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FERMIONIC VACUUM STRESSES IN MODELS WITH TOROIDAL COMPACT DIMENSIONS

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We investigate vacuum expectation value of the energy-momentum tensor for a massive Dirac field in flat spacetime with a toroidal subspace of a general dimension. Quasiperiodicity conditions with arbitrary phases are imposed on the field operator along compact dimensions. These phases are interpreted in terms of magnetic fluxes enclosed by compact dimensions. The equation of state in the uncompact subspace is of the cosmological constant type. It is shown that, in addition to the diagonal components, the vacuum energy-momentum tensor has nonzero off-diagonal components. In special cases of twisted (antiperiodic) and untwisted (periodic) fields the off diagonal components vanish. For untwisted fields the vacuum energy density is positive and the energy-momentum tensor obeys the strong energy condition. For general values of the phases in the periodicity conditions the energy density and stresses can be either positive or negative. The numerical results are given for a Kaluza-Klein type model with two extra dimensions.

Keywords: *topological Casimir effect: Dirac field: toroidal compactification*

1. *Introduction.* The field theoretical models in background spacetimes with compact dimensions appear in a number of theories in fundamental physics like string theories, supergravities and Kaluza-Klein theories. The quantum creation of universe with a compact space has been considered in [1-3]. In this type of models the probability of inflation in the early stages of the universe expansion is not exponentially small. The effects caused by the non-trivial topology of the universe on cosmological scales are discussed, for example, in [4,5]. They include the ghost images of galaxies and quasars, cosmological magnetic fields and observable effects on cosmic microwave background. Physical models formulated on background geometries with nontrivial topology also appear in a number of condensed matter physics systems. Examples are topological structures of graphene, like carbon nanotubes and nanoloops. The long wavelength excitations of the electronic subsystem in those structures are described by an effective field theory (Dirac model, see [6,7]) with 2-dimensional spatial topologies $R^1\times S^1$ and $T^2 = S^1\times S^1$, respectively.

In quantum field theory the nontrivial spatial topology is a source of a number of interesting effects. In particular, the periodicity conditions along compact dimensions modify the spectrum of quantum fluctuations of fields and, as a consequence, the expectation values of the physical characteristics are shifted by an amount that depends on the geometry and topology of the compact subspace. This general phenomenon is known as the topological Casimir effect (see [8-12]). The vacuum energy in the topological Casimir effect depends on the size of compact dimensions and this provides a stabilization mechanism for the corresponding moduli fields. The topological Casimir energy may also appear as the source of the accelerated expansion of uncompact subspace playing the role of the dark energy at recent epoch of the Universe expansion (see, for example, [13-18]).

An important physical characteristic for charged fields is the expectation value of the current density. For a relatively simple model of toroidal compactification in flat spacetime (for quantum field theory in models with toroidal spatial dimensions see, for example, [19]), in references [20-23] it has been shown that the nontrivial phases in the periodicity conditions along compact dimensions give rise to nonzero currents along those dimensions. The phases can be interpreted in terms of magnetic fluxes enclosed by those dimensions. The currents in the compact subspace are sources of magnetic fields having components along uncompactified dimensions. The dependence of the vacuum energy density and diagonal stresses for a massive fermionic field in the same model of flat spacetime with a part of spatial dimensions compactified on a torus has been studied in [24]. In the present paper we show that, in addition to the diagonal components, the vacuum expectation value (VEV) of the energy-momentum tensor may have nonzero off-diagonal components (vacuum stresses) in the compact subspace.

The paper is organized as follows. In the next section we present the problem setup and the eigenmodes for a Dirac field obeying the quasiperiodicity conditions along compact dimensions. The general formulas for the vacuum energy density and stresses are obtained in section 3. The asymptotic and numerical analysis of the VEVs in a model with two compact dimensions is presented in section 4. The main results are summarized in section 5.

2. *Problem setup*. The background geometry we are going to consider is a flat spacetime with topology $M^{p+1}\times T^q$, where M^{p+1} is $(p+1)$-dimensional Minkowski spacetime covered by the Cartesian coordinates $\left(z^0=t,\mathbf{z}_p\right)=$ $=\left(z^0, z^1,...,z^p\right)$, and $T^q=\left(S^1\right)^q$ is a q-dimensional torus with the coordinates $\mathbf{z}_q=\left(z^{p+1},...,z^D\right)$. The length of the l th compact dimension will be denoted by L_l and, hence, $0\le z^l\le L_l$, $l=p+1$, ..., D. The volume of the compact subspace is expressed as $V_q=L_{p+1}...L_D$. For the uncompact dimensions, as usual, we have $z^l\in\left(-\infty,+\infty\right)$. The line element has the standard Minkowskian form

$$ds^2=\eta_{\mu\nu}dx^\mu dx^\nu=dt^2-d\mathbf{z}^2,\quad \mathbf{z}^2=\left(\mathbf{z}_p,\mathbf{z}_q\right), \tag{1}$$

and $\eta_{\mu\nu}$ is the Minkowski metric tensor in Cartesian coordinates.

We are interested in the vacuum stresses for a massive Dirac field $\psi(x)$, $x=(t,\mathbf{z})$, induced by compactification of a part of spatial dimensions. The field equation reads

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi(x)=0, \tag{2}$$

where γ^{μ}, $\mu=0,1,...,D$, are $N\times N$ Dirac matrices with N given by $N=2^{[(D+1)/2]}$ and $[a]$ stands for the integer part of a. The background geoemtry has non-trivial topology and in order to fix the dynamics uniquely the periodicity conditions along the compact dimensions have to be specified for the field operator. We impose quasiperiodicity conditions

$$\psi\left(t,\mathbf{z}_{p},...,z^{l}+L_{l},..., z^{D}\right)=e^{i\alpha_{l}}\,\psi\left(t,\mathbf{z}_{p},...,z^{l},..., z^{D}\right), \tag{3}$$

with phases $\alpha_{l}=\text{const}$ and $l=p+1$, ..., D. The special cases of periodic and antiperiodic fields correspond to $\alpha_{l}=0$ and $\alpha_{l}=\pi$ (untwisted and twisted fields, respectively).

The VEV of the energy-momentum tensor $T_{\mu\nu}$ for the fermionic field is expressed in terms of the mode sum over a complete set of normal modes $\psi_{\beta}^{(\pm)}(x)$, where β is the set of quantum numbers specifying the solutions to the field equation and upper/lower signs correspond to the positive/negative energy modes. Denoting the VEV by и $\langle T_{\mu\nu}\rangle=\langle 0|T_{\mu\nu}|0\rangle$, with $|0\rangle$ being the vacuum state, the mode sum is expressed as

$$\langle T_{\mu\nu}\rangle=-\frac{i}{4}\sum_{\beta}\sum_{j=+,-}j\left[\overline{\psi}_{\beta}^{(j)}(x)\gamma_{(\mu}\partial_{\nu)}\psi_{\beta}^{(j)}(x)-\left(\partial_{(\mu}\overline{\psi}_{\beta}^{(j)}(x)\right)\gamma_{\nu)}\,\psi_{\beta}^{(j)}(x)\right], \tag{4}$$

where $\gamma_{\mu}=\eta_{\mu\nu}\gamma^{\nu}$, $\overline{\psi}(x)=\psi^{\dagger}(x)\gamma^{0}$ is the Dirac adjoint and the parentheses including the indices mean symmetrization over those indices. The symbolic notation $\sum_{\beta}$ stands for the summation over discrete components of the collective index β and the integration over the continuous ones. The problem under consideration has planar symmetry and it is natural to take the normal modes corresponding to plane waves.

Taking the chiral representation of the Dirac matrices,

$$\gamma^{0}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \gamma^{l}=\begin{pmatrix}0 & \sigma_{l}\\ -\sigma_{l}^{\dagger} & 0\end{pmatrix}, \tag{5}$$

the positive and negative energy wave functions with momentum $\mathbf{k}=(k_1, ..., k_D)$ and energy $\varepsilon_{\mathbf{k}}=\sqrt{k^{2}+m^{2}}$ have the form [22]

$$\psi_{\beta}^{(\pm)}(x)=\left[\frac{1+m/\varepsilon_{\mathbf{k}}}{2(2\pi)^{p}\,V_{q}}\right]^{1/2}e^{i\mathbf{kz}\mp i\varepsilon_{\mathbf{k}}t}\begin{pmatrix}\left(-\dfrac{\mathbf{k}\boldsymbol{\sigma}}{\varepsilon_{\mathbf{k}}+m}\right)^{(1\mp1)/2}w_{\chi}^{(\pm)}\\ \left(\dfrac{\mathbf{k}\boldsymbol{\sigma}^{\dagger}}{\varepsilon_{\mathbf{k}}+m}\right)^{(1\pm1)/2}w_{\chi}^{(\pm)}\end{pmatrix}, \tag{6}$$

where $\sigma=(\sigma_1,...,\sigma_D)$. Here, the quantum number $\chi=1,2,...,N/2$ enumerates the polarization degrees of freedom, $w_\chi^{(\pm)}$ are one-column matrices with $N/2$ rows and l th element $w_{\chi l}^{(\pm)}=\delta_{\chi l}$. We will decompose the momentum into two parts, $\mathbf{k}=(\mathbf{k}_p, \mathbf{k}_q)$, where $\mathbf{k}_p=(k_1, ..., k_p)$ and $\mathbf{k}_q=(k_{p+1}, ..., k_D)$ are the parts in uncompact and compact subspaces. For the components k_l, $l=1, 2, ..., p$, one has $-\infty<k_l<+\infty$. The eigenvalues of the components k_l along compact dimensions are quantized by the periodicity conditions (3):

$$k_l=\frac{2\pi n_l+\alpha_l}{L_l},\quad n_l=0,\pm 1,\pm 2,..., \tag{7}$$

for $l=p+1, ..., D$.

The phases in the conditions (3) can be interpreted in terms of the magnetic flux for a vector gauge field enclosed by compact dimensions. The representation described above corresponds to the gauge with zero vector potential, $(\psi, A_\mu)=(\psi,0)$. Let us pass to a new gauge with the fields (ψ', A'_μ), where A'_μ has nonzero constant components along compact dimensions: $A'_\mu=0$, $\mu=0,1,...,p$ and $A'_l=\text{const}$ for $l=p+1, ..., D$. The gauge transformation has the form

$$\psi'=\psi e^{-ie\lambda},\quad A'_\mu=A_\mu+\partial_\mu\lambda,\quad \lambda=b_\mu x^\mu,$$

with constant b_μ and e being the charge of the Dirac field. Choosing $b_\mu=0$, $\mu=0,1,...,p$ and $b_l=A'_l$ for $l=p+1, ..., D$, for $A_\mu=0$ in the new gauge we get $(\psi', A'_\mu)=\left(\psi e^{-ieA'_\mu x^\mu}, A'_\mu\right)$. Taking $A'_l=\alpha_l/(eL_l)$, we see that the new field obeys the periodicity condition with $\alpha'_l=0$. Hence, the initial problem with a zero gauge field and quasiperiodicity condition (3) is transformed to a gauge with periodic boundary conditions for the field and with a gauge field having constant components along compact dimensions. In this interpretation the phases can be expressed as $\alpha_l=-2\pi\Phi_l/\Phi_0$, where $\Phi_0=2\pi/e$ is the flux quantum and $\Phi_l=-A'_l L_l$ is the formal magnetic flux enclosed by the compact dimension x^l. That flux takes on real meaning in models where the space under consideration is embedded in a space of higher dimension. Examples are the braneworld models and carbon nanotubes. In the latter case the Dirac field describing the electronic subsystem of graphene lives in a 2-dimensional space with topology $R^1\times S^1$ and that space is embedded in a 3-dimensional Euclidean space. The magnetic flux is located inside the tube.

3. *Vacuum energy density and stresses*. With the mode functions (6), the VEV of the energy-momentum tensor is evaluated by using the formula (4). The mode sums for the energy density and vacuum stresses are transformed to

$$\langle T_{00}\rangle=-\frac{N}{2V_q}\int\frac{d\mathbf{k}_p}{(2\pi)^p}\sum_{\mathbf{n}_q\in\mathbf{Z}_q}\varepsilon_{\mathbf{k}},\quad \langle T_{\mu\nu}\rangle=-\frac{N}{2V_q}\int\frac{d\mathbf{k}_p}{(2\pi)^p}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\frac{k_\mu k_\nu}{\varepsilon_{\mathbf{k}}}, \tag{8}$$

for $\mu, \nu = 1, 2, ..., D$, and и $\langle T_{0\nu} \rangle = 0$. The component $\langle T_{00} \rangle$ corresponds to the energy density and it is presented as the sum of the zero-point energies for elementary oscillators. The expressions (8) for the diagonal components have been considered in [24]. The expressions in (8) are divergent and in [24] two different methods have been used in order to find the expressions for the renormalized VEVs. The first one is based on the Abel-Plana summation formula and the second one uses the zeta function technique [9,25]. We will follow the second approach.

For the regularization of the mode sums we introduce the zeta function of a complex variable s as

$$\zeta(s) = \frac{1}{V_q} \int \frac{d\mathbf{k}_p}{(2\pi)^p} \sum_{\mathbf{n}_q \in \mathbf{Z}^q} \varepsilon_{\mathbf{k}}^{-2s}, \tag{9}$$

where the term $\mathbf{n}_q = 0$ has to be excluded from the sum in the special case $\alpha_l = 0$, $l = p + 1, ..., D$. After integration over $\mathbf{k}_p$ and by using the generalized Chowla-Selberg formula [26,27] for the resulting series, the zeta function is decomposed as [24] $\zeta(s) = \zeta_M(s) + \zeta_t(s)$, where $\zeta_M(s)$ is the corresponding function for the Minkowski spacetime with trivial topology and the contribution $\zeta_t(s)$ is induced by nontrivial topology. Introducing the vectors $\boldsymbol{\alpha}_q = (\alpha_{p+1}, ..., \alpha_D)$ and $\mathbf{L}_q = (L_{p+1}, ..., L_D)$ in the compact subspace, the topological part is expressed as

$$\zeta_t(s) = \frac{2^{1-s} m^{D-2s}}{(2\pi)^{D/2} \Gamma(s)} \sum_{\mathbf{n}_q \in \mathbf{Z}^q}{}' \cos(\mathbf{n}_q \boldsymbol{\alpha}_q) f_{D/2-s}(mg(\mathbf{L}_q, \mathbf{n}_q)), \tag{10}$$

where the prime on the summation sign means that the term $\mathbf{n}_q = 0$ should be excluded and we have introduced the functions

$$f_\nu(x) = \frac{K_\nu(x)}{x^\nu}, \quad g(\mathbf{L}_q, \mathbf{n}_q) = \left(\sum_{i=p+1}^{D} L_i^2 n_i^2 \right)^{1/2}, \tag{11}$$

with $K_\nu(x)$ being the modified Bessel function of the second kind.

An alternative representation for the zeta function, convenient in the asymptotic analysis of the off-diagonal stress, is obtained from (10) by using the formula (2.40) from [22]. It is given by the formula

$$\begin{aligned} \zeta_t(s) = \zeta_{p+2,q-2}(s) + \frac{2^{1-s} m^{D-2s}}{(2\pi)^{p/2+1} \Gamma(s) V_{q-2}} \sum_{\mathbf{n}_q \in \mathbf{Z}^q}{}' \cos(n_\mu \alpha_\mu) \\ \times \cos(n_\nu \alpha_\nu) \varepsilon_{q-2}^{p-2s+2} f_{p/2-s+1}\left(\sqrt{n_\mu^2 L_\mu^2 + n_\nu^2 L_\nu^2} \, \varepsilon_{\mathbf{n}_{q-2}} \right), \end{aligned} \tag{12}$$

where $\zeta_{p+2,q-2}(s)$ is the zeta function in the model of topology $M^{p+3} \times T^{q-2}$ with decompactified dimensions x^μ and x^ν. Here, M^{p+3} stands for $(p+3)$-dimensional Minkowski spacetime with trivial topology. The prime on the summation sign in (12) means that the term $n_\mu = n_\nu = 0$ is excluded from the summation and we have defined

$$\varepsilon_{\mathbf{n}_{q-2}} = \left(\sum_{l=p+1,\neq\mu,\nu}^{D} k_l^2 + m^2 \right)^{1/2}. \tag{13}$$

The function $\zeta_{p+2,q-2}(s)$ is given by a formula similar to (10) with the summation over $\mathbf{n}_{q-2} \in \mathbf{Z}^{q-2}$.

The background geometry is flat and the renormalization is reduced to the subtraction of the Minkowskian VEV. The renormalized energy density is expressed as $\langle T_0^0 \rangle_t = -N\zeta_t(-1/2)/2$. For the diagonal components of the vacuum stresses along uncompact dimensions one gets (no summation over μ) $\langle T_\mu^\mu \rangle_t = \langle T_0^0 \rangle_t$, $\mu = 1, 2, ..., p$. The diagonal vacuum stresses in the compact subspace are found by using the relation (no summation over μ) $\langle T_\mu^\mu \rangle_t = (L_\mu / V_q) \partial_{L_\mu} \left(V_q \langle T_0^0 \rangle_t \right)$, $\mu = p+1, ..., D$. In this way, from (10) for the diagonal component we find [24] (no summation over μ)

$$\langle T_\mu^\mu \rangle_t = \frac{Nm^{D+1}}{(2\pi)^{(D+1)/2}} \sum_{\mathbf{n}_q \in \mathbf{Z}^q}{}' \cos(\mathbf{n}_q \boldsymbol{\alpha}_q) F_{(\mu)}(mg(\mathbf{L}_q, \mathbf{n}_q)), \tag{14}$$

with the functions

$$F_{(\mu)}(x) = \begin{cases} f_{(D+1)/2}(x), & \mu = 0, 1, 2, ..., p \\ f_{(D+1)/2}(x) - m^2 L_\mu^2 n_\mu^2 f_{(D+3)/2}(x), & \mu = p+1, ..., D. \end{cases} \tag{15}$$

The corresponding expressions in the case of periodic conditions, $\alpha_l = 0$, $l = p+1$, ..., D, are obtained from (14) with $\cos(\mathbf{n}_q \boldsymbol{\alpha}_q) = 1$. In this case the vacuum energy density is positive and for the diagonal stresses one has (no summation over l) $\langle T_l^l \rangle_t < \langle T_0^0 \rangle_t$, $l = p+1$, ..., D. As it will be shown below, for periodic conditions the off-diagonal components vanish. By using the relation

$$x^2 f_{\nu+1}(x) = f_{\nu-1}(x) + 2\nu f_\nu(x), \tag{16}$$

it can be seen that $\sum_{l=1}^{D} \langle T_l^l \rangle_t < \langle T_0^0 \rangle_t$. Hence, the vacuum energy-momentum tensor for a fermionic field with periodic conditions obey the strong energy condition. For twisted fields with $\alpha_l = \pi$, $l = p+1$, ..., D, one has $\cos(\mathbf{n}_q \boldsymbol{\alpha}_q) = (-1)^{n_{p+1}+...+n_D}$.

The result (14) shows that the vacuum stresses in the uncompact subspace are equal to the energy density, $\langle T_\mu^\mu \rangle_t = \langle T_0^0 \rangle_t$, $\mu = 1, ..., p$ (no summation over μ). Of course, this is a consequence of the Lorentz invariance in that subspace. By taking into account that for the vacuum effective pressure along the direction x^μ one has $P_\mu = -\langle T_\mu^\mu \rangle_t$, we see that the equation of state for the vacuum in the uncompact subspace is of the cosmological constant type. The models with the topological Casimir energy as the source of the accelerated expansion are based on this property.

For a massless fermionic field the general result (14) is reduced to (no summation over μ)

$$\left\langle T_{\mu}^{\mu} \right\rangle_t = \frac{N}{(2\pi)^{(D+1)/2}} \Gamma \frac{D+1}{2} \sum_{\mathbf{n}_q \in \mathbf{Z}^q}{}' \frac{\cos(\mathbf{n}_q \boldsymbol{\alpha}_q)}{g^{D+1}(\mathbf{L}_q, \mathbf{n}_q)} F_{(\mu)}^{(0)}(\mathbf{L}_q, \mathbf{n}_q), \tag{17}$$

where $F_{(\mu)}^{(0)}(\mathbf{L}_q, \mathbf{n}_q) = 1$ for $\mu = 0, 1, 2, ..., p$, and

$$F_{(\mu)}^{(0)}(\mathbf{L}_q, \mathbf{n}_q) = 1 - \frac{(D+1) n_\mu^2 L_\mu^2}{g^2(\mathbf{L}_q, \mathbf{n}_q)}, \tag{18}$$

for $\mu = p+1, ..., D$. In this special case the vacuum energy-momentum tensor is traceless $\left\langle T_{\mu}^{\mu} \right\rangle_t = 0$.

Here we are interested in the off-diagonal components. For $\mu = 0, 1, 2, ..., p$ and $\nu \neq \mu$ one gets $\left\langle T_{\mu\nu} \right\rangle_t = 0$. The possible nonzero components $\left\langle T_{\mu\nu} \right\rangle_t$ correspond to $\mu, \nu = p+1, ..., D$. In order to use the zeta function, we note that the following relation takes place

$$\frac{k_\nu k_\mu}{\varepsilon_{\mathbf{k}}} = \frac{L_\mu L_\nu}{3(2\pi)^2} \partial_{\alpha_\mu} \partial_{\alpha_\nu} \varepsilon_{\mathbf{k}}^3 . \tag{19}$$

This allows to write the off-diagonal components in the form

$$\left\langle T_{\mu\nu} \right\rangle = -\frac{N L_\mu L_\nu}{6(2\pi)^2} \partial_{\alpha_\mu} \partial_{\alpha_\nu} \frac{1}{V_q} \int \frac{d\mathbf{k}_p}{(2\pi)^p} \sum_{\mathbf{n}_q \in \mathbf{Z}^q} \varepsilon_{\mathbf{k}}^3 = -\frac{N L_\mu L_\nu}{6(2\pi)^2} \partial_{\alpha_\mu} \partial_{\alpha_\nu} \zeta\left(-\frac{3}{2}\right). \tag{20}$$

By using the formula (10), for the topological part we get

$$\left\langle T_{\mu\nu} \right\rangle_t = \frac{m^{D+3} N L_\mu L_\nu}{(2\pi)^{(D+1)/2}} \sum_{\mathbf{n}_q \in \mathbf{Z}^q}{}' n_\mu n_\nu \cos(\mathbf{n}_q \boldsymbol{\alpha}_q) f_{(D+3)/2}(m g(\mathbf{L}_q, \mathbf{n}_q)). \tag{21}$$

Note that in this representation we can make the replacement

$$\cos(\mathbf{n}_q \boldsymbol{\alpha}_q) \to \sin(n_\mu \alpha_\mu) \sin(n_\nu \alpha_\nu) \cos(\mathbf{n}_{q-2} \boldsymbol{\alpha}_{q-2}), \tag{22}$$

where $\mathbf{n}_{q-2} \boldsymbol{\alpha}_{q-2} = \sum_{l \neq \mu, \nu} n_l \alpha_l$. This replacement explicitly shows that the off-diagonal component $\left\langle T_{\mu\nu} \right\rangle_t$, is an even periodic function of the phases $\alpha_l, l \neq \mu, \nu$, with the period equal to 2π, and an odd periodic function of α_μ and α_ν with the same period. Hence, without the loss of generality, we can assume that $|\alpha_\mu| \leq \pi$. For a massless field, by taking into account that $f_\nu(x) = 2^{\nu-1} \Gamma(\nu) x^{-2\nu}$ for $x << 1$, one obtains

$$\left\langle T_{\mu\nu} \right\rangle_t = \frac{N L_\mu L_\nu}{\pi^{(D+1)/2}} \Gamma \frac{D+3}{2} \sum_{\mathbf{n}_q \in \mathbf{Z}^q}{}' n_\mu n_\nu \frac{\cos(\mathbf{n}_q \boldsymbol{\alpha}_q)}{g^{D+3}(\mathbf{L}_q, \mathbf{n}_q)}. \tag{23}$$

An equivalent expression for the off-diagonal stresses is obtained by using the representation (12) for the zeta function in (20). The first term in the right-hand side of (12) does not depend on α_μ and α_ν and, hence, it does not contribute to the stress. The following expression is obtained:

$$\left\langle T_{\mu\nu}\right\rangle_t = -\frac{NL_\mu^2 L_\nu^2}{(2\pi)^{(p+3)/2} V_q} \sum_{\mathbf{n}_q \in \mathbf{Z}^q}{}' n_\mu n_\nu \sin(n_\mu \alpha_\mu) \times \sin(n_\nu \alpha_\nu) \varepsilon_{\mathbf{n}_{q-2}}^{p+5} f_{(p+5)/2}\left(\sqrt{n_\mu^2 L_\mu^2 + n_\nu^2 L_\nu^2}\, \varepsilon_{\mathbf{n}_{q-2}}\right). \quad (24)$$

For $q=2$ $(p=D-2)$ one has $\varepsilon_{\mathbf{n}_{q-2}} = m$ and the formula is reduced to

$$\left\langle T_{\mu\nu}\right\rangle_t = -\frac{Nm^{D+3} L_\mu L_\nu}{(2\pi)^{(D+1)/2}} \sum_{n_\mu, n_\nu=-\infty}^{+\infty}{}' n_\mu n_\nu \sin(n_\mu \alpha_\mu) \sin(n_\nu \alpha_\nu) f_{(D+3)/2}\left(m\sqrt{n_\mu^2 L_\mu^2 + n_\nu^2 L_\nu^2}\right). \quad (25)$$

In the special case under consideration this coincides with (21).

The special case of the results corresponding to $D=2$ describes the properties of the ground state for the electronic subsystem in graphene nanotubes and nanoloops (toroidal nanotubes) described by the effective Dirac model. For nanotubes one has $(p, q) = (1, 1)$ and for nanoloops $(p, q) = (0,2)$. For metallic nanotubes and in the absence of the threading magnetic flux the phase along the periodic condition is zero, $\alpha_l = 0$. Depending on the chiral vector in semiconductor nanotubes two values of the phases are realized with $\alpha_l = \pi/3$ and $\alpha_l = 2\pi/3$. The corresponding analysis for the diagonal components of the ground state energy-momentum tensor can be found in [24]. The off-diagonal component for nanoloops is obtained from (25) with $D=2$ and $N=2$. In this special case one has $f_{5/2} = \sqrt{\pi/2}\, x^{-5} e^{-x}\left(3+3x+x^2\right)$.

4. *Asymptotic analysis and numerical results*. Let us consider some asymptotics of general formulae. For large values of the length of the compact dimension z^l, $l \neq \mu,\nu$, $L_l >> L_\mu, L_\nu$, the dominant contribution in (21) comes from the term with $n_l=0$ and the leading order term coincides with the off-diagonal stress in the model where the coordinate z^l is decompactified. In the opposite limit $L_l << L_\mu, L_\nu$, it is more convenient to use the the representation (24). The behavior of the stress is essentially different depending on whether the phase α_l is zero or not. For $\alpha_l = 0$ the main contribution to the VEV $\left\langle T_{\mu\nu}\right\rangle_t$ comes from the modes with $n_l=0$. To the leading order we get

$$\left\langle T_{\mu\nu}\right\rangle_t \approx \frac{N\left\langle T_{\mu\nu}\right\rangle_t^{\left(M^{p+1}\times T^{q-1}\right)}}{N_{D-1} L_l}, \quad (26)$$

where $N_{D-1} = 2^{[D/2]}$ and $\left\langle T_{\mu\nu}\right\rangle_t^{\left(M^{p+1}\times T^{q-1}\right)}$ is the corresponding VEV in D-dimensional spacetime with topology $M^{p+1}\times T^{q-1}$ which is obtained from the geometry described by (1) excluding the compact dimension x^l. For $\alpha_l \neq 0$ and assuming that $|\alpha_l| < \pi$, again, the dominant contribution give the modes n_l=0. The argument of the function $f_{(p+5)/2}(x)$ is large and we can use the corresponding asymptotic of the modified Bessel function. This shows that in the limit under consideration the off-diagonal stress $\langle T_{\mu\nu}\rangle_t$ is suppressed by the factor $\exp\left(-|\alpha_l|\sqrt{L_\mu^2+L_\nu^2}/L_l\right)$.

For large values of the lengths L_μ and L_ν compared to the other length scales $1/m$ and L_l, $l \neq \mu,\nu$, by using (24) we can see that the topological contribution $\langle T_{\mu\nu}\rangle_t$ is exponentially suppressed by the factor $\exp\left(-\varepsilon_{(0)}\sqrt{L_\mu^2+L_\nu^2}\right)$, where $\varepsilon_{(0)} = \varepsilon_{\mathbf{n}_{q-2}}\Big|_{\mathbf{n}_{q-2}=0}$ and $0<|\alpha_l|<\pi$. For small values of L_μ and L_ν the dominant contribution in (24) comes from large values of $|n_l|$ and we can replace the corresponding summations by the integration in accordance with

$$\frac{L_\mu L_\nu}{V_q}\sum_{\mathbf{n}_{q-2}\in\mathbf{Z}^{q-2}}{}' \varepsilon_{\mathbf{n}_{q-2}}^{p+5} f_{(p+5)/2}\left(b\varepsilon_{\mathbf{n}_{q-2}}\right) \to \int \frac{d^{q-2}\mathbf{y}}{(2\pi)^{q-2}}\left(\mathbf{y}^2+m^2\right)^{(p+5)/2} f_{(p+5)/2}\left(b\sqrt{\mathbf{y}^2+m^2}\right), \quad (27)$$

where $b=\sqrt{n_\mu^2L_\mu^2+n_\nu^2L_\nu^2}$ and $\mathbf{y}=(y_1,\ldots,y_{q-2})$ with $-\infty<y_l<+\infty$. After integration over the angular part, the integral over $|\mathbf{y}|$ is evaluated by using the formula from [28]. In this way it can be seen that the leading term in the expansion of $\langle T_{\mu\nu}\rangle_t$ coincides with (25). Additionally assuming that $m\sqrt{L_\mu^2+L_\nu^2} \ll 1$, in the leading approximation we get

$$\langle T_{\mu\nu}\rangle_t \approx -\frac{NL_\mu L_\nu}{\pi^{(D+1)/2}}\Gamma\frac{D+3}{2}\sum_{n_\mu,n_\nu=-\infty}^{+\infty}{}' n_\mu n_\nu \frac{\sin(n_\mu\alpha_\mu)\sin(n_\nu\alpha_\nu)}{\left(n_\mu^2L_\mu^2+n_\nu^2L_\nu^2\right)^{(D+3)/2}} \quad (28)$$

Note that the right-hand side presents the off-diagonal component of the vacuum energy-momentum tensor for a massless fermionic field in the model $(p,\ q)=(D-2,\ 2)$ with compact dimensions x^μ and x^ν.

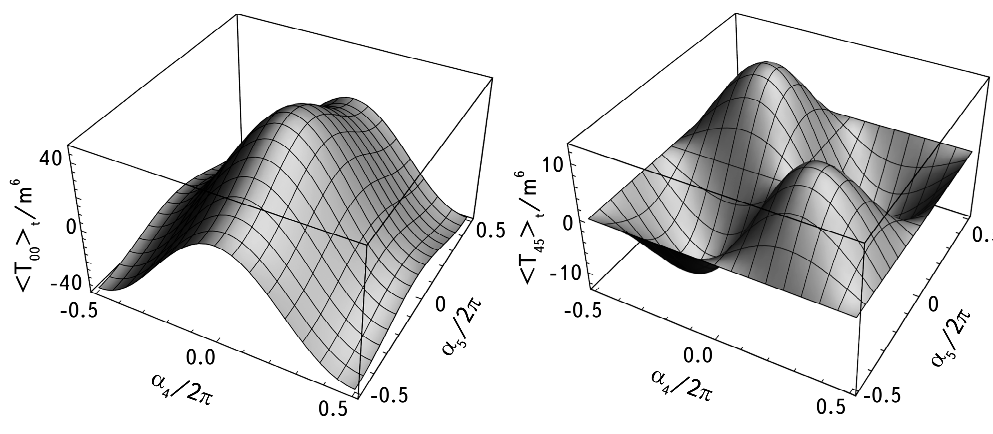

Fig.1. The expectation values of the vacuum energy density and off-diagonal stress on the phases of the periodicity conditions in the model $(p,\ q)=(3,\ 2)$ with $mL_4=0.5$, $mL_5=0.6$.

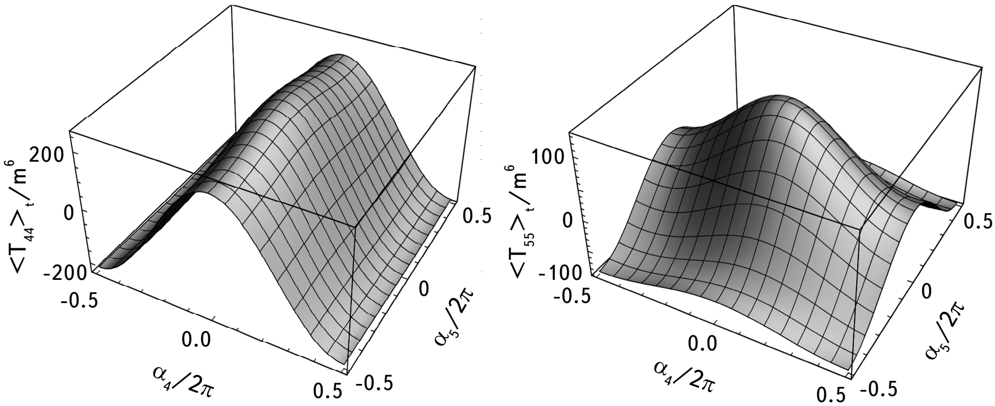

Fig.2. The same as in Fig.1 for the stresses along the compact dimensions x^4 (left panel) and x^5 (right panel).

We will present the numerical analysis for the $D=5$ model with two compact dimensions x^4 and x^5. This corresponds to the set $(p, q)=(3, 2)$. By taking into account that in [24] the numerical results for the energy density and diagonal stresses are given for the model $D=4$ with a single compact dimension, the analysis will be given for those quantities as well. We start from the dependence of the expectation values on the phases α_4 and α_5. Fig.1 presents that dependence of the energy density (left panel) and off-diagonal stress $\langle T_{45}\rangle_t$ (right panel) for $mL_4=0.5$ and $mL_6=0.6$.

The corresponding results for the diagonal stresses along compact dimensions are given in Fig.2. As already mentioned above, the energy density and the diagonal stresses are even periodic functions of α_4 and α_5, whereas the off-

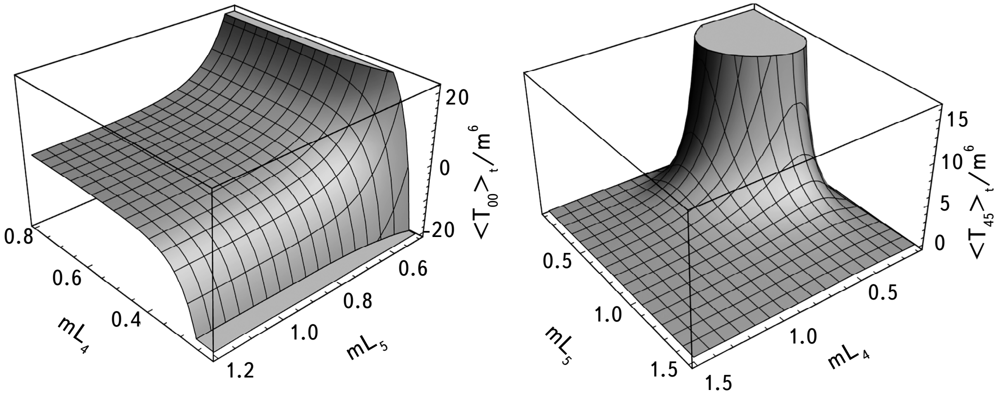

Fig.3. The expectation values of the vacuum energy density and off-diagonal stress on the length of compact dimensions in the model $(p, q)=(3, 2)$. For the left panel we have taken $\alpha_4=\pi/2$, $\alpha_5=0$ and for the right panel $\alpha_4=\pi/2$, $\alpha_5=-0.6\pi$.

diagonal component is an odd periodic function of those phases. Depending on the specific values of the phases, all the components can be either positive or negative. The energy density $\langle T_{00}\rangle_t$ and the vacuum pressures $\langle T_{44}\rangle_t$ and $\langle T_{55}\rangle_t$ are positive for the values of the phases near $(\alpha_4, \alpha_5) = (0, 0)$ (periodic conditions) and negative near $(\alpha_4, \alpha_5) = (\pi, \pi)$ (antiperiodic conditions).

The dependence of the VEVs of the components for the energy-momentum tensor on the lengths of compact dimensions is presented in Fig.3 (energy density and off-diagonal component) and Fig.4 (stresses along compact dimensions). For

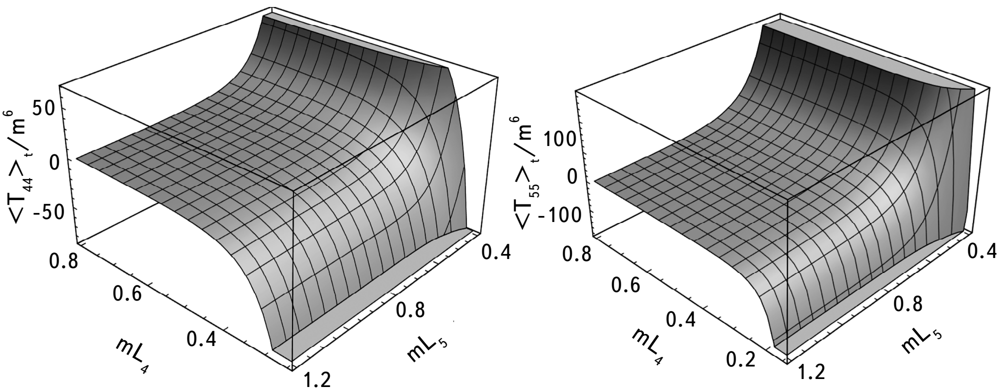

Fig.4. The vacuum stresses along the compact dimensions x^4 (left panel) and x^5 (right panel) versus the lengths of those dimensions. The graphs are plotted for $\alpha_4/2\pi = 0.25$, $\alpha_5/2\pi = 0$.

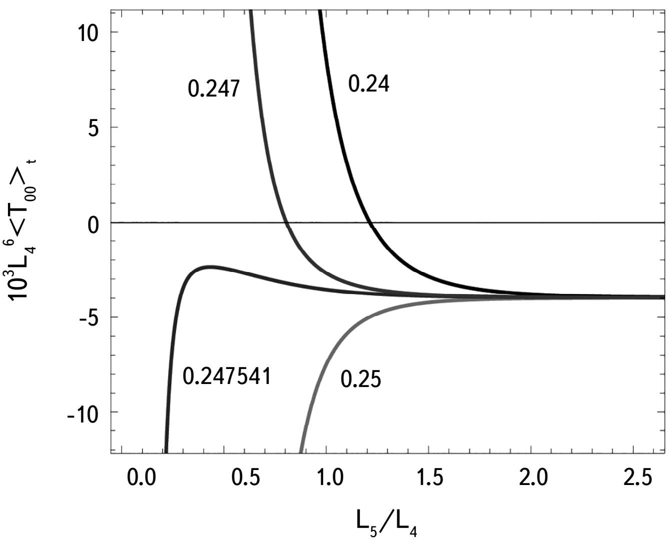

Fig.5. The energy density for a massless fermionic field in the model $(p, q) = (3, 2)$ as a function of the ratio L_5/L_4 for fixed value $\alpha_4 = \pi/2$. The numbers near the graphs are the values of $\alpha_5/2\pi$.

the diagonal components we have taken the phases $\alpha_4 = \pi/2$ and $\alpha_5 = 0$ and the off-diagonal component is plotted for $\alpha_4 = \pi/2$, $\alpha_5 = -0.6\pi$.

From the given graphs, one can get the impression that the energy density is a monotonic function of the lengths of the compact dimensions. However, this is not the case even for a massless field. In order to demonstrate that and by taking into account that the VEVs for a massless field approximate the results for massive fields in the limit of small values of the lengths of compact dimensions, in Fig.5 we have plotted the dimensionless quantity $L_4^6 \langle T_{00} \rangle_t$ as a function of the ratio L_5/L_4. The corresponding expression is given by the right-hand side of (28). The graphs are plotted for $\alpha_4 = \pi/2$ and the numbers near the curves are the values of the ratio $\alpha_5/2\pi$. For large values of L_5/L_4 all the curves tend to the corresponding result for the energy density in the model where the direction x^5 is decompactified $L_5 \to \infty$.

5. *Conclusions*. Continuing the investigations started in [24] we have studied the effects of nontrivial topology on the local characteristics of the fermionic vacuum. A toroidal compactification of a part of spatial dimensions in $(D+1)$-dimensional flat spacetime is considered. In addition to the diagonal components, studied in [24], the vacuum energy-momentum tensor has an off-diagonal components having indices along compact dimensions. Those components vanish for periodic ($\alpha_l = 0$) and antiperiodic ($\alpha_l = \pi$) conditions. In the first case the vacuum energy-momentum tensor for a fermionic field obeys the strong energy condition. For general values of the phases that is not the case. The phases in the periodicity conditions can be interpreted in terms of magnetic fluxes enclosed by compact dimensions. The VEVs are periodic functions of magnetic fluxes with the period of flux quantum. The diagonal components are even functions of the phases α_l. The off-diagonal component $\langle T_{\mu\nu} \rangle_t$, $\mu \neq \nu$, $\mu, \nu = p+1, ..., D$, is an even function of α_l with $l \neq \mu, \nu$, and odd function of the phases α_μ and α_ν. The vacuum stresses in the uncompact subspace are isotropic and the corresponding equation of state is of the cosmological constant type. Depending on the values of the phases the components of the vacuum energy-momentum tensor can be either positive or negative. For small values of the lengths L_μ and L_ν, the off-diagonal component is approximated by the corresponding result for a massless field in the model with $q=2$ and compact subspace (x^μ, x^ν) (see (28)). The numerical analysis of the obtained results is presented for the $D=5$ with $(p, q) = (3, 2)$.

We have considered the effects of the nontrivial topology on the local properties of the fermionic vacuum. In the presence of boundaries additional contributions are induced in the VEVs of physical observables (the boundary-induced Casimir effect). The effects of two planar boundaries with the bag boundary conditions on the Dirac field in the geometry under consideration have been discussed in [29,30]. The results in the special case of 2-dimensional space are applied to finite length carbon nanotubes. The fermionic condensate and the VEV of the energy-momentum tensor in toroidally compactified de Sitter spacetime are studied in [31].

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ФЕРМИОННЫЕ ВАКУУМНЫЕ НАТЯЖЕНИЯ В МОДЕЛЯХ С ТОРОИДАЛЬНО КОМПАКТНЫМИ ИЗМЕРЕНИЯМИ

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Исследовано вакуумное среднее тензора энергии-импульса массивного поля Дирака в плоском пространстве-времени с тороидальным подпространством произвольной размерности. На оператор поля вдоль компактных измерений накладываются условия квазипериодичности с произвольными фазами. Эти фазы интерпретируются в терминах магнитных потоков, пронизывающих компактные измерения. Уравнение состояния в некомпактном подпространстве имеет тип космологической постоянной. Показано, что вакуумный тензор энергии-импульса помимо диагональных компонентов содержит ненулевые недиагональные компоненты. В частных случаях скрученных (антипериодических) и нескрученных (периодических) полей недиагональные компоненты обращаются в нуль. Для нескрученных полей плотность энергии вакуума положительна, а тензор энергии-импульса удовлетворяет сильному энергетическому условию. При общих значениях фаз в условиях периодичности плотность энергии и натяжения могут быть как положительными, так и отрицательными. Численные результаты приведены для модели типа Калуцы-Клейна с двумя дополнительными измерениями.

Ключевые слова: *топологический эффект Казимира: поле Дирака: тороидальная компактификация*

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