mat.*, vol. 40, pp. 1 – 26, 2002.
14. E. Borel, *Leçons sur les Fonctions Monogènes Uniformes d'une Variable Complexe*, Gauthier-Villars, Paris, 1917.
15. J. Bram, "Subnormal operators", *Duke Math. J.*, vol. 22, pp. 75 – 94, 1955.
16. J. E. Brennan, "Invariant subspaces and rational approximation", *J. Funct. Anal.*, vol. 7, pp. 285 – 310, 1971.
17. J. E. Brennan, "Invariant subspaces and weighted polynomial approximation", *Ark. Mat.*, vol. 11, pp. 167 – 189, 1973.
18. J. E. Brennan, "Approximation in the mean and quasianalyticity", *J. Funct. Anal.*, vol. 12, pp. 307 – 320, 1973.
19. J. E. Brennan, "Approximation in the mean by polynomials on non-Carathéodory domains", *Ark. Mat.*, vol. 15, pp. 117 – 168, 1977.
20. J. E. Brennan, "The integrability of the derivative in conformal mapping", *J. London Math. Soc.*, vol. 18, pp. 261 – 272, 1978.
21. J. E. Brennan, "Point evaluations, invariant subspaces and approximation in the mean by polynomials", *J. Funct. Anal.*, vol. 34, pp. 407 – 420, 1979.
22. J. E. Brennan, "Weighted polynomial approximation, quasianalyticity and analytic continuation", *J. Reine Angew. Math.*, vol. 357, pp. 23 – 50, 1985.
23. J. E. Brennan, "The Cauchy integral and analytic continuation", *Math. Proc. Cambridge Philos. Soc.*, vol. 97, pp. 491 – 498, 1985.
24. J. E. Brennan, "Weighted polynomial approximation and quasianalyticity for general sets", *Algebra i Analiz*, vol. 6, pp. 69 – 89, 1994; [English transl. *St. Petersburg Math. J.*, vol. 6, pp. 763 – 779, 1995].
25. J. E. Brennan, "Thomson's theorem on mean square polynomial approximation", (preprint).
26. T. Carleman, "Über die Approximation analytische Funktionen durch lineare Aggregate von vorgegebenen Potenzen", *Ark. Mat. Astr. Fys.*, vol. 17, pp. 1 – 30, 1923.
27. L. Carleson, "Mergelyan's theorem on uniform polynomial approximation", *Math. Scand.*, vol. 15, pp. 167 – 175, 1965.
28. L. Carleson, "On the distortion of sets on a Jordan curve under conformal mapping", *Duke Math. J.*, vol. 40, pp. 547 – 559, 1973.
29. L. Carleson, N. G. Makarov, "Some results connected with Brennan's conjecture", *Ark. Mat.*, vol. 32, pp. 33 – 62, 1994.
30. L. Carleson, N. G. Makarov, "Aggregation in the plane and Loewner's equation", *Comm. Math. Phys.*, vol. 216, pp. 583 – 607, 2001.
31. L. Carleson, N. G. Makarov, "Laplacian path models", *J. Anal. Math.*, vol. 87, pp. 103 – 150, 2002.
32. J. Deny, "Les potentials d'énergie finie", *Acta Math.*, vol. 82, pp. 107 – 183, 1950.
33. P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
34. E. M. Dynkin, "Functions with a given estimate for $\partial f / \partial \bar{z}$ and a theorem of N. Levinson" (in Russian), *Mat. Sb.*, vol. 89, no. 2, pp. 182 – 190, 1972; [English transl. : *Math. USSR-Sb.* vol. 18, pp. 181 – 189, 1972].
35. M. M. Džrbashian, "Metric theory of completeness and representation of analytic functions" (in Russian), Thesis, Yerevan, 1948.
36. O. J. Farrell, "On approximation to an analytic function by polynomials", *Bull. Amer. Math. Soc.*, vol. 40, pp. 908 – 914, 1934.

37. H. Federer, "Curvature measures", *Trans. Amer. Math. Soc.*, vol. 93, pp. 418 – 491, 1959.
38. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, 1969.
39. R. F. Gariepy, L. C. Evans, *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.
40. P. R. Halmos, "Normal dilations and extensions of operators", *Summa Brasilien-sis Mathematicae*, vol. 2, pp. 125 – 134, 1950.
41. F. Hartogs and A. Rosenthal, "Über Folgen analytischer Funktionen", *Math. Ann.*, vol. 104, pp. 606 – 610, 1931.
42. V. P. Havin, "Approximation in the mean by analytic functions" (in Russian), *Dokl. Akad. Nauk SSSR*, vol. 178, pp. 1025 – 1028, 1968; [English transl. : *Soviet Math. Dokl.* vol. 9, pp. 245 – 248, 1968].
43. V. P. Havin, "Approximation in the mean by polynomials on certain non-Carathéodory domains I and II" (in Russian), *Izv. Vysh. Uchebn. Zaved. Mat.*, vol. 76, pp. 86 – 93, 1968 and vol. 77, pp. 87 – 94, 1968.
44. V. P. Havin, "A generalization of the Privalov-Zygmund theorem on the modulus of continuity of the conjugate function" (in Russian), *Izv. Akad. Nauk Armjan. SSR, Matem.*, vol. 6, pp. 252 – 258 and 265 – 287, 1971.
45. V. P. Havin, F. A. Shamoyan, "Analytic functions with Lipschitzian modulus of the boundary values" (in Russian), *Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI)*, vol. 19, pp. 237 – 239, 1970.
46. S. Ya. Havinson, "Representations and approximation of functions on thin sets", *Contemporary Problems in the Theory of Analytic Functions (International Conf., Yerevan 1965)*, pp. 314 – 318, Nauka, Moscow, 1966.
47. L. I. Hedberg, "Approximation in the mean by analytic functions", *Trans. Amer. Math. Soc.*, vol. 163, pp. 157 – 171, 1972.
48. L. I. Hedberg, "Non-linear potentials and approximation in the mean by analytic functions", *Math. Z.*, vol. 129, pp. 299 – 319, 1972.
49. H. Hedenmalm, "The dual of a Bergman space on simply connected domains", *J. Anal. Math.*, vol. 88, pp. 311 – 335, 2002.
50. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, 1962.
51. S. Jacobs, "Ett extremal problem för analytiska funktioner i multipelt samman-hängande omården" (in Swedish), *Uppsala Univ. Math. Dept. report*, vol. 2, 1969.
52. M. V. Keldysh, "Sur l'approximation en moyenne quadratique des fonctions analytiques", *Mat. Sb.*, vol. 5(47), pp. 391 – 401, 1939.
53. M. V. Keldysh, "Sur l'approximation en moyenne par polynomes des fonctions d'une variable complexe", *Mat. Sb.*, vol. 16, pp. 1 – 20, 1945.
54. O. D. Kellogg, "Harmonic functions and Green's integral", *Trans. Amer. Math. Soc.*, vol. 13, pp. 109 – 132, 1912.
55. S. V. Khrushchev (Hruščev), "The Brennan alternative for measures with finite entropy" (in Russian), *Izv. Akad. Nauk ArmSSR, Matem.*, vol. 14, pp. 184 – 191, 1979.
56. P. Koosis, *The Logarithmic Integral*, vol. 1, Cambridge Univ. Press, Cambridge, 1988.
57. M. A. Lavrent'ev, "Sur les fonctions d'une variable complexe représentables par des series de polynomes", Hermann, Paris 1936.

58. A. I. Markushevich, "Conformal mapping of regions with variable boundary and application to the approximation of analytic functions by polynomials" (in Russian), Thesis, Moscow, 1934.
59. V. G. Maz'ja, V. P. Havin, "Approximation in the mean by analytic functions" (in Russian), *Vestnik Leningrad Univ.*, vol. 13, pp. 62 – 74, 1968; [English transl.: *Vestnik Leningrad Univ. Math.*, vol. 1, pp. 231 – 245, 1974].
60. V. G. Maz'ja, V. P. Havin, "Non-linear potential theory" (in Russian), *Uspehi Mat. Nauk*, vol. 27, no. 6, pp. 67 – 138, 1972; [English transl.: *Russian Math. Surveys*, vol. 27, no. 6, pp. 71 – 148, 1972].
61. V. G. Maz'ja, V. P. Havin, "Applications of (p, l) -capacity to some problems in the theory of exceptional sets" (in Russian), *Mat. Sb.* vol. 90, pp. 558 – 591, 1973; [English transl.: *Math. USSR-Sb.*, vol. 19, pp. 547 – 580, 1973].
62. M. S. Mel'nikov, "On the Gleason parts of the algebra $R(X)$ " (in Russian), *Mat. Sb.*, vol. 101, pp. 293 – 300, 1976; [English transl.: *Math. USSR Sbornik*, vol. 30, pp. 261 – 268, 1976].
63. M. S. Mel'nikov, S. O. Sinanian, "Questions in the theory of approximation of functions of one complex variable" (in Russian), in *Contemporary Problems of Mathematics*, vol. 4, pp. 143 – 250, *Itogi Nauki i Tekhniki*, VINITI, Moscow, 1975; [English transl.: *J. Soviet Math.*, vol. 5, pp. 688 – 752, 1976].
64. S. N. Mergelian, "Uniform approximations to functions of a complex variable" (in Russian), *Uspehi Mat. Nauk*, vol. 7, pp. 31 – 122, 1952; [English transl.: *Amer. Math. Soc. Translations*, vol. 101, pp. 294 – 391, 1954].
65. S. N. Mergelian, "On the completeness of systems of analytic functions" (in Russian), *Uspehi Mat. Nauk*, vol. 8, pp. 3 – 63, 1953; [English transl.: *Amer. Math. Soc. Translations*, vol. 19, pp. 109 – 166, 1962].
66. S. N. Mergelian, "General metric criteria for the completeness of systems of polynomials" (in Russian), *Dokl. Akad. Nauk SSSR*, vol. 105, pp. 901 – 904, 1955.
67. S. N. Mergelian, "Weighted approximation by polynomials" (in Russian), *Uspehi Mat. Nauk*, vol. 11, pp. 107 – 152, 1956; [English transl.: *Amer. Math. Soc. Translations*, vol. 10, pp. 59 – 106, 1958].
68. S. N. Mergelian, A. P. Tamadian, "On completeness in a class of non-Jordan regions" (in Russian), *Izv. Akad. Nauk ArmSSR, Matematika*, vol. 7, pp. 1 – 17, 1954; [English transl.: *Amer. Math. Soc. Translations*, vol. 35, pp. 79 – 94, 1964].
69. N. G. Meyers, "A theory of capacities for potentials of functions in Lebesgue classes", *Math. Scand.*, vol. 26, pp. 255 – 292, 1970.
70. H. Pajot, *Notes on Analytic Capacity, Rectifiability, Menger Curvature and Cauchy Operator*, *Lecture Notes in Math.* vol. 1799, Springer, 2002.
71. Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
72. Ch. Pommerenke, "The growth of the derivative of a univalent function", in *The Bieberbach Conjecture*, *Math. Surveys Monogr.*, vol. 21, Amer. Math. Soc., 1986.
73. Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer, Berlin, 1992.
74. Ch. Pommerenke, "The integral means spectrum of univalent functions", *Zap. Nauchn. Sem. S.-Peterburg Otdel. Mat. Inst. Steklov (POMI)*, vol. 237, pp. 119 – 128, 1997; [English transl.: *J. Math. Sci.*, vol. 95, pp. 2249 – 2255, 1999].

75. C. Runge, "Zur Theorie der eindeutigen analytischen Funktionen", *Acta Math.*, vol. 6, pp. 229 – 244, 1885.
76. A. L. Shahinian, "On a criterion for the incompleteness of a system of analytic functions" (in Russian), *Dokl. Akad. Nauk ArmSSR*, vol. 5, pp. 97 – 100, 1946.
77. A. L. Shahinian, "A problem in the theory of approximation in the complex domain" (in Russian), *Sibirsk. Mat. Zhur.*, vol. 1, pp. 523 – 543, 1960.
78. H. S. Shapiro, "Some remarks on weighted polynomial approximations of holomorphic functions" (in Russian), *Mat. Sb.*, vol. 73, pp. 320 – 330, 1967; [English transl. : *Math. USSR Sbornik*, vol. 2, pp. 285 – 293, 1967].
79. N. A. Shirokov, "Properties of the moduli of analytic functions that are smooth up to the boundary" (in Russian), *Dokl. Akad. Nauk SSSR*, vol. 269, pp. 1320 – 1323, 1983; [English transl. : *Soviet Math. Dokl.*, vol. 27, pp. 507 – 510, 1983].
80. N. A. Shirokov, *Analytic Functions Smooth up to the Boundary*, *Lecture Notes in Math.*, vol. 1312, Springer, 1988.
81. S. O. Sinanian, "The uniqueness property of analytic functions on closed sets without interior points" (in Russian), *Sibirsk Mat. Z.*, vol. 6, pp. 1365 – 1381, 1965.
82. S. O. Sinanian, "Approximation by analytic functions and polynomials in the mean with respect to area" (in Russian), *Mat. Sb.*, vol. 69, pp. 546 – 578, 1966; [English transl. : *Amer. Math. Soc. Translations*, vol. 74, pp. 91 – 124, 1968].
83. S. O. Sinanian, "Approximation by polynomials in the mean with respect to area" (in Russian), *Mat. Sb.*, vol. 82, pp. 444 – 455, 1970; [English transl. : *Math. USSR Sbornik*, vol. 11, pp. 411 – 421, 1970].
84. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.
85. A. P. Tamadian, "On a theorem of M. V. Keldysh" (in Russian), *Izv. Akad. Nauk Armjan. SSR, Matematika*, vol. 6, pp. 5 – 11, 1953.
86. J. E. Thomson, "Approximation in the mean by polynomials", *Ann. of Math.* vol. 133, pp. 477 – 507, 1991.
87. X. Tolsa, "On the capacity γ^+ ", *Indiana Univ. Math. J.*, vol. 51, pp. 317 – 343, 2002.
88. X. Tolsa, "Painlevé's problem and the semiadditivity of analytic capacity", *Acta Math.*, vol. 190, pp. 105 – 149, 2003.
89. I. N. Vekua, *Generalized Analytic Functions* (in Russian), 2nd Ed., Nauka, Moscow, 1988.
90. A. G. Vitushkin, "Analytic capacity of sets and problems in approximation theory" (in Russian), *Uspehi Mat. Nauk*, vol. 22, pp. 141 – 199, 1967; [English transl. : *Russian Math. Surveys*, vol. 22, pp. 139 – 200, 1967].
91. A. L. Vol'berg, "The logarithm of an almost analytic function is summable" (in Russian), *Dokl. Akad. Nauk SSSR*, vol. 265, pp. 1297 – 1301, 1982; [English transl. : *Soviet Math. Dokl.*, vol. 26, pp. 238 – 243, 1982].
92. A. L. Vol'berg, B. Jöricke, "The summability of the logarithm of an almost analytic function and a generalization of a theorem of Levinson–Cartwright" (in Russian), *Mat. Sb.*, vol. 130, pp. 335 – 348, 1986; [English transl. : *Math. USSR-Sb.*, vol. 58, pp. 337 – 349, 1987].
93. J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain* (5th Edition), *Math. Surveys Monogr.*, vol. 20, Amer. Math. Soc., 1969.

94. S. E. Warschawski, "Über einen Satz von O. D. Kellogg", Nach. Akad. Wiss. Göttingen, pp. 73 – 86, 1932.
95. K. Weierstrass, "Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen", Sitzungsberichte der Akademie zu Berlin, pp. 633 – 639 and 789 – 805, 1885; [French transl. : J. Math. Pures Appl., vol. 2, pp. 105 – 138, 1886].
96. J. Wermer, "Report on subnormal operators", Report of an International Conference on Operator Theory and Group Representations, Arden House, New York (Nat. Res. Council publ. vol. 387), pp. 1 – 3, 1955.
97. J. H. C. Whitehead, "Manifolds with transverse fields in Euclidean space", Ann. of Math., vol. 73, pp. 154 – 212, 1961.

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ЦЕНТРАЛЬНЫЕ ПРЕДЕЛЬНЫЕ ТЕОРЕМЫ ДЛЯ ТЕПЛИЦЕВЫХ КВАДРАТИЧНЫХ ФУНКЦИОНАЛОВ ОТ СТАЦИОНАРНЫХ ПРОЦЕССОВ

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Резюме. Пусть $X(t), t \in \mathbb{R}$ – центрированный вещественнозначный стационарный гауссовский процесс со спектральной плотностью $f(\lambda)$. В статье рассматривается вопрос об асимптотическом распределении теплицева квадратичного функционала Q_T от процесса $X(t)$. Получены достаточные условия в терминах $f(\lambda)$ и ядра, гарантирующие выполнение центральной предельной теоремы для стандартно нормированных квадратичных функционалов Q_T , распространяя результаты Аврама (1986), Фокса и Такку (1987), Гиравитиса и Сургайлиса (1990), Гиновяна и Саакяна (2003) для процессов с дискретным временем на процессы с непрерывным временем.

§1. ВВЕДЕНИЕ

Пусть $X(t), t \in \mathbb{R}$ – центрированный, вещественнозначный стационарный гауссовский процесс со спектральной плотностью $f(\lambda)$ и ковариационной функцией $r(t)$, т.е.

$$r(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} f(\lambda) d\lambda. \quad (1.1)$$

Рассмотрим вопрос об асимптотическом распределении (когда $T \rightarrow \infty$) следующего теплицевого квадратичного функционала от процесса $X(t)$:

$$Q_T = \int_0^T \int_0^T \widehat{g}(t-s) X(t) X(s) dt ds, \quad (1.2)$$

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где

$$\widehat{g}(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} g(\lambda) d\lambda, \quad t \in \mathbb{R} \quad (1.3)$$

есть преобразование Фурье некоторой чётной функции $g(\lambda)$, $\lambda \in \mathbb{R}$. Назовём $g(\lambda)$ порождающей функцией функционала Q_T .

Предельное распределение функционала (1.2) определяется спектральной плотностью $f(\lambda)$ и порождающей функцией $g(\lambda)$, и в зависимости от свойств последних, может быть гауссовским (т.е. для Q_T с соответствующей нормировкой имеет место центральная предельная теорема), или не гауссовским. Рассмотрим следующие вопросы :

а) При каких условиях на $f(\lambda)$ и $g(\lambda)$ предельное распределение будет гауссовским ?

б) Описать предельное распределение Q_T , если оно не гауссовское ?

В настоящей работе мы касаемся вопроса а). Для процессов с дискретным временем этот вопрос был впервые рассмотрен в классической монографии Гренадера и Сегё [11], как применение их теории об асимптотическом поведении следа произведения усечённых теплицевых матриц. Позже эта проблема была изучена И. А. Ибрагимовым [13] и М. Розенблатом [16], в связи со статистическими оценками для спектральной ($F(\lambda)$) и ковариационной ($r(t)$) функций, соответственно. С 1986 года интерес к вопросам а) и б) возрос в связи со статистическими выводами для сильно зависимых процессов (см., например, Аврам [1], Фокс и Такку [4], Гиравитис и Сургайлис [9], Террин и Такку [17], Танигучи [20], Танигучи и Казизава [21], Гиновян и Саакян [8], и библиографию в них). В частности, Аврамом [1], Фоксом и Такку [4], Гиравитисом и Сургайлисом [9], Гиновяном и Саакяном [8] были получены достаточные условия для выполнения центральной предельной теоремы для квадратичных форм Q_T .

Для процессов с непрерывным временем вопрос а) частично изучен Ибрагимовым [13] и Гиновяном [5], [7].

В этой работе получены достаточные условия на функции $f(\lambda)$ и $g(\lambda)$, гарантирующие выполнение центральных предельных теорем для стандартно нормированных квадратичных функционалов Q_T , распространяя результаты Аврама [1], Фокса и Такку [4], Гиравитиса и Сургайлиса [9], Гиновяна и Саакяна [8] для процессов с дискретным временем на процессы с непрерывным временем.

Введём следующие обозначения : через $\tilde{Q}_T$ обозначим стандартно нормированный квадратичный функционал

$$\tilde{Q}_T = \frac{1}{\sqrt{T}} (Q_T - EQ_T). \quad (1.4)$$

Запись

$$\bar{Q}_T \Longleftrightarrow N(0, \sigma^2) \quad \text{при } T \rightarrow \infty \quad (1.5)$$

будет означать, что распределение случайной величины $\bar{Q}_T$ стремится (при $T \rightarrow \infty$) к центрированному нормальному распределению с дисперсией σ^2 . Через $B_T(\psi)$ обозначим усечённый теплицевый оператор, порождённый функцией $\psi \in L^1(\mathbb{R})$ (см. [11], [13]) :

$$[B_T(\psi)u](\lambda) = \int_0^T \hat{\psi}(\lambda - \mu)u(\mu)d\mu, \quad u(\lambda) \in L^2[0, T], \quad (1.6)$$

где $\hat{\psi}(\cdot)$ – преобразование Фурье функции $\psi(\cdot)$.

Наше исследование асимптотического поведения квадратичного функционала (1.2) основано на хорошо известном представлении кумулянта k -того порядка $\chi_k(\cdot)$ функционала $\bar{Q}_T$ (см., например, [11], [13]) :

$$\chi_k(\bar{Q}_T) = \begin{cases} 0, & \text{при } k = 1 \\ T^{-k/2} 2^{k-1} (k-1)! \operatorname{tr} [B_T(f)B_T(g)]^k, & \text{при } k \geq 2, \end{cases} \quad (1.7)$$

где $\operatorname{tr}[A]$ означает след оператора A .

Через C, M, C_k, M_k обозначим постоянные, которые могут быть разными в разных формулах.

Статья организована следующим образом : В §2 формулируются основные результаты статьи – Теоремы 2.1 – 2.5. В §3 доказываем некоторые предварительные результаты, а §4 посвящён доказательствам основных результатов.

§2. РЕЗУЛЬТАТЫ

Основными результатами данной статьи являются следующие теоремы, где без ограничения общности предполагаем, что $f, g \in L^1(\mathbb{R})$ и $g \geq 0$. Положим

$$\sigma_0^2 = 16\pi^3 \int_{-\infty}^{\infty} f^2(\lambda)g^2(\lambda) d\lambda. \quad (2.1)$$

Теорема 2.1. Предположим, что $f \cdot g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ и при $T \rightarrow \infty$

$$\chi_2(\bar{Q}_T) = \frac{2}{T} \operatorname{tr} [B_T(f)B_T(g)]^2 \rightarrow 16\pi^3 \int_{-\infty}^{+\infty} f^2(\lambda)g^2(\lambda) d\lambda. \quad (2.2)$$

Тогда $\bar{Q}_T \Longleftrightarrow N(0, \sigma_0^2)$ при $T \rightarrow \infty$.

Теорема 2.2. Если функция

$$\varphi(x_1, x_2, x_3) = \int_{-\infty}^{+\infty} f(u)g(u - x_1)f(u - x_2)g(u - x_3) du \quad (2.3)$$

принадлежит $L^2(\mathbb{R}^3)$ и непрерывна в точке $(0, 0, 0)$, то $\bar{Q}_T \Longleftrightarrow N(0, \sigma_0^2)$ при $T \rightarrow \infty$.

Теорема 2.3. Предположим, что $f(\lambda) \in L^p(\mathbb{R})$ ($p \geq 1$) и $g(\lambda) \in L^q(\mathbb{R})$ ($q \geq 1$) с $1/p + 1/q \leq 1/2$. Тогда $\tilde{Q}_T \Rightarrow N(0, \sigma_0^2)$ при $T \rightarrow \infty$.

Теорема 2.4. Пусть $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$, $fg \in L^2(\mathbb{R})$ и

$$\int_{-\infty}^{+\infty} f^2(\lambda)g^2(\lambda - \mu) d\lambda \rightarrow \int_{-\infty}^{+\infty} f^2(\lambda)g^2(\lambda) d\lambda \quad \text{при } \mu \rightarrow 0. \quad (2.4)$$

Тогда $\tilde{Q}_T \Rightarrow N(0, \sigma_0^2)$ при $T \rightarrow \infty$.

Пусть $SV(\mathbb{R})$ – класс медленно меняющихся в нуле функций $u(\lambda)$, $\lambda \in \mathbb{R}$, удовлетворяющих условиям

$$u(\lambda) \in L^\infty(\mathbb{R}), \quad \lim_{\lambda \rightarrow 0} u(\lambda) = 0, \quad u(\lambda) = u(-\lambda), \quad 0 < u(\lambda) < u(\mu) \quad \text{для } 0 < \lambda < \mu.$$

Теорема 2.5. Пусть функции f и g интегрируемы на $\mathbb{R}$ и ограничены на $\mathbb{R} \setminus (-\pi, \pi)$ и пусть

$$f(\lambda) \leq |\lambda|^{-\alpha} L_1(\lambda), \quad |g(\lambda)| \leq |\lambda|^{-\beta} L_2(\lambda), \quad \lambda \in [-\pi, \pi], \quad (2.5)$$

где $L_1(\lambda)$ и $L_2(\lambda)$ – медленно меняющиеся в нуле функции и

$$\alpha < 1, \quad \beta < 1, \quad \alpha + \beta \leq 1/2 \quad \text{и} \quad L_i \in SV, \quad \lambda^{-(\alpha+\beta)} L_i(\lambda) \in L^2(\mathbb{T}), \quad i = 1, 2. \quad (2.6)$$

Тогда $\tilde{Q}_T \Rightarrow N(0, \sigma_0^2)$ при $T \rightarrow \infty$.

Замечание 2.1. Для $p = 2$, $q = \infty$ Теорема 2.3 была доказана Ибрагимовым [13], а общий случай установлен Гиновяном [7]. Здесь мы приводим другое доказательство, основанное на Теореме 2.2.

Замечание 2.2. Аналоги Теорем 2.1 и 2.4 для процессов с дискретным временем были доказаны Гиравитисом и Сургайлисом в [9], дискретные аналоги Теорем 2.2 и 2.5 доказали Гиновян и Саакян в [8], а дискретный аналог Теоремы 2.3 доказал Аврам в [1].

§3. ПРЕДВАРИТЕЛЬНЫЕ РЕЗУЛЬТАТЫ

Лемма 3.1. Пусть $B_T(f)$ и $B_T(g)$ – усечённые теплицевы операторы, порождённые функциями $f \in L^1(\mathbb{R})$ и $g \in L^1(\mathbb{R})$, соответственно. Тогда

$$\begin{aligned} \text{tr} [B_T(f)B_T(g)]^2 &= \int_{\mathbb{R}^4} G_T(x_1 - x_3)G_T(x_3 - x_2)G_T(x_2 - x_4)G_T(x_4 - x_1) \times \\ &\times f(x_1)f(x_2)g(x_3)g(x_4) dx_1 dx_2 dx_3 dx_4. \end{aligned} \quad (3.1)$$

где

$$G_T(u) = \int_0^T e^{itu} dt = \frac{e^{iT u} - 1}{iu} = e^{iT u/2} \cdot D_T(u) \quad (3.2)$$

а D_T – ядро Дирихле, задаваемое формулой

$$D_T(u) = \frac{\sin(Tu/2)}{u/2}. \quad (3.3)$$

Доказательство. Используя (1.6), нетрудно проверить, что $[B_T(f)B_T(g)]^2$ является интегральным оператором с ядром

$$K(t, s) = \int_0^T \int_0^T \int_0^T r(t - u_1) \widehat{g}(u_1 - u_2) r(u_2 - u_3) \widehat{g}(u_3 - s) du_1 du_2 du_3,$$

где $r(t)$ и $\widehat{g}(t)$ определены в (1.1) и (1.3). Согласно формуле о следах интегральных операторов (см. [10], §3.10), имеем

$$\begin{aligned} \text{tr} [B_T(f)B_T(g)]^2 &= \int_0^T K(t, t) dt = \\ &= \int_0^T \int_0^T \int_0^T \int_0^T r(t - u_1) \widehat{g}(u_1 - u_2) r(u_2 - u_3) \widehat{g}(u_3 - t) du_1 du_2 du_3 dt. \end{aligned} \quad (3.4)$$

Учитывая (1.1), (1.3) и (3.2), из (3.4) получим (3.1). Лемма 3.1 доказана.

Рассмотрим ядро

$$\Phi_T(u) = \Phi_T(u_1, u_2, u_3) = \frac{1}{8\pi^3 T} \cdot D_T(u_1) D_T(u_2) D_T(u_3) D_T(u_1 + u_2 + u_3). \quad (3.5)$$

Лемма 3.2. Ядро $\Phi_T(u)$ обладает следующими свойствами :

- a) $\sup_T \int_{\mathbb{R}^3} |\Phi_T(u)| du = C_1 < \infty$;
 - b) $\int_{\mathbb{R}^3} \Phi_T(u) du = 1$;
 - c) $\lim_{T \rightarrow \infty} \int_{\mathbb{I}_\delta^c} |\Phi_T(u)| du = 0$ для любого $\delta > 0$;
 - d) для любого $\delta > 0$ существует постоянная $M_\delta > 0$ такая, что
- $$\int_{\mathbb{I}_\delta^c} \Phi_T^2(u) du \leq M_\delta \quad \text{для } T > 0,$$

где $\mathbb{I}_\delta^c = \mathbb{R}^3 \setminus \mathbb{I}_\delta$ и $\mathbb{I}_\delta = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : |u_i| \leq \delta, i = 1, 2, 3\}$.

Доказательство. Доказательство свойств а) - с) можно найти в [2] (Лемма 3.2). Чтобы доказать d) сначала заметим, что

$$\int_{\mathbb{R}} D_T^2(u) du \leq CT \quad \text{и} \quad |D_T(u)| \leq C_\delta \quad \text{для} \quad |u| > \delta, \quad T > 0. \quad (3.7)$$

Для $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ имеем

$$\int_{\mathbb{R}^3} \Phi_T^2(u) du \leq \int_{|u_1| > \delta} \Phi_T^2(u) du + \int_{|u_2| > \delta} \Phi_T^2(u) du + \int_{|u_3| > \delta} \Phi_T^2(u) du =: I_1 + I_2 + I_3. \quad (3.8)$$

Достаточно оценить I_1 :

$$\begin{aligned} I_1 \leq & \int_{|u_1| > \delta, |u_2| > \delta/3} \Phi_T^2(u) du + \int_{|u_1| > \delta, |u_3| > \delta/3} \Phi_T^2(u) du + \\ & + \int_{|u_1| > \delta, |u_2| \leq \delta/3, |u_3| \leq \delta/3} \Phi_T^2(u) du =: I_1^{(1)} + I_1^{(2)} + I_1^{(3)}. \end{aligned} \quad (3.9)$$

Согласно (3.7)

$$\begin{aligned} I_1^{(1)} & \leq C_\delta \cdot \frac{1}{T^2} \int_{|u_2| > \delta/3} D_T^2(u_2) D_T^2(u_3) D_n^2(u_1 + u_2 + u_3) du_1 du_3 du_2 \leq \\ & \leq C_\delta \int_{|u_2| > \delta/3} \frac{1}{u_2^2} du_2 \leq M_\delta. \end{aligned} \quad (3.10)$$

Аналогично,

$$I_1^{(2)} \leq M_\delta. \quad (3.11)$$

Заметим, что в интеграле $I_1^{(3)}$ имеем $|u_1 + u_2 + u_3| > \delta/3$, следовательно, согласно (3.7)

$$I_1^{(3)} \leq C_\delta \cdot \frac{1}{T^2} \int_{|u_1| > \delta} D_T^2(u_1) D_T^2(u_2) D_n^2(u_3) du_2 du_3 du_1 \leq C_\delta \int_{|u_1| > \delta} \frac{1}{u_1^2} du_1 \leq M_\delta. \quad (3.12)$$

Из (3.8) – (3.12) получим (3.6). Лемма 3.2 доказана.

Лемма 3.3. Если функция $\Psi(u) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ непрерывна в точке $(0, 0, 0)$, то

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^3} \Psi(u) \Phi_T(u) du = \Psi(0, 0, 0), \quad (3.13)$$

где $u = (u_1, u_2, u_3)$ и $\Phi_T(u)$ определено в (3.5).

Доказательство. В силу Леммы 3.2 b) имеем

$$R_T := \int_{\mathbb{R}^3} \Psi(u) \Phi_T(u) du - \Psi(0) = \int_{\mathbb{R}^3} [\Psi(u) - \Psi(0)] \Phi_T(u) du. \quad (3.14)$$

Для любого $\varepsilon > 0$ можно найти $\delta > 0$ такое, что

$$|\Psi(u) - \Psi(0)| < \frac{\varepsilon}{C_1}, \quad (3.15)$$

где C_1 – постоянная из Леммы 3.2 а). Рассмотрим разложение $\Psi = \Psi_1 + \Psi_2$ такое, что

$$\|\Psi_1\|_2 \leq \frac{\varepsilon}{\sqrt{M_\delta}} \quad \text{и} \quad \|\Psi_2\|_\infty < \infty, \quad (3.16)$$

где M_δ определена в Лемме 3.2 d). Применив Лемму 3.2 и (3.14) – (3.16), для достаточно больших T будем иметь :

$$\begin{aligned} |R_T| &\leq \int_{\mathbb{E}_\delta} |\Psi(u) - \Psi(0)| |\Phi_T(u)| du + \int_{\mathbb{E}_\delta^c} |\Psi_1(u)| |\Phi_T(u)| du + \\ &+ \int_{\mathbb{E}_\delta^c} |\Psi_2(u) - \Psi(0)| |\Phi_T(u)| du \leq \frac{\varepsilon}{C_1} \int_{\mathbb{E}_\delta} |\Phi_T(u)| du + \\ &+ \|\Psi_1\|_2 \left[\int_{\mathbb{E}_\delta^c} \Phi_T^2(u) du \right]^{1/2} + C_2 \int_{\mathbb{E}_\delta^c} |\Phi_T(u)| du \leq 3\varepsilon. \end{aligned}$$

Отсюда и из (3.14) получим (3.13). Лемма 3.3 доказана.

Положим

$$\mu_T(A) = \frac{1}{T} \int_A G_T(\lambda_1 - \lambda_3) G_T(\lambda_3 - \lambda_2) G_T(\lambda_4 - \lambda_1) G_T(\lambda_2 - \lambda_4) d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4 \quad (3.17)$$

и пусть $C_{loc}(\mathbb{R}^n)$ – пространство непрерывных на $\mathbb{R}^n$ функций с ограниченным носителем.

Лемма 3.4. Если $f \in C_{loc}(\mathbb{R}^4)$, то

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^4} f d\mu_T = 8\pi^3 \int_{\mathbb{R}} f(u, u, u, u) du. \quad (3.18)$$

Доказательство. Сделав замену переменных

$$\lambda_1 = u, \quad \lambda_3 - \lambda_2 = u_1, \quad \lambda_4 - \lambda_1 = u_2, \quad \lambda_2 - \lambda_4 = u_3,$$

из (3.2) – (3.5), (3.17) и равенства

$$G_T(u_1) G_T(u_2) G_T(u_3) G_T(-u_1 - u_2 - u_3) = D_T(u_1) D_T(u_2) D_T(u_3) D_T(u_1 + u_2 + u_3)$$

получим

$$\begin{aligned} \int_{\mathbb{R}^4} f d\mu_T &= \frac{1}{T} \int_{\mathbb{R}^3} \int_{\mathbb{R}} f(u, u + u_2 + u_3, u + u_1 + u_2 + u_3, u + u_2) du \times \\ &\times G_T(u_1) G_T(u_2) G_T(u_3) G_T(-u_1 - u_2 - u_3) du_1 du_2 du_3 =: 8\pi^3 \int_{\mathbb{R}^3} \Psi(u) \Phi_T(u) du, \end{aligned} \quad (3.19)$$

где $u = (u_1, u_2, u_3)$ и

$$\Psi(u) = \int_{\mathbb{R}} f(u, u + u_2 + u_3, u + u_1 + u_2 + u_3, u + u_2) du.$$

Легко проверить, что функция Ψ удовлетворяет условиям Леммы 3.3 и

$$\lim_{u \rightarrow (0,0,0)} \Psi(u) = \int_{\mathbb{R}} f(u, u, u, u) du. \quad (3.20)$$

Следовательно применив Лемму 3.3, из (3.19) и (3.20) получим (3.18). Лемма 3.4 доказана.

Лемма 3.5. Если $f \in L^1(\mathbb{R})$, то

$$1) \int_{\mathbb{R}^4} |f(x_i)| d|\mu_T| \leq C_1 \|f\|_{L^1(\mathbb{R})}, \quad i = 1, 2, 3, 4, \quad (3.21)$$

$$2) \int_{\mathbb{R}^4} |f(x_i) f(x_j)| d|\mu_T| \leq C_2 \|f\|_{L^2(\mathbb{R})}^2, \quad i, j = 1, 2, 3, 4, \quad i \neq j, \quad (3.22)$$

где C_1 и C_2 суть абсолютные постоянные.

Доказательство. Для $T > 0$ имеем

$$|G_T(x)| \leq 2T\psi_T(x), \quad \text{where} \quad \psi_T(x) = \frac{1}{1 + T|x|}, \quad x \in \mathbb{R}. \quad (3.23)$$

Применив неравенство

$$T \int_{\mathbb{R}} \psi_T(x + u) \psi_T(x + v) dx \leq C_\delta \psi_T^{1-\delta}(u - v), \quad \delta > 0, \quad (3.24)$$

получим (при $i = 1$, $\delta = 1/4$)

$$\begin{aligned} \int_{\mathbb{R}^4} |f(x_1)| d|\mu_T| &\leq CT^3 \int_{\mathbb{R}^4} |f(x_1)| \psi_T(x_1 - x_3) \psi_T(x_3 - x_2) \psi_T(x_4 - x_1) \times \\ &\times \psi_T(x_2 - x_4) dx_1 dx_2 dx_3 dx_4 \leq CT \int_{\mathbb{R}} |f(x_1)| \int_{\mathbb{R}} \psi_T^{1,5}(x_1 - x_2) dx_2 dx_1 \leq \\ &\leq C_1 \|f\|_{L^1(\mathbb{R})}, \end{aligned}$$

что и доказывает (3.21) (при $i = 1$). Для доказательства (3.22) при $i = 1, j = 2$ (например), применим (3.24) и неравенство Коши :

$$\begin{aligned} \int_{\mathbb{R}^4} |f(x_1)f(x_2)|d\mu_T &\leq CT^3 \int_{\mathbb{R}^4} |f(x_1)f(x_2)|\psi_T(x_1-x_3)\psi_T(x_3-x_2)\psi_T(x_4-x_1) \leq \\ &\leq CT \int_{\mathbb{R}^2} |f(x_1)f(x_2)|\psi_T^{1,5}(x_1-x_2)dx_1dx_2 \leq \\ &\leq C \left\{ T \int_{\mathbb{R}^2} f^2(x_1)\psi_T^{1,5}(x_1-x_2)dx_1dx_2 \right\}^{1/2} + \\ &+ \left\{ T \int_{\mathbb{R}^2} f^2(x_2)\psi_T^{1,5}(x_1-x_2)dx_1dx_2 \right\}^{1/2} \leq C_2 \int_{\mathbb{R}} f^2(x)dx. \end{aligned}$$

Лемма 3.5 доказана.

Лемма 3.6. Пусть $f(u_1, u_2, u_3, u_4) = \prod_{i=1}^4 f_i(u_i)$, где $f_i \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $i = 1, 2, 3, 4$. Тогда имеет место (3.18).

Доказательство. Допустим $\|f_i\|_\infty \leq M < \infty$, $i = 1, 2, 3, 4$. Применив теорему Лузина, для любого $\varepsilon > 0$ можно найти функции h_i, g_i , $i = 1, 2, 3, 4$ такие, что

$$f_i = g_i + h_i, \quad g_i \in C_{loc}(\mathbb{R}), \quad \|h_i\|_{L^1(\mathbb{R})} \leq \varepsilon, \quad \|g_i\|_C \leq M, \quad i = 1, 2, 3, 4. \quad (3.25)$$

Тогда

$$\int_{\mathbb{R}^4} f d\mu_T = \int_{\mathbb{R}^4} \prod_{i=1}^4 (g_i + h_i) d\mu_T = \int_{\mathbb{R}^4} \prod_{i=1}^4 g_i d\mu_T + I_T, \quad (3.26)$$

где в силу (3.25) и Леммы 3.5

$$\begin{aligned} |I_T| &\leq \sum_{j=1}^4 \int_{\mathbb{R}^4} |h_j| \prod_{i=1, i \neq j}^4 (|g_i| + |h_i|) d\mu_T \leq \\ &\leq C_M \sum_{j=1}^4 \int_{\mathbb{R}^4} |h_j| d\mu_T \leq C_M \sum_{j=1}^4 \|h_j\|_{L^1(\mathbb{R})} \leq C_M \varepsilon. \end{aligned} \quad (3.27)$$

В силу Леммы 3.4 имеем

$$\begin{aligned} \lim_{T \rightarrow \infty} \int_{\mathbb{R}^4} \prod_{i=1}^4 g_i(u_i) d\mu_T &= \int_{\mathbb{R}} \prod_{i=1}^4 g_i(u) du = \\ &= \int_{\mathbb{R}} \prod_{i=1}^4 [f_i(u) - h_i(u)] du = \int_{\mathbb{R}} f(u, u, u, u) du + J, \end{aligned} \quad (3.28)$$

где

$$|J| \leq \sum_{j=1}^4 \int_{\mathbb{R}} |h_j(u)| \prod_{i=1, i \neq j}^4 (|f_i(u)| + |g_i(u)|) du \leq C_M \varepsilon. \quad (3.29)$$

Из (3.26) – (3.29) получим (3.18). Лемма 3.6 доказана.

Лемма 3.7. Пусть $\psi(u) \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, $1 < p < \infty$. Тогда

$$\lambda_T := \|B_T(\psi)\|_{\infty} = o(T^{1/p}) \quad \text{при } T \rightarrow \infty. \quad (3.30)$$

Доказательство. Пусть N_T – положительное число, зависящее от T , которое мы определим ниже. Положим

$$M_T = \{\lambda \in \mathbb{R}; |\psi(\lambda)| > N_T\}. \quad (3.31)$$

Имеем

$$\begin{aligned} \lambda_T &= \|B_T(\psi)\|_{\infty} = \sup_{u \in L^2(\mathbb{R}), \|u\|_2=1} |(B_T(\psi)u, u)| = \\ &= \sup_{u \in L^2(\mathbb{R}), \|u\|_2=1} \left| \int_0^T \int_0^T \hat{\psi}(t-s) u(t) u(s) dt ds \right| = \\ &= \sup_{u \in L^2(\mathbb{R}), \|u\|_2=1} \left| \int_0^T u(t) dt \int_0^T u(s) ds \int_{-\infty}^{+\infty} e^{i\lambda(t-s)} \psi(\lambda) d\lambda \right|. \end{aligned} \quad (3.32)$$

Квадратично интегрируемая функция $u(t)$ также интегрируема на $[0, T]$. Поэтому, меняя порядок интегрирования в (3.32), получим

$$\begin{aligned} \lambda_T &= \sup_{u \in L^2(\mathbb{R}), \|u\|_2=1} \left| \int_{-\infty}^{+\infty} \psi(\lambda) d\lambda \int_0^T u(t) e^{it\lambda} dt \int_0^T u(s) e^{-is\lambda} ds \right| \leq \\ &\leq \sup_{u \in L^2(\mathbb{R}), \|u\|_2=1} \int_{-\infty}^{+\infty} |\psi(\lambda)| \left| \int_0^T u(t) e^{it\lambda} dt \right|^2 d\lambda. \end{aligned} \quad (3.33)$$

Так как $u(t) \in L^2[0, T]$ с $\|u\|_2 = 1$, имеем $\left| \int_0^T u(t) e^{it\lambda} dt \right|^2 \leq T$, по теореме Планшереля из (3.33) находим, что

$$\lambda_T \leq 2\pi N_T + T \int_{M_T} |\psi(t)| dt, \quad (3.34)$$

где M_T определена в (3.31). Покажем, что для любого p ($1 < p < \infty$)

$$\int_{M_T} |\psi(t)| dt \leq \gamma_T^p N_T^{(1-p)}, \quad (3.35)$$

где

$$\gamma_T = \left(\int_{M_T} |\psi(t)|^p dt \right)^{1/p}. \quad (3.36)$$

Действительно, в силу неравенства Гёлдера

$$\int_{M_T} |\psi(t)| dt \leq \gamma_T [m(M_T)]^{(p-1)/p}, \quad (3.37)$$

где $m(M_T)$ – мера Лебега множества M_T . Далее, в силу неравенства Чебышева, получаем

$$m(M_T) \leq \gamma_T^p N_T^{-p}. \quad (3.38)$$

Из (3.37) и (3.38) следует (3.35). Теперь из (3.34) и (3.35) будем иметь

$$\lambda_T \leq 2\pi N_T + T \gamma_T^p N_T^{1-p}. \quad (3.39)$$

Если $\psi \in L^\infty(\mathbb{R})$, то беря $N_T = \|\psi\|_\infty$ для всех $T > 0$, получим $\gamma_T = 0$ и из (3.39) вытекает $\lambda_T = O(1)$.

Допустим теперь $\psi \notin L^\infty(\mathbb{R})$ и для фиксированного $T > 0$ рассмотрим функцию

$$F(N) = N - T^{1/p} \left(\int_{\{\lambda: |\psi(\lambda)| > N\}} |\psi(\lambda)|^p d\lambda \right)^{1/p}, \quad N \in [0, \infty).$$

Так как $F(0) < 0$ и $\lim_{N \rightarrow \infty} F(N) = +\infty$, то существует положительное число $N = N_T$ такое, что $F(N_T) = 0$, т.е.

$$N_T = T^{1/p} \left(\int_{\{\lambda: |\psi(\lambda)| > N_T\}} |\psi(\lambda)|^p d\lambda \right)^{1/p} = T^{1/p} \gamma_T. \quad (3.40)$$

Легко видеть, что при условии $\psi \notin L^\infty(\mathbb{R})$ из равенства (3.40) вытекает $\lim_{T \rightarrow \infty} N_T = \infty$. Следовательно $\gamma_T = o(1)$ и из (3.39) и (3.40) получим, что $\lambda_T < 8T^{1/p} \gamma_T = o(T^{1/p})$ при $T \rightarrow \infty$. Лемма 3.7 доказана.

Лемма 3.8. Пусть $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Тогда

$$\frac{1}{T} \|B_T(\psi)\|_2^2 \longrightarrow 2\pi \|\psi\|_2^2 \quad \text{при } T \rightarrow \infty. \quad (3.41)$$

Доказательство. Ввиду (1.6), вычисляя стандартную норму Гильберта–Шмидта (см. [10]), получаем

$$\begin{aligned} \frac{1}{T} \|B_T(\psi)\|_2^2 &= \frac{1}{T} \int_0^T \int_0^T |\widehat{\psi}(t-s)|^2 dt ds = \\ &= \frac{1}{T} \int_{-T}^T (T-|t|) |\widehat{\psi}(t)|^2 dt = \int_{-T}^T \left(1 - \frac{|t|}{T}\right) |\widehat{\psi}(t)|^2 dt. \end{aligned} \quad (3.42)$$

Переходя к пределу при $T \rightarrow \infty$ и используя равенство Планшереля, из (3.42) получим (3.41). Лемма 3.8 доказана.

Лемма 3.9. Пусть $Y(t)$, $t \in \mathbb{R}$ – вещественнозначный, центрированный, сепарабельный гауссовский процесс со спектральной плотностью $f_Y(\lambda) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Тогда распределение нормированной квадратичной формы

$$L_T := \frac{1}{\sqrt{T}} \left(\int_0^T Y^2(t) dt - \mathbb{E} \int_0^T Y^2(t) dt \right) \quad (3.43)$$

стремится (при $T \rightarrow \infty$) к нормальному распределению $N(0, \sigma_Y^2)$ с дисперсией

$$\sigma_Y^2 = 4\pi \int_{-\infty}^{+\infty} f_Y^2(\lambda) d\lambda. \quad (3.44)$$

Доказательство. Пусть $R(t)$ – ковариационная функция $Y(t)$. Для $T > 0$ обозначим через $\lambda_j = \lambda_j(T)$, $j \geq 1$ собственные значения оператора $B_T(f_Y)$ (см. (1.6)), и пусть $e_j(t) = e_j(t, T) \in L_2[0, T]$, $j \geq 1$, – соответствующие ортонормальные собственные функции, т.е.

$$\int_0^T R(t-s)e_j(s) ds = \lambda_j e_j(t), \quad j \geq 1. \quad (3.45)$$

В силу теоремы Мерсера (см., например, [10], §3.10), имеем

$$R(t-s) = \sum_{j=1}^{\infty} \lambda_j e_j(t) e_j(s) \quad (3.46)$$

с положительными и суммируемыми собственными значениями $\{\lambda_j\}$. Итак имеет место разложение Карунна–Лоева (см., например, [15], §34.5)

$$Y(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j e_j(t), \quad (3.47)$$

где $\{\xi_j\}$ суть независимые $N(0, 1)$ случайные величины. Следовательно, в силу (3.43) и (3.47)

$$L_T = \frac{1}{\sqrt{T}} \sum_{j=1}^{\infty} \lambda_j (\xi_j^2 - 1). \quad (3.48)$$

Обозначим через $\chi_k(L_T)$ кумулянт k -го порядка L_T . В силу (3.48) (ср. с (1.7)) получаем

$$\chi_k(L_T) = \begin{cases} 0, & \text{при } k = 1, \\ (k-1)! 2^{k-1} T^{-k/2} \text{tr}[B_T(f_Y)]^k, & \text{при } k \geq 2. \end{cases} \quad (3.49)$$

Из (3.49) и Леммы 3.8 имеем

$$\chi_2(L_T) = \frac{2}{T} \|B_T(f_Y)\|_2^2 \rightarrow 4\pi \int_{-\infty}^{+\infty} f_Y^2(\lambda) d\lambda \quad \text{при } T \rightarrow \infty. \quad (3.50)$$

Далее, согласно (3.49) для $k \geq 3$

$$|\chi_k(L_T)| \leq C \frac{1}{T} \|B_T(f_Y)\|_2^2 T^{1-k/2} \lambda_T^{k-2}, \quad (3.51)$$

где $\lambda_T = \|B_T(f_Y)\|_{\infty}$. В силу Лемм 3.7 и 3.8 правая часть неравенства (3.51) стремится к нулю при $T \rightarrow \infty$. Лемма 3.9 доказана.

§4. ДОКАЗАТЕЛЬСТВА ОСНОВНЫХ РЕЗУЛЬТАТОВ

Доказательство Теоремы 2.1. В силу Теоремы 16.7.2 из [14], процесс $X(t)$ допускает представление скользящего суммирования

$$X(t) = \int_{-\infty}^{+\infty} \widehat{a}(t-s) d\xi(s), \quad (4.1)$$

где

$$\int_{-\infty}^{+\infty} |\widehat{a}(t)|^2 dt < \infty, \quad (4.2)$$

а $\xi(s)$ – процесс с ортогональными приращениями такой, что $\mathbb{E}[d\xi(s)] = 0$ и $\mathbb{E}|d\xi(s)|^2 = ds$. Более того, спектральная плотность $f(\lambda)$ имеет вид

$$f(\lambda) = 2\pi |a(\lambda)|^2, \quad (4.3)$$

где $a(\lambda)$ – обратное преобразование Фурье функции $\widehat{a}(t)$. Положим

$$a_1(\lambda) = (2\pi)^{1/2} a(\lambda) \cdot [g(\lambda)]^{1/2}, \quad (4.4)$$

где $g(\lambda)$ – порождающая функция, и рассмотрим процесс $Y(t)$ ($t \in \mathbb{R}$), определённый равенством

$$Y(t) = \int_{-\infty}^{+\infty} \widehat{a}_1(t-s) d\xi(s), \quad (4.5)$$

где $\widehat{a}_1(t)$ – преобразование Фурье $a_1(\lambda)$, а $\xi(s)$ определён в (4.1). Так как $fg \in L^1(\mathbb{R})$, то согласно тождеству Парсеваля будем иметь

$$\int_{-\infty}^{+\infty} |\widehat{a}_1(t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} |a_1(\lambda)|^2 d\lambda = 4\pi^2 \int_{-\infty}^{+\infty} f(\lambda) g(\lambda) d\lambda < \infty. \quad (4.6)$$

Итак, $Y(t)$ – стационарный процесс со спектральной плотностью

$$\varphi(\lambda) := |a_1(\lambda)|^2 = 2\pi f(\lambda) g(\lambda). \quad (4.7)$$

Так как по предположению $f(\lambda)g(\lambda) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, то процесс $Y(t)$, определённый в (4.5), удовлетворяет условиям Леммы 3.9. Поэтому, из Леммы 3.9 и Леммы 4.1 ниже, следует Теорема 2.1.

Лемма 4.1. В условиях Теоремы 2.1

$$\text{Var}(Q_T - L_T) = o(T) \quad \text{при} \quad T \rightarrow \infty, \quad (4.8)$$

где Q_T и L_T определены в (1.2) и (3.43), соответственно.

Доказательство. Согласно (1.2) и (4.1) имеем

$$Q_T = \int_{\mathbb{R}^2} \left[\int_0^T \int_0^T \widehat{g}(t-s) \widehat{a}(t-u_1) \widehat{a}(s-u_2) dt ds \right] d\xi(u_1) d\xi(u_2). \quad (4.9)$$

Аналогично, в силу (3.43) и (4.5)

$$L_T = \int_{\mathbb{R}^2} \left[\int_0^T \widehat{a}(t-u_1) \widehat{a}(s-u_2) dt \right] d\xi(u_1) d\xi(u_2). \quad (4.10)$$

Положив

$$d_{1T}(u_1, u_2) = \int_0^T \int_0^T \widehat{g}(t-s) \widehat{a}(t-u_1) \widehat{a}(s-u_2) dt ds \quad (4.11)$$

и

$$d_{2T}(u_1, u_2) = \int_0^T \int_0^T \widehat{\delta}(t-s) \widehat{a}(t-u_1) \widehat{a}(s-u_2) dt ds = \int_0^T \widehat{a}(t-u_1) \widehat{a}(s-u_2) dt, \quad (4.12)$$

из (4.9)–(4.12) получаем

$$Q_T - L_T = \int_{\mathbb{R}^2} [d_{1T}(u_1, u_2) - d_{2T}(u_1, u_2)] d\xi(u_1) d\xi(u_2). \quad (4.13)$$

Так как $X(t)$ является гауссовским процессом, то

$$\text{Var}(Q_T - L_T) = 2 \int_{\mathbb{R}^2} [d_{1T}(u_1, u_2) - d_{2T}(u_1, u_2)]^2 du_1 du_2. \quad (4.14)$$

Положим

$$p_1(\lambda_1, \lambda_2, \mu) = a(\lambda_1) a(\lambda_2) g(\mu), \quad (4.15)$$

$$p_2(\lambda_1, \lambda_2, \mu) = a_1(\lambda_1) a_1(\lambda_2) \delta(\mu) = a(\lambda_1) a(\lambda_2) [g(\lambda_1)]^{1/2} [g(\lambda_2)]^{1/2}. \quad (4.16)$$

Согласно равенству Планшереля

$$\begin{aligned} \int_{\mathbb{R}^2} d_{iT}^2(u_1, u_2) du_1 du_2 &= \\ &= (2\pi)^2 \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} G_T(\lambda_1 - \mu) G_T(\mu - \lambda_2) p_i(\lambda_1, \lambda_2, \mu) d\mu \right| d\lambda_1 d\lambda_2 = \\ &= (2\pi)^2 T \|p_i\|_T^2, \quad i = 1, 2, \end{aligned} \quad (4.17)$$

где $G_T(u)$ определён в (3.2), $\|p\|_T^2 = (p, p)_T$,

$$(p, p')_T = \int_{\mathbb{R}^4} p(\lambda_1, \lambda_2, \lambda_3) \overline{p'(\lambda_1, \lambda_2, \lambda_4)} d\mu_T, \quad (4.18)$$

а мера μ_T определена в (3.17).

Так же как в (4.17) (см. также (4.14)), будем иметь

$$\text{Var}(Q_T - L_T) = 8\pi^2 T \|p_1 - p_2\|_T^2. \quad (4.19)$$

Для любого $K > 0$ рассмотрим множества

$$E_1^K = \{u \in \mathbb{R} : |a(u)| < K\}, \quad E_2^K = \{u \in \mathbb{R} : g(u) < K\}, \quad (4.20)$$

и положим

$$\begin{aligned} p_1^K(u) &= p_1(u) \chi_1^K(u_1) \chi_1^K(u_2) \chi_2^K(u_3), \\ p_2^K(u) &= p_2(u) \chi_1^K(u_1) \chi_1^K(u_2) \chi_2^K(u_1) \chi_2^K(u_2), \end{aligned} \quad (4.21)$$

где $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, а $\chi_j^K(u)$ – характеристическая функция множества E_j^K , $j = 1, 2$. Тогда

$$\|p_1 - p_2\|_T^2 \leq 3 (\|p_1^K - p_2^K\|_T^2 + \|p_1 - p_1^K\|_T^2 + \|p_2 - p_2^K\|_T^2). \quad (4.22)$$

Теперь, из (4.15), (4.16) и (4.21) следует, что $\|p_1^K - p_2^K\|_T^2 = \int_{\mathbb{R}} \Gamma d\mu_T$, где $\Gamma = \Gamma(u_1, u_2, u_3, u_4)$ является суммой четырёх функций, удовлетворяющих условиям Леммы 3.6. Поскольку $\Gamma(u, u, u, u) = 0$ при $u \in \mathbb{R}$, то применив Лемму 3.6 получим

$$\lim_{T \rightarrow \infty} \|p_1^K - p_2^K\|_T = \int_{\mathbb{R}} \Gamma(u, u, u, u) du = 0. \quad (4.23)$$

Далее, согласно (4.18)

$$\|p_1\|_T^2 = \|p_1^K + (p_1 - p_1^K)\|_T^2 = \|p_1^K\|_T^2 + 2(p_1^K, p_1 - p_1^K)_T + \|p_1 - p_1^K\|_T^2.$$

Следовательно

$$\|p_1 - p_1^K\|_T^2 \leq |\|p_1\|_T^2 - \|p_1^K\|_T^2| + 2|(p_1^K, p_1 - p_1^K)_T|. \quad (4.24)$$

В силу (2.1), (4.17) и Леммы 3.1

$$\|p_1\|_T^2 = (2\pi)^{-2} \frac{1}{T} \text{tr} [B_T(f) B_T(g)]^2 \rightarrow 2\pi \int_{\mathbb{R}} f^2(u) g^2(u) du, \quad (4.25)$$

а согласно Лемме 3.6 и (4.17)

$$\|p_1^K\|_T^2 \rightarrow 2\pi \int_{H_K} f^2(u) g^2(u) du, \quad \text{где } H_K := \{u \in \mathbb{R} : f(u) < K, g(u) < K\}. \quad (4.26)$$

Из (4.25) и (4.26) находим, что

$$\lim_{T \rightarrow \infty} (\|p_1\|_T^2 - \|p_1^K\|_T^2) = \int_{\mathbb{R} \setminus H_K} f^2(u) g^2(u) du = o(1) \quad \text{при } K \rightarrow \infty. \quad (4.27)$$

Оценим последний член в правой части (4.24). Обозначим

$$\Gamma_K(u_1, u_2, u_3, u_4) = p_1^K(u_1, u_2, u_3) [p_1(u_1, u_2, u_4) - p_1^K(u_1, u_2, u_4)].$$

Из (4.20) и (4.21) имеем

$$|a(u_1)| < K, |a(u_2)| < K, g(u_3) < K, g(u_4) > K, \text{ if } \Gamma_K(u_1, u_2, u_3, u_4) \neq 0. \quad (4.28)$$

Далее, для любого $L > K$ и $u = (u_1, u_2, u_3, u_4)$ имеем

$$(p_1^K, p_1 - p_1^K)_T = \int_{\mathbb{R}^4} \Gamma_K(u) d\mu_T = \int_{\mathbb{R}^4} \Gamma_K(u) \chi_2^L(u_4) d\mu_T + I, \quad (4.29)$$

где с некоторой постоянной C_K , зависящей от K

$$|I| \leq C_K \int_{\mathbb{R}^4} g(u_4) (1 - \chi_2^L(u_4)) d|\mu_T|. \quad (4.30)$$

Из (4.15), (4.16) и (4.21) следует, что $\Gamma_K(u) \chi_2^L(u_4)$ является линейной комбинацией функций, удовлетворяющих условиям Леммы 3.6. Применив Лемму 3.6 и учитывая, что $\Gamma_K(u, u, u, u) = 0$ при $u \in \mathbb{R}$ (см. (4.28)), получим

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^4} \Gamma_K(u) \chi_2^L(u_4) d\mu_T = \int_{\mathbb{R}} \Gamma_K(u, u, u, u) \chi_2^L(u) du = 0. \quad (4.31)$$

Для заданного $\varepsilon > 0$ и при достаточно большом L , с учётом (3.21) имеем

$$C_K \int_{\mathbb{R}^4} g(u) (1 - \chi_2^L(u)) d|\mu_T| \leq C_K \int_{\{u: g(u) > L\}} g(u) du \leq \varepsilon. \quad (4.32)$$

Из (4.29) – (4.32) получаем

$$\lim_{T \rightarrow \infty} (p_1^K, p_1 - p_1^K)_T = 0.$$

Отсюда и из (4.24) и (4.27) следует

$$\lim_{T \rightarrow \infty} \|p_1 - p_1^K\|_T = 0. \quad (4.33)$$

Наконец, докажем, что

$$\lim_{T \rightarrow \infty} \|p_2 - p_2^K\|_T = 0. \quad (4.34)$$

Действительно, согласно (4.16), (4.21) и (3.22)

$$\begin{aligned} \|p_2 - p_2^K\|_T &\leq \int_{\mathbb{R}^4} [1 - \chi_1^K(u_1)] f(u_1) g(u_1) f(u_2) g(u_2) d|\mu_T| + \\ &+ \int_{\mathbb{R}^4} [1 - \chi_1^K(u_2)] f(u_1) g(u_1) f(u_2) g(u_2) d|\mu_T| \leq \\ &\leq \int_{\{u: |f(u)| > \sqrt{K}\}} f^2(u) g^2(u) du + \int_{\{u: |g(u)| > K\}} f^2(u) g^2(u) du = o(1), \end{aligned}$$

когда $K \rightarrow \infty$ (равномерно по T). Из (4.19), (4.23), (4.33) и (4.34) вытекает (4.8).

Доказательство Леммы 4.1 завершено. Теорема 2.1 доказана.

Доказательство Теоремы 2.2. Сделав замену переменной $x_1 = u$, $x_1 - x_3 = u_1$, $x_3 - x_2 = u_2$ и $x_2 - x_4 = u_3$, из (3.1) получаем

$$\begin{aligned} \operatorname{tr} [B_T(f)B_T(g)]^2 &= \int_{\mathbb{R}^4} G_T(u_1)G_T(u_2)G_T(u_3)G_T(-u_1 - u_2 - u_3) \times \\ &\times f(u)g(u - u_1)f(u - u_1 - u_2)g(u - u_1 - u_2 - u_3) du du_1 du_2 du_3, \end{aligned} \quad (4.35)$$

где $G_T(u)$ определено в (3.2). Учитывая равенство

$$e^{iu_1(T+1)/2} \cdot e^{iu_2(T+1)/2} \cdot e^{iu_3(T+1)/2} \cdot e^{-i(u_1+u_2+u_3)(T+1)/2} = 1$$

и чётность функции $D_T(u)$, из (4.35) получим

$$\operatorname{tr} [B_T(f)B_T(g)]^2 = 8\pi^3 \int_{\mathbb{R}^3} \Psi(u_1, u_2, u_3) \Phi_T(u_1, u_2, u_3) du_1 du_2 du_3, \quad (4.36)$$

где функция $\Phi_T(u_1, u_2, u_3)$ определена в (3.5), $\Psi(u_1, u_2, u_3) = \varphi(u_1, u_1 + u_2, u_1 + u_2 + u_3)$, а $\varphi(u_1, u_2, u_3)$ определена в (2.3). По Теореме 2.1 и (4.36) нужно доказать, что

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^3} \Psi(u) \Phi_T(u) du = \int_{\mathbb{R}} f^2(x) g^2(x) dx. \quad (4.37)$$

Теперь, поскольку функции $\varphi(u_1, u_2, u_3)$ и $\Psi(u_1, u_2, u_3) = \varphi(u_1, u_1 + u_2, u_1 + u_2 + u_3)$ интегрируемы с квадратом и непрерывны в точке $(0, 0, 0)$, и

$$\Psi(0, 0, 0) = \int_{\mathbb{R}} f^2(x) g^2(x) dx,$$

из Леммы 3.3 получаем (4.37). Теорема 2.2 доказана.

Доказательство Теоремы 2.3 : Согласно Теореме 2.2 достаточно доказать, что если

$$f_i \in L^1(\mathbb{R}) \cap L^{p_i}(\mathbb{R}), \quad 1 \leq p_i \leq \infty, \quad i = 0, 1, 2, 3, \quad \sum_{i=0}^3 \frac{1}{p_i} \leq 1, \quad (4.38)$$

то функция

$$\varphi(t) := \int_{\mathbb{R}} f_0(u) f_1(u - t_1) f_2(u - t_2) f_3(u - t_3) du, \quad t = (t_1, t_2, t_3) \in \mathbb{R}^3 \quad (4.39)$$

принадлежит $L^2(\mathbb{R}^3)$ и непрерывна в точке $(0, 0, 0)$.

Из неравенства Гёлдера и (4.38) следует, что

$$|\varphi(t)| \leq \prod_{i=0}^3 \|f_i\|_{L^{p_i}(\mathbb{R})} < \infty, \quad t = (t_1, t_2, t_3) \in \mathbb{R}^3.$$

Следовательно, $\varphi \in L^\infty(\mathbb{R}^3)$. С другой стороны, из условия $f_i \in L^1(\mathbb{R})$ и (4.39) следует, что $\varphi \in L^1(\mathbb{R}^3)$, значит $\varphi \in L^2(\mathbb{R}^3)$.

Для доказательства непрерывности функции $\varphi(t)$ в точке $(0, 0, 0)$ рассмотрим три случая.

Случай 1. $p_i < \infty$, $i = 0, 1, 2, 3$.

Для любого $\varepsilon > 0$ можно найти $\delta > 0$ такое, что (см. (4.38))

$$\|f_i(u - t) - f_i(u)\|_{L^{p_i}(\mathbb{R})} \leq \varepsilon, \quad i = 1, 2, 3, \quad \text{if } |t| \leq \delta. \quad (4.40)$$

Фиксируем $t = (t_1, t_2, t_3)$ с $|t| < \delta$ и положим

$$\bar{f}_i(u) = f_i(u + t_i) - f_i(u), \quad i = 1, 2, 3.$$

Тогда из (4.40) следует $\|\bar{f}_i\|_{p_i} \leq \varepsilon$, $i = 1, 2, 3$. Итак, в силу (4.39)

$$\varphi(t) = \int_{\mathbb{R}} f_0(u) \prod_{i=1}^3 (\bar{f}_i(u) + f_i(u)) du = \varphi(0, 0, 0) + W.$$

Каждый из пяти интегралов, составляющих W , содержит по крайней мере одну функцию $\bar{f}_i$ и может быть оценён следующим образом :

$$\left| \int_{\mathbb{R}} f_0(u) \bar{f}_1(u) f_2(u) f_3(u) du \right| \leq \|f_0\|_{L^{p_0}} \|\bar{f}_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}} \leq A\varepsilon.$$

Случай 2. $p_i \leq \infty$, $i = 0, 1, 2, 3$, $\sum_{i=0}^3 \frac{1}{p_i} < 1$.

Существуют конечные числа $p'_i < p_i$, $i = 0, 1, 2, 3$ такие, что $\sum_{i=0}^3 1/p'_i \leq 1$. Тогда согласно (4.38) имеем $f_i \in L^{p'_i}$, $i = 0, 1, 2, 3$ и функция φ непрерывна в точке $(0, 0, 0)$ так же, как и в случае 1.

Случай 3. $p_i \leq \infty$, $i = 0, 1, 2, 3$, $\sum_{i=0}^3 \frac{1}{p_i} = 1$.

Сначала заметим, что по крайней мере одно из чисел p_i конечно. Допустим, без ограничения общности, что $p_0 < \infty$. Для любого $\varepsilon > 0$ можно найти функции f'_0, f''_0 такие, что

$$f_0 = f'_0 + f''_0, \quad f'_0 \in L^\infty(\mathbb{R}), \quad \|f''_0\|_{L^{p_0}} < \varepsilon. \quad (4.41)$$

Следовательно

$$\varphi(t) = \varphi'(t) + \varphi''(t), \quad (4.42)$$

где функции φ' и φ'' определяются так же, как функция φ в (4.39), заменив f_0 на f'_0 и f''_0 , соответственно. Из (4.41) следует, что φ' непрерывна в точке $(0, 0, 0)$ (см. Случай 2), а по неравенству Гёльдера $|\varphi''(t)| \leq A\varepsilon$. Поэтому, при достаточно малом $|t|$ получаем

$$|\varphi(t) - \varphi(0, 0, 0)| \leq |\varphi'(t) - \varphi'(0, 0, 0)| + |\varphi''(t) - \varphi''(0, 0, 0)| \leq (A + 1)\varepsilon,$$

откуда следует результат. Теорема 2.3 доказана.

Замечание 4.1. Заметим, что мы доказали, что при условии (4.38)

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^4} f_0(u_1) f_1(u_2) f_2(u_3) f_3(u_4) d\mu_T = 8\pi^3 \int_{\mathbb{R}} f_0(u) f_1(u) f_2(u) f_3(u) du.$$

Доказательство Теоремы 2.4. По Теореме 2.2 достаточно показать, что если f и g удовлетворяют условиям Теоремы 2.4, т.е. $f \in L_2(\mathbb{R})$, $g \in L_2(\mathbb{R})$, $fg \in L_2(\mathbb{R})$ и имеет место (2.4), то функция

$$\varphi(t) = \int_{\mathbb{R}} f(u) g(u - t_1) f(u - t_2) g(u - t_3) du, \quad t = (t_1, t_2, t_3) \in \mathbb{R}^3$$

принадлежит $L^2(\mathbb{R}^3)$ и непрерывна в точке $(0, 0, 0)$.

Поскольку

$$\varphi^2(t) = \int_{\mathbb{R}^2} f(u) f(v) g(u - t_1) g(v - t_1) f(u - t_2) f(v - t_2) g(u - t_3) g(v - t_3) du dv,$$

имеем

$$\begin{aligned} \int_{\mathbb{R}^3} \varphi^2(t) dt &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} g(u - t_1) g(v - t_1) dt_1 \int_{\mathbb{R}} f(u - t_2) f(v - t_2) dt_2 \times \right. \\ &\quad \left. \times \int_{\mathbb{R}} g(u - t_3) g(v - t_3) dt_3 \right] f(u) f(v) du dv \leq \\ &\leq \|f\|_{L^2(\mathbb{R})}^2 \|g\|_{L^2(\mathbb{R})}^4 \int_{\mathbb{R}^2} f(u) f(v) du dv < \infty. \end{aligned}$$

Теперь докажем непрерывность $\varphi(t)$ в точке $(0, 0, 0)$. Пусть $\varepsilon > 0$ – произвольное число. Обозначим

$$E_K = \{u \in \mathbb{R} : |f(u)| \leq K\}, \quad f_1(u) = \chi_{E_K}(u) f(u), \quad f_2(u) = f(u) - f_1(u),$$

где $K > 0$ выбрано так, что $\int_{\mathbb{R} \setminus E_K} f^2(u) g^2(u) du \leq \varepsilon$. Тогда

$$f = f_1 + f_2, \quad \|f_1\|_{\infty} \leq K, \quad \int_{\mathbb{R}} f_2^2(u) g^2(u) du \leq \varepsilon. \quad (4.43)$$

Далее имеем

$$\begin{aligned} \varphi(t) &= \int_{\mathbb{R}} f_1(u) g(u - t_1) f_1(u - t_2) g(u - t_3) du + \\ &+ \int_{\mathbb{R}} f_2(u) g(u - t_1) f(u - t_2) g(u - t_3) du + \\ &+ \int_{\mathbb{R}} f_1(u) g(u - t_1) f_2(u - t_2) g(u - t_3) du =: \varphi_1(t) + \varphi_2(t) + \varphi_3(t). \end{aligned} \quad (4.44)$$

Оценим функции $\varphi_k(t)$, $k = 1, 2, 3$. Имеем

$$\begin{aligned} \varphi_1(t) &= \int_{\mathbb{R}} f_1(u)g(u-t_1)f_1(u-t_2)[g(u-t_3)-g(u)]du + \\ &+ \int_{\mathbb{R}} f_1(u)g(u)f_1(u-t_2)[g(u-t_1)-g(u)]du + \int_{\mathbb{R}} f_1(u)g^2(u)f_1(u-t_2)du =: \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (4.45)$$

Применив неравенство Гёльдера, из (4.43) получим

$$|I_1| \leq K^2 \|g\|_2 \cdot \|g(u+t_3)-g(u)\|_2 = o(1) \quad \text{при} \quad t_3 \rightarrow 0. \quad (4.46)$$

Аналогично

$$|I_2| = o(1) \quad \text{при} \quad t_1 \rightarrow 0. \quad (4.47)$$

Из (4.43) следует, что

$$\begin{aligned} |I_3 - \varphi(0,0,0)| &= \left| \int_{\mathbb{R}} f_1(u+t_2)g^2(u+t_2)f_1(u)du - \int_{\mathbb{R}} f_1^2(u)g^2(u)du \right| + \\ &+ \left| \int_{\mathbb{R}} f_2^2(u)g^2(u)du \right| \leq K \|f_1(u+t_2)g^2(u+t_2) - f_1(u)g_1^2(u)\|_1 + \varepsilon = o(1) + \varepsilon \end{aligned} \quad (4.48)$$

при $t_2 \rightarrow 0$. Из (4.45)–(4.48) при достаточно малом $|t|$ получим

$$|\varphi_1(t) - \varphi(0,0,0)| \leq 2\varepsilon. \quad (4.49)$$

Далее,

$$\begin{aligned} |\varphi_2(t)|^2 &\leq \int_{\mathbb{R}} f_2^2(u)g^2(u-t_1)du \int_{\mathbb{R}} f_2^2(u-t_2)g^2(u-t_3)du = \\ &= \left| \int_{\mathbb{R}} f^2(u)g^2(u-t_1)du - \int_{\mathbb{R}} f_1^2(u)g^2(u-t_1)du \right| \times \int_{\mathbb{R}} f^2(u)g^2(u+t_2-t_3)du \\ &\rightarrow \left| \int_{\mathbb{R}} f^2(u)g^2(u)du - \int_{\mathbb{R}} f_1^2(u)g^2(u)du \right| \int_{\mathbb{R}} f^2(u)g^2(u)du \end{aligned}$$

при $|t| \rightarrow 0$. Следовательно, с учётом (4.43), при достаточно малом $|t|$ будем иметь

$$|\varphi_2(t)| \leq \varepsilon \int_{\mathbb{R}} f^2(u)g^2(u)du. \quad (4.50)$$

Аналогично можно доказать, что при достаточно малом $|t|$

$$|\varphi_3(t)| \leq \varepsilon \int_{\mathbb{R}} f^2(u)g^2(u)du. \quad (4.51)$$

Из (4.44) и (4.49)–(4.51) вытекает, что

$$\lim_{t \rightarrow 0} \varphi(t) = \varphi(0,0,0).$$

Теорема 2.4 доказана.

Доказательство Теоремы 2.5. В силу (3.1) и (3.17), нужно доказать, что из (2.5) и (2.6) вытекает

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{\mathbb{R}^4} f(x_1) f(x_2) g(x_3) g(x_4) d\mu_T = 8\pi^3 \int_{\mathbb{R}} f^2(x) g^2(x) dx. \quad (4.52)$$

Если $\alpha, \beta \geq 0$, тогда из (2.5), (2.6) следует, что $f \in L^{1/\alpha}(\mathbb{R})$, $g \in L^{1/\beta}(\mathbb{R})$, и результат следует из Теоремы 2.3. Предположив, что $\beta < 0$, из (2.5) имеем $g \in L^\infty(\mathbb{R})$.

Обозначим

$$\bar{f}(x) = \begin{cases} 0, & \text{если } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ f(x), & \text{в противном случае,} \end{cases} \quad \bar{g}(x) = \begin{cases} 0, & \text{если } x \in [-\pi, \pi] \\ g(x), & \text{в противном случае,} \end{cases}$$

и пусть $\underline{f} = f - \bar{f}$, $\underline{g} = g - \bar{g}$. Тогда

$$\begin{aligned} \frac{1}{T} \int_{\mathbb{R}^4} f(x_1) f(x_2) g(x_3) g(x_4) d\mu_T &= \frac{1}{T} \int_{\mathbb{R}^4} \bar{f}(x_1) f(x_2) g(x_3) g(x_4) d\mu_T + \\ &+ \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1) \bar{f}(x_2) g(x_3) g(x_4) d\mu_T + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1) \underline{f}(x_2) g(x_3) g(x_4) d\mu_T = \\ &=: I_T^1 + I_T^2 + I_T^3. \end{aligned} \quad (4.53)$$

Так как $\bar{f}, g \in L^\infty(\mathbb{R})$ и $f \in L^1(\mathbb{R})$ (см. Замечание 4.1), то получим

$$\lim_{T \rightarrow \infty} I_T^1 = 8\pi^3 \int_{\mathbb{R}} \bar{f}(x) f(x) g^2(x) dx = 8\pi^3 \int_{|x| > \frac{\pi}{2}} f^2(x) g^2(x) dx \quad (4.54)$$

$$\lim_{T \rightarrow \infty} I_T^2 = 8\pi^3 \int_{\mathbb{R}} \underline{f}(x) \bar{f}(x) g^2(x) dx = 0.$$

Далее,

$$\begin{aligned} I_T^3 &= \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1) \underline{f}(x_2) \underline{g}(x_3) \underline{g}(x_4) d\mu_T + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1) \underline{f}(x_2) \underline{g}(x_3) \bar{g}(x_4) d\mu_T \\ &+ \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1) \underline{f}(x_2) \bar{g}(x_3) \underline{g}(x_4) d\mu_T + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1) \underline{f}(x_2) \bar{g}(x_3) \bar{g}(x_4) d\mu_T =: \sum_{i=1}^4 J_T^i. \end{aligned} \quad (4.55)$$

Учитывая, что

$$J_T^1 = \frac{1}{T} \int_{[-\pi, \pi]^4} \underline{f}(x_1) \underline{f}(x_2) \underline{g}(x_3) \underline{g}(x_4) d\mu_T,$$

так же как для равенства (4.3) из [8], можно доказать, что

$$\lim_{T \rightarrow \infty} J_T^1 = 8\pi^3 \int_{-\pi}^{\pi} \underline{f}^2(x) \underline{g}^2(x) dx = 8\pi^3 \int_{-\pi/2}^{\pi/2} f^2(x) g^2(x) dx. \quad (4.56)$$

Так как $f(x_1)f(x_2) \in L^1(\mathbb{R}^2)$ для любого $\varepsilon > 0$ можно найти $\delta > 0$ такое, что

$$\int_{|x_1 - x_2| < \delta} |f(x_1)f(x_2)| dx_1 dx_2 < \varepsilon.$$

Так как $g \in L^\infty(\mathbb{R})$, ввиду (3.23) и (3.24) при достаточно большом T получим

$$\begin{aligned} |J_T^2| &\leq C \cdot T \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} |f(x_1)f(x_2)| \int_{-\pi/2}^{\pi/2} \psi_T(x_1 - x_3) \psi_T(x_2 - x_3) \times \\ &\times \int_{|x_4| > \pi} \frac{1}{x_4^2} dx_4 dx_3 dx_1 dx_2 \leq C \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} |f(x_1)f(x_2)| (1 + T|x_1 - x_2|)^{-1/2} dx_1 dx_2 \leq \\ &\leq C \int_{|x_1 - x_2| < \delta} |f(x_1)f(x_2)| dx_1 dx_2 + (1 + T\delta)^{-1/2} \int_{|x_1 - x_2| \geq \delta} |f(x_1)f(x_2)| dx_1 dx_2 \leq 2\varepsilon. \end{aligned}$$

Это означает, что

$$\lim_{T \rightarrow \infty} J_T^2 = 0. \quad (4.57)$$

Аналогично

$$\lim_{T \rightarrow \infty} J_T^3 = 0. \quad (4.58)$$

Для оценки интеграла J_T^4 в (4.55) заметим, что в случае $|x_i - x_j| > \frac{\pi}{2}$, $i = 1, 2$, $j = 3, 4$. Следовательно

$$|J_T^4| \leq \frac{C}{T} \int_{\mathbb{R}^4} \underline{f}(x_1) \underline{f}(x_2) \bar{g}(x_3) \bar{g}(x_4) dx_1 dx_2 dx_3 dx_4 \leq \frac{C}{T} \|f\|_{L^1(\mathbb{R})}^2 \|g\|_{L^1(\mathbb{R})}^2 \rightarrow 0, \quad (4.59)$$

при $T \rightarrow \infty$. Из (4.55) – (4.59) получим $\lim_{T \rightarrow \infty} I_T^3 = 0$. Отсюда и из (4.53) и (4.54) вытекает (4.52). Теорема 2.5 доказана.

Abstract. Let $X(t)$, $t \in \mathbb{R}$, be a centered real-valued stationary Gaussian process with spectral density $f(\lambda)$. The paper considers the probability distribution of Toeplitz type quadratic functional Q_T of the process $X(t)$. Sufficient conditions in terms of $f(\lambda)$ and the kernel ensuring central limit theorems for standard normalized quadratic functionals Q_T are obtained, extending the results of Avram (1986), Fox and Taqqu (1987), Giraitis and Surgailis (1990), Ginovian and Sahakian (2003) for discrete time processes.

Л И Т Е Р А Т У Р А

1. F. Avram, "On bilinear forms in Gaussian random variables and Toeplitz matrices", *Probab. Th. Rel. Fields*, vol. 79, pp. 37 – 45, 1988.
2. R. Bentkus, "On the error of the estimate of the spectral function of a stationary process", *Litovskii Mat. Sb.*, vol. 12, No. 1, pp. 55 – 71, 1972.
3. J. Beran, "Estimation, Testing and Prediction for Self-similar and Related Processes", *Diss. ETH No. 8074*, Zurich, 1986.
4. R. Fox, M. S. Taqqu, "Central limit theorem for quadratic forms in random variables having long-range dependence", *Probab. Th. Rel. Fields*, vol. 74, pp. 213 – 240, 1987.
5. M. S. Ginovian, "On estimate of the value of the linear functional in a spectral density of stationary Gaussian process", *Theory Probab. Appl.*, vol. 33, pp. 777 – 781, 1988.
6. М. С. Гиновян, "Заметка о центральной предельной теореме для квадратичных форм типа Теплица в стационарных гауссовских переменных", *Изв. АН Армении, Математика*, том 28, стр. 78 - 81, 1993.
7. M. S. Ginovian, "On Toeplitz type quadratic functionals in Gaussian stationary process", *Probab. Th. Rel. Fields*, vol. 100, pp. 395 – 406, 1994.
8. M. S. Ginovian, A. A. Sahakian, "On the Central Limit Theorem for Toeplitz Quadratic Forms of Stationary Sequences", *Preprint No. 2003 – 01*, Institute of Mathematics, Yerevan, September, 2003.
9. L. Giraitis, D. Surgailis, "A central limit theorem for quadratic forms in strongly dependent linear variables and its application to asymptotical normality of Whittle's estimate", *Probab. Th. Rel. Fields*, vol. 86, pp. 87 – 104, 1990.
10. I. Z. Gohberg, M. G. Krein, "Introduction to a Theory of Linear non-self-conjugate Operators in Hilbert Space", Moscow, Nauka, 1965.
11. U. Grenander, G. Szegő, *Toeplitz Forms and Their Applications*, University of California Press, 1958.
12. R. Z. Hasminskii, I. A. Ibragimov, "Asymptotically efficient nonparametric estimation of functionals of a spectral density function", *Probab. Th. Rel. Fields*, vol. 73, pp. 447 – 461, 1986.
13. I. A. Ibragimov, "On estimation of the spectral function of a stationary Gaussian process", *Theory Probab. and Appl.*, vol. 8, No. 4, pp. 391 – 430, 1963.
14. И. А. Ибрагимов, Ю. В. Линник, "Независимые и стационарно связанные переменные", Москва, Наука, 1965.
15. М. Лоёве, *Теория Вероятностей*, Москва, ИЛ, 1962.
16. M. Rosenblatt, "Asymptotic behavior of eigenvalues of Toeplitz forms", *Journal of Math. and Mech.*, vol. 11, No. 6, pp. 941 – 950, 1962.
17. N. Terrin, M. S. Taqqu, "A noncentral limit theorem for quadratic forms of Gaussian stationary sequences", *Journal of Theoretical Probability*, vol. 3, No. 3, pp. 449 – 475, 1990.
18. N. Terrin, M. S. Taqqu, "Convergence in distributions of sums of bivariate Appel polynomials with long-range dependence", *Probab. Th. Rel. Fields*, vol. 90, pp. 57 - 81, 1991.
19. N. Terrin, M. S. Taqqu, "Convergence to a Gaussian limit as the normalization exponent tends to $1/2$ ", *Statistics and Probability Letters*, vol. 11, pp. 419 - 427, 1991.

20. M. Taniguchi, "Berry-Esseen Theorems for Quadratic Forms of Gaussian Stationary Processes, Probab. Theory Relat. Fields, vol. 72, pp. 185 – 194, 1986.
21. M. Taniguchi, Y. Kakizawa, "Asymptotic Theory of Statistical Inference for Time Series", New York : Springer-Verlag, 2000.

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ОЦЕНКА МАКСИМАЛЬНЫХ ОПЕРАТОРОВ СВЯЗАННЫХ С ОБОБЩЁННЫМИ ЛАКУНАРНЫМИ МНОЖЕСТВАМИ

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Резюме. Пусть Ω – произвольное множество направлений (единичных векторов) на плоскости. Обозначим через $\mathcal{R}_\Omega$ множество всех прямоугольников, имеющих сторону параллельную некоторому направлению из Ω . В работе исследуются максимальные операторы, определённые равенствами $M_\Omega f(x) = \sup_{R \in \mathcal{R}_\Omega, x \in R} \frac{1}{|R|} \int_R |f(y)| dy$. Для заданного натурального N , определяются N -лакунарные множества Ω и для них доказывается неравенство $\|M_\Omega f(x)\|_2 \leq N \|f\|_2$.

§1. ВВЕДЕНИЕ

Пусть Ω – множество направлений (единичных векторов) на плоскости. Обозначим через $\mathcal{R}_\Omega$ – множество всех прямоугольников, имеющих сторону параллельную некоторому направлению из Ω . В работе изучаются максимальные операторы на плоскости $\mathbb{R}^2$, определённые равенствами

$$M_\Omega f(x) = \sup_{R \in \mathcal{R}_\Omega, x \in R} \frac{1}{|R|} \int_R |f(y)| dy. \quad (1.1)$$

А. Нагель, И. Стейн и С. Вайнгер [19], используя метод преобразования Фурье, доказали ограниченность операторов $M_\Omega f(x)$ в пространствах L^p , $1 < p < \infty$ для любого лакунарного множества направлений $\Omega = \{\theta_k\}$, $\arg \theta_{k+1} < \lambda \arg \theta_k$, $\lambda < 1$.

Для данного натурального $N \in \mathbb{N}$ ($\mathbb{N}$ – множество натуральных чисел) мы по индукции определяем N -лакунарные множества. 1-лакунарное множество определяется как последовательность направлений $\Omega_1 = \{v_k \mid k \in \mathbb{N}\}$, удовлетворя-

ющих условию

$$\arg(v_{k+1} - v_\infty) < \lambda \arg(v_k - v_\infty), \quad \lambda < 1 \quad (1.2)$$

для некоторого направления v_∞ . Каждое $(N+1)$ -лакунарное множество получается из некоторого N -лакунарного множества Ω_N добавлением точек в Ω_N следующим образом: между любыми двумя соседними точками $a, b \in \Omega_N$ добавляется 1-лакунарная последовательность (конечная или бесконечная). Итак, для любого N -лакунарного множества Ω существует последовательность k -лакунарных множеств Ω_k таких, что $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_{N-1} \subset \Omega_N = \Omega$. Ограниченность максимального оператора M_Ω в L^p ($p > 1$) для N -лакунарного Ω была доказана П. Шегреном и П. Шелином в [20]. Мы оцениваем рост норм операторов M_Ω для N -лакунарного Ω при $N \rightarrow \infty$.

Теорема 1. Для любого N -лакунарного множества Ω имеет место неравенство

$$\|M_\Omega f(x)\|_2 \lesssim N \|f\|_2.$$

Запись $A \lesssim B$ означает существование абсолютной постоянной K такой, что $A \leq KB$. Легко проверить, что любое множество направлений мощности N является $(C \log N)$ -лакунарным. Отсюда следует, что для любого конечного множества Ω , имеет место неравенство

$$\|M_\Omega f\|_2 \lesssim (\log \#\Omega) \|f\|_2. \quad (1.3)$$

Это неравенство, доказанное Н. Кацом в [18] является окончательным. Следовательно, и оценка из Теоремы 1 также окончательная. Как неравенство (1.3) так и Теорему 1 можно вывести из одного общего результата Альфонсека, Сория и Варгаса [3] приводимого ниже. В данной работе, применяя новый подход, приведено короткое доказательство Теоремы 1. Идеи, изложенные в настоящей работе, формировались в работе С. Вайнгера [24].

Недавно А. Альфонсека, Ф. Сория и А. Варгас ([2], [3], см. также Альфонсека [1]) доказали интересный принцип ортогональности для максимальных функций. Пусть $\Omega_0 = \{v_k \mid k \in \mathbb{N}\}$ есть множество направлений, а множества Ω_k — произвольные множества между двумя соседними точками v_k и v_{k+1} . Тогда для множества $\Omega = \bigcup_{k=0}^{\infty} \Omega_k$ имеем (см. [3])

$$\|M_\Omega\|_{2 \rightarrow 2} \leq C \|M_{\Omega_0}\|_{2 \rightarrow 2} + \sup_k \|M_{\Omega_k}\|_{2 \rightarrow 2}. \quad (1.4)$$

Теорема 1 вытекает из (1.4). Действительно, пусть $\eta(N)$ есть максимум величин $\|M_{\Omega_N}\|_{2 \rightarrow 2}$ по всевозможным N -лакунарным множествам направлений. Из

неравенства (1.4) следует, что $\eta(N) \leq C\eta(1) + \eta(N - 1)$. Сделав итерацию (1.4) $(N - 1)$ раз, получим неравенство Теоремы 1.

Мы заканчиваем Введение более подробным изложением истории этого вопроса. В 1977, А. Кордоба [7] рассмотрел максимальные функции, образованные по прямоугольникам со сторонами 1 и N , и доказал медленный рост их норм в L^2 . Используя геометрический подход А. Кордоба и Р. Фейфферман [9] доказали, что эти максимальные функции эквивалентны слабой версии оператора M_Ω , когда множество Ω состоит из N равномерно распределённых точек, а стороны прямоугольников $R \in \mathcal{R}_\Omega$ имеют длины 1 и N . Оценка (1.3) в случае равномерно распределённых направлений была доказана Ж. Стромбергом [22] в 1978. В связи с этим возникла задача об распространении результата Стромберга на случай произвольных N различных направлений. Частный результат был получен Баррионево [5], [4]. Окончательный результат был получен Н. Кацом в [18]. Метод, использованный в [18] основан на двойственности операторов и теореме Джона-Ниренберга. Случай фиксированного отношения сторон был рассмотрен в работах А. Кордоба [7] и Н. Каца [17]. С другой стороны, как показал Ж. Стромберг в [21], в случае лакунарных направлений можно установить более точные оценки. Оценка таких операторов в L^p для произвольного $1 < p < \infty$ была установлена методом Фурье в работе А. Нагеля, И. Стейна и С. Вайнгера [19]. Как показали А. Кордоба и Р. Фейфферман в [10] и А. Карбери в [6], эти результаты тесно связаны с результатами о мультипликаторах.

Максимальная функция, определённая относительно канторова множества направлений А. Варгасом в [23] и Н. Кацом в [16], является неограниченной в L^2 . К. Хейр в [12] использовал подход Каца для более общих канторовых множеств. Пока ничего неизвестно об ограниченности такого максимального оператора в L^p при $p > 2$.

Ж. Дуондикоекса и А. Варгас в [11] получили необходимые и достаточные условия ограниченности оператора M_Ω . Некоторые обобщения этого результата были получены К. Хейром и Ж. Геоннингом в [14], [15].

Работа М. Криста [8] содержит примеры множеств направлений Ω , и частные результаты об ограниченности оператора M_Ω . К. Хейр и Ф. Риччи [13] рассмотрели интересные модификации направленных, лакунарных максимальных функций.

§2. ОБОЗНАЧЕНИЯ И ЗАМЕЧАНИЯ

Обозначим через $\hat{f}(\xi)$ преобразование Фурье функции f , т.е.

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{ix \cdot \xi} dx.$$

Обозначим

$$P_\alpha f(x_1, x_2) = \sup_{\delta_1, \delta_2} \frac{1}{4\delta_1\delta_2} \int_{x_1-\delta_1}^{x_1+\delta_1} \int_{x_2-t_1\alpha-\delta_2}^{x_2-t_1\alpha+\delta_2} |f(t_1, t_2)| dt_2 dt_1. \quad (2.1)$$

Отметим, что (2.1) является максимальной функцией по параллелограммам, одна сторона которого параллельна оси x , а другая сторона составляет с осью x угол $\arctan \alpha$. Чтобы доказать теорему, достаточно установить неравенство

$$\| \sup_{\alpha \in \Omega} P_\alpha f \|_2 \leq CN \|f\|_2,$$

где Ω есть N -лакунарное множество из $(0, 1)$. Рассмотрим также максимальную функцию

$$\bar{P}_\alpha f(x_1, x_2) = \sup_{\delta_1} \frac{1}{2\delta_1} \int_{x_1-\delta_1}^{x_1+\delta_1} \sup_{\delta_2} \frac{1}{2\delta_2} \int_{x_2-t_1\alpha-\delta_2}^{x_2-t_1\alpha+\delta_2} |f(t_1, t_2)| dt_2 dt_1. \quad (2.2)$$

Очевидно, что $P_\alpha f(x) \leq \bar{P}_\alpha f(x)$. Через $K_r(x)$ обозначим ядро Фейера :

$$K_r(x) = \int_{-r}^r \left(1 - \frac{|t|}{r}\right) e^{-itx} dt = \frac{4 \sin^2 \frac{rx}{2}}{rx^2}.$$

Для любых r, R с $0 \leq r < R/2$ определим следующие функции

$$\psi_r(x) = 2K_{2r}(x) - K_r(x), \quad \psi_{r,R}(x) = \psi_R(x) - \psi_r(x).$$

В некоторых местах будем писать $\psi_{0,r}$ вместо $\psi_r(x)$. Имеем

$$\hat{\psi}_{r,R}(\xi) = \begin{cases} 1 & \text{при } |\xi| \in [2r, R], \\ 0 & \text{при } 0 \leq |\xi| \leq r \text{ или } |\xi| > 2R, \\ \text{линейна} & \text{на интервалах } \pm [r, 2r], \pm [R, 2R]. \end{cases} \quad (2.3)$$

Из свойств ядра Фейера получаем

$$|\psi_{r,R}(x)| \leq C \left(\max \left\{ \frac{1}{Rx^2}, R \right\} + \max \left\{ \frac{1}{rx^2}, r \right\} \right).$$

Следовательно, для некоторой последовательности интервалов $\omega_k = \omega_{k,r,R}$ с центрами в 0 имеем

$$|\psi_{r,R}(x)| \leq C \sum_k \gamma_k \frac{I_{\omega_k}(x)}{|\omega_k|} = \zeta_{r,R}(x), \quad \gamma_k > 0, \quad \sum_k \gamma_k < 1, \quad \omega_k \supset (1/R, 1/R). \quad (2.4)$$

Пусть ϕ – функция Шварца с

$$\phi \geq 0, \quad \text{supp } \hat{\phi} \subset [-1, 1]. \quad (2.5)$$

Очевидно, что

$$\int_{\mathbb{R}} \lambda(x) dx \leq C, \quad \text{где } \lambda(x) = \max\{|\phi(x)|, |x\phi(x)|\}. \quad (2.6)$$

Определим аналоги средних по параллелограммам :

$$\Gamma_{r,R,h}^{\alpha} f(x) = (\psi_{r,R}(x_2 - x_1 \alpha) \phi_h(x_1)) * f(x), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad (2.7)$$

где

$$\phi_h(x) = \frac{1}{h} \phi\left(\frac{x}{h}\right).$$

Из (2.7) и (2.1) следует, что

$$P_{\alpha} f(x) \leq C \sup_{R,h} \Gamma_{R,h}^{\alpha} f(x) \quad (\Gamma_{R,h}^{\alpha} = \Gamma_{0,R,h}^{\alpha}).$$

Следовательно, чтобы доказать теорему достаточно проверить неравенство

$$\left\| \sup_{R,h,\alpha \in \Omega} \Gamma_{R,h}^{\alpha} f(x) \right\|_2 \leq C N \|f\|_2. \quad (2.8)$$

Применив преобразование Фурье, из (2.7) получим

$$\hat{\Gamma}_{r,R,h}^{\alpha} f(\xi) = \hat{\phi}(h(\xi_2 + \xi_1 \alpha)) \hat{\psi}_{r,R}(\xi_1) \hat{f}(\xi). \quad (2.9)$$

§3. ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ

Сперва докажем две леммы.

Лемма 1. Пусть $\alpha, \beta \in (0, 1)$ – любые числа и $0 < r < R$, $h > 0$. Для оператора $\Gamma_{r,R,h}^{\alpha} f(x)$, определённого по (2.7), имеют место следующая оценка

$$|\Gamma_{r,R,h}^{\alpha} f(x)| \leq C (hR|\alpha - \beta| + 1) \bar{P}_{\beta} f(x), \quad x \in \mathbb{R}^2. \quad (3.1)$$

Доказательство. Из (2.4) следует, что

$$\psi_{r,R}(x_2 - x_1 \alpha) \leq C \sum_k \frac{\gamma_k}{|\omega_k|} \mathbf{I}_{\omega_k}(x_2 - x_1 \alpha),$$

где имеем $|\omega_k| > 2/R$. Обозначим $\lambda(x_1) = 2Rx_1|\alpha - \beta| + 2$, и допустим, что для некоторого k

$$x_2 - x_1 \alpha \in \omega_k. \quad (3.2)$$

Тогда, учитывая (3.2) получим

$$\left| \frac{x_2 - x_1 \beta}{\lambda(x_1)} \right| = \left| \frac{x_2 - x_1 \alpha + x_1(\alpha - \beta)}{\lambda(x_1)} \right| \leq \left| \frac{x_2 - x_1 \alpha}{2} \right| + \frac{1}{2R} \leq \frac{|\omega_k|}{2}. \quad (3.3)$$

Следовательно,

$$\frac{x_2 - x_1 \beta}{\lambda(x_1)} \in \omega_k. \quad (3.4)$$

Так как из (3.2) следует (3.4),

$$\Pi_{\omega_k}(x_2 - x_1 \alpha) \leq \Pi_{\omega_k} \left(\frac{x_2 - x_1 \beta}{\lambda(x_1)} \right).$$

Отсюда вытекает

$$\psi_{r,R}(x_2 - x_1 \alpha) \leq C \sum_k \frac{\gamma_k}{|\omega_k|} \Pi_{\omega_k} \left(\frac{x_2 - x_1 \beta}{\lambda(x_1)} \right) = \zeta_{r,R} \left(\frac{x_2 - x_1 \beta}{\lambda(x_1)} \right).$$

Учитывая, что $\phi(x) \leq \lambda(x)/x$ при $x > h$ и $\phi(x) \leq \lambda(x)$ при $x \leq h$ (см. (2.6)), получаем

$$\frac{1}{h} \phi \left(\frac{x_1}{h} \right) \psi_{r,R} \left(\frac{x_2 - x_1 \beta}{\lambda(x_1)} \right) \leq C (hR|\alpha - \beta| + 1) \frac{1}{h} \xi \left(\frac{x_1}{h} \right) \frac{1}{\lambda(x_1)} \zeta_{r,R} \left(\frac{x_2 - x_1 \beta}{\lambda(x_1)} \right), \quad (3.5)$$

что даёт оценку ядра оператора $\Gamma_{r,R,h}$ (см. (2.7)). Комбинируя (3.5) и (2.7), получим (3.1). Лемма 1 доказана.

Для произвольного интервала $J = (a, b)$ обозначим через S_J сектор $\{(x_1, x_2) : x_1 > 0, a \leq x_2/x_1 \leq b\}$. Для сектора S с углом θ через γS ($\gamma > 0$) обозначим сектор с углом $\gamma\theta$, имеющий общую биссектрису с S . Через $T_S f(x)$ обозначим оператор, определённый равенством $\widehat{T_S f} = \Pi_S \widehat{f}$.

Лемма 2. Пусть $J_1 \supset J_2 \supset \dots \supset J_n$ – последовательность интервалов такая, что

$$J_k = [\alpha_k, \beta_k] \subset (0, 1), \quad \text{dist}((J_k)^c, J_{k+1}) \leq |J_{k+1}|, \quad 1 \leq k \leq n. \quad (3.6)$$

Тогда для любого числа $\theta \in \bigcap J_k$ и любой функции $f \in L^2(\mathbb{R}^2)$

$$P_\theta f \lesssim P_0 f + P_\theta(T_{\frac{4}{3}S(J_n)} f) + \sum_{k=1}^{n-1} \bar{P}_{\alpha_k}(T_{\frac{4}{3}S(J_k)} f) + \bar{P}_{\beta_k}(T_{\frac{4}{3}S(J_k)} f). \quad (3.7)$$

Доказательство. Пусть $\theta \in \bigcap J_k$. Для любых R и h имеем

$$\widehat{\Gamma}_{R,h}^\theta f(\xi) = \widehat{\psi}_R(\xi_1) \widehat{\phi}(h(\xi_2 + \xi_1 \theta)) \widehat{f}(x). \quad (3.8)$$

Полагая

$$r_0 = 0, \quad r_k = \frac{6}{h|J_k|}, \quad 1 \leq k \leq n, \quad (3.9)$$

из (2.3) получаем

$$\widehat{\psi}_R(\xi_1) = \sum_{k=1}^m \widehat{\psi}_{2r_{k-1}, r_k}(\xi_1) + \widehat{\psi}_{2r_m, R}(\xi_1), \quad (3.10)$$

где $m = \max\{2k : r_k < R\}$. Обозначим

$$\Gamma_k f(x) = \Gamma_{2r_k, r_{k+1}, h}^\theta f(x), \quad 0 \leq k < m,$$

$$\Gamma_m f(x) = \Gamma_{2r_m, R, h}^\theta f(x).$$

Тогда ввиду (2.9) имеем

$$\widehat{\Gamma}_k f(\xi) = \widehat{\psi}_{2r_{k-1}, r_k}(\xi_1) \widehat{\phi}(h(\xi_2 + \xi_1 \theta)) \widehat{f}(x), \quad 1 \leq k < m,$$

$$\widehat{\Gamma}_m f(x) = \widehat{\psi}_{2r_m, R}(\xi_1) \widehat{\phi}(h(\xi_2 + \xi_1 \theta)) \widehat{f}(x),$$

и, следовательно, из (3.10) получим

$$\Gamma_{R, h}^\theta f = \sum_{k=0}^m \Gamma_k f. \quad (3.11)$$

Чтобы доказать $\Gamma_k f = \Gamma_k(T_{\frac{4}{3}S(J_k)} f)$, $1 \leq k \leq m$, достаточно установить, что

$$\begin{aligned} \text{supp } \widehat{\psi}_{2r_k, r_{k+1}}(\xi_1) \widehat{\phi}(h(\xi_2 + \xi_1 \theta)) &\subset \frac{4}{3}S(J_k), \quad 1 \leq k < m, \\ \text{supp } \widehat{\psi}_{2r_m, R}(\xi_1) \widehat{\phi}(h(\xi_2 + \xi_1 \theta)) &\subset \frac{4}{3}S(J_m). \end{aligned} \quad (3.12)$$

Действительно, из (2.5) и (2.3) следует, что

$$\text{supp } \widehat{\psi}_{2r_k, r_{k+1}}(\xi_1) \widehat{\phi}(h(\xi_2 + \xi_1 \theta)) = \{(\xi_1, \xi_2) : r_k \leq \xi_1 \leq 2r_{k+1}, |\xi_2 + \xi_1 \theta| < \frac{1}{h}\}.$$

Последнее множество есть параллелограмм с вершинами $(r_k, r_k \theta \pm \frac{1}{h})$ и $(2r_{k+1}, 2r_{k+1} \theta \pm \frac{1}{h})$. Эти вершины лежат в секторе $\frac{4}{3}S(J_k)$, так как

$$\frac{r_k \theta \pm \frac{1}{h}}{r_k} = \theta \pm \frac{|J_k|}{6}, \quad \frac{2r_{k+1} \theta \pm \frac{1}{h}}{2r_{k+1}} = \theta \pm \frac{|J_{k+1}|}{6}.$$

Отсюда вытекает (3.12).

Используя Лемму 1, заключаем, что при $1 \leq k < m$, имеем

$$|\Gamma_k f| \lesssim (hr_{k+1} \min\{|\theta - \alpha_k|, |\theta - \beta_k|\} + 1)(\overline{P}_{\alpha_k}(T_{\frac{4}{3}S(J_k)} f) + \overline{P}_{\beta_k}(T_{\frac{4}{3}S(J_k)} f)). \quad (3.13)$$

Заметим также, что

$$|\Gamma_0 f| \leq P_0 f, \quad (3.14)$$

$$|\Gamma_m f| \leq P_\theta T_{\frac{1}{3}S(J_m)} f. \quad (3.15)$$

Так как $\theta \in J_{k+1} \subset J_k$, из (3.6) получим

$$\min\{|\theta - \alpha_k|, |\theta - \beta_k|\} \leq 2|J_{k+1}|.$$

Последнее вместе с (3.9) даёт

$$hr_{k+1} \min\{|\theta - \alpha_k|, |\theta - \beta_k|\} \leq 12.$$

Учитывая (3.13), получаем

$$|\Gamma_k f| \lesssim \bar{P}_{\alpha_k}(T_{\frac{1}{3}S(J_k)} f) + \bar{P}_{\beta_k}(T_{\frac{1}{3}S(J_k)} f), \quad 1 \leq k < m.$$

Теперь (3.14) и (3.15) завершают доказательство Леммы 2. Отметим, что (3.15) используется только в случае $m = n$.

Доказательство Теоремы 1. Не умоляя общности можно предположить, что $\Omega \subset (0, 1/4)$. Сначала заметим, что достаточно доказать теорему в случае, когда условие (1.2) в определении N -лакунарных множеств заменено на

$$\frac{1}{4}|v_k - v_\infty| \leq |v_{k+1} - v_\infty| < \frac{1}{2}|v_k - v_\infty|. \quad (3.16)$$

Можно также предполагать, что $v_\infty \in \Omega_{k-1}$ при $\{v_i\} \in \Omega_k \setminus \Omega_{k-1}$. Фиксируем множества $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_{N-1} \subset \Omega_N = \Omega$ из определения N -лакунарности. Фиксируем угол $\theta \in \Omega$ и $R, h > 0$. Допустим, что

$$\theta \in \Omega_m \setminus \Omega_{m-1} \quad \text{при некотором } m \leq N. \quad (3.17)$$

Обозначим через G_k множество интервалов, концами которых являются соседние точки из Ω_k . Можно выбрать последовательность интервалов $J_k = [\alpha_k, \beta_k] \in G_k$, $k = 1, 2, \dots, m$ такую, что

$$\theta \in \bigcap_{1 \leq k \leq m} J_k, \quad \theta = \alpha_m \quad (\text{или } \theta = \beta_m).$$

Очевидно, что последовательность J_k удовлетворяет условиям Леммы 2. Следовательно,

$$\begin{aligned} |M_\theta f|^2 &\lesssim \left\{ M_0 f + \sum_{k=1}^m (M_{\alpha_k}(T_{\frac{1}{3}S(J_k)} f) + M_{\beta_k}(T_{\frac{1}{3}S(J_k)} f)) \right\}^2 \\ &\lesssim |M_0 f|^2 + m \sum_{k=1}^m |M_{\alpha_k}(T_{\frac{1}{3}S(J_k)} f)|^2 + |M_{\beta_k}(T_{\frac{1}{3}S(J_k)} f)|^2. \end{aligned}$$

Суммируя по всем интервалам $J = (\alpha, \beta) \in G_k$, получим

$$\sup_{\theta \in \Omega} |M_\theta f|^2 \lesssim |M_0 f|^2 + N \sum_{k=1}^N \sum_{J=(\alpha, \beta) \in G_k} |M_\alpha(T_{\frac{1}{3}S(J)} f)|^2 + |M_\beta(T_{\frac{1}{3}S(J)} f)|^2. \quad (3.18)$$

Учитывая, что максимальный оператор имеет тип $(2, 2)$, для любого $1 \leq k \leq N$,

$$\begin{aligned} \int_{\mathbb{R}^2} \sum_{J=(\alpha, \beta) \in G_k} |M_\alpha(T_{\frac{1}{3}S(J)} f)|^2 + |M_\beta(T_{\frac{1}{3}S(J)} f)|^2 dx &\lesssim \\ &\lesssim \int_{\mathbb{R}^2} \sum_{J=(\alpha, \beta) \in G_k} \Pi_{\frac{1}{3}S(J)} |\hat{f}|^2 d\xi \lesssim \int_{\mathbb{R}^2} |\hat{f}|^2 d\xi = \int_{\mathbb{R}^2} |f|^2 dx. \end{aligned}$$

Наконец в силу (3.18) получим

$$\int_{\mathbb{R}^2} \sup_{\theta \in \Omega} |M_\theta f|^2 dx \lesssim N^2 \int_{\mathbb{R}^2} |f|^2 dx.$$

Теорема 1 доказана.

Abstract. Let Ω be a set of directions (unit vectors) on the plane. Denote by $\mathcal{R}_\Omega$ the set of all rectangles which have a side parallel to some direction from Ω . The paper studies maximal operators defined by $M_\Omega f(x) = \sup_{x \in R \in \mathcal{R}_\Omega} \frac{1}{|R|} \int_R |f(y)| dy$. Given an integer N , we define N -lacunary sets Ω and prove the inequality $\|M_\Omega f(x)\|_2 \leq N \|f\|_2$.

ЛИТЕРАТУРА

1. A. Alfonseca, "Strong type inequalities and an almost-orthogonality principle for families of maximal operators along directions in $\mathbb{R}^2$ ", J. London Math. Soc., vol. 67, no. 1, pp. 208 – 218, 2003.
2. A. Alfonseca, F. Soria, A. Vargas, "A remark on maximal operators along directions in $\mathbb{R}^2$ ", Math. Res. Lett., vol. 10, no. 1, pp. 41 – 49, 2003.
3. A. Alfonseca, F. Soria, A. Vargas. An Almost-Orthogonality Principle in L^2 for Directional Maximal Functions, Contemp. Math, 2003.
4. J. Barrionuevo, "Estimates for some Kakeya-type maximal operators", Trans. Amer. Math. Soc., vol. 335, no. 2, pp. 667 – 682, 1993.
5. J. Barrionuevo, "A note on the Kakeya maximal operator", Math. Res. Letters, vol. 3, no. 1, pp. 61 – 65, 1996.
6. A. Carbery, "Differentiation in lacunary directions and an extension of the Marcinkiewicz multiplier theorem", Ann. Inst. Fourier (Grenoble), vol. 38, no. 1, pp. 157 – 168, 1988.
7. A. Córdoba, "The Kakeya maximal function and the spherical summation multipliers", Amer. J. Math., vol. 99, no. 1, pp. 1-22, 1977.
8. M. Christ, "Examples of singular maximal functions unbounded on L^p ", Conference on Mathematical Analysis (El Escorial, 1989), Publ. Mat., vol. 35, no. 1, pp. 269 – 279, 1991.

9. A. Córdoba and R. Fefferman, "On differentiation of integrals", *Proc. Nat. Acad. of Sci. USA*, vol. 74, no. 2, pp. 423 – 425, 1977.
10. A. Córdoba and R. Fefferman, "On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis", *Proc. Nat. Acad. Sci. USA*, vol. 74, no. 2, pp. 423 – 425, 1977.
11. J. Duoandikoetxea, A. Vargas, "Directional operators and radial functions on the plane", *Ark. Mat.*, vol. 33, no. 2, pp. 281 – 291, 1995.
12. K. Hare, "Maximal operators and Cantor sets", *Canad. Math. Bull.*, vol. 43, pp. 330 – 342, 2000.
13. K. Hare, F. Ricci, "Maximal functions with polynomial densities in lacunary directions", *Trans. Amer. Math. Soc.*, vol. 335, no. 3, pp. 1135 – 1144, 2003.
14. K. Hare, J.-O. Rönning, "Applications of generalized Perron trees to maximal functions and density bases", *J. Fourier Anal. Appl.*, vol. 4, no. 2, pp. 215 – 227, 1998.
15. K. Hare, J.-O. Rönning, "The size of $\text{Max}(p)$ sets and density bases", *J. Fourier Anal. Appl.*, vol. 8, no. 3, pp. 259 – 268, 2002.
16. N. H. Katz, "A counterexample for maximal operators over a Cantor set of directions", *Math. Res. Lett.*, vol. 3, no. 4, pp. 527 – 536, 1996.
17. N. H. Katz, "Remarks on maximal operators over arbitrary sets of directions", *Bull. London Math. Soc.*, vol. 31, no. 6, pp. 700 – 710, 1999.
18. N. H. Katz, "Maximal operators over arbitrary sets of directions", *Duke Math. J.*, vol. 97, no. 3, pp. 67 – 79, 1999.
19. A. Nagel, E. M. Stein, S. Wainger, "Differentiation in lacunary directions", *Duke Math. J.*, vol. 97, no. 1, pp. 67 – 79, 1979.
20. P. Sjogren and P. Sjolín, "Littlewood-Paley decompositions and Fourier multipliers with singularities on certain sets", *Ann. Inst. Fourier (Grenoble)*, vol. 31, no. 1, pp. 157 – 175, 1981.
21. J.-O. Strömberg, "Weak estimates for maximal functions with rectangles in certain directions", *Arkiv. for Mat.*, vol. 15, pp. 229 – 240, 1976.
22. J.-O. Strömberg, "Maximal functions associated to rectangles with uniformly distributed directions", *Ann. of Math.*, vol. 107, pp. 399 – 402, 1976.
23. V. M. Vargas, "A remark on a maximal function over a Cantor set of directions", *Rend. Circ. Mat. Palermo (2)*, vol. 44, no. 2, pp. 273 – 282, 1995.
24. S. Wainger, "Applications of Fourier transforms to averages over lower-dimensional sets", *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978)*, Part 1, *Proc. Sympos. Pure Math.*, XXXV, Part, Amer. Math. Soc., Providence, R.I., pp. 85 – 94, 1979.

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