Armenian Journal of Physics, 2024, vol. 17, issue 1, pp. 13-16

On the Passage of a Relativistic Spinless Particle through a Potential Barrier

G.G. Karapetyan

Independent scientist, Yerevan, Armenia

E-mail: grigori.g.karapetyan@gmail.com

(Received: February 1, 2024; Revised: April 10, 2024; Accepted: April 21, 2024)

Abstract. Contradictory descriptions of the passage of a relativistic particle through a high potential barrier are investigated. The conventional energy-momentum relation yields a real momentum for the particle within the barrier, suggesting classical particle motion. However, this contradicts the law of conservation of energy, which necessitates that the particle's momentum be imaginary, thereby rendering classical motion impossible. This paradox is resolved through the adoption of a novel energy-momentum relation.

Keywords: relativistic particle, potential barrier, tunneling, new energy-momentum relation, Klein-Fock-Gordon equation, motion back in time, Wheeler-Feynman hypotheses

DOI:10.54503/18291171-2024.17.1-13

1. Introduction

The passage of a particle through a potential barrier, where the barrier height exceeds the particle energy, presents a quantum phenomenon absent in classical mechanics. Within the barrier, the particle's potential energy surpasses its total energy, resulting in negative kinetic energy and imaginary speed and momentum. This precludes classical motion within the barrier. Quantum mechanically, this state is described by an exponentially decaying wave function, leading to a probability of barrier penetration that decreases exponentially with barrier height and length, as per the nonrelativistic Schrödinger equation [1]. One would naturally anticipate similar outcomes in relativistic scenarios: the absence of classical motion and a decaying wave function within high barriers. However, solving this relativistic problem with equations such as the Hamilton-Jacobi, Klein-Fock-Gordon, or Dirac equations reveals that the particle can have real momentum within the barrier. This manifests as a sinusoidal traveling wave function within the barrier, contradicting the law of conservation of energy. Let us study this problem in more detail.

2. Solution of Klein-Fock-Gordon equation

Consider the fall of a relativistic particle with energy E onto a barrier of finite length L (see Fig.1), given by the formula

$$U = \begin{cases} 0 & \text{at} \quad x < 0, \quad x > L \\ U_0 > 0 & \text{at} \quad 0 < x < L \end{cases} \tag{1}$$

Let's solve the problem using the Klein-Fock-Gordon equation, which, as is known, is based on the relativistic relation between energy and momentum [2]

$$\left(cP\right)^2 = \left(E - U\right)^2 - \left(mc^2\right)^2 \tag{2}$$

and has the form

$$\left[c^2\hbar^2\frac{\partial^2}{\partial \mathbf{r}^2}+\left(E-U\right)^2-\left(mc^2\right)^2\right]\psi=0 \tag{3}$$

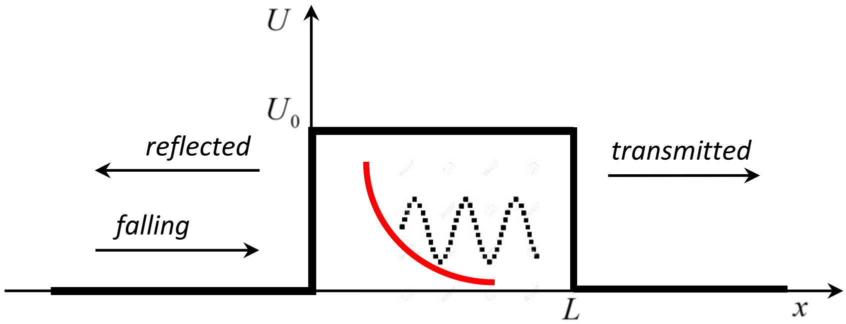

Fig. 1. Potential barrier with height U_0 and length L. Arrows indicate the direction of propagation of incident, reflected and transmitted wave functions. The red solid curve shows the exponentially decreasing amplitude of the wave function calculated by the Schrödinger equation, the dotted black curve shows the sinusoidal real (or imaginary) part of the wave function calculated by the Klein-Fock-Gordon equation.

The momentum of the particle is equal to $p = (E^2 - mc^2)^{1/2}/c$ outside the barrier, and $P = ((E - U_0)^2 - mc^2)^{1/2}/c$ within the barrier. The wave function at $x < 0$ is represented by two counterpropagating waves, incident and reflected with momentums $\pm p$, and at $x > L$ by one transmitted wave with momentum p. Inside the barrier $0 < x < L$ there are two counterpropagating waves with momentums $\pm P$. The amplitudes of the waves are found from the boundary conditions of continuity of the wave function and its derivative with respect to x at $x = 0$ and $x = L$. As a result, the amplitude of the transmitted wave T turns out to be equal to

$$T=\frac{e^{-ipL/\hbar}}{\cos PL/\hbar-i\eta\sin PL/\hbar} \qquad \eta=\frac{1}{2}\left(\frac{P}{p}+\frac{p}{P}\right) \tag{4}$$

and the probability of a particle passing through the barrier is equal to

$$|T|^2=\frac{1}{\cos^2 PL/\hbar+\eta^2\sin^2 PL/\hbar} \tag{5}$$

At high barrier heights, the probability of the passing (tunneling) decreases in inverse proportion to η^2 i.e., the square of the barrier height (since $\eta \sim P/p \sim U_0/mc^2 >>1$). However, there is no exponential decrease in the probability of tunneling depending on the length and the height of the barrier, which is an inalienable characteristic of tunneling. In addition, the real momentum of the particle inside the barrier gives the traveling wave function; this implies classical particle motion inside the barrier. In this case, the law of conservation of energy is violated, since the energy of the particle turns out to be less than its potential energy. Thus, the solution (4) is erroneous, and therefore the behavior of a relativistic particle inside a barrier is paradoxical and has no plausible description either quantum mechanically, according to the Klein-Fock-Gordon equation, or classically.

For a classical solution to this problem, the Wheeler-Feynman concept of “moving back in time” was proposed [3]

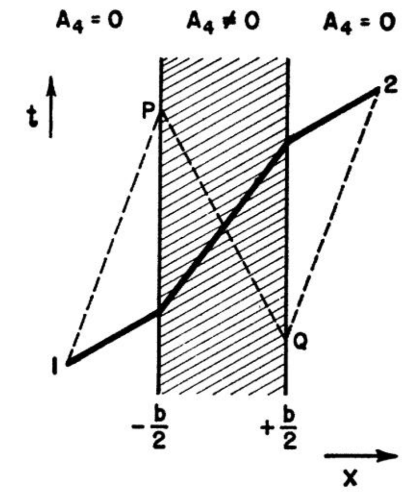

Fig. 2. The passage of a particle through a potential barrier from point 1 to point 2. The solid line shows the usual passage of a fast particle with an energy greater than the height of the barrier. The dashed line has a time-inverted portion and is interpreted as the efficient penetration of a slow particle through the barrier using pair production at point Q and annihilation at point P, with portion PQ representing the motion of the antiparticle. Image taken from [3].

The hypothesis proposes that the movement of a particle within a barrier involves a backward motion in time, which is equivalent to the forward motion in time of an antiparticle born behind the barrier. As the falling particle approaches the barrier, a particle-antiparticle pair is postulated to emerge behind the barrier (point Q in Fig.2). The antiparticle, traversing the barrier from right to left, encounters and annihilates the falling particle at x = 0, whereas the particle of the pair moves rightward at the falling particle's velocity. This motion of the antiparticle within the barrier, from right to left, is interpreted as the falling particle moving backward in time, from left to right. Upon reaching the right boundary of the barrier (point Q in Fig.1), the particle emerges from beneath the barrier, continuing its trajectory even before the falling particle reaches the barrier. Since the antiparticle has an opposite charge, its potential energy inside the barrier is negative, so the law of conservation of energy is not violated.

This exotic theoretical construction, which preserves the law of conservation of energy but violates the principle of causality, was not considered plausible and did not find application in other problems. No alternative explanations have been proposed for the behavior of the particle inside the high potential barrier. Thus, the problem of the passage of a relativistic particle through a high potential barrier is interpreted by the mathematically correct, but physically implausible Wheeler-Feynman hypothesis of motion “back in time.”

3. Solving the paradox in the New theory

These and similar paradoxes were discussed in [4] as examples highlighting the inability of the traditional theory of electrical interactions to explain correctly the motion of particles at high potentials. The New theory of electrical interactions (NTE), introduced in that work, effectively resolves the contradictions. It relies on the following energy-momentum relation:

$$(cP)^2 = E^2 - \left(mc^2 + U\right)^2 \tag{6}$$

This relation initially emerged in the scalar theory of gravity [5,6]. Later, it was adopted in the new classical theory of electromagnetic interactions [4,7]. Based on this equation, a new modified

Klein-Fock-Gordon equation can be obtained, similar to constructing equation (3) from equation (2). This new equation, derived by V. Mekhitarian is written as [7]:

$$\left[c^2\hbar^2\frac{\partial^2}{\partial\mathbf{r}^2}+E^2-\left(mc^2+U\right)^2\right]\psi=0 \tag{7}$$

Applying equation (7) to the problem of a particle passing through a potential barrier instead of (3), we find that inside the high barrier the momentum P is imaginary. Consequently, the formula for the probability of tunneling takes the form:

$$|T|^2=\frac{1}{\left(\mathrm{ch}\,|P|L/\hbar+|\eta|\,\mathrm{sh}\,|P|L/\hbar\right)^2} \tag{8}$$

At large barrier heights, it turns to

$$|T|^2\sim\frac{\exp\left(-\dfrac{2|P|L}{\hbar}\right)}{\left(\dfrac{|P|L}{\hbar}\right)^2} \tag{9}$$

Here, as expected, the tunneling probability decreases exponentially with the barrier length and height, as in the case of the Schrödinger equation for a nonrelativistic particle, depicted by the red solid curve in Fig.1. There is no traveling wave present, due to imaginary momentum P, which indicates that the passage through a high barrier is a quantum process with no classical analog. The commonly accepted interpretation of such a passage is that the particle's energy momentarily exceeds the barrier's height due to vacuum fluctuations, before returning to its original state after passing through the barrier.

4. Conclusions

Solutions of the modified Klein-Fock-Gordon equation (7), based on the new energy-momentum relation (6), reject the notion of the classical motion of a relativistic particle inside a high barrier and provide a plausible formula for the probability of passage, which decreases exponentially with increasing barrier length and height. Consequently, the Wheeler-Feynman hypothesis of "moving back in time," which attempts to explain such classical motion, is rendered flawed. Solution (4) of this problem using the Klein-Fock-Gordon equation is also erroneous.

References

[1] L.D. Landau, E.M. Lifshitz, Quantum Mechanics (Pergamon Press, **3**, 1977).
[2] L.D. Landau, E.M. Lifshitz, The Classical Theory of Fields (Pergamon Press, **2**, 1975).
[3] R.P. Feynman, Phys. Rev. **74** (1948) 939.
[4] G.G. Karapetyan, Intern. J. Fundam. Phys. Sci. **14** (2024) 7;
SSRN https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4769212
[5] O. Bergmann, Am. J. Phys. **24** (1956) 38.
[6] C.M. Andersen, H.C. von Baeyer, Ann. Phys. **62** (1971) 120.
[7] V.M. Mekhitarian, Equations of Relativistic and Quantum Mechanics (without Spin). In Quantum Mechanics, (London, Intech Open, 2020).