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VISCOUS PLANE SYMMETRIC STRING COSMOLOGICAL MODEL IN $f(R)$ GRAVITY

A.DABRE[1], PALLAVI MAKODE[2]

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In this paper, we have used the Plane Symmetric LRS Bianchi type I metric to study bulk viscous fluid coupled to a string of clouds within the framework of the $f(R)$ theory of gravity. To obtain the deterministic solutions, some physically plausible conditions like the weak field limit for a point-like source $f(R)=R^{3/2}$, the very well known expansion-shear scalar proportionality relation and the special form of an average scale factor are taken into account. Furthermore, we have calculated some physical and kinematical parameters along with the energy conditions to study the astrophysical implications of the constructed model and discussed their graphical behaviour, which shows good similarity with recent observational data.

Keywords: *Bianchi type I metric: bulk viscous fluid: cosmic string: energy conditions: $f(R)$ gravity*

1. *Introduction.* To explain the phase transition of the universe theoretically from decelerating to accelerating phase, generally, two approaches are being adopted: one of these methods is to investigate various dark energy candidates playing major roles due to the modification of energy-momentum tensor in the Einstein field equations, and the other is to modify the space-time geometry in the Einstein's equation called modified gravity. Many models of the universe have been introduced to study and explain the accelerating expansion of the universe. In recent years, several probable candidates for dark energy have been proposed and accordingly, the cosmological models are being constructed. As a second approach to explaining accelerating cosmic expansion, many modified theories of gravity have been developed. The modified theories of gravity, generally known as *f*-theories, are those where the gravitational action integral is defined to be a function *f* of specific geometric invariants. The main characteristic of *f*-theories is that the modified field equations explain the cosmological evolution and also recover general relativity.

In recent years, there has been a significant interest in modified theories of gravity. Among various gravitational theories that have been amended, one of the earliest is $f(R)$ theory of gravity, which was first put out by Buchdahl [1].

Numerous scholars have examined and investigated the numerous kinds of cosmological models in $f(R)$ gravity. Baibosunov et al. [2] discussed model of the early universe in $f(R)$ gravity. Nojiri et al. [3] studied unifying inflation with early and late-time dark energy in $f(R)$ gravity. Again Nojiri, Odintsov [4] discussed unified cosmic history in modified gravity from $f(R)$ theory to Lorentz non-invariant models. Dynamics of some cosmological solutions in modified $f(R)$ gravity were examined by Malik, Shamir [5]. Pawar et al. [6] investigated plane symmetric string cosmological model with zero mass scalar field in $f(R)$ gravity. Cognola et al. [7] investigated the class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion. Nojiri, Odintsov [8] studied unifying inflation with ΛCDM epoch in modified gravity consistent with solar system tests. Again modified $f(R)$ gravity unifying R^m inflation with ΛCDM epoch were examined by Nojiri, Odintsov [9].

We have known with the fact that bulk viscosity as well as cosmic strings performs a significant role in cosmology and presents cosmic accelerated expansion popularly known as the inflationary phase. The theoretical development of the universe and the effects of bulk viscosity and/or cosmic strings on cosmic evolution have been examined by numerous cosmologists using the source as a bulk viscous fluid and/or string cloud fluid. Recently, Pawar, Dabre [10,11] studied bulk viscous string cosmological models using power-law volumetric expansion of the universe and constant deceleration parameters in teleparallel gravity. Again, Pawar et al. [12-14] have conducted several string-inspired cosmological investigations. Considering the Kantowski-Sachs metric Reddy et al., [15] have constructed an isotropic bulk viscous string cosmological model showing the special case for the non-validating cosmic strings. Hegazy [16] developed the formula for calculating cosmic entropy in terms of viscosity and tried it to examine the entropy, enthalpy, Gibbs energy, and Helmholtz energy of the constructed model in the presence of viscosity. Nojiri et al. [17] studied string-inspired models, inflation, bounce, and late-time evolution in reference to modified gravity. Freidel et al. [18] discussed the formulation and dynamics of string theory and looked for string solutions. Mishra et al. [19] investigated the string cosmological model using spatially homogeneous and anisotropic Bianchi type V space-time. The viscous string cosmological model explaining the cosmic accelerated expansion has been investigated by Vinutha et al. [20]. Darabi et al. [21] obtained string cosmological solutions via Hojman symmetry using FRW line element. Some recent and important investigations of bulk viscous fluid in the presence of cloud strings have been obtained by several cosmologists [22-28] in different contexts.

Motivated by the situations cited above in this paper, we have considered plane-symmetric LRS Bianchi type I metric to construct a bulk viscous string cosmological model within the context of $f(R)$ gravity. This paper is divided into several sections: Sec. 2 presents $f(R)$ gravity formalism in brief. In Sec. 3 considering Bianchi type I metric, we have obtained the corresponding field equations. In Sec. 4, we obtained the exact solution of highly non-linear differential field equations along with the different physical and kinematical parameters including energy conditions and discussed them with graphs. Lastly, in Sec. 5, we have concluded the investigations.

2. *f(R) gravity formalism*. The $f(R)$ theory of gravitation is a modification of the general theory of relativity. The action for $f(R)$ gravity is given by

$$S = \int \sqrt{-g}\left(f(R) + L_m\right) d^4x , \tag{1}$$

where $f(R)$ is a general function of Ricci Scalar R and L_m is the matter Lagrangian. It is worth mentioning that the standard Einstein-Hilbert action can be recovered when $f(R) = R$.

The corresponding field equations are obtained by varying the action with respect to the metric $g_{\mu\nu}$ as

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \nabla^\mu \nabla_\mu F(R) = T_{\mu\nu} , \tag{2}$$

where $F(R) \equiv df(R)/dR$, ∇_μ denotes covariant differentiation, $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the matter Lagrangian L_m.

3. *Metric and field equations*. We consider the plane-symmetric LRS Bianchi type I metric of the form as

$$ds^2 = dt^2 - A^2 dx^2 - B^2\left(dy^2 + dz^2\right), \tag{3}$$

where A and B are metric potential functions of cosmic time t only.

We consider the source as bulk viscous fluid containing one-dimensional cosmic string given by

$$T_\mu^\nu = (\rho + \bar{p}) u_\mu u^\nu + \bar{p} g_\mu^\nu - \lambda x_\mu x^\nu , \tag{4}$$

$$\bar{p} = p - 3\xi H , \tag{5}$$

where $\rho = \rho_p + \lambda$ is the proper energy density with particles attached to them and λ is the strings tension density, ρ_p is the particle energy density, $3\xi H$ is bulk viscous pressure, $\xi(t)$ is the coefficient of bulk viscosity, H is Hubble's parameter, x^ν denotes a unit space-like vector for the cloud string and u^ν denotes four-velocity vector satisfying the conditions, $u^\nu u_\nu = -1 = -x^\nu x_\nu$ and $u_\nu x^\nu = 0$.

In a co-moving coordinate system, we have

$$u^\nu = (0, 0, 0, 1), \quad x^\nu = \left(A^{-1}, 0, 0, 0\right). \tag{6}$$

Taking consideration of (4) in the field equations (2) for the metric (3) we obtain,

$$\ddot{F}+2\left(\frac{\dot{B}}{B}\right)\dot{F}+\left(\frac{\ddot{A}}{A}+2\frac{\dot{A}\dot{B}}{AB}\right)F-\frac{1}{2}f=p-3\xi H-\lambda, \tag{7}$$

$$\ddot{F}+\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)\dot{F}+\left(\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}+\frac{\dot{A}\dot{B}}{AB}\right)F-\frac{1}{2}f=p-3\xi H, \tag{8}$$

$$\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)\dot{F}+\left(\frac{\ddot{A}}{A}+2\frac{\ddot{B}}{B}\right)F-\frac{1}{2}f=-\rho, \tag{9}$$

where the overhead dot $(\cdot)$ denotes the derivative with respect to cosmic time *t*.

Here we have three non-linear differential field equations with seven unknowns, namely; *f*, *A*, *B*, ρ, λ, *p* and ξ. The solution of these unknowns is discussed in the next section.

Also, we define some kinematical space-time quantities of physical interest in cosmology, as follows.

The average scale factor *a* and the spatial volume *V* are respectively defined as

$$a=\sqrt[3]{AB^2} \quad \text{and} \quad V=a^3. \tag{10}$$

The volumetric cosmic expansion rate is described by the generalized mean Hubble's parameter *H* given by

$$H=\frac{1}{3}\sum_{i=1}^{3}H_i=\frac{1}{3}(H_1+H_2+H_3), \tag{11}$$

in which $H_1=\dot{A}/A$, and $H_2=H_3=\dot{B}/B$ denotes the directional Hubble's parameters.

Using (10) and (11), we have obtained the expansion scalar Θ, the mean anisotropy parameter Δ, the shear scalar σ^2, and the deceleration parameter *q* respectively as

$$\Theta=\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}=3H, \tag{12}$$

$$\Delta=\frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i-H}{H}\right)^2, \tag{13}$$

$$\sigma^2=\frac{1}{2}\left(\sum_{i=1}^{3}H_i^2-\Theta^2\right), \tag{14}$$

$$q=-\frac{a\ddot{a}}{\dot{a}^2}=-1+\frac{d}{dt}\frac{1}{H}. \tag{15}$$

4. *Solution of field equations*. To solve the non-linear differential field Eqs. (7)-(9) completely in order to obtain exact solutions, we consider some physically plausible conditions.

Firstly, we consider the weak field limit for a point-like source of $f(R)$ gravity formalism given by

$$f(R) = R^{3/2} . \tag{16}$$

Using the weak field limit for a point-like source of $f(R)$ gravity model Capozziello et al., [29] briefly reviewed and tested an exact $f(R)$ gravity model at Galactic and local scales. Bisabr [30] investigated local gravity constraints and power law $f(R)$ theories in which he obtained $f(R) = \alpha R^{1+1/n}$ power law gravity model. Lazarov et al. [31] presented the calculations of orbits and periods in R^n gravity in their research geodesic equations in the weak field limit of general $f(R)$ gravity theory. Again Capozziello et al., [32] assumed R^m model and discussed the energy conditions in $f(R)$ gravity.

For the deterministic solutions, we consider the expansion scalar Θ is proportional to the shear scalar σ^2 which leads to the following analytic relation

$$A = B^{\gamma} , \tag{17}$$

where γ is a constant.

Finally, we consider the special form of an average scale factor in the form

$$a(t) = \left(t^2 + \frac{\delta}{\mu} \right)^{1/2\mu} , \tag{18}$$

where δ and μ are constant.

We obtained the metric coefficients A and B as

$$A = \left(t^2 + \frac{\delta}{\mu} \right)^{3\gamma/2\mu(\gamma+2)} \quad \text{and} \quad B = \left(t^2 + \frac{\delta}{\mu} \right)^{3/2\mu(\gamma+2)} . \tag{19}$$

Substituting values of A and B from (19) in (3), we get

$$ds^2 = dt^2 - \left(\frac{\mu t^2 + \delta}{\mu} \right)^{3\gamma/\mu(\gamma+2)} dx^2 - \left(\frac{\mu t^2 + \delta}{\mu} \right)^{3/\mu(\gamma+2)} \left(dy^2 + dz^2 \right). \tag{20}$$

The metric potential functions of the derived model in (20) are constant for any type of finite *t*, and hence it is free from any type of singularity in cosmic evolution until $t \to \infty$.

The spatial volume V becomes

$$V = \left(t^2 + \frac{\delta}{\mu} \right)^{3/2\mu} . \tag{21}$$

The spatial volume V of the universe is a time-dependent function, and its graphical behaviour has been shown in Fig.1, which depicts the exponential expansion of the universe according to the increasing time *t*.

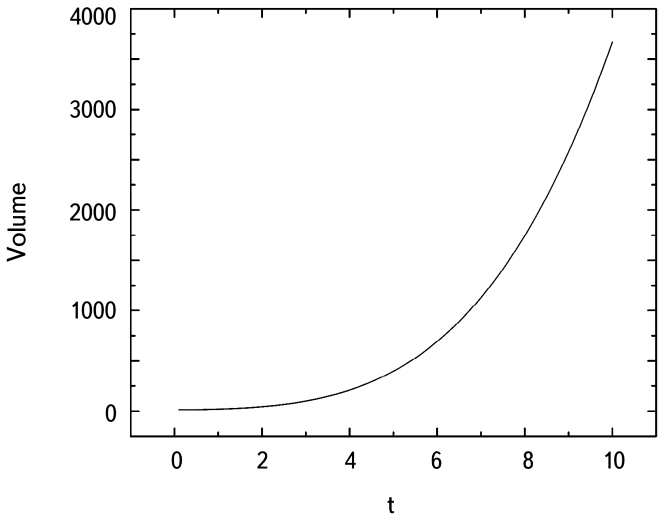

Fig.1. Variation of volume V vs. time t for $\delta = 4$, $\mu = 0.85$.

The mean Hubble's parameter H and the expansion scalar Θ are obtained as

$$H = \frac{t}{\mu t^2 + \delta}, \tag{22}$$

$$\Theta = \frac{3t}{\mu t^2 + \delta}. \tag{23}$$

The graphical behaviour of mean Hubble's parameter and expansion scalar vs.

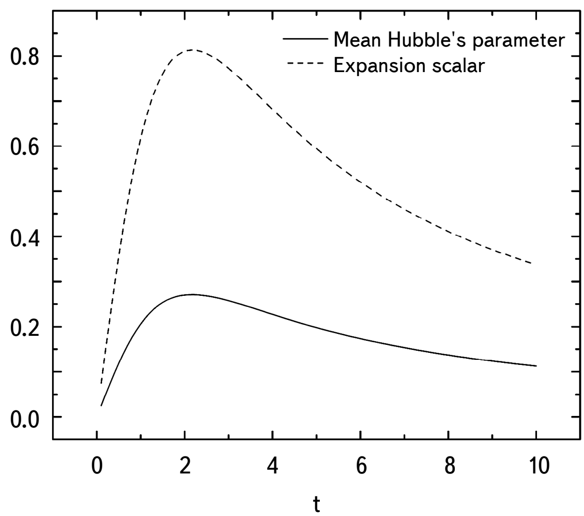

Fig.2. Variation of mean Hubble's parameter H and expansion scalar Θ vs. time t for $\delta = 4$, $\mu = 0.85$.

cosmic time t is shown in Fig.2, where it can be seen that both parameters steadily decline with increasing time after first growing to a minor amount. It demonstrates how the cosmos expands faster at first before slowing down as time passes.

The mean anisotropy parameter Δ and the shear scalar σ^2 are obtained as

$$\Delta = \frac{2(\gamma-1)^2}{(\gamma+2)^2}, \tag{24}$$

$$\sigma^2 = \frac{3t^2(\gamma-1)^2}{(\gamma+2)^2(\mu t^2+\delta)^2}. \tag{25}$$

The ratio of shear scalar to expansion scalar turns out to be

$$\frac{\sigma^2}{\Theta^2} = \frac{(\gamma-1)^2}{3(\gamma+2)^2}. \tag{26}$$

From the expressions (24)-(26), it is been observed that the mean anisotropy parameter, the shear scalar and its ratio are non-zero throughout the universe's evolution except for $\gamma = 1$ which describes that the cosmos is anisotropic and holds shear.

The deceleration parameter is obtained as

$$q = (\mu-1) - \frac{\delta}{t^2}. \tag{27}$$

We are well known with the fact that the deceleration parameter symbolizes the inflation for $q < 0$, deflation for $q > 0$ and constant rate expansion for $q = 0$. The expression (27) demonstrates the value of the deceleration parameter whose

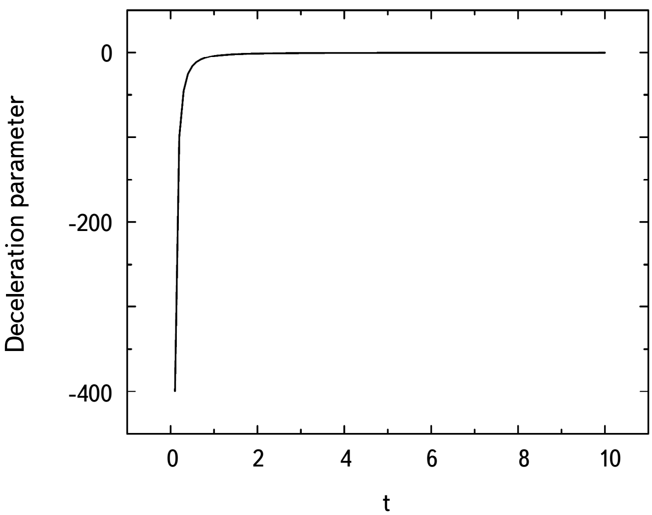

Fig.3. Variation of deceleration parameter q vs. time t for $\delta = 4$, $\mu = 0.85$.

graphical behaviour has been depicted in Fig.3 for the appropriate choice of constants. As can be seen from the figure, the deceleration parameter q rapidly increases in negative during the brief time interval $0 < t < 1$ providing $q < -1$ indicating the super-exponential cosmic expansion leading to the occurrence of Big Rip. However, later on, q develops with time continuously to approach -0.15. It confirms the inflationary cosmic accelerating phase. This observations are supported with [33].

The energy density

$$\rho = -\frac{6\sqrt{6}\left\{\begin{array}{l}\mu^2(\gamma+2)^4 t^4 - \frac{15}{4}\mu(\gamma+2)^2 t^2\left[\frac{11}{15}\delta(\gamma+2)^2 + \frac{t^2}{5}\left(5\gamma^2+4\gamma+12\right)\right] \\ +\frac{1}{4}\delta^2(\gamma+2)^4 + \frac{3}{4}\delta(\gamma+2)^2 t^2\left(5\gamma^2+4\gamma+12\right) + \frac{9}{4}\gamma(\gamma-4)t^4\left(\gamma^2+2\gamma+3\right)\end{array}\right\}}{(\gamma+2)^3\left(\mu t^2+\delta\right)^3\sqrt{\delta(\gamma+2)^2 - \left[\mu(\gamma+2)^2 - 3\gamma(\gamma+2) - 9\right]t^2}}. \tag{28}$$

The energy density is determined to be negative in an initial epoch during $0 < t < 1$, but it thereafter develops in positive gradually to some amount and then drops with increasing time and diminishes when $t \to \infty$ (Fig.4). A similar type of behaviour has been observed by Baro et al. [34].

We assume that the coefficient of viscosity should vary with the expansion scalar in such a way that

$$\xi\theta = \xi_0 = \text{constant}. \tag{29}$$

From (29) we have obtained the coefficient of bulk viscosity as

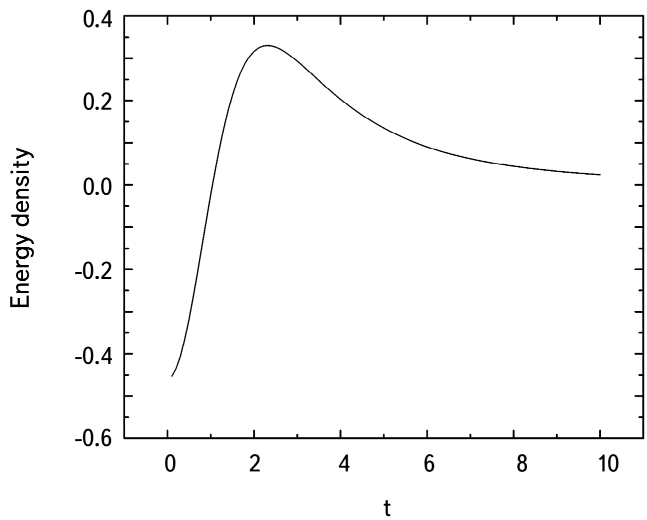

Fig.4. Variation of energy density vs. time t for $\delta = 4$, $\gamma = 0.9$ and $\mu = 0.85$.

$$\xi = \frac{\xi_0\left(\mu t^2+\delta\right)}{3t}. \tag{30}$$

The pressure can be obtained as

$$p=-\frac{\left\{\begin{array}{l}\xi_0(\gamma+2)^3\left(\mu t^2+\delta\right)^3\left\{\left[\begin{array}{l}(\mu-3)\gamma^2\\+2(2\mu-3)\gamma\\+4\mu-9\end{array}\right]t^2-\delta(\gamma+2)^2\right\}\sqrt{\begin{array}{l}\delta(\gamma+2)^2\\-\left[\mu(\gamma+2)^2-3\gamma(\gamma+2)-9\right]t^2\end{array}}\\-3\sqrt{6}\left\{\begin{array}{l}\frac{1}{2}[(2\mu+3)\gamma+4\mu][(\mu-2)\gamma+2\mu-1]\left[(\mu-3)\gamma^2+2(2\mu-3)\gamma+4\mu-9\right]^2t^6\\-\frac{3}{2}\delta\left[\begin{array}{l}\left(5\mu^2-8\mu-3\right)\gamma^2\\+\mu(20\mu-19)\gamma\\+20\mu^2-24\mu+3\end{array}\right]\left[(\mu-3)\gamma^2+2(2\mu-3)\gamma+4\mu-9\right](\gamma+2)^2t^4\\+\frac{6}{4}\delta^2(\gamma+2)^3t^2\left[\begin{array}{l}\left(4\mu^2-7\mu-3\right)\gamma^3+3\left(8\mu^2-10\mu-2\right)\gamma^2\\+\left(48\mu^2-52\mu-3\right)\gamma+4\mu(8\mu-11)+3\end{array}\right]\\-\frac{1}{2}\delta^3(\gamma+2)^4\left[(3\mu-1)\gamma^2+(12\mu-1)\gamma+12\mu-7\right]\end{array}\right\}\end{array}\right\}}{(\gamma+2)^3\left(\mu t^2+\delta\right)^3\left\{\delta(\gamma+2)^2-\left[\mu(\gamma+2)^2-3\gamma(\gamma+2)-9\right]t^2\right\}^{3/2}} \tag{31}$$

Fig.5 depicts the graph of the coefficient of bulk viscosity, which rapidly decreases in the beginning but then gradually begins to increase, whereas the pressure of the universe increases asymptotically in positive and approaches constant with the time elevation (Fig.6).

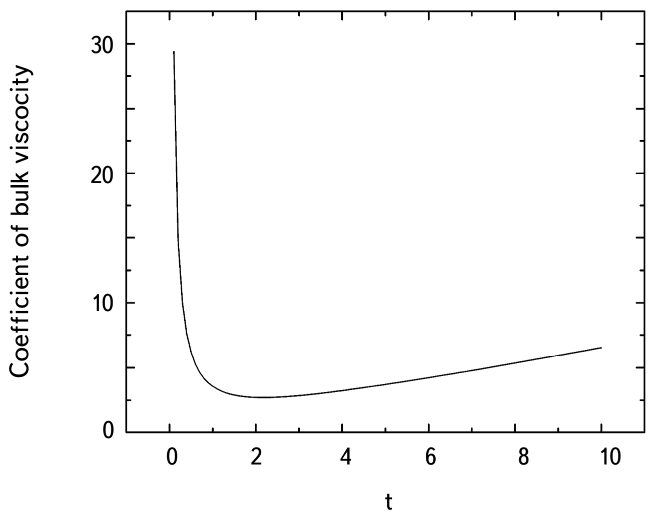

Fig.5. Variation of coefficient of bulk viscosity vs. time t for $\delta = 4$, $\xi_0 = 2.2$ and $\mu = 0.85$.

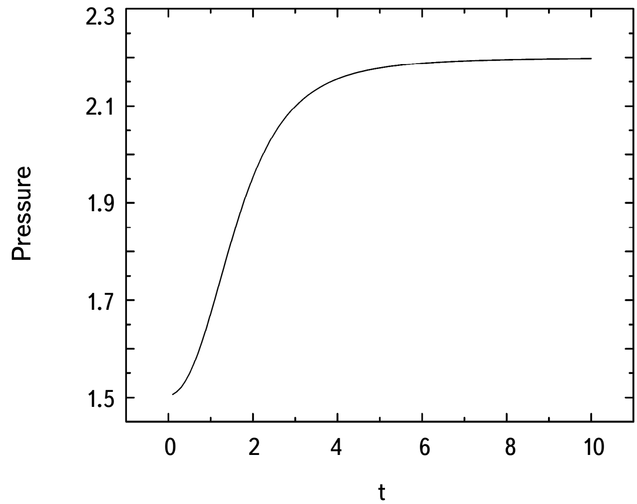

Fig.6. Variation of pressure vs. time t for $\delta = 4$, $\gamma = 0.9$, $\xi_0 = 2.2$ and $\mu = 0.85$.

The string's tension density can be obtained as

$$\lambda = \frac{27\sqrt{6}(\gamma-1)\left\{\begin{array}{l}\left[(\mu-3)\gamma^2 + 2(2\mu-3)\gamma + 4\mu - 9\right]t^4 \\ -\frac{1}{3}\delta(\mu+3)(\gamma+2)^2 t^2 - \frac{1}{3}\delta^2(\gamma+2)^2\end{array}\right\}}{2(\gamma+2)^2\left(\mu t^2+\delta\right)^3\sqrt{\delta(\gamma+2)^2 - \left[\mu(\gamma+2)^2 - 3\gamma(\gamma+2) - 9\right]t^2}}. \tag{32}$$

The particle density turn out to be

$$\rho_p = -\frac{6\sqrt{6}\left\{\begin{array}{l}\frac{1}{4}\left[2(2\mu+3)\gamma^2 + (16\mu+21)\gamma + 16\mu - 18\right]\left[(\mu-3)\gamma^2 + 2(2\mu-3)\gamma + 4\mu - 9\right]t^4 \\ -\frac{1}{4}\delta(\gamma+2)^2\left[2(7\mu-3)\gamma^2 + (47\mu-3)\gamma + 38\mu - 54\right]t^2 - \frac{1}{4}\delta^2(\gamma+2)^3(2\gamma-5)\end{array}\right\}}{(\gamma+2)^3\left(\mu t^2+\delta\right)^3\sqrt{\delta(\gamma+2)^2 - \left[\mu(\gamma+2)^2 - 3\gamma(\gamma+2) - 9\right]t^2}}. \tag{33}$$

A positive value of string tension density shows the existence of the universe's string phase, whereas a negative value of λ suggests the disappearance of the universe's string phase, meaning that the universe is dominated by the cosmological constant [35]. Fig.7 presents the graphical representation of particle and tension density vs. time. As can be seen from Fig.7 the tension density is entirely positive and gradually lowers and diminishes, so our constructed model supports the existence of the universe's string phase. However, the particle density rapidly switches from negative to positive in the beginning to some amount and subsequently decreases and diminishes with infinite time. This leads to the observation

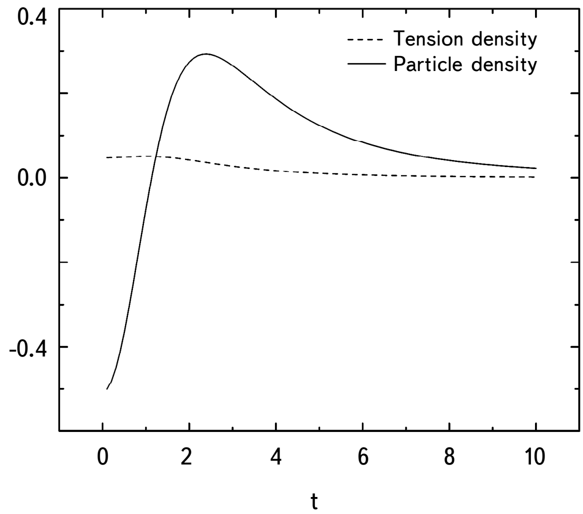

Fig.7. Variation of particle and tension density vs. time t for $\delta = 4$, $\gamma = 0.9$, $\xi_0 = 2.2$ and $\mu = 0.85$.

that the cosmos first appears to be string-dominated, but that later on, the string-dominated phase vanishes and the particle-dominated era begins and rests to end.

The energy conditions:

Energy conditions are mathematically enforced boundaries that strive to claim positive energy. There are several energy conditions associated with physical situations; the most prevalent one is the impact of DE which defies the strong energy condition. The characteristics shared by all non-gravitational fields and states of matter are described by the energy conditions. It eliminates Einstein's field equations' irrational solutions. In this section, we discuss the energy conditions that are respectively defined by Capozziello et al., [32] in which the authors have discussed the role of energy conditions in $f(R)$ cosmology.

The energy conditions are given by

(i) Null Energy Condition (NEC) $\rho + p \geq 0$

(ii) Weak Energy Condition (WEC) $\rho \geq 0$ and $\rho + p \geq 0$

(iii) Strong Energy Condition (SEC) $\rho + 3p \geq 0$ and $\rho + p \geq 0$

(iv) Dominant Energy Condition (DEC) $\rho \geq 0$ and $\rho \pm p \geq 0$.

From (28) and (31) we can express the conditions as

$$\rho+p=-\frac{\left\{\begin{array}{l}\xi_0(\gamma+2)^3\left(\mu t^2+\delta\right)^3\left\{\left[\begin{array}{l}(\mu-3)\gamma^2\\+2(2\mu-3)\gamma\\+4\mu-9\end{array}\right]t^2-\delta(\gamma+2)^2\right\}\sqrt{\begin{array}{l}\delta(\gamma+2)^2\\-\left[\mu(\gamma+2)^2-3\gamma(\gamma+2)-9\right]t^2\end{array}}\\ -3\sqrt{6}\left\{\begin{array}{l}\left[(\gamma+2)^2\mu-3\gamma^2-6\gamma-9\right]^2\left[(\gamma+2)^2\mu^2+\frac{3}{2}(\gamma+2)^2\mu-\frac{9}{2}\gamma^2+\frac{9}{2}\gamma\right]t^6\\ -\frac{3}{2}\delta(\gamma+2)^2t^4\left[(\gamma+2)^2\mu-3\gamma^2-6\gamma-9\right]\left[\begin{array}{l}5(\gamma+2)^2\mu^2-\left(3\gamma^2-\gamma+4\right)\mu\\-9\gamma^2-9\end{array}\right]\\ +6\delta^2(\gamma+2)^3t^2\left\{\begin{array}{l}(\gamma+2)^3\mu^2-\frac{3}{4}(\gamma+2)\left(\gamma^2+2\right)\mu\\-\frac{3}{4}\left[\gamma^2(3\gamma+8)+10\gamma+9\right]\end{array}\right\}\\ -\frac{3}{2}\delta^3(\gamma+2)^4\left[(\gamma+2)^2\mu+\gamma-1\right]\end{array}\right\}\end{array}\right\}}{(\gamma+2)^3\left(\mu t^2+\delta\right)^3\left\{\delta(\gamma+2)^2-\left[\mu(\gamma+2)^2-3\gamma(\gamma+2)-9\right]t^2\right\}^{3/2}}. \quad (34)$$

$$\rho+3p=-\frac{3\left\{\begin{array}{l}\xi_0(\gamma+2)^3\left(\mu t^2+\delta\right)^3\left\{\left[\begin{array}{l}(\mu-3)\gamma^2\\+2(2\mu-3)\gamma\\+4\mu-9\end{array}\right]t^2-\delta(\gamma+2)^2\right\}\sqrt{\begin{array}{l}\delta(\gamma+2)^2\\-\left[\mu(\gamma+2)^2-3\gamma(\gamma+2)-9\right]t^2\end{array}}\\ -3\sqrt{6}\left\{\begin{array}{l}\left[(\gamma+2)^2\mu-3\gamma^2-6\gamma-9\right]^2\left[(\gamma+2)^2\mu^2+\frac{1}{6}(\gamma+2)^2\mu-\frac{7}{2}\gamma^2+\frac{1}{2}\gamma\right]t^6\\ -\frac{1}{2}\delta(\gamma+2)^2t^4\left[(\gamma+2)^2\mu-3\gamma^2-6\gamma-9\right]\\ \times\left[15(\gamma+2)^2\mu^2-\left(19\gamma^2+37\gamma+52\right)\mu-15\gamma^2-3\right]\\ +6\delta^2(\gamma+2)^3t^2\left\{\begin{array}{l}(\gamma+2)^3\mu^2-\left[\gamma^2\left(\frac{17}{12}\gamma+\frac{11}{2}\right)+\frac{19}{2}\gamma+\frac{25}{3}\right]\mu\\-\left[\gamma^2\left(\frac{5}{4}\gamma+3\right)+3\gamma+\frac{7}{4}\right]\end{array}\right\}\\ -\frac{1}{6}\delta^3(\gamma+2)^4\left[(9\mu-2)\gamma^2+(36\mu+1)\gamma+36\mu-17\right]\end{array}\right\}\end{array}\right\}}{(\gamma+2)^3\left(\mu t^2+\delta\right)^3\left\{\delta(\gamma+2)^2-\left[\mu(\gamma+2)^2-3\gamma(\gamma+2)-9\right]t^2\right\}^{3/2}}. \quad (35)$$

$$
\rho-p=\frac{\left\{\begin{array}{l}
\xi_0(\gamma+2)^3\left(\mu t^2+\delta\right)^3\left\{\left[\begin{array}{l}(\mu-3)\gamma^2\\ +2(2\mu-3)\gamma\\ +4\mu-9\end{array}\right]t^2-\delta(\gamma+2)^2\right\}\sqrt{\begin{array}{l}\delta(\gamma+2)^2\\ -\left[\mu(\gamma+2)^2-3\gamma(\gamma+2)-9\right]t^2\end{array}}\\
-3\sqrt{6}\left\{\begin{array}{l}
\left[(\gamma+2)^2\mu-3\gamma^2-6\gamma-9\right]^2\left[(\gamma+2)^2\mu^2-\frac{5}{2}(\gamma+2)^2\mu-\frac{3}{2}\gamma(\gamma+5)\right]t^6\\
-\frac{3}{2}\delta(\gamma+2)^2t^4\left[(\gamma+2)^2\mu-3\gamma(\gamma+2)-9\right]\left[\begin{array}{l}5(\gamma+2)^2\mu^2\\ -(13\gamma(\gamma+3)+44)\mu\\ +3\gamma^2+15\end{array}\right]\\
+6\delta^2(\gamma+2)^3t^2\left\{\begin{array}{l}(\gamma+2)^3\mu^2-\frac{1}{4}\left(\gamma^2(11\gamma+54)+102\gamma+76\right)\mu\\ +\frac{3}{4}\left[\gamma^2(\gamma+4)+8\gamma+11\right]\end{array}\right\}\\
-\frac{3}{2}\delta^3(\gamma+2)^4\left[(\gamma+2)^2\mu-\frac{1}{3}(\gamma(2\gamma+5)+11)\right]
\end{array}\right\}
\end{array}\right\}}{(\gamma+2)^3\left(\mu t^2+\delta\right)^3\left\{\delta(\gamma+2)^2-\left[\mu(\gamma+2)^2-3\gamma(\gamma+2)-9\right]t^2\right\}^{3/2}}. \quad (36)
$$

The graphical behaviour of energy conditions is depicted in Fig.8. From the figure, it is been observed that SEC and NEC are completely satisfied, DEC is completely violated, and because energy density is initially negative i.e. $\rho<0$ for this amount, WEC is violated, but as soon as energy density becomes positive i.e. $\rho\geq 0$, WEC is perfectly satisfied. In the cosmological context, recent research highlights the significance of violating the DEC (i.e. violation of the Phantom Matter field). Naturally, the violation of DEC raises the concerns about causality and stability of the system. While the DEC satisfying matter guarantees the system's causality and stability, but violation of DEC does not always mean that the system is unstable or violates causality [36].

5. *Concluding remarks*. In this paper, we have investigated the plane symmetric LRS Bianchi type I metric in the presence of bulk viscous fluid coupled to a string of cloud fluid in the context of $f(R)$ gravity. The exact solutions to the field equations have been obtained by using some physically plausible situations like the weak field limit for a point-like source $f(R)=R^{3/2}$, the very well known expansion-shear scalar proportionality relation and the special form of an average scale factor. A discussion of the energy conditions and the values of several geometrical and physical parameters has been conducted utilising their time-varying graphs.

It is been observed that the constructed cosmos is singularity-free for any type

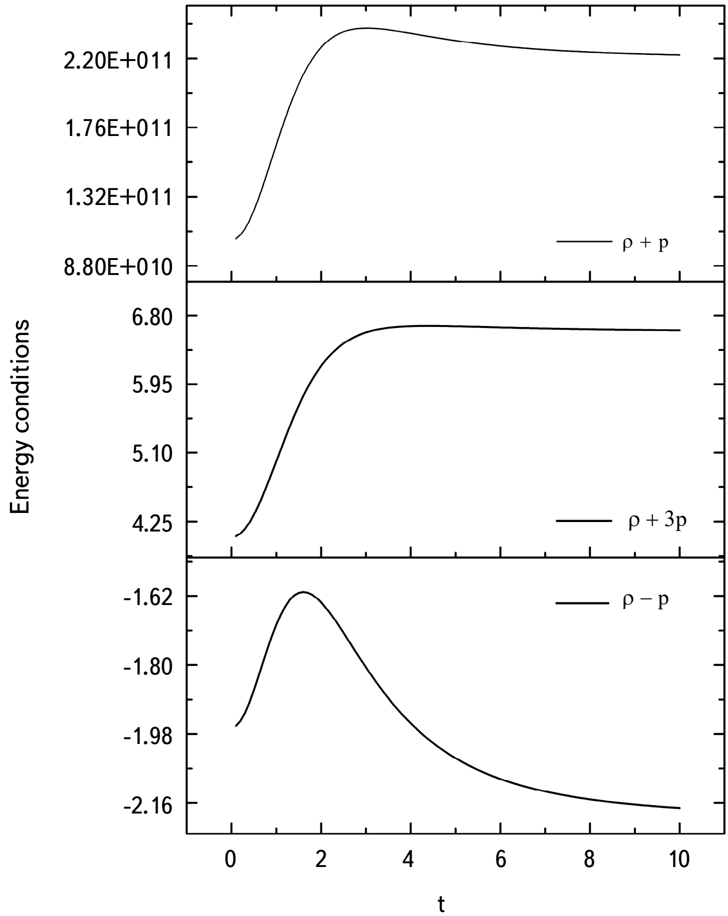

Fig.8. Variation of energy conditions vs. time t for $\delta = 4$, $\gamma = 0.9$, $\xi_0 = 2.2$ and $\mu = 0.85$.

of finite t until $t \to \infty$. The derived cosmos is expanding, but not at the same rate as before, it began expanding more quickly in the beginning and slowed down later time. The cosmos is anisotropic and has held shear throughout its existence, with the exception of $\gamma = 1$. It is also discovered that the cosmic model is accelerating for the appropriate choice of constants showing inflationary phase. Hence the investigations resemble the recent observations of an accelerating, anisotropic and expanding universe.

The energy density is found to be negative during the first epoch, but it later shifts to positive and grows for a while before decreasing and diminishing. Baro et al. [34] have noted a similar kind of behaviour. The universe's pressure grows asymptotically in a positive direction and approaches constantly with time elevation, in contrast to the bulk viscosity coefficient, which first declines quickly before gradually continuing to climb. The constructed model supports the existence of the universe's string phase. Furthermore, the behaviour of particle density and string density leads to the observation that the cosmos first appears to be string-dominated, but that later on, the string-dominated phase vanishes and the particle-dominated era begins and rests to end. It is been observed that SEC and NEC are completely satisfied, DEC is completely violated, and because energy density is initially negative i.e. $\rho < 0$ for this amount, WEC is violated, but as soon as energy density becomes positive i.e. $\rho \geq 0$, WEC is perfectly satisfied.

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[1] Department of Mathematics, Shri R. R. Lahoti Science College, Morshi, Dist. Amravati (M.S.), India, e-mail: ankitdabre@live.com

[2] Department of Mathematics, Government Polytechnic College, Murtizapur, Dist. Amravati (M.S.), India, e-mail: pallavimakode@hotmail.com

ВЯЗКАЯ ПЛОСКО-СИММЕТРИЧНАЯ КОСМОЛОГИЧЕСКАЯ МОДЕЛЬ СТРУНЫ В $f(R)$ ГРАВИТАЦИИ

А.ДАБРЕ[1], П.МАКОДЕ[2]

В данной работе мы использовали метрику плоско-симметричной LRS типа Бьянки I для изучения объемной вязкой жидкости, связанной с облаком струн в рамках теории гравитации $f(R)$. Для получения детерминированных решений учтены некоторые физически обоснованные условия, такие как: предел слабого поля для точечного источника $f(R) = R^{3/2}$ и хорошо известное соотношение пропорциональности между расширением и сдвигом скаляра, а также специальная форма среднего масштабного коэффициента. Кроме того, вычислены некоторые физические и кинематические параметры, а также энергетические условия для изучения астрофизических последствий построенной модели и обсудили их графическое поведение, которое хорошо согласуется с недавними наблюдательными данными.

Ключевые слова: *метрика Бьянки типа I: объемная вязкая жидкость: космическая струна: энергетические условия: гравитация* $f(R)$

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