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RE-EXAMINING BERMAN'S PARAMETRIZATION OF THE HUBBLE PARAMETER IN THE CONTEXT OF LATE-TIME ACCELERATION

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In this paper, we have revisited the Berman's idea of the variation of Hubble parameter. While previously explored in the context of Λ-varying cosmologies, where scale factor variations yield linear universe expansion, this parametrization has undergone extensive scrutiny. Our investigation, however, explores into its implications in the context of late-time cosmic acceleration, within the framework of classical general relativity, adopting the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime as our background metric. Our analysis offers a precise solution to Einstein's field equations (EFEs) in a model-independent way, affording a thorough assessment of both geometrical and physical model parameters. Additionally, this study supplements its findings with graphical representations of the evolving cosmological parameters across flat, closed, and open universe scenarios, all subject to constraints derived from the model parameters. In synthesizing these results, we shed light on the intricate interplay between cosmic acceleration, dark energy, and the parametrization of the Hubble parameter, thereby providing valuable insights into the fundamental mechanics of our universe.

Keywords: *cosmic acceleration: dark energy: cosmological parametrization: Hubble parameter*

1. *Introduction*. Before 1916, the prevailing belief was that gravity constituted an intrinsic quality of objects, exerting a consistent, immediate force over extensive distances. Nonetheless, Einstein's theory of general relativity (GR) marked a significant shift in scientific understanding. GR addressed the enigma of Mercury's precise behavior by revealing that gravity was not a mysterious force acting remotely in the backdrop of space and time. Instead, it emerged as a consequence of the curvature of the underlying space-time framework. The fundamental tenet of GR asserts that the shortest distance between any two remote objects in space is invariably curved, forming the basis for GR's framework built upon this curved geometry.

Over the past 100 years, the perspective of scientists about the universe has completely changed as a result of Einstein's theory of gravity. Many phenomena may be described analytically using Einstein's field equations (EFEs), and this theory, that had been a mystery for decades, suddenly fitted the evidence. After

100 years, there are still several problems with Einstein's general relativity, including its failure to explain the Big Bang, the age of the universe, the singularity within black holes, and many others [1]. Understanding the curvature singularity, geodesic incompleteness, and b-incompleteness is one of GR's toughest hurdles. Several individual develops of the universe have generated a great deal of implications and hypothesis in the area of GR. Therefore, developing a better theory is one of the main goals of physics. Throughout the last century, there have been numerous theoretical and observational problems with Einstein's theory. However, new gravitational wave observations and a black hole picture improve GR foundation. So, we are motivated here just to discover late expansion of the universe in the context of GR.

During the previous many years, one of the important problems in theoretical physics and, more broadly, cosmology has been determining the mysterious nature of the universe's two dominant components, dark energy and dark matter. The physical cause of the late-time cosmic acceleration is the greatest open challenge in cosmology today. Explaning the various statistical observational data sets revealed the physical mechanism [2-11]. Many models of dark energy consider the presence of an additional, undetected field that is perhaps responsible for the universe's rapid expansion. However, some reasonable hypotheses also include an infrared modification to the theory of general relativity [12-14]. The evolution of the current cosmos is consequently governed by dark energy, which makes up about two third of the total energy density of the universe.

According to the literature, Einstein's cosmological constant Λ, which was first proposed in 1917, serves as the best and most straightforward candidate among these various research options for dark energy. This implies that the repulsive nature of Λ is responsible for the universe's acceleration with the equation of state $\omega = -1$. This genuine candidate, however, suffers from a long-standing cosmological constant problem as well as the constant equation of state. In Einstein field equations, the term cosmological constant Λ describes the intrinsic energy density of the vacuum, which is the most interesting candidate of dark energy (DE). The mathematical expression Λ, on the other hand, indicates a significant difference between theoretical and observational predictions [15]. As a result of the variety caused by the fine-tuning issue and the cosmic coincidence problem associated with CDM, many DE models [16-18] have been developed.

As is well known, the EoS parameter is the relationship between energy density and pressure i.e. $\omega = p/\rho$. The decelerated and accelerated expansion of the universe are described by the EoS parameter. It classifies the various cosmological phases as follows: If $\omega = 1/3$ the model denotes the radiation-dominated phase, while $\omega = 0$ denotes the matter-dominated phase. The present study makes an effort to address late-time cosmic acceleration on a Friedmann-Lemaitre-Robertson-

Walker (FLRW) background. The Einstein field equations in the FLRW background contain two independent equations with three unknowns (energy density ρ, pressure p, and scale factor a) that can be resolved by assuming the equation of state. The system becomes insecure when DE, an additional degree of freedom, is added. This inconsistency in the literature can be resolved in a variety of ways. Here, we use a model-independent method, also referred to as cosmological parametrization, to find the exact solution of the field equations.

To fit data to the cosmic evolution of the universe, the model-independent technique (or cosmological parametrization) of reconstructing a cosmological model with or without dark energy has been used in the literature. Nowadays, there is a lot of interest in the model-independent approach used in the framework of some DE candidates, which was first discussed by Starobinsky. In the literature, a wide range of parametrization schemes [19] have been suggested to describe the evolution of universe, including the transition from early deceleration to late acceleration. There are also other parametrization schemes, such as density, pressure, deceleration, Hubble and scale factor parametrization and others. As a result, the goal of this paper is to represent a specific parametrization of the Hubble parameter that better explains cosmic dynamics and provides simpler constraints than any other cosmological parameter.

The structure of the work is as follows: In Section 1, a brief introduction is presented, addressing issues related to GTR and dark energy. Section 2 covers the derivation of field equations, solution techniques, and offers a geometric interpretation of the model obtained in the same section. In Section 3, we discuss into the dynamics of the model, analyzing physical parameters and describing the evolution during the RD and MD eras of the universe. Additionally, graphical representations of the evolution of cosmological parameters are provided. In Section 4.1, we also discussed some kinematic properties of model. The work concludes with our findings in Section 5.

2. *Field equations and solution*. Let us first assume that the universe is homogeneous and isotropic. So, as a background metric, we will use the FLRW spacetime in the following form:

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{1}$$

where $a(t)$ denotes the universe's scale factor, k is the curvature parameter assumes the values 0, 1, or -1, which corresponds to flat, closed and open universe respectively, r, θ and ϕ are spherical polar coordinates, t is the cosmic time. Here, we choose units in such a way that $8\pi G = c = 1$.

The matter source in the universe is provided by the total energy-momentum tensor (EMT) given by the equation,

$$T_{\mu\nu}^{Total} = (\rho_{Total} + p_{Total})u_\mu u_\nu + p\rho_{Total}\, g_{\mu\nu}\,. \tag{2}$$

The energy momentum tensor $T_{\mu\nu}^{Total}$ represents the combined energy momentum of the two energy components in the universe. These components consist of the total energy density ρ_{Total}, which is the sum of the energy densities ρ and ρ_{de} corresponding to ordinary matter and dark energy, respectively. Additionally, the total pressure p_{Total} is the sum of the pressures p and p_{de}, where p represents the combined pressure of radiation and matter. Again, $\rho = \rho_r + \rho_m$ and $p = p_r + p_m$, where the suffixes r and m denote radiation and matter components, respectively. The suffix *de* signifies dark energy in these expressions.

The equation that incorporates the total energy-momentum tensor within the framework of Einstein's field theory is,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -T_{\mu\nu}^{Total} \tag{3}$$

yield two independent equations as follows,

$$\rho_{Total} = 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2}, \tag{4}$$

$$p_{Total} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}, \tag{5}$$

where an overhead dot $(\cdot)$ represents ordinary derivative with respect to cosmic time t only. We believe that the interaction between two matter components are natural. From equations (4) and (5), one can easily derive the equation of continuity as

$$\dot{\rho}_{Total} + 3\frac{\dot{a}}{a}(\rho_{Total} + p_{Total}) = 0\,. \tag{6}$$

The matter content in the univere is not properly known but it can be categorized with the equation of state. Here, we consider the usual barotropic equation of state for normal (/ordinary) matter

$$p = w\rho, \tag{7}$$

where, $w = 1/3$ for radiation component and $w=0$ for pressure-less dust component in the universe.

We address the cosmic history for different phases of evolution by solving these equations, specifically examining the early RD era subsequent to the late MD era. The system of equations presented in equations (4)-(7) yields only three independent equations, while we have four variables in play: a, ρ, ρ_{de}, and p_{de}. To attain a deterministic solution, we require an additional equation. In the scientific literature, numerous methods have been proposed to resolve the field equations and introduce the necessary supplementary equation. One such approach is the model-independent approach, which entails considering a functional form for any cosmological parameter as a supplementary condition. Within this framework, a wide array of parameterization schemes [19] have been explored in the past few decades.

In the subsequent section, we delve into this approach, highlighting the idea of cosmological parametrization that has been under consideration. We then focus on one well-established parameterization to tackle the field equations and conduct a more in-depth analysis.

2.1. *Berman's law of variation of H.* The issue of the Hubble parameter dependence and its implications for cosmological models is of paramount importance in contemporary cosmology. For example, the problem of Hubble tension underscores the need for novel approaches and theoretical frameworks to reconcile observational data with theoretical predictions. According to Alan Sandage, "cosmology is the search of two parameters H_0 and q_0". Also, in a particular model, the Hubble function regulates the dynamics of the universe. Moreover, the complicacy of getting exact solution to the complicated field equations can be made simple without violating the background physics is the model-independent way, where any cosmological parameter (e.g. H, q, a, p, ρ) are allowed to consider as functions of time or redshift with some free parameters (model parameters). Detailed idea is discussed in some literature [19-22]. Our research aims to contribute to this ongoing dialogue by providing a simplified yet insightful model that can shed light on potential solutions to this discrepancy. Moreover, the reference to the work of Bisnovatyi-Kogan and Nikishin [23] highlights the diversity of approaches within the field and the richness of possible avenues for exploration. By incorporating their insights and building upon existing research, we aspire to develop a more comprehensive understanding of cosmological phenomena. In literature, numerous physical justifications and incentives exist for exploring the dynamics of dark energy models in a manner independent of specific models [24-30]. Additionally, it aids in investigating dark energy without relying on any specific cosmological model, apart from adhering to the cosmological principle. The scientific literature contains numerous instances and pieces of evidence for examining the behavior of dark energy models in a model-independent way. In this section, we follow the identical concept of cosmological parametrization and explicitly address the field equations while discussing the universe's behavior during different stages of its evolution. Many researchers have explored various parametrizations of cosmological parameters to describe specific phenomena in the universe, such as the transition from early inflation to deceleration and from deceleration to late-time acceleration. These parametrizations allow model parameters to be constrained by observational data. The Hubble parameter, denoted as H, stands out as one of the most vital cosmic parameters for understanding the rate of cosmic expansion, offering comprehensive insights into cosmic history. In this context, we examine a straightforward parametrization of the Hubble parameter, as referenced in source [31].

$$H(a) = Da^{-n}, \tag{8}$$

where $D > 0$, $n \geq 0$ are constants (call them model parameter).

Using the defitition of Hubble parameter $H = \dot{a}/a$, equation (8) yield the scale factor as an explicit time variation as;

$$a(t) = (Dnt + C)^{1/n}, \tag{9}$$

where C is constant of integration. We observe that the scale factor behaves as a linear function and is influenced by two model parameters, namely, n and D, which govern its evolution. As time approaches zero ($t \to 0$), we can establish that $a(0)$ equals $C^{(1/n)}$. Let us denote this as a^i (where i represents the initial value at $t \to 0$). This signifies a nonzero initial value for the scale factor.

2.2. *Geometrical interpretation of model.* In cosmology, the scale factor represents the relative size of the universe at different times. It is a crucial parameter in describing the expansion of the universe in models like the FLRW metric, which is a fundamental solution to Einstein's equations in general relativity. The first derivative of the scale factor, $\dot{a}$ represents the rate at which the universe is expanding at a given time. A positive value for $\dot{a}$ indicates an expanding universe, while a negative value suggests a contracting universe. The second derivative of the scale factor, describes how the rate of expansion (or contraction) is changing with time. This parameter is crucial in understanding the dynamics of the universe. In the context of the standard cosmological model (ΛCDM model), the behavior of $\ddot{a}$ determines the acceleration or deceleration of the cosmic expansion.

The first and second derivatives of scale factor with respect to time are given by

$$\dot{a} = D(Dnt + C)^{1/n-1} \tag{10}$$

and

$$\ddot{a} = (1-n)D^2(Dnt + C)^{1/n-2}. \tag{11}$$

At the initial time, denoted as $t = 0$, the universe possesses velocities and accelerations represented by $\dot{a}^{(i)} = D(c)^{1/n-1}$ and $\ddot{a}^{(i)} = (n-1)D^2(c)^{1/n-2}$. These values indicate that the model under consideration begins with a finite volume, a finite velocity, and a finite acceleration. Expressions for the Hubble parameter and deceleration parameter in cosmic time t can be derived from equation (9).

$$H(t) = \frac{\dot{a}}{a} = D(Dnt + C)^{-1} \tag{12}$$

$$q(t) = -\frac{\ddot{a}a}{\dot{a}^2} = n-1. \tag{13}$$

Equation (13) demonstrates that the deceleration parameter, as cited in [32], remains constant over time, signifying time-independence throughout the evolutionary process. A negative deceleration parameter ($q<0$) suggests a rapid expansion of the universe, while a positive value ($q>0$) indicates a slowdown. The acceleration observed in the later stages of the universe aligns with the explanation of SNeIa data, whereas the deceleration phase plays a crucial role in the cosmic evolution responsible for structure formation. In our considered model $q=0$ implies a coasting universe (an expanding universe without any acceleration and deceleration). This type of model also capable of explaining some observational data to a certain redshift.

From equation (12), we can see that as $t \to 0$, $H^{(i)} = D/C$, which is constant. Also, $H(t)$ is a decreasing function of time as $t \to \infty$, $H(t)$ becomes zero.

3. *Physical interpretation of model.* Equations (4) and (5) with the help of (7) can be written as

$$\rho + \rho_{de} = 3H^2 + 3\frac{k}{a^2}, \tag{14}$$

$$w\rho + p_{de} = (2q-1)H^2 - \frac{k}{a^2}. \tag{15}$$

We can note that the known functions of cosmic time t in the system of equations mentioned above are on the right-hand side, involving time-dependent functions of a, q, H as specified in (9), (12), (13). On the left-hand side, there are three unknown functions, namely ρ, ρ_{de}, p_{de}. The general equation of state for dark energy can be expressed as,

$$\omega_{de} = \frac{p_{de}}{\rho_{de}}. \tag{16}$$

The parameter ω_{de} can either remain constant or, more commonly, vary with time as the universe expands. The time-dependent nature of ω_{de} has led to the development of numerous dark energy (DE) cosmological models. The characteristics of dark energy, where the equation of state parameter ω_{de} is unknown, still lack a comprehensive understanding. Astrophysical observations indicate that the effective equation of state parameter ω_{eff} for scalar field models falls within the range of $-1.48 < \omega_{eff} < -0.72$ [33-35]. Analyzing observational data, we find a slight preference for dark energy (DE) models in which ω_{eff} has recently crossed the value of -1 [36,37]. For detailed investigations into dark energy and its candidates, please refer to references [38-40]. When considering dark energy within the framework of the ΛCDM model, which aligns with observational data, the parameter ω_{de} remains a constant value of -1. This is why Einstein's cosmological constant serves as a suitable candidate for dark energy. Consequently, we persist in examining the cosmological constant as a potential dark energy candidate. In this scenario, solving equations (14) and (15) yields the expressions for the energy density of matter (including radiation) and the density of dark energy.

$$\rho = \frac{2}{1+w}\left[(1+q)H^2 + \frac{k}{a^2}\right], \tag{17}$$

$$\rho_{de} = \frac{1}{1+w}\left[(1+3w-2q)H^2 + (1+3w)\frac{k}{a^2}\right]. \tag{18}$$

We can now explore the behavior of the acquired model at various stages of the evolution of the universe in two different scenarios within the FLRW geometry: flat ($k=0$), closed ($k=1$), and open ($k=$ -1).

3.1. *Radiation dominated universe*. In this case, we have $\omega = 1/3$ and $\rho \approx \rho_r$. Therefore from equation (17) and (18) the expression for radation energy density and dark energy density can be written as

$$\rho_r = \frac{3}{2}\left[(1+q)H^2 + \frac{k}{a^2}\right] \tag{19}$$

and

$$\rho_{de} = \frac{3}{2}\left[(1-q)H^2 + \frac{k}{a^2}\right]. \tag{20}$$

In view of equation (9), (12) and (13), the above equations can be written as,

$$\rho_r = \frac{3}{2}\left[\frac{D^2 n}{(Dnt+C)^2} + \frac{k}{(Dnt+C)^{2/n}}\right] \tag{21}$$

$$\rho_{de} = \frac{3}{2}\left[\frac{(2-n)D^2}{(Dnt+C)^2} + \frac{k}{(Dnt+C)^{2/n}}\right]. \tag{22}$$

Equations (21) and (22) describe the progression of energy densities during the radiation-dominated era, but their validity does not extend to the Planck epoch. As the cosmic time approaches zero ($t \to 0$), the approximations for ρ_r^i and ρ_{de}^i are given by $\rho_r^i \approx \frac{3}{2}\left[\frac{D^2 n}{C^2} + \frac{k}{C^{2/n}}\right]$ and $\rho_{de}^i \approx \frac{3}{2}\left[\frac{(2-n)D^2}{C^2} + \frac{k}{C^{2/n}}\right]$. These expressions imply that $\rho_r^{(i)} > 0$ at the outset, provided $C \neq 0$, and $\rho_{de}^{(i)} > 0$ under the conditions $n < 2$ and $C \neq 0$ for a flat or closed universe.

Examining equations (21) and (22), it becomes evident that the positivity condition for ρ and ρ_{de} holds true for the specified values of *n*, *C*, and *D* in the cases of flat ($k=0$) and closed ($k=1$) geometries. However, for an open geometry ($k=$ -1), the positivity condition for ρ and ρ_{de} does not hold with the given values of n, C, and D. Fig.1, 2 illustrate the dynamic evolution of physical parameters, specifically the radiation energy density ρ_r and dark energy density ρ_{de}, for particular model parameter choices: $n = 0.86$, $n = 1.23$, $D = 0.1$, and an integration constant $C = 1.2$. We have chosen some particular values of these model parameters as an exemplification. Although they can be constrained through some data analysis (e.g. [41]), we here try to figure out the early and late evolution of the cosmological parameters graphically.

The correlation between radiation energy density ρ_r and temperature T is given by

$$\rho_r = \frac{\pi^2}{30} N(T) T^4 , \tag{23}$$

in the units with $k_B = c = 1$. At a temperature T, the effective quantity of spin degrees of freedom $N(T)$ can be expressed as $N(T) = 7 N_f(T)/8 + N_b(T)$, where $N_f(T)$ and $N_b(T)$ denote the degrees of freedom for fermions and bosons, respectively. It is assumed that the value of $N(T)$ remains constant during this

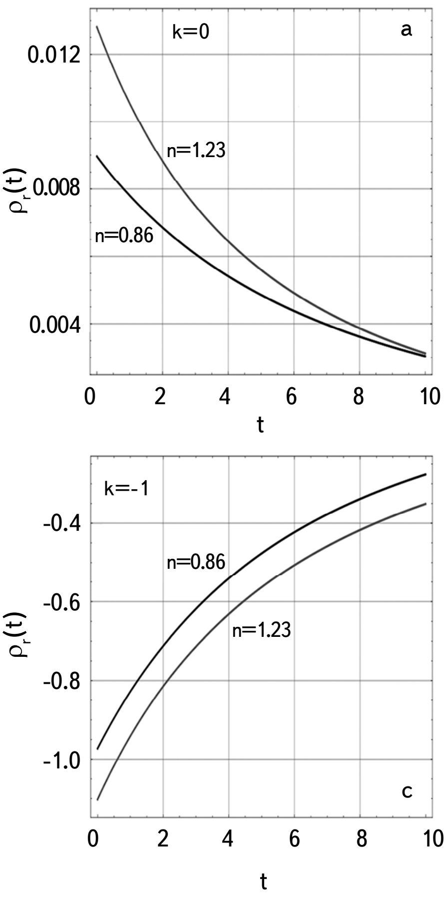

Fig.1. The graph illustrates the time progression of radiation energy density ρ_r for scenarios (k = 0, 1, -1) across panels (a), (b), and (c) correspondingly, each using appropriate units of cosmic time t .

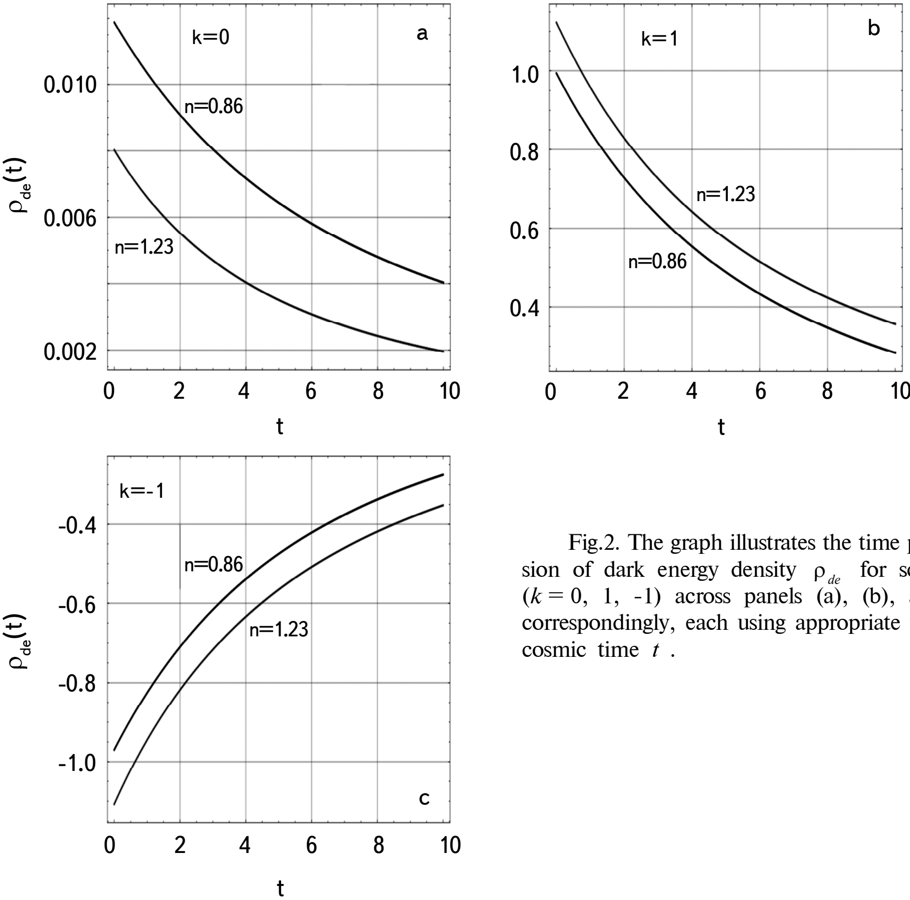

Fig.2. The graph illustrates the time progression of dark energy density ρ_{de} for scenarios ($k = 0,\ 1,\ -1$) across panels (a), (b), and (c) correspondingly, each using appropriate units of cosmic time t .

period. By utilizing equations (21) and (23), we derive the following expression;

$$T = \left(\frac{45}{\pi^2 N}\right)^{1/4}\left[\frac{D^2 n}{(Dnt + C)^2} + \frac{k}{(Dnt + C)^{2/n}}\right]^{1/4}. \tag{24}$$

From EqeS (24) we can notice that as $t \to 0$, we have $T^i \approx \left(\frac{45}{\pi^2 N}\right)^{1/4}\left[\frac{D^2 n}{C^2} + \frac{k}{C^{2/n}}\right]^{1/4}$ showing that radation temperature also attains a finite value initially. The graphs presented in Fig.3 depict the changes in radiation temperature during the early stages of the universe using the identical set of model parameters. However, it should be noted that these specific numerical model parameters are not appropriate for the open $k = -1$ scenario in this context as well.

3.2. *Matter dominated universe*. In this case, we have $\omega = 0$ and $\rho \approx \rho_m$. Therefore from equation (17) and (18) the expression for matter and dark energy density can be written as

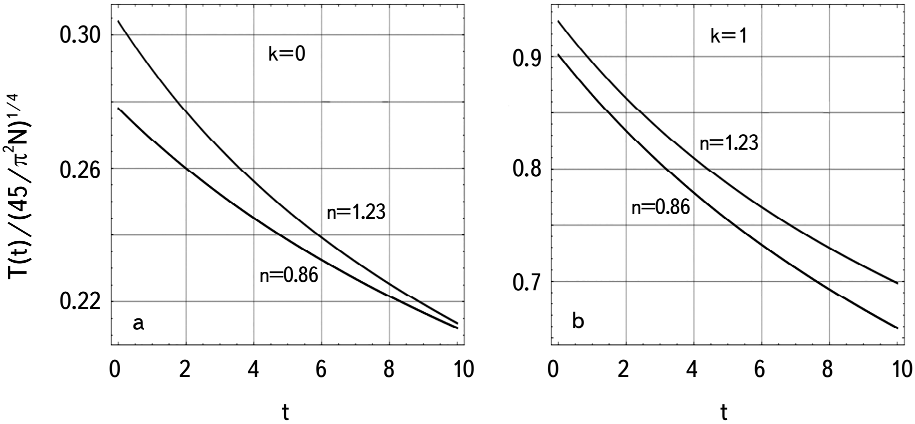

Fig.3. The figure illustrates the variation of radiation temperature T as a function of t for the scenarios corresponding to ($k=0,\ 1$) in panels (a) and (b) respectively.

$$\rho_m = 2\left[(1+q)H^2 + \frac{k}{a^2}\right] \tag{25}$$

$$\rho_{de} = \left[(1-2q)H^2 + \frac{k}{a^2}\right]. \tag{26}$$

Now, using the equations (9), (12) and (13) in equations (25) and (26), we get the expression for matter and dark energy density

$$\rho_m = 2\left[\frac{D^2 n}{(Dnt+C)^2} + \frac{k}{(Dnt+C)^{2/n}}\right] \tag{27}$$

$$\rho_{de} = \left[\frac{(2-n)D^2}{(Dnt+C)^2} + \frac{k}{(Dnt+C)^{2/n}}\right]. \tag{28}$$

In order to understand late-time cosmic acceleration, we can estebilish the ($t-z$) relationship for which, we consider the relation between redshift and the scale factor of the universe, with the standardized unit ($a_0 = 1$) and is given by;

$$a(z) = (1+z)^{-1}. \tag{29}$$

Now, the $t-z$ relationshipis in this case is obtained as;

$$t(z) = \frac{(1+z)^{-n} - C}{Dn}. \tag{30}$$

Now eliminating the inegration C with the help of $t-z$ relationship and we write all the parameters in terme of redshift z and model parameter only. The scale factor and Hubble parameter in terms of redshift can be written as

$$H(z) = D(1+z)^n \tag{31}$$

or

$$H(z) = H_0(1+z)^n, \tag{32}$$

where $H_0 = D$ be the present value of Hubble parameter.

Further, in view of Eq. (29), Eqs. (27) and (28) can be written as

$$\rho_m = 2\left[\frac{D^2 n}{(1+z)^{-2n}} + k\left((1+z)^{-n}\right)^{-2/n}\right] \tag{33}$$

$$\rho_{de} = \left[\frac{D^2(2-n)}{(1+z)^{-2n}} + k\left((1+z)^{-n}\right)^{-2/n}\right]. \tag{34}$$

From Eqs. (33) and (34), we observe that as $z \to 0$, $\rho_m(z) \to 2\left[nD^2 + k\right]$ and $\rho_{de}(z) \to \left[(2-n)D^2 + k\right]$, which is constant. The Fig.4 and 5 show the dynamical behaviour of evolution of matter energy density $\rho_m(z)$ and dark energy density $\rho_{de}(z)$ with respect to redshift z with some particular choice of model parameter $n = 0.86$, $n = 1.23$, $D = 0.1$.

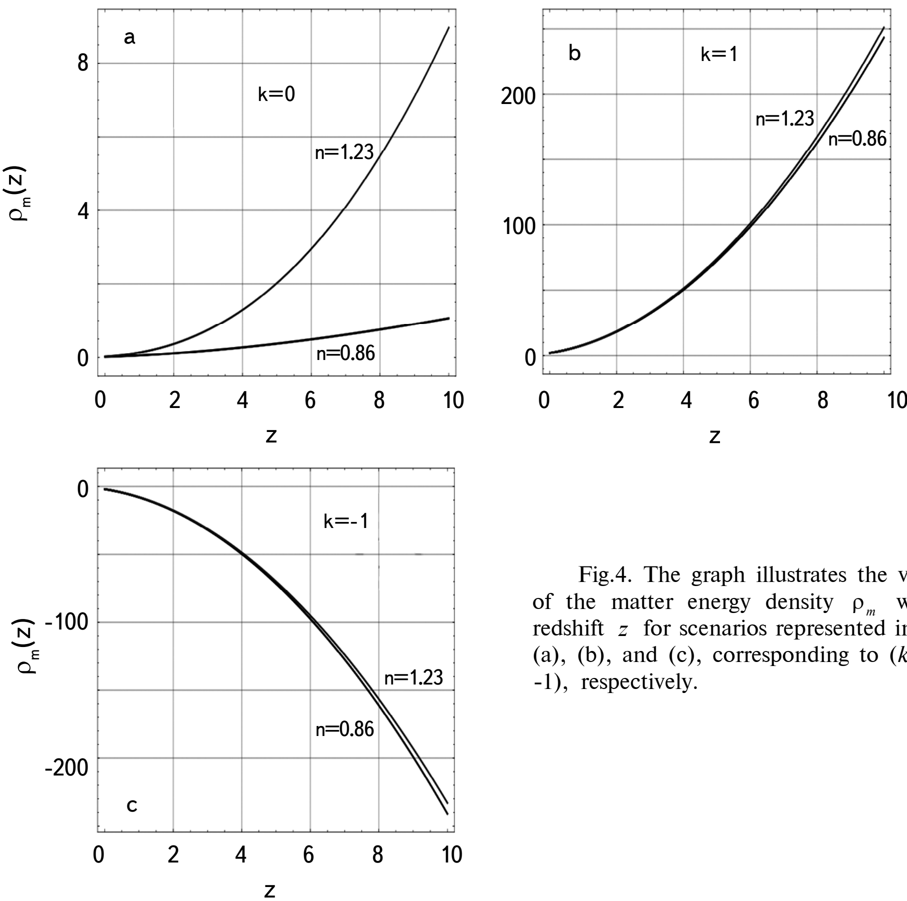

Fig.4. The graph illustrates the variation of the matter energy density ρ_m with the redshift z for scenarios represented in panels (a), (b), and (c), corresponding to ($k = 0$, 1, -1), respectively.

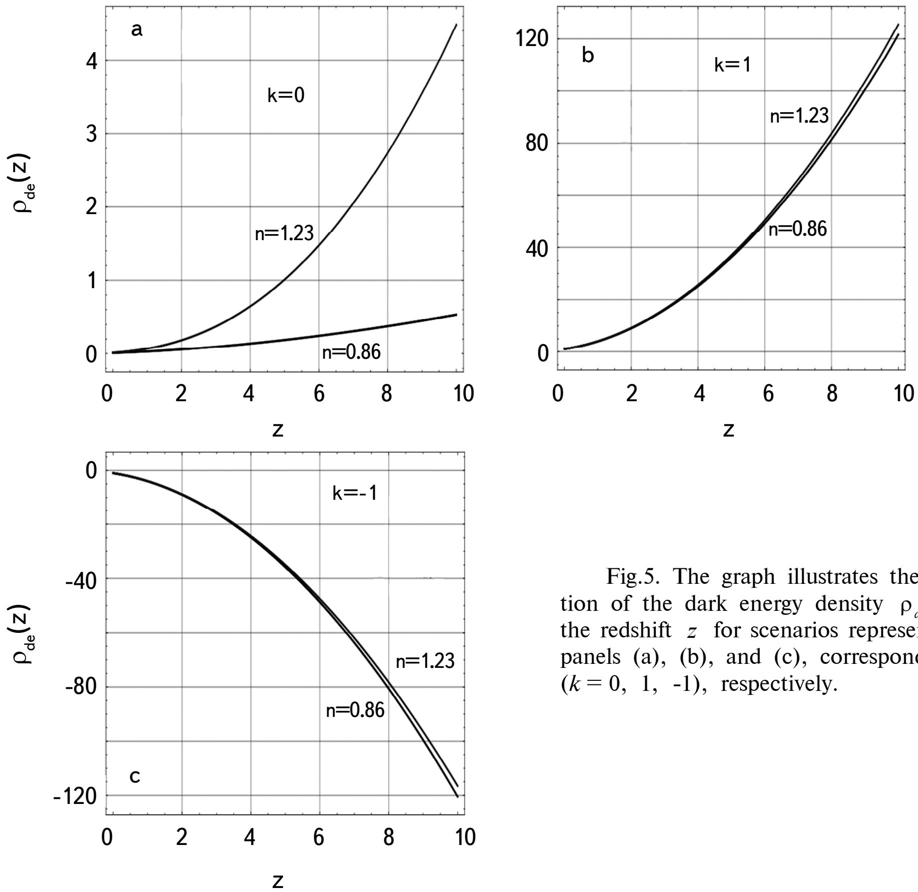

Fig.5. The graph illustrates the variation of the dark energy density ρ_{de} with the redshift z for scenarios represented in panels (a), (b), and (c), corresponding to ($k = 0,\ 1,\ -1$), respectively.

4. *Distance measures in this model.*

4.1. *Lookback time and proper time.* There are various approaches to express the separation between two points in cosmology, specifically in cosmography, the study of the universe. This is because, during the expansion of the universe, the distances among comoving objects are in constant flux, and observers on Earth perceive a backward progression in time as they observe distant objects. The common thread among all distance measurements lies in their estimation of the distances between events along radial null trajectories, which are essentially the paths of photons that reach the observer. The lookback time, denoted as t_L, for an object is the duration between the detection of light today (at redshift $z = 0$) and the emission of photons at a specific redshift z.

$$t_L = t_0 - t(z) = \int_{a}^{a(0)} \frac{dt}{\dot{a}}. \tag{35}$$

The proper distance between two occurrences is determined by measuring it at the instant of observation, representing the distance within the frame of reference where they occur simultaneously. The proper distance is expressed as $d(z) = a(0)r(z)$, where $r(z)$ denotes the radial distance of the object, which is given by

$$r(z) = \int_t^{t_0} \frac{dt}{a(t)}. \tag{36}$$

Therefore, the proper distance $d(z)$ can be written as

$$d(z) = a_0 \left[\frac{(c + Dnt)^{(-1+n)/n}}{D(-1+n)} \right]_t^{t_0} = C^{(1/n)} \left[\frac{(1+z)^{1-n}}{D(1-n)} \right].$$

The luminosity distance, denoted as d_l, for a source exhibiting a redshift of z is formally described as follows:

$$d_l^2 = \frac{l}{4\pi L}, \tag{37}$$

where L is the observed flux and l is the intrinsic luminosity of the object. The luminosity distance is given by

$$d_l = (1+z)d(z) = C^{(1/n)} \left[\frac{(1+z)^{2-n}}{D(1-n)} \right]. \tag{38}$$

The angular diameter distance is defined by

$$d_A = \frac{l_1}{\theta}, \tag{39}$$

where l_1 is intrinsic physical size and θ is the observed angular size of an object, the angular diameter distance d_A of an object in terms of redshift z is

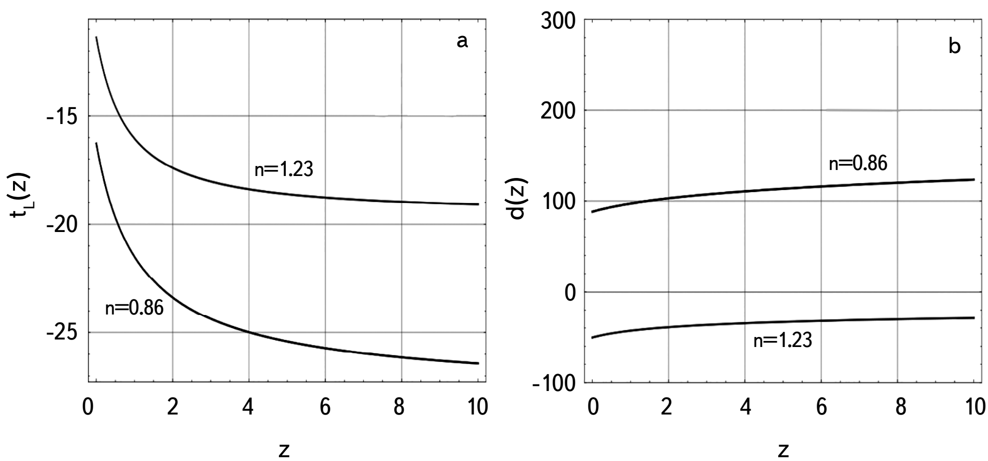

Fig.6. The plots of look back time t_l and proper distance $d(z)$ vs redshift z for the model in the panel (a) and (b) respectively.

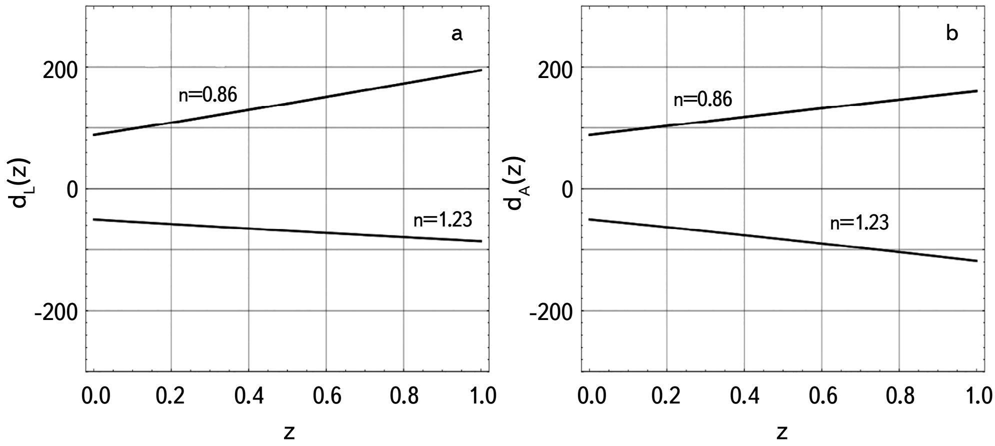

Fig.7. The plots of luminosity distance d_l and angular diameter distance d_A vs redshift z for the model in the panel (a) and (b) respectively.

$$d_A = \frac{d(z)}{1+z} = \frac{d_l}{(1+z)^2} = C^{(1/n)}\left[\frac{(1+z)^n}{D(1-n)}\right]. \tag{40}$$

4.2. *Age of the universe*. The present age of the universe refers to the current elapsed time since the cosmic event known as the Big Bang, which is considered the starting point of our universe. According to the prevailing scientific understanding, the universe is approximately 13.8 billion years old. This estimation is derived from observations of cosmic phenomena, such as the cosmic microwave background radiation and the redshift of distant galaxies. Over the course of these billions of years, the universe has undergone significant transformations, including the formation of galaxies, stars, and planets. The study of the present age of the universe plays a crucial role in our comprehension of its evolution and helps scientists unravel the mysteries of cosmic processes and phenomena. Age estimations derived from alternate sources such as galaxies and the Hubble constant often exhibit discrepancies, posing a persistent challenge in the field of cosmology. The dynamical age of the universe is indicated by this constant.

$$t_0 = \int_0^\infty \frac{dz}{(1+z)H(z)} = H_0^{-1}\int_0^\infty \frac{dz}{(1+z)^{n+1}} = H_0^{-1}\frac{1}{n}. \tag{41}$$

If n is not equal to zero, the value deviates from the current estimate, denoted as $t_0 = H_0^{-1}$, which is approximately 14 billion years. However, when n is set to 1, the model aligns well with the present age of the universe.

5. *Results and conclusion*. This work explores a cosmological model grounded in the general theory of relativity within the framework of FLRW space-

time. To derive an exact solution for the cosmological field equations and accommodate the currently observed cosmic acceleration, we introduce a straightforward parametrization for the Hubble parameter, H which results in a time-independent deceleration parameter, $q(t)$, equal to $n - 1$ (as in [32]). This parametrization also leads to a linear-type evolution of the scale factor. The study thoroughly investigates the behavior of various geometrical parameters $a(t)$, $H(t)$, and $q(t)$ and physical parameters, such as energy densities of radiation, matter, and dark energy (including the cosmological constant). In FLRW space-time, Berman's special law for Hubble's parameter variation (as mentioned in reference [31]) yields a constant value of the deceleration parameter, $q(t) = n - 1$, which results in accelerating universe models when $0 \leq n < 1$ and decelerating ones when $n > 1$, offering an explanation for the current universe's acceleration. This model suggests that the universe originated with finite volume, velocity, and acceleration, in contrast to the standard big bang scenario.

In Section 3, we extensively examine the dynamics of the model we have derived. We discuss how the physical parameters have evolved throughout the history of the universe, accounting for the cosmological constant as a dark energy candidate with an equation of state represented by $\omega_{de} = -1$. The requirement for positive energy densities is satisfied only when considering flat and closed universe geometries, as indicated by the expressions for radiation energy density ρ_r and dark energy density ρ_{de}. However, when it comes to an open universe, the selected nu- merical values for the model parameters n, D, and C do not meet the condition for positive energy densities in both ρ_r and ρ_{de}. In Fig.1 and 2, we depict the profiles of radiation and dark energy energy densities for a range of model parameter values, while keeping the model parameter D constant and varying n. This analysis is conducted for flat, closed, and open universe geometries. For the cases $k = 0$ and $k = 1$, we can clearly see that in Fig.1 and 2 the evolution of radiation and dark energy densities showing similar nature and is decreasing over time, where as for the case (k = -1) is incompatible in this scenario. The Fig.3 illustrates how radiation temperature changes over cosmic time t in the early universe for different universe geometries (flat and closed) with specific model parameter values. Radiation temperature follows a pattern similar to radiation energy density, starting with high temperatures and gradually decreasing over time, eventually reaching a constant value in the late stages.

To gain a deeper understanding of how structures form in the universe and the behavior of cosmological parameters in the late universe, we established a relationship between time and redshift ($t-z$) and expressed the physical parameters like matter and dark energy density in terms of redshift. Upon converting these parameters to redshift z, it became apparent that they are all related to the variable n. We specifically examined the MD era, where dust pressure approaches to minimal value. Fig.4 and 5 depict the red-shift evolution of matter and DE energy densities in various geometries, with n being a variable and D being a constant. In the existing models, both the matter energy density and dark energy density maintain positive values. As a result, the weak and null energy conditions are met, indicating that the resulting models are indeed physically plausible. We have examined the lookback time, proper distance, luminosity distance, and angular diameter distance for our derived model by analyzing the plots presented in Fig.6 and 7.

In this analysis, we present a model designed to address the cosmic acceleration observed in the late-time universe, situated within the framework of an FLRW background. The approach involves adopting a parametrization of H suggested by Berman in 1983. Notably, this model has the flexibility to extend its scope to include anisotropic and inhomogeneous backgrounds. Moreover, it proves versatile in tackling diverse challenges such as big bang nucleosynthesis, structure formation, and inflation within the specified framework. A recent investigation [42] has devised a robust methodology that capitalizes on the redshift dependence of the Alcock-Paczynski test to gauge the expansion history of the universe. This technique harnesses the isotropy of the galaxy density gradient field, leading to more stringent constraints on cosmological parameters with heightened precision. This innovative approach has been extensively explored in a series of papers by Li et al. [43-45]. The proposed model, along with analogous parameterized models [46], holds promise for further scrutiny to refine and augment constraints on model parameters by incorporating additional datasets. However, the detailed examination of these possibilities is reserved for our forthcoming research endeavors.

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ПЕРЕСМОТР ПАРАМЕТРИЗАЦИИ БЕРМАНА ПАРАМЕТРА ХАББЛА В КОНТЕКСТЕ ПОЗДНЕГО УСКОРЕНИЯ

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В этой статье мы вернулись к идее Бермана об изменении параметра Хаббла. Ранее эта параметризация подверглась тщательному изучению в рамках Λ-переменных космологий, где изменения масштабного фактора приводят к линейному расширению Вселенной. В данной работе изучены его последствия в контексте космического ускорения позднего времени в рамках классической общей теории относительности, принимая в качестве основы пространство-время Фридмана-Леметра-Робертсона-Уокера (FLRW). Представлено точное решение уравнений поля Эйнштейна (EFE) способом, независимым от модели, обеспечивая тщательную оценку как геометрических, так и физических параметров модели. Кроме того, это исследование дополнено графическими представлениями эволюционирующих космологических параметров в сценариях плоской, закрытой и открытой Вселенной, причем все они подлежат ограничениям, вытекающим из параметров модели. Синтез этих результатов проливает свет на сложное взаимодействие между космическим ускорением, темной энергией и параметризацией параметра Хаббла, тем самым предоставляя ценную информацию о фундаментальной механике нашей Вселенной.

Ключевые слова: *космическое ускорение: темная энергия: космологическая параметризация: параметр Хаббла*

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