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МАТЕМАТИКА

ПРЕДИСЛОВИЕ РЕДАКТОРА СЕРИИ

Настоящий выпуск представляет собой № 3 тематической серии, посвящённой близким предметам ИНТЕГРАЛЬНОЙ ГЕОМЕТРИИ (в смысле Бляшке–Сантало–Хадвигера) и СТОХАСТИЧЕСКОЙ ГЕОМЕТРИИ. Статьи в этой области, имеющие прикладную или статистическую природу, публикуются в ряде периодических изданий. Однако работы в той же области, посвящённые математическому анализу, несмотря на большой интерес к ним, не имеют определённого признанного места публикации. Настоящая тематическая серия быть может изменит эту ситуацию.

Список публикаций по Интегральной и Стохастической Геометрии в "Известиях Академии Наук Армении", серия Математика довольно значителен. В частности, не считая тематической серии, имеются три специальных выпуска (4-ые номера за 1994, 1996 и 1998 годы) под общим названием "Аналитические Результаты Комбинаторной Интегральной Геометрии". Эти три выпуска заслужили самых высоких оценок (так, профессор Дитер Баум из Трирского Университета писал : "Букет блестящих, впечатляющих аналитических результатов ... очень важные, успешные и глубокие математические исследования"). Эти выпуски составляли хорошую начальную основу для тематической серии.

Выпуски тематической серии не имеют строгого графика. Их публикация зависит от наличия и готовности материала. Предусматривается строгая процедура рецензирования. Планируются несколько дополнительных сборников статей, представляющих лучшие исследования по Интегральной и Стохастической Геометрии, ранее опубликованных в нашем журнале.

Редакционная коллегия надеется на положительный резонанс интернационального математического сообщества, включая представление статей к публикации, участие в процессе рецензирования и т.д.. Журнал готов к любому сотрудничеству, направленному на улучшение качества тематической серии.

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РАДИУСЫ КРИВИЗНЫ ПЛОСКИХ ПРОЕКЦИЙ ВЫПУКЛЫХ ТЕЛ В $\mathbb{R}^n$

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Резюме. В статье изучаются радиусы кривизны плоских проекций центрально-симметричных гладких выпуклых тел в $\mathbb{R}^n$. Среди результатов имеются некоторые новые формулы, включающие опорную функцию и порождающую плотность выпуклых тел. Они обсуждаются в контексте гипотезы Вейля о зональной характеризации зоноидов в пространствах чётной размерности.

§1. ВВЕДЕНИЕ

В работе рассматриваются центрально-симметричные выпуклые тела $\mathbf{K} \subset \mathbb{R}^n$, в частности, **зонотопы** (конечные суммы Минковского прямолинейных отрезков) и **зоноиды** (пределы зонотопов в метрике Хаусдорфа).

Некоторые аналитические характеризации зоноидов в терминах их опорных функций получены Амбарцумяном [2], Леви [19], Шоке [11], Гуди, Лутвак, Вейль, [15] и Вейль [28], [29].

Задача дифференциально-геометрической характеризации зоноидов впервые была поставлена Бляшке [8] и позднее переформулирована многими авторами (Болкер [10], Вейль и Шнайдер [26], Гуди и Вейль [16]). В случае многогранников задача была решена Александровым [1], Болкером [9], Шнайдером [25], Бляшке [7]. Вейлем [27] было показано, что не существует локальной характеризации зоноидов.

Одним из основных результатов теории является уравнение

$$\sum_{i=1}^{n-1} R_i(\omega) = 2 \int_{\mathbf{S}_\omega} h_\omega(u)\, \lambda_{n-2}(du), \tag{1.1}$$

связывающее радиусы главных кривизн $R_i(\omega)$ на поверхности тела $\partial\mathbf{K}$ в точке с нормалью $\omega \in \mathbf{S}^{n-1}$ ($\mathbf{S}^{n-1}$ обозначает единичную $(n-1)$–мерную сферу в $\mathbb{R}^n$), причём

$$h_\omega(u) = \quad \text{сужение порождающей плотности} \quad h(\Omega) \quad \text{тела} \quad \mathbf{K} \quad \text{на} \quad \mathbf{S}_\omega.$$

Подчеркнём, что $h(\Omega)$ может принимать как положительные, так и отрицательные значения. В (1.1), $\lambda_{n-2}(du)$ – сферическая мера Лебега на $\mathbf{S}_\omega$ = большая $(n-2)$-подсфера с полюсом $\omega \in \mathbf{S}^{n-1}$. Обобщённая форма уравнения (1.1), не требующая дополнительных условий гладкости, полученная в [14], использует понятие порождающих распределений. В действительности, (1.1) обобщает известный результат Бляшке [8] для $n = 3$ на выпуклые тела в пространстве $\mathbb{R}^n$. Целью настоящей работы является описание подхода к исследованию уравнения (1.1), основанному на новом выражении для радиусов кривизны проекций. Для двух различных взаимно перпендикулярных направлений $\omega, \xi \in \mathbf{S}^{n-1}$, $\omega \perp \xi$ определим $R(\omega, \xi)$ = радиус кривизны $\partial\mathbf{K}(\omega, \xi)$ в точке, с направлением нормали ω, где $\mathbf{K}(\omega, \xi)$ = проекция $\mathbf{K}$ на плоскость, содержащую начало координат и направления ω и ξ. Это выражение имеет вид

$$R(\omega, \xi) = 2 \int_{\mathbf{S}_\omega} \cos^2(\widehat{u, \xi}) h_\omega(u)\, \lambda_{n-2}(du). \tag{1.2}$$

Отметим, что интеграл

$$\int_{\mathbf{S}_\omega} \cos^2(\widehat{u, \xi}) h_\omega(u)\, \lambda_{n-2}(du)$$

часто используется в теории выпуклости, например, Вейль [30], где неотрицательность интеграла играет существенную роль. Выражение (1.2) даёт ясную геометрическую интерпретацию этого интеграла.

Уравнение (1.1) даёт начало рассмотрению задачи Функа на $\mathbf{S}^{n-1}$ для порождающей плотности $h(\Omega)$. Решение этой задачи имеет различную природу для чётных и нечётных значений n (см. Хелгасон [18]). Для чётного n и гладкой границы $\partial\mathbf{K}$ имеем (см. Теорему 4.3)

$$h(\Omega) = c_n P_n(L)\Big(\int_{\mathbf{S}_\Omega} \sum_{i=1}^{n-1} R_i(\omega)\, \lambda_{n-2}(d\omega)\Big), \tag{1.3}$$

где интегрирование распространяется на $(n-2)$-мерный экватор с полюсом Ω, L – оператор Лапласа–Бертрами, действующий на $\mathbf{S}^{n-1}$, P_n – некоторый многочлен, а c_n – постоянная. Мы получаем характеризационное условие для зоноидов : правая сторона уравнения (1.2) должна быть неотрицательной для всех $\Omega \in \mathbf{S}^{n-1}$. Аналогичный вопрос рассматривался в [14] в более общей формулировке.

Очевидно, что для чётных размерностей, из (1.3) следует, что значение порождающей плотности $h(\Omega)$ в направлении Ω зависит только от поведения поверхности $\partial\mathbf{K}$ внутри произвольной узкой полосы, соответствующей экваториальной зоне на $\mathbf{S}^{n-1}$ с полюсом в Ω. Это замечание подтверждает (в гладком случае) гипотезу Вейля [18] о зональной характеризации зоноидов для чётных размерностей, доказанную ранее Паниной в [22], Гуди и Вейлем в [14]. Возможность использования аппроксимации гладкими телами для получения подтверждения гипотезы Вейля в общем случае остаётся интересной открытой проблемой.

§2. ОБОЗНАЧЕНИЯ И ПРЕДВАРИТЕЛЬНЫЕ РЕЗУЛЬТАТЫ

Обозначим через $\mathbb{R}^n$ n-мерное евклидово пространство, через $\mathbf{S}^{n-1}$ – единичную сферу в $\mathbb{R}^n$, а через $\mathcal{K}_o^n$ – класс выпуклых тел $\mathbf{K} \subset \mathbb{R}^n$, имеющих центр симметрии в начале координат $O \in \mathbb{R}^n$. Выпуклое тело $\mathbf{K} \in \mathcal{K}_o^n$ единственным образом определяется своей опорной функцией $H(\xi) : \mathbf{S}^{n-1} \to \mathbb{R}^1$. Напомним, что $H(\xi) = \max\{(y,\xi) : y \in \mathbf{K}\}$, где $\xi \in \mathbf{S}^{n-1}$, $(\cdot,\cdot)$ обозначает стандартное скалярное произведение в $\mathbb{R}^n$. Отметим, что можно однородно продолжить определение функции $H(\xi)$ на $\mathbb{R}^n : H(x) = |x|\, H(\xi)$, где $x = |x|\,\xi,\ \ \xi \in \mathbf{S}^{n-1}$.

Ниже $C_e^n(\mathbf{S}^{n-1})$ обозначает пространство n раз непрерывно дифференцируемых чётных функций в $\mathbf{S}^{n-1}$. Пусть λ_{n-1} – мера Лебега на $\mathbf{S}^{n-1}$ и $\sigma_{n-1} = \lambda_{n-1}(\mathbf{S}^{n-1})$. Выпуклое тело $\mathbf{K} \in \mathcal{K}_o^n$ назовём k-гладким, если $H(\xi) \in C_e^k(\mathbf{S}^{n-1})$. Рассмотрим выпуклые тела в $\mathbb{R}^n$ с положительной кривизной Гаусса–Кронекера $k_1(\omega)k_2(\omega)\cdots k_{n-1}(\omega) > 0$, где $k_i(\omega)$ суть главные кривизны в точке поверхности с направлением нормали ω (см. Сантало [23]).

Опорная функция произвольного $(n+3)$-гладкого выпуклого тела $\mathbf{K} \in \mathcal{K}_o^n$ имеет представление (см. [7], [16])

$$H(\xi) = \int_{\mathbf{S}^{n-1}} \mid (\xi,\Omega) \mid h(\Omega)\, \lambda_{n-1}(d\Omega), \quad \xi \in \mathbf{S}^{n-1} \tag{2.1}$$

с некоторой единственной, необязательно неотрицательной, чётной функцией $h(\Omega)$.

Ниже $h(\Omega)$ назовем **порождающей плотностью** тела $\mathbf{K}$. Выпуклые тела, опорная функция которых имеет представление (2.1) с четнозначной мерой $h(\Omega)\lambda_{n-1}(d\Omega)$ называются обобщёнными зоноидами. Если $h(\Omega) \geq 0$, то соответствующее выпуклое тело является зоноидом.

§3. РАДИУСЫ КРИВИЗНЫ ПРОЕКЦИЙ

Для $\omega,\ \xi \in \mathbf{S}^{n-1}$ обозначим :

$e(\omega,\xi)$ = 2-мерная плоскость, содержащая начало координат в $\mathbb{R}^n$ и сферические направления (ξ,ω), $\xi \neq \omega$,
(ω,ξ) = угол между двумя направлениями ξ и ω,
$\mathbf{K}(\omega,\xi)$ = проекция $\mathbf{K} \in \mathcal{K}_o^n$ на $e(\omega,\xi)$,
$R(\omega,\xi)$ = радиус кривизны $\partial\mathbf{K}(\omega,\xi)$ в точке, с направлением нормали ω.
Так как $R(\omega,\xi_1) = R(\omega,\xi_2)$, где $\omega,\xi_1,\xi_2 \in \mathbf{S}^{n-1}$ и $\xi_2 \in e(\omega,\xi_1)$, то в случае необходимости будем предполагать, что ξ ортогонально ω. Частный случай $n=3$ нижеследующей теоремы был получен автором в [5]. Другое представление для $H(\xi)$ в $\mathbb{R}^n$ было найдено Паниной [21].

Теорема 3.1. *Опорная функция 2-гладкого выпуклого тела* $\mathbf{K} \in \mathcal{K}_o^n$ *имеет следующее представление :*

$$H(\xi) = \frac{1}{2\sigma_{n-2}} \int_{\mathbf{S}^{n-1}} \frac{R(\omega,\xi)}{\sin^{n-3}(\omega,\xi)}\, \lambda_{n-1}(d\omega), \quad \xi \in \mathbf{S}^{n-1}. \tag{3.1}$$

Доказательство. Пусть $u \in \mathbf{S}^{\xi}$ – направление, перпендикулярное $\xi \in \mathbf{S}^{n-1}$. Приблизим $\mathbf{K}(u,\xi) \subset e(\omega,\xi)$ изнутри многоугольниками, имеющими вершины на поверхности $\partial\mathbf{K}(u,\xi)$. Стороны аппроксимирующего многоугольника обозначим через a_i, а направление нормали к a_i, лежащей в плоскости $e(u,\xi)$, через ν_i (ν_i – угол между направлением нормали и ξ). $H_{\mathbf{K}(u,\xi)}$ – опорная функция $\mathbf{K}(u,\xi)$ в плоскости $e(u,\xi)$. Имеем

$$4H(\xi) = 4H_{\mathbf{K}(u,\xi)}(\xi) = \lim \sum_i |a_i| \sin(\xi,\nu_i) =$$

$$= \lim \sum_i R_u(\nu_i)|\nu_{i+1}-\nu_i| \sin(\xi,\nu_i) = 2\int_0^{\pi} R_u(\nu)\sin\nu\, d\nu, \tag{3.2}$$

где $R_u(\nu)$ – радиус кривизны $\mathbf{K}(u,\xi)$ в точке с нормалью ν. Интегрируя обе части (3.2) по $\lambda_{n-2}(du)$ по сфере $\mathbf{S}_\xi$, используя стандартную формулу $\lambda_{n-1}(d\omega) = \sin^{n-2}\nu\, d\nu\, \lambda_{n-2}(du)$, где $\omega = (\nu,u)$, получим

$$2\sigma_{n-2}H(\xi) = \int_{\mathbf{S}_\xi}\int_0^{\pi} R_u(\nu)\sin\nu\, d\nu\, \lambda_{n-2}(du) =$$

$$= \int_{\mathbf{S}_\xi}\int_0^{\pi} \frac{R_u(\nu)}{\sin^{n-3}\nu} \sin^{n-2}\nu\, d\nu\, \lambda_{n-2}(du) = \int_{\mathbf{S}^{n-1}} \frac{R(\omega,\xi)}{\sin^{n-3}(\omega,\xi)}\, \lambda_{n-1}(d\omega).$$

Заметим, что заменяя в (3.1) $2H(\cdot)$ функцией ширины $W(\cdot)$, получим формулу для функции ширины для всех выпуклых тел (не только центрально-симметричных).

В Теореме 3.2 рассматривается трансляционно-инвариантная мера μ в пространстве гиперплоскостей $\mathbf{E}$ в $\mathbb{R}^n$ и применяя факторизацию (см. Амбарцумян [3]) : $\mu(de) = dp \times \lambda_{n-1}(d\xi)$, где p – расстояние от плоскости e до начала координат в $\mathbb{R}^n$, а $\xi \in \mathbf{S}^{n-1}$ – направление нормали к e.

Теорема 3.2. *Для 2-гладкого выпуклого тела* $\mathbf{K}$ *имеем*

$$\mu([\mathbf{K}]) = \frac{1}{n-1} \int_{\mathbf{S}^{n-1}} \sum_{i=1}^{n-1} R_i(\omega)\ \lambda_{n-1}(d\omega), \tag{3.3}$$

где $[\mathbf{K}] = \{e \in \mathbf{E} : e \cap \mathbf{K} \neq \emptyset\}$, *а* $R_i(\omega)$ *– главные радиусы кривизны поверхности* $\partial\mathbf{K}$ *в точке с направлением нормали* ω.

Доказательство Теоремы 3.2 опирается на следующий результат.

Пусть $E(\omega)$ – гиперплоскость в $\mathbb{R}^n$ с направлением нормали ω, содержащая $t(\omega)$ = точка на $\mathbf{S}^{n-1} \in \mathbb{R}^n$ с направлением нормали ω. Пусть X_i, $i = 1, .., n-1$ – главные направления в точке поверхности $\partial\mathbf{K}$ с направлением нормали ω. В $E(\omega)$ рассмотрим ортогональную систему координат с направлениями осей X_i, $i = 1, .., n-1$. Пусть $\xi = (x_1, .., x_{n-1})$ – соответствующие координаты.

Теорема 3.3. *Для 2-гладкого выпуклого тела* $\mathbf{K}$, *и любых* $\omega \in \mathbf{S}^{n-1}$ *и* $\xi \in \mathbf{S}_\omega$ *имеем*

$$R(\omega, \xi) = \sum_{i=1}^{n-1} R_i(\omega)\, x_i^2, \tag{3.4}$$

где $R_i(\omega)$ *суть главные радиусы кривизны поверхности* $\partial\mathbf{K}$ *в точке с направлением нормали* ω.

Доказательство Теоремы 3.3. Опорная функция тела $\mathbf{K}(\omega, \xi)$ является сужением $H(\xi)$ (опорной функции тела $\mathbf{K}$) на окружность $\mathbf{S}^{n-1} \cap e(\omega, \xi)$. Направление ϕ в плоскости $e(\omega, \xi)$ отождествляется с углом между ϕ и ω. Пусть $t(\omega)$ – точка на окружности с направлением нормали ω. Рассмотрим сужение опорной функции $H(\xi)$ на луч $r(t(\omega), \xi)$, исходящий из точки $t(\omega)$ и имеющий направление ξ. Пусть $x(\phi)$ – точка пересечения луча $r(t(\omega), \xi)$ с лучом $r(O, \phi)$, исходящим из O и имеющим направление ϕ. Для расстояния $\rho = |x(\phi) - t(\omega)|$ имеем $\rho = \tan\phi$. Для луча $r(t(\omega), \xi)$ при фиксированном $\xi \in \mathbf{S}_\omega$ определим функцию

$$W(\rho) = H(x(\phi)) = |x(\phi)|\, H(\phi) = \frac{1}{\cos\phi} H(\phi).$$

Дифференцируя по ρ, получим

$$W'(\rho) = \left[\frac{H(\phi)}{\cos\phi}\right]'_\phi \cdot \phi'_\rho = H'(\phi)\cos\phi + H(\phi)\,\sin\phi,$$

$$W''(\rho) = [H''(\phi)\cos\phi - H'(\phi)\sin\phi + H'(\phi)\sin\phi + H(\phi)\cos\phi]\cos^2\phi.$$

Согласно Бляшке [8] имеем

$$W''(\rho)\,|_{\rho=0} = H''(\phi) + H(\phi)\,|_{\phi=0} = R(\omega, \xi). \tag{3.5}$$

Изменяя ξ в $\mathbf{S}_\omega$, распространим функцию W на гиперплоскость $E(\omega)$. Дифференцируя W в направлении $\xi = (x_1, ..., x_{n-1})$ в гиперплоскости $E(\omega)$, получаем следующее разложение :

$$W'_\xi = W'_1 \cdot x_1 + \cdots + W'_{n-1} \cdot x_{n-1},$$

$$W''_{\xi\xi} = \sum_{i=1}^{n-1} W''_{ii} x_i^2 + 2 \sum_{\substack{i \leq j \\ i,j=1,..,n-1}} W''_{ij} x_i x_j, \tag{3.6}$$

где W'_i – производная по направлению X_i.

Можно доказать, что $W''_{ij}(t(\omega)) = 0$, $i \neq j$. Докажем утверждение для случая $n = 3$. Действительно, пусть при $n = 3$ (ν, φ) – стандартные сферические координаты относительно полюса $t(\omega)$. Имеем

$$W(x_1, x_2) = \sqrt{1 + x_1^2 + x_2^2}\; H(\nu, \varphi),$$

где $\tan \nu = \sqrt{x_1^2 + x_2^2}$ и $\cos\varphi = \frac{x_1}{\sqrt{x_1^2+x_2^2}}$. Следовательно, $W''_{12}(0,0) = 0$. Принимая во внимание (3.6), (3.5) и $R_i(\omega) = R(\omega, x_i)$, получаем (3.4). Отметим, что при $n = 3$ формула (3.4) совпадает с формулой для радиусов кривизны проекций (см. Бляшке [8]).

Следствие 3.1. *Для 2-гладкого выпуклого тела* $\mathbf{K}$ *и для каждого* $\omega \in \mathbf{S}^{n-1}$ *имеем*

$$\int_{\mathbf{S}_\omega} R(\omega, \xi)\, \lambda_{n-2}(d\xi) = a_n \sum_{i=1}^{n-1} R_i(\omega), \tag{3.7}$$

где

$$a_n = \sigma_{n-3} \int_0^\pi \cos^2 \nu \, \sin^{n-3} \nu \, d\nu \tag{3.8}$$

является постоянной, зависящей от n.

Доказательство. Интегрируя обе части (3.4) относительно $\lambda_{n-2}(d\xi)$ по $\mathbf{S}_\omega$ и используя соотношение

$$a_n = \int_{\mathbf{S}_\omega} x_i^2 \, \lambda_{n-2}(d\xi) = \int_{\mathbf{S}^{n-2}} \cos^2(u, \xi)\, \lambda_{n-2}(d\xi) = \sigma_{n-3} \int_0^\pi \cos^2 \nu \, \sin^{n-3} \nu \, d\nu,$$

где $u \in \mathbf{S}^{n-2}$ – произвольное направление, получаем (3.7).

Доказательство Теоремы 3.2. Интегрирование (3.1) относительно меры Лебега $\lambda_{n-2}(d\xi)$ по сфере $\mathbf{S}^{n-1}$, даёт

$$\mu([\mathbf{K}]) = \frac{1}{2\sigma_{n-2}} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \frac{R(\omega, \xi)}{\sin^{n-3}(\omega, \xi)} \, \lambda_{n-1}(d\omega) \lambda_{n-1}(d\xi). \tag{3.9}$$

В сферических координатах относительно ω положим $\xi = (\tau, u)$ (где $\tau \in (0,\pi)$, $u \in \mathbf{S}_\omega$). Используя теорему Фубини и учитывая, что $\lambda_{n-1}(d\xi) = \sin^{n-2}\tau\, d\tau\, \lambda_{n-2}(du)$ и $R(\omega,\xi) = R(\omega,u)$, получим

$$\mu([\mathbf{K}]) = \frac{1}{2\sigma_{n-2}} \int_{\mathbf{S}^{n-1}} \left[\int_0^\pi \sin\tau\, d\tau \times \int_{\mathbf{S}^{n-2}} R(\omega,u)\,\lambda_{n-2}(du)\right] \lambda_{n-1}(d\omega). \quad (3.10)$$

Подставляя (3.7) в (3.10), получаем

$$\mu([\mathbf{K}]) = \frac{a_n}{\sigma_{n-2}} \int_{\mathbf{S}^{n-1}} \sum_{i=1}^{n-1} R_i(\omega)\,\lambda_{n-1}(d\omega),$$

где (см. (3.8))

$$\frac{a_n}{\sigma_{n-2}} = \frac{\sigma_{n-3}\int_0^\pi \cos^2\nu \sin^{n-3}\nu\, d\nu}{\sigma_{n-3}\int_0^\pi \sin^{n-3}\nu\, d\nu} = \frac{1}{n-1}.$$

Теорема 3.2 доказана.

Отметим, что при $n = 3$ формула (3.3) совпадает с известной формулой Минковского для выпуклых тел в $\mathbb{R}^3$ (см. [20]). Другим методом Теорема 3.2 впервые была доказана Паниной [21].

§4. ПОРОЖДАЮЩАЯ ПЛОТНОСТЬ

Теорема 4.1. *Для любого $(n+3)$-гладкого тела $\mathbf{K} \in \mathcal{K}_o^n$ и $\xi, \omega \in \mathbf{S}^{n-1}$, $\xi \perp \omega$, имеем*

$$R(\omega,\xi) = 2\int_{\mathbf{S}_\omega} \cos^2(u,\xi) h_\omega(u)\,\lambda_{n-2}(du), \quad (4.1)$$

где $h_\omega(u)$ – сужение порождающей плотности $h(\Omega)$ тела $\mathbf{K}$ на $\mathbf{S}_\omega$.

Для доказательства Теоремы 4.1 требуется некоторое специальное представление для элемента $(n-1)$-мерной лебеговой меры на $\mathbf{S}^{n-1}$. Пусть в полярных координатах $x = (\rho, \omega)$, где $x \in \mathbb{R}^n$, $\rho \in \mathbb{R}^+$, $\omega \in \mathbf{S}^{n-1}$. Для заданной ортонормальной системы $(z_1, z_2, x_1, x_2, ..., x_{n-2})$ в $\mathbb{R}^n$ представим $\omega \in \mathbf{S}^{n-1}$ в виде : $\omega = (\nu, \varphi, u)$, где ν – угол между ω и $e(z_1,z_2)$, φ – угол между z_1 и проекцией ω на $e(z_1,z_2)$, u – направление проекции ω на $(n-2)$-мерное подпространство, содержащее оси $x_1, x_2, ..., x_{n-2}$.

Лемма 4.1. *При $n \geq 4$ имеем*

$$\lambda_{n-1}(d\omega) = \sin^{n-3}\nu \cos\nu\, d\nu\, d\varphi\, \lambda_{n-3}(du). \quad (4.2)$$

Доказательство. Пусть $x = (z_1, z_2, x_1, ..., x_{n-2}) \in \mathbb{R}^n$ имеет полярное представление $x = (\rho, \nu, \varphi, \theta_{n-2}, ..., \theta_2)$, где θ_{n-2} – угол между осью x_{n-2} и проекцией

$\bar{x}$ точки x на $(n-2)$-плоскость, содержащую оси $x_1, ..., x_{n-2}$. Через θ_{n-3} обозначим угол между осью x_{n-3} и проекцией $\bar{x}$ на $(n-3)$-плоскость, содержащую оси $x_1, ..., x_{n-3}$. Пусть $\theta_{n-4}, \theta_{n-5}, ..., \theta_2$ имеют тот же смысл. Имеем

$$\begin{aligned} z_1 &= \rho \cos\nu \cos\varphi \\ z_2 &= \rho \cos\nu \sin\varphi \\ x_{n-2} &= \rho \sin\nu \cos\theta_{n-2} \\ x_{n-3} &= \rho \sin\nu \sin\theta_{n-2}\cos\theta_{n-3} \\ &\ldots\ldots \\ x_2 &= \rho \sin\nu \sin\theta_{n-2}\cdots\sin\theta_3\cos\theta_2 \\ x_1 &= \rho \sin\nu \sin\theta_{n-2}\cdots\sin\theta_3\sin\theta_2. \end{aligned}$$

Соответствующий Якобиан имеет вид :

$$J = \rho^{n-1} \sin^{n-2}\nu\cos\nu\sin^{n-4}\theta_{n-2}\cdots\sin^2\theta_4\sin\theta_3.$$

Учитывая, что $\lambda_{n-3}(du) = \sin^{n-4}\theta_{n-2}\cdots\sin^2\theta_4\sin\theta_3\, d\theta_{n-2}\cdots d\theta_2$, где $u = (\theta_{n-2}, ..., \theta_2)$, приходим к (4.2).

Доказательство Теоремы 4.1. Через $\mathbf{K}(\omega,\xi)$ обозначим проекцию $\mathbf{K}$ на 2-плоскость $e(\omega,\xi)$. Опорная функция $\mathbf{K}(\omega,\xi)$ является сужением $H(\xi)$ на окружность $\mathbf{S}^{n-1}\cap e(\omega,\xi)$. Рассмотрим координатную систему $(z_1, z_2, x_1, ..., x_{n-2})$, где (z_1, z_2) изменяются в $e(\omega,\xi)$, а z_1 на ω. Измеряя ϕ как и выше, получаем $\omega = \omega(\phi) = (\cos\phi, \sin\phi, 0, ..., 0)$. Согласно Бляшке [8] имеем

$$R(\omega,\xi) = R(\phi) = H(\phi) + H''(\phi)\,|_{\phi=0}, \tag{4.3}$$

где $H(\phi) = H(\omega(\phi))$. Из (2.1) получаем

$$H(\phi) = \int_{\mathbf{S}^{n-1}} |\,(\omega(\phi),\Omega)\,|\; h(\Omega)\,d\Omega = 2\int_{\{\Omega:\ (\omega,\Omega)\geq 0\}} (\Omega_1\cos\phi + \Omega_2\sin\phi)\,h(\Omega)\,d\Omega, \tag{4.4}$$

где $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)$. Дифференцируя, получим

$$H'(\phi) = 2\int_{\{\Omega:\ (\omega,\Omega)\geq 0\}} (-\Omega_1\sin\phi + \Omega_2\cos\phi)\,h(\Omega)\,d\Omega.$$

Пусть $\Omega = (\nu,\varphi,\delta)$, где $\delta\in\mathbf{S}^{n-3}$, ν – угол между Ω и $e(\omega,\xi)$, φ – направление в $e(\omega,\xi)$ (см. (4.2)). Используя (4.2), имеем

$$H''(\phi) = 2\int_{\{\Omega:\ (\omega,\Omega)\geq 0\}} (-\Omega_1\cos\phi - \Omega_2\sin\phi)\,h(\Omega)\,d\Omega +$$

$$+2\int_{\mathbf{S}_{\omega(\phi)}}(-\Omega_1\sin\phi+\Omega_2\cos\phi)\,h(\Omega(\nu,\phi+\frac{\pi}{2},\delta))\,\sin^{n-3}\nu\,\cos\nu\,d\nu\lambda_{n-3}(d\delta).$$

Подставляя в (4.3), получаем

$$R(\omega,\xi)=2\int_{\mathbf{S}_\omega}\Omega_2\,h(\Omega(\nu,\frac{\pi}{2},\delta))\,\sin^{n-3}\nu\,\cos\nu\,d\nu\lambda_{n-3}(d\delta).$$

Учитывая, что $\sin^{n-3}\nu\,d\nu\,\lambda_{n-3}(d\delta)=\lambda_{n-2}(du)$ где $u=(\nu,\delta)$, $u\in\mathbf{S}_\omega$ и $\Omega_2=\cos\nu=\cos(u,\xi)$, получим (4.1). Доказательство завершено.

Теорема 4.2. *Для каждого $(n+3)$-гладкого выпуклого тела $\mathbf{K}\in\dot{\mathcal{K}}_o^n$ имеем*

$$\sum_{i=1}^{n-1}R_i(\omega)=2\int_{\mathbf{S}_\omega}h_\omega(u)\,\lambda_{n-2}(du),\tag{4.5}$$

где $h_\omega(u)$ – сужение порождающей плотности $h(\Omega)$ тела $\mathbf{K}$ на $\mathbf{S}_\omega$, $R_i(\omega)$ – главные радиусы кривизны в точке на поверхности $\mathbf{K}$ с направлением нормали ω.

Доказательство. В $(n-1)$-мерной плоскости с направлением нормали $\omega\in\mathbf{S}^{n-1}$ рассмотрим координатную систему $\{x_i\}$, где x_i – главные направления в точке поверхности $\partial\mathbf{K}$ с направлением нормали ω. Согласно (4.1) имеем

$$R_i(\omega)=R(\omega,X_i)=2\int_{\mathbf{S}_\omega}\cos^2(u,X_i)\,h_\omega(u)\,\lambda_{n-2}(du)=2\int_{\mathbf{S}_\omega}x_i^2\,h_\omega(u)\,\lambda_{n-2}(du),$$

где $u=(x_1,..,x_{n-1})$. Следовательно,

$$\sum_{i=1}^{n-1}R_i(\omega)=2\int_{\mathbf{S}_\omega}\sum_{i=1}^{n-1}x_i^2\,h_\omega(u)\,\lambda_{n-2}(du)=2\int_{\mathbf{S}_\omega}h_\omega(u)\,\lambda_{n-2}(du).$$

Доказательство завершено.

Представляет интерес обобщить представление (4.5) на симметрические многочлены от $R_i(\omega)$ высших порядков.

Заметим, что в случае $n=3$ результат Теоремы 4.2 можно найти в [8] и [2]. Обобщённую форму (4.5) можно найти в [14], где используется понятие порожденных распределений.

Теорема 4.2 сводит задачу реконструкции порождающей плотности к задаче Функа в $\mathbf{S}^{n-1}$. Задача Функа в $\mathbf{S}^{n-1}$ была исследована Хелгасоном [18], Гельфандом, Граевым и Роусом [13], Гринбергом [17]. В чётном случае задача Функа решается с помощью оператора $P_n(L)$, причём $P_n(z)$ является многочленом

$$P_n(z)=[z-(n-3)]\,[z-3\,(n-5)]\cdots[z-(n-3)]$$

степени $\frac{n-2}{2}$, а L – оператор Лапласа-Бельтрами в $\mathbf{S}^{n-1}$ (см. [18]).

Теорема 4.3. *Пусть n – чётное число. Порождающая плотность $(n+2)$-гладкого выпуклого тела $\mathbf{K} \in \mathcal{K}_o^n$ имеет представление вида*

$$h(\Omega) = c_n P_n(L) \left(\int_{\mathbf{S}_\Omega} \sum_{i=1}^{n-1} R_i(\omega)\, \lambda_{n-2}(d\omega) \right), \tag{4.6}$$

где постоянная

$$c_n = \frac{\sqrt{\pi}}{4\sigma_{n-2}\Gamma(\frac{n-1}{2})(-4\pi)^{\frac{n-2}{2}}}$$

зависит от n.

Из Теоремы 4.3 следует зональная характеризация зоноидов в пространствах четных размерностей.

Теорема 4.4. *Пусть n – чётное число. Для того, чтобы $(n+2)$-гладкое тело $\mathbf{K} \in \mathcal{K}_o^n$ являлось бы зоноидом необходимо и достаточно (см. Теорему 4.3)* выполнение следующего условия :

$$(-1)^{\frac{n-2}{2}} P_n(L) \left(\int_{\mathbf{S}_\Omega} \sum_{i=1}^{n-1} R_i(\omega)\, \lambda_{n-2}(d\omega) \right) \geq 0 \quad \text{для всех} \quad \Omega \in \mathbf{S}^{n-1}. \tag{4.7}$$

В [6] подход (называемый методом согласования) позволяет решать интегральные уравнения, обобщающие известное интегральное уравнение Функа (например, интегральное уравнение (4.1)). Этот метод проверен в [6] для случая классического уравнения Функа.

Другое следствие относится к гипотезе Вейля [27] (см. Введение настоящей статьи). Возможность зональной характеризации зоноидов для чётных размерностей впервые продемонстрировала Панина [22], а затем Гуди и Вейль [14].

Для заданного $\Omega \in \mathbf{S}^{n-1}$ окрестность E_Ω экватора $\mathbf{S}_\Omega$ на $\mathbf{S}^{n-1}$ называется **экваториальной зоной**. Для заданного гладкого $\mathbf{K} \in \mathcal{K}_o^n$ **экватором тела** $\mathbf{K}$ с полюсом Ω называется множество всех точек поверхности $\partial\mathbf{K}$, направления нормалей которых принадлежат экватору с полюсом Ω.

Следствие 4.1. *Пусть n – чётное число. Значение порождающей плотности $h(\Omega)$ в направлении $\Omega \in \mathbf{S}^{n-1}$ для любого $(n+2)$-гладкого выпуклого тела (обобщённый зоноид) $\mathbf{K} \in \mathcal{K}_o^n$ зависит только от главных радиусов кривизны и их производных на экваторе тела $\mathbf{K}$ с полюсом Ω.*

Гипотеза Вейля [27] для гладких тел непосредственно вытекает из Следствия 4.1.

Выражаю свою признательность профессору Р. В. Амбарцумяну и рецензенту за важные замечания.

Abstract. The article studies the curvature radii of planar projections of centrally symmetric smooth convex bodies in $\mathbb{R}^n$. Among the results are some new formulas involving the support function and the generating density of convex bodies. They are discussed in the context of Weil conjecture on zonal characterization of zonoids in even dimensions.

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ОБОБЩЕНИЕ КОМБИНАТОРНЫХ ФОРМУЛ АМБАРЦУМЯНА

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Резюме. В статье приведены обобщенные комбинаторные формулы Амбарцумяна для вырожденного и невырожденного случаев. Формулы интерпретируют версии обратного преобразования Радона относительно эйлеровой характеристики.

ВВЕДЕНИЕ

Около 20 лет назад Р. В. Амбарцумяном был введён [1] комбинаторный подход к проблемам интегральной геометрии и геометрических вероятностей, основанный на комбинаторных формулах для пространств гиперплоскостей в евклидовых пространствах. Эти формулы суть комбинаторные обобщения формулы Гаусса-Бонне для сферического треугольника. Они приводят к важным результатам в различных областях, таких как задача равносоставленности в проективном пространстве [7] или случайных геометрических процессах [2].

Необходимые предпосылки для настоящей статьи включают формулы для любого чётно-размерного действительного пространства, доказанные автором в [6], см. ниже §1, пункт 1.4. Комбинаторные алгоритмы (найденные в [6], но отсутствующие в настоящей статье) вычисляют коэффициенты формул шаг за шагом, начиная с членов высших размерностей.

Эти формулы выполняются только для невырожденных случаев, и их комбинаторные коэффициенты не единственны, т.е. существуют различные множества комбинаторных коэффициентов, для которых выполняются эти формулы. С другой стороны, версия комбинаторных формул (для невырожденного случая) с однозначно определёнными коэффициентами получены в [7].

при частичной поддержке грантом RFBR № 02-01-00908

Два вопроса остаются открытыми :

– найти общий канонический вид для вырожденного и невырожденного случаев с однозначно определёнными коэффициентами ;

– не использовать итерационных алгоритмов для коэффициентов.

Настоящая статья даёт ответ на эти вопросы в Теореме 4. Эта теорема интерпретирует эти формулы, как частный случай обратного преобразования Радона относительно эйлеровой характеристики, и получает топологическую интерпретацию формул Амбарцумяна.

§1. ПРЕДВАРИТЕЛЬНЫЕ РАССУЖДЕНИЯ

1.1. Политопные функции. Рассмотрим n-мерное векторное пространство $\mathbb{R}^n$ с фиксированным началом координат O. **Политоп** в $\mathbb{R}^n$ предполагается выпуклым, непустым, ограниченным и замкнутым. Множество всех политопов в $\mathbb{R}^n$ обозначается через $\mathcal{P}$. Под **размерностью** политопа подразумевается размерность его аффинной оболочки. Политоп K с $dim\ K < n$ называется **вырожденным**.

Функция $F : \mathbb{R}^n \longmapsto \mathbb{R}$ называется **политопной**, если она представима в виде конечной суммы $F = \sum_i a_i I_{K_i}$, где $a \in \mathbb{R}$, а I_{K_i} является характеристической функцией политопа K_i, т.е.

$$I_{K_i}(x) = \begin{cases} 1 & \text{если } x \in K_i, \\ 0 & \text{в противном случае.} \end{cases}$$

Соответствие $K \longmapsto I_K$ отождествляет политоп K и его характеристическую функцию I_K. Пространство политопных функций обозначается через $\mathcal{M}(\mathbb{R}^n) = \mathcal{M}$. Напомним, что $\mathcal{M}$ может быть снабжена структурной алгеброй (см. [4], [5]). Обозначим через $\mathcal{M}_0 = \mathcal{M}_0(\mathbb{R}^n)$ подпространство $\mathcal{M}$, порождённое функциями типа $I_{\{P\}}$, где $P \in \mathbb{R}^n$.

Пусть $\mathcal{P} \subset \mathbb{R}^n$ – конечное множество точек. Обозначим через $\mathcal{M}(\mathcal{P})$ подпространство $\mathcal{M}$, порождённое политопами, все вершины которого лежат в $\mathcal{P}$. Обозначим через $\mathcal{M}_0(\mathcal{P})$ подпространство $\mathcal{M}(\mathcal{P})$, порождённое всеми $I_{\{P\}}$, где $P \in \mathcal{P}$. Очевидно, $\mathcal{M}_0(\mathcal{P}) = \mathcal{M}(\mathcal{P}) \cap \mathcal{M}_0(\mathbb{R}^n)$.

1.2. Бюффоновые множества и функции. Пусть $\mathcal{E}$ – пространство гиперплоскостей в $\mathbb{R}^n$. Пусть $\mathcal{P} \subset \mathbb{R}^n$ – конечное множество точек. $\mathcal{P}$ называется **невырожденным**, если ни какие $k+1$ точки из $\mathcal{P}$ не лежат в одной k-плоскости ($k = 1, ..., n$). Две гиперплоскости $e_1, e_2 \in \mathcal{E}$ называются **эквивалентными относительно** $\mathcal{P}$, если

1. $e_1 \cap \mathcal{P} = e_2 \cap \mathcal{P}$,

2. $P, Q \in \mathcal{P}$, e_1 разделяет P от $Q \iff e_2$ разделяет P от Q.

Множество эквивалентных классов обозначается через $Cl(\mathcal{P})$. Эквивалентный класс $c \in Cl(\mathcal{P})$ называется **ограниченным**, если он ограничен относительно расстояния от начала координат O.

Класс $c \in Cl(\mathcal{P})$ называется **атомом**, если он ограничен и если из условия $e \in c$ вытекает, что $e \cap \mathcal{P} = \emptyset$. Множество всех атомов обозначается через $\mathcal{A}(\mathcal{P})$.

Объединение ограниченных классов эквивалентности называется **бюффоновым множеством**.

Пространство действительнозначных функций, порождённое функциями I_c, где $c \in Cl(\mathcal{P})$ и c ограничено, обозначим через $\mathcal{B}(\mathcal{P})$. Обозначим также

$$\mathcal{B}_0(\mathcal{P}) = \{f \in \mathcal{B}(\mathcal{P}) : e \cap \mathcal{P} = \emptyset \Longrightarrow f(e) = 0\}.$$

Мы должны рассмотреть кольцо $\mathcal{B}(\mathcal{P})/\mathcal{B}_0(\mathcal{P})$. Его элементы называются **бюффоновыми функциями.**

Пусть $F \in \mathcal{M}(\mathbb{R}^n)$. **Преобразование Радона функции F относительно эйлеровой характеристики** (введённое в [8]) есть функция

$$F^* : \mathcal{E} \longmapsto \mathbb{R}, \quad F^*(e) = \int_e F(x)\, d\chi(x).$$

Здесь используем понятие интеграла относительно характеристики Эйлера. Его детальное описание можно найти в [8].

Очевидно, $F \in \mathcal{M}(\mathcal{P})$ влечёт $F^* \in \mathcal{B}(\mathcal{P})$. Заметим, что гомоморфизм $* : \mathcal{M}(\mathcal{P}) \longmapsto \mathcal{B}(\mathcal{P})$ индуцирует гомоморфизм

$$* : \mathcal{M}(\mathcal{P})/\mathcal{M}_0(\mathcal{P}) \longmapsto \mathcal{B}(\mathcal{P})/\mathcal{B}_0(\mathcal{P})$$

(не возникнет путаницы в связи с двойным использованием знака $*$).

Преобразование Радона $* : \mathcal{B}(\mathcal{P}) \longmapsto \mathcal{M}(\mathcal{P})$ определяется формулой

$$f^*(x) = \int_{x \in e} f(e)\, d\chi(e).$$

1.3. Симплексы. Пусть $\mathcal{P} \subset \mathbb{R}^n$ – конечное множество точек. Пусть $0 \leq k \leq n$ и $P_0, \ldots, P_k \in \mathcal{P}$ (все P_i разные). Выпуклая оболочка $conv\,\{P_0, \ldots, P_k\}$ называется **k-мерным симплексом с вершинами в $\mathcal{P}$.** Рассмотрим следующие множества симплексов с вершинами в $\mathcal{P}$:

$\mathcal{S}(\mathcal{P})$ = все симплексы,

$\mathcal{S}_k(\mathcal{P})$ = все k -мерные симплексы,

$\mathcal{S}_{even}(\mathcal{P}) =$ все чётно-мерные симплексы,

$\mathcal{S}_{odd}(\mathcal{P}) =$ все нечётно-мерные симплексы.

Симплекс называется **вырожденным** если множество $\{P_i\}$ вырождено. Для симплекса $\theta \in \mathcal{S}(\mathcal{P})$ обозначим

$$[\theta] = \{e \in \mathcal{E} :\ e \cap \theta \neq \emptyset\}.$$

Подпространство $\mathcal{M}(\mathcal{P})$, порождённое всеми θ для $\theta \in \mathcal{S}_{odd}(\mathcal{P})$, обозначается через $\mathcal{M}_{odd}(\mathcal{P})$.

1.4. Комбинаторные формулы Амбарцумяна. Ниже (Теоремы 1 и 2) напоминаем две более общие версии комбинаторных формул.

Теорема 1. *(Представление относительно $\mathcal{S}(\mathcal{P})$, см. [1] и [6]). Пусть $\mathcal{P} \subset \mathbb{R}^n$ – конечное множество точек с чётным n, а A – бюффоновое множество относительно $\mathcal{P}$. Тогда функция I_A допускает разложение*

$$I_A \equiv \frac{1}{2}\sum_{k=1}^{n-1}(-1)^{k+1}\sum_{\mathcal{S}_k(\mathcal{P})} c_A(\theta) I_{[\theta]} \quad (mod\mathcal{B}_0(\mathcal{P})).$$

Это представление не единственно, т.е. существуют разные совокупности коэффициентов, для которых формула выполняется. Следующая Теорема даёт разложение относительно нечётно-мерных симплексов, которое единственно в невырожденном случае.

В случае вырожденного $\mathcal{P}$, совокупности коэффициентов не определяются однозначно. Хотя можно получить некоторые из них, аппроксимируя $\mathcal{P}$ невырожденными множествами, однако этот метод часто не удобен для практических целей.

Теорема 2. *(Разложение относительно $\mathcal{S}_{odd}(\mathcal{P})$, см. [7]). Пусть $\mathcal{P} \subset \mathbb{R}^n$ – конечное невырожденное множество точек, и пусть n чётно, а A – множесто Бюффона относительно $\mathcal{P}$. Тогда сущестует единственная совокупность коэффициентов r_θ, удовлетворяющих условию*

$$I_A \equiv \frac{1}{2}\sum_{\mathcal{S}_{odd}(\mathcal{P})} r_A(\theta) I_{[\theta]} \quad (mod\ \mathcal{B}_0(\mathcal{P})).$$

Нам необходима следующая версия Теоремы 2.

Теорема 3. *Пусть $\mathcal{P} \subset \mathbb{R}^n$ (n чётно) является конечным невырожденным множеством точек. Для заданной функции $f \in \mathcal{B}(\mathcal{P})/\mathcal{B}_0(\mathcal{P})$ существует единственная функция $F \in \mathcal{M}_{odd}(\mathcal{P})/\mathcal{M}_0$ такая, что $f \equiv F^*$ $(mod\mathcal{B}_0(\mathcal{P}))$.*

Основной результат статьи содержится в Теореме 4. Она справедлива для любой размерности n и любого $\mathcal{P}$ (вырожденного или невырожденного). Коэффициенты вместе с алгоритмом для их вычисления заменяются на явную коэффициентную функцию. Имеет место также единственность комбинаторного разложения.

Теорема 4. *(Обобщённая формула Амбарцумяна) Пусть n – целое число, а $\mathcal{P} \subset \mathbb{R}^n$ – конечное множество точек. Определим $\overline{\mathcal{M}_{odd}(\mathcal{P})}$ как подмножество $\mathcal{M}(\mathcal{P})$, порождённое всеми функциями типа*

$$\theta - \frac{1}{2}\partial\theta, \quad \theta \in \mathcal{S}_{odd}(\mathcal{P}), \quad \theta \text{ вырождено},$$

где $\partial\theta$ – граница θ, взятая в аффинной оболочке θ. Тогда для заданной функции $f \in \mathcal{B}(\mathcal{P})$ существует единственная функция $F \in \overline{\mathcal{M}_{odd}(\mathcal{P})}/\mathcal{M}_0$ такая, что $f \equiv F^*$ $(mod\mathcal{B}_0(\mathcal{P}))$. Функция F определяется формулой

$$F = (-1)^{n+1}\Delta_\chi\left((\overline{f})^*\right)$$

(см. §2 и §3 ниже).

Функция F называется **коэффициентной функцией для** f. Она играет роль коэффициентов r_θ или c_θ.

§2. ОПЕРАТОР Δ_χ

Пусть $x \in \mathbb{R}^n$, $\varepsilon \geq 0$. Пусть $B_\varepsilon(x)$ – открытый ε-шар с центром в точке x. Определим гомоморфизм

$$\Delta_\chi : \mathcal{M} \longmapsto \mathcal{M}, \quad \Delta_\chi F(x) = \frac{1}{2}F(x) - \frac{1}{2}\lim_{\varepsilon\to 0}\int_{B_\varepsilon(x)} F(t)\, d\chi(t).$$

Так как $\Delta_\chi(\mathcal{M}_0) = 0$, гомоморфизм Δ_χ индуцирует гомоморфизм

$$\Delta_\chi : \mathcal{M}/\mathcal{M}_0 \longmapsto \mathcal{M}/\mathcal{M}_0$$

(для простоты используется тот же знак).

Пример. Пусть $K \in \mathcal{P}$, $dim\ K \neq 0$. Тогда

$$\Delta_\chi K = \begin{cases} K - 1/2\,\partial K, & \text{если } dim\ K \text{ чётно}, \\ 1/2\,\partial K, & \text{в противном случае.} \end{cases}$$

Предложение.

1. *Из $F \in \mathcal{M}(\mathcal{P})$ вытекает $F^* \equiv (\Delta_\chi F)^*$ $(mod\mathcal{B}_0(\mathcal{P}))$,*
2. $\Delta_\chi^2 = \Delta_\chi$,
3. $\Delta_\chi(\mathcal{M}(\mathcal{P})) = \overline{\mathcal{M}_{odd}(\mathcal{P})}$,
4. *Из $F, G \in \mathcal{M}(\mathcal{P})$ и $F^* \equiv G^*$ $(mod\mathcal{B}_0(\mathcal{P}))$ вытекает $\Delta_\chi F = \Delta_\chi G$.*

Доказательство. 1 и 2 следуют из Примера. Из утверждения 2 вытекает, что $\Delta_\chi \overline{\mathcal{M}_{odd}(\mathcal{P})} = \overline{\mathcal{M}_{odd}(\mathcal{P})}$. С другой стороны, для каждого $F \in \mathcal{M}$ имеем $\Delta_\chi F \in \overline{\mathcal{M}_{odd}(\mathcal{P})}$ (достаточно проверить это для политопа K с вершинами в $\mathcal{P}$). Доказательство пункта 4 практически повторяет доказательство Теоремы 4.4 из [7]. Оно основано на следующей лемме.

Лемма 1 (см. [7]). *Если $\mathcal{P}$ вырождено, то card $\mathcal{A}(\mathcal{P}) =$ card $\mathcal{S}_{odd}(\mathcal{P})$.*

Из Леммы 1 вытекает, что утверждение 4 выполняется для невырожденных случаев. Действительно, имеем

$$dim\ \mathcal{B}(\mathcal{P})/\mathcal{B}_0(\mathcal{P}) = card\ \mathcal{A}(\mathcal{P}) = card\ \mathcal{S}_{odd}(\mathcal{P}) = dim\ \overline{\mathcal{M}_{odd}(\mathcal{P})}.$$

Рассмотрим вырожденное множество $\mathcal{P}$ и аппроксимирующее невырожденное множество $\mathcal{P}'$. Для конечного множества X обозначим через $\mathbb{R}[X]$ пространство формальных сумм

$$\sum_{x \in X} a_x x, \quad \text{где} \quad a_x \in \mathbb{R}.$$

Рассмотрим внутренние отображения :

инъекция $\alpha : \mathcal{A}(\mathcal{P}) \longmapsto \mathcal{A}(\mathcal{P}')$,

изоморфизмы $\sigma : \mathbb{R}[\mathcal{A}(\mathcal{P})] \longmapsto \mathcal{B}/\mathcal{B}_0(\mathcal{P})$ и $\sigma' : \mathbb{R}[\mathcal{A}(\mathcal{P}')] \longmapsto \mathcal{B}/\mathcal{B}_0(\mathcal{P}')$,

изоморфизм $\rho' : \mathbb{R}[\mathcal{S}_{odd}(\mathcal{P}')] \longmapsto \overline{\mathcal{M}_{odd}(\mathcal{P}')}$,

эпиморфизм $\rho : \mathbb{R}[\mathcal{S}_{odd}(\mathcal{P})] \longmapsto \overline{\mathcal{M}_{odd}(\mathcal{P})}$,

изоморфизм $\lambda' : \mathcal{B}/\mathcal{B}_0(\mathcal{P}') \longmapsto \mathbb{R}[\mathcal{S}_{odd}(\mathcal{P}')]$,

изоморфизм $\pi : \mathbb{R}[\mathcal{S}_{odd}(\mathcal{P}')] \longmapsto \mathbb{R}[\mathcal{S}_{odd}(\mathcal{P})]$.

Имеем следующую цепочку гомоморфизмов. Первые три являются изоморфизмами.

$$\mathbb{R}[\mathcal{A}(\mathcal{P}')] \overset{\sigma'}{\longmapsto} \mathcal{B}(\mathcal{P}')/\mathcal{B}_0(\mathcal{P}') \overset{\lambda'}{\longmapsto} \mathbb{R}[\mathcal{S}_{odd}(\mathcal{P}')] \overset{\pi}{\longmapsto} \mathbb{R}[\mathcal{S}_{odd}(\mathcal{P})] \overset{\rho}{\longmapsto} \overline{\mathcal{M}_{odd}(\mathcal{P})}.$$

Рассмотрим множесто $\mathcal{A}^+ = \mathcal{A}(\mathcal{P}') \setminus \alpha\mathcal{A}(\mathcal{P})$. Из существования изоморфизмов σ и σ' следует, что $card\ \mathcal{A}^+ = dim\ \mathcal{B}(\mathcal{P}')/\mathcal{B}_0(\mathcal{P}') - dim\ \mathcal{B}(\mathcal{P})/\mathcal{B}_0(\mathcal{P})$. С другой стороны, элементы $\{\pi \circ \lambda' \circ \sigma'(a)\}_{a \in \mathcal{A}^+}$ линейно независимы в $\mathbb{R}[\mathcal{S}_{odd}(\mathcal{P})]$ и принадлежат $ker\ \rho$. Таким образом, $card\ \mathcal{A}(\mathcal{P}') - card\ \mathcal{A}(\mathcal{P}) = card\ \mathcal{A}^+ \leq dim\ ker\ \rho$. Поэтому $card\ \mathcal{A}(\mathcal{P}) \geq card\ \mathcal{A}(\mathcal{P}') - dim\ ker\ \rho = dim\ \mathbb{R}[\mathcal{S}_{odd}(\mathcal{P})] - dim\ ker\ \rho = dim\ \overline{\mathcal{M}_{odd}(\mathcal{P})}$, откуда следует, что $dim\ \overline{\mathcal{M}_{odd}(\mathcal{P})} \leq card\ \mathcal{A}(\mathcal{P})$. Таким образом, имеем $dim\ ker\ \Delta_\chi \geq dim\ ker(*)$. (Как и выше, $*$ означает преобразование Радона $* : \mathcal{M}(\mathcal{P})/\mathcal{M}_0 \longmapsto \mathcal{B}(\mathcal{P})/\mathcal{B}_0(\mathcal{P})$.) Поскольку $ker\ \Delta_\chi \subset ker(*)$, имеем $dim\ ker\ \Delta_\chi \geq dim\ ker\ (*)$, и, следовательно, $ker\ \Delta_\chi = ker(*)$, что и завершает доказательство пункта 4. Предложение доказано.

§3. ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ 4

Начнём с леммы, которая легко следует из обратной формулы преобразования Радона (см. [4], [8]).

Лемма 2. *Предположим, что $f \in \mathcal{B}(\mathcal{P})$. Тогда следующие два условия эквивалентны :*

1. *Существует функция $F \in \mathcal{M}(\mathcal{P})$ такая, что $F^* = f$.*
2. *Значение интеграла*

$$\int_{e\|e_0} f(e)\, d\chi(e)$$

не зависит от гиперплоскости e_0, а f обращается в нуль на гиперплоскостях, достаточно удалённых от O.

Пусть $f \in \mathcal{B}\mathcal{P}/\mathcal{B}_0(\mathcal{P})$. Определим функцию $\bar{f}$ следующим образом. Если $e \cap \mathcal{P} = \emptyset$, то множество $\bar{f}(e) = f(e)$. Допустим, что $e \cap \mathcal{P} \neq \emptyset$. Рассмотрим две гиперплоскости e^+ и e^-, параллельные e, близкие к e и лежащие по разные стороны от e. Тогда получаем $\bar{f}(e) = 1/2\,(f(e^+) + f(e^-))$. Очевидно, имеем $f \equiv \bar{f}$ $(mod\ \mathcal{B}_0(\mathcal{P}))$. Легко проверить, что функция $\bar{f}$ удовлетворяет условию 2 Леммы 2. Следовательно, существует функция $F \in \mathcal{M}(\mathcal{P})$ такая, что $F^* = f$.

Теперь очень просто доказать Теорему 4. Пусть $f \in \mathcal{B}(\mathcal{P})/\mathcal{B}_0(\mathcal{P})$. Положим $F = \Delta_\chi((\bar{f})^*)$. Имеем $((-1)^n \Delta_\chi[(\bar{f})^*])^* = ((-1)^n(\bar{f})^*)^* = \bar{f} \equiv f \quad (mod\ \mathcal{B}_0(\mathcal{P}))$. Откуда имеем $F^* \equiv f$ $(mod\ \mathcal{B}_0(\mathcal{P}))$. Единственность такой функции следует из Предложения 2.2. Теорема 4 доказана.

Abstract. The paper presents a generalized form of Ambartzumian's combinatorial formulae, valid for both degenerate and non-degenerate cases. The formulae can be interpreted as a version of inverse Radon transformation with respect to Euler characteristic.

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INEQUALITIES IN THE SENSE OF BRUNN–MINKOWSKI–VITALE FOR RANDOM CONVEX BODIES

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Abstract. The well-known Brunn–Minkowski inequality concerning convex addition of measurable sets was generalized by R. A. Vitale for the case of random sets. The paper presents a new proof in the special case of random convex bodies, which does not employ the law of large numbers for random sets, but the mixed area measure. In this way, inequalities for mixed volumes and intrinsic volumes of random convex bodies are also obtained. Finally, consequences for stationary random hyperplane processes are discussed.

§1. INTRODUCTION

Nonempty compact convex subsets of $\mathbb{R}^d$ are called *convex bodies*. Let us denote by $K + L$ the Minkowski sum of the convex bodies $K, L \subset \mathbb{R}^d$ and by $V_d(K)$ the volume of a convex body K. The formula

$$V_d^{1/d}(pK + (1-p)L) \geq pV_d^{1/d}(K) + (1-p)V_d^{1/d}(L),$$

where $0 \leq p \leq 1$, is known as the Brunn–Minkowski inequality for convex bodies K, L [7]. By iteration, for convex bodies $K_1, ..., K_n$,

$$V_d^{1/d}(p_1 K_1 + \cdots + p_n K_n) \geq p_1 V_d^{1/d}(K_1) + \cdots + p_n V_d^{1/d}(K_n), \tag{1}$$

where $p_1, ..., p_n \geq 0$, $p_1 + ... + p_n = 1$.

This formula may be interpreted in a stochastic manner : Let $\mathbf{K}$ be a **random** convex body with range $\{K_1, ..., K_n\}$ and $Prob(\mathbf{K} = K_i) = p_i$, $i = 1, ..., n$. Then (1) can be written in the form

$$V_d^{1/d}(\mathbb{E}\mathbf{K}) \geq \mathbb{E}V_d^{1/d}(\mathbf{K}), \tag{2}$$

where $\mathbb{E}\mathbf{K}$ means the set-valued expectation of $\mathbf{K}$ [9], [10]. On the right we have the usual expectation of the real random variable $V_d^{1/d}(\mathbf{K})$.

The question arises, whether (2) is valid for arbitrary random convex bodies not necessarily discrete. R. Vitale [10] proved (2) for a large class of random sets **K** including random convex bodies. His proof relies on the strong law of large numbers for random sets [1], cf. [11].

The main results of the paper are as follows.

– Generalization of (2) for intrinsic volumes $V_m(\mathbf{K})$ of random convex bodies **K** in $\mathbb{R}^d$ (Theorem 2)

$$V_m^{1/m}(\mathbb{E}\mathbf{K}) \geq \mathbb{E}V_m^{1/m}(\mathbf{K}), \quad m = 2, ..., d-1. \tag{3}$$

– A new method of proof for (2) and (3), which makes no use of the law of large numbers for random sets.

– An inequality for mixed volumes of random convex bodies (Theorem 1), interesting in its own right. The inequalities (2) and (3) are obtained as special case.

– Consequences of the above results for the intersection densities of stationary Poisson hyperplane processes (their distribution can be described by a convex body, the so-called Steiner compact [3]).

§2. BASIC NOTIONS AND NOTATIONS

Let us denote the set of all non-empty compact convex subsets in $\mathbb{R}^d$, $d \geq 2$ by $\mathcal{K}$, and the σ-algebra of subsets of $\mathcal{K}$ as defined in [3] by $\mathbb{K}$. A **random convex body** is a random variable **K** with range $[\mathcal{K}, \mathbb{K}]$ (random element in $[\mathcal{K}, \mathbb{K}]$). Every $K \in \mathcal{K}$ is described by its support function $h(K, \cdot) : \mathbb{R}^d \longmapsto [0, \infty)$ defined by

$$h(K, \mathbf{u}) = \sup\{\langle \mathbf{x}, \mathbf{u} \rangle : \mathbf{x} \in K\},$$

where $\langle \mathbf{x}, \mathbf{u} \rangle$ denotes the inner product of $\mathbf{x}, \mathbf{u} \in \mathbb{R}^d$. The norm of an element $K \in \mathcal{K}$ is

$$||K|| = \max\{||x|| : \ x \in K\},$$

where $||x||$ denotes the usual norm in $\mathbb{R}^d$.

Let **K** be a random convex body. Assuming $\mathbb{E}||\mathbf{K}|| < \infty$, it can be shown [1], [9], [11], that there exists a convex body $\mathbb{E}\mathbf{K} \in \mathcal{K}$ with the support function

$$h(\mathbb{E}\mathbf{K}, \mathbf{u}) = \mathbb{E}h(\mathbf{K}, \mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^d. \tag{4}$$

The convex body $\mathbb{E}\mathbf{K}$ is called **set-valued expectation** of the random convex body **K**.

The volume $V_d(K)$ of a convex body K and the $(d-1)$-content $O_d(K)$ of its boundary ∂K are invariant under Euclidean motions and, as functions on $\mathcal{K}$, have several properties of continuity and additivity. In convex geometry they are considered as special cases of the so-called **intrinsic volumes** $V_m : \mathcal{K} \longmapsto [0, \infty)$, possessing similar properties ($m = 0, 1, ..., d$). They are closely related to the Minkowski-functionals [7]. A more general notion is that of **mixed volume** $V(K_1, ..., K_d)$ of convex bodies $K_1, ..., K_d \in \mathcal{K}$ [7]. Let B be the unit ball in $\mathbb{R}^d$, and α_k be the volume of the k-dimensional unit ball, i.e.

$$\alpha_k = \frac{\pi^{k/2}}{\Gamma(1 + k/2)}, \quad k = 0, 1, ..., d.$$

The intrinsic volume $V_m(K)$ of $K \in \mathcal{K}$ can be expressed as a special mixed volume [7] :

$$V_m(K) = \binom{d}{m}(\alpha_{d-m})^{-1} V\left(\underbrace{K, ..., K}_{m}, \underbrace{B, ..., B}_{d-m}\right), \quad m = 0, 1, ..., d. \tag{5}$$

Then, $V_d(K)$ is the usual volume of K, $V_{d-1}(K) = O_d(K)/2$, $V_1(K)$ is proportional to the mean width of K, and $V_0(K) = 1$.

§3. INEQUALITIES FOR MIXED VOLUMES

Let $\mathbf{S}$ be the σ-algebra of Borel subsets of the unit sphere S^{d-1} in $\mathbb{R}^d$, $d \geq 2$ let be fixed. There exists a function $S : \mathcal{K}^{d-1} \times \mathbf{S} \longmapsto [0, \infty)$, the so-called mixed area measure [7], with the properties

(i) For fixed $K_2, ..., K_d \in \mathcal{K}$, $S(K_2, ..., K_d, \cdot)$ is a finite measure on $\mathbf{S}$.

(ii) For all $K_1, ..., K_d \in \mathcal{K}$ the equation

$$V(K_1, ..., K_d) = \frac{1}{d} \int_{S^{d-1}} h(K_1, \mathbf{u}) S(K_2, ..., K_d, d\mathbf{u}) \tag{6}$$

holds.

The following formula plays a key role in the proofs of our main results.

Proposition 1. *Let* $\mathbf{K}$ *be a random convex body with* $\mathbb{E}||\mathbf{K}|| < \infty$. *If* $K_2, ..., K_d \in \mathcal{K}$, *then*

$$V(\mathbb{E}\mathbf{K}, K_2, ..., K_d) = \mathbb{E}V(\mathbf{K}, K_2, ..., K_d).$$

Proof. Applying formulae (4), (6) and Fubini's theorem we get

$$dV(\mathbb{E}\mathbf{K}, K_2, ..., K_d) = \int h(\mathbb{E}\mathbf{K}, \mathbf{u}) S(K_2, ..., K_d, d\mathbf{u}) =$$

$$= \mathbb{E} \int h(\mathbf{K}, \mathbf{u}) S(K_2, ..., K_d, d\mathbf{u}).$$

Finally, (6) implies

$$\int h(\mathbf{K}, \mathbf{u}) S(K_2, ..., K_d, d\mathbf{u}) = dV(\mathbf{K}, K_2, ..., K_d).$$

Proposition 1 is proved.

We learn from Schneider [7] the following special case of a generalized Alexandrov–Fenchel inequality (formula (6.4.5) in [7] for $i = 0$, $j = m - 1$ and $k = m$) :

Proposition 2. *For $2 \leq m \leq d$ and $Y, Z, K_{m+1}, ..., K_d \in \mathcal{K}$, we have*

$$V\left(Y, \underbrace{Z, ..., Z}_{m-1}, K_{m+1}, ..., K_d\right) \geq V^{1-1/m}\left(\underbrace{Z, ..., Z}_{m}, K_{m+1}, ..., K_d\right) \times$$
$$\times V^{1/m}\left(\underbrace{Y, ..., Y}_{m}, K_{m+1}, ..., K_d\right).$$

(Cf. also [2], exercise p. 321.) That particular inequality can also be verified by repeated application of the Alexandrov–Fenchel inequality.

The next theorem contains our main result.

Theorem 1. *Let $\mathbf{K}$ be a random convex body with $\mathbb{E}||\mathbf{K}|| < \infty$. If $2 \leq m \leq d$ and $K_{m+1}, ..., K_d \in \mathcal{K}$, then*

$$V^{1/m}\left(\underbrace{\mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}}_{m}, K_{m+1}, ..., K_d\right) \geq \mathbb{E}V^{1/m}\left(\underbrace{\mathbf{K}, ..., \mathbf{K}}_{m}, K_{m+1}, ..., K_d\right).$$

Proof. Proposition 2 implies

$$V(\mathbf{K}, \mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}, K_{m+1}, ..., K_d) \geq V^{1-1/m}(\mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}, K_{m+1}, ..., K_d) \times$$
$$\times V^{1/m}(\mathbf{K}, ..., \mathbf{K}, K_{m+1}, ..., K_d).$$

Forming the expectation on both sides and applying Proposition 1, we obtain

$$V(\mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}, K_{m+1}, ..., K_d) \geq V^{1-1/m}(\mathbb{E}\mathbf{K}, ..., \mathbb{E}\mathbf{K}, K_{m+1}, ..., K_d) \times$$
$$\times \mathbb{E}V^{1/m}(\mathbf{K}, ..., \mathbf{K}, K_{m+1}, ..., K_d).$$

The assertion follows immediately.

§4. INEQUALITIES FOR INTRINSIC VOLUMES

The next theorem is derived from Theorem 1 and (5), putting $K_{m+1} = ... = K_d = B$.

Theorem 2. *Let $\mathbf{K}$ be a random convex body with $\mathbb{E}||\mathbf{K}|| < \infty$. If $2 \leq m \leq d$, then*

$$V_m^{1/m}(\mathbb{E}\mathbf{K}) \geq \mathbb{E}V_m^{1/m}(\mathbf{K}).$$

These formulae are called **generalized Brunn–Minkowski–Vitale inequalities**. In the case $m = d$ the result was already proved by R. Vitale in [10], and quoted by Weil and Weacker in [11].

If $K \in \mathcal{K}$ and D is a random rotation about the origin, then DK is a random convex body.

Corollary 1. *Let D be a random rotation about the origin. Then for all $K \in \mathcal{K}$ and $m = 2, ..., d$ the inequalities $V_m(\mathbb{E}DK) \geq V_m(K)$ are fulfilled.*

Proof. From Theorem 2 we obtain $V_m^{1/m}(\mathbb{E}DK) \geq \mathbb{E}V_m^{1/m}(DK)$. The rotation invariance of V_m implies $V_m(DK) = V_m(K)$.

§5. ALTERNATIVE PROOF FOR INTRINSIC VOLUMES

Theorem 2 might as well be proved in the same way Vitale [10] proved (2). We make use of a generalized Brunn-Minkowski inequality concerning intrinsic volumes [2], [7].

Proposition 3. *For $K_1, K_2 \in \mathcal{K}$, $0 \le \lambda \le 1$ and $m = 2, ..., d$ the inequality*

$$V_m^{1/m}(\lambda K_1 + (1-p)K_2) \ge \lambda V_m^{1/m}(K_1) + (1-\lambda)V_m^{1/m}(K_2) \tag{7}$$

holds.

Now, given a random convex body $\mathbf{K}$, such that $\mathbb{E}||\mathbf{K}|| < \infty$, we consider a sequence $X_1, X_2, ...$ of mutually independent random sets, each distributed like $\mathbf{K}$. By iteration (7) is transformed into

$$V_m^{1/m}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \ge \frac{1}{n}\sum_{i=1}^{n} V_m^{1/m}(X_i), \quad n \ge 2. \tag{8}$$

For a constant c depending only on m and d, we have $V_m^{1/m}(L) \le c||L||$ for any $L \in \mathcal{K}$. Since $\mathbb{E}||\mathbf{K}|| < \infty$, the Kolmogorov strong law of large numbers implies, that the right-hand side of (8) tends to $\mathbb{E}V_m^{1/m}(\mathbf{K})$ as $n \to \infty$.

Regarding the left-hand side of (8), there is the a.s. convergence of $\frac{1}{n}\sum\limits_{i=1}^{n} X_i$ to $\mathbb{E}\mathbf{K}$ by a strong law of large numbers for random convex sets (Artstein and Vitale [1], cf. [11]). Due to the continuity of V_m (cf. e.g. [2]), we have $V_m^{1/m}(\mathbb{E}\mathbf{K}) \ge \mathbb{E}V_m^{1/m}(\mathbf{K})$, which is the assertion of Theorem 2.

§6. SOME CONSEQUENCES FOR HYPERPLANE PROCESSES

Let Φ be a stationary Poisson hyperplane process (SPHP) in $\mathbb{R}^d$ [3], [4]. The mean $(d-1)$-content of Φ per unit volume is called **intensity** and denoted by λ. The direction of a hyperplane is described by the perpendicular line through the origin (1-subspace). We denote the set of all 1-subspaces by $\mathcal{H}$ and the σ-algebra of Borel subsets of $\mathcal{H}$ by $\mathcal{B}$ [3], [4].

Given $A \in \mathcal{B}$, let us denote by Φ_A the set of all hyperplanes from Φ with a direction in A. Then Φ_A is again a SPHP, the intensity of which is denoted by $h(A)$. In this way, a finite measure h on $[\mathcal{H}, \mathcal{B}]$ is established, the so-called **directional measure** of Φ. The distribution of a SPHP Φ is completely determined by h. The intensity λ of Φ is equal to the total mass of $h : \lambda = h(\mathcal{H})$.

Every m-tuple of hyperplanes from Φ in general position has for the set of intersection points a $(d-m)$-dimensional affine subspace ($m = 1, ..., d$). For fixed m, all these intersection $(d-m)$-flats form a stationary (non-Poisson) $(d-m)$-dimensional flat process Φ_m. Note that $\Phi_1 = \Phi$ and Φ_d is the point process of vertices of the tessellation formed by Φ. The $(d-m)$-content of Φ_m per unit volume is said to be **the m-intersection density** and denoted by $\rho_m(h)$. Note that $\rho_1 = \lambda$ and ρ_d equals the intensity of the point process of vertices.

To every finite measure h on $[\mathcal{H}, \mathcal{B}]$ different from the zero-measure there corresponds a convex body (more precisely a zonoid), the so-called Steiner compact $\mathcal{S}[h]$ [3], [4]. It is known, that

$$\rho_m(h) = V_m(\mathcal{S}[h]), \quad m = 1, ..., d, \tag{9}$$

if h is the directional measure of Φ [3], [4].

Let D be a random rotation about the origin. It transforms h in a random measure D_h and the Steiner compact $\mathcal{S}[h]$ in a random convex body $D\mathcal{S}[h] = \mathcal{S}[D_h]$. We define a measure $\mathbb{E}D_h$ on $[\mathcal{H}, \mathcal{B}]$ by $(\mathbb{E}D_h)(A) = \mathbb{E}(D_h)(A), \quad A \in \mathcal{B}$. It can be shown that

$$\mathcal{S}[\mathbb{E}D_h] = \mathbb{E}D\mathcal{S}[h]. \tag{10}$$

Corollary 1 and (10) lead to

$$V_m(\mathcal{S}[\mathbb{E}D_h]) \geq V_m(\mathcal{S}[h]), \quad m = 2, ..., d. \tag{11}$$

Combining (9) with (11), we get

$$\rho_m(\mathbb{E}D_h) \geq \rho_m(h), \quad m = 2, ..., d. \tag{12}$$

For $m = d$ the result can be found in [5], [6].

If the distribution of D is a Haar measure on the group of rotations about the origin, we say that D is **uniformly distributed**. In this case, $\mathbb{E}D_h$ is proportional to the uniform distribution γ on $[\mathcal{H}, \mathcal{B}]$: $\mathbb{E}D_h = \lambda\gamma$. A SPHP with directional measure $\lambda\gamma$ is called **isotropic**.

For an arbitrary random rotation D, it seems to be reasonable to say, that $\mathbb{E}D_h$ is "more isotropic" than h. We say also, that a SPHP $\tilde{\Phi}$ with directional measure $\mathbb{E}D_h$ is "more isotropic" than the SPHP Φ with directional measure h. Note that $\tilde{\Phi}$ and Φ have the same intensity λ.

In this context, formula (12) means that the intersection densities of a stationary Poisson hyperplane process Φ are not greater, than the corresponding intersection densities of a stationary Poisson hyperplane process $\tilde{\Phi}$, which is "more isotropic" than Φ, but has the same intensity.

As a special case, the result of Thomas [8], cf. [4] is reestablished : for fixed intensity λ, the intersection densities of stationary Poisson hyperplane processes take their maximal value in the isotropic case.

Резюме. Хорошо известное неравенство Брунна-Минковского, относящееся к выпуклому сложению измеримых множеств было обобщено Р. А. Витале для случая случайных множеств. В настоящей статье приводится его новое доказательство в частном случае случайных выпуклых тел, где вместо закона больших чисел для случайных множеств используется смешанная мера соответствующая площади. Этим путём получены неравенства для смешанных обёмов и внутренних обёмов случайных выпуклых тел. Обсуждаются следствия для стационарных случайных процессов гиперплоскостей.

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ON INTEGRAL GEOMETRY IN PROJECTIVE FINSLER SPACES

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Abstract. We prove integral geometric formulae of Crofton type for Holmes-Thompson areas of rectifiable Borel sets in smooth projective Finsler spaces.

§1. INTRODUCTION

A classical result of integral geometry in Euclidean spaces, usually associated with the name of Crofton (although he obtained only the first very special cases), expresses the area of a submanifold as an integral of the number of intersection points with the affine subspaces of complementary dimension. More precisely, let M be a k-dimensional C^1 submanifold of Euclidean space $\mathbb{R}^n$ ($n \geq 2$, $k \in \{1, \ldots, n-1\}$). Let λ_k denote the k-dimensional differential-geometric surface area, and let μ_j be a rigid motion invariant measure on the affine Grassmannian $A(n, j)$ of j-flats (j-dimensional affine subspaces) of $\mathbb{R}^n$. Then

$$\int_{A(n,n-k)} \operatorname{card}(M \cap E)\, d\mu_{n-k}(E) = a_{nk} \lambda_k(M), \tag{1}$$

with a constant a_{nk} depending on the normalization of the measure μ_{n-k}. More generally, if $j \in \{n-k, \ldots, n-1\}$, then

$$\int_{A(n,j)} \lambda_{k+j-n}(M \cap E)\, d\mu_j(E) = a_{nkj} \lambda_k(M), \tag{2}$$

with a constant a_{nkj}. For proofs and further references, we refer to the books of Santaló [30] (p. 245, (14.69)) and of Sulanke and Wintgen [36] (p. 252, (5)).

In the present paper, we extend formula (2) to Holmes-Thompson areas and rectifiable Borel sets in smooth projective Finsler spaces. The role of the Haar measures μ_j on the affine Grassmannians $A(n,j)$ is then played by suitable signed measures. (For $j = 1$, these are positive measures, and they exist also in general, not necessarily smooth, projective Finsler spaces, see Schneider [34]). Before stating the main result, we want to put these investigations in a wider context and explain some background.

The beauty of formula (1) is an invitation for various generalizations in different directions. Starting from the left side of (1) as a definition, for more general sets M, one is led to the notion of integral geometric (or Favard) measures ; see, e.g., Mattila [26], Section 5.14, and the references given there. In the following, we look at (1) from an opposite point of view : suppose some notion of k-dimensional area is given instead of λ_k ; does there exist a measure or signed measure on $A(n, n-k)$, replacing μ_{n-k}, so that (1) holds for a reasonably large class of sets M ? One may also think of replacing the affine Grassmannians by more general systems of sets with similar properties. We list some work that can be subsumed under this general program. There is a clear distinction between the possibilities and results in dimension two and in higher dimensions.

In dimension two, the program concerns notions of length and measures on the system $A(2,1)$ of lines or on similar curve systems. A very satisfactory result has been obtained in the course of the solution of Hilbert's fourth problem in the plane by Pogorelov [28], Ambartzumian [7], Alexander [1]. Let μ be a Borel measure on $A(2,1)$. For $p,q \in \mathbb{R}^2$, let $d(p,q)$ be the μ measure of the set of lines weakly separating p and q, and suppose that $d(p,p) = 0$ and $0 < d(p,q) < \infty$ for $p \neq q$. Then d is a continuous metric on $\mathbb{R}^2$ which is additive along lines, and the induced curve length has the property that the lines are geodesics. The quoted papers establish, through different approaches, the converse : every metric with these properties, and hence every notion of curve length for which the lines are geodesics, is obtained in the described way from a measure. A related investigation of Ambartzumian [8] replaces the lines by certain axiomatically defined systems of curves. For sufficiently smooth two-dimensional Finsler or Riemannian manifolds, densities on sets of geodesics leading to Crofton formulae were considered by Blaschke [12], Santaló [29], Owens [27] (who was apparently unaware of Blaschke's work). An elementary treatment of a Crofton formula in Minkowski planes was given by Chakerian [15]. For the classical Crofton formula in the Euclidean plane, an elementary proof for rectifiable curves can be found in a paper by Ayari and Dubuc [9].

About dimensions greater than two, we mention first that (1) holds also in spaces of constant curvature, with flats replaced by totally geodesic submanifolds, see Santaló [30]. The investigation of general versions of Crofton formulae began with Busemann [13], [14]. Generalizing Hilbert's fourth problem, he suggested to study axiomatically defined k-dimensional areas in affine spaces for which flats minimize area. Closely related is the question about the validity of Crofton formulae with positive measures, and then the consideration of Crofton formulae involving signed measures is a natural generalization. Concrete Crofton formulae were obtained for Minkowski spaces (finite-dimensional real normed spaces), in special cases by El-Ekhtiar [18] and more systematically by Schneider and Wieacker [35]. The latter paper contains a version of (2) for Holmes-Thompson areas of rectifiable Borel sets in hypermetric Minkowski spaces, with suitable positive measures on $A(n,j)$. In Minkowski geometry, there are different notions of area, see Thompson [37], but only the Holmes-Thompson area seems generally suitable for this type of integral geometric formulae. This was made clear in [32] (Theorem 1) and [33]. The mentioned results of [35] do not require any smoothness assumptions. On the other hand, under smoothness assumptions, there are quite general investigations about Crofton type results for densities, due to Gelfand and Smirnov [22] and to Álvarez, Gelfand and Smirnov [6]. The work of Álvarez and Fernandes [3], [4], [5] and of Fernandes [21] combines a tool from this theory, double fibrations and the Gelfand transform, with other methods, in part from symplectic geometry, to obtain Crofton formulae for Holmes-Thompson areas of smooth submanifolds of smooth projective Finsler spaces. In particular, [3] and [21] extend (1) to this situation, as well as (2) for the special case $k = n-1$. In Section 4 below, we generalize this to all $k \in \{1, \dots, n-1\}$ and to $(\mathcal{H}^k, k)$-rectifiable Borel sets. It turns out that the methods used in [35] for the special case of hypermetric Minkowski spaces are sufficiently general to be adaptable to smooth projective Finsler spaces, thus yielding the following theorem (explanations and precise definitions are given in Section 2).

Theorem 1. *Let $(\mathbb{R}^n, F)$ be a smooth projective Finsler space, and let vol_k denote the corresponding k-dimensional Holmes-Thompson area. For $j \in \{1, \dots, n-1\}$, there exists a signed measure η_j on the affine Grassmannian $A(n,j)$ such that, for $k \in \{n-j, \dots, n\}$ and for every $(\mathcal{H}^k, k)$-rectifiable Borel set $M \subset \mathbb{R}^n$,*

$$\int_{A(n,j)} vol_{k+j-n}(M \cap E)\, d\eta_j(E) = a_{nkj}\, vol_k(M) \tag{3}$$

with a constant a_{nkj}.

Theorem 1 can be considered as giving, for projective Finsler spaces, a positive answer to the first of the three open problems formulated by Chakerian [16] (p. 50). The second of his problems was solved in [34], and the third one in [33] (Theorem 1).

The standard classical examples of projective Finsler spaces are the Minkowski spaces and the Hilbert geometries. In the latter case, the Finsler metric is not defined on all of $\mathbb{R}^n$, but on the interior of a convex body in $\mathbb{R}^n$. In both cases, a single convex body determines the whole geometry, and the smoothness properties of the induced Finsler metric depend on the smoothness of that convex body. In order that arbitrary convex bodies can be admitted, one has to consider general Finsler metrics, which satisfy the usual convexity and continuity, but no smoothness assumptions. In [34], a version of (1), with Holmes-Thompson areas, was obtained for general Finsler metrics F on $\mathbb{R}^n$ such that $(\mathbb{R}^n, F)$ is a hypermetric projective Finsler space. For $k = n-1$, the assumption 'hypermetric' can be deleted ([34], Theorem 2). The corresponding (positive) measure on the space $A(n,1)$ of lines was obtained by approximation and was, therefore, not described in any explicit way. The existence of this line measure can also be proved for Hilbert geometries. For Hilbert geometries in planar polygons, the line measure is known explicitly, see Alexander [1] and Alexander, Berg and Foote [2]. We mention here that for the special case of the Hilbert geometry in an n-dimensional simplex, an explicit description of the line measure can be obtained, using the fact, established by de la Harpe [17], that this metric space is isometric to a certain Minkowski space. We hope to treat the line measure in a polytopal Hilbert geometry somewhere else.

§2. FINSLER SPACES AND AREAS

We restrict ourselves here to Finsler metrics on $\mathbb{R}^n$; the case of an open convex subset instead of $\mathbb{R}^n$ requires only obvious modifications. For convenience, we always assume that $\mathbb{R}^n$ is equipped with its standard scalar product $\langle\cdot,\cdot\rangle$, for $n \geq 2$. One reason for this is that it allows us to talk of Lipschitz mappings $f : \mathbb{R}^k \to \mathbb{R}^n$, of the k-dimensional Hausdorff measure $\mathcal{H}^k$ on $\mathbb{R}^n$ (for $k \geq 0$), and of $(\mathcal{H}^k, k)$-rectifiable sets. A set $M \subset \mathbb{R}^n$ is called $(\mathcal{H}^k, k)$-rectifiable (for $k \in \{1, \dots, n\}$) if $\mathcal{H}^k(M) < \infty$ and there exist Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$, $i \in \mathbb{N}$, such that $\mathcal{H}^k(K \setminus \bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^k)) = 0$. The notion of Lipschitz map, the classes of sets of zero or finite Hausdorff measure, and the notion of $(\mathcal{H}^k, k)$-rectifiable sets do not depend on the choice of the Euclidean metric.

We canonically identify the tangent space $T_x\mathbb{R}^n$ of $\mathbb{R}^n$ at x with $\mathbb{R}^n$. By a **Finsler metric on $\mathbb{R}^n$** we understand here a continuous function $F: \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ such that $F(x,\cdot)$ is a norm on $\mathbb{R}^n$, for each fixed $x \in \mathbb{R}^n$. The Finsler metric F is said to be **smooth** if F is of class C^∞ on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. (The additional assumption made in the differential geometry of Finsler spaces, that $F^2(x,\cdot)$ has positive definite Hessian on $\mathbb{R}^n \setminus \{0\}$, is not needed in the following.) If F is a (smooth) Finsler metric on $\mathbb{R}^n$, we say that $(\mathbb{R}^n, F)$ is a (smooth) Finsler space. In such a space, the length of a parameterized C^1 curve $\gamma : [a,b] \to \mathbb{R}^n$ is defined by $\int_a^b F(\gamma(t), \gamma'(t))\, dt$. The Finsler space $(\mathbb{R}^n, F)$ is called **projective** if line segments are shortest curves (not necessarily the only ones) connecting their endpoints. The metric d_F **induced by** F is defined by letting $d_F(p,q)$ be the infimum of the lengths of the piecewise C^1 curves connecting the points $p, q \in \mathbb{R}^n$. If $(\mathbb{R}^n, F)$ is projective, the segment $[p,q]$ with endpoints p,q has length $d_F(p,q)$.

We assume that a Finsler metric F on $\mathbb{R}^n$ is given. For $x \in \mathbb{R}^n$, we write

$$F(x,\cdot) =: \|\cdot\|_x \qquad \text{and} \qquad B_x := \{\xi \in \mathbb{R}^n : \|\xi\|_x \le 1\}.$$

This convex body, the unit ball of the Minkowski space $(T_x\mathbb{R}^n, \|\cdot\|_x)$, is called the **indicatrix** of the Finsler metric F at x. Since we have identified $T_x\mathbb{R}^n$ with $\mathbb{R}^n$, each $\|\cdot\|_x$ is a norm on $\mathbb{R}^n$, and B_x is a convex body in $\mathbb{R}^n$ which is centrally symmetric with respect to the origin.

Let B_x^o be the polar body of B_x with respect to the chosen scalar product, thus

$$B_x^o = \{\eta \in \mathbb{R}^n : \langle \xi, \eta \rangle \le 1 \quad \text{for all} \quad \xi \in B_x\}.$$

This body is called the **figuratrix** of the Finsler metric F at x (for its role in the calculus of variations, see Blaschke [11]).

The metric d_F induces, in the usual way, an s-dimensional Hausdorff measure $\mathcal{H}_F^s$, for each $s \ge 0$. We recall its definition. Let diam $_F$ denote the diameter in terms of d_F. For a subset $A \subset \mathbb{R}^n$ and for $\delta > 0$, let

$$\Omega_\delta(A) := \{(C_i)_{i\in\mathbb{N}} : C_i \subset \mathbb{R}^n, \quad \operatorname{diam}_F C_i < \delta \quad \text{for all} \quad i,\ A \subset \bigcup_{i\in\mathbb{N}} C_i\}$$

and

$$\mathcal{H}_F^s(A) := \frac{\pi^{s/2}}{2^s\Gamma(1+s/2)} \sup_{\delta>0} \inf_{(C_i)\in\Omega_\delta(A)} \sum_{i\in\mathbb{N}} (\operatorname{diam}_F C_i)^s.$$

This yields a metric outer measure $\mathcal{H}_F^s$ on $\mathbb{R}^n$, and its restriction to the Borel sets is a measure.

Similarly, for each $x \in \mathbb{R}^n$, an s-dimensional Hausdorff measure on $\mathbb{R}^n$ is defined with respect to the metric induced by the norm $\|\cdot\|_x$. This Hausdorff measure is denoted by $\mathcal{H}^s_{F,x}$. In particular, $\mathcal{H}^n_{F,x}$ is the translation invariant Haar measure satisfying $\mathcal{H}^n_{F,x}(B_x) = \kappa_n$, where κ_n denotes the volume of the n-dimensional Euclidean unit ball ([35], p. 235).

Recall that the s-dimensional Hausdorff measure on $\mathbb{R}^n$ that is induced by the auxiliary Euclidean metric, coming from the scalar product $\langle \cdot, \cdot \rangle$, is denoted by $\mathcal{H}^s$. In particular, $\mathcal{H}^n$ coincides with the Lebesgue (outer) measure. It is easy to see, using the continuity of the Finsler metric, that the outer measures $\mathcal{H}^s_F, \mathcal{H}^s_{F,x}, \mathcal{H}^s$ all have the same classes of null sets and of measurable sets.

For a $(\mathcal{H}^k, k)$-rectifiable Borel set M in $\mathbb{R}^n$ (where $k \in \{1, \dots, n\}$), the **Busemann k-area** of M is defined as the Hausdorff measure $\mathcal{H}^k_F(M)$. The **Holmes-Thompson k-area** of M can be defined by

$$\mathrm{vol}_k(M) = \frac{1}{\kappa_k^2} \int_M vp(B_x \cap T_x M)\, d\mathcal{H}^k_F(x). \tag{4}$$

Here $T_x M$ is the approximate tangent space of M at x (a linear subspace of $\mathbb{R}^n$, since $T_x \mathbb{R}^n$ was identified with $\mathbb{R}^n$) ; it exists and is unique for $\mathcal{H}^k_F$-almost all $x \in M$ and is measurable in dependence on x. The functional vp is the volume product, that is, $vp(K)$ is the product of the (Euclidean) volumes of K and K^o ; this definition does not depend on the choice of the scalar product.

The definitions of Busemann and Holmes-Thompson area as given here for rectifiable Borel sets in Finsler spaces are the natural extensions of these notions for smooth submanifolds of Minkowski spaces. In a sense which can be made precise, these two area notions are dual to each other. Areas in Minkowski spaces are thoroughly discussed in the book of Thompson [37]. The Holmes-Thompson area appears also in a natural way as a symplectic volume ; see Álvarez and Fernandes [3].

The auxiliary Euclidean structure on $\mathbb{R}^n$ has been introduced for two additional reasons. First, the introduction below of signed measures on the affine Grassmannians $A(n, j)$, which replace the motion invariant measures in Euclidean spaces and yield Crofton formulae for the Holmes-Thompson areas, rely on results of Pogorelov, which are conveniently formulated in Euclidean terms. Second, we will have to use results from the Euclidean geometry of convex bodies. For notions from this theory which are used below without explanation, we refer to the book [31].

We introduce some Euclidean terminology referring to the scalar product $\langle \cdot, \cdot \rangle$. The unit sphere is given by

$$S^{n-1} := \{u \in \mathbb{R}^n : \langle u, u \rangle = 1\},$$

and its spherical Lebesgue measure is denoted by σ. If $(\cdot)|E$ denotes orthogonal projection from $\mathbb{R}^n$ to the linear subspace E of $\mathbb{R}^n$, then $(B_x \cap E)^o = B_x^o|E$. We will show in Section 3 that in a projective Finsler space (4) can be replaced by

$$\mathrm{vol}_k(M) = \frac{1}{\kappa_k} \int_M \mathcal{H}^k(B_x^o|T_xM)\, d\mathcal{H}^k(x). \tag{5}$$

(Note that $\mathcal{H}^k$ and the orthogonal projection depend on the auxiliary scalar product ; the integral, however, is independent of the choice of the Euclidean structure.) Analogously, the Busemann k-area can be represented by

$$\mathcal{H}_F^k(M) = \kappa_k \int_M \frac{1}{\mathcal{H}^k(B_x \cap T_xM)}\, d\mathcal{H}^k(x). \tag{6}$$

Special cases of (6) for general Finsler spaces are contained in Theorem 4.1 of Belletini, Paolini and Venturini [10].

Defining the 'local scaling function' of the Holmes-Thompson k-area (with respect to the chosen auxiliary Euclidean structure) by

$$\sigma_k(x, E) := \frac{1}{\kappa_k} \mathcal{H}^k(B_x^o|E) \quad \text{for} \quad x \in \mathbb{R}^n \quad \text{and} \quad E \in G(n,k) \tag{7}$$

($G(n,k)$ is the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$), we write (5) in the form

$$\mathrm{vol}_k(M) = \int_M \sigma_k(x, T_xM)\, d\mathcal{H}^k(x). \tag{8}$$

Now we assume that $(\mathbb{R}^n, F)$ is a smooth projective Finsler space. It follows from the work of Pogorelov [28] (see [34] for a brief sketch of the relevant parts) that there exists a continuous function $g : S^{n-1} \times \mathbb{R} \to \mathbb{R}$ such that, for each $x \in \mathbb{R}^n$, the support function $h(B_x^o, \cdot)$ of the figuratrix can be represented by

$$h(B_x^o, \xi) = \int_{S^{n-1}} |\langle \xi, u\rangle| g(u, \langle x, u\rangle)\, d\sigma(u) \tag{9}$$

for $\xi \in \mathbb{R}^n$. Since the integral depends only on the even part of the function $u \mapsto g(u, \langle x, u\rangle)$, one can assume that $g(u,t) = g(-u,-t)$ for $(u,t) \in S^{n-1} \times \mathbb{R}$. Parameterizing hyperplanes of $\mathbb{R}^n$ by

$$H_{u,t} := \{y \in \mathbb{R}^n : \langle y, u\rangle = t\}$$

with $u \in S^{n-1}$ and $t \in \mathbb{R}$, we can consider the function g as a function on the space of hyperplanes, via $g(H_{u,t}) = g(u,t)$. If this function is considered as a density with respect to the (Euclidean) Haar measure on $A(n,n-1)$, it defines a signed measure η on $A(n,n-1)$. This signed measure is given by

$$\int_{A(n,n-1)} f\,d\eta = \int_{S^{n-1}} \int_{\mathbb{R}} f(H_{u,t})g(u,t)\,dt\,d\sigma(u) \tag{10}$$

for nonnegative measurable functions f on the space $A(n,n-1)$ of hyperplanes. Let $k \in \{1,\dots,n\}$. The signed measure η induces a signed measure η_{n-k} on the space $A(n,n-k)$ of $(n-k)$-flats by means of

$$\int_{A(n,n-k)} f\,d\eta_{n-k} = c_k \int_{A(n,n-1)} \cdots \int_{A(n,n-1)} f(H_1 \cap \ldots \cap H_k)\,d\eta(H_1)\cdots d\eta(H_k) \tag{11}$$

for nonnegative measurable functions f on $A(n,n-k)$; here

$$c_k := \frac{2^k}{k!\kappa_k}$$

is a convenient normalizing factor. (Observe that $H_1 \cap \ldots \cap H_k \in A(n,n-k)$ for $\eta^{\otimes k}$-almost all k-tuples $(H_1,\dots,H_k) \in A(n,n-1)^k$.) In terms of hyperplane parameters, this reads

$$\int_{A(n,n-k)} f\,d\eta_{n-k} = c_k \int_{S^{n-1}} \cdots \int_{S^{n-1}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f\left(H_{u_1,t_1} \cap \ldots \cap H_{u_k,t_k}\right) \times \tag{12}$$

$$\times\, g(u_1,t_1)\cdots g(u_k,t_k)\,dt_1\cdots dt_k\,d\sigma(u_1)\cdots d\sigma(u_k).$$

The measures η_j thus defined appear in the Crofton formulae of Theorem 1. In the Euclidean case, where $F(x,\xi) = \langle \xi,\xi \rangle$, they coincide with the Haar measures μ_j in the classical formula (1). This construction of the measures η_{n-k} on $A(n,n-k)$ appeared first, for the case of Minkowski spaces, in [35], Theorem 7.1.

For the proof of Theorem 1 in Section 4, we need the following preparations. For each $x \in \mathbb{R}^n$, we define a signed measure ρ_x on S^{n-1} by

$$\rho_x(A) := \int_A g(u,\langle x,u \rangle)\,d\sigma(u) \tag{13}$$

for Borel sets $A \subset S^{n-1}$. Then we can write (9) as

$$h(B_x^o, \xi) = \int_{S^{n-1}} |\langle \xi, u \rangle| \, d\rho_x(u). \tag{14}$$

It is known from the theory of generalized zonoids that this formula, which can be considered as giving (half of) the Euclidean lengths of the one-dimensional orthogonal projections of B_x^o, extends to higher-dimensional projections. For affine subspaces $E, L \in A(n,k)$, let $[E, L^\perp] = |\langle E, L \rangle|$ be the absolute value of the determinant (in dimension k) of the orthogonal projection from E to L. By $L(u_1, \dots, u_k)$ and $[u_1, \dots, u_k]$ we denote, respectively, the linear subspace spanned by the vectors $u_1, \dots, u_k$ and the k-dimensional volume of the parallelepiped spanned by them. Let $k \in \{1, \dots, n\}$ and $E \in A(n,k)$. Then, for $x \in \mathbb{R}^n$,

$$\mathcal{H}^k(B_x^o|E) = \frac{2^k}{k!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} [E, L(u_1, \dots, u_k)^\perp] \, [u_1, \dots, u_k] \, d\rho_x(u_1) \cdots d\rho_x(u_k); \tag{15}$$

see Weil [38], p. 176. It is important to notice that this follows from (14) even if ρ_x is only a signed measure. Defining the signed measure $\rho_x^{(k)}$ on $G(n,k)$ by

$$\rho_x^{(k)}(A) := c_k \int_{S^{n-1}} \cdots \int_{S^{n-1}} \mathbf{1}_A(L(u_1, \dots, u_k))[u_1, \dots, u_k] \, d\rho_x(u_1) \cdots d\rho_x(u_k) \tag{16}$$

for Borel sets $A \subset G(n,k)$, we can write (7) and (15) in the form

$$\sigma_k(x, E) = \int_{G(n,k)} [E, L^\perp] \, d\rho_x^{(k)}(L) \quad \text{for} \quad x \in \mathbb{R}^n, E \in G(n,k). \tag{17}$$

(Essentially, the definition (16) goes back to Matheron [25], p. 101 ; later uses of this 'projection generating measure', as it has been called, begin with Goodey and Weil [23].) With these notations, we have

$$\int_{G(n,n-j)} \sigma_{k+j-n}(x, E \cap L^\perp)[E, L^\perp] \, d\rho_x^{(n-j)}(L) = \frac{c_{k+j-n} c_{n-j}}{c_k} \sigma_k(x, E) \tag{18}$$

for $k, j \in \{1, \dots, n-1\}$ with $k + j - n \geq 0$ and for every $E \in G(n,k)$. The proof given in [35], Lemma 7.2, for this equation in the case of a measure ρ carries over without change to the signed measure ρ_x, for every $x \in \mathbb{R}^n$.

§3. THE AREA FORMULA FOR PROJECTIVE FINSLER SPACES

For the proof of (5), we need an extension of Federer's area formula ([20], p. 243), which holds for Lipschitz mappings from $\mathbb{R}^k$ to $\mathbb{R}^n$ ($k \leq n$), to Lipschitz mappings into a projective Finsler space $(\mathbb{R}^n, F)$. This could be deduced from a general version for metric spaces (Kirchheim [24], Corollary 8), but for convenience we give here an elementary proof, by just adapting the proof in [35] for Minkowski spaces to the present situation.

In the following, E^k denotes the Euclidean unit ball of $\mathbb{R}^k$. For a Lipschitz mapping $f : \mathbb{R}^k \to \mathbb{R}^n$, the differential of f at $z \in \mathbb{R}^k$ exists for $\mathcal{H}^k$-almost all z and is denoted by Df_z.

Theorem 2. *Let $(\mathbb{R}^n, F)$ be a projective Finsler space. Let $k \in \{1, \dots, n\}$, and let $f : \mathbb{R}^k \to \mathbb{R}^n$ be a Lipschitz map. Then*

$$\kappa_k \int_{\mathbb{R}^n} \text{card}\,(A \cap f^{-1}(\{x\}))\, d\mathcal{H}^k_F(x) = \int_A \mathcal{H}^k_{F,f(z)}(Df_z(E^k))\, d\mathcal{H}^k(z)$$

for every $\mathcal{H}^k$-measurable subset A of $\mathbb{R}^k$.
If h is a nonnegative $\mathcal{H}^k$-measurable function on $\mathbb{R}^k$, then

$$\kappa_k \int_{\mathbb{R}^n} \sum_{y \in f^{-1}(\{x\})} h(y)\, d\mathcal{H}^k_F(x) = \int_{\mathbb{R}^k} h(z) \mathcal{H}^k_{F,f(z)}(Df_z(E^k))\, d\mathcal{H}^k(z).$$

The metric d_F in a Finsler space $(\mathbb{R}^n, F)$ was introduced in Section 2. For a norm N on $\mathbb{R}^k$ and functions $f : \mathbb{R}^k \to \mathbb{R}^n$ and $g : C \to \mathbb{R}^k$ with $C \subset \mathbb{R}^n$ we use the notation

$$\text{Lip}\,(N, d_F, f) := \sup_{x \neq y} \frac{d_F(f(x), f(y))}{N(x-y)} \quad \text{and} \quad \text{Lip}\,(d_F, N, g) := \sup_{x \neq y} \frac{N(g(x) - g(y))}{d_F(x, y)}.$$

By $[x, y]$ we denote the closed segment in $\mathbb{R}^n$ with endpoints x, y.

Now suppose that a projective Finsler space $(\mathbb{R}^n, F)$, a positive integer $k \leq n$ and a Lipschitz map $f : \mathbb{R}^k \to \mathbb{R}^n$ are given, as in Theorem 2. The proof of Theorem 2 requires the following lemma.

Lemma. *Let A be a Borel subset of $\{x \in \mathbb{R}^k : f$ is differentiable at x and Df_x is injective$\}$ and let $t > 1$. Then there is a countable Borel covering C of A such that,* *for each $C \in \mathcal{C}$, the restriction $f|_C$ is injective and there is a norm N (depending on C) on $\mathbb{R}^k$ satisfying*

$$\mathrm{Lip}\,(N, d_F, f|_C) \le t, \qquad \mathrm{Lip}\,(d_F, N, (f|_C)^{-1}) \le t$$

and

$$t^{-k}\mu_N(E^k) \le \mathcal{H}^k_{F,f(x)}(Df_x(E^k)) \le t^k \mu_N(E^k) \qquad \text{for } x \in C,$$

where μ_N is the k-dimensional Hausdorff measure induced on $\mathbb{R}^k$ by the norm N.

Proof. We extend the proof of Lemma 5.1 in [35]. Choose $\epsilon > 0$ such that $t^{-1} + \epsilon < 1 < t - \epsilon$, further a countable, dense subset D of $\mathbb{R}^k$ and a countable family $\mathcal{N}$ of norms on $\mathbb{R}^k$ such that, for each norm N' on $\mathbb{R}^k$, there is a norm $N \in \mathcal{N}$ satisfying $(t^{-1}+\epsilon)N \le N' \le (t-\epsilon)N$. For $z \in D$, $N \in \mathcal{N}$ and $i \in \mathbb{N}$ let $C(z, N, i)$ be the set of all $b \in E(z, i^{-1})$ (where $E(z,r)$ is the Euclidean ball in $\mathbb{R}^k$ with centre z and radius r) such that, for all $a \in E(z, i^{-1})$ and all $p \in [f(a), f(b)]$,

$$(t^{-1}+\epsilon)N \le \|Df_b(\cdot)\|_p \le (t-\epsilon)N, \tag{19}$$

$$\|f(a) - f(b) - Df_b(a-b)\|_p \le \epsilon N(a-b). \tag{20}$$

For $a, b \in C(z, N, i)$ we infer from (19) and (20) that

$$t^{-1}N(a-b) \le \|f(a)-f(b)\|_p \le tN(a-b) \qquad \text{for all} \quad p \in [f(a), f(b)]. \tag{21}$$

In particular, $f|_{C(z,N,i)}$ is injective. We assert that

$$\mathrm{Lip}\,(N, d_F, f|_{C(z,N,i)}) \le t, \tag{22}$$

$$\mathrm{Lip}\,(d_F, N, (f|_{C(z,N,i)})^{-1}) \le t. \tag{23}$$

For the proof, let $a, b \in C(z, N, i)$. Since the Finsler space $(\mathbb{R}^n, F)$ is projective, the distance $d_F(f(a), f(b))$ is given by the Finsler length of the segment $[f(a), f(b)]$, thus

$$d_F(f(a), f(b)) = \int_0^1 \|f(b) - f(a)\|_{(1-\tau)f(a)+\tau f(b)}\, d\tau \le tN(a-b)$$

by (21). This gives (22), and (23) is obtained similarly (only here we use the fact that the Finsler space is projective).

The inequalities (19) for the norms $(t^{-1}+\epsilon)N$, $\|Df_b(\,)\|_p$, $(t-\epsilon)N$ imply for the induced k-dimensional Hausdorff measures the estimates

$$t^{-k}\mu_N(E^k) \le (t^{-1}+\epsilon)^k \mu_N(E^k) \le \mathcal{H}^k_{F,f(b)}(Df_b(E^k)) \le (t-\epsilon)^k \mu_N(E^k) \le t^k \mu_N(E^k).$$

We show that $\{C(z,N,i) : z \in D,\ N \in \mathcal{N},\ i \in \mathbb{N}\}$ is a covering of A. Let $b \in A$. For $c \in E(b,2)$, consider the norm $N_{b,c} := \|Df_b(\cdot)\|_{f(c)}$. Since F is continuous, we can choose a number $i \in \mathbb{N}$ and a norm $N \in \mathcal{N}$ so that

$$(t^{-1} + \epsilon)N \leq N_{b,c} \leq (t - \epsilon)N \qquad \text{for all} \quad c \in E(b, 2i^{-1}).$$

Also by continuity of F, and by compactness of $E(b,2)$, there is a number M so that $\|\cdot\|_{f(c)} \leq M\|\cdot\|_{f(b)}$ for all $c \in E(b,2)$. Since f is differentiable at b, we can further choose i so small that, for all $a \in E(b, 2i^{-1})$,

$$\|f(a) - f(b) - Df_b(a-b)\|_{f(b)} \leq M^{-1}\epsilon N(a-b).$$

This implies

$$\|f(a) - f(b) - Df_b(a-b)\|_{f(c)} \leq \epsilon N(a-b) \qquad \text{for all} \quad a, c \in E(b, 2i^{-1}).$$

Now we choose $z \in D$ with $z \in E(b, i^{-1})$. Then (20) is satisfied for all $a \in E(z, i^{-1})$ and all $p \in [f(a), f(b)]$, hence $b \in C(z,N,i)$.
Finally, we choose $\{x_j : j \in \mathbb{N}\}$ dense in $\mathbb{R}^k$ and $\{\tau_j : j \in \mathbb{N}\}$ dense in $[0,1]$ and put

$$\begin{aligned} A_{j,m} := \{b \in E(z, i^{-1}) : (t^{-1} + \epsilon)N(x_j) \leq \\ \leq \|Df_b(x_j)\|_{(1-\tau_m)f(x_j)+\tau_m f(b)} \leq (t-\epsilon)N(x_j)\} \end{aligned}$$

and

$$B_{j,m} := \{b \in E(z,i^{-1}) : \|f(x_j) - f(b) - Df_b(x_j - b))\|_{(1-\tau_m)f(x_j)+\tau_m f(b)} \leq \epsilon N(x_j - b)\}.$$

Then

$$C(z,N,i) = \bigcap_{j,m \in \mathbb{N}} A_{j,m} \cap \bigcap_{\substack{j,m \in \mathbb{N} \\ x_j \in E(z,i^{-1})}} B_{j,m},$$

which shows that $C(z,N,i)$ is a Borel set.

Proof of Theorem 2. This is now a straightforward generalization of the proof of Theorem 5.2 (and of (32), corrected) in [35] : one has merely to replace $\mu^k(f(G))$ in that proof by $\mathcal{H}^k_F(f(G))$ and $\mu^k(Df_z(E^k))$ in the integrands by $\mathcal{H}^k_{F,f(z)}(Df_z(E^k))$.

For the envisaged application, recall that $\mathcal{H}^k$ is the k-dimensional Hausdorff measure induced on $\mathbb{R}^n$ by the auxiliary Euclidean structure. Let $L \subset \mathbb{R}^n$ be a k-dimensional linear subspace. Let $x \in \mathbb{R}^n$. The restriction of the norm $\|\cdot\|_x$ to L has unit ball $B_x \cap L$; the corresponding k-dimensional Hausdorff measure on L is $\mathcal{H}^k_{F,x} \angle L$, and we have $\mathcal{H}^k_{F,x}(B_x \cap L) = \kappa_k$. Since both, $\mathcal{H}^k_{F,x} \angle L$ and $\mathcal{H}^k \angle L$ are Haar measures on L, they are proportional, thus

$$\frac{\mathcal{H}^k_{F,x}(\cdot \cap L)}{\kappa_k} = \frac{\mathcal{H}^k(\cdot \cap L)}{\mathcal{H}^k(B_x \cap L)}. \tag{24}$$

Now suppose that $f : A \to \mathbb{R}^n$ is an injective Lipschitz map, $A \subset \mathbb{R}^k$ is a bounded Borel set, and $f(A) = M$ is a Borel set. Let $g : M \to \mathbb{R}$ be a nonnegative $\mathcal{H}^k$-measurable function. If we apply Theorem 2, equation (24), and then Theorem 2 to the Euclidean metric, we get

$$\int_M g(x)\, d\mathcal{H}^k_F(x) = \int_A g(f(z)) \frac{\mathcal{H}^k_{F,f(z)}(Df_z(E^k))}{\kappa_k}\, d\mathcal{H}^k(z) =$$

$$= \int_A g(f(z)) \frac{\mathcal{H}^k(Df_z(E^k))}{\mathcal{H}^k(B_{f(z)} \cap Df_z(\mathbb{R}^k))}\, d\mathcal{H}^k(z) = \kappa_k \int_M g(x) \frac{1}{\mathcal{H}^k(B_x \cap T_x M)}\, d\mathcal{H}^k(x).$$

Now the choice $g = 1$ on M gives (6), and the choice

$$g(x) = vp(B_x \cap T_x M)/\kappa_k^2 = \mathcal{H}^k(B_x^\circ | T_x M)\mathcal{H}^k(B_x \cap T_x M)/\kappa_k^2$$

gives (5), under our special assumptions on M. However, to this special case the proof for a general $(\mathcal{H}^k, k)$-rectifiable Borel set M can be reduced; see Theorem 3.2.29 in Federer [20]. The same special assumptions on M can be made in the proof of Theorem 1 in the next section. Here one has to observe that both sides of (3) are zero if $\mathcal{H}^k_F(M) = 0$. This is true in the Euclidean case, as follows from Federer [19]; in a smooth projective Finsler space it then follows by observing that the measure η_j has a density with respect to the Euclidean invariant measure μ_j and that in compact subsets of $\mathbb{R}^n$, the Hausdorff measure $\mathcal{H}^s_F$ can be estimated from above by a constant multiple of $\mathcal{H}^s$.

§4. A GENERAL CROFTON FORMULA

Now we can prove Theorem 1. We assume that $(\mathbb{R}^n, F)$ is a smooth projective Finsler space, and the signed measure η_j on $A(n, j)$ is defined as in Section 2. Let

$k \in \{1, \dots, n\}$, $j \in \{1, \dots, n-1\}$ with $q := k + j - n \geq 0$, and put $m := n - j$. Let $M \subset \mathbb{R}^n$ be a $(\mathcal{H}^k, k)$-rectifiable Borel set. We may assume that M is of the special form as assumed at the end of the last section. From (12), we have

$$\int_{A(n,j)} \mathrm{vol}_q(F \cap M)\, d\eta_j(F) = c_m \int_{(S^{n-1})^m} I(u_1, \dots, u_m)\, d\sigma^{\otimes m}(u_1, \dots, u_m)$$

with

$$I(u_1, \dots, u_m) :=$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathrm{vol}_q(H_{u_1,t_1} \cap \dots \cap H_{u_m,t_m} \cap M) g(u_1, t_1) \cdots g(u_m, t_m)\, dt_1 \cdots dt_m.$$

For $i = 1, \dots, m$, let C_i be an $(n-1)$-dimensional unit cube in $u_i^{\perp} := \{x \in \mathbb{R}^n : \langle x, u_i \rangle = 0\}$. As in [35], formula (54) (where no invariance property of τ_q is needed), we get

$$\mathrm{vol}_q(H_{u_1,t_1} \cap \dots \cap H_{u_m,t_m} \cap M) = \int_{u_1^{\perp}} \cdots \int_{u_m^{\perp}} \mathrm{vol}_q(D_q(y,t))\, d\lambda_{n-1}(y_1) \cdots d\lambda_{n-1}(y_m),$$

where λ_{n-1} denotes the $(n-1)$-dimensional Lebesgue measure and where

$$D_q(y,t) := (C_1 + t_1 u_1 + y_1) \cap \dots \cap (C_m + t_m u_m + y_m) \cap M.$$

By (8), this gives

$$I(u_1, \dots, u_m) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{u_1^{\perp}} \cdots \int_{u_m^{\perp}} \int_{D_q(y,t)} \sigma_q(x, T_x D_q(y,t))$$

$$d\mathcal{H}^q(x) d\lambda_{n-1}(y_1) \cdots d\lambda_{n-1}(y_m) g(u_1, t_1) \cdots g(u_m, t_m)\, dt_1 \cdots dt_m =$$

$$= \int_{u_1^{\perp}} \cdots \int_{u_m^{\perp}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{D_q(y,t)} \sigma_q(x, T_x D_q(y,t)) g(u_1, \langle x, u_1 \rangle) \cdots g(u_m, \langle x, u_m \rangle)$$

$$d\mathcal{H}^q(x) dt_1 \cdots dt_m\, d\lambda_{n-1}(y_1) \cdots d\lambda_{n-1}(y_m).$$

Here we have applied Fubini's theorem and then made use of the fact that $x \in D_q(y,t)$ satisfies $x \in u_i^\perp + t_i u_i = H_{u_i,t_i}$, hence $\langle x, u_i\rangle = t_i$, for $i = 1,\dots,m$. Thus we obtain

$$I(u_1,\dots,u_m) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \int_{D_q'(z)} \sigma_q(x, T_x D_q'(z)) g(u_1, \langle x,u_1\rangle) \cdots g(u_m, \langle x,u_m\rangle) \times$$

$$\times\, d\mathcal{H}^q(x)\, d\lambda_n(z_1) \cdots d\lambda_n(z_m)$$

with $D_q'(z) := (C_1 + z_1) \cap \ldots \cap (C_m + z_m) \cap M$. Writing

$$f(x) := \sigma_q\left(x, L(u_1,\dots,u_m)^\perp \cap T_x M\right) g(u_1, \langle x,u_1\rangle) \cdots g(u_m, \langle x,u_m\rangle),$$

we get

$$I(u_1,\dots,u_m) = \int\limits_{\mathbb{R}^n} \cdots \int\limits_{\mathbb{R}^n} \int\limits_{(C_1+z_1)\cap\ldots\cap(C_m+z_m)\cap M} f(x)\, d\mathcal{H}^q(x)\, d\lambda_n(z_1) \cdots d\lambda_n(z_m).$$

Now we use Lemma 6.1 of [35], where we put $p = m$, $M_0 = M$, $M_i = C_i$ for $i = 1,\dots,m$. We obtain

$$I(u_1,\dots,u_m) = \int_{C_1} \cdots \int_{C_m} \int_M f(x_0)[T_{x_0}^\perp M, T_{x_1}^\perp C_1, \dots, T_{x_m}^\perp C_m]\, d\mathcal{H}^k(x_0) \times$$

$$\times\, d\mathcal{H}^{n-1}(x_1) \cdots d\mathcal{H}^{n-1}(x_m) = [u_1,\dots,u_m] \int_M f(x)[L(u_1,\dots,u_m)^\perp, T_x M]\, d\mathcal{H}^k(x).$$

Inserting this and using (13), (16) and (18) we conclude that

$$\int_{A(n,j)} \mathrm{vol}_q(F \cap M)\, d\eta_j(F) =$$

$$= c_m \int_{(S^{n-1})^m} \int_M [u_1,\dots,u_m][L(u_1,\dots,u_m)^\perp, T_x M] \sigma_q(x, L(u_1,\dots,u_m)^\perp \cap T_x M)$$

$$g(u_1, \langle x,u_1\rangle) \cdots g(u_m, \langle x,u_m\rangle)\, d\mathcal{H}^k(x)\, d\sigma^{\otimes m}(u_1,\dots,u_m) =$$

$$= c_m \int_M \int_{(S^{n-1})^m} \sigma_q(x, L(u_1,\dots,u_m)^\perp \cap T_x M)[L(u_1,\dots,u_m)^\perp, T_x M]$$

$$[u_1, \ldots, u_m] g(u_1, \langle x, u_1 \rangle) \cdots g(u_m, \langle x, u_m \rangle)\, d\sigma^{\otimes m}(u_1, \ldots, u_m)\, d\mathcal{H}^k(x) =$$

$$= \int_M \int_{G(n,m)} \sigma_q(x, L^\perp \cap T_x M)[L^\perp, T_x M]\, d\rho_x^{(m)}(L)\, d\mathcal{H}^k(x) =$$

$$= \frac{c_q c_m}{c_k} \int_M \sigma_k(x, T_x M)\, d\mathcal{H}^k(x) = \frac{c_{k+j-n} c_{n-j}}{c_k} \mathrm{vol}_k(M).$$

This completes the proof of Theorem 1.

Резюме. Доказываются интегрально–геометрические ф рмулы типа Крофтона для площадей Холмса-Томпсона спрямляемых борелевских множеств в гладких проективных пространствах Финслера.

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CAUCHY–KUBOTA–TYPE INTEGRAL FORMULAE FOR THE GENERALIZED COSINE TRANSFORMS

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Abstract. The paper considers the action of Radon transforms on Grassmann manifolds for some special functions. These functions are positive powers of the volume of certain parallelepipeds. As a consequence some integral geometric formulas of Cauchy–Kubota–type for the generalized cosine transforms on Grasmannians are proved. Some applications of these results in convex and stochastic geometry, stereology and geometric tomography are discussed.

§1. INTRODUCTION

This section contains the notation most of the definitions and the main results. Their proofs and corollaries are given in sections 4 – 6. Various applications of the generalized cosine transforms in geometry are touched upon in section 2. The history of the problem is discussed in section 3.

1.1. Preliminaries. Let $G(k,d)$ be the Grassmann manifold of all linear k–dimensional subspaces of $\mathbb{R}^d$, $d \geq 3$. The **Radon transform on Grassmannians** and its dual we introduce following the paper of Grinberg [10] : for $1 \leq i < j \leq d-1$

$$(R_{ij}f)(\xi) = \int\limits_{\eta \in G(i,d):\eta\subset\xi} f(\eta)\,\sigma(d\eta), \quad (R_{ji}g)(\eta) = \int\limits_{\xi\in G(j,d):\xi\supset\eta} g(\xi)\,\sigma(d\xi),$$

where $f \in L^1(G(i,d))$, $g \in L^1(G(j,d))$, $\xi \in G(j,d)$, $\eta \in G(i,d)$, $\sigma(\cdot)$ is the unique rotation invariant measure on the appropriate integration space with total mass 1. Such Radon transforms find numerous applications in convex geometry, see [3] – [5], [8], [9].

We make use of the following notation :

$L\{a_1,\ldots,a_k\}$ = the k-flat spanned by the vectors $a_1,\ldots,a_k$,

$Vol^{(k)}(a_1,\ldots,a_k)$ = the non-oriented volume of the parallelepiped spanned by $a_1,\ldots,a_k$ (we shall often omit the dimension of the volume),

$Vol^{(k)}(\xi)$ = the non-oriented volume of the parallelepiped spanned by the orthonormal basis vectors of the k-flat ξ,

$e_1,\ldots,e_d$ = the Cartesian unit basis vectors in $\mathbb{R}^d$,

$\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the $(d-1)$-dimensional sphere $\mathcal{S}^{d-1}$,

$d\eta$ = the rotation invariant measure on a Grassmannian with total mass 1,

$b(\xi)$ = a set of the orthonormal basis vectors $a_1,\ldots,a_k$, such that $\xi = L\{a_1,\ldots,a_k\}$ for $\xi \in G(k,d)$.

1.2. Main results. In Section 4, we study the action of R_{ij} and R_{ji} on the functions of the type $f(\eta) = [\eta,\zeta_0]^\alpha$ ($f(\eta) = [\eta^\perp,\zeta_0]^\alpha$ in the case of the dual Radon transform) for η from $G(i,d)$ (or $G(j,d)$, respectively) and $\alpha > 0$, where

$$[a,b] = Vol^{(2d-n-m)}(a_{n+1},\ldots,a_d,b_{m+1},\ldots,b_d)$$

and $\zeta_0 = L\{e_1,\ldots,e_k\}$. Here $a_{n+1},\ldots,a_d$ and $b_{m+1},\ldots,b_d$ are some orthonormal bases in the orthogonal complements $a^\perp$ and $b^\perp$ for $a \in G(n,d)$, $b \in G(m,d)$.

Theorem 1.1. *Let $\xi \in G(j,d)$, $\zeta_0 = L\{e_1,\ldots,e_k\}$, $i < j$, $i+k \geq d$, $d \geq 3$, $\alpha > 0$. Then*

$$(R_{ij}[\cdot,\zeta_0]^\alpha)(\xi) = \int\limits_{\eta\in G(i,d):\eta\subset\xi} [\eta,\zeta_0]^\alpha\,\sigma(d\eta) = c(\alpha)[\xi,\zeta_0]^\alpha, \tag{1.1}$$

where

$$c(\alpha) = \prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{j-l}{2}\right)\Gamma\left(\frac{i-l+\alpha}{2}\right)}{\Gamma\left(\frac{i-l}{2}\right)\Gamma\left(\frac{j-l+\alpha}{2}\right)}. \tag{1.2}$$

Proposition 1.1. *Let $i < j \leq k < d$, $d \geq 3$. Then for $\alpha > 0$ the following relation holds :*

$$(R_{ji}[\cdot^\perp,\zeta_0]^\alpha)(\eta) = c^*(\alpha)[\eta^\perp,\zeta_0]^\alpha, \tag{1.3}$$

where $\eta \in G(i,d)$,

$$c^*(\alpha) = \prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-j-l+\alpha}{2}\right)}{\Gamma\left(\frac{d-j-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha}{2}\right)}. \tag{1.4}$$

In Section 5, we study some interesting connections between the generalized cosine transforms and Radon transforms. Let $\mathbf{M}(G(i,d))$ be the Banach space of all signed measures on $G(i,d)$ with finite total variation, and $C(G(i,d))$ be the space of all continuous functions on $G(i,d)$. Introduce for $i+j \geq d$ the **generalized cosine transform** $T_{ij}: \mathbf{M}(G(i,d)) \longmapsto C(G(j,d))$,

$$(T_{ij}\theta)(\xi) = \int\limits_{G(i,d)} [\xi,\eta]\,\theta(d\eta), \tag{1.5}$$

where $\theta \in \mathbf{M}(G(i,d))$, $\xi \in G(j,d)$. In particular, if $\theta(d\eta) = f(\eta)d\eta$, meaning that θ is absolute continuous with respect to $d\eta$, we write

$$(T_{ij}f)(\xi) = \int\limits_{G(i,d)} [\xi,\eta] f(\eta)\,d\eta.$$

To achieve greater generality, we imbed the generalized cosine transforms into the new families of operators $\{T^{\alpha}_{ij}\}$, $\{\tilde{T}^{\alpha}_{ij}\}$, where α is a positive parameter (see Section 2 for details) :

$$T^{\alpha}_{ij},\ \tilde{T}^{\alpha}_{ij} : \mathbf{M}(G(i,d)) \longmapsto C(G(j,d)),$$

$$(T^{\alpha}_{ij}\theta)(\xi) = \int\limits_{G(i,d)} [\xi,\eta]^{\alpha}\,\theta(d\eta), \quad \left(\tilde{T}^{\alpha}_{ij}\theta\right)(\xi) = \int\limits_{G(i,d)} [\xi,\eta^{\perp}]^{\alpha}\,\theta(d\eta), \quad j \geq i.$$

On integrable functions the above transforms can be introduced as before. Evidently, these families comprise generalized cosine (T^{1}_{ij}) transforms. It is also clear, that

$$\left(\tilde{T}^{\alpha}_{ij}\theta\right)(\xi) = \left(T^{\alpha}_{d-i,j}\theta^{\perp}\right)(\xi), \tag{1.6}$$

where $\theta^{\perp}(d\nu) = \theta(d\nu^{\perp})$ for $\nu \in G(d-i,d)$.
In Section 5, the first Cauchy-Kubota - type formula for operators T^{α}_{ij} is proved.

Proposition 1.2. *[First Cauchy-Kubota - type formula] For any $\alpha > 0$ and dimensions i,j,k with $i+k \geq d$, $i < j$ the following integral relation is valid on the space $\mathbf{M}(G(k,d))$:*

$$R_{ij}T^{\alpha}_{ki} = c(\alpha)T^{\alpha}_{kj}, \tag{1.7}$$

where the constant $c(\alpha)$ is defined in (1.2).

Two corollaries of the above proposition and their stereological meaning are discussed at the end of the section. Some interesting consequences of the double fibration relation for T^{α}_{ij} are considered in Section 6, of which the more important is the following.

Proposition 1.3. *[Second Cauchy–Kubota – type formula] For all $i < j$, $i + k \geq d$, $\alpha > 0$ and all absolute integrable functions $\varphi \in L^1(G(j,d))$,*

$$\left(T^{\alpha}_{jk}\varphi\right)(\zeta) = c^{-1}(\alpha) T^{\alpha}_{ik}\left(R_{ji}\varphi\right)(\zeta), \quad \zeta \in G(k,d), \tag{1.8}$$

where $c(\alpha)$ is given by (1.2).

Some upper bounds for the weighted images of Radon transforms are given at the end of §6.

§2. THE MEANING OF T^{α}_{ij} IN STOCHASTIC GEOMETRY

The transforms (1.5) generalize the well-known notion of the **spherical cosine transform** :

$$T\theta(u) = \int\limits_{S^{d-1}} |\langle u, v\rangle| \, \theta(dv) = \left(T_{d-1,1}\theta\right)(u), \quad u \in S^{d-1}.$$

There exists exhaustive literature on the spherical cosine transform and its use in geometry (see e.g. [1], [2], [6], [16], [17]).

The generalized cosine transforms find important applications in convex and stochastic geometry as well. Namely, the **i-th projection function** $v_i(K;\cdot)$ of a **zonoid** K is the generalized cosine transform of its **projection generating measure** $\rho_i(K,\cdot)$:

$$v_i(K;\eta) = \left(T_{i,d-i}\rho_i(K,\cdot)\right)(\eta^{\perp}), \quad \eta \in G(i,d) \tag{2.1}$$

(cf. [8]). By definition, $v_i(K;\eta)$ is the i-dimensional volume of the orthogonal projection of K onto η.

Furthermore, in stochastic geometry $(T_{ij}\theta)(\eta)$ means the **rose of intersections** of a stationary stochastic process of i-dimensional manifolds (or affine flats) Φ^d_i in $\mathbb{R}^d$ with unit intensity and directional distribution measure $\theta(\cdot)$ with an arbitrary j-dimensional flat η through the origin.

Roughly speaking, a stochastic process Φ^d_i of affine i-flats can be thought of as a random countable collection of affine i-flats in $\mathbb{R}^d$. Its stationarity means the invariance of its probabilistic properties with respect to all translations. The intensity

of Φ_i^d is the mean i-volume content of its flats in a unit observation window. The directional distribution of Φ_i^d is the probability distribution of the direction $r(\xi)$ of the **typical** i-flat $\xi \in \Phi_i^d$, i.e., of a flat picked up "at random" among the others in any realization of Φ_i^d.
The **direction** $r(\xi)$ of an affine i-flat ξ is the unique i-flat through the origin parallel to ξ (cf. [20] for exact definitions). Sometimes it is necessary to restore the probabilistic characteristics of such spatial stochastic processes, if one observes the process of intersections of the original process with all affine j–flats η. Due to stationarity, it is enough to consider only $\eta \in G(j,d)$.
One of the characteristics of the latter process is the so-called **rose of intersections**, i.e. the mean i-content of intersection planes $\Phi_i^d \cap \eta$ in a unit test window within η for arbitrary $\eta \in G(j,d)$. As shown in [18] and [19], the retrieval of the directional distribution θ of Φ_i^d from its rose of intersections $T_{ij}\theta$ is possible for some particular dimensions i and j.
The transforms of the parametric family $\{T_{ij}^\alpha\}$ can also get an interpretation in terms of stochastic geometry :

$$(T_{ij}^\alpha\theta)\,(\xi) = \int\limits_{G(i,d)} [\xi,\eta]^\alpha\,\theta(d\eta) = E_\theta\left([\xi,\eta]^\alpha\right)$$

is the moment of order α of the quantity $[\xi,\eta]$ = generalized "sine of the angle" between the direction η of the affine planes of the process Φ_i^d and a fixed **test direction** $\xi \in G(j,d)$ with respect to the directional distribution θ. All the results of this paper are valid for arbitrary real $\alpha > 0$. As far as it is known to the author, the present work is the first attempt to treat those moments.
One can also find another interpretation of T_{ij}^α at least in $\mathbb{R}^2$ and $\mathbb{R}^3$, which can be extended analogously to arbitrary dimensions d. It can be shown (see [20], pp. 286 – 303), that

$$A \longmapsto \frac{\int_A [\xi,\eta]\,\theta(d\eta)}{\int_{G(i,d)}[\xi,\eta]\,\theta(d\eta)}, \quad A \text{ measurable}$$

is the distribution of "sine" of the typical intersection angle between the test hyperplane (or line) ξ and Φ_i^d ($d=2$ or $d=3$). If we rewrite T_{ij}^α in the form

$$c \cdot \frac{\int_{G(i,d)}[\xi,\eta]^{\alpha-1}[\xi,\eta]\,\theta(d\eta)}{(T_{ij}\theta)\,(\xi)} = c \cdot E\left([\xi,\eta]^{\alpha-1} | \eta \in \Phi_i^d,\ \eta \cap \xi \neq \emptyset\right),$$

where $c = (T_{ij}\theta)(\xi)$, we get that $\left(T_{ij}^{\alpha}\theta\right)(\xi)$ is proportional to the moment of order $\alpha - 1$ of the absolute value of the "sine" of the typical intersection angle of Φ_i^d with a fixed test flat (line) ξ.

The meaning of $\left(\tilde{T}_{ij}^{1}\theta\right)(\xi)$ also becomes transparent : it is the rose of intersections of a stationary $(d-i)$-flat process with directional distribution $\theta^{\perp}(\cdot)$ and intensity 1 with a j-flat ξ (cf. relation (1.6)).

It would be of some interest to illustrate the use of Proposition 1.3 in stochastic geometry. Namely, for the values $\alpha = 1$, $d = 3$, $i = 1$, $j = 2$ and $k = 2$, by (1.8),

$$(T_{22}\varphi)(\zeta) = \frac{\pi}{2}T_{12}(R_{21}\varphi)(\zeta), \quad \zeta \in G(k,d). \tag{2.2}$$

The transform $(T_{22}\varphi)(\zeta)$ is the rose of intersections of the stationary process of planes Φ_2^3 in three dimensions with a test plane ζ. The process Φ_2^3 has the unit intensity and the directional distribution with density φ. By (2.2), this rose of intersections is equal to the rose of intersections T_{12} of the process Φ_1^3 of lines with the same test plane ζ, where this new process Φ_1^3 has the unit intensity and directional distribution density $R_{21}\varphi$ obtained from φ by integration.

§3. SOME HISTORICAL REMARKS

A special case of Proposition 1.1 for the dual Radon transform with $\alpha = 1$ and dimension $k = j$ can be found in lemma 4.1 of [7]. The argument there uses the connection between volumes $[\cdot,\cdot]$, mixed volumes and projection functions. Then the following Cauchy–Kubota formula (equation (2.3) of [7]) is applied :

$$R_{ij}\left(v_i(K;\cdot)\right)(\xi) = \frac{i!k_i(j-i)!k_{j-i}}{j!k_j}V_i\left(Pr_{\xi}\{K\}\right) \tag{3.1}$$

for a convex body K and all $\xi \in G(j,d)$, where $i < j < d$, $k_d = \dfrac{2\pi^{d/2}}{d\Gamma(d/2)}$ is the volume of the d-dimensional unit ball, $Pr_{\xi}\{a\}$ denotes the orthogonal projection of the vector (plane, convex body) a onto the plane ξ, $v_i(K;\eta)$ is the i-th projection function of K, and $V_i(L)$ is the **i-th intrinsic volume** (cf. [15]) of a convex body L. Our approach differs substantially from that of [7]. It allows us to gain more generality in dimensions of involved linear subspaces and positive powers of the volume. Corollary 5.1 generalizes a well-known relation for the spherical Radon and usual cosine transforms (cf. relation (5.12) of [18] and references therein) :

$$R_{1r}T_{d-1,1} = \frac{2k_{r-1}}{\omega_r}T_{d-1,r}. \tag{3.2}$$

The name "Cauchy–Kubota - type" for the formulas stated in Propositions 1.2, 1.3 and 6.2 is due to the resemblance between the left-hand side of (1.7) and relation (3.1). Indeed, suppose $\alpha = 1$, $i < j$, and let the convex body K be a zonoid. By (2.1) and the duality relation (2.3) of [5], the left-hand side of (3.1) rewrites $\left(R_{d-i,d-j}\left(T^1_{i,d-i}\rho_i(K,\cdot)\right)\right)(\xi^\perp)$, or, equivalently, $\left(R_{ij}\left(\tilde{T}^1_{d-i,i}\rho_i^\perp(K,\cdot)\right)\right)(\xi)$. In spite of this similarity, the Cauchy–Kubota formula does not follow from our results ; nor can they be deduced as a direct corollary of (3.1).
The classical Cauchy–Kubota formulas and their versions can be found in Ch. 13, §1, 2 of [14], [15], p. 295, and [12], p. 126.

§4. RADON TRANSFORMS OF THE POWER OF THE VOLUME

Proof of Theorem 1.1. We fix an orthonormal basis $\xi_1, \ldots, \xi_j$ of $\xi \in G(j,d)$, so that $\xi_1, \ldots, \xi_i$ is an orthonormal basis of $\eta \in G(i,d)$, $i < j$. Let

$$(\eta)^\perp_\xi = L\{\xi_{i+1}, \ldots, \xi_j\} \tag{4.1}$$

be the orthogonal complement of η in ξ. We denote by $\xi_{j+1}, \ldots, \xi_d$ a certain orthonormal basis of $\xi^\perp$. Then the vectors $\xi_{i+1}, \ldots, \xi_d$ form an orthonormal basis in $\eta^\perp$. If $[\xi, \zeta_0] = 0$, i.e. $dim\left(\xi^\perp \cap \zeta_0^\perp\right) > 0$, then $[\eta, \zeta_0] = 0$ for all $\eta \subset \xi$, since $\xi^\perp \subset \eta^\perp$ and $dim\left(\eta^\perp \cap \zeta_0^\perp\right) > 0$. Hence, formula (1.1) holds automatically. It means that in the following it suffices to prove (1.1) for the case $[\xi, \zeta_0] \neq 0$, i.e.

$$\xi^\perp \cap L\{e_{k+1}, \ldots, e_d\} = \{0\}. \tag{4.2}$$

The following relation holds :

$$[\eta, \zeta_0] \equiv Vol(\xi_{i+1}, \ldots, \xi_d, e_{k+1}, \ldots, e_d) =$$

$$= Vol(\xi_{j+1}, \ldots, \xi_d, e_{k+1}, \ldots, e_d) \cdot Vol\left(Pr_{L^\perp\{\xi_{j+1},\ldots,\xi_d,e_{k+1},\ldots,e_d\}}\{\xi_{i+1}, \ldots, \xi_j\}\right),$$

or, briefly,

$$[\eta, \zeta_0] = [\xi, \zeta_0] Q(\xi, \eta), \tag{4.3}$$

where $Q(\xi, \eta)$ denotes the $(j-i)$-dimensional volume of the parallelepiped spanned by projections of $\xi_{i+1}, \ldots, \xi_j$ (cf. (4.1)) onto the plane

$$L^\perp\{\xi_{j+1}, \ldots, \xi_d, e_{k+1}, \ldots, e_d\} = L^\perp(\xi^\perp, \zeta_0^\perp). \tag{4.4}$$

Thus, by (4.3) we have

$$\int\limits_{\eta \in G(i,d):\eta \subset \xi} [\eta, \zeta_0]^\alpha \sigma(d\eta) = c_\alpha(\xi)[\xi, \zeta_0]^\alpha,$$

$$c_\alpha(\xi) = \int\limits_{\eta\in G(i,d):\eta\subset\xi} Q(\xi,\eta)^\alpha \sigma(d\eta). \tag{4.5}$$

We write $Q(\xi,\eta)$ in a different form. First we give another representation to the plane (4.4) :

$$L^\perp(\xi^\perp,\zeta_0^\perp) = \xi\cap\zeta_0. \tag{4.6}$$

Indeed, if $\tau\in L^\perp(\xi^\perp,\zeta_0^\perp)$, then $\tau\perp\{\xi_{j+1},\dots,\xi_d\}$ and $\tau\perp\{e_{k+1},\dots,e_d\}$. Hence, $\tau\in L^\perp\{\xi_{j+1},\dots,\xi_d\}=\xi$, $\tau\in L^\perp\{e_{k+1},\dots,e_d\}=\zeta_0$, or $\tau\in\xi\cap\zeta_0$. Thus, $L^\perp(\xi^\perp,\zeta_0^\perp)\subseteq\xi\cap\zeta_0$. Since $dim\left(L^\perp(\xi^\perp,\zeta_0^\perp)\right)=dim(\xi\cap\zeta_0)$ as we use the obvious formula

$$dim(a\cap b)\geq dim(a)+dim(b)-d. \tag{4.7}$$

The relation (4.6) is proved. Now show that

$$Q(\xi,\eta)=Vol^{(d-k+j-i)}\left(\xi_{j+1},\dots,\xi_d,b\Big((\xi\cap\zeta_0)_\xi^\perp\Big)\right)\stackrel{def}{=}[\xi\cap\zeta_0,\eta]_\xi. \tag{4.8}$$

By definition of $Q(\xi,\eta)$, owing to (4.6),

$$Q(\xi,\eta)=Vol^{(j-i)}\left(Pr_{\xi\cap\zeta_0}(\eta)_\xi^\perp\right). \tag{4.9}$$

The following formula holds for any flats a and c in arbitrary ambient space :

$$[a^\perp,c]\stackrel{def}{=}Vol\left(b(a),b(c^\perp)\right)=Vol(Pr_c(a))=Vol\left(Pr_{a^\perp}\left(c^\perp\right)\right).$$

Hence, equality (4.9) yields

$$Q(\xi,\eta)=Vol^{(j-i)}\left(Pr_{\xi\cap\zeta_0}(\eta)_\xi^\perp\right)=Vol\left(Pr_\eta(\xi\cap\zeta_0)_\xi^\perp\right)=[\xi\cap\zeta_0,\eta]_\xi$$

(here the ambient space is ξ). Thus, relation (4.8) is proved, and one obtains by (4.5)

$$c_\alpha(\xi)=\int\limits_{\eta\in G(i,d):\eta\subset\xi}[\xi\cap\zeta_0,\eta]_\xi^\alpha\,\sigma(d\eta).$$

We prove that $c_\alpha(\xi)$ does not depend on ξ. According to (4.2), it is sufficient to consider only the case

$$\xi^\perp\cap\zeta_0^\perp=\{0\}. \tag{4.10}$$

For any $\xi \in G(j,d)$ there exists a rotation $\gamma \in SO(d)$, such that $\xi = \gamma\xi_0$, $\xi_0 = L\{e_1,\dots,e_j\}$. Then

$$c_\alpha(\xi) = \int\limits_{\eta\in G(i,d):\eta\subset\xi_0} [\gamma\xi_0 \cap \zeta_0, \gamma\eta]^\alpha_{\gamma\xi_0}\,\sigma(d\eta) = \int\limits_{\eta\in G(i,d):\eta\subset\xi_0} [\xi_0 \cap \gamma^{-1}\zeta_0, \eta]^\alpha_{\xi_0}\,\sigma(d\eta),$$

and (4.10) has the form $(\gamma\xi_0)^\perp \cap \zeta_0^\perp = \gamma\xi_0^\perp \cap \zeta_0^\perp = \xi_0^\perp \cap \gamma^{-1}\zeta_0^\perp = \{0\}$. Without loss of generality, one can substitute γ for γ^{-1}. Thus, we have to show that

$$\bar{c}_\alpha(\gamma) \stackrel{def}{=} \int\limits_{\eta\in G(i,d):\eta\subset\xi_0} [\xi_0 \cap \gamma\zeta_0, \eta]^\alpha_{\xi_0}\,\sigma(d\eta) \tag{4.11}$$

is constant on the set

$$G_{jk} = \{\gamma \in SO(d) : \xi_0^\perp \cap \gamma\zeta_0^\perp = \{0\}\}. \tag{4.12}$$

First, let us first prove that $\dim(\xi_0 \cap \gamma\zeta_0) = j+k-d, \quad \gamma \in G_{jk}$. By (4.7), the dimension of $\xi_0 \cap \gamma\zeta_0$ can not be less than $j+k-d$. Let us prove that it also can not be greater than $j+k-d$. Suppose, ex adverso, that $dim(\xi_0 \cap \gamma\zeta_0) = m > j+k-d$. Let $\tau_1,\dots,\tau_m$ be the basis in $\xi_0 \cap \gamma\zeta_0$. Amplify it to the bases in ξ_0 and $\gamma\zeta_0$: $\xi_0 = L\{\tau_1,\dots,\tau_m,\tau_{m+1},\dots,\tau_j\}$, $\gamma\zeta_0 = L\{\tau_1,\dots,\tau_m,\bar{\tau}_{m+1},\dots,\bar{\tau}_k\}$. The number of distinct unit vectors in ξ_0, $\gamma\zeta_0$ is equal to $j+k-m$. As $m > j+k-d$, that number is less than d. So there exists at least one unit vector $x \in \mathbb{R}^d$, that does not belong to the linear hull of the bases in ξ_0 and $\gamma\zeta_0$. Then $x \in \xi_0^\perp \cap (\gamma\zeta_0)^\perp$. We arrived at the contradiction with (4.12).
Thus, we proved that any transform $\gamma \in G_{jk}$ preserves the dimension of $\beta \stackrel{def}{=} \xi_0 \cap \gamma\zeta_0 \subset \xi_0$. Identifying ξ_0 with $\mathbb{R}^j$, we can rewrite the relation (4.11) as follows :

$$\bar{c}_\alpha(\beta) = \int\limits_{\eta\in G(i,d):\eta\subset\mathbb{R}^j} [\beta,\eta]^\alpha_{\mathbb{R}^j}\,\sigma(d\eta) = \int\limits_{G(i,j)} [\beta,\eta]^\alpha_{\mathbb{R}^j}\,d\eta,$$

$\beta \subset \xi_0$, $dim(\beta) = j+k-d$. Now prove that $\bar{c}_\alpha(\beta)$ does not depend on $\beta \in G(j+k-d,j)$. Indeed, by rotation invariance (since $dim(\beta)$ does not depend on $\gamma \subset G_{jk}$), $\bar{c}_\alpha(\beta) = \bar{c}_\alpha(\beta_0)$, $\beta_0 = L\{e_1,\dots,e_{j+k-d}\}$. Thus, we have proved that

$$c(\alpha) = \int\limits_{G(i,j)} [\beta_0,\eta]^\alpha_{\mathbb{R}^j}\,d\eta$$

is a constant, $\beta_0 = L\{e_1, \dots, e_{j+k-d}\} \in G(j+k-d, j)$. Now our aim is to calculate $c(\alpha)$. For $n + r \geq m$, $\beta_0 = L\{e_1, \dots, e_r\}$ we put

$$b_\alpha(n, m, r) = \int\limits_{G(n,m)} [\beta_0, \eta]^\alpha_{\mathbb{R}^m}\, d\eta.$$

Then $c(\alpha) = b_\alpha(i, j, j + k - d)$. We calculate $b_\alpha(n, m, r)$ for all n, m, r, such that $n + r \geq m \geq 2$. At the first step, we prove that

$$b_\alpha(k, d, i) = b_\alpha(i, j, i + j - d) \cdot b_\alpha(k, d, j) \tag{4.13}$$

for $i + k \geq d$, $i < j$, $i \geq d/2$, $d \geq 2$. Integrate the equality

$$\int\limits_{\eta \in G(i,d):\eta \subset \xi} [\eta, \zeta]^\alpha\, \sigma(d\eta) = c(\alpha)[\xi, \zeta]^\alpha \tag{4.14}$$

with respect to ζ, where $\zeta \in G(k, d)$ and $\xi \in G(j, d)$ ((4.14) follows from (1.1) by rotation invariance). By Fubini's theorem,

$$\int\limits_{\eta \in G(i,d):\eta \subset \xi} \left(\int\limits_{G(k,d)} [\eta, \zeta]^\alpha\, d\zeta \right) \sigma(d\eta) = c(\alpha) \int\limits_{G(k,d)} [\xi, \zeta]^\alpha\, d\zeta.$$

By rotation invariance, the integrand in parentheses in the left-hand side does not depend on η, and is equal to $b_\alpha(k, d, i)$, so we can write

$$b_\alpha(k, d, i) \int\limits_{\eta \in G(i,d):\eta \subset \xi} \sigma(d\eta) = c(\alpha) b_\alpha(k, d, j).$$

As the total mass of the measure σ is one, the above relation completes the proof of (4.13).

By lemma 2.2 (a) of [13] (where the operators A and A^* are applied to a constant function),

$$b_\alpha(d-1, d, d-k) = b_\alpha(d-k, d, d-1) = \frac{\omega_{d-k}\omega_k}{\omega_d} \int\limits_0^1 (1-t^2)^{(k-2)/2} t^{d-k-1+\alpha} dt. \tag{4.15}$$

The integral in the right-hand side of (4.15) is equal to

$$\frac{1}{2}\int_0^1 (1-u)^{k/2-1}u^{\frac{d-k+\alpha}{2}-1}du = \frac{1}{2}\frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)}.$$

By (4.15),

$$b_\alpha(d-1,d,d-k) = \frac{1}{2}\frac{\omega_{d-k}\omega_k}{\omega_d}\frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)} = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d-k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}.$$

Thus, we have proved that

$$b_\alpha(d-1,d,d-k) = b_\alpha(d-k,d,d-1) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d-k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}. \tag{4.16}$$

Now we show that for all k, $b_\alpha(d-k,d,k) = b_\alpha(k,d,d-k)$. By definition, for some $\eta \in G(k,d)$,

$$b_\alpha(d-k,d,k) = \int\limits_{G(d-k,d)} [\beta_0,\eta]^\alpha\, d\eta = \int\limits_{G(d-k,d)} [\beta_0^\perp,\eta^\perp]^\alpha\, d\eta =$$

$$= \int\limits_{G(k,d)} [\beta_0^\perp,\eta^\perp]^\alpha\, d\eta^\perp = \int\limits_{G(k,d)} [\beta_0^\perp,\nu]^\alpha\, d\nu = b_\alpha(k,d,d-k).$$

The above is true since $[\beta,\eta] = [\beta^\perp,\eta^\perp]$ for all $\beta \in G(d-k,d)$, $\eta \in G(k,d)$. Furthermore, by (4.16) we have for all $1 \le r \le d-1$,

$$b_\alpha(d-1,d,r) = b_\alpha(r,d,d-1) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{r+\alpha}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}. \tag{4.17}$$

By (4.15) and (4.17), the following equality is true :

$$b_\alpha(k,d,d-2) = b_\alpha(d-2,d-1,k-1)b_\alpha(k,d,d-1) =$$

$$= \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{k+\alpha}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}\cdot\frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{k-1+\alpha}{2}\right)}{\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{d-1+\alpha}{2}\right)}.$$

In the same way, by induction on r, for $\alpha > 0$

$$b_\alpha(k,d,d-r) = \prod_{l=0}^{r-1} \frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{k-l+\alpha}{2}\right)}{\Gamma\left(\frac{k-l}{2}\right)\Gamma\left(\frac{d-l+\alpha}{2}\right)},$$

or, more generally,

$$b_\alpha(k,d,r) = \prod_{l=0}^{d-r-1} \frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{k-l+\alpha}{2}\right)}{\Gamma\left(\frac{k-l}{2}\right)\Gamma\left(\frac{d-l+\alpha}{2}\right)}, \quad k+r \geq d. \tag{4.18}$$

Then as $c(\alpha) = b_\alpha(i,j,j+k-d)$ and by (4.18) we get (1.2). Theorem 1.1 is proved.

Proof of Proposition 1.1. Let $\eta \subset \xi \in G(j,d)$. Then $\xi^\perp \subset \eta^\perp$, and using the duality relation (2.3) of [5] for the Radon transform, one can write $(R_{ji}[\cdot^\perp, \zeta_0]^\alpha)(\eta) = (R_{d-j,d-i}[\cdot, \zeta_0]^\alpha)(\eta^\perp)$. Above $d-j < d-i$, $\eta^\perp \in G(d-i,d)$, so we can use the result of Theorem 1.1 : $(R_{d-j,d-i}[\cdot, \zeta_0]^\alpha)(\eta^\perp) = c^*(\alpha)[\eta^\perp, \zeta_0]^\alpha$, where $c^*(\alpha) = b_\alpha(d-j, d-i, d-i+k-d) = b_\alpha(d-j, d-i, k-i)$ (cf. proof of Theorem 1.1). By (4.18), we get

$$c^*(\alpha) = \prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-j-l+\alpha}{2}\right)}{\Gamma\left(\frac{d-j-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha}{2}\right)},$$

and the assertion is proved.

§5. THE PROOF OF THE CAUCHY–KUBOTA – TYPE FORMULAS FOR THE GENERALIZED COSINE TRANSFORMS

Proof of Proposition 1.2. We integrate both sides of (4.14) with respect to some measure $\theta \in \mathrm{M}(G(k,d))$ and use Fubini's theorem :

$$\int\limits_{\eta \in G(i,d):\eta\subset\xi} \int\limits_{G(k,d)} [\eta,\zeta]^\alpha\, \theta(d\zeta)\sigma(d\eta) = c(\alpha) \int\limits_{G(k,d)} [\xi,\zeta]^\alpha\, \theta(d\zeta).$$

The comparison of both sides of the above equation with the expression for T_{ij}^α completes the proof.

The following corollary is an easy consequence of Proposition 1.2 for $\alpha = 1$.

Corollary 5.1. *For any i, j, k with $i + k \geq d$, $i < j$,*

$$R_{ij}T_{ki} = \frac{\omega_{i+1-d+k}\omega_{j+1}}{\omega_{j+1-d+k}\omega_{i+1}}T_{kj}.$$

The connection between generalized cosine transforms of different orders can be used in tomography or stereology, in estimation of the statistical parameters of the shape of a geometric structure (e.g. porous media, microscopic shots of tissues, fiber collections, etc.) basing on the experimental data gained from sections or, in our terms, from intersections with flats of different dimensions. The structures in question are often modeled as k-dimensional manifold processes in $\mathbb{R}^d$, which can be seen locally as k-flat processes with directional distribution θ.

The characteristics obtained from the intersections with i-flats are often the roses of intersections $T_{ki}\theta$ (or the corresponding moments $T^{\alpha}_{ki}\theta$). Proposition 1.2 states that passage from the lower-dimensional sections ($T^{\alpha}_{ki}\theta$) to higher-dimensional sections ($T^{\alpha}_{kj}\theta$, $j > i$) can be obtained by simple integration (R_{ij}).

The proof of Theorem 1.1 yields the well-known result below (cf. formula (4.18) with $\alpha = 1$), that we would like to emphasize : it is the value of the rose of intersections $T_{kr}1$ of the stationary isotropic (i.e., with the uniform distribution of directions) Poisson k-flat process (cf. [20]) with arbitrary r-flats. It coincides with the rose of intersections of the stationary isotropic Poisson r–flat process with k–planes :

Corollary 5.2. *For any k and r, such that $k + r \geq d$*

$$T_{kr}1 = T_{rk}1 = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{k+r-d+1}{2}\right)} = \frac{\omega_{d+1}\omega_{k+r-d+1}}{\omega_{k+1}\omega_{r+1}}.$$

§6. DOUBLE FIBRATION FOR $\{T^{\alpha}_{ij}\}$ AND $\{\tilde{T}^{\alpha}_{ij}\}$

The following double fibration relation (cf. [11], p. 168) for R_{ij} holds for all absolutely integrable $f \in L^1(G(i,d))$, $\varphi \in L^1(G(j,d))$:

$$\int\limits_{G(j,d)} (R_{ij}f)(\xi)\varphi(\xi)\,d\xi = \int\limits_{G(i,d)} f(\eta)(R_{ji}\varphi)(\eta)\,d\eta. \tag{6.1}$$

Here R_{ji} is dual to the transform R_{ij}. Now we investigate this relation for functions $f(\eta) = [\eta, \zeta_0]^{\alpha}$, $\eta \in G(i,d)$, $i < j$.

Proof of Proposition 1.3. By Theorem 1.1, one gets from (6.1), that

$$c(\alpha) \int_{G(j,d)} [\xi, \zeta_0]^\alpha \varphi(\xi)\, d\xi = \int_{G(i,d)} [\eta, \zeta_0]^\alpha (R_{ji}\varphi)(\eta)\, d\eta, \tag{6.2}$$

and rotation invariance completes the proof.

Proposition 6.1. *The adjoint operator of T^α_{ij} on $L^1(G(i,d))$ is the operator T^α_{ji} on $L^1(G(j,d))$:*

$$(T^\alpha_{ij})^* = T^\alpha_{ji}.$$

Proof. The desired relation

$$\int_{G(i,d)} f(\eta)\, (T^\alpha_{ji}\varphi)\,(\eta)\, d\eta = \int_{G(j,d)} (T^\alpha_{ij} f)\,(\xi)\varphi(\xi)\, d\xi$$

for $f \in L^1(G(i,d))$ and $\varphi \in L^1(G(j,d))$ follows easily from the more general relation

$$\int_{G(i,d)} (T^\alpha_{ji}\theta)\,(\eta)\,\mu(d\eta) = \int_{G(j,d)} (T^\alpha_{ij}\mu)\,(\xi)\,\theta(d\xi)$$

for any $\theta \in \mathbf{M}(G(j,d))$, $\mu \in \mathbf{M}(G(i,d))$, which can be seen directly by Fubini's theorem. Proposition 6.1 is proved.

Now we state the result for the operator family $\{\tilde{T}^\alpha_{ij}\}$ similar to Proposition 1.3.

Proposition 6.2. *[Third Cauchy-Kubota – type formula] For all $i < j \le k < d$, $\alpha > 0$, and all absolute integrable functions $g \in L^1(G(i,d))$,*

$$\left(\tilde{T}^\alpha_{ik} g\right)(\zeta) = (c^*(\alpha))^{-1}\, \tilde{T}^\alpha_{jk}\,(R_{ij}g)\,(\zeta), \quad \zeta \in G(k,d), \tag{6.3}$$

where $c^*(\alpha)$ is given by (1.4).

Proof. First, it is worth mentioning that by rotation invariance, Proposition 1.1 remains true for any $\zeta \in G(k,d)$ instead of ζ_0. One can write by (1.3) and (6.1), that

$$c^*(\alpha) \int_{G(i,d)} [\eta^\perp, \zeta]^\alpha g(\eta)\, d\eta = \int_{G(j,d)} [\xi^\perp, \zeta]^\alpha (R_{ij}g)(\xi)\, d\xi, \tag{6.4}$$

which together with the definition of $\tilde{T}_{ij}^{\alpha}$ completes the proof.
For $\alpha > 0$ we consider the following functionals on the space of absolute integrable functions g on $G(i,d)$ and φ on $G(j,d)$:

$$\|g\|_{(\alpha)} \stackrel{def}{=} \left| \int\limits_{G(i,d)} g(\eta)[\eta,\zeta_0]^{\alpha}\, d\eta \right|, \quad \|\varphi\|_{(\alpha)}^{\perp} \stackrel{def}{=} \left| \int\limits_{G(j,d)} \varphi(\xi)[\xi^{\perp},\zeta_0]^{\alpha}\, d\xi \right|.$$

Let $\|\cdot\|_p$ denote the usual norm in L^p-spaces. The rest of the section is devoted to inequalities between the weighted images of Radon transforms and their duals as above.

Proposition 6.3. *Choose the numbers $p, q > 1$, such that $1/p + 1/q = 1$.*

1) Let $i < j$, $i + k \geq d$, $\alpha > 0$, and $\varphi \in L^p(G(j,d))$. Then

$$\|R_{ji}\varphi\|_{(\alpha)} \leq d(\alpha,q)\|\varphi\|_p, \tag{6.5}$$

where

$$d(\alpha,q) = c(\alpha)\left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{j-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{j-l}{2}\right)\Gamma\left(\frac{d-l+\alpha+q}{2}\right)}\right)^{1/q}, \tag{6.6}$$

and $c(\alpha)$ is defined by (1.2).

2) Let $i < j \leq k < d$, $\alpha > 0$, and $g \in L^p(G(i,d))$. Then

$$\|R_{ij}g\|_{(\alpha)}^{\perp} \leq d^*(\alpha,q)\|g\|_p, \tag{6.7}$$

where

$$d^*(\alpha,q) = c^*(\alpha)\left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-l+\alpha+q}{2}\right)}\right)^{1/q}, \tag{6.8}$$

and $c^*(\alpha)$ is defined by (1.4).

Proof. First, let us prove the upper bound for the image of the dual Radon transform. By (6.2),

$$\|R_{ji}\varphi\|_{(\alpha)} \leq c(\alpha) \cdot |||\varphi|||_{(\alpha)}.$$

Applying Hölder's inequality one gets

$$|||\varphi|||_{(\alpha)} \leq \left(\int\limits_{G(j,d)} |\varphi(\xi)|^p \, d\xi \right)^{1/p} \left(\int\limits_{G(j,d)} [\xi, \zeta_0]^{\alpha+q} \, d\xi \right)^{1/q} = \|\varphi\|_p b^{1/q}_{\alpha+q}(j,d,k),$$

while by (4.18)

$$b^{1/q}_{\alpha+q}(j,d,k) = \left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right) \Gamma\left(\frac{j-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{j-l}{2}\right) \Gamma\left(\frac{d-l+\alpha+q}{2}\right)} \right)^{1/q}$$

The proof of the second statement of Proposition is conducted analogously : by (6.4),

$$\|R_{ij} g\|^{\perp}_{(\alpha)} \leq c^*(\alpha) \cdot |||g|||^{\perp}_{(\alpha)}.$$

By Hölder's inequality, we get

$$|||g|||^{\perp}_{(\alpha)} \leq \left(\int\limits_{G(i,d)} |g(\eta)|^p \, d\eta \right)^{1/p} \left(\int\limits_{G(i,d)} [\eta^{\perp}, \zeta_0]^{\alpha+q} \, d\eta \right)^{1/q} =$$

$$= \|g\|_p \left(\int\limits_{G(d-i,d)} [\nu, \zeta_0]^{\alpha+q} \, d\nu \right)^{1/q} = \|g\|_p b^{1/q}_{\alpha+q}(d-i,d,k),$$

where by (4.18)

$$b^{1/q}_{\alpha+q}(d-i,d,k) = \left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right) \Gamma\left(\frac{d-i-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{d-i-l}{2}\right) \Gamma\left(\frac{d-l+\alpha+q}{2}\right)} \right)^{1/q}$$

Proposition 6.3 is proved.

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Резюме. В статье рассматривается действие преобразований Радона на грассмановских многообразиях для специальных функций равных положительным степеням объёма некоторых параллелепипедов. Как следствие доказываются интегрально–геометрические формулы типа Коши–Кубота для обобщённых косинус преобразований на грассманианах. Указываются применения этих результатов в выпуклой и стохастической геометрии, стереологии и геометрической томографии.

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SPECIFIC INDEX AND CURVATURE OF RANDOM SIMPLICIAL COMPLEXES

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Abstract. The main result is an ergodic theorem for infinitely extended stationary random simplicial complexes, which states the existence of limits almost surely and in $\mathcal{L}^1$. As a consequence we obtain the critical point theorem as well as the Gauss-Bonnet theorem. The existence of the specific Lipschitz–Killing curvature and the specific Einstein–Hilbert energy is shown as well.

§1. INTRODUCTION

The basic objects of a theory of pure gravity formulated in terms of Euclidean path integrals are random Riemannian spaces (M, g), where g denotes the metric of the manifold M. They are realized at random according to some Gibbs measure

$$P(d(M,g)) = \frac{1}{Z} \cdot \exp(-\beta \cdot H(M,g)) \, "d(M,g)", \tag{1.1}$$

where $\beta > 0$ and $"d(M,g)"$ is the analogon of the Lebesgue measure on the space of admissible spaces (M,g). The so-called Einstein–Hilbert action $H(M,g)$ is defined in terms of invariants, such as the volume $Vol(M,g)$ and the curvature $C(M,g)$, which in this model are random variables. The partition function Z is the normalizing constant.

The main difficulties of such a theory are, that H is unbounded, moreover it is not clear how to define the reference measure $"d(M,g)"$ in such a way that $\exp(-\beta H)$ becomes integrable.

Taking the point of view of modern statistical mechanics, as we do, the realizations (M,g) should be non-bounded. Then it is even more difficult to construct P.

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Furthermore, one of the first questions is how to define the volume or the curvature for such infinitely extended random spaces (M, g). One approach is to start from a lattice description and then to perform some scaling limit, as done by Regge [13]. In the present paper we give a formulation in terms of stationary random simplicial complexes μ embedded in a Euclidean space, including for instance simplicial surfaces (see [16]) or networks (see [11]). This concept has been formalized already in a much more general framework in the fundamental paper of M. Zähle [15]. We concentrate on the existence of specific energies, in particular specific Einstein–Hilbert action, specific curvature and Euler characteristic. This means the existence in some sense of ergodic limits of the form

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot E^{\phi}_{\Lambda_n}(\mu), \quad \mu \in \Gamma, \tag{1.2}$$

for suitable sequences $(\Lambda_n)_n$ of bounded Borel exhausting the space in which the random simplicial complex lives. Here $E^{\phi}_{\Lambda}(\mu)$ defines the energy of the complex μ in Λ, given by the interaction ϕ. Now for the first time these notions are made precise in analogy to classical statistical mechanics.
The main result is an ergodic theorem, which states the existence of the limits (1.2) almost surely and in $\mathcal{L}^1$. As a consequence, the critical point theorem and the Gauss–Bonnet theorem for the infinitely extended random simplicial complex, that were known in the mean sense (see [11], [16]), are now proved to be valid almost surely, and for much more general classes of stationary random simplicial complexes.
Finally, the mean specific Einstein–Hilbert energy is expressed by the mean specific volume and the mean specific Lipschitz–Killing curvature.

§2. RANDOM SIMPLICIAL COMPLEXES IN $\mathbb{R}^d$

We consider always cell complexes embedded in $E = \mathbb{R}^d$, $d > 1$, the d-dimensional Euclidean space. Let X denote the set of finite subsets x of E, which are **affinely independent**, i.e. the **affine hull of** x

$$aff\ x = \left\{ \sum_{a\in x} \lambda_a \cdot a : \sum_{a\in x} \lambda_a = 1,\ \lambda_a \in \mathbb{R} \right\}$$

is different from the affine hull of every proper subset of x. A k-**simplex** is given by the convex hull $\langle x \rangle$ of an $x \in X$ with $card\ x = k + 1$. k denotes here the **dimension** of x (resp. $\langle x\rangle$) given by $\dim x = \dim \langle x\rangle = card\, x - 1$. Denote by $\mathcal{M}^{\cdot}(X)$ the set of all **configurations** μ **of elements** $x \in X$, which are locally finite. This means that μ is of the form

$$\mu = \sum_{x \in D} \delta_x, \tag{2.1}$$

where $D \subseteq X$ is countable and locally finite in the sense, that

$$\zeta_B(D) := card\{z \in D : \langle z\rangle \cap B \neq \phi\} < +\infty \quad \text{for any } B \in \mathcal{B}_0(E). \tag{2.2}$$

Here $\mathcal{B}_0(E)$ denotes the set of open, bounded Borel sets in E. (In the following we always identify μ and its support D, thus considering μ as a measure in X as well as a subset of X.)

A configuration $\mu \in \mathcal{M}^{\cdot}(X)$ is called **simplicial complex in** E, if

$$(x \in \mu, y \subseteq x \Rightarrow y \in \mu), \tag{2.2}$$

$$(x, y \in \mu, \langle x\rangle \cap \langle y\rangle \neq \phi \Rightarrow \exists z \in X, z \subseteq x \cap y : \langle z\rangle = \langle x\rangle \cap \langle y\rangle). \tag{2.3}$$

Denote by $\mathcal{S}^{\cdot}(E)$ the set of all simplicial complexes in E. The **dimension of** μ is $\dim \mu = \max_{x \in \mu} \dim x$. It is obvious that $\dim \mu \leq d$ for each $\mu \in \mathcal{M}^{\cdot}(X)$. The **$k$-skeleton of** μ is $\mu_k = \sum_{x \in \mu, \dim x = k} \delta_x$. μ_0 is the set of vertices a of μ; μ_1 the set of edges e of μ. Elements $x \in \mu$ are called **cells**. If $\dim x = k, x$ is called a **k-cell**. A cell $x \in \mu$ is **maximal**, if $(y \in \mu, y \subseteq x \Rightarrow y = x)$. It is clear that each $\mu \in \mathcal{S}^{\cdot}(E)$ contains maximal elements. The **barycenter** is the mapping $b : X \to E$, defined by

$$b(x) = \frac{1}{card\, x} \sum_{a \in x} a, \quad x \in X. \tag{2.4}$$

b induces a transformation $b : \mathcal{S}^{\cdot}(E) \to \mathcal{M}^{\cdot}(E)$, $\mu \longmapsto b\mu$, where $b\mu$ is the image of μ under b. Here $\mathcal{M}^{\cdot}(E)$ denotes the set of all locally bounded point configurations in E. A subset $\nu \subseteq \mu, \mu \in \mathcal{S}^{\cdot}(E)$, is called a subcomplex of μ, if $(x \in \nu, y \subseteq x \Rightarrow y \in \nu)$. We denote this by $\nu \leq \mu$. Let $\mathcal{K}$ be a subset of $\mathcal{S}^{\cdot}(E)$. Take in $\mathcal{K}$ the σ-field $\mathcal{B}_{\mathcal{K}}$ generated by the mapping

$$\zeta : \mathcal{K} \to \cup_{\mu \in \mathcal{K}} \mathbb{N}_0^{\mathcal{B}_0(E)}, \mu \longmapsto (B \longmapsto \zeta_B(\mu)). \tag{2.5}$$

A random simplicial complex (RASC) **in** E **of type** $\mathcal{K}$ is defined as a probability measure P on $(\mathcal{K}, \mathcal{B}_{\mathcal{K}})$. The probability space $(\mathcal{K}, \mathcal{B}_{\mathcal{K}}, P)$ is called **stationary** if $\mathcal{K}$ as

well as P are invariant under the group of transformations, which are induced by the Euclidean translations. We denote them by $\mu \to \mu - a := \sum_{x \in \mu} \delta_{x-a}$, where $x - a$ is the translation of x by means of $a \in E$.
In the sequel we consider only stationary RASCs. The $0 - \infty$ law of stochastic geometry implies for them that a typical configuration consists of an infinity of cells and thus is unbounded in E because it is locally finite.

Remark. Associated with a RASC P of type $\mathcal{K}$ is the notion of **underlying random piecewise linear space** in E defined by the mapping $\langle \cdot \rangle : \mathcal{K} \to \mathcal{F}(E)$, $\mu \longmapsto \bigcup_{\substack{x \in \mu \\ maximal}} \langle x \rangle$. $\mathcal{F}(E)$ denotes the set of closed subsets of E. Thus, $\langle \mu \rangle = \bigcup_{x \in \mu\ maximal} \langle x \rangle$ is considered always as a topological space, its topology being inherited from E. In $\mathcal{F}(E)$ we consider the σ-field $\mathcal{B}_{\mathcal{F}(E)} = \{F \subseteq \mathcal{F}(E) : \langle \cdot \rangle^{-1}(F) \in \mathcal{B}_{\mathcal{K}}\}$. We call the random variable $\langle \cdot \rangle$, defined on $(\mathcal{K}, \mathcal{B}_{\mathcal{K}}, P)$ **random piecewise linear space of** P as well as the image $P_{\langle \cdot \rangle}$ of P under $\langle \cdot \rangle$.

§3. POTENTIAL AND ENERGY

Consider the set

$$\mathcal{D} = \{(\nu, \mu) : \mu \in \mathcal{S}^{\cdot}(E),\ \nu \leq \mu \text{ finite}\} \tag{3.1}$$

provided with the σ-field $\mathcal{B}_{\mathcal{D}}$ induced by $\mathcal{B}_{\mathcal{S}^{\cdot}(E)}$ in a natural way. A measurable function $\phi : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ is called (interaction) **potential**, if ϕ is stationary with respect to translations, i.e.

$$\phi(\nu - a, \mu - a) = \phi(\nu, \mu) \text{ for each } a \in E \text{ and } (\nu, \mu) \in \mathcal{D}. \tag{3.2}$$

$\phi(\nu, \mu)$ is a real number (or $+\infty$), associated to the finite subcomplex ν of μ, which represents an intrinsic metric characteristic of ν in μ. It depends only on μ and not on the space E into which μ is embedded.
We now take the point of view of classical statistical mechanics. There the potential ϕ is the point of departure. All properties of the system have to be deduced from ϕ. What are the additional properties of ϕ which allow such an analysis? First of all one needs that the **energy of a typical cell** converges for a large class Γ of configurations is a suitable sense. This energy is formally defined by the series

$$E^{\phi}(o|\mu) = \sum_{\substack{\nu \leq \mu \\ o \in b\nu}}^{\cdot} \phi(\nu, \mu), \mu \in \mathcal{M}^0, \tag{3.3}$$

where $\mathcal{M}^{o} = \{\mu \in \mathcal{S}^{\cdot}(E) | 0 \in b\mu\}$. ($\sum_{\nu \leq \mu}^{\cdot}$ means the sum over all finite subcomplexes ν in μ.) In general $E^{\phi}(o, \mu), \mu \in \mathcal{M}^0$, will be divergent for a given ϕ. Therefore, we will consider spaces of configurations of the following type : $\Gamma \subseteq \mathcal{S}^{\cdot}(E)$, measurable, translation invariant, satisfying the following two properties :

$$\|\phi\| = \sup_{\mu \in \Gamma \cap \mathcal{M}^{\circ}} \sum^{\bullet}_{\nu \leq \mu, 0 \in b\mu} |\phi(\nu,\mu)| < +\infty; \tag{3.4}$$

for each $\varepsilon > 0$ there exists $\Lambda \in \mathcal{B}_0(E)$, such that

$$\sup_{\mu \in \Gamma \cap \mathcal{M}^{\circ}} \sum^{\bullet}_{\substack{\nu \leq \mu, 0 \in b\nu \\ b\nu \cap \Lambda^c \neq \emptyset}} |\phi(\nu,\mu)| < \varepsilon. \tag{3.5}$$

Within the class of sets Γ having these properties we then choose Γ sufficiently large and take the trace $\mathcal{B}_\Gamma = \Gamma \cap \mathcal{B}_{\mathcal{S}^{\cdot}}$ as a σ-field. We then say that Γ **has been chosen tempered for** ϕ.

Definition. *If ϕ is a potential and Γ is chosen tempered for ϕ, the* **energy** *of μ in $\Lambda \in \mathcal{B}_0(E)$ is*

$$E_\Lambda = E^{\phi}_{\Lambda}(\mu) = \begin{cases} \sum^{\bullet}_{\nu \leq \mu, b\nu \cap \Lambda \neq \emptyset} \phi(\nu,\mu), & \text{if } \mu \in \Gamma, \\ +\infty, & \text{if } \mu \notin \Gamma. \end{cases} \tag{3.6}$$

Because of (3.4) E_Λ is well defined and finite on Γ, because

$$|E_\Lambda(\mu)| \leq \sum_{a \in b\mu \cap \Lambda} \sum^{\bullet}_{\substack{\nu \leq \mu, \\ a \in b\nu}} |\phi(\nu,\mu)| \leq b\mu(\Lambda)\|\phi\| < +\infty, \quad \mu \in \Gamma.$$

In particular, E_Λ is **stable** on Γ, i.e. there exists $B \geq 0$ such that for each $\mu \in \Gamma$ $E_\Lambda(\mu) \geq -B \cdot b\mu(\Lambda)$. This also shows that E_Λ is P-integrable if $P \in \mathcal{P}\Gamma$ (i.e. P is a probability on $(\Gamma, \mathcal{B}_\Gamma)$) and P **is of first order**, i.e. the image P_b of P under b has the property that the associated moment measure

$$\nu^1_b(B) = \int_\Gamma b\mu(B)\, P(d\mu), \quad B \in \mathcal{B}_E \tag{3.7}$$

is locally finite, i.e. finite on $\mathcal{B}_0(E)$.

We now consider examples of potentials. They are given by counting-, volume-, and curvature measures and polynomials of these measures. The first two are based on the work of Banchoff [1], M. Zähle [15] and Cheeger et al. [4]. The third one is due to Regge [13].

Example 1. Index. **The index of a simplicial complex** $\mu \in \mathcal{S}^{\cdot}(E)$ **in a vertex** $a \in \mu_0$ **for the direction** $\mathbf{n} \in S^{d-1}$ is

$$\iota_{\mathbf{n}}(a,\mu) = \sum_{q=0}^{\dim \mu} (-1)^q \cdot \sum_{a \in x \in \mu_q} 1_{H_{\mathbf{n}}}(x-a), \tag{3.8}$$

where $H_{\mathbf{n}} = \{y \in X : \ 0 \in y, \mathbf{n} \cdot b < 0, \quad b \in y \setminus \{0\}\}$ denotes the half space in the set of all $y \in X$, $0 \in y$, determined by $\mathbf{n}$. ($\mathbf{n} \cdot b$ denotes the scalar product of $\mathbf{n}$ and b in E.)

If we define $\iota_{\mathbf{n}}(\nu,\mu) = 0$ for finite subcomplexes $\nu \subsetneq \mu$, which do not reduce to a vertex, $\iota_{\mathbf{n}}$ defines a potential. The energy of a typical vertex is given by $\iota_{\mathbf{n}}(0,\mu), \mu \in \mathcal{M}^0$; it is not bounded as a function of μ. A possible choice of a tempered $\Gamma \subseteq \mathcal{S}^{\cdot}(E)$ is the set of **Delone configurations** in $\mathcal{S}^{\cdot}(E)$, denoted by $\mathcal{S}^{\cdot}_{r,R}(E)$ for given parameters $0 < r < R < +\infty$. Here $\mu \in \mathcal{S}^{\cdot}_{r,R}(E)$ if and only if

$$\mu_0 \in \mathcal{M}_r(E) = \{\eta \in \mathcal{M}^{\cdot}(E) : a, b \in \eta, a \neq b \Rightarrow |a-b| \geq \eta\},$$

and each maximal $x \in \mu$ is contained in a ball of fixed radius R. (The concept of a Delone configuartion goes back to Hilbert/Cohn–Vossen [7]).

We call the associated energy $E_\Lambda^{\iota_{\mathbf{n}}}(\mu)$ **the index of** μ **in** Λ **for the direction** $\mathbf{n}$. It is connected with the Euler characteristic by the following relation : For any pair $(\mathbf{n},\mu)$ for which $\mathbf{n}$ is in general position for μ (i.e. $\forall a \in \mu_0$, $\forall a \in x \in \mu$, $\forall b \in x \setminus \{a\}$: $\mathbf{n} \cdot (b-a) \neq 0$.)

$$E_\Lambda^{\iota_{\mathbf{n}}}(\mu) = \mathcal{X}(\mu_\Lambda) - \sum_{a \in (\mu_\Lambda)_0 \cap \Lambda^c} \iota_{\mathbf{n}}(a,\mu), \tag{3.9}$$

where $\mu_\Lambda = \sum_{x \cap \Lambda \neq \phi, x \in \mu,} \sum_{y \subseteq x} \delta_y$, and the Euler characteristic of this complex is

$$\mathcal{X}(\mu_\Lambda) = \sum_{q=0}^{\dim \mu} (-1)^q \cdot |\mu_{\Lambda,q}|. \tag{3.10}$$

Note that $\mathcal{X}_\Lambda : \mu \longmapsto \mathcal{X}(\mu_\Lambda)$ is an intrinsic quantity whereas $E_\Lambda^{\iota_{\mathbf{n}}}$ is not. Relation (3.9) says that $E_\Lambda^{\iota_{\mathbf{n}}}$ differs from $\mathcal{X}_\Lambda$ only by some boundary term. We observe also that $\mathcal{X}_\Lambda$ can be considered as an energy $E_\Lambda^{\pi}(\mu)$, for which the potential π is concentrated in the vertices and given by

$$\pi(a,\mu) = \sum_{q=0}^{\dim \mu} (-1)^q \cdot \frac{|\mu_q^a|}{|(\mu_q^a)_0 \cap \Lambda|}. \tag{3.11}$$

Here μ_q^a, $a \in \mu_0$ denotes the configuration of all q-cells $x \in \mu$ with $a \in x$.

Example 2. Curvature. Integration of the index ι_n with respect to the normalized surface measure $\sigma^{d-1}(dn)$ on the d-dimensional unit sphere S^{d-1} immediately gives another important example of a potential :

$$\phi_c(\nu,\mu) = \int_{S^{d-1}} \iota_n(\nu,\mu)\, \sigma^{d-1}(dn), \quad \nu \leq \mu \text{ finite.} \tag{3.12}$$

Explicitely : $\phi_c(\nu,\mu) = 0$ if ν is not a vertex of μ, and

$$\phi_c(a,\mu) = \sum_{q=0}^{\dim \mu} (-1)^q \cdot \sum_{a \in x \in \mu_q} \varepsilon(a,x), \quad a \in \mu_0, \tag{3.13}$$

where $\varepsilon(a,x) = \int_{S^{d-1}} 1_{H_n}(x-a)\, \sigma^{d-1}(dn)$. $\varepsilon(a,x)$ is the so called **normalized exterior angle of the q-cell x at its vertex a**. Observe that $\varepsilon(a,\{a\}) = 1$; moreover, if $q = 1$ and $a \in x \in \mu_1$, then $\varepsilon(a,x) = \frac{1}{2}$. Thus, in the case that μ is a **network**, i.e. if $\dim \mu = 1$, then $\phi_c(a,\mu) = 1 - \frac{1}{2}|\mu_1^a|$, if $a \in \mu_0$.

In the case where μ is a **simplicial surface with or without boundary** (for a definition see Zessin [16]), $\dim \mu = 2$ and each maximal cell $x \in \mu$ is 2-dimensional. If $a \in x \in \mu_2$, then $\varepsilon(a,x) = \frac{1}{2\pi}(\pi - \alpha(a,x))$. where $\alpha(a,x)$ denotes the angle of x in a. Thus

$$\phi_c(a,\mu) = 1 - \frac{1}{2}|\mu_1^a| + \left(\frac{1}{2}|\mu_2^a| - \frac{1}{2\pi} \sum_{a \in x \in \mu_2} \alpha(a,x)\right).$$

If a is not a boundary vertex of μ, then $|\mu_1^a| = |\mu_2^a|$. In this case

$$\phi_c(a,\mu) = \frac{1}{2\pi}\left[2\pi - \sum_{a \in x \in \mu_2} \alpha(a,x)\right].$$

If a is a boundary vertex of μ, we use the relation $|\mu_1^a| = |\mu_2^a| + 1$ to get

$$\phi_c(a,\mu) = \frac{1}{2\pi}\left[\pi - \sum_{a \in x \in \mu_2} \alpha(a,x)\right].$$

To summarize : up to the factor $\frac{1}{2\pi}$ the potential $\phi_c(a,\mu)$ is the deficit angle of μ in a. Thus, in the case of networks respectively simplicial surfaces ϕ_c measures the curvature of μ in a. Therefore, the associated energy $E_\Lambda^{\phi_c}(\mu)$ of μ in Λ, considered on the tempered set $\Gamma = \mathcal{S}_{r,R}^{\cdot}(E)$ of Delone configurations, is well defined and is called the **curvature of** μ **in** Λ. It is an intrinsic quantity since the normalized exterior angle has this property as shown in Banchoff [1].

Example 3. Regge curvature and Einstein–Hilbert action.
We present these notions here in a special case and discuss the general situation in §6. Let Γ_2 denote the set $\mathcal{S}_{r,R,2}^{\cdot}(E)$ of Delone configurations $\mu \in \mathcal{S}_{r,R}^{\cdot}(E)$ satisfying the property ($x \in \mu$ maximal $\Rightarrow \dim x = 2$). The **Regge potential** is defined to be

$$\phi_R = -[\phi_c - \phi_v], \tag{3.14}$$

where $\phi_v(\nu,\mu)$ is the volume of the star of ν in μ. More precisely $\phi_v(\nu,\mu) = 0$ if ν is not a vertex of μ, and, if $a \in \mu_0$,

$$\phi_v(a,\mu) = \sum_{x \in \mu_2^a} \lambda^2(\langle x \rangle). \tag{3.15}$$

Γ_2 is tempered for ϕ_R. The associated energy $E_\Lambda^{\phi_R}(\mu)$ is Regge's simplicial version of the **Einstein–Hilbert action** (see Einstein [5], Hilbert [8] and Weyl [14]). It is an intrinsic quantity.

Example 4. Network curvature. In analogy with example 3 one can consider the following network curvature on the tempered set $\Gamma = \mathcal{S}_{r,R;1}^{\cdot}(E)$:

$$\phi_{nc} = -[\phi_c - \phi_l], \tag{3.16}$$

where $\phi_l(\nu,\mu) = 0$ if ν is not a vertex of μ, and, if $a \in \mu_0$,

$$\phi_l(a,\mu) = \sum_{e \in \mu_1^a} \lambda^1(\langle e \rangle). \tag{3.17}$$

4. SPECIFIC ENERGY

Let ϕ be a potential and $\Gamma \subseteq \mathcal{S}^{\cdot}(E)$ tempered for ϕ. Furthermore let P be a stationary probability on $(\Gamma, \mathcal{B}_\Gamma)$; write then $P \in \mathcal{P}_0\Gamma$. On $(\Gamma, \mathcal{B}_\Gamma, P)$ the energy E_Λ^ϕ is a random variable indexed by $\Lambda \in \mathcal{B}_0(E)$. The following main result of the present paper is an ergodic theorem for the energy. Its proof uses the same idea as Nguyen, Zessin [12], Satz 4 (which in turn is based on an idea of Föllmer [6]). In its formulation $(\Lambda_n)_n$ denotes an increasing sequence Λ_n of centered open balls in E satisfying $\Lambda_n \nearrow E$ if $n \to +\infty$.

Theorem. *Under the additional assumption*

$$P_b(\mathcal{M}^0_r(E)) = 1 \quad \textit{for some } r > 0, \tag{4.1}$$

the following limit

$$e = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot E^{\phi}_{\Lambda_n} \tag{4.2}$$

exists P-a.s. as well as in $\mathcal{L}^1(P)$ and is given by

$$e = E_P\left(\int_D g(\mu - a)b(\mu)\,(da)|\mathcal{J}\right).$$

Here $D \in \mathcal{B}_0(E)$ is some fixed domain satisfying $|D| = 1$, and $\mathcal{J}$ denotes the sub-σ-field of $\mathcal{B}_\Gamma$ consisting of all events, which are translation-invariant. In particular, e is invariant under translations and

$$E_P(e) = P^0(g), \tag{4.3}$$

where P^0 denotes the **Palm measure** *of P with respect to the barycenters, and*

$$g(\mu) = \dot{\sum_{\nu\leq\mu, 0\in b\nu}} \frac{\phi(\nu,\mu)}{|\nu|}, \mu \in \Gamma \cap \mathcal{M}^0. \tag{4.4}$$

Proof. 1. Decomposition of $E_\Lambda = E^{\phi}_\Lambda$: Consider

$$E_\Lambda(\mu) = \dot{\sum_{\nu\leq\mu, b\nu\subseteq\Lambda}} \phi(\nu,\mu) + \dot{\sum_{\substack{b\nu\cap\Lambda\neq\emptyset, p\nu\cap\Lambda^c\neq\emptyset \\ \nu\leq\mu}}} \phi(\nu,\mu).$$

The first term on the right hand side can be represented as

$$E^1_\Lambda(\mu) = \sum_{a\in b\mu\cap\Lambda} \dot{\sum_{\nu\leq\mu, a\in b\nu\subseteq\Lambda}} \frac{\phi(\nu,\mu)}{|b\nu|}.$$

The idea now is to approximate E_Λ by means of

$$g_\Lambda(\mu) = E^{(1)}_\Lambda(\mu) + \sum_{a\in b\mu\cap\Lambda} \dot{\sum_{\substack{\nu\leq\mu, a\in b\nu \\ b\nu\cap\Lambda^c\neq\emptyset}}} \frac{\phi(\nu,\mu)}{|b\nu|}.$$

An important observation is that $g_\Lambda(\mu) = \int_\Lambda g(\mu - a)b\,\mu(da)$, where g is defined by (4.4).

2. The distance of g_Λ from E_Λ is

$$\sup_{\mu\in\Gamma}|E_\Lambda(\mu)-g_\Lambda(\mu)|\leq 2\Delta(\Lambda), \tag{4.5}$$

where

$$\Delta(\Lambda)=\sup_{\mu\in\Gamma}\sum_{a\in b\mu\cap\Lambda}\ \dot{\sum_{\substack{\nu\leq\mu,\, a\in b\nu,\\ b\nu\cap\Lambda^c\neq\emptyset}}}|\phi(\nu,\mu)|.$$

3. Asymptotic behaviour of $\frac{1}{|\Lambda|}\cdot\Delta(\Lambda)$ as $\Lambda=\Lambda_n\nearrow E$. For $\varepsilon>0$ choose a ball $\mathcal{K}_h(0)$ of radius h centered at 0, such that

$$\sup_{\mu\in\Gamma\cap\mathcal{M}^0}\ \dot{\sum_{\substack{\nu\leq\mu,\,0\in b\nu\\ b\nu\cap\mathcal{K}_h^c(0)\neq\emptyset}}}|\phi(\nu,\mu)|<\varepsilon.$$

Here we use (3.5). On the other hand, we estimate $\Delta(\Lambda)$, using the decomposition

$$=\sum_{a\in b\nu\cap\Lambda}\left[\ \dot{\sum_{\substack{a\in b\nu,\nu\leq\mu\\ b\nu\cap\Lambda^c\neq\emptyset,\, b\nu\subseteq\mathcal{K}_h(a)}}}|\phi(\nu,\mu)|+\dot{\sum_{\substack{\nu\leq\mu,a\in b\nu\\ b\nu\cap\mathcal{K}_h^c(a)\neq\emptyset,\, b\nu\cap\Lambda^c\neq\emptyset}}}|\phi(\nu,\mu)|\right],$$

where $\mathcal{K}_h(a)=\mathcal{K}_h(0)+a$. This yields

$$\Delta(\Lambda)\leq\sup_{\mu\in\Gamma}\left[b\mu(\partial_h\Lambda)\cdot||\phi||+b\mu(\Lambda)\cdot\varepsilon\right],$$

where $\partial_h\Lambda=\{a\in E|d(a,\partial\Lambda)\leq h\}$. Using assumption (4.1) this can be further estimated from above by

$$\leq\frac{1}{c_0}\cdot\left[\partial_{2r+h}\Lambda\cdot||\phi||+|\Lambda\cup\partial_r\Lambda|\cdot\varepsilon\right].$$

Here $c_0=|\mathcal{K}_{r/2}(0)|$. Since the $\Lambda=\Lambda_n$ are balls exhausting E, we have

$$\frac{|\partial_{2r+h}\Lambda_n|}{|\Lambda_n|}\longrightarrow 0,\quad \frac{|\Lambda_n\cup\partial_r\Lambda_n|}{|\Lambda_n|}\longrightarrow 1,\quad n\to\infty.$$

Thus $\frac{1}{|\Lambda_n|}\Delta(\Lambda_n)\to 0$.

On account of (4.5) this implies that $\frac{1}{|\Lambda_n|}E_{\Lambda_n}$ and $\frac{1}{|\Lambda_n|}g_{\Lambda_n}$ have the same limiting behaviour, if one of them is convergent. It is therefore enough to analyse $\frac{1}{|\Lambda_n|}g_{\Lambda_n}$.

4. The limiting behaviour of $\frac{1}{|\Lambda_n|} g_{\Lambda_n}$.

This is well known and can be found for instance in [12] : Under the condition that $g \in \mathcal{L}^1(P^0)$ the limit

$$e = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} g_{\Lambda_n}$$

exists P-a.s. and in $\mathcal{L}^1(P)$ and is given by $e = E_P(\int_D g(\mu - a) b(\mu)(da) | \mathcal{J})$. Moreover, e is translation invariant and $E_P(e) = P^0(g)$. Thus, it remains to show that $g \in \mathcal{L}^1(P^0)$. But this follows from the fact that P^0 is a finite measure on Γ (on account of (4.1)), combined with the fact that $|g| \leq \|\phi\| < +\infty$.

We remark, that if in the situation of the theorem P is also ergodic with respect to the group of translations (i.e. $\mathcal{J}$ contains only events of probability 0 and 1), then the specific energy e is P-a.s. constant and equals $P^0(g)$.

§5. THE GAUSS–BONNET THEOREM AND THE CRITICAL POINT THEOREM

Here we discuss some consequences of Theorem. Consider the potentials $\iota_{\mathbf{n}}$ and ϕ_c for the tempered set $\Gamma = \mathcal{S}_{r,R}(E)$ of Delone configurations. If $P \in \mathcal{P}^0\Gamma$ is the underlying law, the theorem immediately yields the existence of the **specific index of μ in the direction n**

$$\iota(\mathbf{n}, \mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot E_{\Lambda_n}^{\iota_{\mathbf{n}}}(\mu), \mu \in \Gamma, \mathbf{n} \in S^{d-1}, \tag{5.1}$$

as well as the existence of the **specific curvature**

$$\kappa(\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot E_{\Lambda_n}^{\phi_c}(\mu), \mu \in \Gamma. \tag{5.2}$$

Both limits exist P-a.s. and in $\mathcal{L}^1(P)$ and are invariant under translations. We discuss the relation of these two quantities to the Euler characteristic. To do this we assume in addition that the set $C = \{(\mathbf{n}, \mu) |\ \mathbf{n}$ is in general position with respect to $\mu\}$ is full with respect to $\sigma^{d-1} \otimes P$. In this case (3.9) is valid $\sigma^{d-1} \otimes P$-a. s. in $(\mathbf{n}, \mu)$, and therefore

$$|E_\Lambda^{\iota_{\mathbf{n}}}(\mu) - \mathcal{X}_\Lambda(\mu)| \leq \bar{\Delta}(\Lambda) := \sup_{(\mathbf{n},\mu)\in C} \sum_{a \in (\mu_\Lambda)_0 \cap \Lambda^c} |\iota_{\mathbf{n}}(a, \mu)| \tag{5.3}$$

is true $\sigma^{d-1} \otimes P$-a.s.. Integration with respect to σ^{d-1} yields immediately

$$|E_\Lambda^{\phi_c}(\mu) - \mathcal{X}_\Lambda(\mu)| \subseteq \bar{\Delta}(\Lambda)\ \ P - a.s.[\mu]. \tag{5.4}$$

On the other hand, it is clear that $\lim_{n\to\infty} \frac{1}{|\Lambda_n|}\bar{\Delta}(\Lambda_n) = 0$ because this is a boundary term. Therefore, the theorem implies the existence of the **specific Euler characteristic** of μ :

$$\mathcal{X}(\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \cdot \mathcal{X}_{\Lambda_n}(\mu) \quad P-a.s. \text{ and in } \mathcal{L}^1(P), \tag{5.5}$$

as well as the relations

$$\mathcal{X} = \iota \ \ \sigma^{d-1} \otimes P-a.s. \quad \text{and} \tag{5.6}$$

$$\mathcal{X} = \kappa \quad P-a.s.. \tag{5.7}$$

In particular we have $\kappa = \iota\ \sigma^{d-1} \otimes P$-a.s.. Relation (5.6) is an individual version for infinitely extended random simplicial complexes of the so called **critical point theorem** in global differential geometry, and (5.7) of the **Gauss–Bonnet theorem**. The existence of the mean specific Euler characteristic can be found already in Leistritz/Zähle [9]. Moreover, under the additional condition that $P\{\mu \in \Gamma : x \in \mu$ maximal $\Rightarrow \dim x = p\} = 1$ we get the mean value relations

$$E_P(\mathcal{X}) = \sum_{q=0}^{p} (-1)^q P^0_{\mathcal{M}^0_0} \left(\sum_{0\in x\in\mu_q} 1_{H_n}(x-a) \right) \quad \sigma^{d-1} - a.s.[\mathrm{n}], \quad \text{and} \tag{5.8}$$

$$E_P(\chi) = \sum_{q=0}^{p} (-1)^q P^0_{\mathcal{M}^0_0} \left(\sum_{0\in x\in\mu_q} \varepsilon(0,x) \right). \tag{5.9}$$

They are contained in Theorem 3.3.6 of Zähle [15]. In the special case of a **random network**, i.e. if $\dim\mu = p = 1$ P-a.s., equation (5.9), combined with the corresponding remarks in Example 2, reduces to

$$E_P(\mathcal{X}) = P^0(\mathcal{M}^0_0) - \frac{1}{2} \cdot P^0_{\mathcal{M}^0_0}(|\mu^0_1|). \tag{5.10}$$

Since $\kappa = \mathcal{X}$ P-a.s., we also have in this case that $E_P(\kappa) = P^0(\mathcal{M}^0_0) - \frac{1}{2}P^0_{\mathcal{M}^0_0}(|\mu^0_1|)$. This result has recently been obtained by Mecke/Stoyan [11].

In the case of random simplicial surfaces in the sense of [16], equation (5.9) yields

$$P^0_{\mathcal{M}^0_0} \text{ (deficit angle of } \mu \text{ in its vertex } 0) = 2\pi \cdot E_P(\mathcal{X}), \tag{5.11}$$

if we take into account the observation of Example 2 above. The left hand side of (5.11) was called in [16] the mean specific total curvature of P and denoted there by $\mathcal{K}(P)$. Note also that in view of (3.11) combined with the theorem, $E_P(\mathcal{X})$ equals the mean specific Euler characteristic $\chi(P)$ of P as defined in [16]. Thus equation (5.7) and (5.9) generalize considerably the recent results in [11, 16].

§6. SPECIFIC EINSTEIN–HILBERT ENERGY

Here we consider the Regge potential ϕ_R for the tempered set $\Gamma_2 = \mathcal{S}_{r,R,2}(E)$ of Delone configurations $\mu \in \mathcal{S}_{r,R}(E)$ as defined above. If $P \in \mathcal{P}^0\Gamma$ is the given law then the theorem implies P-a.s. and in $\mathcal{L}^1(P)$ the existence of the **specific Einstein–Hilbert energy**

$$\eta = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} E^{\phi_R}_{\Lambda_n}. \tag{6.1}$$

η is invariant under translations, and its mean is given by

$$P^0_{\mathcal{M}^0_0}(\phi_R(0,.)) = -\left[P^0_{\mathcal{M}^0_0}(\phi_c(0,.)) - P^0_{\mathcal{M}^0_0}(\phi_v(0,.))\right]. \tag{6.2}$$

This expresses the mean specific Einstein–Hilbert energy by means of the mean specific volume and the mean specific curvature.

§7. GENERALIZATIONS

Until now we considered only curvatures which are concentrated in the vertices of a given complex. We now develop the general notion of index and curvature along the lines of Banchoff [1] (see also Cheeger et al. [4] and Budach [2]). Then we indicate shortly some generalizations of results obtained above.

The index of a simplicial complex $\mu \in \mathcal{S}(E)$ in a cell $x \in \mu$ for the direction n$\in S^{d-1}(E)$ is now defined by

$$I_{\mathbf{n}}(x,\mu) = \sum_{x \subseteq y \in \mu} (-1)^{\dim y - \dim x} \cdot 1_{H_{\mathbf{n}}}(y - b(x)), \tag{7.1}$$

where

$$H_{\mathbf{n}} = \{y \in X : 0 \in b(y), \mathbf{n}\cdot b = 0 \text{ for any } b \in x_0(y), \mathbf{n}\cdot b < 0 \text{ for any } b \in y \setminus x_0(y)\}.$$

Here $x_0(y)$ denotes the unique $z \subseteq y$ with $0 \in b(z)$. We then can define a potential $I_{\mathbf{n}}(\nu,\mu), \nu \leq \mu$ finite, as follows : If ν is not of the form $\bar{x} = \sum_{y \subseteq x} \delta_y, x \in X$, then $I_{\mathbf{n}}(\nu,\mu) = 0$. Otherwise $I_{\mathbf{n}}(\bar{x},\mu) = I_{\mathbf{n}}(x,\mu)$. It is obvious that the set Γ of Delone configurations $\mathcal{S}_{r,R}(E)$ is tempered for $I_{\mathbf{n}}$. Thus the associated energy $E_\Lambda^{I_{\mathbf{n}}}(\mu)$ is well defined for $\mu \in \Gamma, \Lambda \in \mathcal{B}_0(E)$.

We then obtain the notion of curvature if we integrate $I_{\mathbf{n}}(\nu,\mu)$ with respect to $\mathbf{n}$ in the right way :

If $\nu \leq \mu$ is a finite subcomplex of $\mu \in \mathcal{S}(E)$, which is not of the form $\bar{x}, x \in \mu$, then we set $\phi_{\mathcal{CK}}(\nu,\mu) = 0$. If $x \in \mu$ with $\dim x \geq 1$, we define the **Chern–Gauss–Bonnet potential** of $\bar{x}$ in μ by

$$\phi_{\mathcal{CK}}(\bar{x},\mu) = \mathcal{H}^{\dim x}(\langle x \rangle) \cdot \sum_{x \subseteq y \in \mu} (-1)^{\dim y - \dim x} \cdot \varepsilon(x,y); \tag{7.2}$$

if $\dim x = 0$ we use also this definition with the convention $\mathcal{H}^0(\langle x \rangle) = 1$ for any x. (Here $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in E.) $\varepsilon(x,y)$ denotes the **normalized exterior angle of the cell y at the face x**. It is defined as the ratio of the area of the set of normals to the support hyperplanes of y at x to the area of the entire surface generated by these normals. To be more precise : if $x \subseteq y$ and $r = \dim x$, $s = \dim y$,

$$\varepsilon(x,y) = \int_{S^{d-1}} 1_{H_{\mathbf{n}}}(y - b(x))\, \sigma^{s-r-1}(d\mathbf{n}). \tag{7.3}$$

This is an intrinsic quantity.

By means of the Chern–Gauss–Bonnet potential we then define curvature as follows : Let Γ_p be the set of Delone complexes $\mu \in \mathcal{S}_{r,R}(E)$ having the property that each maximal cell of μ has the same dimension p. Given $0 \leq k \leq p$ and $\Lambda \in \mathcal{B}_0(E)$, the **Lipschitz–Killing curvature of μ of order** (p,k) in Λ is the energy defined by the **Lipschitz–Killing potential** as follows :

$$\phi_{\mathcal{LK}}^{(p,k)}(\nu,\mu) = 1_{\{\dim = p-k\}}(\nu) \cdot \phi_{\mathcal{CK}}(\nu,\mu), \mu \in \Gamma_p \quad \text{and} \quad \nu \leq \mu \text{ finite}. \tag{7.4}$$

The **Regge potential of order** (p,k) is defined in this context by

$$\phi_R^{(p,k)} = -\left[\phi_{\mathcal{LK}}^{(p,k)} - \phi_v^{(p,k)}\right], \tag{7.5}$$

where $\phi_v^{(p,k)}$ is the **volume potential of order** (p,k) given by $\phi_v^{(p,k)}(\nu,\mu) = 0$ if ν is not of the form $\bar{x}$ for some $x \in \mu_{p-k}$, and otherwise by

$$\phi_v^{(p,k)}(\bar{x},\mu) = \mathcal{H}^{p-k}(\langle x\rangle) \cdot \sum_{x \subseteq y \in \mu_p} \mathcal{H}^p(\langle y\rangle). \tag{7.6}$$

The energy associated to $\phi_R^{(p,k)}$, if considered on Γ_p, is the **Einstein–Hilbert energy of order** (p,k).

Note that the set Γ_p of Delone complexes is tempered for the potentials $I_{\mathbf{n}}$, $\phi_{\mathcal{LK}}^{(p,k)}$, $\phi_v^{(p,k)}$ and $\phi_R^{(p,k)}$. If now $P \in \mathcal{P}^0\Gamma_p$ is an underlying stationary law on Γ_p the ergodic theorem for the energy implies the existence of the corresponding specific energies, i.e. the **specific index** $I(\mathbf{n},\cdot)$ for any $\mathbf{n} \in S^{d-1}$, the **specific volume of order** (p,k) denoted by $\lambda_{p,k}(\cdot)$, and the **specific Lipschitz–Killing curvature of order** (p,k), denoted by $\kappa_{p,k}(\cdot)$.

Recall that these quantities are given by conditional expectations of the form $e = E_P(\int_D g(\mu - a) b(\mu)(da)/\mathcal{J})$, where g is given by means of the underlying potential ϕ by (4.4). Thus they are translation-invariant and their expectation is

$$E_P(e) = P^0_{\mathcal{M}^0}\left(\sum_{\nu \leq \mu, 0 \in b(\nu)} \frac{\phi(\nu,\mu)}{|\nu|}\right). \tag{7.7}$$

The existence of the specific Euler characteristic has already been shown above. But we give another argument to show that the old and new specific indices ι and I coincide $\sigma^{d-1} \otimes P$-almost surely. First of all we observe that $E_\Lambda^{I_{\mathbf{n}}}(\mu)$ is related to $\mathcal{X}(\mu_\Lambda)$ by the following analogon of (3.9). For any $\mathbf{n} \in S^{d-1}$, $E_\Lambda^{I_{\mathbf{n}}}(\mu)$ and $\mathcal{X}(\mu_\Lambda)$ differ from one another only by a boundary term, thereby implying an estimate of the form (5.3) with some error term $\tilde{\Delta}(\Lambda)$ which does not depend on μ and satisfies $\lim_{n\to\infty} \frac{1}{|\Lambda_n|}\tilde{\Delta}(\Lambda_n) = 0$. This follows from the critical point theorem 5 of Banchoff [1] for finite complexes. From this we then obtain for any $\mathbf{n} \in S^{d-1}$,

$$I(\mathbf{n},\cdot) = \mathcal{X}(\cdot) \quad P-a.s., \tag{7.8}$$

which is another version of the critical point theorem for stationary infinitely extended random simplicial complexes. If combined with (5.6) it yields that ι and $\mathcal{X}$ coincide $\sigma^{d-1} \otimes P$-almost surely.

Now consider the mean specific Lipschitz–Killing curvatures and the mean specific volumes. If we evaluate (7.7) for the curvature potential $\phi_{\mathcal{L}K}^{(p,k)}$ we obtain

$$E_P(\kappa_{p,k}) = \frac{1}{2^{p-k+1}-1}\times$$

$$\times P^0_{\mathcal{M}^0}\left[\sum_{x\in\mu, 0\in b(x)\, \dim x = p-k} \mathcal{H}^{p-k}(\langle x\rangle)\sum_{x\subseteq y\in\mu}(-1)^{\dim y-\dim x}\cdot\varepsilon(x,y)\right].$$

This can be written as

$$E_P(\kappa_{p,k}) = \frac{1}{2^{p-k+1}-1}\times$$

$$\times P^0_{\mathcal{M}^0_{p-k}}\left[\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle)\sum_{x_0(\mu_{p-k})\subseteq y\in\mu}(-1)^{\dim y-p+k}\cdot\varepsilon(x_0(\mu_{p-k}),y)\right]. \tag{7.9}$$

Here $\mathcal{M}^0_{p-k} = \{\mu\in\Gamma_p | 0\in b(\mu_{p-k})\}$ and $P^0_{\mathcal{M}^0_{p-k}} = Res_{\mathcal{M}^0_{p-k}} P^0$. In the same way we obtain for the mean specific volume

$$E_P(\lambda_{p,k}) = \frac{1}{2^{p-k+1}-1}\cdot P^0_{\mathcal{M}^0_{p-k}}\left[\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle)\cdot\mathcal{H}^p(\langle \mu_p^{x_0(\mu_{p-k})}\rangle)\right], \tag{7.10}$$

where $\mu_p^{x_0(\mu_{p-k})} = \sum_{x_0(\mu_{p-k})\subseteq y\in\mu_p}\delta_y$.

Finally, the mean specific Einstein–Hilbert energy of order (p,k) i.e. $E_P(\eta_{p,k})$, where

$$\eta_{p,k} = \lim_{n\to\infty}\frac{1}{|\Lambda_n|}E_{\Lambda_n}^{\phi_R^{(p,k)}},$$

is explicitely given by

$$E_P(\eta_{p,k}) = \frac{-P^0_{\mathcal{M}^0_{p-k}}}{2^{p-k+1}-1}\left[\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle)\times\right.$$

$$\left.\times\sum_{x_0(\mu_{p-k})\subseteq y\in\mu_p}((-1)^{\dim y-p+k}\varepsilon(x_0(\mu_{p-k}),y)-\mathcal{H}^p(\langle z\rangle))\right]. \tag{7.11}$$

§8. CONCLUDING REMARKS

If we look at the mean specific Einstein–Hilbert energies of order (p,k) in (7.11) we can ask for their **ground states**. These states are given by the minima of the functional $P \to E_P(\eta_{p,k})$, which is well defined in Γ_p. It is now evident that the important problem of detailed description of the set of ground states for $\phi_R^{(p,k)}$ in the physically interesting case $p = 4, k = 2$ is far from obvious.

We conclude this paper with a remark concerning the intrinsic character of the mean specific energies in (7.9), (7.10) and (7.11). These energies are represented by means of the non-normalized Palm measure P^0, which is not an intrinsic quantity. We obtain them in terms of intrinsic normalized Palm measures, if we suppose also that P is ergodic with respect to translations and if we normalize the energy not by the volume $|\Lambda|$ but by an intrinsic quantity. Consider for instance the sequence $b\mu_{p-k}(\Lambda_n), n \geq 1$, which counts the number of cells of μ_{p-k} with barycenter in Λ_n. By the ergodic theorem

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} b\mu_{p-k}(\Lambda_n) = P^0(\mathcal{M}^0_{p-k}).$$

If we assume that $0 < P^0(\mathcal{M}^0_{p-k}) < +\infty$, then we can divide the energy by $b\mu_{p-k}(\Lambda_n)$ and get for instance the following law of large numbers

$$\lim_{n\to\infty} \frac{1}{b\mu_{p-k}(\Lambda_n)} \cdot E_{\Lambda_n}^{\phi_R^{(p,k)}}(\mu) = -\frac{1}{2^{p-k+1}-1} \cdot \frac{1}{P^0(\mathcal{M}^0_{p-k})} \cdot P^0_{\mathcal{M}^0_{p-k}} \Big[\mathcal{H}^{p-k}((x_0(\mu_{p-k}))) \cdot$$

$$\cdot \sum_{x_0(\mu_{p-k}) \subseteq y \in \mu_p} \left((-1)^{\dim y - p + k} \cdot \varepsilon(x_0(\mu_{p-k}), y) - \mathcal{H}^p((y))\right)\Big] \tag{8.1}$$

P-a.s. and in $\mathcal{L}^1(P)$. Here the limit is an expectation taken with respect to the probability $\frac{1}{P^0(\mathcal{M}^0_{p-k})} \cdot P^0_{\mathcal{M}^0_{p-k}}$.

Instead of $b\mu_{p-k}(\Lambda_n)$, one could have taken also the volume of the random space (μ_{p-k}) in Λ_n. It follows from the ergodic theorem that P-a.s. and in $\mathcal{L}^1(P)$

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathcal{H}^{p-k}((\mu_{p-k}) \cap \Lambda_n) = \frac{1}{2^{p-k+1}-1} \cdot P^0_{\mathcal{M}^0_{p-k}}(\mathcal{H}^{p-k}((x_0(\mu_{p-k})))).$$

If we assume now that $0 < P^0_{\mathcal{M}^0_{p-k}}(\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle)) < +\infty$, then

$$\lim_{n\to\infty} \frac{1}{\mathcal{H}^{p-k}(\langle\mu_{p-k}\rangle \cap \Lambda_n)} \cdot E^{\phi_R^{(p-k)}}_{\Lambda_n}(\mu) = P^0_{\mathcal{M}^0_{p-k}} \Bigg[\frac{1}{P^0_{\mathcal{M}^0_{p-k}}(\mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle))} \times$$

$$\times \mathcal{H}^{p-k}(\langle x_0(\mu_{p-k})\rangle) \cdot \sum_{x_0(\mu_{p-k}) \subseteq y \in \mu_p} ((-1)^{dim\, y-p+k} \cdot \varepsilon(x_0(\mu_{p-k}), y) - \mathcal{H}^p(\langle y\rangle)) \Bigg] . \tag{8.2}$$

Again we obtain a law of large numbers, which is completely intrinsic.

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Резюме. Основным результатом статьи является эргодическая теорема для бесконечно расширяющихся стационарных случайных симплициальных комплексов, которая утверждает почти наверное существование пределов в $\mathcal{L}^1$. Как следствие, получается теорема о критической точке, а также аналог теоремы Гаусса–Боне. Показано также существование удельной кривизны Липшица–Киллинга и удельной энергии Эйнштейна–Гильберта.

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