ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ PROCEEDINGS OF NATIONAL ACADEMY OF SCIENCES OF ARMENIA
ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ

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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 72, №4, 2019 Механика

ГЕВОРКЯН РУБЕН СТЕПАНОВИЧ
(К 80-летию со дня рождения)

Родился 17 ноября 1939г. в селе Азат Ханларского района Аз.ССР, в семье ремесленника.

В 1957 г. окончил армянскую школу №12 г. Кировабада (Гандзак) и поступил на мех.-мат. факультет ЕГУ, окончил по специальности «Механика» в 1962 г. и был назначен лаборантом Кафедры механики.

С января 1968г. Геворкян Р.С. вёл преподавательскую работу в Ереванском Зооветеринарном институте и в Армянском Государственном аграрном университете. Был старшим преподавателем (с 1988 г.), доцентом (с 1990 г.) и профессором кафедры «Высшая математика и теоретическая механика». Проф. Геворкяном Р.С. написаны свыше 20 учебно-методических работ.

Начиная с 1981 г., Геворкян Р.С. стал активно заниматься научной работой. Он защитил кандидатскую (1987 г.) и докторскую (1999 г.) диссертаци, проф. с 2000г. С 2007г. работает в Ин-те механики НАН Армении ведущим научн.сотрудником.

Научная деятельность докт. физ.-мат. наук, проф. Геворкяна Р.С. посвящена получению асимптотических решений неклассических смешанных краевых задач теории термоупругости слоистых балок, пластин и оболочек из сжимаемых и несжимаемых, анизотропных, упругих и вязкоупругих материаловРезультаты его исследований отражены в книге: Л.А.Агаловян, Р.С.Геворкян – Неклассические краевые задачи анизотропных слоистых балок, пластин и оболочек. Ереван: «Гитутюн», 2005. 468с. и в более 70 научных статьях. Неоднократно участвовал с интересными докладами на международных научных конференциях как в Армении, так и за рубежом.

Редакция журнала «Известия НАН Армении. Механика» поздравляет Рубена Степановича Геворкяна со славным юбилеем, желает ему доброго здоровья и новых творческиж успехов.

ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
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САРКИСЯН САМВЕЛ ОГАНЕСОВИЧ

(к 75-летию со дня рождения)

Исполняется 75 лет со дня рождения известного учёного-механика члена-корреспондента НАН Армении, доктора физико-математических наук, профессора Самвела Оганесовича Саркисяна.

Пятьдесят лет его жизни прошли в неустанном и плодотворном служении науке-механике.

С.О. Саркисяну принадлежат существенные результаты в прикладной теории упругости и строительной механике, в частности, в магнитоупругости тонких оболочек и пластин (С.О. Саркисян. Общая двумерная теория магнитоупругости тонких оболочек. Ереван: Изд-во АН Армении. 1992. 260с.); контактные задачи тонкостенных элементов (Доклады АН Армянской ССР. 1977. Т.64. №4); физически нелинейной теории упругих тонких пластин и оболочек (Изв. АН Арм. ССР. Механика. 1972. Т.25. №4; Изв. АН Арм. ССР. Механика. 1972. Т.25. №5; Изв. АН Арм. ССР. Механика. 1973. Т.26. №3).

Весьма важное место в научном творчестве С.О.Саркисяна занимают исследования по обобщённой механике твёрдого деформированного тела. Известна созданная С.О.Саркисяном прикладная теория микрополярных упругих тонких стержней, пластин и оболочек (J. of Materials Science and Engineering. 2012. V.2. №1; Прикладная механика и техническая физика. 2012. Т.53. Вып.2; Физическая мезомеханика. 2011. Т.14. №1; Доклады российской академии наук. 2011. Т.436. №2; J. of Thermal Stresses. 2013. V.36. №11; Advances in Pure Mathematics. 2015. V.5. №10; Advanced Structured

Materials. 2019. Vol.103. Dynamical Processes in Generalized Continua and Structures. Springer.; Вычислительная механика сплошных сред. 2016. Т.9. №3; Materials Physics and Mechanics. 2018. V.35. №1.). Общие принципы упрощения исходных уравнений и граничных условий (асимптотическая теория и формирование гипотез на основе свойств асимптотического решения в тонких областях) представляют собой оригинальный подход к решению проблемы перехода от трёхмерной задачи к одномерной или двумерной задаче прикладной теории микрополярных упругих тонких тел, с доказательствами соответствующих энергетических теорем и вариационных принципов. Существенным достижением в области прикладной теории микрополярной упругости является разработка (совместно с учениками) варианта метода конечных элементов для решения разнообразных прикладных статических и динамических краевых задач микрополярных тонких тел.

К фундаментальным научным достижениям относятся исследования С.О. Саркисяна, посвящённые объединению представлений прикладной теории микрополярных упругих тонких стержней, пластин и оболочек с новейшей технической проблематикой-наномеханикой (Перспективные материалы и технологии. Монография. Том 2. НАН Беларуси. 2019; Труды XII Всероссийского съезда по фундаментальным проблемам теоретической и прикладной механики. Уфа. Россия. РАН. 2019; Физическая мезомеханика. 2019. Т.22. №5.), в результате которого, во-первых, построены дискретные и континуальные моментные модели деформаций графена и других наноматериалов и, во-вторых, определены моментные упругие постоянные, через параметры их атомной структуры. Исследования в этой современной теоретической и прикладной областях механики деформируемого твёрдого тела продолжаются и успешно развиваются.

Всё творчество Самвела Оганесовича носит отпечаток его яркой индивидуальности и таланта, его ученики относятся к нему с преданностью и восхищением.

Вполне естественно, что Самвелу Оганесовичу Саркисяну присуждено звание Заслуженного деятеля науки Армении.

Редакционная коллегия и международный редакционный совет журнала сердечно поздравляют Самвела Оганесовича с семидесятипятилетием со дня рождения и желают ему доброго здоровья, счастья и новых творческих достижений в научной деятельности.

ХАЧАТРЯН АЛЕКСАНДР МОВСЕСОВИЧ
(К 70-летию со дня рождения)

Хачатрян Александр Мовсесович родился 6-го октября 1949г. в селе Хнапат Аскеранского района Республики Арцах, в семье учителя.

В 1967г. окончил среднюю школу села Хнапат и поступил на механико-математический факультет ЕГУ. В 1972 г. по окончании ЕГУ был направлен в Институт механики АН Армянской ССР. С 1973 по 1975гг.– аспирант Института механики АН Арм. ССР под научным руководством академика Л.А.Агаловяна. В 1981г. защитил кандидатскую, а в 2001г. докторскую диссертации.

С сентября 2003г. по 2018г. А.М. Хачатрян работал в Арцахском государственном университете (АрГУ). Был учёным секретарём (2003-2008г), заведующим Кафедрой математики АрГУ (2010-2017гг.). В январе 2005г. ему было присвоено учёное звание профессора. Был избран членом Учёного совета АрГУ, Учёного совета Арцахского Научного центра, является членом редколлегии журнала «Учёные записки АрГУ». В 2014г. приказом Президента Республики НКР А.М. Хачатряну присвоено почётное звание «Заслуженный деятель науки Нагорно-Карабахской Республики». Под его руководством два преподавателя кафедры математики АрГУ защитили диссертации и получили учёную степень канд. физ.-мат. наук.

Научная деятельность профессора А.М. Хачатряна посвящена получению асимптотических решений краевых задач теории упругости однослойных и многослойных анизотропных балок, пластин и оболочек при полном и неполном контакте между слоями в рамках геометрически нелинейной теории упругости. Результаты его исследований опубликованы в более 50 научных статьях в авторитетных республиканских и иностранных журналах и трудах международных конференций. А.М. Хачатрян – автор трёх учебных пособий для ВУЗ-ов.

А.М. Хачатрян отличается трудолюбием, целеустремлённостью и полной отдачей сил научным исследованиям.

Редакция журнала «Известия НАН Армении. Механика» поздравляет Александра Мовсесовича Хачатряна со славным юбилеем, желает ему доброго здоровья и новых творческих успехов.

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AVERAGED CONTROLLABILITY OF EULER-BERNOULLI BEAMS WITH RANDOM MATERIAL CHARACTERISTICS: THE GREEN'S FUNCTION APPROACH

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Keywords: averaged dynamics, mathematical expectation, parameter-dependent systems

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Усреднённая управляемость балок Эйлера-Бернулли со случайными характеристиками материала: метод функции Грина

Ключевые слова: усреднённая динамика, математическое ожидание, системы, зависящие от параметра

Исследуются точная и приближённая усреднённая управляемость балки Эйлера-Бернулли со случайными характеристиками (жёсткость на изгиб и плотность). Рассматриваются случаи, когда материальные характеристиками являются равномерно и нормально распределённые случайные величины. Задача заключается в определении допустимых управлений, точно или приближённо обеспечивающих требуемое усреднё,нное состояние балки за заданное конечное время. Представив общее решение уравнения изгиба балки с помощью функции Грина, получаются явные представления для прогиба и скорости точек балки, тем самым, упрощая анализ усреднённой управляемости балки. Выводятся необходимые и достаточные условия точной усреднённой управляемости балки, а также достаточные условия приближённой усреднённой управляемости балки. Основные результаты подтверждаются численными расчётами. Функции Грина для уравнения изгиба балки при различных граничных условиях и соответствующие усреднённые решения приводятся в приложениях.

Ավետիսյան Ա.Ս., Խուրշուդյան Ա.Ժ.

Նյութի պատահական բնութագրիչներով Էյլեր-Բերնուլիի հեծանների միջինացված ղեկավարելիությունը. Գրինի ֆունկցիայի եղանակը

Հիմնաբառեր. միջինացված դինամիկա, մաթեմատիկական սպասում, պարամետրից կախված համակարգեր

Հետազոտվում է նյութի պատահական բնութագրիչներով (ծռման կոշտություն և խտություն) Էյլեր-Բերնուլիի հեծանի միջինացված ճշգրիտ և գրեթե ղեկավարելիությունը: Դիտարկվում են հեծանի նյութի բնութագրիչների հավասարաչափ և նորմալ բաշխված պատահական մեծություն լինելու դեպքերը: Որոնվում են տրված վերջավոր ժամանակում հեծանի պահանջվող վիճակն (ճշգրիտ և մոտավոր) ապահովող թույլատրելի ղեկավարումները: Միջինացված ղեկավարելիության հետազոտումը պարզեցնելու նպատակով՝ հեծանի ծռման հավասարման ընդհանուր լուծումը ներկայացվում է Գրինի ֆունկցիայի միջոցով: Հեծանի միջինացված ճշգրիտ ղեկավարելիության համար ստացվում են անհրաժեշտ և բավարար, իսկ միջինացված գրեթե ղեկավարելիության համար՝ բավարար պայմաններ: Հետազոտության արդյունքը հաստատվում է օրինակներով: Հավելվածներում բերվում են հեծանի ծռման հավասարման Գրինի ֆունկցիան՝ տարատեսակ եզրային պայմանների համար, ինչպես նաև համապատասխան միջինացված լուծումները:

We examine the Euler-Bernoulli beam with random material characteristics (bending stiffness and mass per unit length) on exact and approximate averaged controllability. Cases when the material characteristics are standard normally and uniformly distributed random variables are considered. The problem is in an appropriate choice of admissible controls providing a required averaged state of the beam (exactly or approximately) within a desired amount of time. Representing the general solution of the beam equation in terms of the Green's function, it becomes possible to derive explicit closed-form representations for the averaged deflection and velocity of the beam. This makes controllability analysis a matter of straightforward computations. Specifically, necessary and sufficient conditions for the exact averaged controllability, as well as sufficient conditions for the approximate averaged

controllability are derived with respect to admissible controls. Numerical analysis allows to make a sensible understanding of theoretical derivations. Green's functions for the main types of boundary conditions and closed-form representation of the averaged dynamics are defined in appendices.

Introduction

The dynamics described by random state constraints (e.g., differential equations accompanied by initial/boundary conditions), generally, is a random function. Therefore, if such a dynamics is controlled, then the control function also must be random. However, dealing with controllability analysis, control function must not contain any randomness. A way of overcoming such situations for systems containing random parameters has been suggested by the prominent mathematician Enrique Zuazua in [1] where a general theory of controllability of finite-dimensional system has been developed. The compromise is found in a smart way by controlling the averaged dynamics or the mathematical expectation of the dynamics over all possible values of the random parameters. This new type of controllability is called averaged controllability. A general theory for infinite-dimensional or distributed parameter system is currently under development by Zuazua and colleagues. See [2-5] for some of existing contributions. See also [4] for a handful of open problems related to controllability of random evolution equations.

Suppose that the controlled state of a dynamic system is described by a function (for the sake of simplicity, we restrict ourselves by the scalar case) $w:\mathcal{U}\times\mathbb{R}^n\times\mathbb{R}^+\times\Omega\to\mathcal{R}$ where $\Omega\in\mathbb{R}^m$ is the domain of random parameters contained in the state constraints on w (imagine, e.g., a differential equation with initial and boundary conditions), and $\mathcal{U}$ is the set of admissible controls. Then, instead of the usual controllability residue [6]

$$\mathcal{R}_T(u)=\left\|w(u,x,T;\boldsymbol{\omega})-w_T\right\|_{\mathbf{W}_T},$$

where T is the control time, $\boldsymbol{\omega}\in\Omega$ is the vector of random variables, w_T is the desired terminal state and $\mathbf{W}_T$ is the space of terminal states, Zuazua suggests to consider the averaged residue [3]

$$\mathcal{R}_T^{av}(u)=\left\|\int_\Omega w(u,x,T;\boldsymbol{\omega})\,\mathrm{d}\mathbb{P}(\boldsymbol{\omega})-w_T\right\|_{\mathbf{W}_T},$$

where the integral of w over Ω would be the averaged state or the mathematical expectation. After computing the expectation and substituting it into $\mathcal{R}_T^{av}$, it will be guaranteed that the control u does no longer need to be dependent on $\boldsymbol{\omega}$.

At this, following to [3], we distinguish two concrete types of averaged controllability. If for any initial and desired states, control time T, $\mathcal{R}_T^{av}(u)=0$ for a $u\in\mathcal{U}$, then the system is exactly averaged controllable. If for any initial and desired states, control time T and a desired accuracy $\varepsilon>0$, $\mathcal{R}_T^{av}(u)<\varepsilon$ for a $u\in\mathcal{U}$, then the system is approximately averaged controllable. Null-averaged (exact and approximate) controllable systems are defined analogously. Admissible controls providing exact (approximate) averaged controllability, are called exactly (approximately) resolving average controls.

In this paper, we study the averaged controllability of Euler-Bernoulli beam with random parameters. The cases of uniformly and normally distributed random variables are considered. Representing the general solution of the Euler-Bernoulli beam equation in terms of corresponding Green's function and making use of the Green's function approach [6, 7], we derive exact and approximate averaged controllability constraints. Numerical analysis reveals non-triviality of theoretical derivations. At this, it is worth mentioning that the averaging process considered in this paper has nothing in common with the averaging of material characteristics widely applied in mechanics of inhomogeneous materials (see, e.g., [8]).

Note that the averaged controllability of Euler-Bernoulli beams has been considered in a recent paper [9]. However, we would like to point out two principal differences between that and the current papers. First of all, in [9] only the flexural stiffness E is considered as a random variable, where E is the Young's modulus, I is the moment of inertia of the cross section of the beam, while in this paper, besides EI, another important characteristic of the beam - μ, the mass per unit length, is considered as a random variable. This becomes important especially when dealing with problems for beams made of specific material with distinct Young's modulus and density. The second distinctive feature is that the average controllability analysis based on the Green's function approach is quite straightforward and it is easier to apply in particular cases, since the Green's function of the beam equations with various boundary conditions has been found explicitly.

1. Governing equation and its Green's function solution

We consider a Euler-Bernoulli beam subject to a distributed control influence. Then, the vertical displacement of the beam is determined from the fourth-order PDE (all variables and quantities are dimensionless)

$$\frac{\omega_1}{\omega_2}\frac{\partial^4 w}{\partial x^4}+\frac{\partial^2 w}{\partial t^2}=\frac{1}{\omega_2}u(t)v(x),\ 0<x<l,\ t>0, \tag{1.1}$$

where $\omega_1 = EJ > 0$ and $\omega_2 = \mu > 0$ are the beam flexural stiffness and mass per unit length, respectively. Control influence is described by u with distribution v. By a proper choice of v, boundary controls can also be considered.

Hereinafter, we limit the consideration by the case when both ω_1 and ω_2 are either standard normally or uniformly distributed independent random variables. Then, their probability density functions are given by

$$\rho(\omega_1,\omega_2)=\frac{1}{2\pi}\exp\left[-\frac{\omega_1^2+\omega_2^2}{2}\right] \tag{1.2}$$

or

$$\rho(\omega_1,\omega_2)=\frac{1}{\mu(\Omega)}\chi_\Omega(\omega_1,\omega_2), \tag{1.3}$$

where χ_Ω is the indicator function, and $\mu(\Omega)$ is the measure of Ω.

Let at $t=0$ the beam is in equilibrium. Then, the general solution of (1.1) can be represented in terms of Green's function [10]

$$w(x,t;\omega_1,\omega_2)=\frac{1}{\omega_2}\int_0^l\int_0^t G(x,\xi,t-\tau;\omega_1,\omega_2)u(\tau)v(\xi)\,\mathrm{d}\tau\mathrm{d}\xi, \tag{1.4}$$

$$G(x,\xi,t;\omega_1,\omega_2)=\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2\|\varphi_n\|^2}\varphi_n(x)\varphi_n(\xi)\sin\left(\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}t\right).$$

For determination of λ_n and φ_n for multiple boundary conditions, see Appendix 1. Therefore, (1.4) can be rewritten as follows:

$$w(x,t;\omega_1,\omega_2)=\frac{1}{\omega_2}\sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda_n^2\|\varphi_n\|^2}\varphi_n(x)\int_0^t u(\tau)\sin\left[\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}(t-\tau)\right]\mathrm{d}\tau, \tag{1.5}$$

$$\alpha_n=\int_0^l\varphi_n(\xi)v(\xi)\,\mathrm{d}\xi.$$

The particle velocity of the beam is determined by differentiating (1.5}) w.r.t. t:

$$\frac{\partial w}{\partial t}(x,t;\omega_1,\omega_2)=\frac{\sqrt{\omega_1}}{\omega_2^{3/2}}\sum_{n=1}^{\infty}\frac{\alpha_n}{\|\varphi_n\|^2}\varphi_n(x)\int_0^t u(\tau)\cos\left[\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}(t-\tau)\right]\mathrm{d}\tau. \tag{1.6}$$

2. Averaged controllability via the Green's function approach

The averaged controllability problem for the beam can now be formulated as follows. Given any $w_T, w_{T1}\in L^2[0,l]$ and control time T, find control functions $u\in\mathcal{U}=\{u\in L^2[0,T],\ \mathrm{supp}(u)\subseteq[0,T]\}$ such that the averaged residue

$$\mathcal{R}_T^{av}(u)=\left\|\mathbb{M}_T^{\Omega}[w]-w_T\right\|_{L^2[0,l]}^2+\left\|\mathbb{M}_T^{\Omega}\left[\frac{\partial w}{\partial t}\right]-w_{T1}\right\|_{L^2[0,l]}^2 \tag{2.1}$$

satisfies $\mathcal{R}_T^{av}(u)=0$ or $\mathcal{R}_T^{av}(u)<\varepsilon$ with a desired accuracy ε. Here,

$$\mathbb{M}_T^{\Omega}[w]=\int_\Omega w(x,T;\omega_1,\omega_2)\,\mathrm{d}\mathbb{P}(\omega_1,\omega_2)=\int_\Omega w(x,T;\omega_1,\omega_2)\rho(\omega_1,\omega_2)\,\mathrm{d}\Omega$$

is the mathematical expectation of w over Ω at $t=T$.

Straightforward calculations provide

$$\mathbb{M}_T^{\Omega}[w]=\sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda_n^2\|\varphi_n\|_{L^2[0,l]}^2}\varphi_n(x)\int_0^T\Psi_n(T-\tau)u(\tau)\,\mathrm{d}\tau:=\sum_{n=1}^{\infty}\beta_n(T,u)\varphi_n(x),$$

$$\mathbb{M}_T^{\Omega}\left[\frac{\partial w}{\partial t}\right]=\sum_{n=1}^{\infty}\frac{\alpha_n}{\|\varphi_n\|_{L^2[0,l]}^2}\varphi_n(x)\int_0^T\Psi_{n1}(T-\tau)u(\tau)\,\mathrm{d}\tau:=\sum_{n=1}^{\infty}\beta_{n1}(T,u)\varphi_n(x),$$

where

$$\beta_n(T,u)=\frac{\alpha_n}{\lambda_n^2\|\varphi_n\|^2_{L^2[0,l]}}\int_0^T\Psi_n(T-\tau)u(\tau)\,\mathrm{d}\tau,$$

$$\Psi_n(t)=\int_\Omega\frac{1}{\omega_2}\sin\left[\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}t\right]\rho(\omega_1,\omega_2)\,\mathrm{d}\Omega,$$

$$\beta_{n1}(T,u)=\frac{\alpha_n}{\|\varphi_n\|^2_{L^2[0,l]}}\int_0^T\Psi_{n1}(T-\tau)u(\tau)\,\mathrm{d}\tau,$$

$$\Psi_{n1}(t)=\int_\Omega\frac{\sqrt{\omega_1}}{\omega_2^{3/2}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_1}{\omega_2}}t\right]\rho(\omega_1,\omega_2)\,\mathrm{d}\Omega.$$

Furthermore, note that

$$\left\|\mathbb{M}_T^\Omega[w]-w_T\right\|^2_{L^2[0,l]}=\sum_{n=1}^\infty\left(\beta_n(T,u)-w_{Tn}\right)^2\|\varphi_n\|^2_{L^2[0,l]},$$

reducing (2.1) to

$$\mathcal{R}_T^{av}(u)=\sum_{n=1}^\infty\left[\left(\beta_n(T,u)-w_{Tn}\right)^2+\left(\beta_{n1}(T,u)-w_{T1n}\right)^2\right]\|\varphi_n\|^2_{L^2[0,l]},\tag{2.2}$$

where w_{Tn} and w_{T1n} are the expansion coefficients of w_T and w_T^1 into series of $\{\varphi_n\}_{n=1}^\infty$

2.1. Exact controllability

Then, from (2.2) we straightforwardly obtain the following result.

Theorem 1. For the beam exact averaged controllability it is necessary and sufficient that for given T,

$$\begin{cases}\beta_n(T,u)-w_{Tn}=0,\\ \beta_{n1}(T,u)-w_{T1n}=0,\end{cases}\quad n=1,2,\dots\tag{2.3}$$

for $u\in\mathcal{U}$.

Remark 1. Note that system (2.3) is linear in u. Therefore, the set of exactly resolving averaged controls can be described by solving (2.3) as a infinite dimensional linear problem of moments, L^p – optimal solution of which for $1\le p\le\infty$ has been explicitly derived in [11]. The heuristic method [12] can be applied, too.

2.2. Approximate controllability

Making use of the triangle inequality, for $\mathcal{R}_T^{av}$ we obtain the following estimate:

$$\mathcal{R}_T^{av} \le \sum_{n=1}^{\infty}\left[\beta_n^2(T,u)+\beta_{n1}^2(T,u)+w_{Tn}^2+w_{T1n}^2\right]\|\varphi_n\|_{L^2[0,l]}^2 .$$

Then, the following assertion holds.

Theorem 2. If for given T,

$$\sum_{n=1}^{\infty}\left[\beta_n^2(T,u)+\beta_{n1}^2(T,u)+w_{Tn}^2+w_{T1n}^2\right]\|\varphi_n\|_{L^2[0,l]}^2 \le \varepsilon \tag{2.4}$$

for $u \in \mathcal{U}$, then the beam is approximately averaged controllable.

Remark 2. Note that (2.4) makes sense only when

$$\tilde{\varepsilon} := \varepsilon - \sum_{n=1}^{\infty}\left[w_{Tn}^2+w_{T1n}^2\right]\|\varphi_n\|_{L^2[0,l]}^2 \ge 0. \tag{2.5}$$

Moreover, making use of the Cauchy-Schwartz inequality implying

$$\left[\int_0^T \Psi_n(T-\tau)u(\tau)\mathrm{d}\tau\right]^2 \le \|u\|_{L^2[0,T]}^2\|\Psi_n\|_{L^2[0,T]}^2 ,$$

we obtain

$$\beta_n^2(T,u)+\beta_{n1}^2(T,u)=\frac{\alpha_n^2}{\lambda_n^4\|\varphi_n\|_{L^2[0,l]}^4}\left[\int_0^T \Psi_n(T-\tau)u(\tau)\mathrm{d}\tau\right]^2+$$

$$+\frac{\alpha_n^2}{\|\varphi_n\|_{L^2[0,l]}^4}\left[\int_0^T \Psi_{n1}(T-\tau)u(\tau)\mathrm{d}\tau\right]^2 \le \frac{\alpha_n^2\|u\|_{L^2[0,T]}^2}{\lambda_n^4\|\varphi_n\|_{L^2[0,l]}^4}\left[\|\Psi_n\|_{L^2[0,T]}^2+\lambda_n^4\|\Psi_{n1}\|_{L^2[0,T]}^2\right].$$

Therefore, the following assertion holds.

Corollary 1. Assume that the desired state w_T, w_{T1} is constrained by inequality (2.5). If for given T,

$$\sigma(T)=\sum_{n=1}^{\infty}\frac{\alpha_n^2}{\lambda_n^4\|\varphi_n\|_{L^2[0,l]}^4}\left[\|\Psi_n\|_{L^2[0,T]}^2+\lambda_n^4\|\Psi_{n1}\|_{L^2[0,T]}^2\right]\ne 0 \tag{2.6}$$

is finite and

$$\|u\|_{L^2[0,T]}^2\sigma(T)<\tilde{\varepsilon}, \tag{2.7}$$

then the Euler-Bernoulli beam is approximate controllable.

Remark 3. Moreover, if the conditions of Corollary 1 hold, then (2.7) defines a subset of approximately resolving controls. Namely, any admissible control $u \in \mathcal{U}$ with

$$\|u\|_{L^2[0,T]}^2 < \frac{\tilde{\varepsilon}}{\sigma(T)}$$

is an approximately resolving control.

Remark 4. When $\Omega=\{\omega_1,\omega_2\in\mathbb{R}^+,\omega_{i0}\le\omega_i\le\omega_{i1},i=1,2\}$ is a rectangle, using the boundedness of Ψ_n and Ψ_{n1} on $[0,T]$ for all $n=1,2,\ldots$, we see that $\|\Psi_n\|_{L^2[0,T]}$, $\|\Psi_{n1}\|_{L^2[0,T]}$ are finite, so that σ is finite for $T\ge t_0>0$. Therefore, as soon as $\Psi_n\not\equiv 0$ or $\Psi_{n1}\not\equiv 0$ for at least one n, then (2.6) holds.

3. Numerical analysis

In this section, we carry out a numerical analysis of a particular case when Ω is a rectangle. For the sake of simplicity, as well as in order to be eligible to involve the Euler-Bernoulli assumptions, we limit the consideration only by metals (see Table 1). Assume that the beam is of unit length, has a square cross section of side $h=l/100$, and is simply supported (for explicit representation of the corresponding Green's function, see Appendix 1, case 2). The beam is subjected to a control concentrated at the mid-point of the beam, i.e., $v(x)=\delta(x-1/2)$, where δ is the Dirac function.

Table 1. Young's modulus and densities of some metals

Metal	E [GPa]	ρ [kg/m^3]
Aluminum	10	2800
Cast iron	13.4	7300
Titanium	15.5	4500
Al-Bronze	17	8200
Monel 400	26	8600
Steel	29.2	7850
Cr-Mo Steel	31.7	8000

Based on the values presented in Table 1, we compute $\omega_{10}=25/3$ (Aluminum), $\omega_{11}=26.4167$ (Cr-Mo Steel), $\omega_{20}=7/25$ (Aluminum), $\omega_{21}=17/20$ (Monel).

3.1. Uniformly distributed random variables

First, let us consider the case of uniformly distributed independent random variables with (1.2). Numerical analysis shows that, in this case, both Ψ_n and Ψ_{n1} are bounded functions of t and tend to 0 very fast as t increases (see Fig.1). For explicit representation of Ψ_n and Ψ_{n1}, see Appendix 2.

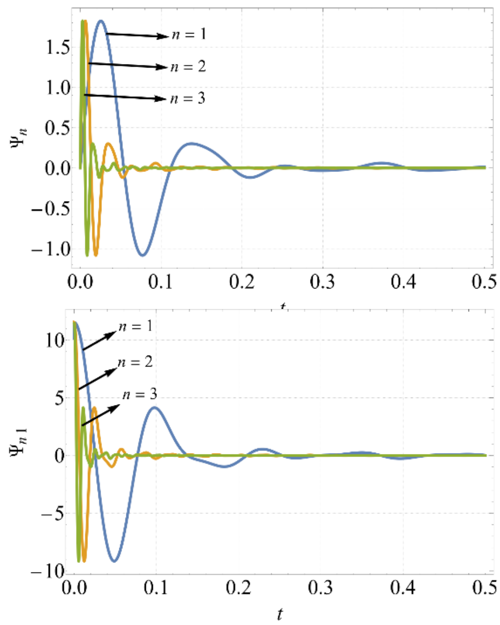

Fig.1. Plots of Ψ_n and Ψ_{n1} against t for $n=1,2,3$: the case of uniformly distributed random variables

Evaluation of σ as in (2.6) shows that as T increases, $\sigma(T)$ increases from 0 and approaches the value of ≈ 3.13125 (see Fig. 2). Therefore, the first condition of Corollary 1 holds. Since in this case, $2\|\varphi_n\|^2_{L^2[0,l]}=1$, then (2.5) holds with $\varepsilon=10^{-4}$, e.g., for $w_T=a\sin(\pi x)$, $w_{T1}\equiv 0$ in $[0,1]$ with $a\le 10^{-2}$. Implying Corollary 1, we see that admissible controls with $\|u\|^2_{L^2[0,T]}<1.66\cdot 10^{-5}$ provide approximate averaged controllability of the beam.

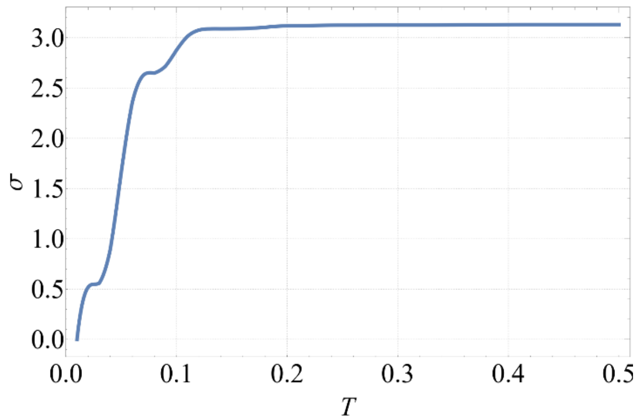

Fig. 2. Plot of σ against T : the case of uniformly distributed random variables

3.2. Normally distributed random variables

Consider now the case when ω_1 and ω_2 are normally distributed independent random variables with (1.2). It seems that it is impossible to express Ψ_n and Ψ_{1n} in an explicit form. Nonetheless, numerical analysis shows that $\left|\Psi_n\right| \leq 3 \cdot 10^{-18}$ and $\left|\Psi_{1n}\right| \leq 8 \cdot 10^{-16}$ on $[0,T]$ for all $n = 1,2,\ldots,$ providing in (2.6), $\sigma \leq 10^{-34}$. Evidently, in this case Corollary 1 does not hold, and for the establishment of approximate averaged controllability of the heat equation, inequality (2.4) must be evaluated.

Conclusions

Using the Green's function approach, necessary and sufficient conditions for exact averaged controllability, as well as sufficient conditions for approximate averaged controllability of a Euler-Bernoulli beam with random material characteristics subjected to multiple boundary conditions are obtained in this paper. At this, both cases of standard normally and uniformly distributed independent random variables are covered. The averaged dynamics of the beam is represented in terms of the random characteristics explicitly, which simplifies the controllability analysis significantly. The determination of exactly resolving average controls is reduced to an infinite-dimensional linear problem of moments, the general solution of which in L^p , $1 \leq p \leq \infty$, is known. A simple inequality on the L^2 -norm of the approximately resolving average controls is derived. Numerical analysis reveals the efficacy in the sense of computational complexity of the derived constraints.

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Appendix 1

In this appendix we define the eigenfunctions φ_n and eigenvalues λ_n for five commonly considered boundary conditions. Below,

$$K_{1n,3n}(x) = \sinh(\lambda_n x) \mp \sin(\lambda_n x),\ K_{2n,4n}(x) = \cosh(\lambda_n x) \mp \cos(\lambda_n x),$$

are the Krylov functions.

1. Both ends are clamped:

$$w = \frac{\partial w}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = l.$$

Then,

$$\frac{1}{\lambda_n^2 \|\varphi_n\|^2} = \frac{4}{l} \frac{\lambda_n^2}{\left[\varphi_{n''}(l)\right]^2},\ \varphi_n(x) = K_{1n}(l) K_{2n}(x) - K_{2n}(l) K_{1n}(x),$$

λ_n are the positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = 1$.

2. Both ends are simply supported:

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } x = l.$$

Then,

$$\frac{1}{\lambda_n^2 \|\varphi_n\|^2} = \frac{2l}{\pi^2 n^2},\ \varphi_n(x) = \sin(\lambda_n x),\ \lambda_n = \frac{\pi n}{l}.$$

3. One end is clamped, one end is simply supported:

$$w = \frac{\partial w}{\partial x} = 0 \text{ at } x = 0 \text{ and } w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = l.$$

Then,

$$\frac{1}{\lambda_n^2 \|\varphi_n\|^2} = \frac{2}{l} \frac{\lambda_n^2}{\left|\varphi_{n'}(l)\varphi_{n''}(l)\right|},\ \varphi_n(x) = K_{1n}(l)K_{2n}(x) - K_{2n}(l)K_{1n}(x),$$

λ_n are the positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$;

4. One end is clamped, one end is free:

$$w = \frac{\partial w}{\partial x} = 0 \text{ at } x = 0 \text{ and } w = \frac{\partial^3 w}{\partial x^3} = 0 \text{ at } x = l.$$

Then,

$$\frac{1}{\lambda_n^2 \|\varphi_n\|^2} = \frac{4}{l} \frac{1}{\lambda_n^2 \varphi_n^2(l)},\ \varphi_n(x) = K_{3n}(l)K_{2n}(x) - K_{4n}(l)K_{1n}(x),$$

λ_n are the positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = -1$;

5. One end is simply supported, one end is free:

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } w = \frac{\partial^3 w}{\partial x^3} = 0 \text{ at } x = l.$$

Then,

$$\frac{1}{\lambda_n^2 \|\varphi_n\|^2} = \frac{4}{l} \frac{1}{\lambda_n^2 \varphi_n^2(l)},\ \varphi_n(x) = \sin(\lambda_n l)\sinh(\lambda_n x) + \sinh(\lambda_n l)\sin(\lambda_n x),$$

λ_n are the positive roots of the transcendental equation $\tan(\lambda l) - \tanh(\lambda l) = 0$;

Appendix 2

In this appendix, we compute functions Ψ_n and Ψ_{n1} explicitly when ω_1 and ω_2 are uniformly distributed random variables.

Let $\Omega = \left\{\omega_1, \omega_2 \in \mathbb{R}^+, \omega_{i0} \le \omega_i \le \omega_{i1}, i = 1, 2\right\}$ be a rectangle. Then, the joint probability density function of ω_1 and ω_2 reads as

$$\rho(\omega_1, \omega_2) = \frac{1}{\mu(\Omega)} \chi_\Omega(\omega_1, \omega_2),$$

where χ_Ω is the indicator function, and $\mu(\Omega) = (\omega_{11} - \omega_{10})(\omega_{11} - \omega_{10})$ is the area of Ω. Then,

$$\Psi_n(t)=\frac{2}{\mu(\Omega)\lambda_n^4t^2}\left[\omega_{20}S_{1n}(t)-\omega_{21}S_{2n}(t)\right]-\frac{2}{\mu(\Omega)\lambda_n^2t}\left[C_{1n}(t)-C_{2n}(t)\right]-$$

$$-\frac{2}{\mu(\Omega)\lambda_n^2t}\left[\omega_{10}S_{3n}(t)-\omega_{11}S_{4n}(t)\right],$$

$$\Psi_{n1}(t)\;=-\frac{4}{\mu(\Omega)\lambda_n^6t^3}\left[\omega_{20}S_{1n}(t)-\omega_{21}S_{2n}(t)\right]+\frac{4}{\mu(\Omega)\lambda_n^4t^2}\left[C_{1n}(t)-C_{2n}(t)\right],$$

where

$$S_{1n}(t)=\sin\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{20}}t}\right]-\sin\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{20}}t}\right],$$

$$S_{2n}(t)=\sin\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{21}}t}\right]-\sin\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{21}}t}\right],$$

$$C_{1n}(t)=\sqrt{\omega_{10}\omega_{20}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{20}}t}\right]-\sqrt{\omega_{11}\omega_{20}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{20}}t}\right],$$

$$C_{2n}(t)=\sqrt{\omega_{10}\omega_{21}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{21}}t}\right]-\sqrt{\omega_{11}\omega_{21}}\cos\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{21}}t}\right],$$

$$S_{3n}(t)=\mathrm{Si}\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{20}}t}\right]-\mathrm{Si}\left[\lambda_n^2\sqrt{\frac{\omega_{10}}{\omega_{21}}t}\right],$$

$$S_{4n}(t)=\mathrm{Si}\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{20}}t}\right]-\mathrm{Si}\left[\lambda_n^2\sqrt{\frac{\omega_{11}}{\omega_{21}}t}\right],$$

and Si is the integral sine.

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ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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LOCALIZED SHEAR WAVES IN PIEZOELECTRIC LAYER COVERED BY ELASTIC THIN COATING

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Key words: localized wave, piezoelectric, thin film, composite.

Казарян К.Б., Папян А.А.

Локализованные сдвиговые волны в пьезоэлектрическом слое с тонким упругим покрытием

Ключевые слова: локализованная волна, пьезоэлектрики, тонкое покрытие, композит.

В статье рассматривается распространение локализованной волны сдвига в двухфазной среде: упругое покрытие и пьезоэлектрический слой. В рамках прикладной модели тонкого слоя получено дисперсионное уравнение для частот связанных электроупругих волн. Обсуждается влияние упругого покрытия и пьезоэффекта на локализованные фазовые скорости волн.

Ղազարյան Կ.Բ., Պապյան Ա.Ա.

Տեղայնացված սահքի ալիքները բարակ առաձգական ծածկույթով պիեզոէլեկտրիկ շերտում

Հիմնաբառեր. տեղայնացված ալիքներ, պիեզոէլեկտրիկ, բարակ ծածկույթ, կոմպոզիտ.

Հոդվածում ուսումնասիրվում է սահքի տեղայնացված ալիքների տարածումը երկշերտ միջավայրում: Բաղկացած առաձգական ծածկույթից և պիեզոէլեկտրիկ շերտից,: Բարակ շերտի կիրառական մոդելի հիման վրա ստացվել է դիսպերսիոն հավասարում կապակցված էլեկտրաառաձգական ալիքների համար: Քննարկվել է առաձգական ծածկույթի և պիեզոէֆեկտի ազդեցությունը տեղայնացված ալիքի փուլային արագության վրա:

The paper focuses on shear localized wave propagation in two phase medium: elastic thin coating and piezoelectric layer. In the framework of an applied model of thin layer a dispersion equation is derived for coupled electro elastic wave phase speeds. The influence of elastic coating and piezo effect on localized wave phase speeds are discussed.

Introduction.

Piezoelectric composites that are made of by two or more of piezoceramic materials are widely studied and discussed in [1-11]. In the problems of wave propagation in the composites the perfect bonding at the interface between two materials is routinely assumed. The composite structures consisting of piezoelectric ceramics with several types of partial contacts at the interface between two materials i.e., the electrically shorted or electrically closed, mechanically compliant (sliding) interfaces are considered in [4-8]. Electro elastic shear surface (localized) waves in composite structures: inhomogeneous piezoelectric layer – piezoelectric substrate, dielectric layer- piezoelectric substrate, piezoelectric layer - inhomogeneous substrate are considered in [8-11]. The localized waves in piezoelectric layer with different electrical and mechanical boundary conditions at walls of layers are studied in [12]. A model of boundary contact for electro-magneto-elastic composites with interface roughness is proposed in [13].

In the paper an analytical solution is given for a problem of shear localized wave propagation in bi-material media constituted by thin elastic thin coating and piezoelectric layer. The interface between elastic coating and piezoelectric layer is considered to be elastically perfect and electrically shorted one. Analogous problem for pure elastic bi-material media is considered in [11]. For thin coating the applied model is used which brings to averaged boundary conditions at contact interface between piezoelectric and thin coating.

Statement of the problem

Let's consider shear wave propagation along a bi-material layer consisting from thin elastic coating and piezoelectric layer. The geometry of the bi-material layer in the Cartesian system (x, y, z), $-a_0 < x < a$, $-\infty < y < \infty$, $-\infty < z < \infty$ is depicted in Fig.1.

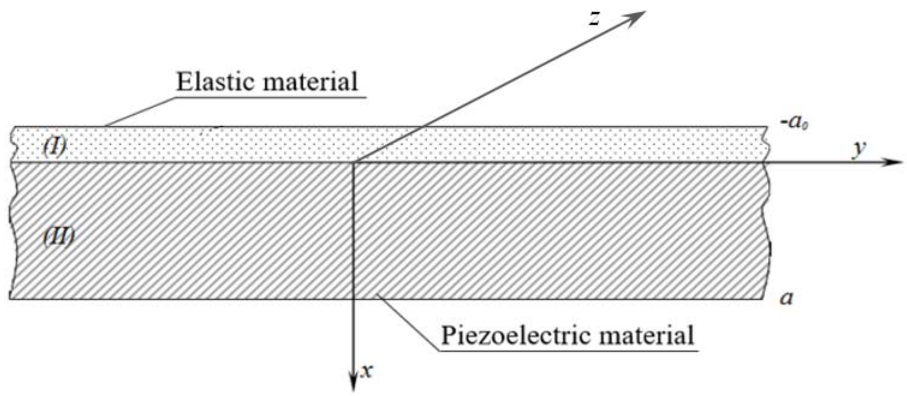

Fig1. Bi-material layer in Cartesian system (x, y, z)

For piezoelectric layer of piezo crystal of 6mm hexagonal class of symmetry with polling axis parallel to z coordinate direction, the anti-plane problem is described by the following equations and the constitutive material relations, based on the decoupled linear dynamic equations of theory of elasticity and quasi- static set of Maxwell equations [1,16]

$$\nabla\cdot\vec{\sigma}=\rho\frac{\partial^2 U}{\partial t^2}, \qquad \nabla\cdot\vec{D}=0, \qquad \vec{E}=-\vec{\nabla}\cdot\varphi$$

$$\vec{\sigma}=\vec{\nabla}\left(GU+e_{15}\varphi\right)\ , \quad \vec{D}=\vec{\nabla}\left(-\varepsilon\varphi+e_{15}U\right) \tag{1}$$

$$\vec{\sigma}=\left(\sigma_{xz}\left(x,y,t\right),\sigma_{yz}\left(x,y,t\right)\right);\vec{E}=\left(E_x\left(x,y,t\right),E_y\left(x,y,t\right),0\right);$$

$$\vec{D}=\left(D_x\left(x,y,t\right),D_y\left(x,y,t\right),0\right),$$

Here, σ_{xz} and σ_{yz} are shear stresses, $\vec{E}$ is the electric field intensity vector, $\vec{D}$ is the electrical displacement vector, $\varphi=\varphi(x,y,t)$ is the electric field potential, ρ is bulk density, $\vec{\nabla}$ is the nabla vector, G is the shear modulus, ε is electrical permittivity coefficient, e_{15} is the piezoelectric modulus.

Using (1) we get the following set of equations

$$c^2\Delta U-\frac{\partial^2 U}{\partial t^2}=0; \qquad \Delta\left(U-\frac{e_{15}}{\varepsilon}\varphi\right)=0\ , \quad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{2}$$

For elastic layer the following equation and material relations are valid

$$\frac{\partial\sigma_{0xz}}{\partial x}+\frac{\partial\sigma_{0yz}}{\partial y}-\rho_0\frac{\partial^2 V}{\partial t^2}=0,\quad \sigma_{0xz}=G_0\frac{\partial V}{\partial x},\quad \sigma_{0yz}=G_0\frac{\partial V}{\partial y};\tag{3}$$

In (2, 3) ρ_0 is the bulk density, G_0 are the shear modulus of piezoelectric material, $c=\sqrt{\left(G+\varepsilon^{-1}e_{15}^2\right)\rho^{-1}}$.

At the bi-material interface $y=0$ we consider the partial contact conditions of electrically shorted contact for electric potential and continuous mechanical displacements and tractions

$$\varphi(0,y,t)=0;\quad \sigma_{xz}(0,y,t)=\sigma_{0xz}(0,y,t);\qquad \mathrm{U}(0,y,t)=V(0,y,t)\tag{4}$$

At the bi-material layer external surfaces we take the following conditions

$$\sigma_{0xz}(-a_0,y,t)=0,\qquad \sigma_{xz}(a,y,t)=0,\qquad \varphi(a,y,t)=0.\tag{5}$$

Assuming that $a_0 << a$ we average [10, 14, 15] the equation (3) by elastic layer thickness along coordinate x

$$\int_{-a_0}^{0}\left(\frac{\partial\sigma_{0xz}}{\partial x}+\frac{\partial\sigma_{0yz}}{\partial y}-\rho_0\frac{\partial^2 V}{\partial t^2}\right)dx=0.\tag{6}$$

Taking in a view of the smallness of the thickness a_0, assuming that the displacement $V(x,y,t)$ do not vary along the thickness of the elastic layer and taking into account the boundary and contact conditions (4, 5) the averaged boundary condition at $x=0$ interface can be cast as

$$\sigma_{xz}(0,y,t)+a_0G_0\frac{\partial^2 U}{\partial y^2}-a_0\rho_0\frac{\partial^2 U}{\partial t^2}=0;\qquad x=0.\tag{7}$$

Presenting the solutions in the form of plane wave propagating along y direction

$$U(x,y,t)=U_0(x)\exp(iky-i\omega t),\varphi(x,y,t)=\varphi_0(x)\exp(iky-i\omega t)\tag{8}$$

we get solutions for $U_0(x),\varphi_0(x)$ as

$$U_0(x)=C_1\cos\left(x\sqrt{\eta^2-1}\right)+C_2\sin\left(x\sqrt{\eta^2-1}\right)\tag{9}$$

$$\varphi_0(x)=A_1\exp(kx)+A_2\exp(-kx)+\frac{e_{15}}{\varepsilon}\left(C_1\cos\left(x\sqrt{\eta^2-1}\right)+C_2\sin\left(x\sqrt{\eta^2-1}\right)\right)$$

Here C_1,C_2,A_1,A_2 are the arbitrary unknown constants, $\eta=\omega(ck)^{-1}$ is the dimensional phase speed of electro elastic vibrations.

Substituting these solutions into the boundary conditions we obtain a homogeneous set of algebraic equations, with respect to the arbitrary constants C_1,C_2,A_1,A_2. For nontrivial solutions, the determinant of this set has to vanish. Equating the determinant to zero we obtain the following dispersion equation determining phase speed η.

$$\left(-\frac{2\left(\chi^2+\chi\right)}{\cosh\left(K\sqrt{1-\eta^2}\right)\cosh(K)}+2\left(\chi^2+\chi\right)\right)+$$
$$+\left(\left(\eta^2-1\right)(\chi+1)^2-\chi^2\right)\frac{\tanh\left(K\sqrt{1-\eta^2}\right)\tanh(K)}{\sqrt{1-\eta^2}}+ \tag{10}$$
$$+\frac{\rho_0}{\rho}K_0\left(\frac{c_0^2}{c^2}-\eta^2\right)\left(\chi\frac{\tanh\left(K\sqrt{1-\eta^2}\right)}{\sqrt{1-\eta^2}}-(\chi+1)\tanh(K)\right)=0$$

Here the following notations are used

$\chi=\dfrac{e_{15}^2}{G\varepsilon}$, $K=ka,\ K_0=ka_0;\ c_0^2=G_0/\rho_0$.

The dispersion equation (10) defines a localised wave phase speed $\eta(K)$ as a function of the dimensionless width K and the elastic and electromechanical coefficients of the layered structure.

Now we shall restrict ourselves to consideration the case of localized vibrations of the piezo elastic layer when $\eta<1$ only.

Elastic layer dispersion equation

When the thin elastic coating is absent ($a_0\to 0$), the equation (10) results in

$$f_1(\eta)\,f_2(\eta)=0$$
$$f_1(\eta)=-(\chi+1)\sqrt{1-\eta^2}\,\tanh\left(\frac{K\sqrt{1-\eta^2}}{2}\right)+\chi\tanh\left(\frac{K}{2}\right); \tag{11}$$
$$f_2(\eta)=\chi\tanh\left(\frac{K\sqrt{1-\eta^2}}{2}\right)-(\chi+1)\sqrt{1-\eta^2}\,\tanh\left(\frac{K}{2}\right)$$

The equations (11) have been obtained first in [8].

Here we will only mention that the first equation of (11) corresponds to a dispersion equation of symmetric vibration of layer with boundary condition

$$\sigma_{xz}(0,y,t)=0,\ \varphi(0,y,t)=0;\ \sigma_{xz}\left(\frac{a}{2},y,t\right)=0,\ D_x\left(\frac{a}{2},y,t\right)=0 \tag{12}$$

and the second equation of (11) corresponds to a dispersion equation of anti-symmetric vibration of layer with boundary condition

$$\sigma_{xz}\left(0,y,t\right)=0,\ \varphi\left(0,y,t\right)=0;\ U\left(\frac{a}{2},y,t\right)=0,\ \varphi\left(\frac{a}{2},y,t\right)=0; \tag{13}$$

The phase speed –wave number (dimension less thickness) dispersion curves $\eta\left(K\right)$ are depicted on the Fig.2 for piezoceramic PZT-5H, where δ_0 is the well-known Bluestein-Gulyaev solution for semi- space with electrically shorted and mechanically free interface [12]., δ_{01} is the limiting value of localized wave speed at $K\to 0$

$$\delta_0=\sqrt{1+2\chi}\big/\left(1+\chi\right)=\sqrt{1-\chi_0^4}\ ,\quad \delta_{01}=\left(1+\chi\right)^{-\frac{1}{2}}<\delta_0 \tag{14}$$

The notation $\chi_0=\sqrt{e_{15}^2\left(G\varepsilon+e_{15}^2\right)^{-1}}$ was used by Bluestein in [12].

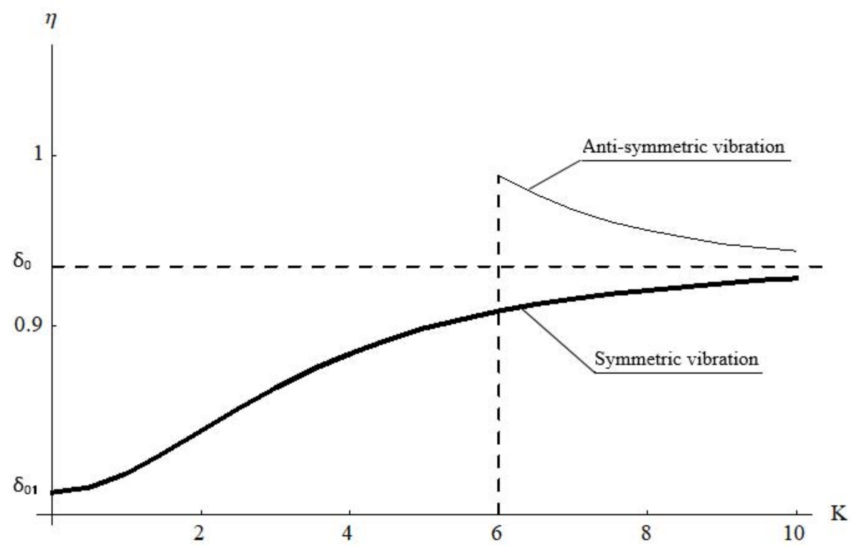

Fig.2. Dispersion curves $\eta\left(K\right)$ for elastic layer, $\delta_0=0.935,\ \delta_{01}=0.803$

Modal structure of the localised wave in the bi-material layer

First it will be noted that the bi-material layer dispersion equation does not decoupled into symmetric and anti-symmetric types of vibration which means that two phase structure may not in some cases support two types of localized waves.
In the case of $K>>1$ the dispersion (10) can be rewritten as

$$\frac{K_0G_0}{G}\left(1-\gamma^{-1}\eta^2\right)+\left(\left(1+\chi\right)\sqrt{1-\eta^2}-\chi\right)=0 \tag{15}$$

For thin layer $K<<1$ instead of dispersion equation (9) we get

$$\eta = \sqrt{\frac{1+\gamma\dfrac{a_0\rho_0}{a\rho}}{(1+\chi)+\dfrac{a_0\rho_0}{a\rho}}} \tag{16}$$

Here $\gamma = c_0^2/c^2$ is the ratio coefficient of phase speeds of bulk shear waves in the elastic layer and piezoelectric layer. The analysis of the dispersion equations (11, 15, 16) reveals that its solutions are very sensitive to coefficient γ. For the «soft» elastic coating $\gamma \le 1$ we have that $\eta < 1$ for any values of K, but in the case of the «hard» elastic coating $\gamma > 1$ the equation (9) may have no localised solutions for small values of K. Moreover, in this case, for some classes of materials, only one localized wave can exist in a layered structure, in contrast to the case of a «soft» elastic layer.

These circumstances and more are illustrated in Table 1, for certain class of materials (η_1, η_2 are the speeds of localized waves).

Table 1

	$\chi=0$		$\chi=0.3$		$\chi=0.5$		$\chi=0$		$\chi=0.3$		$\chi=0.5$	
K	η_1	η_2	η_1	η_2	η_1	η_2	η_1	η_2	η_1	η_2	η_1	η_2
0.5	-	-	-	-	-	-	0.885	-	0.805	-	0.762	-
1.0	-	-	-	-	-	-	0.933	-	0.841	-	0.793	-
3.0	-	-	0.980	-	0.936	-	0.973	-	0.897	-	0.856	-
5.0	-	-	0.973	-	0.938	-	0.981	-	0.924	-	0.890	-
7.0	-	-	0.973	-	0.941	-	0.985	-	0.936	-	0.905	0.962
10	-	-	0.973	-	0.942	-	0.987	-	0.943	0.980	0.911	0.945
$\rho_0/\rho=2,\ c_0/c=4\quad K_0=0.1$							$\rho_0/\rho=2,\ c_0/c=0.2\quad K_0=0.1$					

Conclusion

The localized vibration of layered structure is considered consisting from elastic coating and piezoelectric layer of piezo crystals of 6mm hexagonal symmetry class. Based on an applied model of thin elastic coating the dispersion equation is obtained for phase speed as a function of the layer dimensionless width and the elastic and electromechanical coefficients of the layered structure. It is defined that the solutions of the dispersion equation are very sensitive to the coefficient of ratio of phase speeds of bulk shear waves in the elastic layer and piezoelectric layer. For some classes of materials it is shown that the layered structure with the “hard” elastic coating can support only one localized wave, contrary to the case of the “soft” elastic coating where two localized waves may exist.

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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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ON AXIAL SYMMETRIC STRESS STATE OF UNIFORMLY LAYERED SPACE WITH SYSTEM OF PERIODICAL INNER DISK-SHAPED CRACKS

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Keywords: discontinuous solutions, periodic system of cracks, axial symmetric stress state.

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Осесимметричное напряжённое состояние равномерно слоистого пространства, содержащей периодическую систему внутренних дискообразных трещин

Ключевые слова: разрывные решения, периодическая система трещин, осесимметричное напряжённое состояние.

В статье рассматривается осесимметричное напряжённое состояние кусочно-однородного, равномерно слоистого пространства, образованного чередующимся соединением двух неоднородных слоев одинаковой толщины, которые содержат периодические системы круглых дискообразных параллельных трещин в срединных плоскостях. Получена система определяющих уравнений задачи в виде системы интегральных уравнений типа Фредгольма, решение которой построено методом механических квадратур. Проведён численный анализ и выявлены закономерности изменения коэффициентов интенсивности разрушающих напряжений и раскрытия трещин в зависимости от физико-механических и геометрических характеристик задачи.

Հակոբյան Վ.Ն., Դաշտոյան Լ.Լ., Մուրաշկին Ե.Վ.

Շրջանաձև ներքին ճաքերի համակարգ պարունակող հավասարաչափ շերտավոր տարածության առանցքահամաչափ լարվածային վիճակը

Հիմնաբառեր: խզվող լուծումներ, ճաքերի պարբերական համակարգ, առանցքահամաչափ լարվածային վիճակ:

Աշխատանքում դիտարկված է հավասար հաստություններ ունեցող երկու անհամասեռ շերտերի հաջորդական միացումից ստացված կտոր առ կտոր համասեռ, շերտավոր տարածության առանցքահամաչափ լարվածային վիճակը, երբ շերտերի միջին հարթության մեջ առկա են զուգահեռ, շրջանաձև ճաքերի պարբերական համակարգեր: Ստացվել է խնդրի որոշիչ հավասարումները Ֆրեդհոլմի տիպի երկու ինտեգրալ հավասարումների համակարգի տեսքով, որի լուծումը կառուցված է մեխանիկական քառակուսացման բանաձևերի մեթոդով: Ուսումնասիրված են քայքայող լարումների ինտենսիվության գործակիցների և ճաքերի բացվածքների փոփոխման օրինաչափությունները՝ կախված ֆիզիկամեխանիկական և երկրաչափական բնութագրիչներից:

This paper considers the axisymmetric stress state of piecewise homogeneous, uniformly layered space formed by alternate junction of two heterogeneous layers with same thickness, which contains the periodical systems of circle disk-shaped parallel cracks in median plane. The governing system of integral equations for problem is obtained using the Hankel integral transformation. Using rotation operators the governing system of equations is reduced to the system of Fredholm integral equation of second kind. The system is solved by the method of mechanical quadratures. The numerical analysis is carried out and the regulations for changing of intensity factors of fracture stresses and opening of crack depending on physical and mechanical and geometrical characteristics of problem are revealed.

Introduction

Few research studies are dedicated to the analysis of the axisymmetric stress state of an elastic piecewise-homogeneous, uniformly layered space of two heterogeneous layers with interphase disc-shaped defects. Among them, the work [1], where the discontinuous solutions of the equations of the axisymmetric theory of elasticity are constructed for a piecewise homogeneous, uniformly layered space obtained by alternately junction of two heterogeneous layers of the same thickness, which contain a periodic system of circular disc-shaped parallel interphase defects is noteworthy. The solutions of two certain problems when the defect is an absolutely rigid disk-shaped inclusion and when the defect is a disk-shaped crack are obtained. Also, a complete review of the papers directly related to this article is presented here. We also cite papers [2–4], where many of the main results on axisymmetric contact and mixed boundary value problems of the theory of elasticity are demonstrated. As regards the axial symmetric stress state of a piecewise-homogeneous, layered space with a periodic system of circular disk-shaped parallel internal defects, which is interesting and relevant both from scientific and practical points of view, there are no available studies conducted to our best knowledge.

1. The statement of problem and discontinuous solutions

In a cylindrical coordinate system $Or\varphi z$, we consider the axisymmetric stress state of a piecewise-homogeneous elastic space, obtained by alternately sequenced junction of two heterogeneous layers of thickness $2h$ with Lame coefficients λ_1, μ_1 and λ_2, μ_2, respectively, when on the median planes of heterogeneous layers $z = (2n+1)h \ (n \in Z)$ the space is relaxed by two systems of periodic circular disk-shaped, parallel cracks with radii $a_j \ (j = 1,2)$. We assume that the space is deformed under the influence of the same axisymmetric normal loads $P_j(r) \ (j = 1,2)$ acting respectively on the banks of cracks in heterogeneous layers. Problem is to determine the patterns of change of the opening of crack and the intensity factors of fracture stress at the circles $r = a_j$ depending both on the physical and mechanical as well as geometric characteristics of the heterogeneous layers. It is obvious, with such a formulation of the problem, all the middle planes $z = (2n+1)h \ (n \in Z)$ of heterogeneous layers are planes of symmetry, which allows us to separate the base cell as a two-component layer occupying a region $\Omega\{|z| \le h; 0 \le r < \infty; 0 \le \varphi \le 2\pi\}$ in space and state the problem as a boundary value problem for this layer. Fig.1 shows the axial section of the base cell.

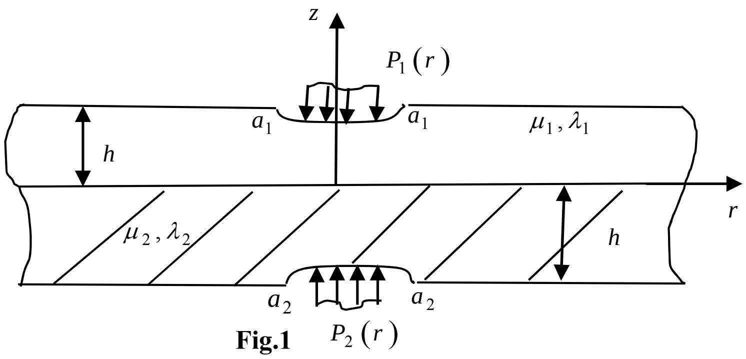

Fig.1

Using indices 1 and 2 for all the parameters describing the stress state of the heterogeneous layers of the base cell, respectively, we write the problem as the following boundary value:

$$\begin{cases} \tau_{rz}^{(j)}(r,(-1)^{j+1}h)=0 \quad (0\le r<\infty) \\ u_z^{(j)}\left(r,(-1)^{j+1}h\right)=0 \quad \left(a_j\le r<\infty\right) \qquad (j=1,2) \\ \sigma_z^{(j)}(r,h)=-P_j(r) \quad \left(0\le r<a_j\right) \end{cases} \tag{1.1a}$$

$$\begin{cases} u_r^{(1)}(r,0)=u_r^{(2)}(r,0) \quad (0\le r<\infty) \\ u_z^{(1)}(r,0)=u_z^{(2)}(r,0) \quad (0\le r<\infty) \\ \sigma_z^{(1)}(r,0)=\sigma_z^{(2)}(r,0) \quad (0\le r<\infty) \\ \tau_{rz}^{(1)}(r,0)=\tau_{rz}^{(2)}(r,0) \quad (0\le r<\infty) \end{cases} \tag{1.1b}$$

Here $u_r^{(j)}(r,z)$ and $u_z^{(j)}(r,z)$ $(j=1,2)$ are respectively the radial and vertical displacements of the points of the layers, and $\sigma_z^{(j)}(r,z)$ and $\tau_{rz}^{(j)}(r,z)$ are the normal and radial stresses acting in the corresponding layers. To construct a solution to the boundary value problem (1.1), we represent the solutions of the Lame equations in the form of Hankel integrals [1]:

$$\begin{aligned} u_r^{(j)}(r,z)&=\int_0^\infty\left[\left(A_j(s)+zB_j^*(s)\right)\mathrm{ch}(zs)+\left(B_j(s)+zA_j^*(s)\right)\mathrm{sh}(zs)\right]sJ_1(rs)ds; \\ u_z^{(j)}(r,z)&=\int_0^\infty\left[\left(C_j(s)-zA_j^*(s)\right)\mathrm{ch}(zs)+\left(D_j(s)-zB_j^*(s)\right)\mathrm{sh}(zs)\right]sJ_0(rs)ds, \end{aligned} \tag{1.2}$$

$$A_j^*(s)=\frac{s}{æ_j}\left(A_j(s)+D_j(s)\right);\ B_j^*(s)=\frac{s}{æ_j}\left(B_j(s)+C_j(s)\right)\ (j=1,2),$$

$J_j(x)$ $(j=0,1)$ – the Bessel functions of the real argument, $A_j(s)$, $B_j(s)$, $C_j(s)$, $D_j(s)$ are the unknown coefficients to be determined, and $æ_j$ $(j=1,2)$ the known constants of Muskhelishvili. To solve the boundary value problem (1.1), we introduce into consideration the unknown functions of the displacements of the crack edges.

$$u_z\left(r,(-1)^{j+1}h\right)=\frac{(-1)^j}{2}w_j(r)\ \left(0<r<a_j\right), \tag{1.3}$$

and solve an auxiliary boundary-value problem consisting of conditions (1.1), in which the last conditions (1.1a) are replaced by conditions (1.3). Then, using Hooke's law and representations (1.2), we determine the stress components, satisfy the conditions of the auxiliary boundary value problem, and determine the unknown coefficients $A_j(s), B_j(s), C_j(s), D_j(s)$ through Hankel trasponants of the functions of the displacements of the crack edges. The following is obtained

$$A_1^*(s) = -\frac{\left[\mu_2 æ_1 \vartheta_2^{(1)} - \mu_1(\mu_1 - \mu_2)\beta \operatorname{th}\beta\right]}{2æ_1 \vartheta_2^{(1)} \Delta_1(\beta)\operatorname{ch}\beta} s\,\overline{w}_1(s) - \frac{\mu_2 s\,\overline{w}_2(s)}{2\Delta_1(\beta)\operatorname{ch}\beta};$$

$$A_2^*(s) = -\frac{\mu_1 s\,\overline{w}_1(s)}{2\Delta_2(\beta)\operatorname{ch}\beta} - \frac{\left[\mu_1 æ_2 \vartheta_2^{(2)} + \mu_1(\mu_1 - \mu_2)\beta \operatorname{th}\beta\right]}{2æ_2 \vartheta_2^{(2)} \Delta_2(\beta)\operatorname{ch}\beta} s\,\overline{w}_2(s);$$

$$A_1(s) = A_2(s) = \frac{1}{\mu_1 - \mu_2}\left[\frac{æ_1 \vartheta_2^{(1)}}{s} A_1^*(s) - \frac{æ_2 \vartheta_2^{(2)}}{s} A_2^*(s)\right];$$

$$C_1(s) = C_2(s) = -\frac{1}{\mu_1 - \mu_2}\left[\operatorname{th}\beta\left(\frac{æ_1 \vartheta_2^{(1)}}{s} A_1^*(s) + \frac{æ_2 \vartheta_2^{(2)}}{s} A_2^*(s)\right) + \frac{\mu_1 \overline{w}_1(s)}{2\operatorname{ch}\beta} + \frac{\mu_2 \overline{w}_2(s)}{2\operatorname{ch}\beta}\right]$$

$$B_1^*(s) = -A_1^*(s)\operatorname{th}\beta - \frac{\mu_1 s \overline{w}_1(s)}{2æ_1 \vartheta_2^{(1)}\operatorname{ch}\beta};\quad B_1^*(s) = A_2^*(s)\operatorname{th}\beta + \frac{\mu_2 s \overline{w}_2(s)}{2æ_2 \vartheta_2^{(2)}\operatorname{ch}\beta};$$

$$\mathrm{B}_j(s) = -C_j(s) + \frac{æ_j}{s} B_j^*(s);\quad D_j(s) = -A_j(s) + \frac{æ_j}{s} A_j^*(s);$$

Here we use the notation:

$$\Delta_j(\beta) = 2æ_j \vartheta_2^{(j)} \Delta_j^*(\beta);\quad \Delta_j^*(\beta) = \operatorname{th}\beta + (-1)^j (\mu_1 - \mu_2) E_j(\beta);$$

$$E_j(\beta) = \frac{1}{2\vartheta_2^{(j)}}\left[\operatorname{th}\beta - \frac{\beta}{æ_j \operatorname{ch}^2\beta}\right];\quad \overline{w}_j(s,z) = \int_0^{a_j} r w_j(r) J_0(sr)\,dr;$$

$$\vartheta_1^{(j)} = \frac{\mu_j^2}{\lambda_j + 3\mu_j};\quad \vartheta_2^{(j)} = \frac{\mu_j(\lambda_j + 2\mu_j)}{\lambda_j + 3\mu_j};\quad (\beta = hs;\ j = 1,2).$$

Further, using the obtained values of the coefficients, we determine the normal stresses acting on the cracks through the unknown dislocation functions $w_j'(r)$ $(j = 1,2)$. The following is found:

$$\sigma_z^{(j)}\left(r, (-1)^{j+1} h\right) = -\vartheta_j L_{0,0}^{(2)}[w_j] - L_{0,0}^{(2,j,1)}[w_1] + L_{0,0}^{(2,j,2)}[w_2] \quad (j = 1,2), \tag{1.4}$$

$$L_{00}^{(2)}[w_j] = \int_0^{a_j} W_{00}^{(2)}(r,\xi)\xi w_j(\xi)\,d\xi;\quad W_{m,n}^{(k)}(r,\xi) = \int_0^{\infty} t^k J_m(tr) J_n(t\xi)\,dt$$

$$W_{m,n}^{(k,i,j)}(r,\xi)=\int_0^\infty K_{i,j}(\mathrm{th})t^k J_m(tr)J_n(t\xi)dt;$$

$$L_{m,n}^{(k,i,j)}\left[w_j\right]=\int_0^\infty s^2 K_{i,j}(\mathrm{sh})\overline{w}_j(s)J_0(sr)ds=\int_0^{a_j} W_{m,n}^{(k,i,j)}(r,\xi)\xi w_j(\xi)d\xi;$$

$$K_{1,1}(\beta)=\frac{\mu_1\mathrm{th}\beta\left[\left(\mu_1(æ_1+1)+æ_1\mu_2\right)\mathrm{sh}^2\beta+(æ_1+1)(\mu_1+\mu_2)/2\right]}{\mu_1(æ_1+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\mathrm{ch}^2(\beta)}+$$

$$+\frac{Q_{1,1}(\beta)}{\mu_1(æ_1+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\mathrm{ch}^2(\beta)}-\frac{\mu_1}{\vartheta_2^{(1)}}\left[\vartheta_1^{(1)}\mathrm{th}\beta+\frac{\mu_1\beta}{æ_1}\right]+$$

$$+\frac{2\beta\mathrm{th}\beta\left[(\mu_1+æ_1\mu_2)\mathrm{sh}^2\beta+\mu_2(æ_1+1)/2\right]}{(æ_1+1)^2\Delta_1^*(\beta)\mathrm{ch}^2(\beta)}-\frac{\mu_1}{2(1-\nu_1)};$$

$$K_{1,2}(\beta)=-\frac{\mu_1\mathrm{th}\beta\left[æ_1\mu_2\mathrm{sh}^2\beta+(æ_1+1)(\mu_1+\mu_2)/2\right]-Q_{1,2}(\beta)}{\mu_1(æ_1+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\mathrm{ch}^2(\beta)}-$$

$$-\frac{\mu_1\beta\mathrm{th}\beta}{(æ_2+1)\Delta_2^*(\beta)\mathrm{ch}^2(\beta)};$$

$$K_{2,1}(\beta)=\frac{\mu_2\mathrm{th}\beta\left[æ_2\mu_1 sh^2\beta+(æ_2+1)(\mu_1+\mu_2)/2\right]+Q_{2,1}(\beta)}{\mu_2(æ_2+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\mathrm{ch}^2(\beta)}+$$

$$+\frac{\mu_2\beta\mathrm{th}\beta}{(æ_1+1)\Delta_1^*(\beta)\mathrm{ch}^2(\beta)};$$

$$K_{2,2}(\beta)=-\frac{\mu_2\mathrm{th}\beta\left[\left(\mu_2(æ_2+1)+æ_2\mu_1\right)\mathrm{sh}^2\beta+(æ_2+1)(\mu_1+\mu_2)/2\right]}{\mu_2(æ_2+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\mathrm{ch}^2(\beta)}+$$

$$+\frac{Q_{2,2}(\beta)}{\mu_2(æ_2+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\mathrm{ch}^2(\beta)}+\frac{\mu_2}{\vartheta_2^{(2)}}\left[\vartheta_1^{(2)}\mathrm{th}\beta+\frac{\mu_2\beta}{æ_2}\right]-$$

$$-\frac{2\beta\mathrm{th}\beta\left[(\mu_2+æ_2\mu_1)\mathrm{sh}^2\beta+\mu_1(æ_2+1)/2\right]}{(æ_2+1)^2\Delta_2^*(\beta)\mathrm{ch}^2(\beta)}-\frac{\mu_2}{2(1-\nu_2)};$$

$$Q_{1,1}(\beta)=-\mu_1\mu_2E_2\left[(\mu_1+æ_1\mu_2)\mathrm{sh}^2\beta+\frac{\mu_2(æ_1+1)}{2}\right]-\frac{\mu_1^3(æ_1+1)E_1}{2}+$$

$$+\mu_1^3(æ_1+1)\mathrm{sh}^2\beta\left[E_2-E_1-(\mu_1-\mu_2)E_1E_2\mathrm{cth}\beta\right];$$

$$Q_{1,2}(\beta)=\mu_1\mu_2E_2\left[(\mu_1+æ_1\mu_2)\operatorname{sh}^2\beta+\frac{\mu_2(æ_1+1)}{2}\right]+\frac{\mu_1^3(æ_1+1)E_1}{2}-$$

$$-\mu_1^2\mu_2(æ_1+1)\operatorname{sh}^2\beta\left[E_2-E_1-(\mu_1-\mu_2)E_1E_2\operatorname{cth}\beta\right];$$

$$Q_{2,1}(\beta)=-\mu_1\mu_2E_1\left[(\mu_2+æ_2\mu_1)\operatorname{sh}^2\beta+\frac{\mu_1(æ_2+1)}{2}\right]-\frac{\mu_2^3(æ_2+1)E_2}{2}-$$

$$-\mu_1\mu_2^2(æ_2+1)\operatorname{sh}^2\beta\left[E_2-E_1-(\mu_1-\mu_2)E_1E_2\operatorname{cth}\beta\right];$$

$$Q_{2,2}(\beta)=\mu_1\mu_2E_1\left[(\mu_2+æ_2\mu_1)\operatorname{sh}^2\beta+\frac{\mu_1(æ_2+1)}{2}\right]+\frac{\mu_2^3(æ_2+1)E_2}{2}+$$

$$+\mu_2^3(æ_2+1)\operatorname{sh}^2\beta\left[E_2-E_1-(\mu_1-\mu_2)E_1E_2\operatorname{cth}\beta\right];\ \left(\vartheta_j=\frac{\mu_j}{2(1-\nu_j)}\right).$$

Note that kernels $W_{m,n}^{(k,i,j)}(r,\xi)$ are regular functions of both arguments. Now we satisfy the last two conditions (1.1a). As a result, to determine the displacements of the points of the edges of cracks, we obtain the following governing system of integral equations:

$$-\vartheta_jL_{0,0}^{(2)}\left[w_j\right]-L_{0,0}^{(2,j,1)}\left[w_1\right]+L_{0,0}^{(2,j,2)}\left[w_2\right]=-P_j(r)\ \left(0<r<a_j,\ \ j=1,2\right),\qquad(1.5)$$

which should be considered taking into account the conditions of continuity of displacements on circles $r=a_j$

$$w_j(a_j)=0\ \ (j=1,2)\qquad(1.6)$$

2. The solution of the governing system of integral equations

To solve the system of integral equations (1.5), using the rotation operator

$$I\left[\varphi(r)\right]=\int_0^x\frac{r\varphi(r)\,dr}{\sqrt{x^2-r^2}}$$

we reduce it to a system of Fredholm integral equations of the second kind. In this order, as in [2], we introduce functions

$$V_j(t)=\frac{2}{\pi}\int_t^{a_j}\frac{\xi w_j(\xi)\,d\xi}{\sqrt{\xi^2-t^2}}\ \ (j=1,2)\qquad(2.1)$$

continue them on the interval $(-a_j,0)$ as an even function and, taking into account the relations,

$$\bar{w}_j(s)=\frac{2}{\pi}\int_0^{a_j}\left[\int_t^{a_j}\frac{rw_j(r)\,dr}{\sqrt{r^2-t^2}}\right]\cos(ts)\,dt=\int_0^{a_j}V_j(t)\cos(ts)\,dt;$$

$$L_{0,0}^{(2)}\left[w_j\right]=\int_0^{a_j}V_j(t)\,dt\int_0^{\infty}s^2J_0(sr)\cos(ts)\,ds=-\int_0^{a_j}V_j'(t)\,dt\int_0^{\infty}sJ_0(sr)\sin(ts)\,ds;$$

$$L_{0,0}^{(2,i,j)}\left[w_j\right]=-\int_0^{a_j}V_j'(t)\,dt\int_0^{\infty}K_{i,j}(s)\,sJ_0(sr)\sin(ts)\,ds,$$

rewrite the system (1.5) in the following form:

$$\vartheta_j\int_0^{a_j}R(t,r)V_j'(t)\,dt+\sum_{i=1}^{2}\int_0^{a_i}R_{j,i}(t,r)V_i'(t)\,dt=-P_j(r);\qquad (0<r<a_j,\quad j=1,2), \tag{2.2}$$

$$R(r,t)=\int_0^{\infty}sJ_0(sr)\sin st\,ds;\quad R_{j,i}(r,t)=(-1)^{i+1}\int_0^{\infty}K_{j,i}(s)\,sJ_0(sr)\sin st\,ds.$$

From representation (2.1) it appears that $V_j(a_j)=0$. In this case, the conditions (1.6) are satisfied automatically, since using the inverse operators of rotation operators [1-3], functions can be written in the following form

$$w_j(r)=-\frac{1}{r}\frac{d}{dr}\int_r^{a_j}\frac{sV_j(s)}{\sqrt{s^2-r^2}}\,ds=-\int_r^{a_j}\frac{V_j'(s)}{\sqrt{s^2-r^2}}\,ds\quad (j=1,2) \tag{2.3}$$

Further, we apply the operator I to both parts of equations (2.2). After calculations, using the values of known integrals

$$\int_0^{x}\frac{J_0(rt)\,r\,dr}{\sqrt{x^2-r^2}}=\frac{\sin(xt)}{t};\quad \int_0^{\infty}\sin(st)\sin(sx)\,ds=\frac{\pi}{2}\left[\delta(t-x)-\delta(t+x)\right],$$

where $\delta(x)$ is the well-known Dirac function, we come to the following system of integral equations of the second kind of Fredholm type:

$$V_j'(x)+\sum_{i=1}^{2}\int_{-a_j}^{a_j}R_{j,i}^{*}(t,x)V_i'(t)\,dt=f_j(x)\quad (j=1,2); \tag{2.4}$$

$$f_j(x)=-\frac{2}{\pi\vartheta_j}I\left[P_j(r)\right];$$

$$R_{j,i}^{*}(t,x)=\frac{(-1)^{i+1}}{\pi\vartheta_j}\int_0^{\infty}K_{j,i}(s)\sin(ts)\sin(xs)\,ds.\quad (i,j=1,2)$$

With the help of the replacements of variables $t=a_j\xi,\ \ x=a_j\eta\ \ (j=1,2)$, we formulate the obtained equations on the interval (-1,1) and denote $\varphi_j(\eta)=V_j'(a_j\eta)/a_j$ we come to the system

$$\varphi_j(\eta)+\sum_{i=1}^{2}\int_{-1}^{1}Q_{j,i}^{*}(\xi,\eta)\varphi_i(\xi)\,d\xi=f_j^{*}(\eta)\quad (-1<\eta<1,\ j=1,2), \tag{2.5}$$

$$Q_{1,1}^{*}(\xi,\eta)=l_1R_{1,1}^{*}(l_1\xi,l_1\,\eta);\ Q_{2,2}^{*}(\xi,\eta)=l_2R_{2,2}^{*}(l_2\xi,l_2\,\eta);$$

$$Q_{1,2}^{*}(\xi,\eta)=\frac{l_2^2}{l_1}R_{1,2}^{*}(l_2\xi,l_1\eta);\ Q_{2,1}^{*}(\xi,\eta)=\frac{l_1^2}{l_2}R_{2,1}^{*}(l_1\xi,l_2\eta);$$

$$f_j^{*}(\eta)=-I\left[P_j\left(a_j\xi\right)\right]/\,\mathrm{a}_j;\ \ \left(l_j=a_j/h;\ \ j=1,2\right).$$

Thus, the solution of the problem was reduced to solving a system of integral equations of the second kind of Fredholm type (2.5), the solution of which can be constructed by the method of successive approximations. It is obvious that the solutions of this system functions $\varphi_j(x)$ $(j=1,2)$ are bounded on circles $r=a_j$, respectively. We will also write formulas with the help of which, after determining the functions $\varphi_j(\eta)$, it is possible to find the crack opening and the intensity factors of fracture stresses on the circles $r=a_j$. To determine the crack opening, we use formulas (2.3), from where we get the formulas for dimensionless crack openings:

$$W_j^{*}(\eta)=\frac{w_j\left(a_j\eta\right)}{a_j}=-\int_{\eta}^{1}\frac{\varphi_j(\xi)}{\sqrt{\xi^2-\eta^2}}d\xi. \tag{2.6}$$

To determine the intensity factors of fracture stresses $K_I^{(j)}\left(a_j\right)$ on the circles $r=a_j$ $(j=1,2)$ the formulas (1.7) with $(r>a)$ are represented in the form:

$$\sigma_z^{(j)}\left(r,(-1)^{j+1}h\right)=\vartheta_j\int_0^{a_j}R(t,r)V_j'(t)\,dt+\sum_{i=1}^{2}\int_0^{a_i}R_{j,i}(t,r)V_i'(t)\,dt \tag{2.7}$$

Further, using known formulas [5]

$$sJ_0(rs)=\frac{1}{r}\frac{d}{dr}\left(rJ_1(rs)\right);\qquad \int_0^{\infty}J_1(sr)\sin ts\,ds=\begin{cases}0 & t>r\\ \dfrac{t}{r}\dfrac{1}{\sqrt{r^2-t^2}} & t<r\end{cases},$$

(2.7) we write in the following form

$$\sigma_z^{(j)}\left(r,(-1)^{j+1}h\right)=\frac{\vartheta_j}{r}\frac{d}{dr}\int_0^{a_j}\frac{tV_j'(t)\,dt}{\sqrt{r^2-t^2}}+\sum_{i=1}^{2}\int_0^{a_i}R_{j,i}(t,r)V_i'(t)\,dt= \tag{2.8}$$

$$=-\frac{\vartheta_jV_j'\left(a_j\right)}{\sqrt{r^2-a_j^{\,2}}}+F_j(r).$$

$$F_j\ (r)=\sum_{i=1}^{2}\int_0^{a_i}R_{j,i}(t,r)V_i'(t)\,dt+\frac{\vartheta_j}{r}\frac{d}{dr}\int_0^{a_j}\frac{t\left[V_j'(t)-V_j'\left(a_j\right)\right]dt}{\sqrt{r^2-t^2}}+\vartheta_jV_j'\left(a_j\right)/\,r,$$

bounded functions respectively on circles $r=a_j$ $(j=1,2)$. From (2.8) we find

$$K_I^{(j)}\left(a_j\right)=\lim_{r\to a+0}\sqrt{2\left(r-a_j\right)}\,\sigma_z^{(j)}\left(r,(-1)^{j+1}h\right)=-\vartheta_j V_j'\left(a_j\right)/\sqrt{a_j}=$$
$$=-\sqrt{a}\,\vartheta_j\,\varphi_j(1)\quad\left(j=1,2\right).$$

Consequently, the dimensionless intensity factors of fracture stresses on circles $r=a_j\ \left(j=1,2\right)$ can be determined by the formulas:

$$K_{I,\mathrm{j}}^*(a)=K_I^{(j)}\left(a_j\right)/\sqrt{a_j}\mu_1=-\left(\frac{\mu_j}{\mu_1}\right)\frac{\varphi_j(1)}{2\left(1-\nu_j\right)}. \tag{2.9}$$

3. Numerical analysis

Using the method of mechanical quadrature, the numerical analysis is carried out and the regulations of change of dimensionless intensity coefficients of fracture stresses $K_{I,\mathrm{j}}^*\left(a_j\right)$ and dimensionless crack opening $W_j^*(x)=w_j\left(a_j x\right)/a_j$ depending on the parameter $l_j=a_j/h$ change in the case where $a_1=a_2=a,\ l_1=l_2=l,\ \ P_j(r)=P_0=\mathrm{const},$ $P_1^*=P_0/\mu_1=0.1,\ P_2^*=P_0/\mu_2=0.2,\ \nu_1=0,25,\quad \nu_2=0,3\quad\left(j=1,2\right)$. At the same time, the right parts of system (2.5) have the following form:

$$f_j^*(\eta)=-\frac{4\left(1-\nu_j\right)}{\pi}P_j^*\eta\quad\left(j=1,2\right).$$

The results of numerical calculations are shown in Table 1 and in Fig. 2a and 2b. Table 1 shows the values of the dimensionless intensity factors $K_{I,j}^*(a)\ \left(j=1,2\right)$ depending on the parameter $l=a/h$. The calculations show that with a decrease of l , which can be interpreted as an increase in the height h of the layers, with a constant a, the intensity factors increase, tending to a certain limit, corresponding to the case of a homogeneous space, made respectively of the first and second materials with one disk-shaped crack. Figures 2a and 2b show the graphs of crack opening depending on the parameter l .

Table 1. Intensity Factors depending on l

l	0.1	0.2	0.5	1	2	10
$K_{I,1}^*(a)$	0.06662	0.06649	0.06543	0.06197	0.05459	0.03327
$K_{I,2}^*(a)$	0.07138	0.07121	0.06981	0.06566	0.05833	0.03832

It can be seen from Figures that when l is decreasing, i.e. when removing cracks, the openings of crack also increases, tending to a certain limit, corresponding to the opening of one disk-shaped crack in a homogeneous space, made respectively of the material of the first and second layers.

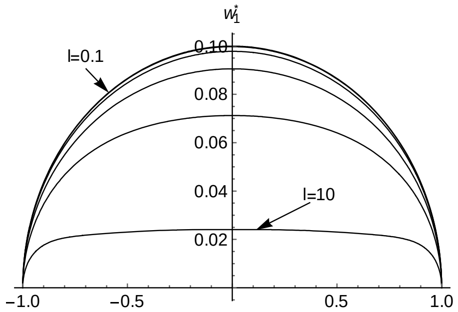

Fig. 2a. The first crack openings

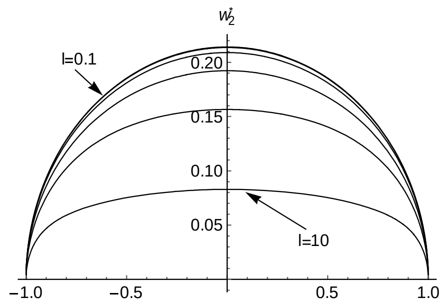

Fig. 2b. The second crack openings

Table 2 and Fig. 3a and 3b show the values of the dimensionless intensity factors $K^*_{I,j}(a)\ (j=1,2)$ and the graphs of crack openings $W_j^*(x)$ as a function of the parameter $\mu=\mu_2/\mu_1$ in the case when $l=2$, $\nu_1=0{,}4$, $\nu_2=0{,}25$ $P_1^*=0.1$.

Table 2. Intensity factors depending on μ

μ	1	2	5	10	50	100
$K^*_{I,1}(a)$	0.0707	0.0662	0.0608	0.0578	0.0544	0.0538
$K^*_{I,2}(a)$	0.0377	0.0349	0.0311	0.0289	0.0261	0.0257

The Tables show that in the this case, with an increase of μ, which can be interpreted as an increase of μ_2, with a constant μ_1, as the intensity factors of the fracture stress on the bounding circles of both cracks, and the opening of crack decrease.

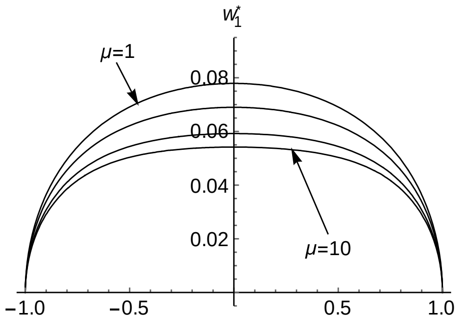

Fig. 3a. The first crack opening

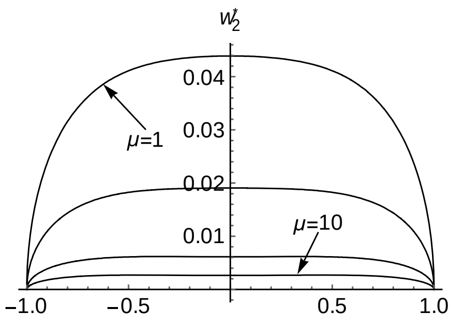

Fig. 3b. The second crack opening

CONCLUSION

Thus, a governing system of integral equations for a piecewise homogeneous, uniformly layered space with periodic systems of circular disk-shaped parallel internal cracks is obtained. The solution of system is constructed by the numerical-analytical method of mechanical quadratures. By numerical analysis, it is shown that the mutual influence of cracks increases when they approach each other. In the case of the removal of cracks from each other, they work as separate, single disk-shaped cracks in homogeneous spaces, made respectively of the material of the first and second layers. It is also shown that with an increase of the rigidity of one of the layers, when the rigidity of the other layer does not change, it leads to a decrease in both the intensity factors of the fracture stresses on the bounding circles of both cracks and their opening.

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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 72, №4, 2019 Механика

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ON THE STRESS-STRAIN STATE OF AN ELASTIC INFINITE PLATE WITH A CRACK EXPANDING BY MEANS OF SMOOTH THIN INCLUSION INDENTATION

Mkhitaryan S.M.

Keywords: elastic plate, crack, inclusion, cracks gap, stress intensity factor (SIF)

Мхитарян С.М.

О напряжённо-деформационном состоянии упругой бесконечной пластины с трещиной, расширяющейся посредством вдавливания в неё гладкого тонкого включения

Ключевые слова: упругая пластина, трещина, включение, раскрытие трещины, коэффициент интенсивности напряжений (КИН).

Рассматривается плоская задача об определении компонент напряжённо-деформированного состояния упругой изотропной бесконечной пластины с прямолинейной трещиной конечной длины, в которой вставлено тонкое абсолютно жёсткое включение с гладкой поверхностью. Предполагается, что эта поверхность обладает центральной и осевой симметрией, имеет форму типа сплюснутого эллипса или форму тонкого стержня прямоугольного сечения и при вдавливании плотно прилегает к берегам трещины, а на образовавшемся при этом контактном участке действуют только нормальные контактные напряжения. Решение обсуждаемой задачи сведено к решению интегрального уравнения Фредгольма первого рода с симметрическим логарифмическим ядром. Построены точное и численно-аналитическое приближённое решения этого уравнения. Рассмотрены частные случаи, проведён их численный анализ и исследованы закономерности изменения характеристик задачи.

Մխիթարյան Ս.Մ.

Ճաքով առաձգական անվերջ սալի լարվածադեֆորմացիոն վիճակի մասին, երբ ճաքն ընդլայնվում է նրան սեղմվող ողորկ բարակապատ ներդրակով

Հիմնաբառեր: առաձգական սալ, ճաք, ներդրակ, ճաքի բացվածք, լարումների ուժգնության գործակից:

Դիտարկվում է վերջավոր երկարության ուղղագիծ ճաքով առաձգական իզոտրոպ սալի լարվածադեֆորմացիոն վիճակի բաղադրիչների որոշման հարթ խնդիրը, երբ ճաքի մեջ ներդրված է ողորկ մակերևույթով բացարձակ կոշտ բարակապատ ներդրակ: Ենթադրվում է, որ այդ մակերևույթը օժտված է կենտրոնական և առանցքային համաչափությամբ և ունի տափակեցրած էլիպսի կամ ուղղագիծ բարակապատ ձողի տեսք և սեղմվելիս քիփ հպվում է ճաքի ափերին, իսկ առաջացած կոնտակտային տեղամասում գործում են միայն նորմալ կոնտակտային լարումներ: Խնդրի լուծումը բերված է սիմետրիկ լոգարիթմական կորիզով Ֆրեդհոլմի առաջին սեռի ինտեգրալ հավասարման լուծման: Կառուցված են այդ հավասարման փակ և մոտավոր թվային-վերլուծական լուծումներ: Դիտարկված են մասնավոր դեպքեր և կատարված են թվային վերլուծություններ, հետազոտված են խնդրի բնութագրիչների փոփոխման օրինաչափությունները:

The plane problem of determining components of the stress-strain state of an elastic isotropic infinite plate with a rectilinear crack of finite length, in which a thin rigid inclusion with a smooth surface is indented, is considered. We assumed that the inclusion surface has a centrally-axial symmetry, its shape is an oblate ellipse part or a thin rectangle, and during the indentation, the inclusion surface is tightly adjoined to edges of the crack, and only normal stresses act in the formed contact region. Solving the problem is reduced to solving the Fredholm integral equation of the first kind with a logarithmic symmetric kernel. The exact analytical and approximate numerical-analytical solutions of this equation convenient for engineering calculations are constructed. The main characteristics of the problem (such as contact normal stresses in the region of contact between inclusion and crack edges, the cracks gap (opening) outside the inclusion, normal breaking stresses outside the crack on its location line, their stress intensity factors (SIF)) are represented by explicit analytical formulas. In particular cases, the numerical analysis of these characteristics is carried out, regularities of their changes are revealed. The phenomenon of infinite increase of SIF is established at infinite approach of the end points of inclusion to the crack tips. As a result, the crack begins to propagate and brittle fracture of the plate occurs. Proceeding from this, as an application of the obtained results, an

estimation of crack resistance of a plate with a crack is given; namely, the critical value of the relative length of the rectilinear inclusion, at which the crack propagation begins, is determined.

1. Introduction. Cracks and foreign inclusions in deformable solid bodies are stress concentrators, around which the local stress fields characterized by large and rapidly changing gradients are formed. The stress concentrations have a significant influence on the strength characteristics of structures. Therefore, the qualitative and quantitative investigation of the stress concentrations as well as the development of ways to reduce them have theoretical and practical significance. Problems of determining the stress strain state of deformable bodies with cracks and inclusions, issues regarding the interaction of cracks and inclusions and their influence on strength properties of structures are often encountered in the mechanics of composites, in geomechanics, in thermoelasticity, and in their engineering applications. These problems are the subject of numerous studies [1-7]. In [3] the problem on the development of a crack in the vicinity of the rigid inclusion top was studied, and SIFs near the crack, on the continuation of the linear rigid inclusion, are determined. Different cases of combination of cracks and rigid inclusions in an elastic matrix were investigated in [8]. Many results on this topic are summarized in the handbooks of SIFs [9, 10].

In the present paper, we consider the plane problem of determining components of stresses and displacements (playing an important role in fracture mechanics) of an elastic isotropic infinite plate with a rectilinear crack of finite length, in which a thin absolutely rigid inclusion with a smooth surface is indented. We assume that the surface has a centrally-axial symmetry, its shape is a part of an ellipse oblate along its length or a thin rectangle. During the indentation, the inclusion surface is tightly adjoined to the crack edges and only normal stresses act on formed contact regions. Solving the problem is reduced to solving the Fredholm integral equation of the first kind with the logarithmic symmetric kernel; its exact solution is obtained (the necessary formulas and transformations for construction of the exact analytical solution are transferred to Appendices A, B, C). Based on the Gauss-type quadrature formulas (like in [11]) in combination with the method of collocation, the approximate numerical-analytical solution of the governing equation is also obtained.

It should be noted that the problem discussed here represents a flat analog of the axisymmetric problem, previously considered in [7], where only SIF is approximately calculated using the model based on the solution of the classical contact problem on the indentation of a round punch with the flat base into the elastic half-space. Such model is applicable only for small relative lengths of inclusion. However, while investigating the problem of crack propagation and clarifying the issue of the maximal inclusion length which an elastic matrix with a crack can withstand, it is necessary to consider exactly the large values of the relative length of the inclusion. On the other hand, these problems are closely related to the problems of wedging elastic bodies by thin, absolutely rigid wedges of various shapes, widely covered in the handbooks [9, 10]. On this concern let us point out also the works close to our subject [12-16]. However, the problem formulations are different: in the problems on wedging elastic bodies, the positions of the end points of cracks and contact zones are unknown in advance and are determined in the course of problem-solving. In our case, these parameters are given in advance.

The main characteristics of the problem mentioned above are represented by explicit analytical formulas. In particular cases, regularities of change in these characteristics depending on the specific parameters are revealed by the numerical analysis.

As an application of the obtained results, an estimation of crack resistance of a plate with a crack is given for the rectilinear inclusion. For this case, the critical value of the relative length of the inclusion, at which the crack propagates occurs, is determined. Thus, a plate with a crack cannot withstand inclusion of any length; especially the inclusion with a length equal to the length of the crack, i.e. when there is a complete contact of the inclusion with the crack edges.

2. Problem formulation. Assume that an infinite elastic plate with the elastic modulus E, the Poisson's ratio ν and the thickness h in a rectangular coordinate system Oxy has a crack along the abscissa axis (mathematical cut) of finite length $2l$, $L=\{y=0,\ -l\le x\le l\}$ (Fig. 1). It is thought that the plate is isotropic, homogeneous, and is in the generalized plane stress state. Suppose that a thin absolutely rigid inclusion of length $2a\ (a<l)$ with a smooth surface is indented into the crack. The surface is described by equations $y=\pm f(x)\ (-a\le x\le a)$, where $f(x)$ is an even nonnegative function $(f(-x)=f(x))$; both the function $f(x)$ and its first derivative are continuous on the interval $[-a,a]$ and $\Delta=f(0)=\max\limits_{|x|\le a} f(x)<<a(a<l)$.

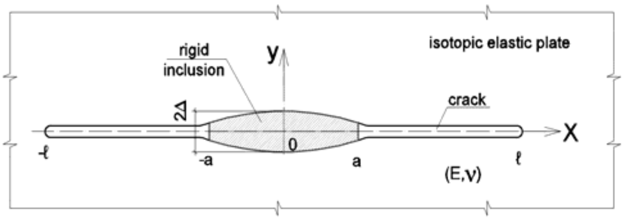

Fig.1. Smooth absolutely rigid oblate along its length inclusion with an upper surface $y=f(x)$ and lower surface $y=-f(x)$ putting pressure on crack edges $L=\{y=0,-l\le x\le l\}$ in an elastic infinite plate.

In particular, the thin inclusion may be in the form of a strongly oblate along its length ellipse part

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1 \quad (b_1<<a,-a\le x\le a,\ a<a_1<l).$$

The minor semi-axis b_1 of the ellipse is much shorter than the major semi-axis a_1. Whence

$$f(x)=\frac{b_1}{a_1}\sqrt{a_1^2-x^2}\ (-a\le x\le a;\ a<a_1\le l). \tag{1}$$

Such a choice of the shape of the inclusion is due to the fact that according to [17–19] the crack is considered as the limiting case of an ellipse when $b_1\to 0\ (a_1=l)$; in this sense geometric forms of the thin inclusion and crack are compatible. In addition, under this limiting transition, the stress state of an elastic infinite plate with a thin elliptic hole becomes a stress state of the plate caused by a Griffiths crack [19] (pp. 308-309).

Besides the described form, a thin inclusion can also be in the form of a rectilinear segment with length $2a$ and height 2δ, where $\delta<<a$ (Fig.2).

Hereafter we consider the possibility of approaching the inclusion endpoints to the crack tips $(a\to l)$. In accordance with the crack interpretation as a limiting case of the ellipse (1) when $b_1\to 0$ [17–19], and taking into account well-known asymptotic formulas for displacements near the crack tips, it will be assumed that the function $f(x)$ has the following behavior

$f(x) = 0\left((l \pm x)^{-1/2}\right) \quad (x \to \mp l)$.

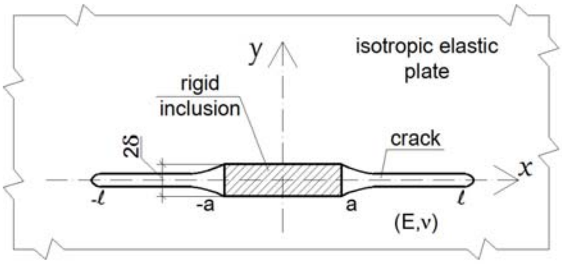

Fig.2. Smooth absolutely rigid linear inclusion with length $2a$ and height $2\delta \ (\delta << a,\ a < l)$ putting pressure on crack edges $L = \{y = 0,\ -l \le x \le l\}$ in an elastic infinite plate.

Then we suppose that the thin inclusion with a smooth surface, indenting into the crack edges, tightly adjoins them along its entire length, i.e. contact region is the line segment $-a \le x \le a$. Because of the smoothness of the inclusion surface, we assume that only normal stresses arise in the contact region.

Under these assumptions, it is required to determine the pressure of the inclusion surface on the crack edges or normal contact stresses, crack opening outside the inclusion, normal breaking stresses outside the crack along the line of its location and SIFs.

Due to symmetry about the x-axis, within a well-known approximation [20, pp. 114-115], according to which the boundary conditions from the walls of the inclusion can be transferred to its midline, the posed problem can be formulated as the following mixed boundary-value problem of the mathematical theory of elasticity for the elastic upper half-plane $y > 0$:

$$\left.\begin{array}{l} \mathrm{v}(x,y)\big|_{y=+0} = f(x) \ \left(-a \le x \le a,\ f(-x) = f(x) \ge 0\right), \quad \mathrm{v}(x,y)\big|_{y=+0} = 0 \ \left(|x| \ge l\right), \\ \tau_{xy}\big|_{y=+0} = 0 \quad (-\infty < x < \infty), \quad \sigma_y(x,y)\big|_{y=+0} = 0 \qquad \left(a < |x| < l\right), \\ \sigma_x(x,y),\ \sigma_y(x,y)\ \tau_{xy}(x,y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \end{array}\right\} \quad \text{(2a-e)}$$

Here $\mathrm{v}(x,y)$ is a vertical displacement of the point $M(x,y)$ of the upper elastic half-plane and σ_x, σ_y, τ_{xy} are components of normal and tangential stresses, respectively.

Reduce solving the problem (2a-e) to solving an integral equation. For this purpose, we use the solution of the auxiliary problem from Appendix A. Namely, with the help of (A7) fulfilling the boundary condition (2a), we arrive at the following Fredholm governing integral equation (IE) of the first kind with a symmetric kernel for unknown pressure $p(x)\ \left(p(-x) = p(x)\right)$ of the rigid inclusion on the crack edges:

$$\frac{1}{\pi E}\int_{-a}^{a} \ln\frac{l^2 - xs + \sqrt{\left(l^2 - x^2\right)\left(l^2 - s^2\right)}}{l^2 - xs - \sqrt{\left(l^2 - x^2\right)\left(l^2 - s^2\right)}}\, p(s)\,ds = f(x) \quad \left(|x| < a\right) \tag{3}$$

Further, in (3), (A8) – (10) we proceed to dimensionless coordinates and values, assuming

$$\xi = x/a,\quad \eta = s/a,\quad p_0(\xi) = p(a\xi)/E,\quad f_0(\xi) = f(a\xi)/a,$$
$$\delta_0 = \delta/a,\quad \rho = l/a\quad (\rho > 1),\quad \Psi_0(\xi) = \Psi(a\xi)/a, \tag{4}$$
$$K_I^0 = \sqrt{\pi l}\,K_I/aE,\quad \sigma_0(\xi) = \sigma(a\xi)/E,\quad \sigma_y^0(\xi) = \sigma_y(a\xi,+0)/E.$$

As a result, these formulas are transformed to the followings:

$$\frac{1}{\pi}\int_{-1}^{1} \ln\frac{\rho^2 - \xi\eta + \sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2)}}{\rho^2 - \xi\eta - \sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2)}}\, p_0(\eta)\,d\eta = f_0(\xi)\quad (-1<\xi<1); \tag{5}$$

$$\sigma_y^0(\xi) = -\sigma_0(\xi) = -\frac{1}{\pi\sqrt{\xi^2-\rho^2}}\int_{-1}^{1}\frac{\sqrt{\rho^2-\eta^2}\,p_0(\eta)\,d\eta}{\eta-\xi}\quad (\xi>\rho); \tag{6}$$

$$K_I = \left(aE/\sqrt{\pi l}\right)K_I^0;\ K_I^0 = \int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\,p_0(\eta)\,d\eta; \tag{7}$$

$$\Psi_0(\xi) = \frac{2}{\pi}\int_{-1}^{1} \ln\frac{\rho^2 - \xi\eta + \sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2)}}{\rho^2 - \xi\eta - \sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2)}}\, p_0(\eta)\,d\eta\quad (1\le\xi\le\rho). \tag{8}$$

The formulas (5)–(8) are the basic equations and relationships of the posed problem.

3. Method of analytical solution. We proceed to the solution of the governing IE (5) and first transform it to a simpler trigonometric form, assuming that

$$\xi = \rho\cos\vartheta,\quad \eta = \rho\cos\varphi;\quad \alpha = \arccos(1/\rho) = \arccos(a/l);$$
$$\omega_0(\vartheta) = p_0(\rho\cos\vartheta)\sin\vartheta;\ g_0(\vartheta) = f_0(\rho\cos\vartheta); \tag{9}$$
$$\alpha<\vartheta,\ \varphi<\beta;\ \beta = \pi-\alpha\ (0<\alpha<\pi/2).$$

As a result, after simple manipulations the equation (5) gets the following form:

$$\frac{2\rho}{\pi}\int_{\alpha}^{\beta}\ln\left[\sin\frac{(\vartheta+\varphi)}{2}\Big/\left|\sin\frac{(\vartheta-\varphi)}{2}\right|\right]\omega_0(\varphi)\,d\varphi = g_0(\vartheta)\quad (\alpha<\vartheta<\beta). \tag{10}$$

Represent the solution of the integral equation (10) equivalent to the original equation (5) in the form of an infinite series with unknown coefficients $x_n\ (n=0,1,2,...)$ as follows (see Appendix B):

$$\omega_0(\vartheta) = \frac{1}{\sqrt{\cos 2\alpha - \cos 2\vartheta}}\sum_{n=0}^{\infty} x_n T_n(\mathrm{X})\quad (\alpha<\vartheta<\pi-\alpha) \tag{11}$$

where X is given in (B9), $T_n(\mathrm{X})$ are the Chebyshev polynomials of the first kind. Further, we substitute (11) into (10), change the order of integration and summation, and use spectral relationships (B9). As a result, we get

$$2\rho\sum_{n=0}^{\infty}\mu_n x_n T_n(\mathrm{X}) = g_0(\vartheta) \qquad (\alpha < \vartheta < \pi - \alpha).$$

Whence using the orthogonality conditions (B11), we find

$$x_n = \frac{\sqrt{2}\cos^2\dfrac{\alpha}{2}\, h_n}{\rho K' k_n \mu_n};\ h_n = \int_{\alpha}^{\pi-\alpha}\frac{g_0(\vartheta)T_n(\mathrm{X})d\vartheta}{\sqrt{\cos 2\alpha - \cos 2\vartheta}};\ k_n = \begin{cases} 2\ (n=0); \\ 1\ (n \neq 0); \end{cases} (n = 0,1,2,...) \quad (12)$$

where μ_n is given by the formula (B9). Thus, the coefficients x_n are determined by formulas (12).

To solve the integral equation (5) for $\rho = 1$ $(a = l)$, according to (B19), we represent its solution as an infinite series with unknown coefficients y_n $(n = 1,2,...)$

$$p_0(\xi) = \sum_{n=1}^{\infty} y_n U_{n-1}(\xi) \qquad (-1 < \xi < 1), \tag{13}$$

where $U_{n-1}(\xi)$ are Chebyshev polynomials of the second kind.

Further, as above, we substitute (13) into equation (5) for $\rho = 1$, change the order of integration and summation, and then use the relations (B19). As a result, we obtain the equality

$$\sqrt{1-\xi^2}\sum_{n=1}^{\infty} n^{-1} y_n U_{n-1}(\xi) = 2^{-1} f_0(\xi) \qquad (-1 < \xi < 1).$$

Now multiply both sides of this equality by $U_{m-1}(\xi)$ $(m = 1,2,...)$ and integrate with respect of ξ from -1 to 1. Using the orthogonality condition of the Chebyshev polynomials of the second kind, we find

$$y_n = 2^{-1} n g_n;\ \ g_n = 2\pi^{-1}\int_{-1}^{1} f_0(\xi) U_{n-1}(\xi) d\xi \quad (n = 1,2,...). \tag{14}$$

If we assume that the function $f_0(\xi)$ is a twice continuously differentiable function on the interval $(-1,1)$ and we take into account that $f_0(\pm 1) = 0$, then after integration by parts it becomes possible to get the estimation of the Fourier coefficients:

$g_n = O\left(1/n^{2+\varepsilon}\right)$ $(\varepsilon > 0)$ for $n \to \infty$

and, consequently, the series (13) or corresponding Fourier sine series converges uniformly on any interval $[-r, r]$ $(r < 1)$.

4. Solution of IE (5) by numerical-analytical method. We will also construct approximate solutions of the discussed integral equations using the numerical-analytical method based on the Gaussian quadrature formula for calculating definite integrals in conjunction with the collocation method. The same approach is in the basis of the well-known numerical-analytical method for solving singular integral equations (SIE) proposed in [21] and [22]. Based on these considerations the solution of equation (5) can be represented in the form

$$p_0(\xi) = \Omega(\xi, \rho) \Big/ \sqrt{1-\xi^2} \quad (-1 < \xi < 1),$$

where $\Omega(\xi, \rho)$ is the Hölder function in the interval $-1 \le \xi \le 1$. Further, following the well-known procedure, solving the equation (5) is reduced to solving the following finite set of linear algebraic equations

$$\left.\begin{aligned} &\sum_{m=1}^{N} K_{rm} \mathrm{X}_m = a_r \quad \left(r = \overline{1, N}\right) \\ &K_{rm} = \frac{1}{N} \ln \frac{\rho^2 - \xi_r \eta_m + \sqrt{\left(\rho^2 - \xi_r^2\right)\left(\rho^2 - \eta_m^2\right)}}{\rho^2 - \xi_r \eta_m - \sqrt{\left(\rho^2 - \xi_r^2\right)\left(\rho^2 - \eta_m^2\right)}}, \quad \mathrm{X}_m = \Omega(\eta_m, \rho), \quad a_r = f_0(\xi_r), \end{aligned}\right\}$$

$$\eta_m = \cos\left[(2m-1)\pi/2N\right], \quad \xi_r = \cos\left[\pi r/(N+1)\right] \quad \left(m, r = \overline{1, N}\right). \tag{15}$$

Here η_m and ξ_r are Chebyshev knots, roots of Chebyshev polynomials of the first kind $T_N(\eta)$ and the second kind $U_N(\xi)$, respectively, where N is any natural number. After solving the system of equations (15), the solution of (5) at Chebyshev knots, η_m, will be determined by the formula

$$p_0(\eta_m) = \mathrm{X}_m \Big/ \sqrt{1-\eta_m^2} \quad \left(m = \overline{1, N}\right). \tag{16}$$

In a similar manner, the solution of the same equation (5) for $\rho = 1$ reduces to that of the system of equations (15), in which, however, one should put

$$\rho = 1; \quad \mathrm{X}_m = \sqrt{1-\eta_m^2}\, \Omega(\eta_m, \rho) \quad \left(m = \overline{1, N}\right).$$

5. Analytical results. Let us calculate the basic mechanical characteristics of the problem under consideration in the explicit analytical form. Turn first to the formula (11), from which the dimensionless pressure of the rigid inclusion on the crack edges is determined. In this formula, we return to the former variables and quantities. From (4) and (9) we get

$$x = a\xi, \quad \xi = \rho\cos\vartheta, \quad (\rho = l/a), \text{ i.e. } x = l\cos\vartheta \quad (\alpha < \vartheta < \pi - \alpha).$$

In the light of the above, we transform

$$\sqrt{\cos 2\alpha - \cos 2\vartheta} = \sqrt{1 + \cos 2\alpha - (1 + \cos 2\vartheta)} = \sqrt{2}\sqrt{\cos^2\alpha - \cos^2\vartheta} =$$

$$= \frac{1}{\rho\sqrt{2}}\sqrt{1-\cos^2\alpha}\,(\rho + \xi)\sqrt{\left(\frac{\rho+1}{\rho-1} - \frac{\rho-\xi}{\rho+\xi}\right)\left(\frac{\rho-\xi}{\rho+\xi} - \frac{\rho-1}{\rho+1}\right)} = \frac{2\sqrt{a^2 - x^2}}{l\sqrt{2}}$$

$$(-a < x < a).$$

On the other hand, in formula (B9) assuming that

$$\vartheta = \arccos\frac{x}{l}, \quad t = \arccos\frac{s}{l} \quad (-a < x, s < a),$$

after simple transformations we get

$$\mathrm{X}=\cos\Theta,\quad \Theta=\frac{\pi}{2K'}(a+l)\int_x^a\frac{ds}{\sqrt{(a^2-s^2)(l^2-s^2)}}\qquad(-a<x<a). \tag{17}$$

Then from (9)

$$\omega_0(\vartheta)=p_0(\rho\cos\vartheta)\sin\vartheta=p_0(\xi)\sqrt{1-\xi^2/\rho^2}=p(x)\sqrt{l^2-x^2}\Big/El.$$

Taking into consideration these transformations, the formula (11) for the pressure of the inclusion on the crack edges is represented as

$$p(x)=\frac{El^2}{\sqrt{2}}\frac{1}{\sqrt{(a^2-x^2)(l^2-x^2)}}\sum_{a=0}^{\infty}x_nT_n(\mathrm{X})\quad(-a<x<a), \tag{18}$$

where X is defined by (17).

Now in the expression for the pressure $p(x)$ we replace x by s and X by V, and substitute it into the right-hand side of (A8). Using the relations (B17), we get

$$\sigma_y\big|_{y=0}=\frac{l^2E}{\sqrt{2(x^2-a^2)(x^2-l^2)}}\sum_{n=0}^{\infty}(-1)^n x_n\frac{\mathrm{ch}(\pi nuK')}{\mathrm{ch}(\pi nK/K')}\quad(x>l), \tag{19}$$

where coefficients x_n are defined by formulas (12) and the variable u $(0<u<K)$ is given by the formula (B17).

Let us find the crack opening on the interval $a<x<l$. In the variables of (9), the formula (8) for dimensionless opening takes the form

$$\Psi_0(\rho\cos\vartheta)=\frac{4\rho}{\pi}\int_{\alpha}^{\pi-\alpha}\ln\left[\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right]\omega_0(\varphi)d\varphi\qquad\left(0<\vartheta<\alpha;\ \alpha=\arccos\frac{a}{l}\right).$$

Substituting in this expression $\omega_0(\vartheta)$ from (11) and taking into account the relations (B10) for $0<\vartheta<\alpha$ as well as (B7) where $\sin\varphi=sn(u,k)$, we get

$$\Psi_0(\rho\cos\vartheta)=4\rho\sum_{n=0}^{\infty}\frac{x_n}{n}\nu_n\,\mathrm{sh}\left(\frac{\pi nu}{K'}\right)\qquad(0\le\vartheta\le\alpha\quad\text{or}\quad 0\le u\le K),$$

$$u=\int_0^{\sin\varphi}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=F(\varphi,k);\ k=\mathrm{tg}^2\frac{\alpha}{2};\ \varphi=\arcsin\left(\frac{\mathrm{tg}(\vartheta/2)}{\mathrm{tg}(\alpha/2)}\right), \tag{20}$$

where $F(\varphi,k)$ is an incomplete elliptic integral of the first kind (B3a).

The SIF is determined by the formula (A9) (the dimensionless SIF is determined by the formula (7)) where the pressure $p(x)$ is given by (18).

Let us express characteristics of the problem through the solution of the finite system of equations (15). Then the solution of the equation (5), i.e. the dimensionless pressure of rigid inclusion on the crack edges, will be determined by the formula (16) in terms of the solution X_m $(m=\overline{1,N})$ of the system (15). However, using the Gauss quadrature formula for integrals with the Cauchy kernel and for definite integrals, from (6)–(8) we have the following formulas for the dimensionless normal stresses, the dimensionless SIF, and dimensionless opening:

$$
\left.
\begin{array}{l}
\sigma_y^0(\xi) = -\sigma_0(\xi) = -\dfrac{1}{N\sqrt{\xi^2-\rho^2}}\displaystyle\sum_{m=1}^{N}\dfrac{\sqrt{\rho^2-\eta_m{}^2}\,\mathrm{X}_m}{\eta_m-\xi} \quad (\xi>\rho), \quad K_I^0 = \dfrac{\pi}{N}\displaystyle\sum_{m=1}^{N}\sqrt{\dfrac{\rho+\eta_m}{\rho-\eta_m}}\mathrm{X}_m, \\
\Psi_0(\xi) = \dfrac{2}{N}\displaystyle\sum_{m=1}^{N}\mathrm{X}_m \ln\dfrac{\rho^2-\xi\eta_m+\sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2{}_m)}}{\rho^2-\xi\eta_m-\sqrt{(\rho^2-\xi^2)(\rho^2-\eta^2{}_m)}} \quad (1\le\xi\le\rho).
\end{array}
\right\} \tag{21}
$$

Calculate also the resultant pressure on the crack edges

$$
P=\int_{-a}^{a} p(x)\,dx \;\Rightarrow\; P_0=\int_{-1}^{1} p_0(\xi)\,d\xi \;\; (P_0=P/aE); \qquad P_0=\frac{\pi}{N}\sum_{m=1}^{N}\mathrm{X}_m. \tag{22}
$$

6. Particular cases. Let us consider two important particular cases of the discussed problem where the thin rigid inclusion has the form of a line segment, more precisely, the form of a thin rectangle of length $2a$ and height $2\delta\,(\delta << a)$, and where it has the form of a strongly oblate along its major axis half-ellipse part (1).

In the case of a rectangular inclusion (Fig.2), $f(x)=\delta$ and the right-hand side of the governing integral equation (5) is $f_0(\xi)=\delta_0 \; (\delta_0=\delta/a)$. Then for the right-hand side of equation (10), equivalent to (5), we also have $g_0(\vartheta)=\delta_0$. Hence by the orthogonality of the Chebyshev polynomials of the first kind (B11), it follows from (12) and (B9) that

$$
x_0=\left(\delta_0\cos^4(\alpha/2)/\rho KK'\right)h_0, \quad h_0=\int_{\alpha}^{\pi-\alpha}\frac{d\vartheta}{\sqrt{\cos 2\alpha-\cos 2\vartheta}} \quad x_n=h_n=0 \quad (n=1,2,...).
$$

Calculate the integral

$$
h_0=\int_{\alpha}^{\pi-\alpha}\frac{d\vartheta}{\sqrt{1+\cos 2\alpha-(1+\cos 2\vartheta)}}=\frac{1}{2\sqrt{2}}\int_{\alpha}^{\pi-\alpha}\left(\sin\frac{\vartheta+\alpha}{2}\sin\frac{\vartheta-\alpha}{2}\cos\frac{\vartheta+\alpha}{2}\cos\frac{\vartheta-\alpha}{2}\right)^{-1/2}d\vartheta=
$$

$$
=\frac{1}{\sqrt{2}\sin\alpha}\int_{\alpha}^{\pi-\alpha}\left[\left(\mathrm{ctg}^2\frac{\alpha}{2}-\mathrm{t}g^2\frac{\alpha}{2}\right)\left(\mathrm{ctg}^2\frac{\vartheta}{2}-\mathrm{tg}^2\frac{\alpha}{2}\right)\right]^{-1/2}\frac{d\vartheta}{\cos^2(\vartheta/2)}.
$$

Passing to the variable y and parameters c and d by formulas (B4), we get from the above expression

$$
h_0=\frac{\sqrt{2}}{\sin\alpha}\int_{c}^{d}\frac{dy}{\sqrt{(d^2-y^2)(y^2-c^2)}}.
$$

Using the value of this integral [23] (p.260, f.-la 3.152.10), we finally have

$$
h_0=K'/\sqrt{2}\cos^2(\alpha/2).
$$

As a result,

$$
x_0=\frac{(1+\rho)\delta_0}{2\sqrt{2}\rho^2 K}=\frac{(a+l)\delta}{2\sqrt{2}K(k)l^2} \quad \left(k=\frac{\rho-1}{\rho+1}=\frac{l-a}{l+a}\right), \quad x_n=0 \quad (n=1,2,...). \tag{23}
$$

Now taking into account (23), we obtain from (18)

$$p(x)=\frac{E(a+l)\delta}{4K(k)\sqrt{(a^2-x^2)(l^2-x^2)}}\quad(-a<x<a) \tag{24}$$

or in the dimensionless form

$$p_0(\xi)=\frac{(1+\rho)\delta_0}{4K(k)\sqrt{(1-\xi^2)(\rho^2-\xi^2)}}\quad(-1<\xi<1).$$

Then with the help of (24) we calculate SIF K_I by the formula (A9):

$$K_I=\frac{1}{\pi}\sqrt{\frac{\pi}{l}}\frac{E(a+l)\delta}{4K(k)}\int_{-a}^{a}\frac{ds}{(l-s)\left(\sqrt{a^2-s^2}\right)}$$

Again, using the value of the well-known integral [24] (p. 175, f.-la (21)), we get

$$K_I=\sqrt{\frac{\pi}{l}}\frac{E\delta}{4K(k)}\sqrt{\frac{l+a}{l-a}}=\frac{aE}{\sqrt{\pi l}}K_I^0 \tag{25}$$

where K_I^0 is the dimensionless SIF:

$$K_I^0=\frac{\pi\delta_0}{4K(k)}\sqrt{\frac{\rho+1}{\rho-1}}. \tag{26}$$

By the formula (19) using (23), we find immediately the normal breaking stresses outside the crack

$$\sigma_y\Big|_{y=0}=\frac{(a+l)\delta E}{4K(k)\sqrt{(x^2-a^2)(x^2-l^2)}}\quad(x>l) \tag{27}$$

or in the dimensionless form

$$\sigma_y^0(\xi)=\frac{(1+\rho)\delta_0}{4K(k)\sqrt{(\xi^2-1)(\xi^2-\rho^2)}}\quad(\xi>\rho).$$

Now SIF K_I can be also calculated by the formula (27) that once again will lead to (25).

Finally, the dimensionless opening of crack edges outside the rigid inclusion according to (20) and (23) is determined by the formula

$$\Psi_0(\rho\cos\vartheta)=\frac{(1+\rho)\delta_0}{\rho K(k)\cos^2(\alpha/2)}u\qquad(0\le\vartheta\le\alpha),$$

$$u=\int_0^{\varphi}\frac{d\tau}{\sqrt{1-k^2\sin^2\tau}}=\int_0^{\sin\varphi}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=F(\varphi,k);\ \varphi=\arcsin\left(\frac{\mathrm{tg}(\vartheta/2)}{\mathrm{tg}(\alpha/2)}\right), \tag{28}$$

where $F(\varphi,k)$ like that in (B3a) is the incomplete elliptic integral of the first kind of the modulus k.

Calculate also the resultant pressure from (24):

$$P=\int_{-a}^{a}p(x)\,dx=aE\int_{-1}^{1}p_0(\xi)\,d\xi=\frac{aE(1+\rho)\delta_0}{2K(k)}\int_{0}^{1}\frac{d\xi}{\sqrt{(1-\xi^2)(\rho^2-\xi^2)}}.$$

The last integral is the complete elliptic integral of the first kind of module $\chi=a/l$. Hence,

$$P=\frac{E(a+l)\delta}{2lK(k)}K(\chi)\qquad\left(k=\frac{\rho-1}{\rho+1}=\frac{1-\chi}{1+\chi}\right). \tag{29}$$

Relationship (29) establishes the dependence between the resultant pressure P and the half-width of inclusion δ, i.e. the measure of the vertical settlement of longitudinal sides of a rectangular inclusion into an elastic matrix. In the dimensionless form, we get

$$P_0=(1+\chi)\delta_0 K(\chi)/2K(k)\ (P_0=P/aE). \tag{30}$$

Consider the second particular case. Equation of the upper oblate semi-ellipse (1) is represented in the form

$$f_0(\xi)=f(a\xi)/a=\varepsilon\sqrt{a_0^2-\xi^2}\qquad(\varepsilon=b_1/a_1;\ a_0=a_1/a;\ \varepsilon<<1). \tag{31}$$

Then all mechanical characteristics of the problem may be calculated by formulas (16) and (21) - (22).

Let us discuss the limiting case $a\to l$, assuming that a thin absolutely rigid elliptical inclusion, indenting into a crack across its entire surface, is closely adjacent to the edges of the crack along its entire length. In this case, in (31) $\varepsilon=b_1/l$, $a_0=1$ should be taken and based on the integral relationship (B19), the solution to the governing integral equation (5), where $\rho=1$, should be represented in the form of an infinite series (13) of the Chebyshev polynomials of the second kind with unknown coefficients y_n. These coefficients are expressed by (14), from which for the function (31) we have

$$y_1=\varepsilon/2,\quad y_n=0\quad(n=1,2,...).$$

Hence by (13) $p_0(\xi)=\varepsilon/2\quad(-1\le\xi\le1)$ or

$$p(x)=\varepsilon E/2\quad(-l\le x\le l). \tag{32}$$

Substituting (32) into (A8) and taking into account the value of the well-known integral [24] (p.175, f.-la (19)), we easily obtain

$$\sigma_y\big|_{y=0}=\frac{\varepsilon E}{2\sqrt{x^2-l^2}}\left(x-\sqrt{x^2-l^2}\right)\ (x>l). \tag{33}$$

Calculate the SIF K_I from (33):

$$K_I=\lim_{x\to l+0}\left[\sqrt{2\pi(x-l)}\,\sigma_y\big|_{y=0}\right]=\varepsilon E\sqrt{\pi l}/2 \tag{34}$$

or in the dimensionless form

$$K_I^0=\varepsilon/2;\quad K_I^0=K_I/E\sqrt{\pi l}.$$

The same result (34) can be obtained directly by means of (32) and (A9) for $a=l$.

For a comparative analysis of the analytical expression of SIF (25) and the other known similar expressions, we consider the following cases.

1) As a comparison, we consider the case of a normal opening of a crack by a dipole of concentrated at the origin forces with a magnitude P determined by (29). In this case, for SIF we have [20]

$$\tilde{K}_I = P/\sqrt{\pi l}\,.$$

Then writing the SIF (25) with the help of (29) as

$$K_I = \pi P/2\sqrt{\pi l}K(\chi)\sqrt{1-\chi^2}\,,$$

we get for their ratio

$$k_1 = k_1(\chi) = K_I/\tilde{K}_I = \pi/2\sqrt{1-\chi^2}K(\chi). \tag{35}$$

2) With the same case of dipole forces we compare SIF (7) when the inclusion shaped as a strongly oblate half-ellipse (31). In this case, taking into account (22), we have

$$k_2 = k_2(\chi) = \frac{K_I}{\tilde{K}_I} = \sum_{m=1}^{N}\sqrt{\frac{1+\chi\eta_m}{1-\chi\eta_m}}X_m\left(\sum_{m=1}^{N}X_m\right)^{-1}, \tag{36}$$

where X_m is the solution of the system (15).

3) We also compare SIF (25) with SIF calculated by the model proposed in [7]. We assume that normal forces $p(x)$ are symmetrically applied to the crack edges along the segment $(-a,a)$. In accordance with Sadovsky's solution to the problem on the indentation of a punch with a flat base into the elastic half-plane [25] we have:

$$p(x) = P/\pi\sqrt{a^2-x^2} \quad (-a < x < a).$$

Then by (7)

$$\tilde{K}_I = \frac{1}{\sqrt{\pi l}}\frac{P}{\pi}\int_{-a}^{a}\sqrt{\frac{l+s}{l-s}}\frac{ds}{\sqrt{a^2-s^2}} = \frac{2P}{\pi\sqrt{\pi l}}K(\chi)$$

and hence

$$k_3 = k_3(\chi) = \frac{K_I}{\tilde{K}_I} = \pi^2/4K^2(\chi)\sqrt{1-\chi^2}. \tag{37}$$

7. On estimation of crack resistance of a plate with a crack. Now proceeding from (25), we note that $\lim\limits_{a\to l}K_I = \infty$, i.e. when the inclusion ends are approaching the crack tips, the SIF K_I increases infinitely, taking on also its critical value K_{IC} at which the crack starts to propagate. It follows that the crack will propagate before inclusion ends reach the tips of the crack. This phenomenon is quite similar to the phenomenon of cracking the brittle elastic bodies during their wedging by an absolutely rigid thin wedge [12]. The quantity K_{IC} is called the crack resistance or the limit of the ductile fracture of materials during normal separation at the maximum constraint of plastic deformation and is an important characteristic of materials. The values of K_{IC} for a large number of materials are given in [26].

With the help of (25), we evaluate the critical value of the parameter $\chi = \rho^{-1} = a/l$, at which the crack begins to propagate. For this, we require the fulfillment of the condition

$$K_I \geq K_{Ic} \Rightarrow \sqrt{\frac{\pi}{l}}\frac{E\delta}{4K(k)}\sqrt{\frac{l+a}{l-a}} \geq K_{Ic}.$$

From this

$$\sqrt{\frac{l+a}{l-a}} \ge K_{IC}\frac{\sqrt{l}}{\sqrt{\pi}}\frac{4K(k)}{E\delta}. \tag{38}$$

Evaluate the complete elliptic integral of the first kind $K(k)$. It is evident that

$$k' = \sqrt{1-k^2} \le \sqrt{1-k^2t^2} \le 1 \quad (0 \le t \le 1) \quad \Rightarrow \qquad 1 \le \frac{1}{\sqrt{1-k^2t^2}} \le \frac{1}{k'}.$$

Hence

$$\frac{\pi}{2} \le K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \le \frac{\pi}{2k'}.$$

If we now require the fulfillment of the condition

$$\sqrt{\frac{l+a}{l-a}} \ge \frac{4K_{IC}\sqrt{l}}{\sqrt{\pi}E\delta}\frac{\pi}{2k'},$$

then the condition (38) is a fortiori fulfilled. Further elementary transformation of this inequality leads to a quadratic inequality for a:

$$K_0^2a^2 + a - l^2K_0^2 \ge 0, \quad K_0 = \sqrt{\pi}K_{IC}/E\delta.$$

The solution to this inequality has the forms

$$a \ge \frac{-1+\sqrt{1+4l^2K_0^4}}{2K_0^2} \quad \Rightarrow \quad \frac{a}{l} \ge \frac{-1+\sqrt{1+4l^2K_0^4}}{2K_0^2 l}.$$

Finally, we have

$$\chi = a/l \ge \chi_c,\ \chi_c = \left[-1+\sqrt{1+\left(K_{IC}^0\right)^2}\right] \Big/ K_{IC}^0 = \\ = K_{IC}^0 \Big/ \left[1+\sqrt{1+\left(K_{IC}^0\right)^2}\right]; K_{IC}^0 = 2\pi l K_{IC}^2 / E^2\delta^2. \tag{39}$$

We will call the dimensionless quantity K_{Ic}^0 the reduced crack resistance.

At $\chi \ge \chi_c$ the crack propagation begins and brittle fracture of the plate occurs. Hence, it follows that according to condition (39), a plate with a crack of a given length can withstand a thin rectilinear inclusion indented into crack edges only if the inclusion has proper length. Consequently, a complete contact of the rectilinear thin inclusions with the crack edges along its entire length from the point of view of fracture mechanics is impossible. Therefore, it is necessary to introduce corresponding corrections in the formulation of the Sherman-Muskhelishvili classical mixed boundary-value problem [19] (pp. 444-446), where full contact is considered. This issue was studied in detail in [6].

8. Numerical analysis of the mechanical characteristics. To determine changes in the basic mechanical quantities of the discussed problem and to reveal regularities of their changes depending on specific geometrical and physical parameters, the numerical implementation of the obtained analytical results were carried out.

In the case of a thin rectilinear inclusion (Fig. 2), values of the dimensionless SIF K_I^0 are calculated by the exact formula (26) at $\delta_0 = 0,05$, and for different values of the parameter χ (29), the relative distance from the inclusion right end to the crack right tip $(\chi = a/l)$. At the same time for the same values of parameters, K_I^0 has been also calculated by the high-accuracy approximate formula (21), where X_m is the solution to the linear system of equations (15). The calculation results obtained by both formulas, which practically coincide for large N, are represented in Table 1.

Table 1. Values of K_I^0 (exact and approximate)

χ	0,03	0,05	0,09	0,1	0,2	0,5	0,8	0,9
K_I^0	0,0161 0,0161	0,0179 0,0179	0,0207 0,0207	0,0214 0,0214	0,0266 0,0266	0,0421 0,0421	0,0748 0,0751	0,1089 0,1096

The Table of exact and approximate values of the dimensionless SIF K_I^0 for different values of the parameter $\chi = a/l$ and for a fixed value of $\delta_0 = \delta/a = 0,05$ in case of the thin linear smooth rigid inclusion, wherein the upper row shows the exact values of K_I^0, and the bottom row – approximate values of K_I^0.

Here, in the first row values calculated by the formula (26), in the second row those calculated by (21) are given. According to these values, for the sake of visual illustration of the change of dimensionless SIF, the graph of K_I^0 is plotted (Fig.3). It shows that the value of K_I^0 increases significantly as $\chi \to 1$.

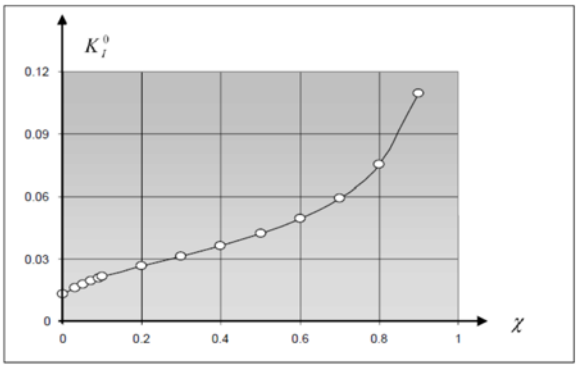

Fig.3. The graph of dimensionless SIF, K_I^0 depending on the parameter χ $(\chi = a/l)$ and for a fixed value of δ_0 $(\delta_0 = \delta/a = 0,05)$.

Turn to the quantity χ_c defined by the formula (39) and consider a specific calculation example to determine the order of magnitude of K_I^0. Let the elastic plate be made of extruded aluminum strip alloys for which according to [26] (p.113, Table 2.2) $K_{Ic} = 410 MPa \cdot \sqrt{cm}$ and according to [27] (p.63, Table 1) $E = 0,7 \cdot 10^6 \text{kg/cm}^2 = = 6,9 \cdot 10^4 MPa$. Assume that $2l = 40cm$ and $\delta = 0,1cm$; 0,3cm; 0,5cm; 0,7cm; 1cm; 2cm; 5cm; 10cm. For these values of physical constants and geometrical parameters, the critical value of χ at which the crack propagates, χ_c, as well as the reduced crack resistance K_{Ic}^0 are calculated from the formulas (39). According to the results of calculations, Table 2 was compiled. As it follows from Table 2, the values of K_{Ic}^0 are small numbers (e.g. for $\delta = 1cm$ $K_{Ic}^0 = 4,435 \cdot 10^{-3}$), and therefore, we can simplify (39) by taking $1 + \left(K_{Ic}^0\right)^2 \approx 1$. As a result, we can practically assume that $\chi_c \approx K_{Ic}^0/2$ which is confirmed by the first row of Table 2.

Table 2. Values of χ_c and K_{Ic}^0

δ/l	0,005	0,015	0,035	0,05	0,1	0,25	0,5
χ_c	0.211885	0.02463	0.00453	0.00222	0.00055	0.00008	0.00002
K_{Ic}^0	0.44369	0.04929	0.00905	0.00443	0.00111	0.00018	0.00004

Table of values of the parameter χ_c, the critical value of the relative distance from the right end of linear inclusion to the right crack tip ($\chi = a/l$) and the reduced crack resistance K_{Ic}^0 for different values of δ/l .

Note that the results of calculations of the dimensionless resultant P_0 according to the formulas (22) and (30) also coincide with high accuracy, and the value of P_0 increases appreciably with increasing of χ .

Next, the numerical realization of (35)-(37) is carried out and results of calculating the deviations of the compared SIFs k_j and their errors $|1-k_j|$ $(j = 1,2,3)$ depending on the values of the characteristic parameter are given in Tables 3 and 4 ($a_0 = 1.2$ and $a_0 = 2$ are taken when calculating k_2)

Table 3. Values of k_j and errors $|1-k_j|$ $(j = 1,2,3;\ a_0 = 1.2)$

χ	k_1	$\|1-k_1\|$	k_2	$\|1-k_2\|$	k_3	$\|1-k_3\|$
0.01	0.997542	0.00245798	0.999983	0.0000165878	0.99504	0.00495967
0.1	0.97908	0.0209196	0.999947	0.0000532126	0.953793	0.0462066
0.2	0.965995	0.0340055	1.00176	0.00176355	0.914292	0.085708
0.3	0.960763	0.0392368	1.00682	0.00681802	0.880549	0.119451
0.4	0.964197	0.0358027	1.01642	0.0164202	0.852062	0.147938
0.5	0.978277	0.0217226	1.03251	0.0325148	0.828809	0.171191
0.6	1.00714	0.00714397	1.05857	0.0585711	0.811471	0.188529
0.7	1.05984	0.0598409	1.10182	0.101817	0.80217	0.19783
0.8	1.15984	0.159839	1.18098	0.180976	0.807135	0.192865
0.9	1.3978	0.397799	1.3702	0.370199	0.85166	0.14834

Table of the values of compared SIFs k_j and their absolute errors $|1-k_j|$ $(j = 1,2,3)$ for different values of the parameter χ , where $a_0 = 1.2$.

As follows from (35) - (37), $k_j(0)=1$ and, therefore, the quantities $\left|1-k_j\right|$ $(j=1,2,3)$ are the absolute deviation k_j from the unit. These values simultaneously give the relative errors of the compared

SIFs, since according to the cases discussed above $\left|1-k_j\right|=\left|K_I-\tilde{K}_I\right|/\tilde{K}_I$.

Analysis of the data in Table 3 show that for small values of χ, the errors $\left|1-k_j\right|$ $(j=1,2,3)$ are very small numbers. For instance, for $\chi=0,5$ these errors are about 3%, for $\chi<0,5$ – even less. The highest accuracy is provided in cases 1) and 2) (section 6). The values of $\left|1-k_j\right|$ $(j=1,2,3)$ increase significantly with the increase of χ and a_0.

We also present the results of the numerical analysis that illustrate the course of the change in the crack opening, which can be used in deformation theories of cracks propagation. Values of the dimensionless crack opening on the interval $a \le x \le l$ or on the corresponding intervals $\rho \le \xi \le 1$ and $0 \le \vartheta \le \alpha$ $(\alpha=\arccos\alpha=\arccos\alpha/l)$ are calculated by the exact formulas (28) and by the approximate formula (21) $(\xi=\rho\cos\vartheta)$ at nodal points $\vartheta_j=j\alpha/n$ $\left(j=\overline{0,n}\right)$ of the interval $0 \le \vartheta \le \alpha$. The graph of the change of $\Psi_0(\xi)$ depending on ξ for $\delta_0=0.05$ and $\chi=0.5$ is shown in Fig. 4, in which points corresponding to exact and approximate solutions practically merged. $\Psi_0(\xi)$ reaches its highest value at the point $\xi=1$ $(x=a)$, i.e. at the right end of the inclusion.

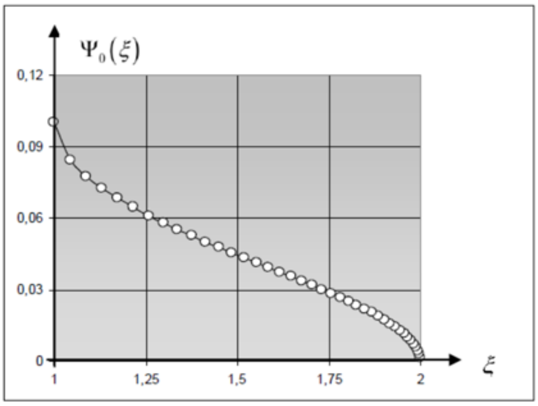

Fig.4. Graph of the dimensionless crack opening $\Psi_0(\xi)$ on the interval $1 \le \xi \le \rho = l/a$ $(a \le x \le l)$ in case of a linear inclusion for fixed values $\delta_0=\delta/a=0,05$ and $\chi=a/l=0,5$.

We turn now to the second particular case when a thin inclusion has the shape of a strongly oblate along its length ellipse (31). In this case, by solving a linear system of algebraic equations (15) the main mechanical characteristics are expressed by formulas (21)–(22). For different values of the parameter χ and for $\varepsilon=0.05$, $a_0=1.4$, the values of dimensionless SIF K_I^0 are calculated by the formula (21) and using these values, the graph of SIF change is plotted in Fig. 5.

Note that values of K_I^0 increase appreciably with the increase of the parameter χ. It is also confirmed analytically. Indeed, it follows from formulas (C1) - (C2) of Appendix C that $\lim_{\rho\to 1} K_I^0 = \lim_{\chi\to 1}\left[BK(\chi)\right] = \infty$, i.e. the values of K_I^0 as $\chi \to 1$ not only increase but increase infinitely. Now, it should be emphasized that at the end of section 6 the passage to the limit was formally carried out, $\chi \to 1(a \to l)$, i.e. it was formally assumed that the rigid thin elliptical inclusion of length $2l$ entirely fits into the crack along its entire length, which is also equal to $2l$. This problem as a mixed boundary value problem of the mathematical theory of elasticity is correct and has a simple solution with characteristics (32)-(34). However, from the point of view of fracture mechanics, this solution is unfounded and devoid of real physical content since $K_I^0 \to \infty$ as $\rho \to 1$ and therefore when $\rho \to 1$ the crack propagates earlier than the end points of the thin inclusion reach the tips of the crack.

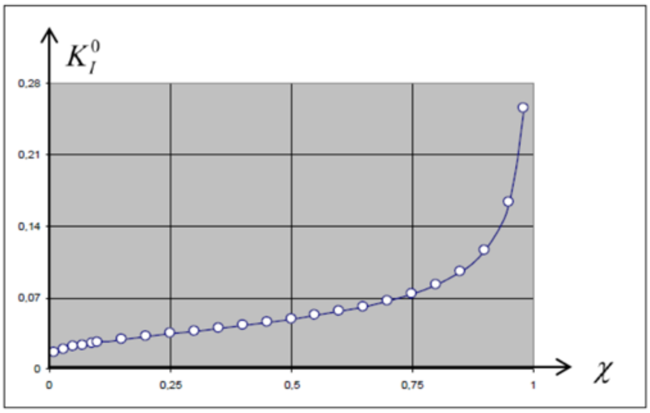

Fig.5. The graph of the change of dimensionless SIF, K_I^0, depending on the parameter χ for $\varepsilon = 0.05,\ a_0 = 1.4$ in case of a rigid strongly oblate elliptical inclusion

8. Conclusions. In the paper, by the method of integral equations an exact solution to the plane problem of the theory of elasticity on the stress state of an infinite elastic plate with a finite rectilinear crack, in which a thin smooth absolutely rigid inclusion shaped as an oblate along its length ellipse is indented, is obtained.

At the same time, an approximate solution of the problem is obtained by reduction of the governing integral equation to a system of linear algebraic equations. The main mechanical quantities and characteristics of fracture mechanics are represented by explicit analytical formulas of simple structures.

The important special cases of rigid inclusions shaped as a thin rectangle and as a strongly oblate ellipse are considered; to reveal and promote understanding the regularities of the change in the characteristics of fracture mechanics, their numerical analysis is carried out.

It is established that the crack extension occurs when the inclusion end points are approaching the crack tips, while the crack propagates earlier than the end points of inclusion will reach the crack tips. Proceeding from this phenomenon, an estimate for the crack resistance of a plate with a crack is given; namely, the critical value of the relative length of the inclusion at which the crack starts to spread is determined.

It is shown that the model of the complete contact of the inclusion along the entire crack length, adopted in the classical boundary-value problems of the theory of elasticity, is unacceptable from the point of view of fracture mechanics.

Appendix A. Let us consider an auxiliary boundary-value problem of the stress state of an infinite elastic plate with a crack along the line segment $-l \le x \le l$, to the upper and lower edges of which normal distributed forces of intensity $p(x)$, equal in magnitude but opposite in direction, are applied and $p(-x) = p(x)$, $p(x) \equiv 0$ for $a < |x| < l$.

Due to the symmetry about the abscissa axis, this auxiliary problem is equivalent to the following mixed boundary-value problem for the upper elastic half-plane:

$$\left.\begin{aligned} &\sigma_y\big|_{y=+0} = -p(x)\ \ (-l<x<l);\ \ \tau_{xy}\big|_{y=+0} = 0\ \ (-\infty<x<\infty);\ \ \mathrm{v}\big|_{y=+0} = 0\ \ (|x| \ge l); \\ &\sigma_x,\ \sigma_y,\ \tau_{xy} \to 0\ \ as\ \ x^2 + y^2 \to \infty \end{aligned}\right\} \tag{A1}$$

A solution of the problem (A1) in displacement can be immediately obtained using the solution of the well-known Flamant's problem for an elastic half-plane in combination with the linear superposition principle. Namely, introducing the notation

$$\sigma_y\big|_{y=+0} = -\sum(x) = \begin{cases} -p(x) & (-l < x < l), \\ -\sigma(x) & (|x| > l) \end{cases} \tag{A2}$$

and using the known formula [25, pp.95-96] for vertical displacements of boundary points of the upper elastic half-plane (upper semi-infinite plate) we have

$$\mathrm{v}(x,+0) = \frac{2}{\pi E}\int_{-\infty}^{\infty} \ln\frac{1}{|x-s|}\sum(s)\,ds + \text{const} \qquad (-\infty < x < \infty).$$

By differentiation of both sides of the equation with respect to x we get

$$\frac{E}{2}\psi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sum(s)\,ds}{s-x}, \quad \psi(x) = \frac{d\mathrm{v}(x,+0)}{dx} \qquad (-\infty < x < \infty). \tag{A3}$$

From (A3) by Hilbert's inversion formula, we will come to the key equation of the problem taking into account the boundary condition (2b):

$$\sum(x) = -\frac{E}{2\pi}\int_{-\infty}^{\infty}\frac{\psi(s)\,ds}{s-x} = -\frac{E}{2\pi}\int_{-l}^{l}\frac{\psi(s)\,ds}{s-x} \quad (-\infty < x < \infty). \tag{A4}$$

Now considering the key equation (A4) on the interval $(-l,l)$, in accordance with (A2) we can write

$$p(x) = -\frac{E}{2\pi}\int_{-l}^{l}\frac{\psi(s)\,ds}{s-x} \quad (-l < x < l). \tag{A5}$$

Then the equation (A5) is treated as singular integral equation (SIE) and the following well-known formula from [28] (pp.445-446) is used:

$$\psi(x) = \frac{d\mathrm{v}(x,+0)}{dx} = \frac{2}{\pi E}\frac{1}{\sqrt{l^2-x^2}}\int_{-a}^{a}\frac{\sqrt{l^2-s^2}\,p(s)\,ds}{s-x} + \frac{C}{\sqrt{l^2-x^2}}. \tag{A6}$$

Integrating both sides of this equation and using the well-known integral expression given in [29] (p.111), we obtain

$$\mathrm{v}(x,+0)=\frac{1}{\pi E}\int_{-a}^{a}\ln\frac{l^2-xs+\sqrt{(l^2-x^2)(l^2-s^2)}}{l^2-xs-\sqrt{(l^2-x^2)(l^2-s^2)}}p(s)\,ds+$$

$$+C\arcsin\frac{x}{a}+C_1\quad(-l\le x\le l),$$

where C and C_1 are constants. Since $\mathrm{v}(\pm l,+0)=0$, then $C=C_1=0$.

Consequently,

$$\mathrm{v}(x,+0)=\frac{1}{\pi E}\int_{-a}^{a}\ln\frac{l^2-xs+\sqrt{(l^2-x^2)(l^2-s^2)}}{l^2-xs-\sqrt{(l^2-x^2)(l^2-s^2)}}p(s)\,ds\quad(-l\le x\le l). \tag{A7}$$

Formula (A7) coincides with the result of the monograph [30, p.33] obtained earlier by the same author with the help of complex potentials of the plane theory of elasticity.

Considering the key equation (A4) outside the interval $(-l,l)$, we get an expression of normal stresses outside the crack along its location line:

$$\sigma_y\big|_{y=+0}=-\sigma(x)=\frac{E}{2\pi}\int_{-l}^{l}\frac{\psi(s)\,ds}{s-x}\quad(|x|>l).$$

Substituting the expression $\psi(x)$ from (A6) where $C=0$, after simple transformations we obtain

$$\sigma_y\big|_{y=+0}=-\sigma(x)=-\frac{\mathrm{sign}x}{\pi\sqrt{x^2-l^2}}\int_{-a}^{a}\frac{\sqrt{l^2-s^2}\,p(s)\,ds}{s-x}\quad(|x|>l). \tag{A8}$$

Here, the expression for the known integral [24] (p.175) was used.

Proceeding from (A8), we calculate the SIF K_I and due to the symmetry we restrict our consideration only by its value at the right tip of the crack, $x=l$:

$$K_I=\lim_{x\to l+0}\left[\sqrt{2\pi(x-l)}\sigma_y(x,+0)\right]=\frac{1}{\sqrt{\pi l}}\int_{-a}^{a}\sqrt{\frac{l+s}{l-s}}p(s)\,ds. \tag{A9}$$

Formula (A9) coincides with the known result obtained in [10].

Finally, again taking into account the symmetry, we get from (A7) the following expression for the crack opening out of inclusion, calculated only for the right segment $a\le x\le l$:

$$\Psi(x)=2\mathrm{v}(x,+0)=\frac{2}{\pi E}\int_{-a}^{a}\ln\frac{l^2-xs+\sqrt{(l^2-x^2)(l^2-s^2)}}{l^2-xs-\sqrt{(l^2-x^2)(l^2-s^2)}}p(s)\,ds\quad(a\le x\le l). \tag{A10}$$

Appendix B. To construct the exact solution of Eq. (10), let us find eigenfunctions and eigenvalues of the integral operator

$$Kh(\vartheta)=\frac{1}{\pi}\int_{\alpha}^{\beta}\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)h(\varphi)\,d\varphi.$$

For this purpose, we use the results obtained in [31] and in [32], where by the methods of logarithmic potential and with the help of conformal mapping of the complex plane with two identical and symmetrically located cuts onto an annular ring, the spectral relationships are established by means of Jacobi elliptic sine functions:

$$\left.\begin{aligned}
&\frac{1}{\pi}\int_{c}^{d}\ln\frac{y+\mathrm{v}}{|y-\mathrm{v}|}\frac{T_n(\mathrm{V})\,d\mathrm{v}}{\sqrt{\left(d^2-\mathrm{v}^2\right)\left(\mathrm{v}^2-c^2\right)}}=\lambda_n T_n(Y)\quad \left(c<y<d;\ 0<c<d;\ n=0,1,2,\ldots\right)\\
&\mathrm{V}=\cos\Phi,\quad \Phi=\frac{\pi}{K'}\int_{1}^{\mathrm{v}/c}\frac{dt}{\sqrt{\left(t^2-1\right)\left(1-k^2t^2\right)}};\quad k=\frac{c}{d};K=K(k)=\int_{0}^{1}\frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}}\\
&Y=\cos\Theta,\quad \Theta=\frac{\pi}{K'}\int_{1}^{y/c}\frac{dt}{\sqrt{\left(t^2-1\right)\left(1-k^2t^2\right)}};\lambda_n=\begin{cases}\dfrac{1}{\pi n\,d}K'\,\mathrm{th}\left(\pi nK/K'\right)(n=1,2,\ldots);\\ K/d \qquad\qquad (n=0).\end{cases}
\end{aligned}\right\}\quad \text{(B1)}$$

Here $K(k)$ is the complete elliptic integral of the first kind of the modulus k, $K'=K(k')$, where $k'=\sqrt{1-k^2}$ is the complementary modulus; $T_n(\mathrm{X})$ are Chebyshev polynomials of the first kind of the argument X; Φ and Θ are incomplete elliptic integrals of the first kind. In addition, the following integral relations cognate with (B1) were also obtained in [31,32]:

$$\frac{1}{\pi}\int_{c}^{d}\ln\frac{y+\mathrm{v}}{|y-\mathrm{v}|}\frac{T_n(\mathrm{V})\,d\mathrm{v}}{\sqrt{\left(d^2-\mathrm{v}^2\right)\left(\mathrm{v}^2-c^2\right)}}=$$

$$=\frac{K'}{\pi dn\,\mathrm{ch}\left(\pi nK/K'\right)}\left[H(c-y)+(-1)^n H(y-d)\right]\mathrm{sh}\left(\pi nu/K'\right)\quad \text{(B2)}$$

$$\left(y\in(0,c)\cup(d,\infty);\ n=0,1,2,\ldots\ 0<u<K\right)$$

where $H(x)$ is the well-known Heaviside function. If in (B2) $y\in(0,c)$, then $y=c\,\mathrm{sn}(u,k)$ and if $y\in(d,\infty)$, then $y=d/\mathrm{sn}(u,k)$ where $\mathrm{sn}(u,k)$ is the Jacobi elliptic sine function of the modulus k.

Note that in the theory of elliptic functions [23] the quantity u, called the argument $(u=\arg\varphi)$, is given by the formula

$$u=\int_{0}^{\varphi}\frac{d\tau}{\sqrt{1-k^2\sin^2\tau}}\qquad(0<k<1)$$

where the limit φ, called the amplitude $(\varphi=am\ u)$, is considered as a function of u.

Supposing $t=\sin\tau$, we get

$$u=\int_0^{\sin\varphi}\frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}}=F\left(\varphi,k\right),\tag{B3a}$$

where $F\left(\varphi,k\right)$ is the incomplete elliptic integral of the first kind. However, by definition $\sin\varphi=\mathrm{sn}\left(u,k\right)=snu$. As a result, from (B3a) the well-known formula is obtained [23] (p.924, f.-la 8.141):

$$u=\int_0^{snu}\frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}}.\tag{B3b}$$

Now, in the relations (B1) we pass to new variables ϑ,φ and parameters α,β

$$y=\mathrm{tg}\frac{\vartheta}{2},\quad \mathrm{v}=\mathrm{tg}\frac{\varphi}{2};\quad c=\mathrm{tg}\frac{\alpha}{2},\quad d=\mathrm{tg}\frac{\beta}{2}\ \left(\alpha<\vartheta,\quad \varphi<\beta\right).\tag{B4}$$

By corresponding replacements in (B1) and slightly changing notations, after simple transformations we come to the following spectral relationships:

$$\left.\begin{aligned}
&\frac{1}{\pi}\int_\alpha^\beta \ln\left[\frac{\sin\left(\vartheta+\varphi\right)}{2}\Big/\left|\frac{\sin\left(\vartheta-\varphi\right)}{2}\right|\right]\frac{T_n\left(Y\right)d\varphi}{\sqrt{\left(\cos\varphi-\cos\beta\right)\left(\cos\alpha-\cos\varphi\right)}}=\\
&=\lambda_n\sec\frac{\alpha}{2}\sec\frac{\beta}{2}T_n\left(\mathrm{X}\right)\\
&\mathrm{X}=\cos\Theta,\quad \Theta=\frac{\pi}{K'}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\int_\alpha^\vartheta\frac{dt}{\sqrt{\left(\cos\alpha-\cos t\right)\left(\cos t-\cos\beta\right)}},\\
&Y=\cos\Phi,\quad \Phi=\frac{\pi}{K'}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\int_\alpha^\varphi\frac{dt}{\sqrt{\left(\cos\alpha-\cos t\right)\left(\cos t-\cos\beta\right)}}\\
&\left(n=0,1,2,...\right)\qquad \left(\alpha<\vartheta,\ \varphi<\beta,\ k=\mathrm{tg}\frac{\alpha}{2}\mathrm{ctg}\frac{\beta}{2}\right).
\end{aligned}\right\}\tag{B5}$$

Passing to the variables (B4) in the integral relations (B2), we get

$$\begin{aligned}
&\frac{1}{\pi}\int_\alpha^\beta \ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)\frac{T_n\left(Y\right)d\varphi}{\sqrt{\left(\cos\alpha-\cos\varphi\right)\left(\cos\varphi-\cos\beta\right)}}=\\
&=\frac{\sec\left(\alpha/2\right)\mathrm{cosec}\left(\beta/2\right)K'}{\pi n\,\mathrm{ch}\left(\pi nK/K'\right)}\left[H\left(\alpha-\vartheta\right)+\left(-1\right)^n H\left(\vartheta-\beta\right)\right]\mathrm{sh}\left(\pi nu/K'\right)\\
&\left(\vartheta\in\left(0,\alpha\right)\cup\left(\beta,\pi\right);\quad n=0,1,2...\quad 0<u<K\right)
\end{aligned}\tag{B6}$$

Here according to the above

$$\vartheta=\begin{cases}2\operatorname{arctg}\left(\operatorname{tg}\dfrac{\alpha}{2}\operatorname{sn}(u,k)\right) & (0<\vartheta<\alpha);\\ 2\operatorname{arctg}\left(\operatorname{tg}\dfrac{\beta}{2}\Big/\operatorname{sn}(u,k)\right) & (\beta<\vartheta<\pi).\end{cases}\tag{B7}$$

Then we write the orthogonality conditions of Chebyshev polynomials of the first kind entered in the relations (B5)

$$\int_c^d T_n(Y)T_m(Y)\frac{dy}{\sqrt{\left(d^2-y^2\right)\left(y^2-c^2\right)}}==\frac{K'}{\pi d}\int_0^{\pi}\cos n\Phi\cos m\Phi d\Phi=\frac{K'}{\pi d}\begin{cases}\pi & (m=n=0);\\ \dfrac{\pi}{2} & (m=n\neq 0);\\ 0 & (m\neq n).\end{cases}$$

$$(m,n=0,1,2,...).$$

Here we used the integral value from [23], (p.260, f-la 3.152.10).

Hence, the orthogonality conditions are given in the form

$$\int_c^d T_n(Y)T_m(Y)\frac{dy}{\sqrt{\left(d^2-y^2\right)\left(y^2-c^2\right)}}=\begin{cases}K'/d & (m=n=0),\\ K'/2d & (m=n\neq 0),\\ 0 & (m\neq n).\end{cases}$$

Transforming these conditions into variables (B4), we obtain

$$\int_\alpha^\beta T_n(\mathrm{X})T_m(\mathrm{X})\frac{d\vartheta}{\sqrt{(\cos\alpha-\cos\vartheta)(\cos\vartheta-\cos\beta)}}=\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\begin{cases}K'/d & (m=n=0);\\ K'/2d & (m=n\neq 0);\\ 0 & (m\neq n).\end{cases}\tag{B8}$$

Note that the relations (B5)–(B7) take place on a more general interval (α,β) $(0<\alpha<\beta<\pi)$ (α,β) $(0<\alpha<\beta<\pi)$ than the interval in the integral governing IE (10) where $\beta=\pi-\alpha$. Therefore, these relations should be modified in the equation (10) assuming $\beta=\pi-\alpha$. Then after simple transformations, the spectral relations (B5) turn to the followings:

$$\left.\begin{array}{l}\dfrac{1}{\pi}\displaystyle\int_\alpha^{\pi-\alpha}\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)\frac{T_n(Y)d\varphi}{\sqrt{\cos 2\alpha-\cos 2\varphi}}=\mu_n T_n(\mathrm{X});(\alpha<\vartheta<\pi-\alpha;n=0,1,2...)\\ \mathrm{X}=\cos\Theta,\ \Theta=\dfrac{\pi\sqrt{2}}{K'}\cos^2\dfrac{\alpha}{2}\displaystyle\int_\alpha^\vartheta\frac{dt}{\sqrt{\cos 2\alpha-\cos 2t}};\ Y=\cos\Phi,\ \Phi=\dfrac{\pi\sqrt{2}}{K'}\cos^2\dfrac{\alpha}{2}\displaystyle\int_\alpha^\varphi\frac{dt}{\sqrt{\cos 2\alpha-\cos 2t}}\\ \mu_n=\begin{cases}\dfrac{K'\sec^2\frac{\alpha}{2}}{\pi\sqrt{2}\,n}\operatorname{th}\left(\dfrac{\pi nK}{K'}\right) & (n=1,2,...);\\ \dfrac{K\sec^2\frac{\alpha}{2}}{\sqrt{2}\,n}\operatorname{th}\left(\dfrac{\pi nK}{K'}\right) & (n=0);\end{cases}\quad\left(k=\operatorname{tg}^2\dfrac{\alpha}{2};\ \alpha<\vartheta,\ \ \varphi<\pi-\alpha\right)\end{array}\right\}\tag{B9}$$

and integral relations (B6) turn to:

$$\frac{1}{\pi}\int_{\alpha}^{\pi-\alpha}\ln\left[\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right]\frac{T_n(Y)\,d\varphi}{\sqrt{\cos 2\alpha-\cos 2\varphi}}=$$

$$=\frac{v_n}{n}\left[H(\alpha-\vartheta)+(-1)^n H(\vartheta-\pi+\alpha)\right]\mathrm{sh}\left(\frac{\pi nu}{K'}\right)\ \left(\vartheta\in(0,\alpha)\cup(\pi-\alpha,\pi);=0,1,2...\right),$$

$$v_n=K'\sec^2\frac{\alpha}{2}\Big/\pi\sqrt{2}\,\mathrm{ch}\left(\pi nK/K'\right), \tag{B10}$$

where the case $n=0$ is replaced by the limiting case $n\to 0$ and the orthogonality conditions (B8) become:

$$\int_{\alpha}^{\pi-\alpha}T_m(\mathrm{X})T_n(\mathrm{X})\frac{d\vartheta}{\sqrt{\cos 2\alpha-\cos 2\vartheta}}=\frac{\sec^2\frac{\alpha}{2}}{2\sqrt{2}}K'\cdot\begin{cases}2 & (m=n=0),\\ 1 & (m=n\neq 0),\\ 0 & (m\neq n)\end{cases} \tag{B11}$$

and X is expressed by formula (B9). Note that in (B10) according to (B7)

$$\vartheta=\begin{cases}2\,\mathrm{arctg}\left(\mathrm{tg}\frac{\alpha}{2}\mathrm{sn}(u,k)\right) & (0<\vartheta<\alpha),\\ 2\,\mathrm{arctg}\left(\mathrm{ctg}\frac{\alpha}{2}\Big/\mathrm{sn}(u,k)\right) & (\pi-\alpha<\vartheta<\pi)\end{cases}\qquad (0<u<K) \tag{B12}$$

where $k=c/d=\mathrm{tg}^2(\alpha/2)\quad(0<\alpha<\pi/2)$ and hence $k'=\sqrt{1-\mathrm{tg}^4(\alpha/2)}$.

Spectral and related to them integral relations for the integral operator in (5) can be obtained directly from the relations (B9) – (B10) by returning to the previous variables (9) and taking into account formulas (B12). However, the relations (B9) - (B10) in the trigonometric forms are somewhat simpler.

We also transform the integral relations (B2). We get

$$\frac{1}{\pi}\int_c^d\ln\frac{y+\mathrm{v}}{|y-\mathrm{v}|}\frac{T_n(\mathrm{V})\,d\mathrm{v}}{\sqrt{(d^2-\mathrm{v}^2)(\mathrm{v}^2-c^2)}}=\frac{(-1)^n K'\,\mathrm{sh}(\pi nu/K')}{\pi nd\,\mathrm{ch}(\pi nK/K')}$$

$$\left(n=0,1,2,...,\ 0<u<K,\ y>d\right).$$

After differentiation of both sides of this relation with respect to y and taking into account that according to (B3b) in case $y>d$, we will come to equality

$$\frac{1}{\pi}\int_c^d\frac{T_n(\mathrm{V})2\mathrm{v}d\mathrm{v}}{(\mathrm{v}^2-y^2)\sqrt{(d^2-\mathrm{v}^2)(\mathrm{v}^2-c^2)}}=(-1)^{n+1}\frac{\mathrm{ch}(\pi nu/K')}{\mathrm{ch}(\pi nK/K')}\frac{1}{\sqrt{(x^2-d^2)(x^2-c^2)}}\quad(n=0,1,2,...,\ y>d).$$

Introduce new variables ζ,τ:

$$y^2=\zeta+\left(c^2+d^2\right)/2,\quad \mathrm{v}^2=\tau+\left(c^2+d^2\right)/2\quad\left(-b<\zeta,\tau<b;b=\left(d^2-c^2\right)/2\right).$$

After elementary transformations we obtain

$$\frac{1}{\pi}\int_{-b}^{b}\frac{T_n(V)\,d\tau}{(\tau-\zeta)\sqrt{b^2-\tau^2}}=(-1)^{n+1}\frac{\operatorname{ch}(\pi nu/K')}{\operatorname{ch}(\pi nK/K')}\frac{1}{\sqrt{\zeta^2-b^2}}\quad (n=0,1,2,...,\ \zeta>b),\quad \text{(B13)}$$

where according to (B1)

$$\mathrm{V}=\cos\Phi,\quad \Phi=\frac{\pi}{K'}\int_{1}^{\gamma(\tau)}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}},\ \gamma(\tau)=\frac{1}{c\sqrt{2}}\sqrt{2\tau+c^2+d^2}\qquad \text{(B14)}$$

and for variable u $(0<u<K')$ again by (B3b) we have

$$u=\int_{0}^{snu}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_{0}^{\chi(\zeta)}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}};\ \chi(\zeta)=\frac{d\sqrt{2}}{\sqrt{2\zeta+c^2+d^2}}.\qquad \text{(B15)}$$

Since here $b=(d^2-c^2)/2=(\operatorname{ctg}^2\alpha/2-\operatorname{tg}^2\alpha/2)/2=2\cos\alpha/\sin^2\alpha$ $(0<\alpha<\pi/2)$, then $0<b<\infty$ and hence in (B13)–(B15), we can formally replace b by a; ζ by x; τ by s and express other parameters in terms of a. Namely, it is easy to find that

$$\left.\begin{aligned}&c=\sqrt{\sqrt{1+a^2}-a},\quad d=\sqrt{a+\sqrt{1+a^2}},\quad k=\sqrt{1+a^2}-a;\ \cos\alpha=\frac{a}{1+\sqrt{1+a^2}};\\&\frac{c^2+d^2}{2}=\sqrt{1+a^2};\ \cos^2\frac{\alpha}{2}=\frac{1}{2a}\left(a-1+\sqrt{1+a^2}\right);\quad \cos^2\alpha=\frac{1}{a^2}\left(2+a^2-2\sqrt{1+a^2}\right).\end{aligned}\right\}\ \text{(B16)}$$

Now taking into account these changes, we rewrite the relationships (B13)–(B15):

$$\left.\begin{aligned}&\frac{1}{\pi}\int_{-a}^{a}\frac{T_n(\mathrm{V})\,ds}{(s-x)\sqrt{a^2-s^2}}=(-1)^{n+1}\frac{\operatorname{ch}(\pi nu/K')}{\operatorname{ch}(\pi nK/K')}\frac{1}{\sqrt{x^2-a^2}}\ \ (n=0,1,2,...;\ x>a),\\&\mathrm{V}=\cos\Phi,\ \Phi=\frac{\pi}{K'}\int_{1}^{\gamma(s)}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}},\ \gamma(s)=\frac{1}{a}\sqrt{s+\sqrt{1+a^2}},\\&u=\int_{0}^{\chi(x)}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},\quad \chi(x)=\sqrt{\frac{a+\sqrt{1+a^2}}{x+\sqrt{1+a}}},\end{aligned}\right\}\ \text{(B17)}$$

where the value of modulus k is given in (B16).

Let us consider also the limiting case of the discussing problem as $a\to l$. In this case, $\rho=1$ should be substituted into the equations (5) - (7), the relation (8) is excluded from consideration, and the integral equation (10) takes the form

$$\frac{2}{\pi}\int_{0}^{\pi}\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)\omega_0(\varphi)\,d\varphi=g_0(\vartheta)\quad (0<\vartheta<\pi)\qquad \text{(B18)}$$

The eigenfunctions and eigenvalues of the kernel of the equation (B18) can be easily found using well-known Fourier series [23] (p.52, f-la 1.441.2):

$$\ln\left(1\big/2\left|\sin\frac{x}{2}\right|\right)=\sum_{n=1}^{\infty}\frac{\cos nx}{n}\left(-2\pi<x<2\pi\right).$$

Whence we get

$$\ln\left(\sin\left(\frac{\vartheta+\varphi}{2}\right)\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)=2\sum_{n=1}^{\infty}\frac{\sin n\vartheta\sin n\varphi}{n}\qquad\left(0<\vartheta,\ \varphi<\pi\right).$$

Now if we multiply both sides of this expansion by $\sin m\varphi\ \left(m=1,2,...\right)$ and integrate the resulting equality with respect to φ over the interval $\left(0,\pi\right)$, we come to the following spectral relations $\left(m\to n\right)$

$$\frac{1}{\pi}\int_0^{\pi}\ln\left(\frac{\sin\left(\vartheta+\varphi\right)}{2}\Big/\left|\sin\left(\frac{\vartheta-\varphi}{2}\right)\right|\right)\sin n\varphi d\varphi=\frac{1}{n}\sin n\vartheta\quad\left(n=1,2,...,0<\vartheta<\pi\right).$$

This relation in the former variables (9) with $\rho=1$ takes the form [29]

$$\frac{1}{2\pi}\int_{-1}^{1}\ln\frac{1-\xi\eta+\sqrt{\left(1-\xi^2\right)\left(1-\eta^2\right)}}{1-\xi\eta-\sqrt{\left(1-\xi^2\right)\left(1-\eta^2\right)}}U_{n-1}\left(\eta\right)d\eta=\frac{1}{n}\sqrt{1-\xi^2}U_{n-1}\left(\xi\right)\ \left(-1<\xi<1;\ n=1,2,...\right)\tag{B19}$$

Appendix C. To analytically explore the behavior of K_I^0 as $\chi\to1$ or $\rho\to1$, we substitute in the formula (7) (as in section 4)

$p_0\left(\xi\right)=\Omega\left(\xi,\rho\right)\big/\sqrt{1-\xi^2}\quad\left(-1<\xi<1\right)$,

where $\Omega\left(\xi,\rho\right)$ is the even Hölder function on the interval $\left[-1,1\right]$. Then we transform this formula as follows:

$$K_I^0=\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{\Omega\left(\eta,\rho\right)d\eta}{\sqrt{1-\eta^2}}=\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{A\eta+B}{\sqrt{1-\eta^2}}d\eta+\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{\Omega\left(\eta,\rho\right)-A\eta-B}{\sqrt{1-\eta^2}}d\eta$$

where A and B are not yet known constants. Set then $\Omega_0\left(\eta,\rho\right)=\Omega\left(\eta,\rho\right)-A\eta-B$ $\left(-1\le\eta\le1\right)$ and determine the constants A and B from the conditions $\Omega_0\left(\pm1,\rho\right)=0$. As a result,

$A=\left[\Omega_0\left(1,\rho\right)-\Omega_0\left(-1,\rho\right)\right]\big/2;\quad B=\left[\Omega_0\left(1,\rho\right)+\Omega_0\left(-1,\rho\right)\right]\big/2$

and due to the parity of the function $\Omega_0\left(\eta,\rho\right)\quad A=0,\ B\ne0$. Then we can write

$$K_I^0=B\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{d\eta}{\sqrt{1-\eta^2}}+\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{\Omega_0\left(\eta,\rho\right)}{\sqrt{1-\eta^2}}d\eta,\ \ \Omega_0\left(\eta,\rho\right)=\Omega\left(\eta,\rho\right)-B.\tag{C1}$$

Since $\Omega_0\left(\pm1,\rho\right)=0$, the second integral in (65a) is limited when $\rho\to1$, and the first integral is easily calculated. Namely,

$$I\left(\rho\right)=\int_{-1}^{1}\sqrt{\frac{\rho+\eta}{\rho-\eta}}\frac{d\eta}{\sqrt{1-\eta^2}}=\int_{-1}^{1}\frac{\left(\rho+\eta\right)d\eta}{\sqrt{\left(1-\eta^2\right)\left(\rho^2-\eta^2\right)}}=\rho\int_{-1}^{1}\frac{d\eta}{\sqrt{\left(1-\eta^2\right)\left(\rho^2-\eta^2\right)}}=2K\left(\chi\right)\left(\chi=1/\rho\right)\tag{C2}$$

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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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ОБ УСТАЛОСТНОЙ ПРОЧНОСТИ КОМПОЗИТНЫХ МАТЕРИАЛОВ

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Ключевые слова: композитные материалы (КМ), микротрещина, циклическое нагружение, усталостная прочность, выносливость, предел прочности**.**

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About fatigue strength of composite materials

Keywords: composite materials, micro crack, cyclic loading, fatigue strength, endurance, strength limit.

Composite materials are classified as anisotropic, and then the analysis of their fatigue strength requires special approaches. In this work, a deformation theory of fatigue penetration CM based on a review of the deforming process by cyclic loading has been developed, as an elastic-plastic material. A differential equation for uniaxial oscillations with variable, coefficients of the Mathieu-Hill type was obtained. For low-cycle loading an equation for the fatigue curve was obtained.

Գևորգյան Ա.Վ., Շեկյան Հ.Գ.

Կոմպոզիտային նյութերի հոգնածային ամրության մասին

Հիմնաբառեր: կոմպոզիտային նյութեր, միկրոճաք, ցիկլիկ բեռնավորվածություն, դիմացկունություն, հոգնածային ամրություն:

Կոմպոզիտային նյութերը պատկանում են անիզատրոպ նյութերի շարքին և դրա համար նրա հոգնածային ամրության վերլուծության հարցերը պահանջում են յուրահատուկ մոտեցում: Տվյալ աշխատանքում մշակված է կոմպոզիտային նյութերի հոգնածային ամրության դեֆորմացիոն տեսություն, որտեղ ցիկլիկ բեռնավորված նյութը դիտարկվում է որպես առաձգա-պլաստիկ: Ստացված է միառանցք տատանման փոփոխական գործակիցներով Մատե-Հիլլի տիպի դիֆերենցիալ հավասարում: Ցածր հաճախականությամբ փոփոխվող ցիկլիկ լարվածային վիճակի համար ստացված է հոգնածային կորի հավասարումը:

Композитные материалы относятся к анизотропным материалам, поэтому вопросы анализа их усталостной прочности требуют особых подходов. В данной работе разработана деформационная теория усталостной прочности КМ, где циклически нагружённый материал подвергнут упруго-пластической деформации.

Получено дифференциальное уравнение с переменными коэффициентами типа Матье-Хилля для случая одноосного колебания системы. Получено также уравнение кривой усталости для состояния низко-частотных циклически изменяющегося напряжённого состояния.

Введение: Одна из основных трудностей современной теории прочности твёрдых при циклическом нагружении состоит в том, что пока ещё недостаточно ясно понимают механизм «усталостной» прочности. Разрушение деталей при переменном режиме изменения нагружения происходит при напряжениях, намного меньших предела прочности и даже предела текучести, если только эти изменения повторяются достаточно большое число раз. Вследствие этого «усталостные» поломки деталей, изготовленных из композитных материалов, происходят обычно без внешних проявлений пластической деформации и носят характер внезапных разрушений. Кривые, выражающие зависимости между числами циклов и напряжениями, устанавливаются экспериментально и строятся в координатах $\sigma - N$ или $\sigma - lgN$. Кривые позволяют определить наибольшее напряжение цикла выносливости, при котором элемент конструкции не разрушается. Основной особенностью переменного деформирования КМ является неодновременность разрыва армированных волокон из-за существующего первоначального натяга отдельных волокон. Очевидно, что предельное состояние для предварительно натянутых волокон достигается раньше, чем для остальных волокон. Это приводит к образованию новых перенапряжений в волокнах. Экспериментальные исследования показывают, что такой материал ещё в состоянии воспринимать нагрузки до тех пор, пока не разрушатся перезагружённые остальные волокна. При этом, после разрушения этих волокон нагрузка перераспределяется между остальными волокнами равномерно и при повторных нагружениях постепенно накапливаются дефекты, связанные с разрушением связывающего и появляются новые трещины в волокнах. Это вызывает потери несущей способности поперечного сечения. Тогда действующее среднее напряжение будет увеличиваться и когда оно по величине станет равным пределу статической прочности (σ_b или σ_τ), произойдет полное разрушение. В существующей теории не объясняется причина роста пор и трещин в КМ. Процесс развития трещин в испытуемых образцах из КМ могут начинаться в нескольких участках одновременно и дальнейшее развитие в других новых участках непредсказуема. Следовательно, определение предела выносливости для КМ требует большого количества испытаний и значительного времени. Идею теоретического вывода функциональной связи роста напряжений в зависимости от числа циклов нагружения, суммированием усталостных повреждений и микротрещин, чрезвычайно трудно, и если учесть, что часто разрушения происходят не там, где большая микротрещина, а совершенно в другом месте, а объёмы работ по изучению микро- и макротрещин огромны, то значимость такого подхода становится неэффективным [4,5]. Для исключения подобных ситуаций представляется необходимость разработки специальной теории, позволяющей выполнять объективные расчётные оценки усталостной прочности и долговечности элементов конструкции из КМ. В настоящей работе по результатам анализа сформулированы основные положения деформационной теории усталостной прочности элементов из КМ, которые могут использоваться при инженерных расчётах на этапах эскизного и рабочего проектирования.

<u>Постановка задачи и решение.</u> Проблемы формирования прочностных параметров КМ при циклически изменяющихся нагружениях в системах машин, механизмов и летательных аппаратах должны быть решены на основе тех теоретических подходов, которые разработаны и доведены до инженерного приложения. Необходимо синтезировать аналитические подходы и на их базе получить приемлемые результаты.

Известно, что предел выносливости для любого материала зависит как от числа циклов и характера изменения нагружения, так и от типа напряжённого состояния (характера изменения напряжения во времени), т.е. от степени асимметрии цикла. В большинстве случаев испытания на выносливость выполняются для симметричного цикла изменения напряжения, в то время, как в большинстве случаев изменения напряжения происходят по асимметричному циклу. Первый экспериментатор этой области Веллер показал, что диапазон напряжения $\sigma_{\min} \div \sigma_{\max}$, необходимого для определения коэффициента асимметрии «r», будет:

$$\sigma_r = \sigma_a + \sigma_m; \quad r = \frac{\sigma_{\min}}{\sigma_{\max}}. \tag{1}$$

Здесь σ_a – амплитуда напряжения, σ_m – среднее напряжение.

На фиг.1 приведена схематизированная диаграмма зависимости σ_a от σ_m. При $\sigma_m = 0$ имеем симметричный цикл, где предельное напряжение равно σ_{-1}.

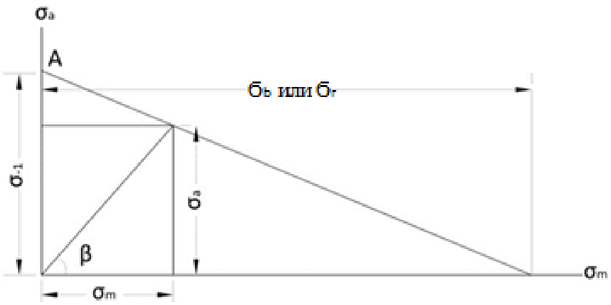
Из фиг.1 имеем:

$$\operatorname{tg}\beta = \frac{\sigma_a}{\sigma_m} = \frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} = \frac{1-r}{1+r}. \tag{2}$$

Из (1) и (2) $\sigma_r = \sigma_m + \sigma_a = \dfrac{2\sigma_m}{1+r}.$ (3)

Тогда уравнение прямой АС можно представить в виде:

$$\frac{\sigma_a}{\sigma_{-1}} + \frac{\sigma_m}{\sigma_b} = 1. \tag{4}$$



Фиг.1. Диаграмма предельных напряжений

С учётом (4) σ_r примет вид:

$$\sigma_r = \frac{2\sigma_{-1}\sigma_b}{\sigma_b\left(1-r\right) + \left(1+r\right)\sigma_{-1}}. \tag{5}$$

Отсюда следует, что зная предел статической прочности $\sigma_b\left(\sigma_r\right)$ и предел выносливости σ_{-1}, можно определить предел усталостной прочности для любого асимметричного цикла нагружения.

При оценке прочности деталей из композитных материалов, значения предела выносливости σ_{-1} можно определить на основании данных испытаний с корректировкой результатов в связи с учётом влияний ряда известных факторов на прочность (формы и размеры деталей, состояние поверхностей, наличие различных концентратов, изменения режима нагружения и т.д.) [1,2,3].

Очень важно знать, как быстро кривая $\sigma - N$ приближается к асимптоте, т.к. число циклов, необходимых для установления предела выносливости, характеризует долговечность детали. Опыты показывают, что для чёрных металлов предел выносливости установлен и при числе циклов $N_\sigma = 10^7$ $\sigma_r = 0,5\sigma_b$ (для симметричного цикла).

Для цветных металлов и композитных материалов нет определённого предела выносливости, и ординаты кривой $\sigma - \lg N$ уменьшаются до нуля с возрастанием числа циклов.

На наш взгляд, суммированием повреждений и микротрещин чрезвычайно затруднительно осуществить идею теоретического вывода функциональной связи роста напряжений в КМ в зависимости от числа циклов нагружения. В выражениях интегральной суммы повреждений присутствуют величины микротрещин, которые могут быть определены с помощью корреляционного анализа результатов изучения микроструктуры образца. Если учесть, что разрушения часто происходят не там, где большая микротрещина, а совершенно в другом месте, то значимость этого подхода становится абсурдным.

Для описания упруго-пластической деформации композитных материалов используем формы записи, часто применяемые в электротехнике для учёта сдвига между напряжением и силой тока. Тогда, для упруго-пластической деформации КМ при одноосном растяжении-сжатии можно написать:

$$\sigma = E\varepsilon e^{\alpha}, \tag{6}$$

где ε и σ – упругая деформация и напряжение, α – некоторая постоянная, зависящая от свойства композита, характера деформирования и вида напряжённого состояния $(\alpha << 1)$. Разложив e^{α} в ряд Тейлора и пренебрегая малыми членами, выражение (6) можно представить в виде:

$$\sigma = E\varepsilon(\mathrm{I} + \alpha), \tag{7}$$

здесь $\alpha\varepsilon$ – пластическая деформация. При многократном повторении циклов напряжения, пластическая деформация будет расти, что приведёт к изменению жёсткости исследуемого образца. Если начальная жёсткость рассматриваемого элемента конструкции была C_0, то после определённого количества циклов изменения напряжений жёсткость уменьшается до значения C_0-C_k, где $C_k = \alpha C_0 n$, n – количество чисел нагружения в долях $\lg$ N, т.е.

$$n = \lg N, \ N = 10^n. \tag{8}$$

Поскольку пластическая деформация растёт по экспоненциальному закону $\left(\varepsilon_n = \varepsilon e^{a}\right)$, то жёсткость образца также уменьшится по экспоненциальному закону. Тогда, жёсткость испытуемого образца после воздействия циклических нагрузок можно представить в виде:

$$C'_k\left(1 - a\lg N\right), \qquad C'_k = C_0 - C_k.$$

Если испытуемый образец с изменяющейся по времени нагрузкой рассмотреть как колебательную систему с возбуждающей силой $P(t)$=P$\sin\omega t$, где ω – угловая чистота возбуждения, $\left(\omega = 2\pi\dfrac{N}{t}\right)$, то одноосное колебательное движение образца можно описать дифференциальным уравнением:

$$\ddot{y} + \frac{C'_k}{m^x}y = \frac{P}{m^x}\sin\omega t \ \text{ или } \ \ddot{y} + \frac{C_0}{m^x}(1 - \alpha\lg N)y = \frac{P}{m^x}\sin\omega t, \tag{9}$$

где y – амплитуда колебания (растяжения-сжатия), m^x – массы единицы объёма образца, P – амплитуда нагрузки.

Уравнение (9) с переменным коэффициентом при "y", зависящим от числа циклов и времени, является уравнением Матье-Хилл и не имеет аналитического решения, его можно решить приближёнными методами.

При изменении нагрузки с частотой меньше 20гц (малоцикловое напряжённое состояние) инерционными силами можно пренебречь и тогда будем иметь:

$$\omega_0\left(1-a\lg N\right)y=\frac{P}{m^x}\sin\omega t \text{ или}$$

$$P_{\max}(1-\alpha\lg N)y=P\sin\omega t, \tag{10}$$

где $y=y_{\max}\sin\omega \mathrm{t}$, а $C_0\left(1-a\lg N\right)y_{\max}=P$. (11)

Приняв $C_o=\dfrac{P_{\max}}{y_{\max}}$, где $P_{\max}$ – максимальная сила, которая при начальном сечении F_o образца может вызывать напряжения, равные σ_b или σ_r, получим:

$$P_{\max}\left(1-a\lg N\right)=P. \tag{12}$$

Разделив обе части уравнения (12) на F_o, будем иметь:

$$\sigma_b\left(1-a\lg N\right)=\sigma_a, \tag{13}$$

где $\dfrac{P_{\max}}{F_0}=\sigma_b$, $\dfrac{P}{F_0}=\sigma_a$

или $\sigma_a+a\sigma_b\cdot\lg N=\sigma_b$, $\alpha\sigma_b=m$, тогда $\sigma_a+m\lg N=\sigma_b$. (14)

Для определения коэффициента «α», испытуемый образец рассмотрим как прямоугольный брус с размерами a_0, b_0 и l_0 (a_0 – толщина, b_0 – ширина, l_0 – начальная длина).

Тогда можем написать:

$$\left.\frac{da}{a}\right|_{a_0}^{a}=\mu_1\left.\frac{dz}{z}\right|_{l_0}^{l};\quad \left.\frac{db}{b}\right|_{b_0}^{b}=\mu_2\left.\frac{dz}{z}\right|_{l_0}^{l}. \tag{15}$$

После интегрирования будем иметь:

$$\frac{a}{d_0}=\left(\frac{\ell_0}{\ell}\right)^{\mu_1};\frac{b}{b_0}=\left(\frac{\ell_0}{\ell}\right)^{\mu_2}\ \frac{ab}{a_0b_0}=\left(\frac{\ell_0}{\ell}\right)^{\mu_1+\mu_2};\ \frac{F}{F_0}=\left(\frac{\ell_0}{\ell}\right)^{\mu_1+\mu_2}, \tag{16}$$

здесь μ_1 и μ_2 – коэффициенты Пуасона во взаимно-перпендикулярных направлениях в образце. F_0 – начальная площадь сечения образца, F– площадь сечения образца после деформирования.

Величина уменьшения площади сечения будет:

$$\Delta F=F_0-F \quad \text{или} \quad \Delta F=F_0\left[1-\left(\frac{\ell_0}{\ell}\right)^{\mu_1+\mu_2}\right]. \tag{17}$$

Поскольку $\varepsilon=\dfrac{\sigma_1}{E}=\dfrac{\sigma}{E}+\dfrac{\sigma}{E}\alpha$, получим $\varepsilon=\varepsilon_1+\varepsilon_n=\varepsilon_1+\alpha\varepsilon_1$, где $\varepsilon = \dfrac{\sigma_1}{E}$; $\varepsilon_y = \dfrac{\sigma}{E}$; $\sigma_1 = \dfrac{P}{F}$; $\sigma_a = \dfrac{P}{F_0}$, (18)

$$\frac{P}{EF} = \frac{P}{EF_0} + \alpha\frac{P}{EF_0}; \quad \frac{F_0}{EF} = \frac{1}{E} + \frac{\alpha}{E}; \quad \frac{F_0}{F} = \left(\frac{l}{l_0}\right)^{\mu_1+\mu_2}.$$

Из (16), (17) и (18) будем иметь:

$$\alpha = \left(\frac{\ell}{\ell_0}\right)^{\mu_1+\mu_2} - 1; \tag{19}$$

Имея в виду, что $\ell = \ell_0 + \Delta$; $\dfrac{\Delta}{\ell_0} = \varepsilon_y = \dfrac{\sigma}{E}$, будем иметь:

$$\alpha = \left(\frac{E+\sigma}{E}\right)^{\mu_1+\mu_2} - 1 \text{ или } \alpha = \left(\frac{KE+\sigma_b}{KE}\right)^{\mu_1+\mu_2} - 1, \tag{20}$$

где $K = \dfrac{\sigma_b}{\sigma_a}$.

Тогда уравнение (14) примет вид:

$$\sigma_a + m\lg N = \sigma_b; \tag{21}$$

где $m = \sigma_b\left[\left(\dfrac{KE+\sigma_b}{KE}\right)^{\mu_1+\mu_2} - 1\right]$. (22)

Выводы: На основании проведённых исследований и опыта мировой практики впервые получены:

– формулы расчёта предельных напряжений КМ для любого асимметричного цикла изменения напряжения при известном пределе выносливости симметричного цикла;

–уравнение кривой выносливости, учитывающее физико-механические характеристики КМ и характера напряжённого состояния.

–формулы расчёта предельных напряжений усталости КМ с учётом анизотропности во взаимно перпендикулярных направлениях.

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CONTENTS Mechanics 2019, vol. 72

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CONTENTS v. 72, №4, 2019

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