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МАТЕМАТИКА

ПРЕДИСЛОВИЕ РЕДАКТОРОВ СЕРИИ

Настоящий номер журнала продолжает публикацию (начало в № 4 за 2003) оригинальных статей, представленных на международной конференции "МАТЕМАТИКА в АРМЕНИИ: Достижения и перспективы" сентябрь 30 - октябрь 7, 2003, Цахкадзор, Армения. Основной целью конференции, посвящённой 60 летию Армянской Национальной Академии Наук, было обсуждение текущей работы и перспективы математики в Армении. Конференция привлекла внимание многих участников из-за границы, включая многих математиков из армянской диаспоры.

Научная программа конференции охватила следующие направления современной математики: комплексный анализ, вещественный анализ, теория аппроксимации, теория вероятностей и математическая статистика, дифференциальные и интегральные уравнения, математическая физика, алгебра, геометрия, топология.

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ON X–DECOMPOSABLE FINITE GROUPS

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Abstract. Let G be a finite group, A be a normal subgroup of G and $ncc(A)$ = the number of G–conjugacy classes of A, A is called n–decomposable, if $ncc(A) = n$. One can speak about decomposability of a group with respect to a non-empty subset X of positive integers. The present paper reports on the problem of finding all finite subsets X of positive integers, such that there is a X-decomposable finite group.

§1. INTRODUCTION

Let G be a finite group and let $\mathcal{N}_G$ be the set of normal subgroups of G. An element K of $\mathcal{N}_G$ is said to be n**-decomposable**, if K is a union of $n = ncc(K)$ distinct conjugacy classes of G. Let $\mathcal{K}_G = \{ncc(A) : A \in \mathcal{N}_G\}$ and X be a non-empty subset of positive integers. A group G is called X -decomposable, if $\mathcal{K}_G = X$.

In [19] Wujie Shi defined the notion of a complete normal subgroup of a finite group, which we call 2-decomposable. He proved that if G is a group and N a complete normal subgroup of G, then N is a minimal normal subgroup of G and it is an elementary Abelian p-group. Moreover, $N \subseteq Z(O_p(G))$, where $O_p(G)$ is a maximal normal p-subgroup of G, and $|N|(|N| - 1)$ divides $|G|$ and in particular, $|G|$ is even. Next, Wang Jing, a student of Wujie Shi, defined the notion of a sub-complete normal subgroup of a group G [20], which we call 3-decomposable. She proved that if N is a sub-complete normal subgroup of a finite group G, then N is a group in which every element has prime power order. Throughout this paper, as usual, G' denotes the derived subgroup of G, Z_n denotes the cyclic group of order n, $E(p^n)$ denotes an elementary Abelian p–group of order p^n, for a prime p, and $Z(G)$ is the center of G.

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A group G is called non-perfect, if $G' \neq G$. Also, $D(n)$ denotes the set of positive divisors of n and $SmallGroup(n,i)$ is the i-th group of order n in the small group library of GAP, [16]. All groups considered are assumed to be finite. Our notation is standard and taken mainly from [8], [9] and [14].

§2. MAIN RESULTS

In this section we first present some examples of X-decomposable finite groups and consider open questions. Then we consider the structure of non-perfect X-decomposable finite groups, for some finite subset X of positive integers. We first begin with the finite Abelian groups.

Lemma 1. *Let G be a finite Abelian group. Set $X = D(n)$, where $n = |G|$. Then G is $X-$decomposable.*

Proof follows from the fundamental theorem of finite Abelian groups.

The following lemmas consider the normal subgroups of some non-Abelian finite groups.

Lemma 2. *Let p be a prime, $q|p-1$ and $F_{p,q}$ be a group presented by :*

$$F_{p,q} = \langle a, b : a^p = b^q = 1,\ b^{-1}ab = a^u \rangle,$$

where u is an element of order q in Z_p^*, the multiplicative group of integers modulo p. Then the group $F_{p,q}$ is $\left\{1, 1+\frac{p-1}{q}, q+\frac{p-1}{q}\right\}-$decomposable.

Proof. It is not hard to show that, up to isomorphism, $F_{p,q}$ does not depend on which integer u of order q we choose and that $F_{p,q}$ is a Frobenius group. Let S be the subgroup of Z_p^* consisting of the powers of u, so $|S| = q$. Write $r = \frac{p-1}{q}$, and choose coset representatives $v_1, \cdots, v_r$ in Z_p^*. Then by [10], Proposition 25.9, the conjugacy classes of $F_{p,q}$ are

$$\{1\}, \qquad (a^{v_i})^{F_{p,q}} = \{a^{v_i s} : s \in S\}\ (1 \le i \le r),$$
$$(b^n)^{F_{p,q}} = \{a^m b^n : 0 \le m \le p-1\}\ (1 \le n \le q-1).$$

Suppose H is the unique Sylow $p-$subgroup of G. Then the mentioned information on conjugacy classes of $F_{p,q}$ show that H is $\left(1+\frac{p-1}{q}\right)$-decomposable. Finally, we can see that H is the unique non-trivial proper normal subgroup of $F_{p,q}$. Therefore, $F_{p,q}$ is $\left\{1, 1+\frac{p-1}{q}, q+\frac{p-1}{q}\right\}$ $-$decomposable. Lemma 2 is proved.

Lemma 3. *Every non-Abelian $p-$group of order p^3 is $\{1, p, 2p-1, p^2+p+1\}$-decomposable.*

Proof. It is a well-known fact that every non-Abelian $p-$group of order p^3 has p^2+p-1 conjugacy classes. Since the length of every conjugacy class of G is p, G is $\{1, p, 2p-1, p^2+p+1\}$-decomposable.

Lemma 4. *Suppose*

$$X = \left\{\frac{d+1}{2} : d|n\right\} \cup \left\{\frac{n+3}{2}\right\}, \quad A = Y \cup \left\{\frac{n}{4}+2, \frac{n}{2}\right\}, \quad B = Y \cup \left\{\frac{n}{2}+1, \frac{n}{2}\right\},$$

$$Y = \left\{\frac{d+2}{2} : d|n,\ 2|d\right\} \cup \left\{\frac{d+1}{2} : d|n,\ 2 \nmid d\right\} \cup \{n+3\},$$

and $n \geq 3$ is a positive integer. If n is odd, then the dihedral group D_{2n} of order $2n$ is X−decomposable. Moreover, if $4|n$, then D_{2n} is A−decomposable and if $4 \nmid n$ then D_{2n} is B−decomposable.

Proof. This group can be presented by $D_{2n} = \langle a, b : a^n = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$. We first assume that n is odd. In this case every proper normal subgroup of D_{2n} is contained in $\langle a\rangle$ and so D_{2n} is X−decomposable. Next we assume that n is even. In this case, we can see that D_{2n} has exactly two other normal subgroups $H = \langle a^2, b\rangle$ and $K = \langle a^2, ab\rangle$. To complete the example, we must compute $ncc(H) = ncc(K)$. If $4|n$, then $ncc(H) = \frac{n}{4} + 2$ and so it is enough to consider the case that $4 \nmid n$. In this case, we can see that $ncc(H) = \frac{n+6}{4}$. So, if $4|n$, then D_{2n} is A−decomposable and if $4 \nmid n$, then the dihedral group D_{2n} is B−decomposable. Lemma 4 is proved.

Lemma 5. *Suppose*

$$X = \left\{\frac{d+1}{2} : d|n,\ 2 \nmid d\right\} \cup \left\{\frac{d+2}{2} : d|2n,\ 2|d\right\} \cup \{n+3\}, \quad Y = X \cup \left\{\frac{n+4}{2}, n+3\right\}$$

and $n \geq 2$ is a positive integer. If n is odd, then the generalized quaternion group Q_{4n} of order $8n$ is X−decomposable, and if n is even, then Q_{4n} is Y−decomposable.

Proof. The generalized quaternion group Q_{4n}, $n \geq 2$, can be presented by

$$Q_{4n} = \langle a, b : a^{2n} = 1,\ b^2 = a^n,\ b^{-1}ab = a^{-1}\rangle.$$

It is a well-known fact that Q_{4n} has $n + 3$ conjugacy classes, as follows :

$$\{1\}, \quad \{a^n\}, \quad \{a^r, a^{-r}\}, \quad 1 \leq r \leq n-1,$$

$$\{a^{2j}b : 0 \leq j \leq n-1\}, \quad \{a^{2j+1}b : 0 \leq j \leq n-1\}.$$

We consider two separate cases of n odd or even. If n is odd then every normal subgroup of Q_{4n} is contained in the cyclic subgroup $\langle a\rangle$. Thus, in this case Q_{4n} is X−decomposable. If n is even, we have two other normal subgroups $\langle a^2, b\rangle$ and $\langle a^2, ab\rangle$ which are both $\frac{n+4}{2}$−decomposable. Therefore, Q_{4n} is Y−decomposable. Lemma 5 is proved. Now it is natural to generally ask about the set $\mathcal{K}_G = \{ncc(A) : A \trianglelefteq G\}$. In fact we have the following open question.

Question 1. Suppose X is a finite subset of positive integers containing 1. Does an $X-$decomposable finite group G exist ?.

We continue our classification of $\{1, n, m\}-$decomposable, $1 < n < m$, finite group and first look at the solvable case.

Theorem 1. *Let G be a $\{1, n, m\}-$decomposable finite group, $1 < n < m$. If G is solvable then G' is Abelian, $\mathcal{K}_G = \{1, G', G\}$, $G' \cong E(p^r)$ is maximal in G, and G is a Frobenius group with kernel G' and its complement is a cyclic group of prime order q with $p^r - 1 = (n-1)q$. Moreover, if G is non-solvable and non-perfect finite group with $n \leq 8$, then $n \geq 5$ and G' is simple.*

Proof. By Lemma 1, every finite Abelian group of order n is $D(n)-$ decomposable. Hence without loss of generality, we can assume that G is non-Abelian. Suppose G' is not Abelian. Then $1 < G'' < G' < G$ and so $|Ncc(G)| \geq 4$, i.e. a contradiction. Thus G' is Abelian. If N is a proper non-trivial normal subgroup of G different from G' then we must have $G = NG'$ and $N \cap G' = 1$. Therefore, $G \cong N \times G'$ and G is Abelian. But this is impossible. So $\mathcal{N}_G = \{G'\}$.

Since G' is a maximal subgroup of G, $|G : G'| = q$ with q prime. Since G' is a minimal normal subgroup of G, G' is an elementary Abelian subgroup of order, say p^r. Thus, $|G| = p^r q$. Since G is non-Abelian, $q \neq p$ and $C_G(x) = G'$, for any $1 \neq x \in G'$. Therefore, by [11], Theorem 1.2, p. 1136, G is a Frobenius group with kernel G'. Since G' is Abelian, by [11], Theorem 5.1, p. 1160, $n-1 = \frac{|G'|-1}{q}$. Therefore, $p^r - 1 = (n-1)q$, as desired. If G is non-solvable and non-perfect, then by [2], Lemma 2 and [4], Lemma 2.1 $n \geq 5$ and G' is simple. Theorem 1 is proved.

Lemma 6. *Let G be a $n-$decomposable non-solvable non-perfect finite group and $|\mathcal{N}_G| \geq 2$. Then $|\mathcal{N}_G| = 2$, n is a prime number and $G \cong Z_n \times B$, where B is a non-Abelian simple group with exactly n conjugacy classes.*

Proof. Let A and B be elements of $\mathcal{N}_G$. Then by [1], Theorem 2, $G \cong A \times B$. It is easy to see that A and B are simple groups. By [14], p. 88, A and B are the only proper non-trivial normal subgroups of G. So $|\mathcal{N}_G| = 2$. If A and B are non-Abelian simple groups then $G' = G$, i.e. a contradiction. Therefore, one of A or B, say A, is Abelian. Since A is simple, n is a prime number and $A \cong Z_n$, proving the lemma.

Theorem 2. *Let G be a non-perfect non-solvable $\{1, n, m\}-$decomposable finite group. Then G is isomorphic to $Z_5 \times A_5$, S_5, S_6, $A_6.2_2$, $A_6.2_3$, $Aut\ (PSL(2, q))$, for $q = 7, 8$. Here, $A_6.2_2$ and $A_6.2_3$ are non-isomorphic split extensions of the alternating group A_6, in the small group library of GAP [17].*

Proof. If $n = 2$ then according to [17], Theorem 2.1, G' is the unique non-trivial proper normal subgroup of G and is elementary Abelian. This implies that G is solvable, a contradiction. Suppose $n = 3$ and H is a non-trivial proper normal subgroup of G. As H is 3-decomposable, it follows from [18] that H is solvable, but this is impossible. Thus $n \geq 4$ and the theorem follows from [1], Theorem 5, [2], Theorems 5 and 6, and [4], Theorems 2.5 and 2.6. Theorem 2 is proved.

Lemma 7. *Suppose G is a X−decomposable finite group with $X = \{1, 2, ..., n\}$. Then $n \leq 3$ and $G \cong 1, Z_2$ or S_3.*

Proof. If G is Abelian then by fundamental theorem of finite Abelian groups, $|G| \leq 2$. Suppose G is non-Abelian. Then by [12], Proposition 2.1, G is a Frobenius group with kernel N of odd order $\frac{|G|}{2}$. On the other hand, by [12], Theorem 2.2 if $G \ncong S_3$ then G has a normal or Abelian 2-complement M. Suppose M is normal. Then $G \cong Z_2 \times M$. This imlies that $n \neq 4$. Suppose $n \geq 5$ and H is a $(n-2)$−decomposable subgroup of G. Then H is a normal subgroup of M. Since M is $(n-1)$−decomposable, by [12], Proposition 2.1 $|M|$ must be even, but this is impossible. Lemma 7 is proved.

Theorem 3. *Suppose G is a non-perfect $\{1, 2, 3, n\}$−, $\{1, 3, 4, m\}$− or $\{1, 2, 4, m\}$-decomposable finite group, for $n > 3$ and $m > 4$. Then in the first case $n = 5, 6, 7$ and G is isomorphic to Z_6, D_8, Q_8, S_4, $SmallGroup(20, 3)$ or $SmallGroup(24, 3)$. In the second case, G is isomorphic to $Small\ Group(36, 9)$, a metAbelian group of order $2^n(2^{\frac{n-1}{2}} - 1)$, in which n is odd positive integer and $2^{\frac{n-1}{2}} - 1$ is a Mersenne prime or a metAbelian group of order $2^n(2^{\frac{n}{3}} - 1)$, where $3|n$ and $2^{\frac{n}{3}} - 1$ is a Mersenne prime. Finally, in the third case, G is isomorphic to an Abelian group of order 8, $SmallGroup(12, 1)$, $SmallGroup(24, 13)$, $SmallGroup(168, 43)$, a metAbelian group of order 2^{n+1} $(2^n - 1)$ or a metAbelian group of order $2^{2n}(2^n - 1)$, where $2^n - 1$ is a Mersenne prime.*

Proof. The proof follows from the main results of [5] – [7] and some elementary calculation with GAP, [16]. We end this paper with the following question :

Question 2 : Is there any classification of perfect X−decomposable finite groups ?

Резюме. Пусть G – конечная группа, а A – нормальная подгруппа группы G и $ncc(A)$ = число G−сопряжённых классов A. Подгруппа A называется n−представимой, если $ncc(A) = n$. Можно говорить о представимости группы относительно непустого подмножества X положительных целых чисел. В настоящей статье решается задача нахождения всех конечных подмножеств X положительных целых чисел таких, что существует X−представимая конечная группа.

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ZERO-$\sqrt{3}$ AND ZERO-2 LAWS FOR REPRESENTATIONS OF LOCALLY COMPACT ABELIAN GROUPS

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Abstract. A well-known "zero-2 law" states that if $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of bounded operators on a Banach space X, and if $\limsup_{t\to 0^+}\|I-T(t)\|<2$, then $\lim_{t\to 0^+}\|I-T(t)\|=0$. We discuss here analogous problems for unitary representations θ of a general topological group $\mathcal{U}$ on a unitary Banach algebra A. Let 1 be the unit element of $\mathcal{U}$, and let I be the unit element of A. Elementary geometric considerations show that the situation with the spectral radius $\rho(I-\theta(u))$ as $u\to 1$ is quite simple, since there are only four possibilities : $\lim_{u\to 1}\rho(I-\theta(u))=0$, $\limsup_{u\to 1}\rho(I-\theta(u))=\sin(\frac{n\pi}{2n+1})\geq\sqrt{3}$ for some $n\geq 1$, $\limsup_{u\to 1}\rho(I-\theta(u))=2$ and $\limsup_{u\to 1}\rho(I-\theta(u))=+\infty$. If the group $\mathcal{U}$ admits "continuous division by 2," the second case is impossible and a "zero-2 law" holds for $\limsup_{u\to 1}\rho(I-\theta(u))$. Another phenomenon holds for unitary representations of an Abelian locally compact group $(H,+)$ on a Banach algebra A. Using a classical result of Gelfand, which is equivalent to the fact that points are sets of synthesis for the algebra of absolutely convergent Fourier series, it is possible to show that if $\lim_{h\to 0}\rho(I-\theta(h))=0$, then either $\lim_{h\to 0}\|I-\theta(h)\|=0$, i.e. the representation is continuous with respect to the norm of A, or $\limsup_{h\to 0}\|I-\theta(h)\|=+\infty$. So if we consider any unitary representation of $(H,+)$, then either $\lim_{h\to 0}\|I-\theta(h)\|=0$, or $\limsup_{h\to 0}\|I-\theta(h)\|\geq\sqrt{3}$. If a locally compact Abelian group admits continuous division by 2, then either $\lim_{h\to 0}\|I-\theta(h)\|=0$, or $\limsup_{h\to 0}\|I-\theta(h)\|\geq 2$. If we restrict attention to representations which are bounded on some neighborhood of 0, we obtain a more precise result : either the representation is continuous with respect to the norm of A, or

$$\limsup_{h\to 0}\|p(\theta(u))\|\geq\limsup_{h\to 0}\rho(p(\theta(u))=\max_{|z|=1}|p(z)|$$

for every polynomial p.

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§1. INTRODUCTION

A well-known "zero-2 law", see [14], p. 60, shows that if $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of bounded operators on a Banach space X, and if $\lim\sup_{t\to 0^+}\|I - T(t)\| < 2$, then $\lim_{t\to 0^+}\|I - T(t)\| = 0$, which means that the infinitesimal generator A of the semigroup is bounded, so that the map $z \longmapsto e^{zA}$ defines a holomorphic extension of the semigroup on the complex plane. This is a consequence of a theorem of Kato [9] which shows that if a C_0-semigroup $(T(t))_{t>0}$ of bounded operators satisfies $\lim\sup_{t\to 0^+}\|I - T(t)\| < 2$, then the semigroup admits a holomorphic semigroup extension to some angular sector

$$\Omega_\theta = \{z \in \mathbf{C} \setminus \{0\} \mid |arg(z)| < \theta\}.$$

Analogous results for exponentially bounded weakly measurable semigroups have been obtained by Beurling [1].

Another approach, due to Neuberger [12] and Pazy [13], consists in showing that if a C_0- semigroup $(T(t))_{t>0}$ of bounded operators satisfies $\lim\sup_{t\to 0^+}|I - T(t)| < 2$, or the weaker condition $|I - T(t)| \leq 2 - t\,\log(\frac{1}{t})u(t)$ for t sufficiently small, where $\lim_{t\to 0^+} u(t) = +\infty$, then $AT(t)$ is bounded for every $t > 0$, which implies that A is bounded if $T(t)$ is invertible for some, or equivalently all, $t > 0$.
In a recent paper [4], the author observed that if $(T(t))_{t>0}$ is any one-parameter group in a unitary Banach algebra A, and if $\rho(x)$ denotes the spectral radius of $x \in A$, then there are only three possibilities : either $\lim_{t\to 0^+}\rho(I - T(t)) = 0$, or $\lim\sup_{t\to 0^+}\rho(I - T(t)) = 2$, or $\lim\sup_{t\to 0^+}\rho(I - T(t)) = +\infty$, which leads to an elementary proof of the "zero-2 law" for strongly continuous groups of bounded operators. For general one-parameter groups bounded near 0 a standard renorming argument which goes back do Feller [7] gives a smaller equivalent norm on the closed subalgebra generated by the semigroup for which $\lim_{t\to 0}\|T(t)\| = 1$ (in which case $\lim\sup_{t\to 0}\|I-T(t)\| = \lim\sup_{t\to 0^+}\|I-T(t)\|$), and a very elementary computation, based on the identity

$$(I + T(h))^2 - (I - T(h))^2 = 4\,T(h)$$

shows that either $\lim_{t\to 0^+}\|I - T(t)\| = 0$, or $\lim\sup_{t\to 0^+}\|I - T(t)\| \geq \sqrt{3}$. (A similar "zero-$\sqrt{3}$ law holds for $\lim\inf_{t\to 0^+}\Delta(t)$, where $\Delta(t) = \lim\sup_{h\to 0^+}\|T(t+h) - T(t)\|$, and the elementary tricks used in [4] were suggested to the author by Baxter's proof [2] of the inequality $\Delta(s+t) \leq \Delta(s)\Delta(t)$, stated in [2] for C_0-semigroups but in fact valid for arbitrary semigroups in a Banach algebra). So it is a natural question to ask whether a stronger "zero-2 law" holds for arbitrary one-parameter groups. For groups of the form $(T(s))_{s\in S}$, where S is a dense additive subgroup of $\mathbb{R}$, the "zero-$\sqrt{3}$" law holds, and a nice observation of Borichev [3], see Example 2.4 below, shows that $\sqrt{3}$ turns out to be optimal. However we will see, that the "zero-2 law" holds for all semigroups $(T(t))_{t\in\mathbb{R}}$. In fact, we completely clarify here the situation for representations of locally compact Abelian groups. There are two distinct phenomena. The first phenomenon concerns unitary representations $\theta = \mathcal{U} \longmapsto A$ of a topological group $\mathcal{U}$ on a Banach algebra A, i.e. maps $\theta = \mathcal{U} \longmapsto A$ such that $\theta(1) = I$, where 1 denotes the unit element of $\mathcal{U}$ and I the unit element of A, and such that $\theta(uv) = \theta(u)\theta(v)$ for $u, v \in \mathcal{U}$. We assume that they are locally spectrally bounded, in the sense that there exists a neighborhood U of 1 and $M > 0$ satisfying $\rho(\theta(u)) \leq M$ for every $u \in U$. We introduce in Section 2 the set $\Gamma(\theta)$ is equal to $\{\lambda \in \mathbf{C} \mid \liminf_{u\to 1} \text{dist}(\lambda, \text{spec}(\theta(u)) = 0\}$. Elementary observations show that either $\Gamma(\theta) =$ the unit circle $\mathbf{T}$, or there exists a finite family $p_1, ..., p_k$ of positive integers such that $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$, where $\Gamma_{p_j} := \{z \in \mathbf{C} \mid z^{p_j} = 1\}$. The case $p_1 = 1, k = 1$ gives $\Gamma(\theta) = 1$. In the case $\Gamma(\theta) \neq \mathbf{T}$ we obtain a finite union of vertices of regular polygons containing 1 and contained in the unit circle $\mathbf{T}$. This shows that we have the following possibilities :

1) $\lim_{u\to 1}\rho(I - \theta(u)) = 0$,

2) $\limsup_{u\to 1}\rho(I - \theta(u)) = 2\sin(\frac{n\pi}{2n+1}) \geq \sqrt{3}$ for some $n \geq 1$,

3) $\limsup_{u\to 1}\rho(I - \theta(u)) = 2$,

4) $\limsup_{u\to 1}\rho(I - \theta(u)) = +\infty$.

For representations of some compact Abelian groups and of some dense additive subgroups of $\mathbb{R}$ the case 2 can occur for any $n \geq 1$. But if $\mathcal{U}$ admits "continuous division by 2" (which means the possibility to define square roots in a reasonable way near the unit element), then case 2 cannot occur and the "zero-2 law" holds for $\limsup_{u\to 1} \rho(I - \theta(u))$.

The second phenomenon, completely described below in the case of locally compact Abelian groups, concerns the behavior of $\limsup_{u\to 0}\|(I - \theta(u))\|$ for representations θ of Abelian groups satisfying condition 1), i.e, in additive notation, $\lim_{u\to 0}\rho(I - \theta(u)) = 0$. In this case we show in Section 3 that we have a "zero-∞ law" : either $\lim_{u\to 0}\|I - \theta(u)\| = 0$, or $\limsup_{u\to 0}\|(I - \theta(u))\| = +\infty$. This result is based on a general structure theorem for locally compact Abelian groups and on a classical theorem of Gelfand [8], improved later by Hille [10] : if an invertible element a in a Banach algebra A satisfies $\text{spec}(a) = \{1\}$ and $\sup_{n\in Z} \|a^n\| < +\infty$, (or, more generally, $\|a^n\| = o(n)$ as $|n| \to \infty$), then a is the unit element of A. This well-known result is not elementary : Gelfand's original version of this statement is equivalent to the fact that functions of zero exponential type which are bounded on the real line are constant, and to the fact that points are sets of synthesis for the algebra of absolutely convergent Fourier series.

Other results concerning the behavior of a one-parameter semigroup $(T(t))_{t>0}$ in a Banach algebra A which imply the existence of an element $P \in A$ satisfying $\lim_{t\to 0^+} \|P - T(t)\| = 0$ appeared recently. It was shown in [6] that such an element P exists if

$$\limsup_{t\to 0^+} \|T(t) - T((n+1)t)\| < \frac{n}{(n+1)^{1+\frac{1}{n}}}$$

for some $n \geq 1$, and more general results of this type involving the behavior of $\|T(s) - T(t)\|$ near the origin are obtained in [4] in the case of strongly continuous semigroups.

§2. BEHAVIOR OF THE SPECTRAL RADIUS OF A GROUP REPRESENTATION

Definition 2.1. Let $(\mathcal{U}, \cdot)$ be a topological group. We will say that $\mathcal{U}$ admits continuous division by 2 if there exists an open subset U of $\mathcal{U}$ containing the unit element 1 and a map $\phi : U \to \mathcal{U}$, which is continuous at 1 and satisfies $\phi(1) = 1$ and $\phi^2(u) = u$ for every $u \in U$.

Of course if $\mathcal{U}$ is an additive Abelian group the conditions $\phi(1) = 1, \phi^2(u) = u$ are to be replaced by the conditions $\phi(0) = 0$, $2\phi(u) = u$.

Proposition 2.2. *Let $\mathcal{U}$ be a topological group, and let $\theta : \mathcal{U} \to A$ be a locally spectrally bounded unitary representation of $\mathcal{U}$ on a Banach algebra. Set*

$$\Gamma(\theta) = \{\lambda \in \mathbf{C} \mid \liminf_{u\to 1} dist(\lambda, spec(\theta(u)) = 0\}.$$

Then either $\Gamma(\theta) = \mathbf{T}$, or there exists a finite family $p_1, ..., p_k$ of positive integers such that $\Gamma(\theta) = \cup_{1\leq j\leq k} \Gamma_{p_j}$, where $\Gamma_{p_j} := \{z \in 1\}$. If, further, $\mathcal{U}$ admits continuous division by 2, then either $\Gamma_\theta = \{1\}$ or $\Gamma_\theta = \mathbf{T}$.

Also lim $\sup_{u\to 1} \rho(p(\theta(u)) = \max_{z\in\Gamma(\theta)} |p(z)|$ for every polynomial p, and for every open subset Ω of $\mathbf{C}$ containing $\Gamma(\theta)$ there exists a neighborhood V of 1 such that $spec(\theta(u)) \subset \Omega$ for every $u \in V$.

Proof : It follows from the definition of $\Gamma(\theta)$ that $\Gamma(\theta)$ is closed, and $\lambda^n \in \Gamma(\theta)$ for $\lambda \in \Gamma(\theta)$, $n \in \mathbf{Z}$. If $|\lambda| \neq 1$ for some $\lambda \in \Gamma(\theta)$, then taking if necessary λ^{-1} instead of λ we can assume that $|\lambda| > 1$. We obtain

$$\limsup_{u\to 0} \rho(I - \theta(u)) \geq |\lambda|^n \qquad \text{for} \qquad n \geq 1,$$

which contradicts the fact that θ is locally spectrally bounded. Hence $\Gamma(\theta) \subset \mathbf{T}$. If there exists $\lambda \in \Gamma(\theta)$ such that $\lambda^n \neq 1$ for $n \geq 1$, then the set $\{\lambda^n\}_{n\in\mathbf{Z}}$ is dense in $\mathbf{T}$ and $\Gamma(\theta) = \mathbf{T}$. Otherwise there exists $F \subset \mathbb{N}^*$ such that $\Gamma(\theta) = \cup_{n\in F}\{z \in \mathbf{C} \mid z^n = 1\}$, and F is clearly finite if $\Gamma(\theta) \neq \mathbf{T}$.

Assuming that $\mathcal{U}$ admits a continuous division by 2, let $\lambda \in \Gamma_\theta$, and let μ_1 and $\mu_2 = -\mu_1$ be the square roots of $\lambda \in \mathbf{C}$. There exists a net $(u_\tau)_{\tau\in\mathcal{T}}$ in $\mathcal{U}$ such that $\lim_\tau u_\tau = 1$ and $\lim_\tau \alpha_\tau = \lambda$ for some net $(\alpha_\tau)_{\tau\in\mathcal{T}}$ of complex numbers satisfying $\alpha_\tau \in \operatorname{spec}(\theta(u_\tau))$ for $\tau \in \mathcal{T}$. We can assume that $u_\tau \in U$ for $\tau \in \mathcal{T}$, where U is an open subset of $\mathcal{U}$ on which there exists a function $\phi : u \longmapsto u^{\frac{1}{2}}$ satisfying the conditions of Definition 2.1. There exists $\beta_\tau \in \operatorname{spec}(\theta(u_\tau^{1/2}))$ such that $\beta_\tau^2 = \alpha_\tau$. We have $\lim_\tau u_\tau^{1/2} = 1$ and

$$\limsup_\tau \min(|\beta_\tau - \mu_1|, |\beta_\tau - \mu_2|) \leq \lim_\tau |\alpha_\tau - \lambda|^{1/2} = 0.$$

This shows that there exists $i \in \{1,2\}$ such that μ_i is the limit of a subnet of the net $(\beta_\tau)_{\tau\in\mathcal{T}}$. Hence $\mu_i \in \Gamma_\theta$ and the equation $z^2 = \lambda$ admits at least a solution in Γ_θ for every $\lambda \in \Gamma_\theta$. This leaves only the possibilities $\Gamma_\theta = \{1\}$ or $\Gamma_\theta = \mathbf{T}$.
Now let $p \in \mathbf{C}[x]$, and let $\lambda \in \Gamma(\theta)$ such that $|p(\lambda)| = \max_{z\in\Gamma(\theta)} |p(z)|$. Let $\epsilon > 0$ and let $\delta > 0$ be such that $|p(z)| > |p(\lambda)| - \epsilon$ for $|z - \lambda| < \eta$. It follows from the definition of $\Gamma(\theta)$ that for every neighborhood U of 1 there exists $u \in U$ such that $\operatorname{Spec}(\theta(u))$ contains some $z \in \mathbf{C}$ such that $|z - \lambda| < \eta$, and so $\rho(p(\theta) > |p(\lambda)| - \epsilon$, and

$$\limsup_{u\to 1} \rho(p(\theta(u))) \geq \max_{z\in\Gamma(\theta)} |p(z)|.$$

Let Ω be an open subset of $\mathbf{C}$ containing $\Gamma(\theta)$. There exists a neighborhood U_0 of 1 such that $\rho(\theta(u)) \leq M$ for every $u \in U_0$, and we can assume that $M > 1$ and $\Omega \subset D(0, M) := \{z \in \mathbf{C} \mid |z| \leq M\}$. Set $K := D(0, M) \setminus \Omega$. A routine compactness argument shows that there exists a finite open covering $W_1, ..., W_k$ of K and a family $U_1, ..., U_k$ of neighborhoods of 1 contained in U_0 such that $\operatorname{spec}(\theta(u)) \cap W_j = \emptyset$ for every $u \in U_j$. Set $U = \cap_{1\leq j\leq k} U_j$. Then $\operatorname{spec}(\theta(u)) \subset \Omega$ for every $u \in U$. This shows in particular that

$$\limsup_{u\to 1} \rho(p(\theta(u)) \leq \max_{z\in\Gamma(\theta)} |p(z)|,$$

and so $\limsup_{u\to 1} \rho(p(\theta(u)) = \max_{z\in\Gamma(\theta)} |p(z)|$ for every polynomial $p \in \mathbf{C}[x]$.

Corollary 2.3. Let $\mathcal{U}$ be a topological group, and let $\theta : \mathcal{U} \to A$ be a locally spectrally bounded unitary representation of $\mathcal{U}$ on a Banach algebra. Then either $\lim_{u\to 1} \rho(\theta(u) - I) = 0$, or $\limsup_{u\to 1} \rho(\theta(u) - I) = 2\sin(\frac{n\pi}{2n+1}) \geq \sqrt{3}$ for some $n \geq 1$, or $\limsup_{u\to 1} \rho(\theta(u) - I) = 2$. If further, $\mathcal{U}$ admits continuous division by 2, then either $\lim_{u\to 1} \rho(\theta(u) - I) = 0$, or $\limsup_{u\to 1} \rho(\theta(u) - I) = 2$.

Proof : Set $p = x - 1$. Then

$$\limsup_{u\to 1} \rho(\theta(u) - I) = \limsup_{u\to 1} \rho(p(\theta(u))) = \max_{\lambda\in\Gamma(\theta)} |\lambda - 1|.$$

If $\Gamma(\theta) = \{1\}$, we get $\lim_{u\to 1}\rho(\theta(u) - I) = 0$. If $\Gamma(\theta) = \mathbf{T}$, or if $\Gamma(\theta)$ contains Γ_{2n} for some $n \geq 1$, we get $\limsup_{u\to 1} \rho(\theta(u) - I) = 2$. Otherwise there exists a strictly increasing finite sequence $(n_1, ..., n_p)$ of positive integers such that $\Gamma(\theta) = \cup_{1\leq j\leq p}\Gamma_{2n_j+1}$. By a standard calculation

$$\max_{\lambda\in\Gamma_{2n+1}} |\lambda - 1| = 2\sin\left(\frac{n\pi}{2n+1}\right)$$

for $n \geq 1$, and so in this situation we have

$$\limsup_{u\to 1} \rho(\theta(u) - I) = 2\sin\left(\frac{n_1\pi}{2n_1+1}\right) \geq 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

Now if $\mathcal{U}$ admits a continuous division by 2 let $\lambda \in \Gamma_\theta$. Then either $\Gamma_\theta = \{1\}$, in which case $\lim_{u\to 1}\rho(\theta(u) - I) = 0$, or $\Gamma_\theta = \mathbf{T}$, in which case $\limsup_{u\to 1}\rho(\theta(u) - I) = 2$.
It is easy to see that all the situations described above can occur with representations of dense subgroups of $\mathbb{R}$ in finite dimensional algebras. The morphisms $\theta : \mathbb{R} \to \mathbf{T}$ for which $\Gamma(\theta) = \mathbf{T}$ are the well-known non-measurable characters of $\mathbb{R}$, i.e. the non-measurable morphisms from $(\mathbb{R}, +)$ onto $(\mathbf{T}, \cdot)$ (see [11]).
We also have the following example, which is a minor modification of an example of Borichev [3].

Example 2.4. Let $(a_\tau)_{\tau\in\mathbb{R}}$ be a Hamel basis of $\mathbb{R}$ viewed as a vector space over the field $\mathbf{Q}$ of rational numbers such that $a_\tau > 0$ for $\tau \in \mathbb{R}$ and $\inf_{\tau\in\mathbb{R}} a_\tau = 0$. Set $G = \oplus_{\tau\in\mathbb{R}}\mathbf{Z}a_\tau$.
Then for every strictly increasing family $(p_1, ..., p_k)$ of integers ≥ 2 there exists a bounded representation $\theta : G \to \mathbf{C}^k$ such that $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$.
To see this consider a strictly increasing family $(p_1, ..., p_k)$ of positive integers. For $1 \leq i \leq k$, we define $\theta_j : G \to \mathbf{C}$ by the formula

$$\theta_j\left(\sum_{\tau\in\mathbb{R}} n(\tau)a_\tau\right) = e^{2\left(\sum_{\tau\in\mathbb{R}} n(\tau)\right)\frac{i\pi}{p_j}}.$$

Let $\theta : G \to \mathbf{C}^k$ be the map $u \longmapsto (\theta_1(u), ..., \theta_k(u))$. Then $\theta_j(a_\tau) = e^{\frac{2i\pi}{p_j}}$ for $\tau \in \mathbb{R}$, and it follows immediately that $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$. In particular the additive group

G constructed above admits for $n \geq 1$ a one- dimensional representation $\theta_{(n)}$ such that

$$\limsup_{u\to 0} \|I - \theta_{(n)}(u)\| = \limsup_{u\to 0} \rho(I - \theta_{(n)}(u)) = \sin(\frac{n\pi}{2n+1}).$$

We conclude this section by an example which shows that the set $\Gamma(\theta)$ can take all the forms described in Proposition 2.2 for bounded representations of compact Abelian groups.

Example 2.5. For a strictly increasing sequence of positive integers $p_1, ..., p_k$ with $p_1 \geq 2$, equip $G_j = \Gamma_{p_j}^{\mathbb{N}}$ with the product topology, and set $G = G_1 \times ... \times G_k$. Then the compact group G admits a representation θ on $\mathbf{C}^k$ for which $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$. To see this pick a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and set $\chi_j(u_j) = \lim_{\mathcal{U}} u_{j,n}$ for $u_j = (u_{j,n})_{n\geq 0}$ in G_j, and $\theta(u) = (\chi_1(u_1), ..., \chi_k(u_k))$ for $u = (u_1, ..., u_k) \in G$. Looking at sequences $u_j^{(m)} = (u_{j,n}^{(m)})_{n\geq 0} \in G_j$ which are constant for $n > m$ and satisfy $u_{j,n}^{(m)} = 1$ for $n \leq m$, we see immediately that the bounded representation θ of G satisfies $\Gamma(\theta) = \cup_{1\leq j\leq k}\Gamma_{p_j}$. Also for $1 \leq j \leq k$ the group G admits a one-dimensional representation θ_j for which

$$\limsup_{u\to 0} \|I - \theta_j(u)\| = \limsup_{u\to 0} \rho(I - \theta_j(u)) = \sin(\frac{p_j\pi}{2p_j+1}).$$

§3. THE ZERO-$\sqrt{3}$ AND ZERO-2 LAWS FOR REPRESENTATIONS OF LOCALLY COMPACT ABELIAN GROUPS

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a discontinues additive map, and let $\chi = e^{\phi}$. Equip $A := \mathbf{C}^2$ with the norm $(u, v) \longmapsto |u| + |v|$ and product $(u_1, v_1)(u_2, v_2) = (u_1u_2, u_1v_2 + v_1u_2)$. We obtain a two dimensional Banach algebra and $I = (1, 0)$ is the unit element of A. Now set $\theta(x) = (1, \chi(x))$ for $x \in \mathbb{R}$. This gives a representation of $(\mathbb{R}, +)$ on A such that $\rho((I - \theta(x)) = 0$ for $x \in \mathbb{R}$, while $\lim\sup_{x\to 0^+} \|\theta(x)\| = +\infty$. We will see that this phenomenon disappears for representations θ of locally compact Abelian groups $(G, +)$ which are locally bounded in the sense that there exists a neighborhood U of 0 in G and $M > 0$ such that $\|\theta(g)\| \leq M$ for every $g \in U$. For such representations the condition $\lim_{g\to 0} \rho(I - \theta(g)) = 0$ implies $\lim_{g\to 0} \|I - \theta(g)\| = 0$.

We begin with a preliminary observation. A well-known theorem of Gelfand [8] shows that if an invertible element of a unitary Banach algebra A satisfies $\sup_{n\in\mathbf{Z}}\|a^n\| < +\infty$, and if $\text{spec}(a) = \{1\}$, then a is the unit element of A. As indicated in the Introduction, this result is equivalent to the fact that points are sets of synthesis for the algebra of absolutely convergent Fourier series, and also is equivalent to the fact that an entire function of zero exponential type which is bounded on the real line is constant.

Let A be a Banach algebra with unit element I, and let $a \in A$ be such that $\rho(I-A) < 1$. We can define $\log(a)$ by the usual holomorphic functional calculus :

$$\log(a) = \log(1+a-1) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}(a-1)^n.$$

It follows from the standard properties of the holomorphic functional calculus that $e^{\log(a)} = a$ and that $\chi(\log(a)) = \log(\chi(a))$ and $\chi(a) = e^{\chi(\log(a))}$ for every character χ on A. Also if two commuting elements $a, b \in A$ satisfy $\rho(a-1) < 1$, $\rho(b-1) < 1$, $\rho(ab-1) < 1$, then we have $\log(ab) = \log(a) + \log(b)$. For a proof of this standard fact set $x = \log(ab), y = \log(a) + \log(b)$. We have $0 = e^x - e^y = (x-y)F(x,y)$, where F is the entire function defined on $\mathbf{C}^2$ by the formula $F(z_1, z_2) = \frac{e^{z_1}-e^{z_2}}{z_1-z_2}$ if $z_1 \neq z_2$, $F(z,z) = e^z$ for $z \in \mathbf{C}$. Hence $F(z_1, z_2) \neq 0$ unless there exists $k \in \mathbf{Z}$ such that $z_1 - z_2 = 2ik\pi$. Now if χ is a character on A, we have

$$\Im(\chi(x)) \in (-\frac{\pi}{2}, \frac{\pi}{2}), \qquad \Im(\chi(y)) \in (-\pi, \pi), \qquad |\chi(x) - \chi(y)| < \frac{3\pi}{2},$$

and so $F(x,y)$ is invertible and $x = y$.

Lemma 3.1. *Let $t = (t_1, ..., t_k) \longmapsto T(t)$ be a unitary representation of $\mathbb{R}^k$ on a Banach algebra A. If $\lim_{t\to 0} \rho(T(t) - I) = 0$, then either $\limsup_{t\to 0} ||(T(t) - I)|| = +\infty$, or the representation is continuous, so that $\lim_{t\to 0} ||(T(t) - I)|| = 0$.*

Proof : Set $|t| = max_{1\leq j\leq k}|t_j|$ for $t \in \mathbb{R}^k$. There exists $\delta > 0$ such that $\rho(I - T(t)) < 1$ for $|t| \leq \delta$. In this situation we can set as above $U(t) = \log(T(t))$ for $|t| \leq \delta$, and we have $U(s+t) = U(s) + U(t)$ if $|s| \leq \frac{\delta}{2}$, $|t| \leq \frac{\delta}{2}$. Also if $n \in \mathbb{N}$, we have $(U(2^n t)) = 2^n U(t)$ for $|t| \leq 2^{-n}\delta$. Let $t \in \mathbb{R}^k$ and let $n, m \in \mathbb{N}$ satisfying $2^n\delta > |t|$, $2^m\delta > |t|$. Then $2^n U(2^{-n}t) = 2^{m+n}U(2^{-m-n}t) = 2^{-m}U(2^{-m}t)$. So we can set $U(t) = 2^n U(2^{-n}t)$ for $t \in \mathbb{R}^k$, where $n \geq 0$ is any integer such that $2^n\delta \geq |t|$, and the map $t \longmapsto T(t)$ is well defined on $\mathbb{R}^k$. Let $s, t \in \mathbb{R}^k$, and let $n \in \mathbb{N}$ satisfying $2^{n-1}\delta \geq \max(|s|, |t|)$. Then $U(s+t) = 2^n U(2^{-n}(s+t)) = 2^{-n}U(s) + 2^{-n}U(t) = U(s) + U(t)$.

We can assume that A is commutative. Let χ be a character on A. The map $t \longmapsto \chi(U(t)) = \log(\chi(T(t))$ is continuous for $|t| \leq \delta$; being additive on $\mathbb{R}^k$, it is continuous on $\mathbb{R}^k$. Set $e_j = (\delta_{j,n})_{1\leq n\leq k}$ for $1 \leq j \leq k$, where $\delta_{j,n}$ denotes the usual Kronecker symbol. Then $\chi(U(t_1, ..., t_n)) = \sum_{j=1}^k t_j \chi(U(e_j))$ for $t = (t_1, ..., t_k) = \mathbf{Q}^k$. By continuity, this equality holds for every $t = (t_1, ..., t_k) \in \mathbb{R}^k$.

Now set $V(t) = \sum_{j=1}^k t_j U(e_j)$ for $t = (t_1, ..., t_k) \in \mathbb{R}^k$ and $R(t) = T(t)e^{-V(t)} = e^{W(t)}$, where $W(t) = U(t) - V(t)$. We have $\chi(W(t)) = 0$ for every character χ on

A, and so spec$(R(t)) = \{1\}$ for $t \in \mathbb{R}^k$. Assume that $\sup_{t\in U} ||T(t)|| < +\infty$ for some neighborhood U of the origin. Then $M := \sup_{|t|\le 1} ||T(t)|| < +\infty$. By construction, $R(e_j) = I$ for $1 \le j \le k$, and so $R(m) = I$ for every $m = (m_1, ..., m_k) \in \mathbf{Z}^k$. This shows that $||R(t)|| \le M$ for $t \in \mathbb{R}^k$. Now let $t \in \mathbb{R}^k$. We have

$$\sup_{n\in\mathbf{Z}} ||R(t)^n|| = \sup_{n\in\mathbf{Z}} ||R(nt)|| \le M.$$

Since spec$(R(t)) = \{1\}$, it follows from Gelfand's theorem that $R(t) = I$. Hence $T(t) = e^{V(t)}$ for $t \in \mathbb{R}^k$, which completes the proof of the lemma.
A similar phenomenon holds for representations of compact Abelian groups.

Lemma 3.2. *Let θ be a unitary representation of a compact Abelian group $(G, +)$ on a Banach algebra A. If $\lim_{g\to 0}\rho(\theta(g) - I) = 0$, then either*

$$\limsup_{g\to 0} ||(\theta(g) - I)|| = +\infty,$$

or the representation is continuous, so that $\lim_{g\to 0}||(\theta(g) - I)|| = 0$.

Proof : We can assume that A is commutative and that span$\{\theta(g)\}_{g\in G}$ is dense in A. Let χ be a character on A. The map $g \longmapsto \chi(\theta(g))$ is continuous on G, and so $\chi(\theta(G))$ is a compact multiplicative subgroup of $\mathbf{C} \setminus \{0\}$. Hence $\chi(\theta(G)) \subset \mathbf{T}$. This shows that $\chi \circ \theta$ is a continuous character of the group G. It follows from the definition of the topologies of the character space $\hat{A}$ and of the dual group $\hat{G}$ that $\tilde{\theta} : \chi \longmapsto \chi \circ \theta$ is a continuous map from $\hat{A}$ into $\hat{G}$.
Hence $\tilde{\theta}(\hat{A})$ is a compact subset of the discrete group $\hat{G}$, which shows that $\tilde{\theta}(\hat{A})$ is finite. Also $\tilde{\theta}$ is one-to-one since span$\{\theta(g)\}_{g\in G}$ is dense in A, and $\hat{A}$ is finite. Let $\chi_1, ..., \chi_k$ be the elements of $\hat{A}$. A standard application of Shilov's idempotent theorem shows that there exists idempotents $e_1, ..., e_k$ in A such that $I = e_1 + ... + e_k$, $e_i e_j = 0$ for $i \neq j$ and $\chi_i(e_j) = \delta_{i,j}$ for $1 \le i \le k$, $1 \le j \le k$. Set $\psi(g) = \sum_{j=1}^{k} \chi_j(g) e_j$. Then $\psi : G \to A$ is a continuous representation of G on A, and $\chi(\psi(g)) = \chi(\theta(g))$ for every $\chi \in \hat{A}$ and every $g \in G$.
Assume that θ is locally bounded. Clearly, $\theta \circ \psi^{-1}$ is a locally bounded representation of G on A. Since G is compact, $\theta \circ \psi^{-1}$ is in fact bounded, and there exists $M > 0$ such that $\theta \circ \psi^{-1}(g) \le M$ for every $u \in G$. Since spec$((\theta \circ \psi^{-1})(g)) = \{1\}$, it follows again from Gelfand's theorem that $(\theta \circ \psi^{-1})(g) = I$ for every $g \in G$. Hence $\theta = \psi$ and θ is continuous.

Theorem 3.3. *Let H be a locally compact Abelian group, and let θ be a unitary representation of H on a Banach algebra. If $\lim_{h\to 0}\rho(I - \theta(h)) = 0$, then either*

$$\lim_{h\to 0} ||I - \theta(h)|| = 0 \quad \text{or} \quad \limsup_{h\to 0} ||I - \theta(h)|| = +\infty.$$

Proof : It follows from a standard result of the theory of locally compact Abelian groups (see, for example, [15]) that H possesses an open (hence also closed) subgroup H_1 which is isomorphic to a direct product $\mathbb{R}^k \times G$, where G is a compact Abelian group. So we can assume that $H = R^k \times G$. Set $\theta_1(t) = \theta(t,0)$ for $t \in \mathbb{R}^k$, and set $\theta_2(g) = \theta(0,g)$ for $g \in G$. If $\limsup_{h\to 0}\|I - \theta(h)\| < +\infty$, then

$$\limsup_{t\to 0} \|I - \theta_1(t)\| < +\infty, \qquad \limsup_{g\to 0} \|I - \theta_2(g)\| < +\infty,$$

and it follows from Lemma 2.1 and Lemma 2.2 that

$$\lim_{h\to 0} \|I - \theta(h)\| = \lim_{(t,g)\to(0,0)} \|I - \theta_1(t)\theta_2(g)\| = 0.$$

Corollary 3.4. Let H be a locally compact Abelian group, and let θ be a unitary representation of H on a Banach algebra. Then either

$$\lim_{h\to 0} \|I - \theta(h)\| = 0 \quad \text{or} \quad \limsup_{h\to 0} \|I - \theta(h)\| \geq \sqrt{3}.$$

If further, H admits continuous division by 2, then either $\lim_{h\to 0}\|I - \theta(h)\| = 0$, or $\limsup_{h\to 0}\|I - \theta(h)\| \geq 2$.

The proof follows immediately from Corollary 1.3 and Theorem 2.3.

In fact we have obtained a more precise result for locally bounded representations of a locally compact Abelian group $(H, +)$ which admits continuous division by 2 : if θ is a locally bounded unitary representation of H into a Banach algebra, then either $\lim_{h\to 0}\|I - T(h)\| = 0$, which means that the representation is continuous with respect to the norm of A, or

$$\limsup_{h\to 0} \|p(T(h))\| \geq \limsup_{h\to 0} \rho(p(T(h)) = \max_{|z|=1} |p(z)|$$

for every polynomial p.

Резюме. Хорошо известный "закон 2-нуля" утверждает, что если $(T(t))_{t\in\mathbb{R}}$ строго непрерывная однопараметрическая группа ограниченных операторов в банаховом пространстве X, и если $\limsup_{t\to 0^+} \|I - T(t)\| < 2$, то $\lim_{t\to 0^+} \|I - T(t)\| = 0$. Мы обсуждаем здесь аналогичные задачи для унитарных представлений θ общей топологической группы $\mathcal{U}$ унитарной банаховой алгебры A. Пусть 1 – единичный элемент группы $\mathcal{U}$, и пусть I – единичный элемент группы A. Элементарные геометрические рассуждения показывают, что ситуация с спектральным радиусом $\rho(I - \theta(u))$ при $u \to 1$ является достаточно простой,

поскольку существуют только четыре возможности : $\lim_{u\to 1} \rho(I - \theta(u)) = 0$, $\limsup_{u\to 1} \rho(I - \theta(u)) = \sin(\frac{n\pi}{2n+1}) \geq \sqrt{3}$ для некоторого $n \geq 1$, $\limsup_{u\to 1} \rho(I - \theta(u)) = 2$ и $\limsup_{u\to 1} \rho(I - \theta(u)) = +\infty$. Если группа $\mathcal{U}$ допускает "непрерывное деление на 2," второй случай не возможен и "закон 2-нуля" выполняется для $\limsup_{u\to 1} \rho(I - \theta(u))$. Другой феномен имеет место для унитарных представлений абелевых локально компактных групп $(H, +)$ на банаховой алгебре A. Используя классический результат Гельфанда, эквивалентный факту, что точки являются множествами синтеза для алгебры абсолютно сходящихся рядов Фурье, в работе показано, что если $\lim_{h\to 0} \rho(I - \theta(h)) = 0$, то $\lim_{h\to 0} \|I - \theta(h)\| = 0$, т.е. представление непрерывно относительно нормы A, или $\limsup_{h\to 0} \|I - \theta(h)\| = +\infty$. Итак, если рассмотреть любое унитарное представление $(H, +)$, то $\lim_{h\to 0} \|I - \theta(h)\| = 0$, или $\limsup_{h\to 0} \|I - \theta(h)\| \geq \sqrt{3}$. Если локально компактная абелева группа допускает непрерывное деление на 2, то $\lim_{h\to 0} \|I - \theta(h)\| = 0$, или $\limsup_{h\to 0} \|I - \theta(h)\| \geq 2$. Если рассмотреть представления, ограниченные в некоторой окрестности точки 0, то получаем более точный результат : либо представление непрерывно относительно нормы алгебры A, либо $\limsup_{h\to 0} \|p(\theta(u))\| \geq \limsup_{h\to 0} \rho(p(\theta(u)) = \max_{|z|=1} |p(z)|$ для любого многочлена p.

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О ЗАДАЧЕ ЧЕБЫШЕВА–СЕГЁ

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Резюме. Статья описывает спектр самосопряжённого оператора, действующего в подпространстве многочленов, порождённых оператором умножения. Результат применяется для исследования задачи Чебышева-Сегё определения экстремумов линейного функционала над множеством положительных многочленов.

§1. ОПИСАНИЕ СПЕКТРА

Пусть w – весовая функция, определённая на отрезке $[a, b]$ с бесконечным числом точек роста, а ϕ – действительнозначная функция, определённая на том же отрезке. Предположим, что моменты

$$w_m = \int_a^b t^m w(t)\, dt, \quad \phi_m = \int_a^b t^m \phi(t) w(t)\, dt, \quad m \in \mathbb{Z}^+$$

существуют. Обозначим через $\mathcal{P}_n$ $(n \in \mathbb{N})$ множество многочленов Q от одной вещественной переменной, удовлетворяющих условию $\deg Q \leq n - 1$. В общем случае, подпространство $\mathcal{P}_n \subset L^2_w(a, b)$ не инвариантно относительно умножения на функцию ϕ, поэтому вместо формы $\langle \phi f, g \rangle$ мы ограничимся формой $\mathcal{P}_n$. Оператор, порожденный этой формой, обозначим через Φ_n.

Предложение 1. *Спектр оператора Φ_n совпадает с множеством корней уравнения*

$$\det\left(\{\phi_{i+k} - \lambda w_{i+k}\}_{i,k=0}^{n-1}\right) = 0, \tag{1.1}$$

а собственные векторы формы Φ_n суть обобщённые собственные векторы пучка

$$\{\phi_{i+k}\}_{i,k=0}^{n-1} - \lambda \{w_{i+k}\}_{i,k=0}^{n-1}.$$

Доказательство. Пусть $\{e_m\}_0^{n-1}$ – совокупность одночленов $e_m(t) = t^m$. Обозначим через G матрицу Грама $G = \{g_{ik}\}$, $g_{ik} = \langle e_k, e_i \rangle = w_{i+k}$. Имеем

$$e_i = \sum_{k=1}^{n-1} \langle e_i, e_k \rangle e_k^*, \tag{1.2}$$

где $\{e_k^*\}$ есть система, биортогональная с $\{e_k\}$. В силу (1.2) оператор $\mathbf{G}$, порождённый матрицей Грама отображает любой элемент e_k^* на e_k, $\mathbf{G}e_k^* = e_k$.

В любом конечномерном линейном пространстве спектр оператора совпадает с множеством собственных значений, поэтому мы должны решить уравнение $(\Phi_n - \lambda I)x = \theta$. Последнее может быть переписано в виде

$$(\Phi_n - \lambda I)\, x = (\Phi_n \mathbf{G} - \lambda \mathbf{G})\, \mathbf{G}^{-1} x = \theta.$$

Матрица оператора $\mathbf{G}$ в базисе $\{e_m^*\}$ совпадает с G, а для нахождения матрицы оператора $\Phi_n \mathbf{G}$ мы должны посчитать скалярные произведения $\langle \Phi_n \mathbf{G} e_m^*, e_k \rangle$. Следовательно, $\langle \Phi_n \mathbf{G} e_m^*, e_k \rangle = \langle \Phi_n e_m, e_k \rangle = \phi_{m+k}$. Предложение 1 доказано.

Для частного случая $\phi(t) = t$, имеем $\phi_m = w_{m+1}$ и

$$\det\left(\{\phi_{i+k} - \lambda w_{i+k}\}_{i,k=0}^{n-1}\right) = (-1)^n \begin{vmatrix} w_0 & w_1 & \cdots & w_n \\ w_1 & w_2 & \cdots & w_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n-1} & w_n & \cdots & w_{2n-1} \\ 1 & \lambda & \cdots & \lambda^n \end{vmatrix} = 0.$$

Следовательно, в этом случае спектр оператора Φ_n совпадает с множеством корней n-ого w–ортогонального многочлена и соответствующие собственные элементы суть фундаментальные многочлены Лагранжа. Этот результат установлен автором в [3] и использован для исследования сдвига частот колебаний механической системы при наложении связей.

§2. ЗАДАЧА ЧЕБЫШЕВА–СЕГЁ

Пусть $\mathcal{P}_n^+ \subset \mathcal{P}_n$ состоит из неотрицательных на $[a, b]$ многочленов. Мы ищем экстремальные значения частного

$$\frac{\int_a^b \phi(t) P(t)\, dt}{\int_a^b w(t) P(t)\, dt}, \quad P \in \mathcal{P}_n^+. \tag{2.1}$$

Эта задача известна как задача Чебышева ([5], гл. 7, 7.72). Чебышев и позднее Сегё исследовали главным образом случай $\phi(t) = tw(t)$. Ниже рассмотрим общий случай.

Согласно известной теореме Маркова-Лукача ([2], гл. 3, Теорема 2.2) любой неотрицательный на $[a,b]$ многочлен P может быть представлен в виде

$$P(t) = \left(\sum_{k=0}^{m} x_k t^k\right)^2 + (b-t)(t-a)\left(\sum_{k=0}^{m-1} y_k t^k\right)^2, \quad n = 2m,$$

$$P(t) = (t-a)\left(\sum_{k=0}^{m} x_k t^k\right)^2 + (b-t)\left(\sum_{k=0}^{m} y_k t^k\right)^2, \quad n = 2m+1. \qquad (2.2)$$

Рассмотрим случай нечётного порядка (случай четного порядка рассматривается аналогичным образом). В этом случае

$$\int_a^b w(t)P(t)\,dt = \left\|\sum_{k=0}^{m} x_k t^k\right\|_1^2 + \left\|\sum_{k=0}^{m} y_k t^k\right\|_2^2, \qquad (2.3)$$

где L^2–нормы $\|\cdot\|_1$ и $\|\cdot\|_2$ определены весовыми функциями $w_1(t) = w(t)(t-a)$ и $w_2(t) = w(t)(b-t)$, соответственно. Для числителя в (2.1) имеем

$$\int_a^b \phi(t)P(t)\,dt = \int_a^b \frac{\phi(t)}{w(t)}\left[\left(\sum_{k=0}^{m} x_k t^k\right)^2 + \left(\sum_{k=0}^{m} y_k t^k\right)^2\right] w(t)(t-a)\,dt.$$

Следовательно (2.1) можно записать в виде $\dfrac{\langle Bu,u\rangle}{\|u\|^2}$, где

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1 = \sum_{k=0}^{m} x_k t^k, \quad u_2 = \sum_{k=0}^{m} y_k t^k, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

B_1, B_2 суть операторы, индуцированные умножением на функцию $w^{-1}(t)\phi(t)$ в пространствах L_1^2 и L_2^2, соответственно. Пусть A – оператор гильбертова пространства. Хорошо известно [4], что множество

$$W(A) = \left\{\frac{\langle Ax,x\rangle}{\langle x,x\rangle},\ x \neq \theta\right\}$$

является множеством значений оператора A. Согласно классической теореме Хаусдорфа - Теплица, для любого самосопряжённого оператора A множество $\overline{W}(A)$ (черта означает замыкание) есть выпуклая оболочка спектра SpA. Если A ортогонально расщеплён $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, то $\overline{W}(A)$ является замкнутой линейной оболочкой $\overline{W}(A_1)$ и $\overline{W}(A_2)$. Очевидно, что максимум (или минимум) $W(A)$ совпадает с максимумом (или минимумом) $W(A_1)$ или $W(A_2)$. Таким образом, $2m+2$–мерная проблема сводится к двум $m+1$–мерным проблемам. Мы предлагаем следующий алгоритм решения предыдущей проблемы : составить четыре матрицы $\{\alpha_{i+j}\}$, $\{\beta_{i+j}\}$, $\{u_{i+j}\}$, $\{v_{i+j}\}$, где

$$\alpha_k = \int_a^b \phi(t)(t-a)t^k\, dt, \quad \beta_k = \int_a^b \phi(t)(b-t)t^k\, dt,$$

$$u_k = \int_a^b w(t)(t-a)t^k\, dt, \quad v_k = \int_a^b w(t)(b-t)t^k\, dt, \quad k = 0, 1, \cdots 2m,$$

и решить два характеристических уравнения

$$\det\left(\{\alpha_{i+j}\}_{i,j=0}^m - \lambda\{u_{i+j}\}_{i,j=0}^m\right) = 0,$$

$$\det\left(\{\beta_{i+j}\}_{i,j=0}^m - \mu\{v_{i+j}\}_{i,j=0}^m\right) = 0, \tag{2.4}$$

а затем найти $\max\{\lambda_1, \cdots, \lambda_{m+1}, \mu_1, \cdots, \mu_{m+1}\}$ и $\min\{\lambda_1, \cdots, \lambda_{m+1}, \mu_1, \cdots, \mu_{m+1}\}$, где $\{\lambda_j\}$ и $\{\mu_j\}$ суть корни характеристических уравнений. Многочлены, реализирующие максимум и минимум выражения (2.1), могут быть найдены как обобщенные собственные векторы указанных характеристических уравнений, возведённые в квадрат и затем умноженные на $(t-a)$ и $(b-t)$, соответственно. Результаты вычислений с использованием программы MatLab показывают, что для весовой функции $w(t) = t(2-t)$, $t \in [0,1]$ и функционала $\phi(t) = (1-t)^2$ (эти функции встречаются в некоторых задачах внешней баллистики), многочлен $P_5(t) = (1-t)\left(8.198\cdots - 31.726\cdots t + 27.371\cdots t^2\right)^2$ реализирует максимум отношения (2.1), равный 5.318..., а многочлен $Q_5(t) = t\left(0.788\cdots - 5.781\cdots t + 8.088\cdots t^2\right)^2$ реализирует минимум отношения (2.1), равный 0.016... .

§3. ПРИМЕР

Пусть $[a,b] = [0,1]$ и $w(t) = 1-t$. Для функционала l, определённого формулой $l(P) = P(0)$, можно получить окончательную формулу. Имеем

$$\alpha_k = l\left(t^{k+1}\right) = 0, \quad \beta_k = l\left((1-t)t^k\right) = \begin{cases} 1, & k = 0, \\ 0, & k > 0, \end{cases}$$

$$u_k = \int_0^1 (1-t)t^{k+1} dt = \frac{1}{(k+2)(k+3)},$$

$$v_k = \int_0^1 (1-t)^2 t^k dt = \frac{2}{(k+1)(k+2)(k+3)}.$$

Первое уравнение в (2.4) даёт $\lambda = 0$, а второе принимает вид

$$\det \begin{pmatrix} 1-\lambda v_0 & -\lambda v_1 & \cdots & -\lambda v_m \\ -\lambda v_1 & -\lambda v_2 & \cdots & -\lambda v_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda v_m & -\lambda v_{m+1} & \cdots & -\lambda v_{2m} \end{pmatrix} =$$

$$= (-\lambda)^m \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ v_1 & v_2 & \cdots & v_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_m & v_{m+1} & \cdots & v_{2m} \end{pmatrix} +$$

$$+(-\lambda)^{m+1} \det \begin{pmatrix} v_0 & v_1 & \cdots & v_m \\ v_1 & v_2 & \cdots & v_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_m & v_{m+1} & \cdots & v_{2m} \end{pmatrix} = 0.$$

Следовательно,

$$\lambda = \frac{\Gamma(t, t^2, \cdots t^m)}{\Gamma(1, t, t^2, \cdots t^m)}, \qquad (3.1)$$

где Γ есть определитель матрицы Грама G. В случае многочлена чётного порядка получим аналогичное выражение, только с весовой функцией $w(t) = 1 - t$. Мы рассмотрим более общий случай, когда $w(t) = (1-t)^k$, $k \in \mathbb{Z}^+$.

Легко видеть, что ортонормированные по отношению к весу $w(t) = (1-t)^k$ многочлены имеют вид

$$q_m(t) = \frac{\sqrt{2m+k+1}}{m!} \cdot \frac{1}{(t-1)^k} \frac{d^m}{dt^m} \left\{ t^m (t-1)^{m+k} \right\}.$$

С другой стороны, они могут быть найдены по формуле ($\Gamma(\emptyset) = 1$)

$$q_m(t) = \frac{1}{\sqrt{\Gamma(1, t, \cdots, t^m)\, \Gamma(1, t, \cdots, t^{m-1})}} \begin{vmatrix} (1,1) & (1,t) & \cdots & (1,t^m) \\ (t,1) & (t,t) & \cdots & (t,t^m) \\ \vdots & \vdots & \ddots & \vdots \\ (t^{m-1},1) & (t^{m-1},t) & \cdots & (t^{m-1},t^m) \\ 1 & t & \cdots & t^m \end{vmatrix}.$$

Коэффициент при старшем члене q_m равен $\sqrt{\Gamma(1, t, \cdots, t^{m-1}) / \Gamma(1, t, \cdots, t^m)}$, откуда

$$\Gamma(1, t, \cdots, t^m) = \left(\frac{m!(m+k)!}{(2m+k)!} \right)^2 \cdot \frac{\Gamma(1, t, \cdots, t^{m-1})}{2m+k+1}.$$

Используя ортогональные по отношению к весу $u(t) = t^2(1-t)^k$ многочлены

$$\frac{1}{t^2\,(t-1)^k} \cdot \frac{d^m}{dt^m} \left\{ t^{m+2}\,(t-1)^{m+k} \right\},$$

можно получить рекуррентное соотношение для $\Gamma\,(t, \cdots, t^m)$:

$$\Gamma\left(t, \cdots, t^m, t^{m+1}\right) = \left(\frac{m!\,(m+k+2)!}{(2m+k+2)!} \right)^2 \cdot \frac{(m+1)\,(m+2)\,\Gamma\,(t, \cdots, t^m)}{(m+k+1)\,(m+k+2)\,(2m+k+3)}.$$

Отсюда следует

$$\frac{\Gamma\,(t, \cdots, t^m)}{\Gamma\,(1, t, \cdots, t^m)} = \frac{(m+1)\,(m+k+1)}{m\,(m+k)} \cdot \frac{\Gamma\left(t, \cdots, t^{m-1}\right)}{\Gamma\left(1, t, \cdots, t^{m-1}\right)}.$$

Индукцией по $m = 0, \cdots$ получаем $\sup\limits_{\|P\|=1} P(0) = \lambda = (m+1)(m+k+1)$. Выражение (3.1) может быть интерпретировано как обратная величина к квадрату расстояния многочлена $p_0(t) \equiv 1$ до подпространства многочленов P с $\deg P \le m$, удовлетворяющих условию $P(0) = 0$, (см. [1], 5.7). Поэтому многочлен $v_m(t) = 1 + a_1 t + \cdots + a_m t^m$, на котором это расстояние достигается, ортогонален любому одночлену t^j, $j = 1, \cdots, m$, так что v_m есть m-ый ортогональный по отношению к весу $\kappa(t) = t(1-t)^k$ многочлен и

$$v_m(t) = \frac{const}{t(t-1)^k} \cdot \frac{d^m}{dt^m} \left\{ t^{m+1}(t-1)^{m+k} \right\}.$$

Постоянная может быть найдена из условия $v_m(0) = 1$, следовательно

$$v_m(t) = \frac{1}{t(t-1)^k(m+1)!} \cdot \frac{d^m}{dt^m} \left\{ t^{m+1}(1-t)^{m+k} \right\}.$$

Abstract. The note describes the spectrum of a selfadjoint operator, acting in a subspace of polynomials, generated by a multiplication operator. The result is applied in the investigation of the Chebshev-Szegö problem of determination of the extrema of a linear functional over the set of positive polynomials.

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ЗАДАЧА ДИРИХЛЕ ДЛЯ ПРАВИЛЬНО ЭЛЛИПТИЧЕСКИХ УРАВНЕНИЙ В КРУГЕ

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Резюме. В статье рассматривается задача Дирихле для класса правильно эллиптических уравнений в круге. В однородном случае найдено необходимое и достаточное условие, когда уравнение имеет только нулевое решение, и получена формула для числа линейно независимых решений.

§1. ЗАДАЧА И ОСНОВНЫЕ РЕЗУЛЬТАТЫ

Пусть $\mathbf{D} = \{z : |z| < 1\}$ – единичный круг в комплексной плоскости с границей $\Gamma = \partial D = \{z : |z| = 1\}$. Рассмотрим уравнение

$$\sum_{k=0}^{2n} a_k \frac{\partial^{2n} u(z)}{\partial x^k \, \partial y^{2n-k}} = 0, \quad z \in \mathbf{D}, \tag{1}$$

с граничными условиями Дирихле

$$\frac{\partial^k u(z)}{\partial r^k} = 0, \quad z \in \Gamma, \quad k = 0, 1, ..., n-1, \tag{2}$$

где $a_k (k = 0, 1, ..., 2n)$ – комплексные постоянные ($a_0 \neq 0$), а $\dfrac{\partial u(z)}{\partial r}$ – производная функции $u(z)$ по радиусу. Ищем решение $u(z)$ в классе функций $C^{2n}(D) \cap C^{2n-1}(\overline{D})$, где $\overline{D} = D \cup T$.

Напомним, что уравнение (1) называется правильно эллиптическим, если количество корней характеристического уравнения

$$a_0 \lambda^{2n} + a_1 \lambda^{2n-1} + ... + a_{2n} = 0 \tag{3}$$

удовлетворяющего $\mathrm{Im}\lambda > 0$ и $\mathrm{Im}\lambda < 0$, равно n. Мы будем предполагать, что уравнение (1) правильно эллиптическое и максимальная кратность корней характеристического уравнения не превышает 2.

В монографии [1] доказано, что однородная задача (1), (2) при $n = 1$ в любой конечной односвязной области имеет только нулевое решение. Если коэффициенты a_k уравнения (1) действительны, то это утверждение верно для любого $n \geq 1$. Если $n = 2$ и коэффициенты a_k уравнения (1) комплексные, то однородная задача (1), (2) может иметь ненулевые решения (см. [3]).

При $n = 2$ формулы для числа линейно независимых решений однородной задачи (1), (2) были получены в [3]–[5]. В работе [3] были получены точные выражения для этих решений, а также условия для единственности решения неоднородной задачи в случае простых корней характеристического уравнения, а также в случае, когда выполнено условие $\lambda_1 = \lambda_2 = i$, $\mathrm{Im}\lambda_3 < 0$, $\mathrm{Im}\lambda_4 < 0$.

Случай $\lambda_1 = \lambda_2 = \lambda \neq i$, $\mathrm{Im}\lambda_3 < 0$, $\mathrm{Im}\lambda_4 < 0$ и $\lambda_3 \neq \lambda_4$ разобран в [6]. Для произвольного n задача (1), (2) в случае простых корней характеристического уравнения была исследована в [7].

Основной целью данной работы является нахождение числа линейно независимых решений однородной задачи Дирихле (1), (2) и описание эффективного метода его решения.

Пусть $\lambda_1, \lambda_2, ..., \lambda_{n_1}, \sigma_1, \sigma_2, ..., \sigma_{m_1}$ – двукратные корни, а λ_{n_1+1}, λ_{n_1+2}, ..., λ_{n_2}, $\sigma_{m_1+1}, \sigma_{m_1+2}, ..., \sigma_{m_2}$ – простые корни уравнения (3), где $m_1 + m_2 = n$, $n_1 + n_2 = n$, и

$$Im\lambda_k > 0, \quad Im\sigma_j < 0, k = 1, 2, ..., n_2, \quad j = 1, 2, ..., m_2. \tag{4}$$

Положим

$$\mu_j = \frac{i - \lambda_j}{i + \lambda_j}, \quad \nu_k = \frac{i + \sigma_k}{i - \sigma_k}, \quad j = 1, 2, ..., n_2, \quad k = 1, 2, ..., m_2. \tag{5}$$

Из (3) следует, что

$$|\mu_j| < 1, \quad |\nu_k| < 1, \quad j = 1, 2, ..., n_2, \quad k = 1, 2, ..., m_2. \tag{6}$$

Рассмотрим столбцы

$$\begin{gathered} \alpha(x) = (1, x, ..., x^{n-1})^T, \quad \beta(x) = (x^{n-1}, ..., x, 1)^T, \\ \alpha_l(x) = x^l \alpha'(x) + l x^{l-1} \alpha(x), \quad \beta_l(x) = x^l \beta'(x) + l x^{l-1} \beta(x), \end{gathered} \tag{7}$$

и $n \times n$ матрицы

$$\begin{aligned} A_{11} &= \|\alpha(\mu_1), \alpha(\mu_2), ..., \alpha(\mu_{n_2}), \alpha'(\mu_1), ..., \alpha'(\mu_{n_1})\|, \\ A_{22} &= \|\beta(\nu_1), \beta(\nu_2), ..., \beta(\nu_{m_2}), \beta'(\nu_1), ..., \beta'(\nu_{m_1})\|, \\ A_{21l} &= \|\mu_1^l \alpha(\mu_1), \mu_2^l \alpha(\mu_2), ..., \mu_{n_2}^l \alpha(\mu_{n_2}), \alpha_l(\mu_1), ..., \alpha_l(\mu_{n_1})\|, \\ A_{12l} &= \|\nu_1^l \beta(\nu_1), \nu_2^l \beta(\nu_2), ..., \nu_{m_2}^l \beta(\nu_{m_2}), \beta_l(\nu_1), ..., \beta_l(\nu_{m_1})\|. \end{aligned} \tag{8}$$

Обозначим через A_l блочные матрицы

$$A_l = \begin{Vmatrix} A_{11} & A_{12l} \\ A_{21l} & A_{22} \end{Vmatrix}, \quad l = 0, 1, \ldots. \tag{9}$$

Основным результатом данной статьи является следующая теорема.

Теорема 1.1. *Число k_0 линейно независимых решений однородной задачи (1), (2) определяется формулой*

$$k_0 = \sum_{l=n+1}^{\infty} (2n - \mathrm{rank} A_l). \tag{10}$$

Статья построена следующим образом : в §2 доказываются некоторые вспомогательные результаты, а §3 содержит доказательство Теоремы 1.1.

§2. ВСПОМОГАТЕЛЬНЫЕ ЛЕММЫ

Лемма 2.1. *Пусть $P(x,y)$ – многочлен, удовлетворяющий условиям*

$$\frac{\partial^k P}{\partial r^k} = 0, \quad (x,y) \in \Gamma, \quad k = 0, 1, \ldots, m-1. \tag{11}$$

Тогда

$$P(x,y) = \left(1 - |z|^2\right)^m Q(x,y), \tag{12}$$

где $Q(x,y)$ – некоторый многочлен.

Доказательство. Сначала рассмотрим случай $m = 1$, т.е.

$$P(x,y) = 0, \quad |z| = 1. \tag{13}$$

Имеем $P(x,y) = P_0(z,\overline{z})$, где $P_0(z,\overline{z})$ – многочлен относительно z и $\overline{z}$. Ясно, что $P_0(z,\overline{z})$ можно записать в виде

$$P_0(z,\overline{z}) = \sum_{j=0}^{l} (1 - z\,\overline{z})^j (\alpha_j(z) + \beta_j(\overline{z})), \tag{14}$$

где $\alpha_j(z)$ и $\beta_j(\overline{z})$ многочлены, удовлетворяющие условию $\deg(\alpha_j(z)), \deg(\beta_j(\overline{z})) \le \deg(P_0(z,\overline{z}))$.

Подставляя $P(x,y) = P_0(z,\overline{z})$ из (14) в (12) получим $\alpha_0(z) + \beta_0(\overline{z}) = 0$, $|z| = 1$. Отсюда, следует $\alpha_0(z) + \beta_0(\overline{z}) \equiv 0$. Следовательно,

$$P_0(z,\overline{z}) = (1 - |z|^2)\, Q(z,\overline{z}), \tag{15}$$

где $Q(z,\overline{z})$ – многочлен относительно z и $\overline{z}$. Для случая $m=1$ Лемма 2.1 доказана. Пусть теперь $m=2$. Многочлен $P(x,y)$ можно записать в виде (15), где $Q(z,\overline{z})$ удовлетворяет условию $Q(z,\overline{z})=0$ при $|z|=1$. Следовательно

$$Q(x,y)=(1-|z|^2)\,Q_1(z,\overline{z}).$$

Подставляя $Q(z,\overline{z})$ в (15) получим требуемый результат при $m=2$. Продолжая аналогично, завершаем доказательство Леммы 2.1 для произвольного m.

Пусть λ – постоянная и $\mathrm{Im}\lambda\neq 0$. Обозначим через $\mathbf{D}_\lambda$ образ области $\mathbf{D}$ при отображении $\zeta=x+\lambda y$ $(x+iy\in\mathbf{D},\zeta\in\mathbf{D}_\lambda)$.

Будем говорить, что функция $\varphi(x+\lambda y)$ аналитична в $\mathbf{D}$ относительно аргумента $x+\lambda y$, если она является суперпозицией аналитической в $\mathbf{D}$ функции $\varphi(\zeta)$ и функции $\zeta=x+\lambda y$ $(\zeta=\xi+i\eta)$.

Пусть $\Phi_1,\Phi_2,\Phi_3,\Psi_1,\Psi_2,\Psi_3$ – аналитические функции в $\mathbf{D}$. Рассмотрим сумму

$$u=\Phi(x+\lambda_1 y)+\Phi_2(x+\lambda_2 y)+y\,\Phi_3'(x+\lambda_1 y)+\Psi_1(x+\sigma_1 y)+\Psi_2(x+\sigma_2 y)+y\,\Psi_3'(x+\sigma_1 y). \tag{16}$$

Лемма 2.2. *Если в (16) $u\equiv 0$, то функции Φ_j, Ψ_j $(j=1,2,3)$ являются многочленами порядка не больше 4.*

Доказательство. Если $u\equiv 0$, то применяя оператор

$$\left(\frac{\partial}{\partial y}-\lambda_1\frac{\partial}{\partial x}\right)^2\left(\frac{\partial}{\partial y}-\lambda_2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}-\sigma_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}-\sigma_2\frac{\partial}{\partial x}\right)$$

к обеим частям (15), получим $\Psi_3^{(5)}(x+\sigma_1 y)=0$, $(x,y\in\mathbf{D})$. Отсюда следует, что Ψ – многочлен порядка не больше 4. Аналогично, можно показать, что Φ есть многочлен порядка не больше 4. Теперь, применяя к обеим частям (16) оператор

$$\left(\frac{\partial}{\partial y}-\lambda_1\frac{\partial}{\partial x}\right)^2\left(\frac{\partial}{\partial y}-\lambda_2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}-\sigma_1\frac{\partial}{\partial x}\right)$$

и используя, что $\deg(\Phi_3)$, $\deg(\Psi_3)\le 4$, получим $\Psi_2^{(4)}(x+\sigma_2 y)+c=0$, $(x,y\in\mathbf{D})$, где c – постоянная. Отсюда следует $\deg(\Psi_2)\le 4$. Аналогично, можно доказать, что Φ_1, Φ_2, Ψ_1 суть многочлены порядка не больше 4. Лемма 2.2 доказана.

Следующая лемма доказана в [1].

Лемма 2.3. *Пусть $\varphi(z+\mu\overline{z})$ – аналитическая функция от аргумента $z+\mu\overline{z}$ при $z\in\mathbf{D}$ и $|\mu|<1$. Тогда для $z\in\mathbf{D}$*

$$\varphi(z+\mu\overline{z})=\Omega\left(\frac{z+\mu\overline{z}+\sqrt{(z+\mu\overline{z})^2-4\mu}}{2}\right)+\Omega\left(\frac{z+\mu\overline{z}-\sqrt{(z+\mu\overline{z})^2-4\mu}}{2}\right), \tag{17}$$

где $\Omega(z)$ – аналитическая функция в круге **D**.

Из (17) получаем

$$\varphi(z+\mu\overline{z}) = \Omega(z) + \Omega(\mu\overline{z}), \quad z \in \Gamma. \tag{18}$$

Обратное также верно, т.е. из (18) следует (17).

Замечание 2.1. Если $\varphi(z+\mu\overline{z})$ – многочлен порядка m, то в силу (17) $\Omega(z)$ также является многочленом порядка m.

Лемма 2.4. *Пусть $\varphi(z+\mu\overline{z})$ – многочлен порядка m, ($|\mu|<1, z \in \mathbf{D}$). Тогда*

$$\psi(z+\mu\overline{z}) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{t}\varphi'(t+\mu\overline{t})\,d(t+\mu\overline{t})}{t+\mu\overline{t}-z-\mu\overline{z}}, \quad z \in \mathbf{D} \tag{19}$$

является многочленом порядка m.

Доказательство. Ясно, что ψ можно записать в виде

$$\psi(z+\mu\overline{z}) = \overline{z}\varphi'(z+\mu\overline{z}) + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi'(\overline{t}-\overline{z})\varphi'(t+\mu\overline{t})\,d(t+\mu\overline{t})}{t+\mu\overline{t}-z-\mu\overline{z}}, \quad z \in \mathbf{D}. \tag{20}$$

Переходя к пределу под знаком интеграла при $z \to t \in \Gamma$ заметим, что (20) выполняется также при $z \in \Gamma$.

Так как $\overline{z} = 1/z$, $\overline{t} = 1/t$ при $t, z \in \Gamma$, равенство (20) можно записать в виде

$$\psi(z+\mu\overline{z}) = \overline{z}\varphi'(z+\mu\overline{z}) - \frac{\overline{z}}{2\pi i}\int_\Gamma \frac{(\overline{t}+\mu\overline{t})\,d(t+\mu\overline{t})}{t-\mu\overline{z}}, \quad z \in \Gamma.$$

Поскольку

$$\frac{(\overline{z}-\overline{t})}{t+\mu\overline{t}-z-\mu\overline{z}} = \frac{\overline{z}}{t-\mu\overline{z}}, \quad t, z \in \Gamma,$$

интегрируя по частям, получаем

$$\psi(z+\mu\overline{z}) = \overline{z}\varphi'(z+\mu\overline{z}) + \frac{\overline{z}}{2\pi i}\int_\Gamma \frac{\varphi(t+\mu\overline{t})\,d(t)}{(t-\mu\overline{z})^2}, \quad z \in \Gamma. \tag{21}$$

Представим функцию $\varphi(z+\mu\overline{z})$ в виде (18) и отметим, что согласно Замечанию 2.1, $\Omega(z)$ является многочленом порядка не выше m. Далее, подставляя (18) в (20), получаем

$$\psi(z+\mu\overline{z}) = \overline{z}\varphi'(z+\mu\overline{z}) + \frac{\overline{z}}{2\pi i}\int_\Gamma \frac{\Omega(t)\,d(t)}{(t-\mu\overline{z})^2} + \frac{\overline{z}}{2\pi i}\int_\Gamma \frac{\Omega(\mu\overline{t})\,d(t)}{(t-\mu\overline{z})^2}, \quad z \in \Gamma. \tag{22}$$

Легко видеть, что

$$\int_\Gamma \frac{\Omega(\mu\overline{t})\,d(t)}{(t-\mu\overline{z})^2} = 0, \qquad \frac{1}{2\pi i}\int_\Gamma \frac{\Omega(t)\,d(t)}{(t-\mu\overline{z})^2} = \Omega'(\mu\overline{z}), \quad z \in \Gamma.$$

Отсюда и из (22), получаем

$$\psi(z+\mu\overline{z}) = \overline{z}\varphi'(z+\mu\overline{z}) + z\Omega'(\mu\overline{z}), \quad z\in\Gamma. \tag{23}$$

Ввиду (14) имеем

$$\overline{z}\varphi'(z+\mu\overline{z}) + z\Omega'(\mu\overline{z}) = \alpha_0(z) + \beta_0(\overline{z}), \quad z\in\Gamma, \tag{24}$$

где α_0, β_0 – многочлены порядка не выше m. Представим ψ в виде (18)

$$\psi(z+\mu\overline{z}) = \Lambda(z) + \Lambda(\mu\overline{z}), \quad z\in\Gamma,$$

где $\Lambda(z)$ – аналитическая функция в $\mathbf{D}$. Следовательно, имеем

$$\alpha_0(z) + \beta_0(\overline{z}) = \Lambda(z) + \Lambda(\mu\overline{z}), \quad z\in\Gamma.$$

Таким образом, Λ является многочленом порядка не выше m. Учитывая Замечание 2.1, завершаем доказательство Леммы 2.4.

Лемма 2.5. *Система функций*

$$(z+\mu_1\overline{z})^{n-1}, ..., (z+\mu_{n_2}\overline{z})^{n-1}, \overline{z}(z+\mu_1\overline{z})^{n-2}, ..., \overline{z}(z+\mu_{n_1}\overline{z})^{n-2} \tag{25}$$

линейно независима над полем комплексных чисел.

Доказательство. Пусть

$$\sum_{k=1}^{n_2} a_k(z+\mu_k\overline{z})^{n-1} + \sum_{k=1}^{n_1} b_k(z+\mu_k\overline{z})^{n-2} = 0. \tag{26}$$

Обозначим через M_l следующий оператор :

$$M_l = \left(\frac{\partial}{\partial\overline{z}} - \mu_1\frac{\partial}{\partial z}\right)\cdots\left(\frac{\partial}{\partial\overline{z}} - \mu_l\frac{\partial}{\partial z}\right).$$

Применяя оператор $M_{n_1-1}\,M_{n_2}$ к обеим частям (26), получим $b_{n_1}=0$. Аналогично имеем $b_j=0$, $j=1,2,...,n_1$. Теперь применяя оператор M_{n_2-1} к обеим частям (26) получим $a_{n_1}=0$. Аналогично $a_j=0$, $j=1,2,...,n_2$. Лемма 2.5 доказана.

Лемма 2.6. $detA_{11}\neq 0$, $detA_{22}\neq 0$.

Доказательство. Представим функции из (24) в виде

$$\begin{aligned} (z+\mu_j\overline{z})^{n-1} &= \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!}\,\mu_j^k\, z^{n-1-k}\,\overline{z}^k, \quad j=1,2,...,n_2,\\ \overline{z}(z+\mu_l\overline{z})^{n-2} &= \sum_{k=1}^{n-1} \frac{(n-2)!}{(k-1)!(n-1-k)!}\,\mu_l^{k-1}\, z^{n-1-k}\,\overline{z}^k, \quad l=1,2,...,n_1. \end{aligned} \tag{27}$$

По Лемме 2.5, детерминант матрицы, образованной коэффициентами $\overline{z}^k\, z^{n-1-k}$ в (27) отличен от нуля. С другой стороны, этот детерминант равен $c\det A_{11}$, $(c\neq 0)$. Аналогично можно доказать, что $\det A_{22}\neq 0$. Лемма 2.6 доказана.

§3. ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 1.1

Для простоты докажем Теорему 1.1 при $n_1=1$, $m_1=1$, $n_2=2$, $m_2=2$ (общий случай доказывается аналогично). В нашем частном случае (1) примет вид

$$\left(\frac{\partial}{\partial y}-\lambda_1\frac{\partial}{\partial x}\right)^2\left(\frac{\partial}{\partial y}-\sigma_1\frac{\partial}{\partial x}\right)^2\left(\frac{\partial}{\partial y}-\lambda_2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}-\sigma_2\frac{\partial}{\partial x}\right)u(z)=0, \tag{28}$$

$z\in\mathbf{D}$, $z=x+iy$, с граничными условиями

$$\frac{\partial u(x,y)}{\partial r^k}=0, \quad (x,y)\in\Gamma, \quad k=0,1,2. \tag{29}$$

Общее решение $u(z)$ уравнения (28) можно представить в виде (см. [1])

$$u=\Phi_1(x+\lambda_1 y)+\Phi_2(x+\lambda_2 y)+y\,\Phi_3'(x+\lambda_1 y)+\Psi_1(x+\sigma_1 y)+\Psi_2(x+\sigma_2 y)+y\,\Psi_3'(x+\sigma_1 y), \tag{30}$$

где $\Phi_1(x+\lambda_1 y)$, $\Phi_2(x+\lambda_2 y)$, $\Phi_3(x+\lambda_1 y)$, $\Psi_1(x+\sigma_1 y)$, $\Psi_2(x+\sigma_2 y)$, $\Psi_3(x+\sigma_1 y)$ – аналитические функции своих аргументов. Используя (4), перепишем (30) в следующем виде

$$\begin{aligned} u &= L_1(z+\mu_1\overline{z})+L_2(z+\mu_2\overline{z})+\overline{z}L_3'(z+\mu_1\overline{z})+\\ &+W\left[R_1(z+\overline{\nu}_1\overline{z})+R_2(z+\overline{\nu}_2\overline{z})+\overline{z}R_3'(z+\overline{\nu}_1\overline{z})\right]+d_0+d_1\,z+d_2\,\overline{z}, \end{aligned} \tag{31}$$

где $W\,\varphi=\overline{\varphi}$, $L_1(z+\mu_1\overline{z})$, а $L_2(z+\mu_2\overline{z})$, $L_3(z+\mu_1\overline{z})$, $R_1(z+\overline{\nu}_1\overline{z})$, $R_2(z+\overline{\nu}_2\overline{z})$, $R_3'(z+\overline{\nu}_1\overline{z})$ – аналитические функции своих аргументов, удовлетворяющие условиям $L_j^{(k)}=0$, $R_j^{(k)}=0$, $k=0,1$, $j=1,2,3$, а d_0, d_1, d_2 – произвольные комплексные постоянные.

Граничные условия (29) эквивалентны условиям

$$\frac{\partial^2 u(z)}{\partial z^k\partial\overline{z}^{2-k}}=0, \quad z\in\Gamma, \quad k=0,1,2, \tag{32}$$

$$u(1,0)=0,\quad \frac{\partial u(1,0)}{\partial z}=0,\quad \frac{\partial u(1,0)}{\partial \overline{z}}=0, \tag{33}$$

где

$$\frac{\partial u}{\partial z}=\frac{1}{2}\left(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right),\qquad \frac{\partial u}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}\right).$$

Положим

$$L_j''(z+\mu_k\overline{z})=P_j(z+\mu_k\overline{z}),\qquad R_j''(z+\overline{\nu}_k\overline{z})=Q_j(z+\overline{\nu}_k\overline{z}),\quad k=0,1,\quad j=1,2,3. \tag{34}$$

Заменим P_i, Q_j, $(j=1,2,3)$ на ϕ_j, ψ_j $(j=1,2,3)$ по формулам

$$P_1(z+\mu_1\overline{z})=\varphi_1(z+\mu_1\overline{z})-\frac{1}{2\pi i}\int_\Gamma \frac{\overline{t}\varphi_3'(t+\mu_1\overline{t})\,d(t+\mu_1\overline{t})}{t+\mu_1\overline{t}-z-\mu_1\overline{z}}, \tag{35}$$

$$Q_1(z+\overline{\nu}_1\overline{z})=\psi_1(z+\overline{\nu}_1\overline{z})-\frac{1}{2\pi i}\int_\Gamma \frac{\overline{t}\psi_3'(t+\overline{\nu}_1\overline{t})\,d(t+\overline{\nu}_1\overline{t})}{t+\overline{\nu}_1\overline{t}-z-\overline{\nu}_1\overline{z}}, \tag{36}$$

$$P_2(z+\mu_2\overline{z})=\varphi_2(z+\mu_2\overline{z}),\qquad Q_2(z+\overline{\nu}_2\overline{z})=\psi_2(z+\overline{\nu}_2\overline{z}), \tag{37}$$

$$P_3(z+\mu_1\overline{z})=\varphi_3(z+\mu_1\overline{z}),\qquad Q_3(z+\overline{\nu}_1\overline{z})=\psi_3(z+\overline{\nu}_1\overline{z}). \tag{38}$$

Используем для функций φ_j, ψ_j, $j=1,2,3$ представления, приведенные в Лемме 2.3, т.е.

$$\varphi_j(z+\mu\overline{z})=\Omega_j(z)+\Omega_j(\mu\overline{z}),\qquad \psi_j(z+\overline{\nu}\overline{z})=\Lambda_j(z)+\Lambda_j(\overline{\nu}\overline{z}),\quad z\in\Gamma,\quad j=1,2,3. \tag{39}$$

Подставляя (31) в (32) и используя (34) – (38), получаем

$$\begin{aligned}&\Omega_1(z)+\Omega_1(\mu_1\overline{z})+\Omega_2(z)+\Omega_2(\mu_2\overline{z})+\overline{z}\Omega_3'(\mu_1\overline{z})+W[\overline{\nu}_1^2(\Lambda_1(z)+\Lambda_1(\overline{\nu}_1\overline{z})+\\&+\overline{z}\Lambda_3'(\overline{\nu}_1\overline{z}))+\overline{\nu}_2^2(\Lambda_2(z)+\Lambda_2(\overline{\nu}_2\overline{z}))+2\overline{\nu}_1(\Lambda_3(z)+\Lambda_3(\overline{\nu}_2\overline{z}))]=0,\quad z\in\Gamma,\end{aligned} \tag{40}$$

$$\begin{aligned}&\mu_1^2(\Omega_1(z)+\Omega_1(\mu_1\overline{z})+\overline{z}\Omega_3'(\mu_1\overline{z}))+\mu_2^2(\Omega_2(z)+\Omega_2(\mu_2\overline{z}))+2\mu_1(\Omega_3(z)+\Omega_3(\mu_1\overline{z}))+\\&+W[\Lambda_1(z)+\Lambda_1(\overline{\nu}_1\overline{z})+\overline{z}\Lambda_3'(\overline{\nu}_1\overline{z}))+\Lambda_2(z)+\Lambda_2(\overline{\nu}_2\overline{z})]=0,\quad z\in\Gamma,\end{aligned} \tag{41}$$

$$\begin{aligned}&\mu_1(\Omega_1(z)+\Omega_1(\mu_1\overline{z})+\overline{z}\Omega_3'(\mu_1\overline{z}))+\mu_2(\Omega_2(z)+\Omega_2(\mu_2\overline{z}))+\Omega_3(z)+\Omega_3(\mu_2\overline{z})+\\&+W[\overline{\nu}_1(\Lambda_1(z)+\Lambda_1(\overline{\nu}_1\overline{z})+\overline{z}\Lambda_3'(\overline{\nu}_1\overline{z}))+\overline{\nu}_2(\Lambda_2(z)+\\&+\Lambda_2(\overline{\nu}_2\overline{z}))+\Lambda_3(z)+\Lambda_3(\overline{\nu}_1\overline{z})]=0,\quad z\in\Gamma.\end{aligned} \tag{42}$$

Представим функции Ω_j, Λ_j $(j=1,2,3)$ в виде сходящихся рядов

$$\Lambda_j(z)=\sum_{l=0}^{\infty}\overline{b}_{jl}\,z^l,\qquad \overline{\Omega}_j(z)=\sum_{l=0}^{\infty}a_{jl}\,z^l,\quad z\in\overline{\mathbf{D}},\quad j=1,2,3. \tag{43}$$

Подставляя (43) в (40) – (42), получим систему алгебраических уравнений

$$A_l\,\chi_l = 0, \qquad l = 1, 2, ..., \tag{44}$$

где χ_l – вектор-столбец $(a_{1l}, a_{2l}, a_{3l}, b_{1l}, b_{2l}, b_{3l})^T$, а A_l – матрица (9), соответствующая уравнению (28). Из условия (33) получим d_0, d_1, d_2, входящие в общее решение (31), однозначно определяющиеся через $L_j(z + \mu_k\overline{z})$, $R_j(z + \overline{\nu}_k\overline{z})$, $(j = 1, 2, 3; k = 1, 2)$.

Из Леммы 2.6 следует, что $\det A_l \to a \neq 0$, $l \to \infty$. Это означает, что при больших l система уравнений (44) имеет только нулевое решение. Таким образом, решение однородной задачи Дирихле (28) – (29) сводится к решению однородной алгебраической системы уравнений (44) при $l = 1, 2, ..., l_0$, где l_0 – достаточно большое натуральное число.

Аналогично, неоднородная задача Дирихле для уравнения (28) сводится к неоднородной системе алгебраических уравнений (44), где правая часть суть некоторые вектор-столбцы, зависящие от граничного условия.

Используя (44) и Леммы 2.1 – 2.4, аналогично случаю простых корней характеристического уравнения ([7], стр. 29 – 39), получим (10). Теорема 1.1 для задачи (28) – (29) доказана. Аналогично доказывается эта теорема в общем случае.

Так как задача Дирихле для правильно эллиптического уравнения фредгольмова [7], из Теоремы 1.1 получим

Следствие 3.1. Пусть (1) есть правильно эллиптическое уравнение и кратность корней характеристического уравнения (3) не превышает 2. Тогда неоднородная задача Дирихле для уравнения (1) имеет единственное решение тогда и только тогда, когда $\det A_l \neq 0$, $l = n+1, n+2, ...$.

Abstract. The paper considers the Dirichlet problem for a class of properly elliptic equations in the disk. In the homogeneous case a necessary and sufficient condition for the problem to have only trivial solution is found, and a formula for the number of linearly independent solutions is obtained.

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ON THE HOMOGENEITY OF PRINCIPAL BUNDLES

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Abstract. The paper considers the principal bundles and gives some results about the structure of geodesics in the base space of the principal bundles.

§1. INTRODUCTION

Let G be a Lie group. A (smooth) **principal bundle with structure group** G is a pair $(\wp, T)$ satisfying

(i) $\wp = (P, \pi, B, G)$ is a smooth fiber bundle.

(ii) $T : P \times G \longmapsto P$ is a right action of G on P.

(iii) $\wp$ admits a coordinate representation (U_α, ψ_α) such that

$$\psi_\alpha(x, ab) = \psi_\alpha(x, a)b, \quad x \in U_\alpha, \quad a, b \in G.$$

The action T is called **principal action** and the coordinate representation (U_α, ψ_α) is called **principal coordinate** (see [5], vol. I, p. 50).

Let G be a connected Lie group and K be a closed subgroup of G. The set G/K of left cosets of K in G possesses a unique differentiable structure, and is called homogeneous manifold.

Let $T : G \times M \longrightarrow M$ be a transitive action of G on a differentiable manifold M, and let K be the invariant subgroup of the point $x_0 \in M$. Then by the map $\phi : G/K \longrightarrow M$ with $\phi(gK) = gx_0$, taking $M = G/K$ reduces M to a homogeneous differentiable manifold. Given an affine connection ∇ on M, we are concerned with geodesics on (M, ∇) with respect to one parameter subgroups of G, called homogeneous geodesics (see [7], and also Definition 2.5).

By $\Im = (G, T, G/K, K)$ we denote the fiber bundle with group structure K (see Definition 2.4). In this paper we prove some results about the structure of homogeneous geodesics on the base space G/K of the fiber bundle $\Im$. First we consider the case where G is a semisimple Lie group, and then we remove this assumption.

§2. PRELIMINARIES

Let $\wp = (P, \pi, B, G)$ be a principal bundle with principal action T. A left action of G on a manifold F we write as

$$S : G \times F \to F.$$

Definition 2.1. A left action Q of G on the product manifold $P \times G$, given by

$$Q_a(z, y) = (z, y).a = (z.a, a^{-1}.y), \quad z \in P, \; y \in F, \; a \in G$$

is called **joint action of G.**

The set of orbits for a joint action we denote by $P \times_G F$ and define a map q by

$$q : P \times F \to P \times_G F.$$

Notice that q determines a map $\rho : P \times_G F \to B$ such that

$$\rho \circ q = \pi \circ \pi_p,$$

where $\pi_p : P \times F \to P$ is the projection and $\pi : P \to B$ is the bundle map.

Definition 2.2. A smooth fiber bundle $\Im = (P \times_G F, \rho, B, F)$ with a unique smooth structure on $P \times_G F$ is called **fiber bundle associated with $\wp$.**

Let P be a representation of a Lie group in a real vector space W. An Euclidean inner product $< \cdot, \cdot >$ in W is said to be invariant with respect to P, if

$$< p(x)u, p(x)v > = < u, v >, \quad x \in G, \quad u, v \in W.$$

Notice that for each $h \in T_eG$ the map $p'(h) : W \longrightarrow W$ is skew.

Let $M = G/K$ be a homogeneous manifold. G/K is **reductive**, if the Lie algebra $\mathcal{G}$ of G can be represented as a direct sum of the Lie algebra $\mathcal{K}$ of the subgroup K and a vector space $\mathcal{M}$ which is $ad_{\mathcal{K}}$-invariant, i.e.

1. $\mathcal{G} = \mathcal{M} + \mathcal{K}; \qquad \mathcal{M} \bigcap \mathcal{K} = \{0\},$
2. $ad_{\mathcal{K}} \mathcal{M} \subset \mathcal{M}.$ (1)

It follows from (1) that

$$[\mathcal{K}, \mathcal{M}] \subset \mathcal{M}. \qquad (2)$$

Observe that if K is connected then (2) implies (1) (see [5], vol. I).

Definition 2.3. Let G be a connected Lie group and K be a closed connected subgroup of G with Lie algebras $\mathcal{G}$ and $\mathcal{K}$, respectively. Then $\mathcal{G} \subseteq \mathcal{K}$ and hence we can write

$$\mathcal{G} = \mathcal{K}^{\perp} \oplus \mathcal{K}.$$

The algebra $\mathcal{K}^{\perp}$ is called the **orthogonal complement** of $\mathcal{K}$ in $\mathcal{G}$ with respect to Euclidean inner product $< \cdot, \cdot >$ in $\mathcal{G}$. We have

$$Ady = Ad^{\perp}y \oplus id_{\mathcal{K}}, \quad y \in \mathcal{G},$$

where $Ad^{\perp}y$ stands for the restriction of Ady to $\mathcal{K}^{\perp}$.

Definition 2.4. Let K be a closed subgroup G. The fiber bundle $\Im = (G, \pi, G/K, K)$ with right multiplication action of K on G is a principal bundle with structure group K, and is called **principal homogeneous bundle** (see [2], p. 45).

Definition 2.5. Let ∇ be an affine connection on $M = G/K$, which is invariant under the action $T : G \times M \longrightarrow M$. Then a geodesic $\gamma : I \longrightarrow M$ is called **homogeneous**, if for some $X \in \mathcal{G} = T_1\mathcal{G}$ there exists an one-parameter subgroup $t \longrightarrow \exp tX$, $t \in R$, such that

$$\gamma(t) = T(\exp tX, x), \quad \gamma(0) = x, \quad t \in I.$$

A connected Riemannian manifold M is homogeneous, if either the isometry group $I(M)$ or a connected subgroup G of $I(M)$ acts transitively on M. In this case, if $x_0 \in M$ and K is the stabilizer of x_0, then $G/K = M$. Moreover, G will act effectively on G/K from the left. The point $x_0 = \{K\}$ is called the origin of the homogeneous Riemannian manifold M.

Now let $M = G/K$ be a reductive homogeneous Riemannian manifold and let $\mathcal{G} = \mathcal{M} + \mathcal{K}$ be its Lie algebra decomposition. The natural map $\phi : G \longrightarrow G/K = M$ will induce a linear epimorphism $(d\phi)_e : T_eG \longrightarrow T_{x_0}M$ and the vector space $\mathcal{M}$ will be identified with $T_{x_0}M$.

For a Riemannian M, the inner product on $T_{x_0}M$ induces an inner product C on $\mathcal{M}$, which is $ad_{\mathcal{K}}$-invariant. According to Definition 2.1, the geodesic γ on M passing through x_0 is homogeneous if and only if for all $t \in R$

$$\gamma(t) = (\exp tX)(x_0), \quad \text{for some} \quad X \in g, \ X \neq 0.$$

Definition 2.6. Let G be a Lie group and $\mathcal{G}$ be its Lie algebra. A vector $X \in \mathcal{G}$ $(X \neq 0)$ is called a **geodesic vector**, if the curve $\gamma(t) = (\exp tX)(x_0)$ is a geodesic on M (see [7]).

In view of Definition 2.6 there is a correspondence between the geodesic vectors and the homogeneous geodesics passing through $x_0 \in M$. Let C be an inner product on $\mathcal{M}$, induced by the inner product on $T_{x_0}M$. The following lemma holds.

Lemma 2.7 ([7]). *Let $X \in \mathcal{G}$ and let $[X,Y]_{\mathcal{M}}$ be the component of $[X,Y]$ in $\mathcal{M}$ with respect to reductive decomposition. Then X is geodesic if and only if*

$$C(X,[X,Y]_{\mathcal{M}}) = 0 \quad \text{for all} \quad Y \in \underline{\mathcal{G}}.$$

Definition 2.8. Let C be a bilinear symmetric form on a finite dimensional vector space V. The **radical** of C is a vector subspace of V, such that

$$rad\, C = \{v \in V; C(v,u) = 0 \quad \text{for all} \quad u \in V\}.$$

Definition 2.9. The **Killing form** B on $\mathcal{G}$ is defined to be a bilinear symmetric form given by $B(X,Y) = Tr(ad_X ad_Y)$.
The **Cartan's criterion for solvability** of $\mathcal{G}$ asserts that $\mathcal{G}$ is solvable if and only if $B(X,Y) = 0$ for every $Y \in [\mathcal{G},\mathcal{G}]$ and $X \in \mathcal{G}$ (see [4], p. 669).
The next result was proved in ([8], p. 55).

Theorem 2.10. *Every Lie algebra $\mathcal{G}$ has largest solvable ideal, which is denoted by $rad\, \mathcal{G}$. Every Lie group G has largest connected normal solvable Lie subgroup such that its Lie algebra is $rad\, \mathcal{G}$.*
The subgroup of the Lie group G in Theorem 2.10 is called **radical of** G and is denoted by $Rad\, G$.

Definition 2.11. A Lie group G is called **semisimple**, if $Rad\, G = \{e\}$. A Lie algebra $\mathcal{G}$ is called **semisimple**, if $rad\, \mathcal{G} = \{0\}$.
If a Lie algebra $\mathcal{G}$ is semisimple, then $[\mathcal{G},\mathcal{G}] = \mathcal{G}$. It is well known, that $\mathcal{G}$ is semisimple iff $B(X,Y)$ is non degenerate for all X and Y from $\mathcal{G}$. In other words $\mathcal{G}$ is semisimple if and only if the radical of the corresponding Killing form is identical zero (see [4], [5]).

§3. MAIN RESULTS

Let G be a connected Lie group, $T : G \times M \longrightarrow M$ be a transitive action of G on a differentiable manifold M and K be the invariant subgroup of the point $x_0 \in M$. The Lie algebras of K and G are denoted by $\mathcal{K}$ and $\mathcal{G}$ respectively. The adjoint representation of G leads to a representation Ad_K of K in $\mathcal{G}$. Since the Lie algebra $\mathcal{K}$ is stable under the map $Ad_K(a)$, $a \in K$, we get a representation $Ad^{\perp}$ of K in $\mathcal{G}/\mathcal{K}$. The sequence

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{G}/\mathcal{K}$$

is a short exact and K-equivariant with respect to the representations Ad, Ad_K and $Ad^{\perp}$ of K.
The next result is known (see, e.g. [2], vol. I, pp. 45, 94).

Proposition 3.1. Under the above hypotheses the vector bundle

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathbf{G}/\mathbf{K}, \mathcal{G}/\mathcal{K})$$

is a fiber bundle associated with

$$\Im = (G, \pi, G/K, K),$$

and is strongly isomorphic to the tangent bundle $\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathbb{R}^m)$, where $m = \dim(G/K)$.

By Proposition 3.1, instead of the tangent bundle $\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathbb{R}^m)$ we can consider the base space G/K of $\Im = (G, \pi, G/K, K)$ and fiber space $\mathcal{G}/\mathcal{K}$ of $\xi = (G \times_K \mathcal{K}, \rho_\xi, \mathbf{G}/\mathbf{K}, \mathcal{G}/\mathcal{K})$.

Every Lie algebra $\mathcal{G}$ possesses largest nilpotent ideal (see 1], p. 58). Moreover, every Lie group G has largest connected normal and nilpotent subgroup, such that its Lie algebra is the largest nilpotent ideal in $\mathcal{G}$ (see [1], p. 59).

Definition 3.2. The largest nilpotent ideal in the Lie algebra $\mathcal{G}$ is called **weak radical of** $\mathcal{G}$ and is denoted by $W_{rad}\mathcal{G}$. The largest Lie subgroup which is normal nilpotent and its Lie algebra is $W_{rad}\mathcal{G}$, is called **weak radical of** G and is denoted by $W_{Rad}G$. According to Definitions 2.8 and 2.9 :

$$W_{rad}\mathcal{G} \subseteq \mathrm{rad}\mathcal{G}, \qquad \mathcal{W}_{\mathcal{R}\dashv\lceil}\mathcal{G} \subseteq \mathrm{Rad}\mathcal{G}.$$

Definition 3.3. A Lie group G (Lie algebra $\mathcal{G}$, respectively) is called weakly semisimple if $W_{Rad}G = \{e\}$ ($W_{rad}\mathcal{G} = \{0\}$, respectively).

Observe that every semisimple Lie group is weakly semisimple. The converse is not always true.

Definition 3.4. Let B be a Killing form on $\mathcal{G}$. The **weak radical of** B is defined to be

$$W_{rad}B = \{X \in [\mathcal{G}, \mathcal{G}], B(X, Y) = 0 \quad \text{for all} \quad Y \in \mathcal{G}\}.$$

Theorem 3.5. *Let G be a connected transitive Lie subgroup of the isometry group $I(M)$ of a Riemannian manifold $M = G/K$. Let $\Im = (G, \pi, G/K, K)$ be a principal homogeneous bundle and $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathbf{G}/\mathbf{K}, \mathcal{G}/\mathcal{K})$ be a bundle associated with $\Im$. If G is semisimple and $m = \dim G/K$, then there are m orthogonal homogeneous geodesics passing through $x_0 = \{K\}$.*

Proof : Let $\Im = (G, \pi, G/K, K)$ be a principal homogeneous bundle and G be a connected Lie subgroup of $I(M)$ acting transitively on M. Let $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathbf{G}/\mathbf{K}, \mathcal{G}/\mathcal{K})$ be a bundle associated with $\Im$ and B be the Killing form on $\mathcal{G}$.

It is known (see [6]) that if $M = G/K$ is a homogeneous Riemannian manifold such that $\subseteq I(M)$ is solvable, then there exists a homogeneous geodesic passing through any point $x_0 \in M$.

Since $\Im = (G, \pi, G/K, K)$ is principal homogeneous bundle, there exists a homogeneous geodesic passing through every point in the base space.

Let $\mathcal{G}/\mathcal{K}$ be the fiber space of ξ and B be the Killing form on $\mathcal{G}$. Since $radB$ is solvable we have $radB \subseteq rad\mathcal{G}$ (see [3], p. 22). On the other hand, since B is non degenerate on $\mathcal{K}$, taking $\mathcal{M}$ the orthogonal complement of $\mathcal{K}$ with respect to B, we conclude that $\mathcal{G} = \mathcal{M} + \mathcal{K}$ is a reductive decomposition. (Here $\mathcal{K}$ is the Lie algebra of K, while $\mathcal{M}$ is a vector subspace of T_eG (see Proposition 3.1)). By Proposition 2 from [7] the Killing form B is non degenerate on $\mathcal{M}$. Hence taking into account that $\mathcal{M}$ is the orthogonal complement of $\mathcal{K}$ with respect to B we get $radB \subseteq \mathcal{M}$.

First we consider the case where $radB \subset \mathcal{M}$. By means of the inner product C on $\mathcal{M}$ we define an endomorphisms $\phi : \mathcal{M} \longrightarrow \mathcal{M}$ by (see [4], p. 669) :

$$B(X,Y) = C(\phi(X), Y), \quad X, Y \in \mathcal{M}.$$

Since the matrices of ϕ and B in the basis orthogonal with respect to C coincide, the matrix of ϕ is symmetric. Hence the eigenvalues $\lambda_1, \cdots, \lambda_m$ are real, the corresponding eigenvectors $v_1, \cdots, v_m$ form an orthogonal basis with respect to C and

$$B(v_i, v_j) = C(\lambda_i v_i, v_j) = \lambda_i C(v_i, v_j) = 0 \quad \text{for} \quad i \neq j.$$

If for some index l we have $B(v_l, v_l) = 0$, then $v_l \in radB$. Let $\lambda_l \in (\mathcal{M} - radB)$, implying $\lambda_l \neq 0$, so for any $Z \in \mathcal{G}$ we have

$$C(v_l, [v_l, Z]) = \frac{1}{\lambda_l} C(\phi(v_l), [v_l, Z]_{\mathcal{M}}) = \frac{1}{\lambda_l} B(v_l, [v_l, Z]_{\mathcal{M}}) =$$

$$= \frac{1}{\lambda_l} B(v_l, [v_l, Z]) = \frac{1}{\lambda_l} B([v_l, v_l], Z) = 0,$$

i.e. v_l is a geodesic vector.

Next, since $\mathcal{G}$ is semisimple we have $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$ and $rad\, B = 0$ (see Definition 2.10). But $B(X,Y) = C(\phi(X), Y)$, hence $\mathrm{Ker}\phi = rad\, B$ and ϕ is isomorphism. Thus, all the eigenvalues $\lambda_i \neq 0$ $1 \leq i \leq m$ and the eigenvectors $v_1, \cdots, v_m$ are geodesic vectors, i.e. there are m orthogonal homogeneous geodesics passing through $x_0 = \{K\}$.

Let now $rad\,B = \mathcal{M}$. There is a solvable Lie group of isometries acting transitively on M. Since $\Im = (G, \pi, G/K, K)$ is a principal homogeneous bundle, there exists a homogeneous geodesic passing through every point of the base space. Taking into account that $\mathcal{G}$ is semisimple, we get $m = dim G/K$ orthogonal homogeneous geodesics passing through $x_0 = \{K\}$. Theorem 3.5 is proved.

Theorem 3.6. *With the hypotheses of Theorem 3.5 let*

$$\Im = (G, \pi, G/K, K)$$

be a principal homogeneous bundle and

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathbf{G/K}, \mathcal{G}/\mathcal{K})$$

be a bundle associated with $\Im$. *Let* $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ *and let* $x_0 = \{K\}$ *be the origin of* M, *where* $\mathcal{G}' = \mathcal{S} + \mathcal{P}$ *is a reductive decomposition of* $\mathcal{G}'$. *If* G *is weakly semisimple, then there are* $r = \dim S$ *orthogonal homogeneous geodesics passing through* x_0.

Proof : Consider the base space G/K of $\Im = (G, \pi, G/K, K)$ and fiber space $\mathcal{G}/\mathcal{K}$ of $\xi = (G \times_K \mathcal{K}, \rho_\xi, \mathcal{G}/\mathcal{K}, \mathcal{G}/\mathcal{K})$. Since $[\mathcal{G}, \mathcal{G}] \subseteq \mathcal{G}$ and $\mathcal{G}$ is weak by semisimple, then $W_{rad}\mathcal{G} = 0$. By Definition 3.4

$$W_{rad}B = \{X \in [\mathcal{G}, \mathcal{G}], B(X, Y) = 0, \quad \text{for all} \quad Y \in \mathcal{G}\}.$$

This implies $W_{rad}B \subseteq rad\,B$, and the equality holds when $\mathcal{G}$ is semisimple ($rad\,B = 0$). If $\mathcal{G}$ is not semisimple, there are elements from $rad\,B$ that belong to neither $W_{rad}B$ nor $[\mathcal{G}, \mathcal{G}]$.

Since G is connected, G' is a normal Lie subgroup of G and its Lie algebra is $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$. In the reductive decomposition of $\mathcal{G}'$ with respect to the restriction of Killing form on $\mathcal{G}'$ we set $\mathcal{G}' = \mathcal{S} + \mathcal{P}$, where S is a subspace of $\mathcal{M}$ with dimension r, while $\mathcal{P}$ is the Lie algebra of the closed subgroup P of G' such that $P = G' \cap K$.

Now if $v_l \in S - (W_{rad}B)$ then $B(v_l, v_l) \neq 0$ and by Theorem 3.5 v_l is a geodesic vector. Since $\mathcal{G}$ is weak semisimple $W_{rad}\mathcal{G} = 0$ and so $W_{rad}B = 0$. Therefore $v_1, v_2, \cdots, v_r$ are independent geodesic vectors passing through the origin x_0.

Using Gram-Schmidt method we can get r orthogonal geodesics passing through x_0. This completes the proof of Theorem 3.6.

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Резюме. В статье рассматриваются главные пучки и приводятся некоторые результаты о структуре геодезических в основном пространстве главных пучков.

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SHARP ASYMPTOTICS FOR POSITIVE SERIES

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Abstract. Suppose that $\sum\limits_{k=0}^{\infty} a_k$ is a series with $a_k \geq 0$. In [3] some asymptotic properties were obtained for series of the type $\sum\limits_{k=0}^{\infty} a_k\varphi(\sum\limits_{\nu=0}^{k} a_\nu)$ and $\sum\limits_{k=0}^{\infty} a_k\varphi(\sum\limits_{\nu=k}^{\infty} a_\nu)$ for a class of functions φ. It has been proved by L. Leindler [2] that these asymptotics are best possible for the case $\varphi(t) = t^{-\alpha}$. In this paper we show that the asymptotics obtained in [3] are best possible for more general $\varphi(t)$.

§1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout this note we consider series $\sum\limits_{k=0}^{\infty} a_k$ with $a_k \geq 0$. As usual

$$S_n := \sum_{k=0}^{n} a_k; \qquad R_n := \sum_{k=n}^{\infty} a_k$$

are the partial sums and the remainders respectively, and suppose that $S_n > 0$ and $R_n > 0$ for all $n \in \mathbb{N}_0$.

In our previous paper [3] we investigated convergence – divergence properties of the series

$$\sum_{k=0}^{\infty} a_k\, \varphi(S_k) \quad \text{and} \quad \sum_{k=0}^{\infty} a_k \psi(R_k),$$

under certain assumptions on the functions φ and ψ. Among others we proved the following results.

Theorem A. *Let $\sum\limits_{k=0}^{\infty} a_k$ be divergent and φ be a positive and decreasing function on $[a_0, \infty)$. The following assertions hold :*

(1) If $\int\limits_{a_0}^{\infty} \varphi(t)\,dt < \infty$, *then* $\sum\limits_{k=0}^{\infty} a_k \varphi(S_k) < \infty$,

(2) If $\int\limits_{a_0}^{\infty} \varphi(t)\,dt = \infty$, *then* $\sum\limits_{k=1}^{\infty} a_k \varphi(S_{k-1}) = \infty$.

Theorem B. *Let* $\sum\limits_{k=0}^{\infty} a_k$ *be convergent and* ψ *be a positive and decreasing function on* $(0, R_0]$. *The following assertions hold :*

(1) If $\int\limits_{0}^{R_0} \psi(t)\,dt < \infty$, *then* $\sum\limits_{k=0}^{\infty} a_k \psi(R_k) < \infty$,

(2) If $\int\limits_{0}^{R_0} \psi(t)\,dt = \infty$, *then* $\sum\limits_{k=0}^{\infty} a_k \psi(R_{k+1}) = \infty$.

For $\varphi(t) = t^{-\alpha}$, $t \in [a_0, \infty)$, Theorem A reduces to the well-known Abel-Dini-theorem (see for instance [1], p. 299, or [5], Theorem 7.9). A classical result of Dini (see for instance [1], p. 301) is obtained for $\psi(t) = t^{-\alpha}$, $t \in (0, R_0]$, as a special case of Theorem B.

For the case $\varphi(t) = t^{-\alpha}$ and $\psi(t) = t^{-\alpha}$ L. Leindler [2] proved that Theorem A and Theorem B are in a certain sense best possible.

At the International Conference on Functions, Series, Operators (Alexits Memorial Conference, Budapest) the problem was set whether Theorem A and Theorem B are also best possible for more general functions φ and ψ. Using Leindler technic [2] we answer this question and prove the following results.

Theorem 1. *Let* φ *be a positive and decreasing function on* $[1, \infty)$.

(1) Suppose that $\int\limits_{1}^{\infty} \varphi(t)dt < \infty$ *and let* $\{\rho_k\}$ *be a sequence of positive factors with* $\lim\limits_{k\to\infty} \rho_k = \infty$. *Then there exists a sequence* $\{a_k\}$ *of positive numbers with* $a_0 = S_0 = 1$ *such that*

a) $\sum\limits_{k=0}^{\infty} a_k = \infty$,

b) $\sum\limits_{k=0}^{\infty} a_k \varphi(S_k) < \infty$,

c) $\sum\limits_{k=0}^{\infty} a_k \rho_k \varphi(S_k) = \infty$.

(2) Suppose that $\int\limits_{1}^{\infty} \varphi(t)dt = \infty$ *and let* $\{\varepsilon_k\}$ *be a sequence of positive factors with* $\lim\limits_{k\to\infty} \varepsilon_k = 0$. *Then there exists a sequence* $\{a_k\}$ *of positive numbers with* $a_0 = S_0 = 1$ *such that*

a) $\sum\limits_{k=0}^{\infty} a_k = \infty$,

b) $\sum\limits_{k=1}^{\infty} a_k \varphi(S_{k-1}) = \infty$, *c)* $\sum_{k=1}^{\infty} a_k \varepsilon_k \varphi(S_{k-1}) < \infty$.

Theorem 2. *Let ψ be a positive and decreasing function on $(0,1]$.*

(1) Suppose that $\int_0^1 \psi(t)dt < \infty$ and let $\{\rho_k\}$ be a sequence of positive factors with $\lim_{k\to\infty} \rho_k = \infty$. Then there exists a sequence $\{a_k\}$ of positive numbers such that

a) $\sum_{k=0}^{\infty} a_k = R_0 = 1$,

b) $\sum_{k=0}^{\infty} a_k \psi(R_k) < \infty$,

c) $\sum_{k=0}^{\infty} a_k \rho_k \psi(R_k) = \infty$.

(2) Suppose that $\int_0^1 \psi(t)\,dt = \infty$ and let $\{\varepsilon_k\}$ be a sequence of positive factors with $\lim_{k\to\infty} \varepsilon_k = 0$. Then there exists a sequence $\{a_k\}$ of positive numbers such that

a) $\sum_{k=0}^{\infty} a_k = R_0 = 1$,

b) $\sum_{k=0}^{\infty} a_k \psi(R_{k+1}) = \infty$,

c) $\sum_{k=0}^{\infty} a_k \varepsilon_k \psi(R_{k+1}) < \infty$.

§2. PROOF OF THEOREM 1

a) We first prove (1). Since $\lim_{k\to\infty} \rho_k = \infty$, there exists a strictly increasing sequence $\{\mu_m\}$ of natural numbers with $\mu_1 := 1$ and

$$\rho_k \geq \frac{1}{\varphi(m+1)} \qquad \text{for all} \quad k \geq \mu_m;\ m = 2, 3, \ldots .$$

Let the sequence $\{a_k\}$ be defined by $a_0 := 1$ and

$$a_k := \frac{1}{\mu_{m+1} - \mu_m} \qquad \text{for} \quad \mu_m \leq k < \mu_{m+1};\ m = 1, 2, \ldots .$$

We obviously obtain $\sum_{k=0}^{\infty} a_k = \infty$ and it follows from [3] that $\sum_{k=0}^{\infty} a_k \varphi(S_k) < \infty$. For $\mu_m \leq k < \mu_{m+1}$ we have $S_k \leq S_{\mu_{m+1}-1} = m+1$ and therefore $\varphi(S_k) \geq \varphi(m+1)$. For $N \geq 2$ this implies

$$\sum_{k=0}^{\mu_{N+1}-1} a_k \rho_k \varphi(S_k) \geq \sum_{m=2}^{N} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \rho_k \varphi(S_k) \geq$$

$$\geq \sum_{m=2}^{N} \frac{1}{\varphi(m+1)} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \varphi(S_k) \geq \sum_{m=2}^{N} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k = N-2$$

Therefore $\sum_{k=0}^{\infty} a_k \rho_k \varphi(S_k) = \infty$.

b) To prove (2) we choose a strictly increasing sequence $\{\mu_m\}$ of natural numbers with $\mu_1 = 1$ and

$$0 < \varepsilon_k < \frac{1}{m^2 \varphi(m)} \qquad \text{for all} \quad k \geq \mu_m;\ m = 2, 3, \ldots .$$

We define the sequence $\{a_k\}$ putting $a_0 := 1$ and

$$a_k := \frac{1}{\mu_{m+1} - \mu_m} \qquad \text{for all} \quad k \geq \mu_m;\ m = 1, 2, \ldots .$$

We again obtain $\sum_{k=0}^{\infty} a_k = \infty$ and it follows now from [3] that $\sum_{k=1}^{\infty} a_k \varphi(S_{k-1}) = \infty$. For $\mu_m \leq k < \mu_{m+1}$ we have $S_{k-1} \geq S_{\mu_m - 1} = m$ and therefore $\varphi(S_{k-1}) \leq \varphi(m)$, implying

$$\sum_{k=\mu_2}^{\infty} a_k \varepsilon_k \varphi(S_{k-1}) = \sum_{m=2}^{\infty} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \varepsilon_k \varphi(S_{k-1}) \leq$$

$$\leq \sum_{m=2}^{\infty} \frac{1}{m^2 \varphi(m)} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \varphi(S_{k-1}) \leq \sum_{m=2}^{\infty} \frac{1}{m^2} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k = \sum_{m=2}^{\infty} \frac{1}{m^2}.$$

Therefore $\sum_{k=1}^{\infty} a_k \varepsilon_k \varphi(S_{k-1}) < \infty$. Theorem 1 is proved.

§3. PROOF OF THEOREM 2

a) We first prove (1). Since $\lim_{k \to \infty} \rho_k = \infty$, there exists a strictly increasing sequence $\{\mu_m\}$ of natural numbers with $\mu_1 := 1$ and

$$\rho_k \geq \frac{2^{m+1}}{\psi(2^{-m})} \qquad \text{for all} \quad k \geq \mu_m; \quad m = 2, 3 \ldots .$$

Let the sequence $\{a_k\}$ be defined by $a_0 := \frac{1}{2}$ and

$$a_k := \frac{1}{2^{m+1} \cdot (\mu_{m+1} - \mu_m)} \qquad \text{for } \mu_m \leq k < \mu_{m+1};\ m = 1, 2, \ldots .$$

A short calculation gives $\sum_{k=0}^{\infty} a_k = 1$ and from [3] we get $\sum_{k=0}^{\infty} a_k \psi(R_k) < \infty$. For $k \geq \mu_m$ we obtain

$$R_k = \sum_{\nu=k}^{\infty} a_\nu \leq \sum_{\nu=\mu_m}^{\infty} a_\nu = \sum_{j=m}^{\infty} \sum_{\nu=\mu_j}^{\mu_{j+1}-1} a_\nu = \sum_{j=m}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^m}$$

and therefore $\psi(R_k) \geq \psi(2^{-m})$. For $N \geq 2$ this implies

$$\sum_{k=0}^{\infty} a_k \rho_k \psi(R_k) \geq \sum_{m=2}^{N} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \rho_k \psi(R_k) \geq \sum_{m=2}^{N} \frac{2^{m+1}}{\psi(2^{-m})} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \psi(R_k) \geq$$

$$\geq \sum_{m=2}^{N} \frac{2^{m+1}}{\psi(2^{-m})} \psi(2^{-m}) \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k = N-2.$$

Therefore $\sum\limits_{k=0}^{\infty} a_k \rho_k \psi(R_k) = \infty$.

b) To prove (2) we choose a strictly increasing sequence $\{\mu_m\}$ of natural numbers with $\mu_1 = 1$ and

$$0 < \varepsilon_k \leq \frac{1}{\psi(2^{-m-1})} \qquad \text{for all} \quad k \geq \mu_m;\ m = 2, 3, \ldots.$$

We define the sequence $\{a_k\}$ putting $a_0 := \frac{1}{2}$ and

$$a_k := \frac{1}{2^{m+1} \cdot (\mu_{m+1} - \mu_m)} \qquad \text{for} \quad \mu_m \leq k < \mu_{m+1};\ m = 1, 2, \ldots.$$

Again we have $\sum\limits_{k=0}^{\infty} a_k = 1$ and from [3] it follows $\sum\limits_{k=0}^{\infty} a_k \psi(R_{k+1}) = \infty$. For $k < \mu_{m+1}$ we get

$$R_{k+1} = \sum_{\nu=k+1}^{\infty} a_\nu \geq \sum_{\nu=\mu_{m+1}}^{\infty} a_\nu = \sum_{j=m+1}^{\infty} \sum_{\nu=\mu_j}^{\mu_{j+1}-1} a_\nu = \sum_{j=m+1}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^{m+1}}$$

and therefore $\psi(R_{k+1}) \leq \psi(2^{-m-1})$, implying

$$\sum_{k=\mu_2}^{\infty} a_k \varepsilon_k \psi(R_{k+1}) = \sum_{m=2}^{\infty} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \varepsilon_k \psi(R_{k+1}) \leq$$

$$\leq \sum_{m=2}^{\infty} \frac{1}{\psi(2^{-m-1})} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k \psi(R_{k+1}) \leq \sum_{m=2}^{\infty} \frac{\psi(2^{-m-1})}{\psi(2^{-m-1})} \sum_{k=\mu_m}^{\mu_{m+1}-1} a_k = \sum_{m=2}^{\infty} \frac{1}{2^{m+1}}.$$

Therefore $\sum\limits_{k=0}^{\infty} a_k \varepsilon_k \psi(R_{k+1}) < \infty$. Theorem 2 is proved.

Резюме. Предположим, что $\sum\limits_{k=0}^{\infty} a_k$ есть ряд с $a_k \geq 0$. В [3] получены некоторые асимптотические свойства для рядов типа $\sum\limits_{k=0}^{\infty} a_k \varphi(\sum\limits_{\nu=0}^{k} a_\nu)$ и $\sum\limits_{k=0}^{\infty} a_k \varphi(\sum\limits_{\nu=k}^{\infty} a_\nu)$ для некоторого класса функций φ. Л. Лейндлером [2] было доказано, что эти асимптотики являются наилучшими возможными для случая $\varphi(t) = t^{-\alpha}$. В настоящей статье доказано, что асимптотики, полученные в [3] являются наилучшими возможными для более общего $\varphi(t)$.

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О СПЕКТРЕ ОДНОГО ВЫРОЖДАЮЩЕГОСЯ ОПЕРАТОРА

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§1. ПОСТАНОВКА ЗАДАЧИ

Рассмотрим дифференциально-операторное уравнение

$$Bu \equiv (-1)^m D_t^m (t^\alpha D_t^m u) + At^{\alpha-2m} u = f, \qquad (1)$$

где $\alpha \geq 0$, $\alpha \neq 1, 3, \ldots, 2m-1$, $t \in [0, b]$, $D_t \equiv d/dt$, $f \in L_{2,2m-\alpha}(\mathcal{H}, (0, b)) \equiv H$, а A – оператор, действующий в гильбертовом пространстве $\mathcal{H}$, коммутирующий с D_t и обладающий полной системой собственных функций $\{\varphi_k\}_{k=1}^\infty$, образующих базис Рисса в $\mathcal{H}$.

Сначала рассмотрим одномерный случай операторного уравнения (1) в весовых пространствах Соболева W_α^m, т.е. когда A является оператором умножения на число a. Для этого случая доказывается, что спектр оператора $\mathbb{B} \equiv t^{2m-\alpha} B$: $L_{2,\alpha-2m} \longmapsto L_{2,\alpha-2m}$ чисто непрерывный и совпадает с некоторым лучом в комплексной плоскости. Затем изучается операторное уравнение (1) и дается полная характеристика спектра соответствующего оператора.

Заметим, что в отличие от [5] и [6], здесь рассматриваем случай произвольного $\alpha \geq 0$, $\alpha \neq 1, 3, \ldots, 2m-1$.

§2. ОДНОМЕРНЫЙ СЛУЧАЙ

Одномерный случай операторного уравнения (1) имеет вид :

$$Su \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} = f, \quad f \in L_{2,2m-\alpha}, \quad \alpha \geq 0, \quad \alpha \neq 1, 3, \ldots, 2m-1, \quad (1')$$

где

$$L_{2,\beta} = \left\{ f : |f, L_{2,\beta}|^2 = \int_0^b t^\beta |f(t)|^2 \, dt < \infty \right\}.$$

Пусть $\dot{C}^m$ – множество m раз непрерывно дифференцируемых функций на $[0,b]$, удовлетворяющих условиям

$$u^{(k)}(t)\Big|_{t=0} = u^{(k)}(t)\Big|_{t=b} = 0, \quad k = 0, 1, \ldots, m-1. \tag{2}$$

Обозначим через W_α^m пополнение $\dot{C}^m$ по норме

$$|u, W_\alpha^m|^2 = \int_0^b t^\alpha |D_t^m u|^2\, dt. \tag{3}$$

Как доказано в работе [1], при $\alpha \neq 1, 3, \ldots, 2m-1$ имеет место непрерывное вложение пространства $W_\alpha^m \subset L_{2,\alpha-2m}$, которое не является компактным. Отметим, что пространство $L_{2,\alpha-2m}$ является предельным в том смысле, что при $\beta > \alpha - 2m$ вложение $W_\alpha^m \subset L_{2,\beta}$ компактно, а при $\beta < \alpha - 2m$ вложение $W_\alpha^m \subset L_{2,\beta}$ нарушается.

Определение 1. Функция $u \in W_\alpha^m$ называется **обобщённым решением уравнения** $(1')$, если для любого $v \in W_\alpha^m$ имеет место равенство

$$\{u, v\}_\alpha \equiv (t^\alpha u^{(m)}, v^{(m)}) = (f, v). \tag{4}$$

Можно доказать, что обобщённое решение уравнения $(1')$ существует и единственно для любой $f \in L_{2,2m-\alpha}$. Отметим также, что пространство $L_{2,2m-\alpha}$ является самым широким пространством, в котором существует скалярное произведение (f, v) в правой части равенства (4) при $v \in W_\alpha^m \subset L_{2,\alpha-2m}$.

Если $g = t^{2mq-\alpha} f$, то очевидно, что $g \in L_{2,\alpha-2m}$ и $|g, L_{2,\alpha-2m}| = |f, L_{2,2m-\alpha}|$. Теперь можем определить оператор, соответствующий определению обобщённого решения.

Определение 2. Будем говорить, что функция $u \in W_\alpha^m$ **принадлежит области определения** $\mathcal{D}(\mathcal{S})$ **оператора** $\mathcal{S}$, если имеет место равенство (4) для некоторого $f \in L_{2,2m-\alpha}$. Тогда будем писать $\mathcal{S}u = g$.

Отметим, что оператор $\mathcal{S}: \ D(\mathcal{S}) \subset W_\alpha^m \longmapsto L_{2,\alpha-2m}$ определяется как $\mathcal{S}u = t^{2m-\alpha} Su$, где S из $(1')$. Легко показать, что $D(\mathcal{S})$ плотно в W_α^m. Следовательно, оператор $\mathcal{S}$ является положительным, самосопряжённым оператором, а $\mathcal{S}^{-1}$: $L_{2,\alpha-2m} \longmapsto L_{2,\alpha-2m}$ является ограниченным (но не компактным) оператором. Используя неравенство Харди [2] можно доказать, что для оператора $\mathcal{S}$ имеет место следующая оценка :

$$(\mathcal{S}u, u) \geq \lambda_\alpha |u, L_{2,\alpha-2m}|^2, \tag{5}$$

где $(.,.)$ означает скалярное произведение в $L_{2,\alpha-2m}$, и

$$\lambda_\alpha = \frac{(\alpha-1)^2\cdot(\alpha-3)^2\cdots(\alpha-(2m-1))^2}{4^m}.$$

Поскольку оператор $\mathcal{S}$ самосопряжённый, из неравенства (5) следует, что его спектр лежит в интервале $[\lambda_\alpha, +\infty)$.

Теорема 1. *Спектр оператора $\mathcal{S}$ чисто непрерывен и совпадает с лучом $[\lambda_\alpha, +\infty)$.*

Доказательство. Сначала докажем, что $\lambda = \lambda_\alpha$ принадлежит непрерывному спектру оператора $\mathcal{S}$. Для этого достаточно показать, что количество решений дифференциального уравнения

$$(t^\alpha u^{(m)})^{(m)} + (-1)^{m-1}\lambda_\alpha t^{\alpha-2m}u = 0, \tag{6}$$

не принадлежащих $L_{2,\alpha-2m}$, больше m (см. [3], [4]). Отметим, что уравнение (6) является уравнением Эйлера, и его характеристическое уравнение имеет вид

$$\mu(\mu-1)\cdots(\mu-(m-1))(\mu+\alpha-m)(\mu+\alpha-m-1)\cdots(\mu+\alpha-2m+1)+(-1)^{m-1}\lambda_\alpha = 0. \tag{7}$$

Если решение уравнения (6) принадлежит $L_{2,\alpha-2m}$, то ему соответствует корень уравнения (7), удовлетворяющий условию

$$\operatorname{Re}\mu > m - \frac{\alpha+1}{2}. \tag{8}$$

Положим $\nu = \mu(\mu+\alpha-2m+1)$. Тогда уравнение (7) примет следующий вид

$$\prod_{k=0}^{m-1}(\nu + k(2m-\alpha-k-1)) + (-1)^{m-1}\lambda_\alpha = 0. \tag{9}$$

Отметим, что уравнение (9) имеет корень $\nu_m = -(m-\frac{\alpha+1}{2})^2$, поскольку при $k = 0, 1, \ldots, m-1$ имеем $\nu_m + k(2m-\alpha-k-1) =$

$$= -\left(m-\frac{\alpha+1}{2}\right)^2 + 2\left(m-\frac{\alpha+1}{2}\right) - k^2 = -\frac{[\alpha-(2(m-k)-1)]^2}{4}.$$

Этому корню соответствует 2-кратный корень $\mu = m - \frac{\alpha+1}{2}$ уравнения (7). Очевидно, что этот корень не удовлетворяет условию (8). Из оставшихся $2m-2$ корней уравнения (7), по крайней мере $m-1$ корней не удовлетворяют условию (8), поскольку они имеют следующий вид

$$\mu_l^\pm = m - \frac{\alpha+1}{2} \pm \sqrt{\left(m-\frac{\alpha+1}{2}\right)^2 + \nu_l}, \quad l = 1, \ldots, m-1. \tag{10}$$

Следовательно, $\lambda = \lambda_\alpha$ принадлежит непрерывному спектру оператора $\mathcal{S}$, поскольку количество корней, не удовлетворяющих условию (8), больше m, и следовательно число решений уравнения (6), принадлежащих $L_{2,\alpha-2m}$, меньше m.
Пусть теперь $\lambda > \lambda_\alpha$. Отметим, что в этом случае уравнение

$$\mu(\mu-1)\cdots(\mu-(m-1))(\mu+\alpha-m)(\mu+\alpha-m-1)\cdots(\mu+\alpha-2m+1)+(-1)^{m-1}\lambda=0$$

будет иметь корень $\nu < -(m-\frac{\alpha+1}{2})^2$, поскольку при возрастании λ от λ_α до ∞, корень ν убывает с $-(m-\frac{\alpha+1}{2})^2$ до $-\infty$. Отсюда и из (10) следует, что соответствующие ему два решения не принадлежат $L_{2,\alpha-2m}$. Дальнейшие рассуждения аналогичны случаю $\lambda = \lambda_\alpha$. Теорема 1 доказана.

Замечание 1. Аналогичный результат для $\alpha = 2$ и $m = 1$ доказан в работе [5], а для $\alpha = 2m$ и произвольного m - в [6]. В обоих случаях имеем $L_{2,\alpha-2m} = L_{2,2m-\alpha} = L_2(0,b)$.

Замечание 2. Мы рассмотрели частный случай $a = 0$ одномерного уравнения (1). Теперь заметим, что согласно Определению 2, число a можно рассматривать как спектральный параметр. В силу Теоремы 1 заключаем, что спектр соответствующего оператора будет чисто непрерывным и совпадает с некоторым лучом $\sigma\mathcal{S} = a + [\lambda_\alpha; +\infty)$ в комплексной плоскости.

§3. ОПЕРАТОРНОЕ УРАВНЕНИЕ

В этом параграфе рассмотрим операторное уравнение (1) для функции $f \in L_{2,2m-\alpha}(\mathcal{H},(0,b)) \equiv H$. Поскольку оператор A обладает полной системой собственных функций $\{\varphi_k\}_{k=1}^\infty$, образующих базис Рисса в $\mathcal{H}$, то мы можем записать

$$u = \sum_{k=1}^\infty u_k(t)\varphi_k, \quad f = \sum_{k=1}^\infty f_k(t)\varphi_k, \quad A\varphi_k = a_k\varphi_k, \quad k \in \mathcal{N},$$

где $\mathcal{N}$ – множество натуральных чисел. Тогда операторное уравнение (1) расщепляется на бесконечную цепочку обыкновенных дифференциальных уравнений

$$B_k u_k \equiv (-1)^m (t^\alpha u_k^{(m)})^{(m)} + a_k t^{\alpha-2m} u_k = f_k, \quad k \in \mathcal{N}. \tag{11}$$

Из $f \in H$ следует, что $f_k \in L_{2,2m-\alpha}$, $k \in \mathcal{N}$. Аналогично Определению 1, для уравнения (11) можно определить обобщённые решения u_k, $k \in \mathcal{N}$. Положим $g_k = t^{2m-\alpha} f_k$.

Определение 3. При $f \in H$, **обобщённое решение уравнения (1)** определяется как функция $u = \sum\limits_{k=1}^\infty u_k(t)\varphi_k \in H$, где $u_k(t)$ являются обобщёнными решениями уравнения (11).

Предложение 1. Операторное уравнение (1) однозначно разрешимо при любых $f \in H$ тогда и только тогда, когда уравнения (11) однозначно разрешимы при любых $f_k \in L_{2,\alpha-2m}$ и равномерно по $k \in \mathcal{N}$ выполняются неравенства

$$|u_k, L_{2,\alpha-2m}| \le c|f_k, L_{2,2m-\alpha}| = c|g_k, L_{2,\alpha-2m}|, \quad k \in \mathcal{N}. \tag{12}$$

Из Замечания 2 следует, что достаточным условием для неравенства (12) являются, например, условия

$$\rho(-a_k; [\lambda_\alpha, \infty)) > \varepsilon, \quad k \in \mathcal{N}, \tag{13}$$

где $\varepsilon > 0$ и ρ – расстояние в комплексной плоскости.

Теорема 2. *При выполнении условия (13), при любой $f \in H$ обобщённое решение уравнения (1) существует и единственно.*

Доказательство следует из (13) и $f = \sum\limits_{k=1}^{\infty} f_k(t)\varphi_k$.

Заметим, что операторы $\mathcal{S}$ и A коммутируют, так как коммутируют операторы A и D_t. Пусть $\sigma\mathcal{S}$ (см. Теорему 1) и σA – спектры операторов $\mathcal{S}$ и A, соответственно. Из теоремы о спектре суммы двух коммутирующих операторов (см. [7]), получим следующее утверждение.

Теорема 3. *Спектр оператора $\mathbb{B}$ совпадает с прямой суммой спектров $\sigma\mathcal{S}$ и σA, т.е.*

$$\sigma\mathbb{B} = \sigma\mathcal{S} + \sigma A \equiv \{\lambda_1 + \lambda_2 : \lambda_1 \in \sigma\mathcal{S},\ \lambda_2 \in \sigma A\}.$$

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CONCIRCULAR CURVATURE TENSOR IN CONTACT METRIC MANIFOLDS

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Abstract. The paper considers (κ, μ)-manifolds that satisfy $Z(\xi, X) \cdot S = 0$, where Z stands for the concircular curvature tensor and S for Ricci tensor. A clasification of $N(\kappa)$-contact metric manifolds satisfying $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$ is proposed, where R is the curvature tensor. There are some applications to concircularly symmetric $N(\kappa)$-contact metric manifolds or manifolds possessing non-vanishing recurrent concircular curvature tensor.

§1. INTRODUCTION

Semisymmetric spaces have been investigated by E. Cartan, they generalize the symmetric spaces ($\nabla R = 0$). A Riemannian manifold M is said to be semisymmetric, if its curvature tensor R satisfies

$$R(X, Y) \cdot R = 0, \quad X, Y \in TM,$$

where $R(X, Y)$ acts on R as a derivative. A Riemannian manifold M is Ricci-semisymmetric (sometimes Ricci-semiparallel), if its Ricci tensor S is semisymmetric, that is, its curvature tensor R satisfies $R(X, Y) \cdot S = 0$, $X, Y \in TM$, where $R(X, Y)$ acts on S as a derivative. Ricci-semisymmetric Riemannian manifolds are natural generalizations of symmetric spaces ($\nabla R = 0$), Einstein spaces, semisymmetric spaces ($R(X, Y) \cdot R = 0$) and Ricci-symmetric Riemannian manifolds ($\nabla S = 0$) (see [13] for more details). In [14], V. Mirzoyan proved a general structure theorem for Ricci-semisymmetric manifolds asserting that a Riemannian manifold is Ricci-semisymmetric if and only if it is 2-dimensional or an Einstein space or a semi-Einstein space or a local product of such spaces. In contact geometry, S. Tanno [20] showed that a semisymmetric K-contact manifold M^{2n+1} is locally isometric to the unit sphere $S^{2n+1}(1)$. He also proved that for a K-contact manifold M the following four conditions are equivalent :

(a) M is an Einstein manifold,

(b) M possesses parallel Ricci tensor (that is, M is Ricci-symmetric),

(c) M satisfies $R(X,Y) \cdot S = 0$ (that is, M is Ricci-semisymmetric) and

(d) M satisfies $R(\xi, X) \cdot S = 0$, where ξ is the structure vector field.

Since a Sasakian manifold is always a K-contact manifold, this result is valid for Sasakian manifolds. Thus, a Ricci-semisymmetric Sasakian manifold is an Einstein manifold. This generalizes a result of M. Okumura [17], which states that any Ricci-symmetric Sasakian manifold is an Einstein manifold.

We remark that a contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature +1 or is 3-dimensional and flat (see [16] or [5], pp. 98-99). A contact metric manifold M^{2n+1} satisfying $R(X,Y)\xi = 0$, where ξ is the characteristic vector field of the contact structure, is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat in dimension 3 (see [5], p. 101).

D. Perrone studied contact metric manifolds satisfying $R(\xi, X) \cdot R = 0$ in [19], where he showed that under additional assumptions the manifold is either Sasakian (and of constant curvature +1) or $R(X,\xi)\xi = 0$. B. J. Papantoniou [18] showed, that a semisymmetric contact metric manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. Both Perrone and Papantoniou also studied manifolds satisfying $R(\xi, X) \cdot S = 0$, where S denotes the Ricci tensor. Perrone shows that if ξ belongs to the κ-nullity distribution and if $R(\xi, X) \cdot S = 0$, then the contact metric manifold is locally isometric to $E^{n+1} \times S^n(4)$ or is Sasakian-Einstein. The author [22] improved the results of B. J. Papantoniou [18] for (κ, μ)-manifolds satisfying $R(\xi, X) \cdot S = 0$.

In [2], C. Baikoussis and T. Koufogiorgos showed that if ξ belongs to the κ-nullity distribution and if $R(\xi, X) \cdot C = 0$, C being the Weyl conformal curvature tensor, the contact metric manifold M^{2n+1} is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. This generalizes a result of Chaki and Tarafdar [10] that a Sasakian manifold M^{2n+1} such that $R(\xi, X) \cdot C = 0$ is locally isometric to $S^{2n+1}(1)$. Moreover, in [15] Murathan and Yildiz studied (κ, μ)-manifolds satisfying $C(\xi, X) \cdot S = 0$.

The present paper is mainly based on the joint work of the author with Professor D. E. Blair and Dr. Jeong-Sik Kim (see [8], [23]). Section 2 contains necessary details about contact metric manifolds. In section 3, we give a brief account of (κ, μ)-manifolds and also present two results. In section 4, we explain the notion of $\mathcal{D}$-homothetic deformation and construct a key example for later use. We give a brief introduction to concircular curvature tensor in section 5. In section 6, we give an example of a non-Sasakian η-Einstein manifold, present a structure theorem for non-Sasakian η-Einstein manifolds, and give classifications of (κ, μ)-manifolds satisfying $Z(\xi, X) \cdot S = 0$. In section 7, we classify $N(\kappa)$-contact metric manifolds satisfying $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$ and point at some applications to $N(\kappa)$-contact metric manifolds, which are concircularly symmetric or are with non-vanishing recurrent concircular curvature tensor. In the end an open problem is proposed.

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§2. CONTACT METRIC MANIFOLDS

An odd-dimensional manifold M^{2n+1} is said to admit an **almost contact structure**, sometimes called a (φ, ξ, η)-**structure**, if it admits a tensor field φ of type $(1, 1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{1}$$

The first and one of the remaining three relations in (1) imply the other two relations in (1). Let g be a compatible Riemannian metric with (φ, ξ, η), that is,

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y)$$

for all $X, Y \in TM$. Then, M becomes an **almost contact metric manifold** equipped with an almost contact metric structure (φ, ξ, η, g). An almost contact metric structure becomes a **contact metric structure**, if

$$g(X, \varphi Y) = d\eta(X, Y), \quad X, Y \in TM.$$

The 1-form η is then a contact form and ξ is its characteristic vector field. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is Levi-Civita connection. A contact metric manifold M is Sasakian if and only if the curvature tensor R satisfies

$$R(X, Y)\xi = R_0(X, Y)\xi, \quad X, Y \in TM, \tag{2}$$

where $R_0(X,Y)U = g(Y,U)X - g(X,U)Y, \quad X,Y,U \in TM$.
A contact metric manifold is called a **K-contact manifold**, if the characteristic vector field ξ is a Killing vector field. An almost contact metric manifold is K-contact if and only if $\nabla\xi = -\varphi$. A K-contact manifold is a contact metric manifold, while the converse is true if $h = 0$, where $2h$ is the Lie derivative of φ in the characteristic direction ξ. A Sasakian manifold is always a K-contact manifold. A 3-dimensional K-contact manifold is a Sasakian manifold. Thus a 3-dimensional contact metric manifold is a Sasakian manifold if and only if $h = 0$. For more details see [5].

§3. (κ,μ)-MANIFOLDS

It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [3]. On the other hand, as we have noted (see (2)), on a Sasakian manifold $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$. As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case ; D. Blair, T. Koufogiorgos and B. J. Papantoniou [6] considered the **(κ,μ)-nullity condition** on a contact metric manifold. The **(κ,μ)-nullity distribution** $N(\kappa,\mu)$ ([6], [18]) of a contact metric manifold M is defined by

$$N(\kappa,\mu) : p \longmapsto N_p(\kappa,\mu) = \{U \in T_pM \mid R(X,Y)U = (\kappa I + \mu h)R_0(X;Y)U\}$$

for all $X,Y \in TM$, where $(\kappa,\mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(\kappa,\mu)$ is called a **(κ,μ)-manifold**. In particular on a (κ,μ)-manifold, we have

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

On a (κ,μ)-manifold $\kappa \le 1$. If $\kappa = 1$, the structure is Sasakian ($h = 0$ and μ is not determined) and if $\kappa < 1$, the (κ,μ)-nullity condition determines the curvature of M^{2n+1} completely [6]. In fact, among (κ,μ)-manifolds, the subclasses of Sasakian manifolds, K-contact manifolds coincide, and are described by $\kappa = 1$ and $h = 0$. Moreover, we have $Q\xi = 2n\kappa\xi, \quad h^2 = (\kappa - 1)\varphi^2$, where Q is Ricci operator. If $\mu = 0$, the (κ,μ)-nullity distribution $N(\kappa,\mu)$ is reduced to the **κ-nullity distribution** $N(\kappa)$ [21], where the **κ-nullity distribution** $N(\kappa)$ of a Riemannian manifold M is defined by

$$N(\kappa) : p \longmapsto N_p(\kappa) = \{U \in T_pM \mid R(X,Y)U = \kappa R_0(X,Y)U\},$$

where κ is a constant. If $\xi \in N(\kappa)$, then a contact metric manifold M we call **$N(\kappa)$-contact metric manifold**. If $\kappa = 1$, an $N(\kappa)$-contact metric manifold is Sasakian and if $\kappa = 0$, an $N(\kappa)$-contact metric manifold is locally isometric to $E^{n+1} \times S^n(4)$. In [1], where $N(\kappa)$-contact metric manifolds were studied, it was shown that $\kappa < 1$

implies that the scalar curvature $r = 2n(2n-2+\kappa)$. For more detail we refer to [1] and [6].

The standard contact metric structure on the tangent sphere bundle T_1M satisfies the (κ,μ)-nullity condition if and only if the base manifold M is of constant curvature. In particular if M has constant curvature c, then $\kappa = c(2-c)$ and $\mu = -2c$. To end the section, we reproduce the following results :

Theorem 3.1 [23]. *A Ricci flat (κ,μ)-manifold is necessarily flat and 3-dimensional.*

Theorem 3.2 [23]. *A non-Sasakian Einstein (κ,μ)-manifold is necessarily 3-dimensional and flat.*

Theorem 3.2 is a generalization of the following

Theorem 3.3 [21] *If an $N(\kappa)$-contact metric manifold of dimension ≥ 5 is Einstein, then it is necessarily Sasakian.*

§4. $\mathcal{D}$–HOMOTHETIC DEFORMATION

For a given contact metric structure (φ,ξ,η,g), a **$\mathcal{D}$-homothetic deformation** is the structure defined by

$$\bar\eta = a\eta, \quad \bar\xi = \frac{1}{a}\xi, \quad \bar\varphi = \varphi, \quad \bar g = ag + a(a-1)\eta\otimes\eta,$$

where a is a positive constant. It preserves the contact metric, K-contact, Sasakian or strongly pseudo-convex CR properties, but destroys the relations like $R(X,Y)\xi = 0$ or $R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y)$. However, the form of the (κ,μ)-nullity condition is preserved under a $\mathcal{D}$-homothetic deformation with

$$\bar\kappa = \frac{\kappa + a^2 - 1}{a^2}, \quad \bar\mu = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian (κ,μ)-manifold M, Boeckx [9] introduced an invariant

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$$

and showed that for two non-Sasakian (κ,μ)-manifolds $(M_i,\varphi_i,\xi_i,\eta_i,g_i)$, $i = 1,2$ we have $I_{M_1} = I_{M_2}$ if and only if up to a $\mathcal{D}$-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Thus we know all non-Sasakian (κ,μ)-manifolds locally as soon as we have for every odd dimension $2n+1$ and for every possible value of the invariant I, one (κ,μ)-manifold (M,φ,ξ,η,g) with $I_M = I$. For $I > -1$ such examples can be built using the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c, where we have $I = \frac{1+c}{|1-c|}$.

Boeckx also gives a Lie algebra construction for any odd dimension and values of $I \leq -1$. Using that invariant, we now construct an example of a $(2n+1)$-dimensional $N\left(1-\frac{1}{n}\right)$-contact metric manifold, $n > 1$.

Example 4.1. Since the Boeckx invariant for a $(1-\frac{1}{n},0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting $\mathcal{D}$-homothetic deformation will be a $(1-\frac{1}{n},0)$-manifold. That is, for $\kappa = c(2-c)$ and $\mu = -2c$ we solve $1-\frac{1}{n} = \frac{\kappa+a^2-1}{a^2}$, $0 = \frac{\mu+2a-2}{a}$ for a and c. For the values $c = \frac{(\sqrt{n}\pm 1)^2}{n-1}$, $a = 1+c$, we obtain a $N\left(1-\frac{1}{n}\right)$-contact metric manifold. Example 4.1 will be used in Theorems 6.3, 7.1 and 7.2.

§5. CONCIRCULAR CURVATURE TENSOR

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a **concircular transformation** ([12], [24]). (A geodesic circle is a curve in M whose first curvature is constant and whose second curvature is identical zero.) A concircular transformation is always a conformal transformation ([12]). Thus, the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [4]). An interesting invariant of a concircular transformation is the **concircular curvature tensor** Z ([24], [25]) :

$$Z = R - \frac{r}{n(n-1)} R_0,$$

where R is the curvature tensor and r is the scalar curvature. From the form of the concircular curvature tensor we conclude that Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. A necessary and sufficient condition that a Riemannian manifold can be reduced to a Euclidean space by a concircular transformation is that its concircular curvature tensor vanishes.

§6. (κ,μ)-MANIFOLDS WITH $Z(\xi,X)\cdot S = 0$

A contact metric manifold M is said to be **η-Einstein** (see [17] or [5], p. 105), if the Ricci tensor S satisfies

$$S = ag + b\eta\otimes\eta, \tag{3}$$

where a and b are some smooth functions on the manifold. In particular if $b=0$, then M becomes an **Einstein manifold**. In dimensions ≥ 5 it is known, that for any η-Einstein K-contact manifold, a and b are constants [20].

We note that a non-Sasakian (κ,μ)-manifold M^{2n+1} is η-Einstein if and only if $\mu = -2(n-1)$. In particular, a 3-dimensional contact metric manifold is η-Einstein if and only if it is an $N(\kappa)$-contact metric manifold [7]. More precisely, in a 3-dimensional

$N(\kappa)$-contact metric manifold

$$S = \left(\frac{r}{2} - \kappa\right) g + \left(3\kappa - \frac{r}{2}\right) \eta \otimes \eta. \tag{4}$$

Example 6.1. A contact metric manifold, obtained by a $\mathcal{D}$-homothetic deformation of the contact metric structure on the tangent sphere bundle of a Riemannian manifold M^{n+1} of constant curvature $\frac{n^2 \pm 2n+1}{n^2-1}$, is a non-Sasakian η-Einstein (κ, μ)-manifold. In a non-Sasakian η-Einstein (κ, μ)-manifold M^{2n+1}, we have

$$S = 2(n^2 - 1)g - 2(n^2 - n\kappa - 1)\eta \otimes \eta. \tag{5}$$

Theorem 6.1 [23]. *Let M^{2n+1} be a non-Sasakian η-Einstein (κ, μ)-manifold. Then the concircular curvature tensor Z satisfies $Z(\xi, X) \cdot S = 0$ if and only if M^{2n+1} is flat and 3-dimensional.*

We close this section with the following theorem.

Theorem 6.2 [23]. *Let M^{2n+1} be a (κ, μ)-manifold. The concircular curvature tensor Z satisfies $Z(\xi, X) \cdot S = 0$ if and only if one of the following conditions is satisfied :*

(a) M^{2n+1} is flat and 3-dimensional.

(b) M^{2n+1} is locally isometric to the Example 4.1.

(c) M^{2n+1} is an Einstein-Sasakian manifold.

§7. $N(\kappa)$-CONTACT METRIC MANIFOLDS SATISFYING $Z(\xi, X) \cdot Z = 0$

We now present a theorem in which Example 4.1 arises naturally in contrast to $E^{n+1} \times S^n(4)$, cf. Theorem 7.3 below.

Theorem 7.1 [8]. *A $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold M satisfies $Z(\xi, X) \cdot Z = 0$, if and only if M is locally isometric to the sphere $S^{2n+1}(1)$, M is locally isometric to the Example 4.1 or M is 3-dimensional and flat.*

The following theorem is a corollary.

Theorem 7.2 [8]. *A $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold M satisfies $Z(\xi, X) \cdot R = 0$, if and only if M is locally isometric to the sphere $S^{2n+1}(1)$, M is locally isometric to Example 4.1 or M is 3-dimensional and flat.*

On the other hand, reversing the order of Z and R gives the following result.

Theorem 7.3 [8]. *A $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold M satisfies $R(\xi, X) \cdot Z = 0$, if and only if M is locally isometric to the sphere $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$.*

A Riemannian manifold is said to be **concircularly symmetric**, if the concircular curvature tensor Z is parallel, that is

$$\nabla Z = 0. \tag{6}$$

Theorem 7.4 [8]. *Let M^{2n+1} be a concircularly symmetric $N(\kappa)$-contact metric manifold. Then M is locally isometric to either $E^{n+1}(0) \times S^n(4)$ or the sphere $S^{2n+1}(1)$.*

Remark. We note that while Z is a concircular invariant, the connection ∇ is not and hence $\nabla Z = 0$ is not a concircular invariant. It can be interesting to study spaces, which are concircularly equivalent to a locally symmetric space.
If we assume that the concircular curvature tensor Z in an $N(\kappa)$-contact metric manifold M^{2n+1} is recurrent, that is

$$\nabla Z = \alpha \otimes Z, \tag{7}$$

where α is an everywhere non-vanishing 1-form, then we have the following theorem.

Theorem 7.5 [8]. *Let M^{2n+1} be an $N(\kappa)$-contact metric manifold with non-vanishing recurrent concircular curvature tensor. Then M^{2n+1} is locally isometric to $E^{n+1}(0) \times S^n(4)$.*
To conclude, we propose the following problem.

Problem. To classify (κ, μ)-manifolds under conditions $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$, $\nabla Z = 0$, and $\nabla Z = \alpha \otimes Z$.

Резюме. В статье рассматриваются (κ, μ)-многообразия, удовлетворяющие $Z(\xi, X) \cdot S = 0$, где Z означает конциркулярный тензор кривизны, а S – тензор Риччи. В работе предлагается классификация $N(\kappa)$-касательных метрических многообразий, удовлетворяющих $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot Z = 0$, где R – тензор кривизны. Имеются несколько применений конциркулярных симметрических $N(\kappa)$-касательных метрических многообразий или многообразий, обладающих ненулевым периодическим конциркулярным тензором кривизны.

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NONSTANDARD TRANSFINITE ELECTRICAL NETWORKS

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Abstract. The idea of a nonstandard, transfinite, linear electrical network is defined. To prove an existence and uniqueness theorem for a hyperreal current-voltage regime, the concept of a transfinite graph has to be lifted into a nonstandard setting. Nonstandard versions of Kirchhoff's laws are also examined. Finally, it is pointed out that certain nonlinear networks have unique hyperreal current-voltage regimes as well.

§1. INTRODUCTION

Our objective in this paper is to lift the ideas of transfinite graphs and electrical networks, as well as the fundamental theorem concerning the existence of a current-voltage regime in a transfinite network, into a nonstandard setting. For the sake of brevity, we shall restrict our attention to the first rank of transfiniteness. Such transfinite graphs are defined in [6], Sec. 3.2 and also in [7], Sec. 2.1, and the fundamental theorem for electrical networks having such graphs is stated by [6], Theorem 3.3-5 and also by [7], Theorem 5.2-8. A simpler version of these ideas for restricted transfinite connections can be found in [8].

Herein, we will state briefly the necessary definitions concerning transfinite graphs, alleviating thereby the need to refer to those prior works. In §2 we construct a nonstandard version of a conventional graph, and in §3 we do the same for the first rank transfinite graph, called a "1-graph". A nonstandard transfinite linear electrical network is defined in §4, wherein the existence of a hyperreal current-voltage regime is also established. Finally, Kirchhoff's laws in a nonstandard setting are examined in §5. In §6 we discuss the extensions to certain nonlinear networks.

We will employ a variety of concepts and results from nonstandard analysis whose definitions can be found in many books, as for example [2] – [5]. We adopt the ultrapower approach to nonstandard analysis and will mention the transfer principle only occasionally. Thus, it is understood that a nonprincipal ultrafilter $\mathcal{F}$ (also called a "free ultrafilter") has been chosen, and equivalence classes of sequences are defined with respect to $\mathcal{F}$. Terminology and symbolism vary somewhat in the literature on nonstandard analysis; we follow those used by [2]. For instance, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{R}$ is the set of real numbers. An infinite sequence $a_0, a_1, a_2, \ldots$ indexed by the natural numbers is denoted by $\langle a_n : n \in \mathbb{N}\rangle$ or simply by $\langle a_n\rangle$. Then, a nonstandard entity is an equivalence class of such sequences whereby two sequences $\langle a_n\rangle$ and $\langle b_n\rangle$ are considered equivalent if $\{n : a_n = b_n\} \in \mathcal{F}$. Any such nonstandard entity is denoted by $[a_n]$, where a_n enters any sequence in the equivalence class.

§2. NONSTANDARD GRAPHS

A graph G is a pair $G = \{X, B\}$, where X is a set and B is a set of two-element subsets of X. Thus, a typical branch b is $b = \{x, y\}$, where $x, y \in X$ and $x \neq y$. X and B may be infinite sets. We will use electrical terminology by referring to the elements of X as **nodes** (instead of "vertices") and to the elements of B as **branches** (instead of "edges"). Given any branch $b = \{x, y\} \in B$, x and b are said to be **incident**, and similarly for y and b.

Next, let $\langle G_n : n \in \mathbb{N}\rangle$ be a given sequence of graphs. The nonstandard graph we shall construct will depend upon this choice of the sequence $\langle G_n : n \in \mathbb{N}\rangle$. For each n, we have $G_n = \{X_n, B_n\}$, where X_n is the set of nodes and B_n is the set of branches. We allow $X_n \cap X_m \neq \emptyset$ so that G_n and G_m may be subgraphs of a larger graph. In fact, we can view each G_n as being a subgraph of the union $G = \{\cup X_n, \cup B_n\}$ of all the G_n. As a special case, we may have $X_n = X_m$ and $B_n = B_m$ for all $n, m \in \mathbb{N}$ so that G_n may be the same graph for all $n \in \mathbb{N}$.

In the following, $\langle x_n\rangle = \langle x_n : n \in \mathbb{N}\rangle$ will denote a sequence of nodes with $x_n \in X_n$ for all $n \in \mathbb{N}$. A **nonstandard node x** is an equivalence class of such sequences of nodes, where two such sequences $\langle x_n\rangle$ and $\langle y_n\rangle$ are taken to be equivalent if $\{n : x_n = y_n\} \in \mathcal{F}$, in which case we write $\langle x_n\rangle = \langle y_n\rangle$ a.e. or say that $x_n = y_n$ "for almost all n." As was stated before, we also write $\mathbf{x} = [x_n]$, where it is understood that x_n enters any sequence in the equivalence class.

Reflexivity and symmetry of this relation being obvious, consider transitivity : Given that $\langle x_n\rangle = \langle y_n\rangle$ a.e. and that $\langle y_n\rangle = \langle z_n\rangle$ a.e., we have $N_{xy} = \{n : x_n = y_n\} \in \mathcal{F}$ and $N_{yz} = \{n : y_n = z_n\} \in \mathcal{F}$. By the properties of the ultrafilter, $N_{xy} \cap N_{yz} \in \mathcal{F}$. Moreover, $N_{xz} = \{n : x_n = z_n\} \supseteq (N_{xy} \cap N_{yz})$. Therefore, $N_{xz} \in \mathcal{F}$. Hence, $\langle x_n\rangle = \langle z_n\rangle$ a.e. ; transitivity holds. We let ${}^*X$ denote the set of nonstandard nodes.

Next, we define the nonstandard branches : Let $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ be two nonstandard nodes. This time, let $N_{xy} = \{n : \{x_n, y_n\} \in B_n\}$ and $N_{xy}^c = \{n : \{x_n, y_n\} \notin B_n\}$. Since $\mathcal{F}$ is an ultrafilter, exactly one of N_{xy} and N_{xy}^c is a member of $\mathcal{F}$. If it is N_{xy}, then $\mathbf{b} = [\{x_n, y_n\}]$ is defined to be a **nonstandard branch** ; that is, $\mathbf{b}$ is an equivalence class of sequences $\langle b_n \rangle$ where $b_n = \{x_n, y_n\}$, $n = 0, 1, 2, \ldots$. In this case, we also write $\mathbf{x}, \mathbf{y} \in \mathbf{b}$ and $\mathbf{b} = \{\mathbf{x}, \mathbf{y}\}$. We let ${}^*B$ denote the set of nonstandard branches. On the other hand, if $N_{xy}^c \in \mathcal{F}$, then $[\{x_n, y_n\}]$ is not a nonstandard branch. We shall now show that this definition is independent of the representatives chosen for the nodes. Let $[\{x_n, y_n\}]$ and $[\{v_n, w_n\}]$ represent the same nonstandard branch. We want to show that, if $\langle x_n \rangle = \langle v_n \rangle$ a.e., then $\langle y_n \rangle = \langle w_n \rangle$ a.e. Suppose $\langle y_n \rangle \neq \langle w_n \rangle$ a.e. Then

$$\{n : x_n = v_n\} \cap \{n : y_n \neq w_n\} \in \mathcal{F}.$$

Thus, there is at least one n for which the three nodes $x_n = v_n$, y_n, and w_n are all incident to the same standard branch – in violation of the definition of a branch. Similarly, if all of $\langle x_n \rangle$, $\langle y_n \rangle$, $\langle v_n \rangle$, $\langle w_n \rangle$ are different a.e., then there would be a standard branch having four incident nodes – again a violation.

Next, we show that we have an equivalence relation for the set of all sequences of standard branches. Reflexivity and symmetry being obvious again, consider transitivity : Let $\mathbf{b} = [\{x_n, y_n\}]$, $\bar{\mathbf{b}} = [\{\bar{x}_n, \bar{y}_n\}]$, $\acute{\mathbf{b}} = [\{\acute{x}_n, \acute{y}_n\}]$, and assume that $\mathbf{b} = \bar{\mathbf{b}}$ and $\bar{\mathbf{b}} = \acute{\mathbf{b}}$. We want to show that $\mathbf{b} = \acute{\mathbf{b}}$. We have

$$N_{b\bar{b}} = \{n : \{x_n, y_n\} = \{\bar{x}_n, \bar{y}_n\}\} \in \mathcal{F}, \quad N_{\bar{b}\acute{b}} = \{n : \{\bar{x}_n, \bar{y}_n\} = \{\acute{x}_n, \acute{y}_n\}\} \in \mathcal{F},$$

Moreover,

$$N_{b\acute{b}} = \{n : \{x_n, y_n\} = \{\acute{x}_n, \acute{y}_n\}\} \supseteq (N_{b\bar{b}} \cap N_{\bar{b}\acute{b}}) \in \mathcal{F}.$$

Therefore, $N_{b\acute{b}} \in \mathcal{F}$. Thus, $\mathbf{b} = \acute{\mathbf{b}}$, as desired.

Finally, we define a **nonstandard graph** ${}^*G$ to be the pair ${}^*G = \{{}^*X, {}^*B\}$; we also write ${}^*G = [G_n]$. As an example, let all the G_n be the same one-way infinite path P. That path is an alternating sequence of nodes x_k and branches b_k :

$$P = \langle x_0, b_0, x_1, b_1, x_2, b_2, \ldots \rangle$$

where all the x_k and b_k are distinct and b_k is incident to x_k and x_{k+1} for every k. We can identify x_k with k. Then, the nonstandard graph ${}^*G = \{{}^*X, {}^*B\}$ has the hypernatural numbers as its nodes, and there is a nonstandard branch connected between each pair of consecutive hypernatural numbers. There are no other nonstandard branches. Note that the nonstandard nodes can be partitioned into galaxies, just as are the hypernatural numbers. Thus, there is no "next galaxy" after the first one consisting of the standard nodes : $x_0, x_1, x_2, \ldots$. In fact, between any two galaxies there is another galaxy.

In general, however, the graphs G_n may be arbitrary and may be different from each other so that the resulting nonstandard graph may have a far more complicated structure than does our simple example. Nevertheless, the nonstandard nodes can still be partitioned into galaxies whereby two such nodes are in the same galaxy if there is a finite nonstandard path connecting them.

A special case arises when almost all the G_n are (possibly different) finite graphs. In conformity with the terminology used for hyperfinite internal subsets of $^*R$, we will refer to the resulting nonstandard graph $^*G$ as a **hyperfinite graph**. As a result, we can lift many theorems concerning finite graphs to hyperfinite graphs. It is just a matter of writing the standard theorem in an appropriate form using symbolic logic and then applying the transfer principle ([9] has several such results). We let $^*G_f$ denote the set of hyperfinite graphs.

§3. NONSTANDARD TRANSFINITE GRAPHS

A "1-graph" is a transfinite graph of the first rank of transfiniteness. Let us briefly define it before turning to its nonstandard generalization. Let $G^0 = \{X^0, B\}$ be conventional graph containing at least one one-way infinite path. The nodes of x^0 will be called "0-nodes". We partition the set of all such paths into equivalence classes by taking two as equivalent if they are identical except for finitely many nodes and branches. Each such equivalence class is a "0-tip" for G^0.

Next, we partition the set of 0-tips for G^0 in an arbitrary fashion. To each set of that partition we may (or may not) assign a 0-node of X^0 with the proviso that, if a 0-node is assigned to a set of the partition, it is not assigned to any other set of the partition. Each set of the partition augmented by thus assigned 0-node (if such exists) is called a "1-node". It can be viewed as a connection among the infinite extremities of G^0 and possibly with a particular 0-node as well.

Thus, if G^0 has many infinite components, the 1-nodes serve as connections among the infinite extremities of those components, yielding thereby a transfinite graph. The resulting 1-graph is denoted by $G^1 = \{X^0, B, X^1\}$, where X^0 and B are the sets of 0-nodes and branches of G^0 and X^1 is the set of 1-nodes. We refer to the 0-tips and the assigned 0-nodes (those occurring in 1-nodes) as the **extremities** of the 0-graph $\{X^0, B\}$. If e and f are extremities in the same 1-node x^1 of G^1, we will say that e and f are **shorted together** by x^1 and will write $e \asymp f$.

We turn now to the definition of a nonstandard 1-graph. We start with a given sequence $\langle G_n^1 : n \in \mathbb{N}\rangle$ of 1-graphs $G_n^1 = \{X_n^0, B_n, X_n^1\}$. $G_n^0 = \{X_n^0, B\}$ is the 0-graph from which G_n^1 was constructed. Our next step is to make an ultrapower construction to get the nonstandard 1-nodes. We consider sequences of extremities of G_n^0, $\langle e_n\rangle$ being one such sequence and e_n being an extremity of G_n^0. Two such sequences $\langle e_n\rangle$ and $\langle f_n\rangle$ are taken to be equivalent if $e_n = f_n$ for almost all n. This partitions the set of all such sequences into equivalence classes. Indeed, reflexivity and symmetry are obvious, and transitivity follows as usual (i.e., if $e_n = f_n$ a.e. and if $f_n = g_n$ a.e., then $e_n = g_n$ a.e.). Each equivalence class is taken to be a nonstandard extremity $\mathbf{e} = [e_n]$ where $\langle e_n\rangle$ is any sequence in that equivalence class.

Given any sequence $\langle e_n\rangle$, let $N_{t^0} = \{n : e_n \text{ is a 0-tip}\}$ and $N_{x^0} = \{n : e_n \text{ is a 0-node}\}$. Thus, $N_{t^0} \cap N_{x^0} = \emptyset$ and $N_{t^0} \cup N_{x^0} = \mathbb{N}$. So, exactly one of N_{t^0} and N_{x^0} is a member of $\mathcal{F}$. If it is N_{t^0} (resp. N_{x^0}), $\langle e_n\rangle$ is a representative of a **nonstandard 0-tip** (resp. a **nonstandard 0-node**).

Now, let $\mathbf{e} = [e_n]$ and $\mathbf{f} = [f_n]$ be two nonstandard extremities, and let $\mathbb{N}_{ef} = \{n : e_n \asymp f_n\}$ and $\mathbb{N}_{ef}^c = \{n : e_n \not\asymp f_n\}$. Exactly one of $\mathbb{N}_{ef}$ and $\mathbb{N}_{ef}^c$ is a member of $\mathcal{F}$. If it is $\mathbb{N}_{ef}$ (resp. $\mathbb{N}_{ef}^c$), we say that $\mathbf{e}$ is **shorted** to $\mathbf{f}$ (resp. $\mathbf{e}$ is **not shorted** to $\mathbf{f}$), and we write $\mathbf{e} \asymp \mathbf{f}$ (resp. $\mathbf{e} \not\asymp \mathbf{f}$). Furthermore, we take it that every $\mathbf{e}$ is shorted to itself : $\mathbf{e} \asymp \mathbf{e}$. This shorting is an equivalence relation for the set of all nonstandard extremities, as can be shown much as before ; indeed, for transitivity, assume $\mathbf{e} \asymp \mathbf{f}$ and $\mathbf{f} \asymp \mathbf{g}$. Since

$$\{n : e_n \asymp f_n\} \cap \{n : f_n \asymp g_n\} \subseteq \{n : e_n \asymp g_n\},$$

we have $\mathbf{e} \asymp \mathbf{g}$. The resulting equivalence classes are the **nonstandard 1-nodes**.

This definition can be shown to be independent of the representative sequences chosen for the nonstandard extremities. To be specific, let $\mathbf{e} = [e_n] = [\bar{e}_n]$ and $\mathbf{f} = [f_n] = [\bar{f}_n]$. Set $\mathbb{N}_e = \{n : e_n = \bar{e}_n\} \in \mathcal{F}$ and $\mathbb{N}_f = \{n : f_n = \bar{f}_n\} \in \mathcal{F}$. Assume $[e_n] \asymp [f_n]$. Thus, $\mathbb{N}_{ef} = \{n : e_n \asymp f_n\} \in \mathcal{F}$. We want to show that $\mathbb{N}_{\bar{e}\bar{f}} = \{n : \bar{e} = \bar{f}\} \in \mathcal{F}$ and thus $[\bar{e}_n] \asymp [\bar{f}_n]$. We have $(\mathbb{N}_e \cap \mathbb{N}_f \cap \mathbb{N}_{ef}) \subseteq \mathbb{N}_{\bar{e}\bar{f}}$, whence our conclusion.

Altogether we have defined a nonstandard 1-node $\mathbf{x}^1$ to be any set in the partition of the set of nonstandard extremities induced by the shorting $\asymp$, with every nonstandard 1-node having at least one nonstandard 0-tip. With ${}^*X^1$ standing for the set of nonstandard 1-nodes, we define the **nonstandard 1-graph** ${}^*\mathcal{G}^1$ to be the triplet ${}^*\mathcal{G}^1 = \{{}^*X^0, {}^*B, {}^*X^1\}$.

Let us observe the each nonstandard 1-node $\mathbf{x}^1$ contains no more than one nonstandard 0-node, and, if it does contain such a 0-node, it does not share that nonstandard 0-node with any other nonstandard 1-node. Indeed, if $\mathbf{x}^1$ had two nonstandard 0-nodes, then, for at least one n, two 0-nodes in G_n^1 would have to be shorted together within a 1-node of G_n^1, a violation of the definition of standard 1-nodes. Our second observation follows in the same way because for no n will a 1-node in G_n^1 share a 0-node with another 1-node in G_n^1.

§4. NONSTANDARD 1-NETWORKS AND THEIR HYPERREAL CURRENT–VOLTAGE REGIMES

A 1-network $\mathbf{N}^1$ is a 1-graph where every branch b is assigned an orientation (that is, a direction from one of the nodes to another node). A resistor r_b is required to be a positive real number, whereas a voltage source e_b can be any real number, possibly 0. Also, b has a current i_b and a voltage v_b measured with respect to b's orientation in such a fashion that Ohm's law holds : $v_b + e_b = r_b i_b$.

To get a **nonstandard 1-network**, we first start with a sequence $\langle \mathbf{N}_n^1 : n \in \mathbb{N} \rangle$ of 1-networks, with each 1-network $\mathbf{N}_n^1$ having $G_n^1 = \{X_n^0, B_n, X_n^1\}$ for its 1-graph and with every branch b_n of G_n^1 having the parameters r_{b_n} and e_{b_n} as well as a current i_{b_n} and voltage v_{b_n}, as stated above. The currents and voltages of b_n are again measured with respect to a given orientation so that

$$v_{b_n} + e_{b_n} = r_{b_n} i_{b_n}. \tag{1}$$

We will now state a previously established theorem concerning the existence and uniqueness of the current i_{b_n} in every branch b_n of $\mathbf{N}_n^1$. To do this, we need to construct for each $\mathbf{N}_n^1$ a solution space $\mathcal{L}_n$ that will be searched for a unique branch-current vector satisfying a form of Tellegen's equation (see (3)). Below we let $\sum_{b_n \in B_n}$ denote the summation over all the branches b_n in B_n. $\mathcal{I}_n$ will denote the linear space over the field $\mathbb{R}$ of all finite-powered branch-current vectors in $\mathbf{N}_n^1$, so $i_n = \{i_{b_n} : b_n \in B_n\} \in \mathcal{I}_n$ whenever

$$\|i_n\|^2 = \sum_{b_n \in B_n} i_{b_n}^2 r_{b_n} < \infty.$$

We assign the norm $\|i_n\|$ to the members of $\mathcal{I}_n$ and make $\mathcal{I}_n$ a real Hilbert space with the inner product $(i_n, s_n) = \sum_{b_n \in B_n} i_{b_n} s_{b_n} r_{b_n}$.

The (unit) **loop current** for a given oriented loop L in $\mathbf{N}_n^1$ is a vector $i_n = \{i_{b_n} : b_n \in B_n\}$ of branch currents i_{b_n} such that $i_{b_n} = 1$ (resp. $i_{b_n} = -1$, resp. $i_{b_n} = 0$) if the branch b_n is in L with the same orientation as L (resp. b_n is in L with the opposite orientation, resp. b_n is not in L). If L is a 0-loop, the loop current for L is a member of $\mathcal{L}_n$, but, if L is a 1-loop, the loop current for L will be in $\mathcal{I}_n$ if and only if the sum of branch resistances r_{b_n} for the branches in L is finite.

Let $\mathcal{L}_n^o$ be the span of all the loop currents in $\mathcal{I}_n$. Finally, let $\mathcal{L}_n$ be the closure of $\mathcal{L}_n^o$ in $\mathcal{I}_n$: $\mathcal{L}_n$ is a subspace of $\mathcal{I}_n$ and is a Hilbert space by itself with the same norm and inner product as those of $\mathcal{I}_n$.

Next, let $e_n = \{e_{b_n} : b_n \in B_n\}$ be the vector of branch voltage sources in $\mathbf{N}_n^1$. We say that e_n is of **finite total isolated power** if

$$\sum_{b_n \in B_n} e_{b_n}^2 g_{b_n} < \infty, \tag{2}$$

where $g_{b_n} = 1/r_{b_n}$. We let $\mathcal{E}_{f,n}$ denote the set of all e_n satisfying (2).

We have the following fundamental theorem for $\mathbf{N}_n^1$. (See [6], Theorem 3.3-5 or [7], Theorem 5.2-8.)

Theorem 1. *If $e_n \in \mathcal{E}_{f,n}$, then there exists a unique branch-current vector $i_n \in \mathcal{L}_n$, such that*

$$\sum_{b_n \in B_n} r_{b_n} i_{b_n} s_{b_n} = \sum_{b_n \in B_n} e_{b_n} s_{b_n} \tag{3}$$

for every $s_n = \{s_{b_n} : b_n \in B_n\} \in \mathcal{L}_n$.

Note that the branch voltages $v_{b_n} = r_{b_n} i_{b_n} - e_{b_n}$ are also determined by this theorem.

We wish to obtain a nonstandard version of this theorem that is applicable to a nonstandard 1-network $^*\mathbf{N}^1$ obtained from the given sequence $\langle \mathbf{N}_n^1 \rangle$ of 1-networks through an ultrapower construction. Upon constructing the nonstandard 1-graph $^*G^1 = \{^*X^0, {}^*B, {}^*X^1\}$ from $\langle G_n^1 \rangle$ as in the preceding section, the branch parameters too undergo an ultrapower construction to become hyperreal parameters. Thus, each nonstandard branch $\mathbf{b}$ of $^*G^1$ has a hyperreal positive resistor $\mathbf{r_b}$ and possibly a nonzero hyperreal branch voltage source $\mathbf{e_b}$. In particular, for $\mathbf{b} = [\{x_n^0, y_n^0\}] \in {}^*B$, we have $\mathbf{r_b} = [r_{b_n}]$, where $r_{b_n} > 0$ is the resistance of the branch $b_n = \{x_n^0, y_n^0\}$ for almost all n, and similarly $\mathbf{e_b} = [e_{b_n}]$, where $e_{b_n} \in \mathbb{R}$ is the branch voltage source for b_n, again for almost all n. All this yields the **nonstandard 1-network** $^*\mathbf{N}^1 = [\mathbf{N}_n^1]$, whose 1-graph is $^*G^1$ and whose branch parameters are the hyperreals $\mathbf{r_b}$ and $\mathbf{e_b}$. Furthermore, i_{b_n} and v_{b_n} denote the branch current and the branch voltage for the branch $b_n \in B_n$, and these yield the hyperreal branch current $\mathbf{i}_b = [i_{b_n}]$ and the hyperreal branch voltage $\mathbf{v}_b = [v_{b_n}]$ for each $\mathbf{b} = [b_n] \in {}^*B$.

Every member $i_n = \{i_{b_n} : b_n \in B_n\}$ of any space $\mathcal{L}_n$ ($n \in \mathbb{N}$) determines a function mapping the set B_n of branches in $\mathbf{N}_n^1$ into $\mathbb{R}$, and thus by means of an ultrapower construction of $[i_n] = \mathbf{i} = \{\mathbf{i}_b : b \in {}^*B\}$ determines an internal function mapping ${}^*B$ into ${}^*\mathbb{R}$ with regard to the nonstandard network ${}^*\mathbf{N}_n^1$. In particular, $\mathbf{i}_b = [i_{b_n}]$, where $\{i_{b_n} : b_n \in B_n\}$ is a member of $\mathcal{L}_n$ for almost all n. All this yields a solution space ${}^*\mathcal{L} = [\mathcal{L}_n]$ consisting of the nonstandard current vectors $\mathbf{i} = \{\mathbf{i}_b : \mathbf{b} \in {}^*B\}$.

In order to invoke Theorem 1, we also assume that, for almost all n, the branch voltage sources e_{b_n} together have finite total isolated power (i.e., (2) is satisfied for almost all n). We let ${}^*\mathcal{E}_f$ denote the set of such nonstandard branch-voltage-source vectors ; that is, each member of ${}^*\mathcal{E}_f$ is a vector $\mathbf{e} = \{\mathbf{e_b} : \mathbf{b} \in {}^*B\}$, where $\mathbf{e_b} = [e_{b_n}]$ and the e_{b_n} satisfy (2) for almost all n. Then, Theorem 1 holds again for almost all n. For the nonstandard 1-network ${}^*\mathbf{N}^1$ this can be restated as follows :

Theorem 2. *If* $\mathbf{e} \in {}^*\mathcal{E}_f$, *then there exists a unique branch-current vector* $\mathbf{i} = \{\mathbf{i_b} : \mathbf{b} \in {}^*B\} \in {}^*\mathcal{L}$ *such that*

$$\sum_{\mathbf{b}\in {}^*B} \mathbf{r_b i_b s_b} = \sum_{\mathbf{b}\in {}^*B} \mathbf{e_b s_b} \tag{4}$$

for every $\mathbf{s} = \{\mathbf{s_b} : \mathbf{b} \in {}^*B\} \in {}^*\mathcal{L}$.

Each side of (4) is well-defined as the hyperreal having the sequence of real numbers given by (3) for a representative sequence. Note that the uniquely determined branch-current vector $\mathbf{i}$ determines a unique branch-voltage vector $\mathbf{v} = \{\mathbf{v_b} : \mathbf{b} \in {}^*B\}$ by means of Ohm's law :

$$\mathbf{v} = \mathbf{r_b i_b} - \mathbf{e_b}.$$

Theorem 2 could also have been obtained from Theorem 1 by appending asterisks in accordance with the transfer principle.

§5. KIRCHHOFF'S LAWS

Kirchhoff's laws can also be lifted in a nonstandard way for the current-voltage regime dictated by Theorem 2. First, consider Kirchhoff's current law. The nonstandard 0-node $\mathbf{x}^0 = [x_n^0]$ is called **maximal** if x_n^0 is maximal in $\mathbf{N}_n^1$ for almost all n (that is, if x_n^0 is not contained in any 1-node of $\mathbf{N}_n^1$). Below $\sum_{b_n \ni x_n^0}$ will denote a summation over all branches b_n that are incident at x_n^0. Also, x_n^0 is called **restraining** if the sum of the conductances $g_b = 1/r_b$ for the branches incident at x_n^0 is finite (in symbols, if $\sum_{b_n \ni x_n^0} g_{b_n} < \infty$). We say that $\mathbf{x}^0$ is **restraining** if x_n^0 is restraining for almost all n.

Under the assumptions on $\mathbf{N}_n^1$ required for Theorem 1, Kirchhoff's current law is satisfied at every restraining maximal 0-node x_n^0 as follows :

$$\sum_{b_n \ni x_n^0} \pm i_{b_n} = 0, \tag{5}$$

where the plus (resp. minus) sign is used if b_n is incident away from (resp. toward) x_n^0. Furthermore, (5) converges absolutely, as established in [6], Theorem 3.4-1 or [7], Theorem 5.3-1.

Turning to the nonstandard case, we first observe again that every branch b_n in $\mathbf{N}_n^1$ has an orientation. So, for $\mathbf{x}^0 = [x_n^0]$ and $\mathbf{b} = [b_n]$, every branch b_n incident at x_n^0 is either oriented away from x_n^0 a.e. or is oriented toward x_n^0 a.e. Thus, $\mathbf{b}$ acquires an orientation either away from $\mathbf{x}^0$ or toward $\mathbf{x}^0$. Also (5) holds for almost all n. We set

$$\sum_{\mathbf{b} \ni \mathbf{x}^0} \pm \mathbf{i}_\mathbf{b} = \left[\sum_{b_n \ni x_n^0} \pm i_{b_n} \right].$$

In this way, we get Kirchhofff's current law for ${}^*\mathbf{N}^1$:

Theorem 3. *If $\mathbf{x}^0$ is a restraining maximal 0-node in ${}^*\mathbf{N}^1$, then under the regime dictated by Theorem 2*

$$\sum_{\mathbf{b} \ni \mathbf{x}^0} \pm \mathbf{i}_\mathbf{b} = 0, \tag{6}$$

where the summation converges absolutely (i.e., $\sum_{\mathbf{b} \ni \mathbf{x}^0} |\mathbf{i}_\mathbf{b}| < \infty$).

Next, we discuss a nonstandard version of Kirchhoff's voltage law for ${}^*\mathbf{N}^1 = [\mathbf{N}_n^1]$. For this purpose we need to define nonstandard loops. Let ${}^*\mathbf{N}^1 = [\mathbf{N}_n^1]$ be a nonstandard 1-network, and let $G_{s,n}$ be a branch-induced subnetwork of $\mathbf{N}_n^1$. Then, the **relative degree** $d_x(G_{s,n})$ of a node x (0-node or 1-node) in $G_{s,n}$ is the cardinality of the set of branches and 0-tips in $G_{s,n}$ that are incident to x. Finally, a loop L (0-loop or 1-loop) in $\mathbf{N}_n^1$ is a connected subgraph $G_{s,n}$ having at least three branches and whose every node x has a relative degree equal to 2 (i.e., $d_x(G_{s,n}) = 2$ for every node x in $G_{s,n}$).

Any sequence $\langle G_{s,n} \rangle$ of subgraphs $G_{s,n}$ in the $\mathbf{N}_n^1$ determines a nonstandard subgraph ${}^*G_s$ of the nonstandard graph ${}^*G^1$ of ${}^*\mathbf{N}^1$ in the same way as $\langle G_n^1 \rangle$ determines the nonstandard 1-graph ${}^*G^1$ of ${}^*\mathbf{N}^1$. (A nonstandard branch $\mathbf{b} = [b_n]$ is in ${}^*G_s$ if and only if b_n is in $G_{s,n}$ for almost all n, and similarly for 0-nodes, 0-tips, and 1-nodes.) Then, ${}^*G_s$ is a nonstandard loop (0-loop or 1-loop) if, for almost all n, $G_{s,n}$ is connected, has at least three branches, and the relative degrees of all its relatively maximal nodes equal 2. In this case, we write $\mathbf{L} = {}^*G_s = [L_n]$, where $L_n = G_{s,n}$ is a loop in $\mathbf{N}_n^1$ for almost all n.

In the following, $\sum_{b_n \dashv L_n}$ denotes a sum over all the branches in the standard loop L_n. L_n is called **permissive** if $\sum_{b_n \dashv L_n} r_{b_n} < \infty$. Furthermore, we assign an orientation to each loop L_n. Under the regime dictated by Theorem 1, Kirchhoff's voltage law is satisfied around every permissive loop L_n in $\mathbf{N}_n^1$. In symbols,

$$\sum_{b_n \dashv L_n} \pm v_{b_n} = 0, \tag{7}$$

where the plus (resp. minus) sign is used if the orientations of b and L_n agree (resp. disagree). This too is a known result; see [6], Theorem 3.4-3 or [7], Theorem 5.3-4. With regard to the nonstandard case, the nonstandard loop $\mathbf{L} = [L_n]$ is called **permissive** if L_n is permissive for almost all n. Also, $\mathbf{L}$ acquires an orientation with regard to its nonstandard branches $\mathbf{b} = [b_n]$ in the following way. For almost all n, b_n is in L_n, and the orientation of b_n either agrees a.e or disagrees a.e with the orientation of L_n. So, if $\mathbf{v_b}$ is the hyperreal voltage of the nonstandard oriented branch $\mathbf{b}$ in $\mathbf{L}$, we have unambiguously the voltage $+\mathbf{v_b}$ or $-\mathbf{v_b}$ measured with respect to this implicitly defined orientation of $\mathbf{L}$. Upon setting

$$\sum_{\mathbf{b} \dashv \mathbf{L}} \pm \mathbf{v_b} = \left[\sum_{b_n \dashv L_n} \pm v_{b_n} \right],$$

we obtain the following nonstandard version of Kirchhoff's voltage law.

Theorem 4. *If* $\mathbf{L}$ *is an oriented permissive loop in* ${}^*\mathbf{N}^1$*, then under the regime dictated by Theorem 2*

$$\sum_{\mathbf{b} \dashv \mathbf{L}} \pm \mathbf{v_b} = 0, \tag{8}$$

where the summation converges absolutely (i.e., $\sum_{\mathbf{b} \dashv \mathbf{L}} |\mathbf{v_b}| < \infty$*)).*

Finally, let us note an immediate corollary. If x_n^0 is a 0-node of finite degree for almost all n, then $\mathbf{x}^0 = [x_n^0]$ is restraining. Also, if L_n is a finite 0-loop for almost all n, then $\mathbf{L} = [L_n]$ is permissive. It follows that Kirchhoff's laws will always hold for nonstandard 0-networks having hyperfinite graphs.

Corollary 5. If the nonstandard 0-network ${}^*\mathbf{N}^0$ has a hyperfinite graph, then Kirchhoff's laws are satisfied at all its nodes and around all its loops.

§6. A FINAL COMMENT

Finally, let us simply take note of the following nonlinear result obtained by transferring Duffin's theorem (see [1] or [8], Sec. 6.4). Let ${}^*\mathbf{N}^0 = [\mathbf{N}_n^0]$ be any nonstandard nonlinear 0-network such that its nonstandard graph ${}^*G^0 = [G_n^0]$ is hyperfinite and, for almost all n, the resistance characteristic $R_{b_n} : i_{b_n} \mapsto v_{b_n}$ for each branch $b_n \in B_n$ in G_n^0 is a continuous, strictly monotonically increasing bijection of $\mathbb{R}$ onto $\mathbb{R}$. Then, the hyperreal current-voltage regime for ${}^*\mathbf{N}^0$ is determined by Kirchhoff's current law at each nonstandard 0-node $\mathbf{x}^0 = [x_n^0]$, by Kirchhoff's voltage law around each nonstandard 0-loop, and by the replacement of Ohm's law by the expression $\mathbf{v_b} = \mathbf{R_b}(\mathbf{i_b})$ for each nonstandard branch $\mathbf{b} = [b_n]$, where now $\mathbf{R_b} = [R_{b_n}]$ is the internal nonlinear resistance characteristic, that is, $v_{b_n} = R_{b_n}(i_{b_n})$ for almost all n.

Резюме. Реализована идея нестандартной, трансконечной, линейной электрической сети. Чтобы доказать теорему существования и единственности для гипервещественного режима напряжения тока, понятие трансконечного графа должно быть распространено на нестандартные положения. Проверяются также нестандартные версии законов Хирчова. Наконец, указывается, что некоторые нелинейные сети также имеют единственные гипервещественные режимы напряжения тока.

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