ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱ
НАЦИОНАЛЬНАЯ АКАДЕМИЯ НАУК АРМЕНИИ
NATIONAL ACADEMY OF SCIENCES OF ARMENIA
ДОКЛАДЫ ԶԵԿՈՒՅՑՆԵՐ REPORTS

Հատոր
Том
Volume 124 2024 № 2

MECHANICS

УДК 531/534+539.21
DOI: 10.54503/0321-1339-2024.124.2-7

K. B. Ghazaryan

Bandgaps, Dynamics and Stability of Multi-Span Beam Rested on Periodically Arranged Exterior Supports

(Submitted by corresponding member of NAS RA Ara S. Avetisyan 13/III 2024)

Keywords: *periodic structures, Floquet theory, multi span beams, bandgaps.*

Introduction. The paper presents a comparative study of bandgap formation mechanisms in infinite homogeneous multi span beam in tension rested on periodically arranged exterior intermediate rigid and elastic supports. The effects in beams caused by frequency bandgaps where flexural waves are forbidden to propagate when the frequencies are in the bandgap (or the stopband) are widely used in vibration energy harvesting devises consisting of meta beams with attached or covered by piezoelectric patches [1-4]. Flexural frequency multiple bandgaps in metamaterial beams can be generated in many ways. Bandgap formation due to Bragg's scattering in periodic beam consisting of two or more kinds of materials are discussed in [5, 6]. Bandgaps in beams with periodic local resonators or rested on periodic foundation are considered in [7-9]. In [10-12] it is shown that periodic exterior supports, interior hinges can open wide bandgaps in homogeneous beams. Effects caused by bandgaps in the piezoelectric periodic meta beams are investigated in [13].

Governing equations and solutions. This section presents the basic dynamics equations, interface relations and solutions of an Euler meta beam tensioned by an axial force and rested on periodically arranged intermediate rigid and elastically constraint supports. Two configurations of a meta beam are considered:

1) beam with elastically constraint supports distanced by d length (ES),
2) beam with elastically constraint supports and rigid supports when the distance between rigid supports and nearby elastically constraint supports is equal to d (ERS).

These configurations are graphically presented on Fig. 1, 2 in dimensionless coordinate system $x = z / d$.

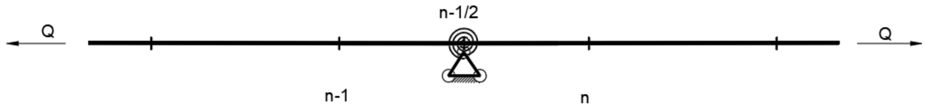

Fig. 1. Basic repeated unit span of a tensioned beam with periodically arranged elastic supports (ES).

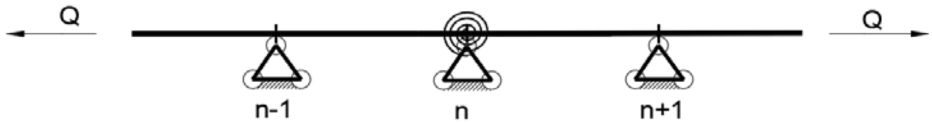

Fig. 2. Basic repeated unit span of a tensioned beam with periodically arranged rigid and elastic supports (ERS).

Transverse vibration of the Euler beam is given by the following equation

$$EI\frac{\partial^4 W}{\partial z^4} - Q\frac{\partial^2 W}{\partial z^2} + \rho A\frac{\partial^2 W}{\partial t^2} = 0; \tag{1}$$

where $W(z,t)$ is the defection of beam, EI, ρ, A denote the flexural rigidity, the mass density per unit volume and the cross-sectional area, respectively, $Q > 0$ is the axial tension force. Notice that the force Q can also be negative (compression) as long as the beam is stable.
Assuming $W(z,t)$ in the form

$$W(z,t) = U(z)\exp(i\omega t) \tag{2}$$

where ω is the circular frequency, $U(x)$ is the amplitude function and introducing the dimensionless coordinate $x = z / d$, solutions for amplitude functions can be written as

$$\begin{gathered} U_{\pm}(x) = A_{\pm 1}\sin(px) + A_{\pm 2}\sinh(qx) + A_{\pm 3}\cos(px) + A_{\pm 4}\cosh(qx) \\ q = \frac{\sqrt{\sqrt{F^2 + 4\Omega^2} + F}}{\sqrt{2}}, p = \frac{\sqrt{\sqrt{F^2 + 4\Omega^2} - F}}{\sqrt{2}}; \\ \Omega^4 = \frac{\omega^2 d^4 \rho A}{EI}; F = \frac{d^2 Q}{EI} \end{gathered} \tag{3}$$

Here Ω and F are dimensionless parameters of frequency and force.

In (3) superscripts $(\pm)$ denote regions:

in beam of ES configuration $(-) \to x \in (n-1, n-1/2), (+) \to x \in (n-1/2, n)$,

in beam of ERS configuration $(-) \to x \in (n-1, n), (+) \to x \in (n, n+1)$.

Beam with periodically arranged exterior elastic supports (ES). Consider the beam with the periodically arranged exterior elastic supports located at points $x = n - 1/2$ of the basic unit span $x \in (n-1, n)$ (Fig. 1).

Contact conditions at points $x = n - 1/2$ can be cast as

$$
\begin{aligned}
&\frac{dU_+(x)}{dx} = \frac{dU_-(x)}{dx}, \quad U_+(x) = U_-(x) = 0 \\
&\frac{d^2U_+(x)}{dx^2} - \frac{d^2U_-(x)}{dx^2} = \gamma \frac{dU_-(x)}{dx},
\end{aligned}
\tag{4}
$$

Here $\gamma = Td/EI$ is the dimensionless parameter, T is the stiffness of the rotational spring attached to an elastic support.

At the end points of the periodic basic unit span the Floquet conditions will be used [6]

$$
\begin{aligned}
&\frac{d^3U_+(n)}{dx^3} = \lambda \frac{d^3U_-(n-1)}{dx^3}, \frac{d^2U_+(n)}{dx^2} = \lambda \frac{d^2U_-(n-1)}{dx^3} \\
&\frac{dU_+(n)}{dx} = \lambda \frac{dU_-(n-1)}{dx}, U_+(n) = \lambda U_-(n-1)
\end{aligned}
\tag{5}
$$

Here $\lambda = \exp(ikd)$, k is the Floquet wave number.

Applying to the solutions (3) the contact (4) and the Floquet conditions (5) we get the equation determining the Floquet wave number

$$
\begin{aligned}
&(1+\lambda^2) f(p,q) + \lambda g(p,q) = 0 \\
&\cos(kd) = \eta(\Omega) \\
&\eta(\Omega) = -\frac{g(p,q)}{2f(p,q)}
\end{aligned}
\tag{6}
$$

Here

$$f(p,q) = (p^2+q^2)(q\sin(p) - p\sinh(q));$$

$$g(p,q) = \gamma\left(2pq(1-\cos(p)\cosh(q)) - (p^2-q^2)\sin(p)\sinh(q)\right) +$$

$$+2(p^2+q^2)(q\sin(p)\cosh(q) - p\cos(p)\sinh(q))$$

The condition $|\eta(\Omega)| > 1$, where values of k are complex, defines the bandgaps of frequencies ranges of eigenfrequencies, in which flexural waves cannot propagate in the infinite periodic beam with periodically repeated elastic supports. The stopband edges of eigenfrequencies are given by condition $|\eta(\Omega)| = 1$ [6].

In the case of the action of a compressed force $F = -P$ we can obtain the beam stability equation from (6) at $\Omega \to 0$

$$\cos(kd) = \eta(P)$$

$$\eta(P) = \frac{\gamma\left(\sqrt{P}\sin\left(\sqrt{P}\right) + 2\cos\left(\sqrt{P}\right) - 2\right) - 2\sqrt{P}\sin\left(\sqrt{P}\right) + 2P\cos\left(\sqrt{P}\right)}{2\left(P - \sqrt{P}\sin\left(\sqrt{P}\right)\right)}; \tag{7}$$

The range of P, where $|\eta(P)| > 1$ (values of k are complex), corresponds to the beam stability region. The range of P, where $|\eta(P)| < 1$ (values of k are real), corresponds to the beam instability region [14]. Since $\eta(0) = -\left(\frac{\gamma}{4} + 2\right)$ the critical values of P will be determined from equation . $\eta(P) = -1$.

$$\gamma\left(\sqrt{P}\sin\left(\sqrt{P}\right) + 2\cos\left(\sqrt{P}\right) - 2\right) + 2\left(P - 2\sqrt{P}\sin\left(\sqrt{P}\right) + P\cos\left(\sqrt{P}\right)\right) = 0 \tag{8}$$

Beam with periodically arranged elastic and rigid exterior supports (ERS). Consider the beam with elastically constraint supports and rigid supports. The distance between rigid supports and nearby elastically constraint supports is equal to d and in the basic unit span $x \in (n-1, n+1)$ the rigid supports are located at points $x = n-1, x = n+1$, the elastic support at point $x = n$., (Fig 2).

Contact conditions at points $x = n$ are as in (3), the Floquet conditions can be cast

$$\frac{dU_+(n+1)}{dx} = \lambda\frac{dU_-(n-1)}{dx}, \frac{d^2U_+(n+1)}{dx^2} = \lambda\frac{d^2U_-(n-1)}{dx^3}$$
$$U_+(n+1) = 0, U_-(n-1) = 0 \tag{9}$$

Applying to the solutions (3) the contact (4) and the Floquet conditions (9) we get the equation determining the Floquet wave number

$$\cos(2kd) = \eta(\Omega)$$
$$\eta(\Omega) = -\frac{g_c(p,q)}{4f_c(p,q)} \tag{10}$$

$$f_c(p,q)=4\left(p^2+q^2\right)(q\sin(p)-p\sinh(q))^2;$$
$$g_c(p,q)=-2\left(p^2+q^2\right)\begin{pmatrix}\left(p^2-q^2\right)\cosh(2q)\cos(2p)-p^2\cos(2p)+q^2\cosh(2q)-p^2\cos(2p)\\+2qp\left(2\sin(p)\sinh(q)-\sin(2p)\sinh(2q)\right)\end{pmatrix}$$
$$+\gamma\begin{pmatrix}p\left(p^2+q^2\right)\sin(2p)-q\left(p^2+q^2\right)\sinh(2q)+q\left(q^2-3p^2\right)\cos(2p)\sinh(2q)\\-p\left(p^2-3q^2\right)\sin(2p)\cosh(2q)+8p^2q\cos(p)\sinh(q)-8pq^2\sin(p)\cosh(q)\end{pmatrix}$$

In the case of the action of a compressed force $F=-P$ we can obtain a beam stability equation approaching $\Omega\to 0$ in (10).

$$\cos(2kd)=\eta(P)$$
$$\eta(P)=\frac{(\gamma+1)\sqrt{P}+4(\gamma+P)\sin\left(\sqrt{P}\right)+(\gamma(P-2)-4P)\sin\left(2\sqrt{P}\right)-4\gamma\sqrt{P}\cos\left(\sqrt{P}\right)+\sqrt{P}(3\gamma+2P-1)\cos\left(2\sqrt{P}\right)}{2\sqrt{P}\left(\sqrt{P}-\sin\left(\sqrt{P}\right)\right)^2}$$

(11)

Since $\eta(0)=7+\gamma$, the critical value of P will be determined from equation $\eta(P)=1$

$$(\gamma+2P)\sin\left(\sqrt{P}\right)-\gamma\sqrt{P}\cos\left(\sqrt{P}\right)=0$$

Note that the stability of beams with rigid and elastic supports has been thoroughly discussed in [14, 15]. The stability analysis of the multi- span beams on periodically arranged exterior rigid and elastic supports is carried out in [16].

Discussion, numerical results. On Fig. 3 the stability curves of the critical minimal values of compressive axial force P above of which the beam is unstable are presented for beam ES and ERS configurations versus parameter γ defining the stiffness of the rotational spring attached to an elastic support. The solid curve corresponds to ES beams, and the dashed curve - to ERS beam.

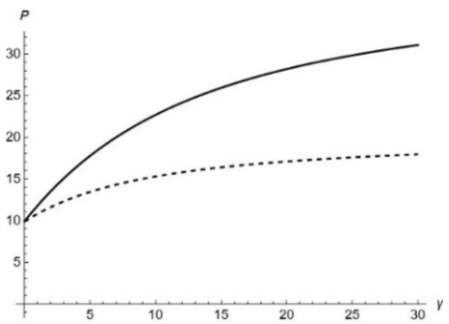

Fig. 3. Stability curves of critical minimal values of force P versus stiffness parameter γ.

The imaginary parts of the Floquet wave number $\mathrm{Im}(kd)$ define the attenuation of the flexible waves whose frequencies are inside the bandgaps, while the real part of the Floquet wave number $\mathrm{Re}(kd)$ defines the dispersion of the flexible waves, whose frequencies are outside the bandgaps.

On Figures 4 the attenuation curves $\mathrm{Im}(kd)$ versus frequency Ω are plotted, illustrating the variation of bandgap widths. The lowest contours of the attenuation curves, where $\mathrm{Im}(kd) \to 0$ define the maps of bandgap frequencies.

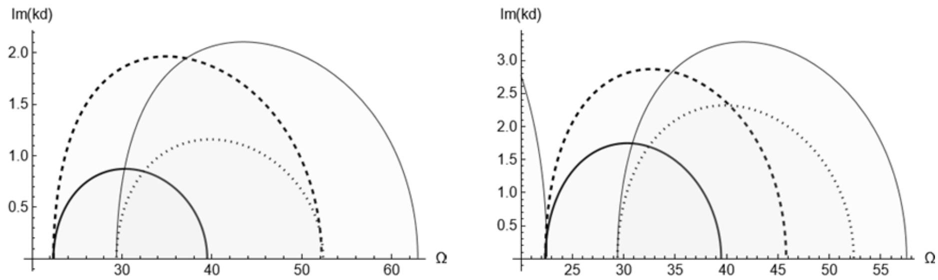

Fig. 4. a – Maps of the first bandgap of ES beam in tension; b – maps of the first bandgap of ERS beam in tension.

On Fig. 4, a, b. the maps of the first bandgap are presented for ES and ERS beams under tension. The thick solid curve corresponds to beams with parameters: $\gamma = 0, F = 0$, dashed curve: $\gamma = 30, F = 0$, dotted curve: $\gamma = 0, F = 30$, thin solid curve : $\gamma = 30, F = 30$

On Fig. 5a the maps of first bandgap are presented for elastically stable ES beam under action of compressed force F. The solid curve corresponds to beam with parameters: $\gamma = 0, F = 0$, dashed curve : $\gamma = 25, F = 25$, dotted curve: $\gamma = 25, F = -25$, respectively (See Fig 3.)

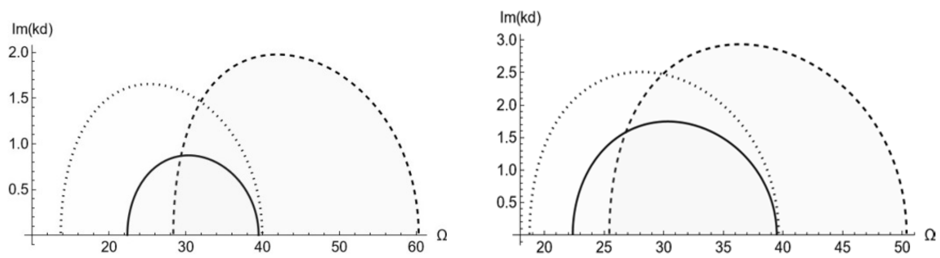

Fig. 5. a – Maps of the first bandgap of compressed ES beam compressed; b – maps of the first bandgap of ERS beam.

On Fig. 5b the bandgap maps are presented for elastic stable ESR beam under action of compressed force F. The solid curve corresponds to beam with parameters: $\gamma = 0, F = 0$, dashed curve to: $\gamma = 25, F = 12$, dotted curve to: $\gamma = 25, F = -12$.

Note that compressive force move the bandgaps to the low frequency range, while tension force and the stiffness of elastic support to high frequency range.

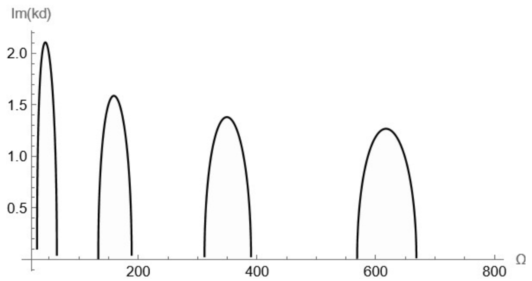

Fig.6. Maps of multiple bandgaps of ES beam , $\gamma = 30, F = 30$.

From Fig. 6 one can conclude that the effect of the bandgap widening takes place also for multiple bandgaps. Moreover, the subsequent bandgaps are much wider than the previous bandgaps.

As it follows from Fig. 4 – 6 the stiffness of the rotational spring attached to elastic periodic support, as well as the tensive and compressive axial forces sufficiently widening and changing locations of the multiple bandgaps.

Conclusions. Based on the Floquet theory it is shown that in a multi-span beam rested on periodic rigid and elastic supports the tensive and compressive axial forces sufficiently widening the multiple bandgaps of the flexural waves. The widening of the bandgaps occurs also with increasing the stiffness of the rotational spring attached to an elastic support. This study lays a certain theoretical foundation for the design and tuning the metabeam bandgap locations and widths by varying the stiffness of the elastic supports as well as the axial force magnitude and direction.

Acknowledgments. The work has been supported by the State Committee of Science of RA, within the framework of the research project 21T-2C299.

Institute of Mechanics, NAS of Armenia
e-mail: kghazaryan@mechins.sci.am

K. B. Ghazaryan

Bandgaps, Dynamics and Stability of a Multi-Span Beams Rested on Periodically Arranged Exterior Supports

In the framework of the Floquet theory formation of flexural frequency bandgaps is considered for a beam in tension rested on periodically arranged rigid or elastically

constraint intermediate exterior supports. In the case of an axial compressive force action the elastic stability of such beams is also discussed.

Կ. Բ. Ղազարյան

Արտաքին հենարաններին պարբերաբար դասավորված բազմաթռիչք հեծանների արգելված գոտիները, դինամիկան և կայունությունը

Ֆլոկեի տեսության շրջանակում դիտարկված են ձգված հեծանի ծռման հաճախությունների արգելված գոտիների ձևավորման խնդիրներ: Հեծանը դրված է պարբերաբար դասավորված կոշտ կամ առաձգական միջանկյալ արտաքին հենարաններին: Քննարկվում է նաև այդպիսի հեծանների կայունությունն առանցքային սեղմող ուժի ազդեցության դեպքում:

К. Б. Казарян

Запрещённые зоны, динамика и устойчивость многопролетных балок на периодически расположенных внешних опорах

В рамках теории Флоке рассмотрены задачи формирования запрещенных зон изгибных частот растянутой балки, покоящейся на периодически расположенных жестких или упругих промежуточных внешних опорах. Обсуждается также устойчивость таких балок в случае действия осевой сжимающей силы.

References

1. *Chen Z., Yang Y., Lu Z. et al.* – Physica B: Condensed Matter. 2013. V. 410. P. 5-12.
2. *Salman E., Lustig S., Elata D.* – Smart Materials and Structures. 2024. V. 33. № 3. P. 34-41.
3. *Wang Q., Wu N.* – Smart Mater. Struct. 2012. V. 21(8). P. 085013.
4. *Erturk A., Hoffmann J., Inman D. J.* – Applied Physics Letters. 2009. V. 94(25).
5. *Adams S., Craster R. V., Guenneau S.* – In: Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences. 2008. V. 464. № 2098.
6. *Hussein M. I., Leamy M. J., Ruzzene M.* – Applied Mechanics Reviews. 2014. V. 66. P. 040802/1-38.
7. *Anigbogu W., Nguyen H., Bardaweel H. K.* – Front. Mech. Eng. 2021. V. 7. P. 768508.
8. *Xiao Y., Wen J., Yu D. et al.* – Journal of Sound and Vibration. 2013. V. 332(4). P. 867-893.
9. *Han W., Wan S.* – 2023. Sustainability. 2023. V. 15(4). P. 3680.
10. *Mead D. J.* – Journal of Sound and Vibration. 1970. V. 11(2). P. 181-197.
11. *Ghazaryan K., Jilavyan S., Piliposyan D. et al.* – In: Solid Mechanics, Theory of Elasticity and Creep. Cham: Springer International Publishing. 2023. P. 101-113.
12. *Ghazaryan K., Piliposyan G., Jilavyan S. et al.* – European Journal of Mechanics - A/Solids. 2024. V. 103. P. 105194.

13. *Zhao Xi-Ning, Yang Xiao-Dong, Zhang Wei et al.* – AIP Advances. 2021. V. 11. № 6.
14. *Timoshenko S. P., Gere J. M.* Theory of elastic stability. Courier Corporation. 2009. 541 p.
15. *Elishakoff E., Li Y., Starnes Jr., J. H. et al.* – Applied Mechanics Reviews. 2001. V. 54(5), B86-B86.
16. *Avetisyan A., Ghazaryan K., Marzocca P.* – International Journal of Solids and Structures. 2023. V. 281. P. 112410.