АСТРОФИЗИКА

ТОМ 67 НОЯБРЬ, 2024 ВЫПУСК 4

DOI: 10.54503/0571-7132-2024.67.4-567

THERMODYNAMICS AND ITS QUANTUM CORRECTION OF VACUUM NONSINGULAR BLACK HOLE

D.MA, T.HUO, C.LIU

Recieved 23 July 2024
Accepted 6 December 2024

This paper investigates the thermodynamic properties of vacuum nonsingular black holes. Considering the energy characteristics of the regularity spacetime, we use the modified first law of black hole thermodynamics to calculate the black holes' Hawking temperature, entropy and heat capacity. The obtained temperature is the same as that obtained by the surface gravity and tunneling methods. Also, the entropy is satisfied with the Bekenstein-Hawking area law. Notably, the heat capacity of large-mass black holes diverges, while that of small-mass black holes tends to zero, with a phase transition point existing. Additionally, we consider the quantum gravity effect by using the generalized uncertainty principle to study the quantum corrections of the thermodynamic properties for the vacuum nonsingular black holes. The generalized uncertainty principle introduces a logarithmic correction term to the black hole entropy. Also, the temperature and heat capacity are modified.

Keywords: *thermodynamics of black holes: vacuum nonsingular black holes: generalized uncertainty principle: the modified first law of black hole thermodynamics*

1. *Introduction*. In general, black holes are characterized by the presence of singularities within their interiors, the existence of singularities seems to be unavoidable for almost all of the physically acceptable solutions of Einstein's equation. However, there are some attempts to remove these singularities. The idea of replacing a Schwarzschild singularity with a de Sitter vacuum goes back to the 1968 papers of Sakharov who considered $p_r = -\rho$ as the equation of state for super high density and of Gliner who interpreted $p_r = -\rho$ as corresponding to a vacuum [1,2]. In 1968 Bardeen presented the spherically symmetric metric of the same form as the Schwarzschild and Reisner-Nordstrom metric, describing a non-singular black hole [3]. "Non-singular" black holes refer to those where singularities are absent. Specifically, these black holes' metric and curvature invariants remain non-singular throughout spacetime. The regular black holes of Bardeen are spherically symmetric and violate the strong energy condition. The violation of an energy condition is the origin of the regularity of Bardeen's black hole [4]. With a de Sitter core, the internal region of regular geometries violates one

condition of Hawking-Penrose theorems of singularity. Then, with such a violation, the existence of a singular point is not a necessary consequence of the theorems of singularity. Later, other regular solutions with spherical symmetry were proposed by Dymnikova, Bronnikov, and Hayward [5-9]. Now, many regular black holes have been proposed and their intriguing properties including thermodynamics that are different from those of singular black holes were studied (see [10] as a review and references therein). The vacuum nonsingular black hole, provided by Dymnikova, is an important and feasible example of a black hole with a regular center, where the de Sitter core smoothly connects to the exterior geometry of Schwarzschild [5-7]. In the classical framework of general relativity, the vacuum nonsingular black hole provides an important new perspective for understanding static spherically symmetric black holes. This black hole's spacetime structure and thermodynamic properties have garnered attention [11-14].

The discovery of black hole thermodynamics has profoundly influenced our understanding of the relationship between general relativity and quantum field theory. The concept of Hawking radiation has changed our understanding of black holes [15]. It reveals that black holes are not just the result of gravity, but also have thermodynamic properties. Historically, black holes were considered completely irreversible systems without temperature or entropy. However, the theory of Hawking radiation suggests that black holes have a temperature and emit particles over time, gradually losing mass. This discovery links black holes to thermodynamics, indicating that they also follow the laws of thermodynamics [15,16]. In the thermodynamics of black holes, the temperature is obtained from the first law of black hole thermodynamics. The entropy for the black hole is given by Bekenstein's area law [17]. For the acquisition of Hawking radiation, in addition to Hawking's initial method, there is also the tunneling method that provides black hole radiation. A particle may cross the event horizon by quantum tunneling in the tunneling method. Two methods can be employed to derive the tunneling result: the first, pioneered by Parikh and Wilczek, is the null-geodesic method, and the WKB (Wentzel-Kramers-Brillouin) method is used [18,19]; the second, developed by Agheben et al., relies on the Hamilton-Jacobi ansatz [20]. In this paper, we will adopt the former approach to calculate Hawking temperatures for the vacuum nonsingular black holes. Then, based on the first thermodynamics law of the black hole, we calculate the temperature and derive the entropy and heat capacity of the regular black hole. The calculations reveal an inconsistency between the temperature derived from the first law of thermodynamics and that obtained through surface gravity and tunneling methods. In the lectures [21-25], by examining the internal energy characteristics of regular black holes, it is pointed out that the traditional first law of black hole thermodynamics is no longer used and needs to be revised. The correction form of the first law of black hole thermodynamics was provided by Ma et al. [21,22]. The deviation relies on the general structure of the energy-momentum tensor of matter fields. When the black hole mass parameter M is included in the energy-momentum tensor, the traditional form of the first law is modified by an additional factor. Here, in the vacuum nonsingular spacetime, by utilizing the corrected first law of black hole thermodynamics, consistency was established for black hole temperature results obtained through surface gravity, tunneling, and the first law of thermodynamics. Additionally, the applicability of the Beckenstein-Hawking area law was verified.

The quantization of gravity poses a significant challenge in theoretical physics, and a quantum theory of gravity remains to be established. The existence of a minimum observable length scale at the Planck length level is a common prediction of various candidate theories of quantum gravity [26]. The concept of a minimum length has led to numerous quantum gravity effects, one of the most notable being the generalized uncertainty principle (GUP) [27-28], which suggests modifications to the semi-classical description of black holes from an extension of the Heisenberg uncertainty principle, which involves a deformation parameter related to a minimal fundamental length [29]. Introducing the GUP into black hole thermodynamics allows for studying quantum corrections to the thermodynamic properties of black holes [30-39]. The GUP offers high-energy corrections to black hole thermodynamics. Among these, the GUP can affect the quantum tunneling process of particles on the black hole horizon, increasing the probability of tunneling and giving quantum corrections to the thermodynamic quantities of the black hole [40-42]. For the vacuum nonsingular black holes, including the GUP effect in quantum tunneling calculations, reveal quantum corrections to the thermodynamic quantities of the black hole, wherein, the quantum correction to entropy is logarithmic.

The paper is organized as follows: Section 2 briefly introduces vacuum non-singular black holes. Section 3 discusses the thermodynamic properties of regular black holes using the modified first law of black hole thermodynamics. Section 4 discusses the GUP effects on the black holes' temperature, entropy, and hot capacity. The last part is a summary and discussion. Where the speed of light c in vacuum, the gravitational constant G, the Boltzmann constant k_B, and the reduced Planck constant $\hbar$ are set to 1.

2. *Vacuum non-singular black holes*. The general form of a static spherically symmetric metric can be written as

$$ds^2 = -f(r)c^2dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{1}$$

In Dymnikova's work [5], the stress-energy-momentum tensor is assumed to have a specific form, which is given by

$$T_t^t(r) = \varepsilon_0 \exp\left(-\frac{r^3}{r_0^2 r_g}\right), \tag{2}$$

where ε_0 represents the non-zero energy density of the vacuum, $r_g = 2M$ and M is the mass of the black hole as measured by a distant observer. r_0 is connected with ε_0 by the de Sitter relation

$$r_0^2 = \frac{3c^4}{8\pi G\varepsilon_0}. \tag{3}$$

Integrating the Einstein field equations with the assumed form of T_t^t, we obtain the following

$$f(r) = 1 - \frac{R_g(r)}{r}, \tag{4}$$

where

$$R_g(r) = r_g\left(1 - e^{-r^3/r_0^2 r_g}\right), \tag{5}$$

the mass term $m(r)$ given in [5] is of the form

$$m(r) = \frac{4\pi}{c^2}\int_0^r T_t^t r^2 dr. \tag{6}$$

The quadratic invariant of the Riemann tensor $\mathcal{R}^2 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ has the form [5]

$$\mathcal{R}^2 = 4\frac{R_g^2(r)}{r^6} + 4\left(\frac{3}{r_0^2}e^{-r^3/r_0^2 r_g} - \frac{R_g(r)}{r^3}\right) + \frac{2R_g(r)}{r^3} - \frac{9r^3}{r_0^4 r_g}e^{-r^3/r_0^2 r_g}, \tag{7}$$

when $r = 0$, $\mathcal{R}^2$ does not diverge and remains finite, tending towards the de Sitter value $\mathcal{R}^2 = 24/r_0^4$, which should be the limit of spacetime curvature. All other invariants are also finite. So the Dymnikova solution describes a spherically symmetric black hole singularity-free everywhere, hence called a "non-singular black hole".

For $r >> r_0^2 r_g$, the vacuum non-singular black hole (5) is actually consistent with the Schwarzschild solution. For $r << r_0^2 r_g$, it is consistent with the de Sitter solution. It provides an important new perspective for understanding static spherically symmetric black holes.

3. *Thermodynamics of vacuum non-singular black holes*. In this section, we discuss the thermodynamics of non-singular black holes. We calculate the temperature using the surface gravity method, the quantum tunneling method, and the first law of thermodynamics. One will observe discrepancies between the temperature obtained from the first law of thermodynamics and those derived

through two other methods. By employing the corrected first law of black hole thermodynamics as provided in reference [21], we obtain a black hole temperature consistent with surface gravity and quantum tunneling methods, yielding entropy by the Beckenstein-Hawking area law. Further computation of the black hole's heat capacity suggests the possibility of establishing thermodynamically stable vacuum nonsingular black holes, contrary to Schwarzschild black holes.

Firstly, we calculate the temperature using the surface gravity method. The temperature of a black hole is a characteristic quantity of its event horizon, which can be derived from the surface gravity on the event horizon. The surface gravity on the black hole horizon is a conserved quantity determined by the geometric structure of the horizon. The calculation is as follows [15]

$$T_\kappa = \frac{\kappa}{2\pi} = -\frac{1}{4\pi} \frac{\partial_r g_{tt}}{\sqrt{-g_{tt} g_{rr}}}\Bigg|_{r=r_h} = \frac{1}{4\pi} g'(r)\big|_{r=r_h}, \tag{8}$$

here, κ represents the surface gravity, r_h stands for the location of the event horizon. Making a simple change to equation (4) and calculating according to (8), we have

$$T_\kappa = \frac{3r_h^3 - 3r_h^2 + r_0^2 r_g}{4\pi r_h r_g r_0^2}. \tag{9}$$

Using (1) and (4) and setting $[g_{tt}]_{r=r_h} = 0$, we can obtain the black hole's horizon radius, which is

$$r_h = 2M\left(1 - e^{-r_h^3/2Mr_0^2}\right). \tag{10}$$

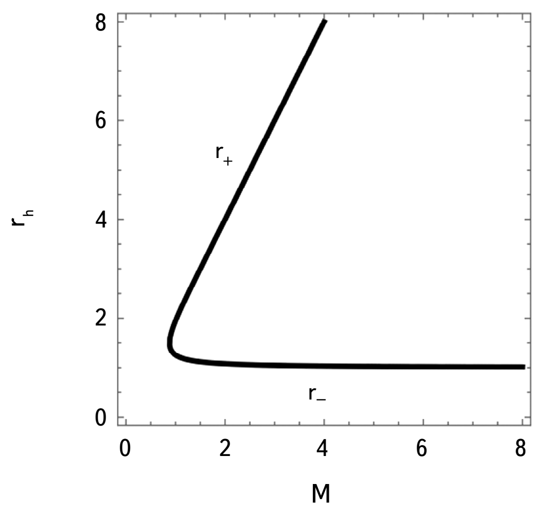

Fig.1. The metric (4) has two event horizons located for $r_0 = 1$.

Equation (10) does not have an analytical solution. Fig.1 shows that r_+ is the external event horizon, and r_- is related to the inner event horizon which is the Cauchy horizon. The metric (4) also showed that two horizons are movable. From equations (9) and (10), it can be seen that neglecting vacuum energy, letting $r_0^2 \to \infty$, one can respectively obtain the Schwarzschild cases of $r_h = 2M$ and $T_\kappa = 1/8\pi M$.

The temperature of the black hole can also be determined using the quantum tunneling method of Hawking radiation [19,20]. For this, the metric is expressed in the Painlevé form to eliminate coordinate singularities, which can be achieved through a suitable transformation [43]

$$t \to t - \int \frac{\sqrt{1-f(r)}}{f(r)} dr, \tag{11}$$

this change (1) into

$$ds^2 = -f(r)dt^2 + 2\sqrt{1-f(r)}\,dtdr + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{12}$$

Form equation (12), the radial null geodesic equation of massless particles can be derived as

$$\dot{r} = \frac{dr}{dt} = \pm 1 - \sqrt{1-f(r)}, \tag{13}$$

the positive and negative signs in the equation describe the outgoing and ingoing, respectively.

The WKB approximation describes semi-classical approximate solutions to wave equations. For particle tunneling through a potential barrier, the WKB approximation expresses the tunneling probability related to the imaginary part of the tunneling action [19,20]. Specifically, the tunneling probability of a particle through a potential barrier under the WKB approximation is given by

$$\Gamma \propto e^{-2\mathrm{Im}(S)/\hbar}, \tag{14}$$

where Γ is the tunneling probability, $\mathrm{Im}(S)$ is the imaginary part of the action. In this expression, the negative exponent indicates the probability of exponential decline with increasing barrier width. The imaginary part of the action is typically expressed as

$$\mathrm{Im}(S) = \mathrm{Im}\int_{r_1}^{r_2} p(r)dr = \mathrm{Im}\int_{r_1}^{r_2}\int_0^{p_r} dp_r' dr, \tag{15}$$

where $p(r)$ is the classical momentum of the particle at position r, and r_1 and r_2 are the positions on either side of the barrier.

Continuing the solution, we use the Hamiltonian canonical equation $\dot{r} = dH/dp_r = d\omega'/dp_r$, where ω' represents the energy of the tunneling particle.

Substituting this into equation (15) and changing the order of integration, we obtain

$$\mathrm{Im}(S) = \mathrm{Im}\int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{d\,\omega'}{\dot{r}}\,dr\,. \tag{16}$$

Expanding equation (13) in a Taylor series around the horizon and retaining only the leading order term, we can substitute this approximation into the radial motion equation (15) for the practice, leading to

$$\mathrm{Im}(S) = \int_0^{\omega}\int_{r_i}^{r_f} \frac{2}{f'(r_h)(r-r_h)}\,drd\,\omega'\,. \tag{17}$$

Using the Residue Theorem, we have the following integral concerning r

$$\mathrm{Im}(S) = \mathrm{Im}\int_0^{\omega} \frac{2\pi}{f'(r_h)}\,d\,\omega' = \frac{2\pi\omega}{f'(r_h)}\,. \tag{18}$$

In the limit of the WKB approximation, the tunneling probability will take the following form

$$\Gamma \propto \exp(-2\mathrm{Im}(S)) = \exp(-\beta\omega). \tag{19}$$

Therefore, Hawking's temperature is

$$T_H = \frac{f'(r_h)}{4\pi} = T_{\kappa}\,. \tag{20}$$

In conclusion, the black hole temperature obtained through the quantum tunneling and surface gravity methods is the same.

The first law of thermodynamics provides another way to calculate the black hole temperature. For a vacuum spacetime with spherical symmetry, the first law of black hole thermodynamics can be expressed as

$$\delta M = T_H\,\delta S\,, \tag{21}$$

where S is the entropy of the black hole and T_H is the thermodynamic temperature of the black hole. Then, by utilizing the properties of the implicit function and equation (10), we have

$$F(M, r_h) = 2M\left(1 - e^{-r_h^3/2Mr_0^2}\right) - r_h = 0\,. \tag{22}$$

Using the derivative property of implicit functions, obtain dM/dr_h through equation (22). In this way we can obtain

$$T_H = \frac{dM}{dS} = \frac{1}{2\pi r_h}\frac{dM}{dr_h} = \frac{r_g\left(3r_h^3 - 3r_h^2 r_g + r_0^2 r_g\right)}{4\pi r_h^2\left(r_h^3 - r_h^2 r_g + r_0^2 r_g\right)}\,. \tag{23}$$

It is seen that the temperature according to the first law of black hole thermodynamics differs from the temperature obtained from the surface gravity method and the tunneling method. The Dymnikova black hole is not a vacuum solution of the Einstein field equations. The three methods yield two different temperatures for the black hole, while the surface gravity and tunneling effects give the same result.

Furthermore, if we require the vacuum non-singular black hole to satisfy the first law, then the entropy is not the Beckenstein-Hawking entropy. Resolving the contradiction between entropy and area requires considering the contribution of matter fields. Since the Dymnikova black hole is a solution of the Einstein equations, we tend to believe that the entropy of the black hole should follow the form of the Beckenstein-Hawking area law and that a modification of the black hole temperature obtained from the traditional laws of thermodynamics is necessary.

In [21], the authors studied the thermodynamics of the regular black hole and pointed out the existence of a correction factor in equation (21). Thus, the modified first law of black hole thermodynamics is [21]

$$C(M, r_h)\delta M = T_h \,\delta\frac{A}{4}, \tag{24}$$

where $C(M, r_h)$ is a corrected factor, and

$$C(M, r_h) = 1 + 4\pi\int_{r_h}^{\infty} r^2 \frac{\partial T_t^t}{\partial M} dr. \tag{25}$$

Then, for regular black holes, the first law of black hole thermodynamics has a correction term compared to the traditional first law of thermodynamics and the entropy satisfies the area law [23-25]. In the case of T_t^t being independent of mass, the correction term will vanish.

For vacuum non-singular black hole (1), substituting equation (2) into equation (25), the correction factor as

$$C(M, r_h) = \frac{r_h^4 + r_h r_g r_0^2 - r_h^3 r_g}{r_0^2 r_g^2}. \tag{26}$$

Thus, substituting the correction factor (26) into equation (24) directly gives the corrected form of the temperature as

$$T_h = C(M, r_h) T_H = \frac{3 r_h^3 - 3 r_h^2\, 2M + r_0^2 r_g}{4\pi\, r_h r_g r_0^2}. \tag{27}$$

This is consistent with the results obtained from the other two methods, that is

$$T_h = T_\kappa = T_t. \tag{28}$$

Furthermore, using the corrected first law of black hole thermodynamics and substituting equations (26) and (27) into equation (29), we verify that the entropy of a vacuum non-singular black hole satisfies the area law

$$S = \int \frac{C(M, r_h)}{T_h} dM = \pi r_h^2 = \frac{A}{4}. \tag{29}$$

The heat capacity of a black hole provides information about the thermodynamic stability of the system. Next, we calculate the heat capacity of a vacuum nonsingular black hole to understand its local stability. The heat capacity of a black hole is defined as

$$C_v = T\frac{\partial S}{\partial T} = T\frac{\partial S}{\partial r_h}\left(\frac{\partial T}{\partial r_h}\right)^{-1}. \tag{30}$$

For a Schwarzschild black hole, substituting the temperature $(T_h)_{Sch} = 1/(8\pi M)$ and the equation (29) into the equation (30), we have

$$(C_v)_{Sch} = -2\pi r_h^2. \tag{31}$$

A negative heat capacity indicates that a Schwarzschild black hole's thermodynamic system is unstable [44]. When the black hole loses energy, its mass and surface area decrease, but its temperature increases. As a result, the increased temperature makes the black hole more prone to radiate energy. As the radiation rate is related to the temperature, the black hole losing energy will radiate faster, leading to instability. Eventually, a radiation explosion occurs as the black hole radius approaches zero.

Substituting the temperature (27) and the entropy (29) into equation (30), we get

$$C_v = -\frac{2\pi r_h^2\left(r_g r_0^2 + r_h^2(-r_g + r_h)\right)\left(r_g r_0^2 + 3r_h^2(-r_g + r_h)\right)}{r_0^4 r_g^2 + 2r_0^2 r_g r_h^2(r_g - r_h) + 3r_h^4\left(-r_g^2 + r_h^2\right)}. \tag{32}$$

We can see that when $r_0^2 \to \infty$ of without considering vacuum energy, this equation gives the result (31) of Schwarzschild black hole.

As shown in Fig.2 (black curve), the heat capacity will approach zero as the black hole tends to the external one and diverges at the point where the temperature of the black hole takes the maximum. In the study referenced as [45], which investigates Bardeen and Hayward geometries, the discontinuity in the heat capacity of rotating black holes (RBHs) indicates a phase transition. We can also observe that the phase transition of the vacuum non-singular black hole occurs at $r_h = 2.242$. For large black holes, the heat capacity is negative, indicating that the thermodynamic system is unstable. In contrast, the heat capacity for small black holes is positive, indicating that the black hole may reach thermodynamic equilibrium with the surroundings. At the critical point of $r_h = 2.242$, there may

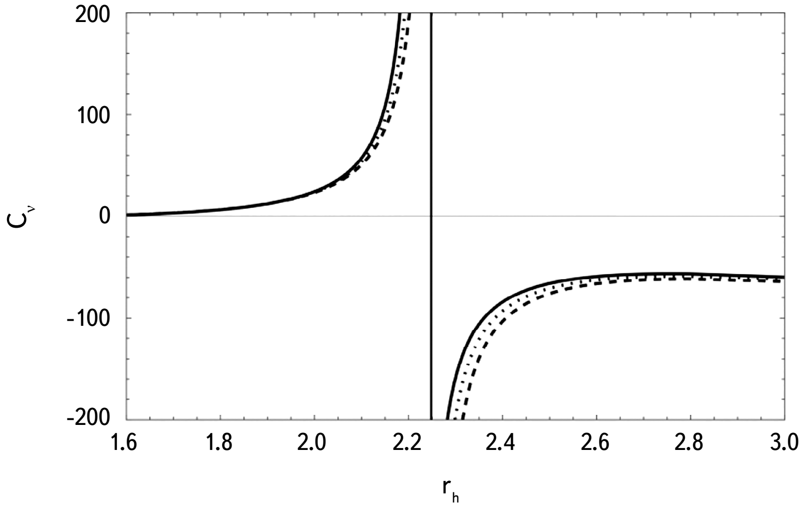

Fig.2. The black curve represents C_v, the black dashed curve represents $(C_v)_{GUP}$ for $\lambda = 0.4$, and the black dotted curve represents $(C_v)_{GUP}$ for $\lambda = 0.8$. Here $l = 1$, $M = 1$ and $r_0 = 1$.

exist a phase transition from a larger black hole to a smaller one, which is a second-order transition.

4. *Quantum corrections to black hole thermodynamics*. In this section, we will consider the GUP effect thermodynamic quantities related to vacuum non-singular black holes. The Heisenberg uncertainty principle can be extended to the GUP in the following form [26-28,30]

$$\Delta x \Delta p \geq 1 + \frac{\lambda^2 l_p^2}{\hbar^2} \Delta p^2 , \tag{33}$$

where λ is a dimensionless quantum gravity parameter, and $l_p \approx 10^{-35}\, m$ corresponds to the Planck length scale. When λ tends to zero, the GUP return to the standard uncertainty relation leads to the results of Hawking's semiclassical method. We can rewrite the GUP as

$$\Delta p \geq \frac{\Delta x}{2\lambda^2 l_p^2} \left(1 - \sqrt{1 - \frac{4\lambda^2 l_p^2}{\Delta x^2}} \right). \tag{34}$$

Considering $l_p << \Delta x$, we expand equation (34) using a Taylor series

$$\Delta p \geq \frac{1}{\Delta x} \left(1 + \frac{2\lambda^2 l_p^2}{\Delta x^2} + \ldots \right). \tag{35}$$

Using the Heisenberg uncertainty principle, we have the particle energy $\omega \geq 1/\Delta x$.

Considering the GUP given by equation (35), the energy correction is

$$\omega_{GUP} \geq \omega\left(1+\frac{2\lambda^2 l_p^2}{\Delta x^2}\right), \tag{36}$$

up to second order in l_p. Substituting equation (36) into equation (19), we obtain the particle tunneling probability corrected by the GUP as

$$\Gamma \simeq \exp\left[-\frac{4\pi\omega_{GUP}}{f'(r_h)}\right] = \exp\left[-\frac{4\pi\omega}{\hbar f'(r_h)}\left(1+\frac{\lambda^2 l_p^2}{\Delta x^2}\right)\right]. \tag{37}$$

Comparing (37) with the Boltzmann factor $\exp(-\omega/T)$, we obtain the quantum-corrected Hawking temperature

$$T_{GUP} = T_h\left(1+\frac{\lambda^2 l_p^2}{4 r_h^2}\right)^{-1}, \tag{38}$$

where T_h is given by equation (27). It can be seen that GUP reduces the temperature of vacuum nonsingular black holes.

Using the modified first law of thermodynamics (24), the entropy of a vacuum non-singular black hole can be expressed as

$$S_{GUP} = \int \frac{C(M, r_h)}{T_{GUP}} dM, \tag{39}$$

ignoring the integration constant, the result can be expressed as

$$S_{GUP} = \pi\left(r_h^2 + \frac{\lambda^2 l_p^2}{2}\ln r_h\right) = \frac{A}{4} + \frac{\lambda^2}{4\pi}\ln A. \tag{40}$$

Where the first term is the contribution from the traditional black hole entropy, and the second term is the GUP correction term. The correction is the logarithm of the black hole's area, consistent with related research findings [46-49]. The correction term reflects the influence of the correlation between position and momentum on the black hole entropy when considering the GUP. Ignoring the GUP effect by letting $\lambda \to 0$, then the quantum-corrected entropy (40) reverts to the Beckenstein-Hawking area law (29).

Taking into account the GUP effects the formula (30) for the black hole's heat capacity can be written as

$$(C_v)_{GUP} = T_{GUP}\frac{\partial S_{GUP}}{\partial T_{GUP}}. \tag{41}$$

Substituting equations (40) and (38) into equations (41), we get

$$(C_\nu)_{GUP} = -\left\{ -\frac{2r_0^4 r_g^2 \left(4r_h^2 - l_p^2 \lambda^2\right)}{\pi\left(r_g r_0^2 + r_h^2(-r_g + r_h)\right)\left(r_g r_0^2 + 3r_h^2(-r_g + r_h)\right)\left(4r_h^2 + l_p^2\lambda^2\right)^2} + \right.$$
$$\frac{4r_g r_0^2 \left(r_g - r_h^2\right) r_h^2 \left(4r_h^2 + 5l_p^2 \lambda^2\right)}{\pi\left(r_g r_0^2 + r_h^2(-r_g + r_h)\right)\left(r_g r_0^2 + 3r_h^2(-r_g + r_h)\right)\left(4r_h^2 + l_p^2\lambda^2\right)^2} - \tag{42}$$
$$\left. \frac{6\left(r_g - r_h\right) r_h^4 \left(4r_g r_h^2 + 4r_h^3 + 3\lambda^2 l_p^2 - r_h \lambda^2 l_p^2\right)}{\pi\left(r_g r_0^2 + r_h^2(-r_g + r_h)\right)\left(r_g r_0^2 + 3r_h^2(-r_g + r_h)\right)\left(4r_h^2 + l_p^2\lambda^2\right)^2} \right\}^{-1} .$$

This is the heat capacity of vacuum non-singular black holes under the influence of the GUP. It can be observed that when $\lambda \to 0$, $(C_\nu)_{GUP}$ can be restored to equation (32), which is consistent with the heat capacity results obtained without considering the GUP correction. Moreover, Fig.2 demonstrates that, in general, there is no significant distinction between $(C_\nu)_{GUP}$ and C_ν.

In addition, by directly setting $r_0^2 \to \infty$ in equation (42), we obtain

$$(C_\nu)_{GUP} = -2\pi \frac{\left(r_h^2 + \lambda^2 l_p^2/4\right)^2}{r_h^2 - \lambda^2 l_p^2/4}, \tag{43}$$

which corresponding to the Schwarzschild black hole heat capacity when considering the GUP effects. Then, setting $\lambda \to 0$, equation (43) can also be reduced to the heat capacity equation (31) of the Schwarzschild black hole.

5. *Conclusion*. This paper investigates the thermodynamic properties of vacuum non-singular black holes. We find that the temperature obtained using the traditional first law of thermodynamics is inconsistent with those obtained using the surface gravity and tunneling methods. By considering the modified form of the black hole thermodynamic given in the [21], we get consistent results and ensure the validity of the Beckenstein-Hawking area law. Furthermore, using the corrected thermodynamic first law, we derive the heat capacity of vacuum non-singular black holes, indicating that these black holes may exhibit thermodynamic behaviors different from traditional Schwarzschild black holes. Under certain conditions, the heat capacity of vacuum non-singular black holes can be positive and then the black hole may reach thermodynamic equilibrium with the surroundings. Finally, we consider the quantum gravity effects manifested by the GUP and study the thermodynamics of vacuum non-singular black holes using the modified first law of black hole thermodynamics. In particular, the GUP introduces a logarithmic correction term to the black hole entropy.

Acknowledgments. The work was supported by the Natural Science Foundation of Zhejiang Province of China (No.LY14A030001).

Department of Physics, Shaoxing University, Shaoxing 312000,
China, e-mail: czlbj20@163.com

ТЕРМОДИНАМИКА И ЕЕ КВАНТОВАЯ КОРРЕКЦИЯ ВАКУУМНОЙ НЕСИНГУЛЯРНОЙ ЧЕРНОЙ ДЫРЫ

Д.МА, Т.ХО, Ч.ЛЮ

В данной статье исследуются термодинамические свойства вакуумных несингулярных черных дыр. Учитывая энергетические характеристики регулярного пространства-времени, использован модифицированный первый закон термодинамики черных дыр для расчета температуры Хокинга, энтропии и теплоемкости черных дыр. Полученная температура совпадает с температурой, полученной методами поверхностной гравитации и туннелирования. Кроме того, энтропия удовлетворяет закону площади Бекенштейна-Хокинга. Примечательно, что теплоемкость черных дыр большой массы расходится, в то время как теплоемкость черных дыр малой массы стремится к нулю, при этом существует точка фазового перехода. Кроме того, эффект квантовой гравитации рассмотрен использованием обобщенного принципа неопределенности для изучения квантовых поправок термодинамических свойств в случае вакуумных несингулярных черных дыр. Обобщенный принцип неопределенности вводит логарифмический поправочный член в энтропию черной дыры. Также модифицируются температура и теплоемкость.

Ключевые слова: *термодинамика черных дыр: вакуумные несингулярные черные дыры: обобщенный принцип неопределенности: модифицированный первый закон термодинамики черных дыр*

REFERENCES

1. *A.D.Sakharov*, Sov. Phys. JETP, **22**, 241, 1966.
2. *E.B.Gliner*, Sov. Phys. JETP, **22**, 378, 1966.
3. *J.C.S.Neves*, *S.Alberto*, Phys. Lett. B, **734**, 44, 2014.
4. *J.M.Bardeen*, in: GR-5 Book of Abstracts, Tbilisi, 1968.

5. *I.Dymnikova*, Gen. Relativ. Gravit., **24**, 235, 1992.
6. *I.G.Dymnikova*, Int. J. Mod. Phys. D, **5**, 529, 1996.
7. *I.Dymnikova*, Int. J. Mod. Phys. D, **12**, 1015, 2003.
8. *K.Bronnikov*, Phys. Rev. D., **63**, 044005, 2001.
9. *S.Hayward*, Phys. Rev. Lett., **96**, 031103, 2006.
10. *C.Lan et al.*, Int. J. Theor. Phys., **62**, 202, 2023.
11. *I.Dymnikova*, *K.Michał*, Entropy, **13**, 1967, 2011.
12. *G.Alencar et al.*, J. Cosmol. Astropart. Phys., **2023**, 100, 2023.
13. *M.H.Macêdo et al.*, arXiv preprint arXiv:2404.02818, 2024.
14. *T. De Lorenzo*, Investigating Static and Dynamic Non-Singular Black Holes., 2014.
15. *S.W.Hawking*, Math. Phys., **43**, 199, 1975.
16. *J.D.Bekenstein*, Phys. Rev. D, **7**, 2333, 1973.
17. *S.W.Hawking*, *N.P.Don*, Commun. Math. Phys., **87**, 577, 1983.
18. *M.K.Parikh*, *W.Frank*, Phys. Rev. Lett., **85**, 5042, 2000.
19. *M.Parikh*, Int. J. Mod. Phys. D, **13**, 2351, 2004.
20. *M.Angheben et al.*, JHEP, **2005**, 014, 2005.
21. *Meng-Sen Ma*, *Z.Ren*, Class. Quantum Gravity, **31**, 245014, 2014.
22. *D.V.Singh*, *S.Sanjay*, arXiv preprint arXiv:1909.11529, 2019.
23. *Zhong-Ying Fan*, *W.Xiaobao*, Phys. Rev. D, **94**, 124027, 2016.
24. *Zhong-Ying Fan*, Eur. Phys. J. C, **77**, 1, 2017.
25. *Y.Guo*, *M.Yan-Gang*, Nucl. Phys. B, **980**, 115839, 2022.
26. *L.J.Garay*, Int. J. Mod. Phys. A, **10**, 145, 1995.
27. *M.Maggiore*, Phys. Lett. B, **304**, 65, 1993.
28. *A.Kempf*, *M.Gianpiero*, *B.M.Robert*, Phys. Rev. D, **52**, 1108, 1995.
29. *E.C.Vagenas*, *M.A.Salwa*, *F.A.Ahmed*, EPL, **120**, 40001, 2018.
30. *L.N.Chang et al.*, Rev. D, **65**, 125028, 2002.
31. *Li Xiang*, Phys. Lett. B, **540**, 9, 2002.
32. *C.Z.Liu*, Sci. China Ser. G-Phys. Mech. Astron., **51**, 113, 2008.
33. *S.Gangopadhyay*, *D.Abhijit*, *F.Mir*, EPL, **112**, 20006, 2015.
34. *Hui-Ling Li*, *Song De-Wen*, *Li Wei*, Gen. Relativ. Gravit., **51**, 1, 2019.
35. *R.Zhao*, *Z.Sheng-Li*, Phys. Lett. B, **641**, 318, 2006.
36. *B.Majumder*, Phys. Lett. B, **703**, 402, 2011.
37. *R.Banerjee*, *G.Sumit*, Phys. Lett. B, **688**, 224, 2010.
38. *R.Casadio*, *N.Piero*, *R. Da Roldão*, Class. Quantum Gravity, **35**, 185001, 2018.
39. *R.V.Maluf*, *C.S.N.Juliano*, Phys. Rev. D, **97**, 104015, 2018.
40. *B.Majumder*, Gen. Relativ. Gravity, **45**, 2403, 2013.
41. *D.Y.Chen*, *H.W.Wu*, *Y.Haitang*, JCAP, **2014**, 036, 2014.
42. *D.Chen*, *W.Houwen*, *Y.Haitang*, Adv. High Energy Phys., **2013**, 432412, 2013.
43. *P.Painlevé*, C.R. Acad. Sci. (serie non specifiee), **173**, 677, 1921.
44. *R.M.Wald*, Living Rev. Relativ., **4**, 1, 2001.
45. *R.Tharanath*, *S.Jishnu*, *V.C.Kuriakos*, Gen. Relativ. Gravity, **47**, 1, 2015.
46. *R.V.Maluf*, *C.S.N.Juliano*, Phys. Rev. D, **97**, 104015, 2018.
47. *R.K.Kaul*, *M.Parthasarathi*, Phys. Rev. Lett., **84**, 5255, 2000.
48. *M.A.Anacleto*, *F.A.Brito*, *E.Passo*, Phys. Lett. B, **749**, 181, 2015.
49. *M.A.Anacleto et al.*, Adv. High Energy Phys., **2016**, 8465759, 2015.