АСТРОФИЗИКА

ТОМ 67 НОЯБРЬ, 2024 ВЫПУСК 4

DOI: 10.54503/0571-7132-2024.67.4-515

TOPOLOGICAL CASIMIR EFFECT IN MODELS WITH HELICAL COMPACT DIMENSIONS

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Received 5 August 2024

We investigate the influence of the helical compactification of spatial dimension on the local properties of the vacuum state for a charged scalar field with general curvature coupling parameter. A general background geometry is considered with rotational symmetry in the subspace with the coordinates appearing in the helical periodicity condition. It is shown that by a coordinate transformation the problem is reduced to the problem with standard quasiperiodicity condition in the same local geometry and with the effective compactification radius determined by the length of the compact dimension and the helicity parameter. As an application of the general procedure we have considered locally de Sitter spacetime with a helical compact dimension. By using the Hadamard function for the Bunch-Davies vacuum state, the vacuum expectation values of the field squared, current density, and energy-momentum tensor are studied. The topological contributions are explicitly separated and their asymptotics are described at early and late stages of cosmological expansion. An important difference, compared to the problem with quasiperiodic conditions, is the appearance of the nonzero off-diagonal component of the energy-momentum tensor and of the component of the current density along the uncompact dimension.

Keywords: *topological Casimir effect: vacuum polarization: helical periodicity conditions: de Sitter spacetime*

1. *Introduction*. The topological effects play an important role in various fields of physics. The latter include high-energy models with compact extra dimensions and different types of condensed matter systems. As examples we mention here the Kaluza-Klein type theories in supergravity and in string theories and different types of topological structures of 2D materials. In field theories formulated on background of spacetimes with nontrivial topology, in addition to the field equations, periodicity conditions have to be imposed on the fields along compact dimensions. As a consequence, the local physical characteristics of fields depend on the global properties of the background geometry. In particular, that is the case for the vacuum state of quantum fields. In models with compact dimensions the influence of the periodicity conditions on the properties of quantum vacuum is similar to that induced by boundary conditions on the field operator in the Casimir effect and is known as the topological Casimir effect (for reviews see [1-7]). It has been investigated for different topological classes and

background geometries. The interest is motivated by applications in theories with extra dimensions as a stabilization mechanism for moduli fields, in cosmology as a possible source of dark energy driving the accelerated expansion of the Universe, and in condensed matter physics as a source of generation of ground state stresses and currents. Among other implications of compact dimensions we can mention here new mechanisms for symmetry breaking, the generation of topological mass in field theories and different types of instabilities (see, for example, references [8-24]).

An interesting feature in theories with compact dimensions is the possibility of inequivalent field configurations with different periodicity conditions [1,25,26]. The different conditions lead to different physical consequences. Among the interesting directions in the studies of the topological Casimir effect is the dependence of the physical characteristics of the vacuum state on the periodicity conditions in the compact subspace. The most popular conditions in the literature correspond to periodic and antiperiodic fields (untwisted and twisted fields). They are special cases of more general periodicity conditions for charged fields with general phases. For the values of the phase different from 0 and π vacuum currents appear along compact dimensions. Those currents have been studied in [27-37] for locally Minkowski, de Sitter (dS) and anti-de Sitter (AdS) spacetimes (for a review see [38]). More general helical conditions include an additional shift along uncompact dimensions [39,40]. The vacuum energy in models with helical conditions along compact dimensions with zero value of the phase has been studied in [41-47]. The current density in the case of general phase is discussed in [48,49].

In the present paper we show that the characteristics of the vacuum in problems with helical periodicity conditions can be generated by using the corresponding results for standard quasiperiodicity conditions by a coordinate transformation depending on the length of compact dimension and the helicity parameter. The organization of the paper is as follows. In the next section the problem setup is presented. The coordinate transformation is described and the connection between the vacuum expectation values (VEVs) is given. As an example of general procedure, in Section 3, a locally dS background geometry is considered. The expressions of the Hadamard function, for the VEVs of the field squared, current density and the energy-momentum tensor are presented. The main results of the paper are summarized in Section 4.

2. *Problem setup and coordinate transformation*. Let us consider the background geometry described by the $(D+1)$-dimensional line element $ds^2 = g_{ik}dx^i dx^k$, where

$$g_{ik} = g_{ik}\left(x_{\perp}^{l}\right), \quad g_{l,D-1} = g_{l,D} = 0, \quad g_{D-1,D-1} = g_{D,D}, \quad x_{\perp}^{l} = \left(x^0, x^1, \ldots, x^{D-2}\right), \qquad (1)$$

with $l = 0, 1, \ldots, D-2$. It will be assumed that the spatial coordinate x^D is compactified to a circle with the length a, $0 \leq x^D \leq a$ and for the coordinate x^{D-1} one has $-\infty < x^{D-1} < +\infty$. No specific conditions will be imposed on the geometry and topology of the subspace covered by the coordinates $x_\perp^l$. We discuss the dynamics of a scalar field $\varphi(x)$ with curvature coupling parameter ξ, governed by the equation

$$\left(g^{ik} D_i D_k + \xi R + m^2\right)\varphi(x) = 0, \tag{2}$$

where $D_i = \nabla_i + ieA_i$ is the gauge-covariant derivative and e is the coupling between the scalar and gauge fields. Since the background space has non-trivial topology, in addition to the field equation one should specify the periodicity conditions along compact dimensions. In the subspace (x^{D-1}, x^D) we impose helical periodicity condition

$$\varphi\left(x_\perp^l, x^{D-1}, x^D + a\right) = e^{i\alpha_p}\, \varphi\left(x_\perp^l, x^{D-1} + h, x^D\right), \tag{3}$$

with the helicity parameter h and constant phase α_p. In the special case $h = 0$ the relation (3) reduces to a generic quasiperiodicity condition.

Here we consider a simple configuration of the gauge field with A_{D-1}, A_D = const. These constant components of the gauge field can be excluded from the field equation by the gauge transformation

$$A_i = A_i' + \partial_i \omega, \quad \varphi(x) = e^{-ie\omega}\, \varphi'(x), \quad \omega = A_{D-1} x^{D-1} + A_D x^D. \tag{4}$$

In the new gauge one has $A_l' = 0$ and $D'_l = \nabla_l$ for $l = D-1$, D. Now the condition (3) takes the form

$$\varphi'\left(x_\perp^l, x^{D-1}, x^D + a\right) = e^{i\tilde{\alpha}_p}\, \varphi'\left(x_\perp^l, x^{D-1} + h, x^D\right), \tag{5}$$

with the new phase

$$\tilde{\alpha}_p = \alpha_p - eA_{D-1} h + A_D a. \tag{6}$$

The physical characteristics will depend on the quantities α_p, A_{D-1}, A_D in the form of the combination.

In quantum field theory the periodicity conditions imposed on the field operator modify the spectrum of vacuum fluctuations and the vacuum expectation values of physical observables are shifted by amount that depends on the parameters of the compactification (the topological Casimir effect [1-7]). These effects for the quasiperiodicity conditions, corresponding to the zero value of the helicity parameter, $h = 0$, have been widely investigated in the literature for different local geometries. Simple geometries with helical conditions in the case of zero phase were discussed in [41-47]. In the discussion below we will show that the results for the helical conditions can be obtained by an appropriate coordinate transformations from the formulas for quasiperiodicity conditions.

The helical condition identifies the spacetime points with the coordinates

$P_{(0,a)} = \left(x_{\perp}^{l}, x^{D-1}, x^{D}+a\right)$ and $P_{(h,0)} = \left(x_{\perp}^{l}, x^{D-1}+h, x^{D}\right)$. Let us introduce new coordinates $\bar{x}^{i}$ in accordance with $\bar{x}^{l} = x^{l}$ for $l = 0, 1, \ldots, D-2$, and

$$\bar{x}^{D-1} = \frac{a}{\bar{a}} x^{D-1} + \frac{h}{\bar{a}} x^{D} + \frac{a}{\bar{a}} h, \quad \bar{x}^{D} = -\frac{h}{\bar{a}} x^{D-1} + \frac{a}{\bar{a}} x^{D} - \frac{h^{2}}{\bar{a}}, \tag{7}$$

where

$$\bar{a} = \sqrt{a^{2}+h^{2}}. \tag{8}$$

The inverse transformation reads

$$x^{D-1} = \frac{a}{\bar{a}} \bar{x}^{D-1} - \frac{h}{\bar{a}} \bar{x}^{D} - h, \quad x^{D} = \frac{h}{\bar{a}} \bar{x}^{D-1} + \frac{a}{\bar{a}} \bar{x}^{D} \tag{9}$$

For the identification points in the coordinates $\bar{x}^{i}$ one has

$$P_{(0,a)} = \left(\bar{x}_{\perp}^{l}, \bar{x}^{D-1}, \bar{x}^{D}+\bar{a}\right), \quad P_{(h,0)} = \left(\bar{x}_{\perp}^{l}, \bar{x}^{D-1}, \bar{x}^{D}\right). \tag{10}$$

The coordinate transformation (7) is a combination of the rotation by angle $\theta = \arctan(h/a)$, $0 \le \theta \le \pi/2$, and the shift of the origin to the point $x^{i} = \left(x_{\perp}^{l}, -h, 0\right)$. The metric tensor is form-invariant under the transformation (7).

Now we can reformulate the problem of the investigation of the VEVs for the field $\varphi(x)$ with helical condition (3) in the coordinate system $\bar{x}^{i}$. For the corresponding metric tensor we still have

$$\bar{g}_{ik} = \bar{g}_{ik}\left(\bar{x}_{\perp}^{l}\right), \quad \bar{g}_{l,D-1} = \bar{g}_{l,D} = 0, \quad \bar{g}_{D-1,D-1} = \bar{g}_{D,D}, \tag{11}$$

with $\bar{x}_{\perp}^{l} = x_{\perp}^{l}$. The field equation has the form (2) with the replacements $g_{ik} \to \bar{g}^{ik}$ for the metric tensor and $D_{i} \to \bar{D}_{i} = \bar{\nabla}_{i} + ie\bar{A}_{i}$ for the covariant derivative, where $\bar{A}_{i} = A_{i}$ for $i = 0, 1, \ldots, D-2$, and

$$\bar{A}_{D-1} = \frac{a}{\bar{a}} A_{D-1} + \frac{h}{\bar{a}} A_{D}, \quad \bar{A}_{D} = -\frac{h}{\bar{a}} A_{D-1} + \frac{a}{\bar{a}} A_{D}. \tag{12}$$

In the new coordinates the periodicity condition takes the form

$$\bar{\varphi}\left(\bar{x}_{\perp}^{l}, \bar{x}^{D-1}, \bar{x}^{D}+\bar{a}\right) = e^{i\alpha_{p}} \bar{\varphi}\left(\bar{x}_{\perp}^{l}, \bar{x}^{D-1}, \bar{x}^{D}\right), \tag{13}$$

which is a standard quasiperiodicity condition. This shows that we can use the results for the VEVs in problems with quasiperiodicity condition (13) in order to find the expectation values in problems with helical conditions. Let us specify this procedure for the current density and the energy-momentum tensor. The renormalized VEVs in the coordinate system $\bar{x}^{i}$ we denote by $\left\langle \bar{j}^{i} \right\rangle = \left\langle \bar{j}^{i}\left(\alpha_{p}, \bar{A}_{l}\right)\right\rangle$ and $\left\langle \bar{T}^{ik} \right\rangle = \left\langle \bar{T}^{ik}\left(\alpha_{p}, \bar{A}_{l}\right)\right\rangle$ for the current density and the energy-momentum tensor, respectively. The corresponding expectation values $\left\langle j^{i} \right\rangle = \left\langle j^{i}\left(\alpha_{p}, A_{l}\right)\right\rangle$ and $\left\langle T^{ik} \right\rangle = \left\langle T^{ik}\left(\alpha_{p}, A_{l}\right)\right\rangle$ in the original problem with helical periodicity condition (3) are obtained by the coordinate transformation $\bar{x}^{i} \to x^{i}$.

We start with the current density. Note that in the coordinate system $\bar{x}^{i}$ we can make a gauge transformation $\overline{A}_i = \overline{A}'_i + \overline{\partial}_i \overline{\omega}$, $\overline{\varphi}(\overline{x}) = e^{-ie\overline{\omega}}\,\overline{\varphi}'(\overline{x})$ with the function $\overline{\omega} = \overline{A}_{D-1}\overline{x}^{D-1}$. In the new gauge one gets $\overline{A}'_{D-1} = 0$. Both the field equation and the periodicity condition (13) are invariant under this gauge transformation and the physical results do not depend on $\overline{A}_{D-1}$. In the gauge $\overline{A}_{D-1} = 0$ the metric tensor, the field equation and the periodicity condition in the coordinate system $\overline{x}^i$ are symmetric under the reflection $\overline{x}^{D-1} \to -\overline{x}^{D-1}$. Assuming that the vacuum state is also symmetric under this reflection, we conclude that the component of the current density along the coordinate direction $\overline{x}^{D-1}$ vanishes by the symmetry, $\left\langle \overline{j}^{D-1} \right\rangle = 0$. In this case the components of the current density in the coordinates x^i are expressed as

$$\left\langle j^i \right\rangle = \left\langle \overline{j}^i \right\rangle =, \quad i = 0,1,...,D-2, \quad \left\langle j^{D-1} \right\rangle = -\frac{h\left\langle \overline{j}^D \right\rangle}{\sqrt{a^2+h^2}}, \quad \left\langle j^D \right\rangle = \frac{a\left\langle \overline{j}^D \right\rangle}{\sqrt{a^2+h^2}}. \tag{14}$$

and the vacuum current density has a nonzero component along the uncompact dimension x^{D-1} as well. The components along compact and uncompact dimensions related by the helical condition are connected by the formula

$$\left\langle j^{D-1} \right\rangle = -\frac{h}{a}\left\langle j^D \right\rangle. \tag{15}$$

This relation for the locally Minkowski bulk was obtained in [48] by direct evaluation of the VEV using the corresponding mode functions.

Another important characteristic of the vacuum state is the expectation value of the energy-momentum tensor. Again, assuming that the vacuum is symmetric with respect to the reflection $\overline{x}^{D-1} \to -\overline{x}^{D-1}$, we conclude that $\left\langle \overline{T}^{i,D-1} \right\rangle = 0$ for $i \neq D-1$. By using the transformation rule for the second rank tensor, for the components of the energy-momentum tensor we get (i, $k = 0$, 1, ..., $D-2$)

$$\left\langle T^{ik} \right\rangle = \left\langle \overline{T}^{ik} \right\rangle, \quad i,k = 0,1,...,D-2, \quad \left\langle T^{iD} \right\rangle = -\frac{a}{h}\left\langle T^{i,D-1} \right\rangle = \frac{a\left\langle \overline{T}^{iD} \right\rangle}{\sqrt{a^2+h^2}}, \tag{16}$$

for the components with one or two indices in the subspace ($x^0, x^1, ..., x^{D-2}$) and

$$\begin{gathered} \left\langle T^{D-1,D-1} \right\rangle = \frac{a^2\left\langle \overline{T}^{D-1,D-1} \right\rangle}{a^2+h^2} + \frac{h^2\left\langle \overline{T}^{DD} \right\rangle}{a^2+h^2}, \\ \left\langle T^{D-1,D} \right\rangle = \frac{ah}{a^2+h^2}\left[\left\langle \overline{T}^{D-1,D-1} \right\rangle - \left\langle \overline{T}^{DD} \right\rangle\right], \quad \left\langle T^{DD} \right\rangle = \frac{h^2\left\langle \overline{T}^{D-1,D-1} \right\rangle}{a^2+h^2} + \frac{a^2\left\langle \overline{T}^{DD} \right\rangle}{a^2+h^2}, \end{gathered} \tag{17}$$

for the components in the subspace (x^{D-1}, x^D).

Note that the condition (3) can also be interpreted as a helical periodicity condition along the compact dimension x^{D-1} with the length h, with the helicity parameter a along the uncompact direction x^D, and with the phase $-\alpha_p$. This shows that there is a duality between the models with the sets (a, h, α_p) and $(h, a, -\alpha_p)$. In the dual models the roles of the dimensions x^{D-1} and x^D are interchanged. The duality is also seen in the VEVs (14), (16), and (17).

3. *Models with locally dS spacetime*. As an application of the general procedure described above let us consider a background spacetime with local dS geometry. The dS spacetime is among the most popular geometries in quantum field theory in curved spacetime. In particular, that is motivated by important applications in inflationary models of the early Universe and in models of accelerating expansion at recent epoch. In inflationary coordinates the corresponding line element reads

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{i=1}^{D} \left(dx^i\right)^2 , \tag{18}$$

where the constant α determines the curvature radius of the spacetime. It is expressed in terms of the cosmological constant Λ by the formula $\alpha^2 = D(D-1)/2\Lambda$. For the remaining spatial dimensions we take $-\infty < x^i < +\infty$, $i = 1, \ldots, D-1$. Introducing a conformal time τ in accordance with $\tau = -\alpha e^{-t/\alpha}$, the line element is written in a conformally flat form

$$ds^2 = g_{ik} dx^i dx^k = \frac{\alpha^2}{\eta^2}\left[d\tau^2 - \sum_{i=1}^{D} \left(dx^i\right)^2\right], \tag{19}$$

where $\eta = |\tau|$. For the scalar curvature in the field equation (2) one has $R = D(D-1)/\alpha^2$. The VEVs of the field squared and energy-momentum tensor in the model with a single compact dimension and periodic condition along it were studied in [50]. The general case of spatial topology $R^p \times \left(S^1\right)^q$, $p + q = D$, has been discussed in [51,52]. The vacuum currents for quasiperiodic conditions with general phases are investigated in [30]. For simplicity here we consider the special case $p = D-1$ and $q = 1$, assuming that the only compact dimension corresponds to the coordinate x^D along which the quantum scalar field obeys the condition (3). In the discussion below we will work in the coordinate system (19).

3.1. *Hadamard function and the VEVs of the field squared and current density*. The local characteristics of the vacuum state $|0\rangle$ for a quantum scalar field $\varphi(x)$ are obtained from the two-point functions. They describe the correlations of zero-point fluctuations at different spacetime points x and x'. As a two-point function we will take the Hadamard function defined as the VEV

$$G(x, x') = \langle 0|\varphi(x)\varphi^\dagger(x') + \varphi^\dagger(x')\varphi(x)|0\rangle . \tag{20}$$

For dS spacetime different vacuum states have been considered in the literature. Among them the Bunch-Davies vacuum is distinguished by the following two properties: it is maximally symmetric and is reduced to the Minkowski vacuum in flat spacetime in the slow expansion limit. Here we assume that the field $\varphi(x)$ is prepared in the Bunch-Davies vacuum state. The Hadamard function $\overline{G}(\overline{x},\overline{x}')$ in the problem at hand for the coordinates $\overline{x}^i$ is obtained from the expression in [30] as a special case. Transforming to the coordinates x^i we get

$$G(x,x')=\frac{4(\eta\eta')^{D/2}}{(2\pi)^{D/2+1}\alpha^{D-1}}\int_0^\infty dzz[I_{-\nu}(\eta z)K_\nu(\eta' z)+K_\nu(\eta z)I_\nu(\eta' z)]$$
$$\times\sum_{n=-\infty}^{\infty}e^{-in\tilde{\alpha}_p}\frac{f_{D/2-1}\left(z\sqrt{|\Delta x_D|^2+n^2(a^2+h^2)+2n(a\Delta x^D-h\Delta x^{D-1})}\right)}{\left[|\Delta x_D|^2+n^2(a^2+h^2)+2n(a\Delta x^D-h\Delta x^{D-1})\right]^{D/2-1}}. \quad (21)$$

where $x_D=(x^1,...,x^D)$, $\Delta x_D=x_D-x'_D$, $\Delta x^i=x^i-x'^i$, $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions [53] with the order

$$\nu=\left[D^2/4-D(D+1)\xi-m^2\alpha^2\right]^{1/2}. \quad (22)$$

The function $f_\mu(y)$ in (21) is defined by $f_\mu(y)=y^\mu K_\mu(y)$. The $n=0$ term in (21) corresponds to the Hadamard function $G_{dS}(x,x')$ in the dS spacetime without compactification and the remaining part is induced by the helical compactification. The expression for $G_{dS}(x,x')$ in terms of the hypergeometric function is well known from the literature.

Given the Hadamard function, the VEVs of physical observables are obtained taking the coincidence limit of the arguments of the Hadamard function or its derivatives. We start with the VEV of the field squared $\langle\varphi\varphi^\dagger\rangle=\langle 0|\varphi\varphi^\dagger|0\rangle$. It is obtained in the limit $\langle\varphi\varphi^\dagger\rangle=\lim_{x'\to x}G(x,x')/2$. This limit is divergent and a renormalization is required. The compactification scheme under consideration does not change the local geometry and the divergences are the same as in the dS spacetime without compactification. The corresponding part in the Hadamard function (21) is presented by the $n=0$ term. Separating the topological contribution and taking the coincidence limit the VEV is decomposed as

$$\langle\varphi\varphi^\dagger\rangle=\langle\varphi\varphi^\dagger\rangle_{dS}+\langle\varphi\varphi^\dagger\rangle_c, \quad (23)$$

where the renormalized VEV $\langle\varphi\varphi^\dagger\rangle_{dS}$ in dS spacetime has been already studied in the literature. By the maximal symmetry of the Bunch-Davies vacuum state, it does not depend on spacetime coordinates. The topological contribution $\langle\varphi\varphi^\dagger\rangle_c$ is directly obtained from the part in (21) with $n\neq 0$ in the coincidence limit:

$$\langle\varphi\varphi^\dagger\rangle_c=\frac{4\alpha^{1-D}\eta^{D-2}}{(2\pi)^{D/2+1}(a^2+h^2)^{D/2-1}}\int_0^\infty dzzF_\nu(z)\sum_{n=1}^{\infty}\frac{\cos(n\tilde{\alpha}_p)}{n^{D-2}}f_{D/2-1}(y_n), \quad (24)$$

with the notations

$$y_n = \frac{nz}{\eta}\sqrt{a^2+h^2}\,, \tag{25}$$

and

$$F_\nu(z) = [I_{-\nu}(z) + I_\nu(z)]K_\nu(z). \tag{26}$$

The VEV (24) is an even function of the phase $\tilde{\alpha}_p$. This corresponds to the periodicity with respect to the magnetic flux enclosed by the compact dimension, with the period equal to the flux quantum. In addition, the mean field squared $\langle\varphi^2\rangle_c$ is invariant under the change $(a, h, \alpha_p) \to (h, a, -\alpha_p)$. This is a manifestation of the duality mentioned above.

For a charged scalar field the operator of the current density is given by

$$j_l = ie\left[\varphi^\dagger D_l\,\varphi - (D_l\,\varphi)^\dagger \varphi\right]. \tag{27}$$

The corresponding VEV can be obtained in two different ways. The first one corresponds to the limiting transition (in the gauge where $A_i = 0$)

$$j_l = \frac{i}{2} e \lim_{x' \to x} (\partial_l - \partial'_l) G(x, x'), \tag{28}$$

by using the Hadamard function (21). Note that the limit in the right-hand side of (28) with the dS Hadamard function vanishes and the renormalization for the current density is not required. In the second way, the vacuum current density is obtained from the corresponding result for quasiperiodic condition, given in [30], by the coordinate transformation (14). For the nonzero components we get

$$\langle j^D\rangle = \frac{8e\alpha^{-D-1} a\eta^D}{(2\pi)^{D/2+1}\left(a^2+h^2\right)^{D/2}} \int_0^\infty dzzF_\nu(z) \sum_{n=1}^\infty \frac{\sin(n\tilde{\alpha}_p)}{n^{D-1}} f_{D/2}(y_n), \tag{29}$$

and $\langle j^{D-1}\rangle = -h\langle j^D\rangle/a$. Here, y_n is defined by (25). The physical components of the current density, denoted here by $\langle j^l_p\rangle$, are connected to the contravariant components by the relation $\langle j^l_{(p)}\rangle = (\alpha/\eta)\langle j^l\rangle$. The components $\langle j^D\rangle$ and $\langle j^{D-1}\rangle$ are odd functions of the phase $\tilde{\alpha}_p$. In particular, the current density vanishes for half-integer values of the parameter $\tilde{\alpha}_p$. In agreement with the duality mentioned at the end of the previous section, the current densities are invariant under the change $(a, h, \alpha_p) \to (h, a, -\alpha_p)$ with the change of the roles of the coordinates $\left(x^{D-1}, x^D\right) \to \left(x^D, x^{D-1}\right)$.

3.2. *Vacuum energy-momentum tensor*. Finally, we turn to the VEV of the energy-momentum tensor. In the gauge with $A_i = 0$ it is obtained from the Hadamard function (21) with the help of the formula (again, in the gauge with zero gauge potential)

$$\langle T_{ik}\rangle = \lim_{x' \to x} \partial_i \partial'_k\, G(x, x') + \left[\left(\xi - \frac{1}{4}\right) g_{ik}\,\nabla_l \nabla^l - \xi \nabla_i \nabla_k - \xi R_{ik}\right] \langle\varphi^2\rangle, \tag{30}$$

where $R_{ik} = Dg_{ik}/\alpha^2$ is the Ricci tensor for the dS spacetime. Alternatively, the VEV is derived by the coordinate transformation from the results in the coordinate system $\bar{x}^i$ with standard periodicity condition. The corresponding formulas in the special case $\tilde{\alpha}_p = 0$ are obtained from the results of [51]. Generalizing for $\tilde{\alpha}_p \neq 0$ and using the transformation rules (16) and (17) one finds

$$\langle T_i^k \rangle = \langle T_i^k \rangle_{dS} + \langle T_i^k \rangle_c , \tag{31}$$

where $\langle T_i^k \rangle_{dS} = \mathrm{const}\,\delta_i^k$ is the corresponding VEV in the dS spacetime without compactification. The topological contribution for the vacuum energy density reads

$$\langle T_0^0 \rangle_c = \frac{2\alpha^{-1-D}\eta^{D-2}}{(2\pi)^{D/2+1}(a^2+h^2)^{D/2-1}} \sum_{n=1}^{\infty} \frac{\cos(n\tilde{\alpha}_p)}{n^{D-2}} \int_0^{\infty} dz z F^{(0)}(z) f_{D/2-1}(y_n), \tag{32}$$

with the notation

$$F^{(0)}(z) = \frac{1}{2} z [z F_\nu'(z)]' + D\left(\frac{1}{2} - 2\xi\right) z F_\nu'(z) + 2(m^2\alpha^2 - z^2) F_\nu(z), \tag{33}$$

where the prime stands for the derivative with respect to z. For the vacuum stresses along the directions x^i, with $i = 1, 2, ..., D-2$, one gets (no summation over i)

$$\begin{aligned} \langle T_i^i \rangle_c &= \frac{2\alpha^{-1-D}\eta^{D-2}}{(2\pi)^{D/2+1}(a^2+h^2)^{D/2-1}} \\ &\times \sum_{n=1}^{\infty} \frac{\cos(n\tilde{\alpha}_p)}{n^{D-2}} \int_0^{\infty} dz z \left[F(z) f_{D/2-1}(y_n) - \frac{2\eta^2 F_\nu(z)}{n^2(a^2+h^2)} f_{D/2}(y_n) \right], \end{aligned} \tag{34}$$

where

$$F(z) = \left(2\xi - \frac{1}{2}\right) z [z F_\nu'(z)]' + \left[2(D+1)\xi - \frac{D}{2}\right] z F_\nu'(z). \tag{35}$$

Now we turn to the components of the topological part in the vacuum energy-momentum tensor with one or two indices in the subspace (x^{D-1}, x^D). The off-diagonal components $\langle T_D^i \rangle_c$, with $i = 1, 2, ..., D-2$, vanish: $\langle T_D^i \rangle_c = 0$. For the diagonal components in the subspace (x^{D-1}, x^D) we find

$$\begin{aligned} \langle T_{D-1}^{D-1} \rangle_c &= \frac{2\alpha^{-1-D}\eta^{D-2}}{(2\pi)^{D/2+1}(a^2+h^2)^{D/2-1}} \sum_{n=1}^{\infty} \frac{\cos(n\tilde{\alpha}_p)}{n^{D-2}} \int_0^{\infty} dz z \Big\{ F(z) f_{D/2-1}(y_n) \\ &- \frac{2\eta^2 F_\nu(z)}{n^2(a^2+h^2)} \left[\left(1 - \frac{Dh^2}{a^2+h^2}\right) f_{D/2}(y_n) - h^2 \frac{y_n^2 f_{D/2-1}(y_n)}{a^2+h^2} \right] \Big\}, \end{aligned} \tag{36}$$

and

$$\left\langle T_D^D \right\rangle_c = \frac{2\alpha^{-1-D}\eta^{D-2}}{(2\pi)^{D/2+1}\left(a^2+h^2\right)^{D/2-1}} \sum_{n=1}^{\infty} \frac{\cos\left(n\tilde{\alpha}_p\right)}{n^{D-2}} \int_0^{\infty} dzz \left\{ F(z) f_{D/2-1}(y_n) \right.$$

$$\left. - \frac{2\eta^2 F_\nu(z)}{n^2\left(a^2+h^2\right)} \left[\left(1 - \frac{Da^2}{a^2+h^2}\right) f_{D/2}(y_n) - a^2 y_n^2 \frac{f_{D/2-1}(y_n)}{a^2+h^2} \right] \right\}, \tag{37}$$

with y_n from (25). In addition to the diagonal components, the helical periodicity condition induces a nonzero off-diagonal component $\left\langle T_D^{D-1} \right\rangle_c$. It is obtained from the diagonal components in the coordinate system $\bar{x}^i$ by the transformation given by (17):

$$\left\langle T_D^{D-1} \right\rangle_c = \frac{-4\alpha^{-1-D}\eta^D ah}{(2\pi)^{D/2+1}\left(a^2+h^2\right)^{D/2+1}} \sum_{n=1}^{\infty} \frac{\cos\left(n\tilde{\alpha}_p\right)}{n^D} \int_0^{\infty} dzzF_\nu(z) \left[y_n^2 f_{D/2-1}(y_n) + Df_{D/2}(y_n) \right]. \tag{38}$$

All the components of the vacuum energy-momentum tensor are even functions of $\tilde{\alpha}_p$. Note that the parameter ν defined by (22) can be either nonnegative real number or purely imaginary. The integral representations given above are valid in the range $\mathrm{Re}\,\nu < 1$. This restriction follows from the condition of the convergence of the integrals over z in the lower limit. Note that off-diagonal components of the vacuum energy-momentum tensor may arise also in models with quasiperiodic conditions (see [54]).

It can be explicitly checked that the topological part of the vacuum energy-momentum tensor obeys the trace relation

$$\left\langle T_i^i \right\rangle_c = \left[D(\xi - \xi_D)\nabla_l \nabla^l + m^2 \right] \left\langle \varphi^2 \right\rangle_c , \tag{39}$$

where $\xi_D = (D-1)/4D$ is the value of the curvature coupling parameter for a conformally coupled scalar field. For a conformally coupled massless field the topological contribution $\left\langle T_i^k \right\rangle_c$ is traceless. The anomaly in the trace is contained in the pure dS part $\left\langle T_D^i \right\rangle_{dS}$.

Note that the parameters a and h are the coordinate lengths. The corresponding physical (proper) lengths measured by an observer at rest in the coordinates x^i are given by $a_{(p)} = \alpha a/\eta$ and $h_{(p)} = \alpha h/\eta$. The VEVs $\left\langle \varphi\varphi^\dagger \right\rangle_c$, $\left\langle j_{(p)}^l \right\rangle$, and $\left\langle T_i^k \right\rangle_c$ depend on a, h, and η in the form of the ratios a/η and h/η. The latter are the proper lengths measured in units of the curvature radius α.

3.3. *Conformally coupled massless field and the asymptotics*. For a conformally coupled massless field one has $\nu = 1/2$ and $F_\nu(z) = 1/z$. The integrals are evaluated by using the formulae [55]

$$\int_0^\infty dy f_{D/2}(y) = \int_0^\infty dy y^2 f_{D/2-1}(y) = (D-1)\int_0^\infty dy f_{D/2-1}(y) = 2^{D/2-1}\sqrt{\pi}\Gamma\left(\frac{D+1}{2}\right). \tag{40}$$

For the VEVs of the field squared and physical component of the current density one gets

$$\langle \varphi\varphi^\dagger \rangle_c = \frac{\Gamma((D-1)/2)(\eta/\alpha)^{D-1}}{2\pi^{(D+1)/2}(a^2+h^2)^{(D-1)/2}}\sum_{n=1}^\infty \frac{\cos(n\tilde{\alpha}_p)}{n^{D-1}},$$

$$\langle j_{(p)}^D \rangle_c = \frac{2e\Gamma((D+1)/2)(\eta/\alpha)^D a}{\pi^{(D+1)/2}(a^2+h^2)^{(D+1)/2}}\sum_{n=1}^\infty \frac{\sin(n\tilde{\alpha}_p)}{n^D}, \quad \langle j_{(p)}^{D-1} \rangle_c = -\frac{h}{a}\langle j_{(p)}^D \rangle_c . \tag{41}$$

The expressions for the energy density and stresses along the directions x^i, $i=1, 2, \ldots, D-2$, are simplified to (no summation over $i=0, 1, \ldots, D$-2)

$$\langle T_i^i \rangle_c = -\frac{\Gamma((D+1)/2)(\eta/\alpha)^{D+1}}{\pi^{(D+1)/2}(a^2+h^2)^{(D+1)/2}}\sum_{n=1}^\infty \frac{\cos(n\tilde{\alpha}_p)}{n^{D+1}}. \tag{42}$$

For the diagonal components of the energy-momentum tensor in the subspace (x^{D-1}, x^D) we find

$$\langle T_{D-1}^{D-1} \rangle_c = \left[1-\frac{(D+1)h^2}{a^2+h^2}\right]\langle T_0^0 \rangle_c, \quad \langle T_D^D \rangle_c = \left[1-\frac{(D+1)a^2}{a^2+h^2}\right]\langle T_0^0 \rangle_c . \tag{43}$$

Finally, the expression for the off-diagonal component is reduced to

$$\langle T_D^{D-1} \rangle_c = \frac{(D+1)ah}{a^2+h^2}\langle T_0^0 \rangle_c . \tag{44}$$

For a conformally coupled massless field the problem on the dS bulk is conformally related to the corresponding problem in the locally Minkowski spacetime, with the same parameters a, h, α_p, and the VEVs are connected by the standard formulae

$$\langle \varphi\varphi^\dagger \rangle_c = \frac{\langle \varphi\varphi^\dagger \rangle_c^{(M)}}{(\alpha/\eta)^{D-1}}, \quad \langle j_{(p)}^D \rangle_c = \frac{\langle j^D \rangle_c^{(M)}}{(\alpha/\eta)^D}, \quad \langle T_i^i \rangle_c = \frac{\langle T_i^i \rangle_c^{(M)}}{(\alpha/\eta)^{D+1}}. \tag{45}$$

In the special cases $\tilde{\alpha}_p = 0$ and $\tilde{\alpha}_p = \pi$ the current density vanishes and the series in the expressions for the field squared and energy-momentum tensor are expressed in terms of the Riemann zeta function. Depending on the values of the parameter $\tilde{\alpha}_p$, the VEVs can be either positive or negative. In particular, the topological contribution to the energy density is negative for an untwisted field ($\tilde{\alpha}_p = 0$) and positive for twisted field ($\tilde{\alpha}_p = \pi$). For some intermediate value of $\tilde{\alpha}_p$ the VEVs become zero. The vacuum pressure along the direction x^i, $i=0, 1, \ldots, D-2$, is given by $-\langle T_i^i \rangle_c$ and it is equal to the energy density with an opposite sign.

This corresponds to the equation of state of the cosmological constant type in the subspace ($x^0, x^1, ..., x^{D-2}$). That is not the case for general conformal coupling and for massive fields.

At the early stages of the cosmological expansion one has $\tau \to -\infty$ and η is large. In order to find the asymptotics of the VEVs in that limit it is convenient to introduce a new integration variable $u = z/\eta$ in the expressions for the VEVs. The function $F_\nu(z)$ becomes $F_\nu(u\eta)$ and its argument is large. By using the asymptotic of the modified Bessel functions for large argument it can be shown that $F_\nu(z) \approx 1/z$ for $z >> 1$. This asymptotic coincides with the exact expression for a conformally coupled massless scalar field. Replacing $F_\nu(z) = 1/z$ in the expressions for the field squared, current density, and off-diagonal component $\langle T_D^{D-1} \rangle_c$, we see that the leading terms in the expansion over $1/\eta$ coincide with the corresponding expressions for a conformally coupled massless field, given by (41) and (44). In the expression (32) for the energy density, in the leading order, one has $F^{(0)}(z) \approx -2z$ and the corresponding asymptotic, again, coincides with the result (42) for $i=0$. In the components (36) and (17) we have $F(z) \approx -2D(\xi - \xi_D)/z$ and the terms involving the function $F_\nu(z) \approx 1/z$ contain additional factor η^2. Hence, the latter term dominates in the asymptotic and the leading terms coincide with (43). We conclude that in the limit $\tau \to -\infty$, corresponding to $t \to -\infty$, the leading asymptotics of the topological contributions of the VEVs coincide with the corresponding result for a conformally coupled massless field and the effects of gravity on those contributions are weak. In the limit under consideration the dominant contribution to the total VEV (31) comes from the topological part.

The late stages of the expansion correspond to $t \to +\infty$ and $\eta \to 0$. Again, introducing a new integration variable $u = z/\eta$, we expand the function $F_\nu(u\eta)$ for small values of the argument. For $\nu > 0$ one has $F_\nu(u\eta) \propto (u\eta)^{-2\nu}$ and the topological terms in the VEVs tend to zero monotonically, like $\eta^{D-2\nu}$ for the VEVs of the field squared and energy-momentum tensor and like $\eta^{D+2-2\nu}$ for the current density. For purely imaginary ν, $\nu = i|\nu|$, and for small η we have $F_\nu(u\eta) \approx \mathrm{Re}[(2/u\eta)^{2\nu}\,\Gamma(\nu)/\Gamma(1-\nu)]$. In this case the topological VEVs tend to zero with oscillating behavior. The amplitudes of the oscillations decay as η^D for the field squared and energy-momentum tensor and as η^{D+2} in the case of the current density.

4. *Conclusion*. We have studied the topological Casimir effect in models with compact dimension along which the field operator obeys helical periodicity condition given by (3). A general background is considered with the metric tensor invariant under the rotations in the plane (x^{D-1}, x^D). In addition, the presence of a gauge field is assumed with constant covariant components A_{D-1} and A_D. We can pass to the new gauge with zero values of those components. In that gauge

the field operator obeys the helical condition (5) with the new phase (6) depending on the components A_{D-1} and A_D. The corresponding contribution can be interpreted in terms of the magnetic flux enclosed by the compact dimension. We have shown that by the coordinate transformation (7) the problem with helical periodicity condition is reduced to the problem with standard quasiperiodicity condition (13) with the same phase. The length of the corresponding compact dimension is expressed as $\sqrt{a^2+h^2}$.

The procedure we have described allows to find the VEVs of physical observables in the topological Casimir effect for helical periodicity conditions by using the corresponding results for quasiperiodic conditions. That is done by the standard transformation of the tensors under the coordinate transformation (9). As important local characteristics of the vacuum state we have considered the VEVs of the current density and energy-momentum tensor. Their transformation laws are given by (14), (16), and (17). As an example of general prescription the locally dS spacetime is considered with a single compact dimension x^D and helicity shift along the direction x^{D-1}. The geometry is described by the line element (19). The corresponding problem with general number of toroidally compactified dimensions has been considered in [30,51]. In [51] the VEVs of the field squared and energy-momentum tensor were studied for periodic and antiperiodic conditions. The VEV of the current density in the case of quasiperiodic conditions with general phases is considered in [30].

In the problem at hand the properties of the vacuum state are encoded in two-point functions describing the correlations of the vacuum fluctuations in different spacetime points. As a two-point function we have taken the Hadamard function. In the problem with helical condition in locally dS spacetime that function is expressed as (21). As local characteristics of the scalar vacuum we have considered the expectation values of the field squared, current density and energy-momentum tensor. In the corresponding expressions the parts induced by the compactification are explicitly separated. The field squared and energy-momentum tensor are even functions of the phase $\tilde{\alpha}_p$ in the periodicity condition, whereas the current density is an odd function. An important difference of the helical compactification is the presence of nonzero off-diagonal component $\left\langle T_D^{D-1} \right\rangle_c$ of the energy-momentum tensor. At the early stages of the dS expansion the VEVs are dominated by the topological contribution and at those stages the influence of gravity on the local characteristics is weak. The corresponding asymptotics are conformally related to the VEVs on the locally Minkowski bulk. At late stages, depending on the parameter ν, the topological parts in the VEVs decay monotonically or oscillatory and the pure dS contributions dominate.

Acknowledgments. The work was supported by the grant No. 21AG-1C047 of the Higher Education and Science Committee of the Ministry of Education, Science, Culture and Sport RA.

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ТОПОЛОГИЧЕСКИЙ ЭФФЕКТ КАЗИМИРА В МОДЕЛЯХ СО СПИРАЛЬНЫМИ КОМПАКТНЫМИ РАЗМЕРНОСТЯМИ

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Исследовано влияние спиральной компактификации пространственной размерности на локальные свойства вакуумного состояния для заряженного скалярного поля с общим параметром связи с кривизной. Рассматривается общая фоновая геометрия с вращательной симметрией в подпространстве с координатами, появляющимися в условии спиральной периодичности. Показано, что с преобразованием координат задача сводится к задаче со стандартным условием квазипериодичности в той же локальной геометрии и с эффективным радиусом компактификации, определяемым длиной компактной размерности и параметром спиральности. В качестве применения общей процедуры рассмотрено локально де Ситтеровское пространство-время со спиральной компактной размерностью. Используя функцию Адамара для вакуумного состояния Банча-Дэвиса, изучаются вакуумные средние квадрата поля, плотности тока и тензора энергии-импульса. Явно выделены топологические вклады, и описаны их асимптотики на ранних и поздних стадиях космологического расширения. Важным отличием по сравнению с задачей с квазипериодическими условиями является появление ненулевой недиагональной компоненты тензора энергии-импульса и компоненты плотности тока вдоль некомпактного измерения.

Ключевые слова: *топологический Казимир эффект: поляризация вакуума: условие спиральной периодичности: время-пространство де Ситтера*

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