ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ PROCEEDINGS OF NATIONAL ACADEMY OF SCIENCES OF ARMENIA
ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ

ՄԵԽԱՆԻԿԱ
МЕХАНИКА
MECHANICS

2017

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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 70, №2, 2017 Механика

УДК 536.21; 539.3

АСИМПТОТИЧЕСКИЕ РЕШЕНИЯ НЕСВЯЗАННЫХ ЗАДАЧ СТАЦИОНАРНОЙ ТЕПЛОПРОВОДНОСТИ И ТЕРМОУПРУГОСТИ ДЛЯ ДВУХСЛОЙНЫХ ПЛАСТИН С НЕКЛАССИЧЕСКИМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

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Ключевые слова: Неклассическая краевая задача, асимптотический метод, внутренняя задача, контакт между слоями.
Keywords: Thenon-classicalboundary value problem,asymptotic method, theinternal task, contact between the layers.
Բանալի բառեր: Ոչ դասական եզրային խնդիր, ասիմպտոտիկական եղանակ, ներքին խնդիր, շերտերի միջև ոչ լրիվ կոնտակտ:

Աղալովյան Լ.Ա., Գևորգյան Ռ.Ս.

Երկշերտ սալերի համար ստացիոնար ջերմահաղորդականության և ջերմաառաձգականության ոչ կապակցված ոչ դասական ե զրային պայմաններով խնդիրների ասիմպտոտիկական լուծումները

Հաշվի առնելով ոչ դասական եզրային խնդիրների կիրառական նշանակությունը, մասնավորապես սեյսմոլոգիայում, երկշերտ օրթոտրոպ սալի համար ասիմպտոտիկ եղանակով կառուցված են Ջերմհաղորդականության և ջերմաառաձգականության ոչ կապակցված ոչ դասական եզրային պայմաններով խնդիրների լուծումները: Շերտերի միջև կոնտակտը կարող է լինել լրիվ և ոչ լրիվ: Բերված են վերլուծական օրինակներ:

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Asymptotic solutions of stationary problems of thermal conductivity and thermoelasticity with nonclassical boundary conditions for the two-layer plates with full and incomplete contact layers

The practical importance of the neoclassical solutions is well known. The solutions of the nonclassical boundary value problems of stationary heat conduction and no connected thermoelasticity theory are constructed for the orthotropic two-layer plates. Examples are given together with their analysis.

В последние годы проявляется повышенный интерес к неклассическим краевым задачам математической физики, когда по какой-либо причине на одной части поверхности области, занимаемой материальным телом, граничных условий задано больше, чем необходимо для краевой задачи данного класса, а на другой части – меньше, чем необходимо или вообще не заданы [1–3]. Возникновение таких задач, в частности, связано с изучением напряжённо-деформированных состояний Литосферных плит Земли [6]. Одним из эффективных методов решения подобных задач является асимптотический метод решения сингулярно-возмущённых дифференциальных уравнений [4-7]. В настоящей работе асимптотическим методом строятся общие интегралы в виде рекуррентных формул для стационарной задачи теплопроводности и несвязанной теории термоупругости. Удовлетворив неклассическим смешанным граничным условиям, однозначно определены все функции интегрирования, позволяющие вычислить температурную функцию, а также компоненты тензора напряжений и вектора перемещения двухслойной ортотропной пластины переменной толщины. Задача, в частности, может моделировать напряжённо-деформированное состояние земной коры в зоне коллизии тектонических плит Земли [1-3,6,7].

1. Постановка краевых задач и общий интеграл разрешающих уравнений. Рассмотрим двухслойный пакет пластин из ортотропных материалов, слои которого ограничены гладкими непересекающимися поверхностями и относительно выбранной прямоугольной системы координат *Oxyz* удовлетворяют условиям

$\varphi_1(x,y) > \varphi_0(x,y) > \varphi_2(x,y),\ h = \mathrm{Sup}\left|\varphi_1 - \varphi_2\right| << l,\ -\infty < (x,y) < \infty$, где $l-$некоторый характерный продольный размер тонкого пакета.

Пусть на лицевой поверхности $z = \varphi_1(x,y)$ двухслойного пакета заданы неклассические граничные условия задачи стационарной теплопроводности (изменение температуры и плотность потока теплоты),

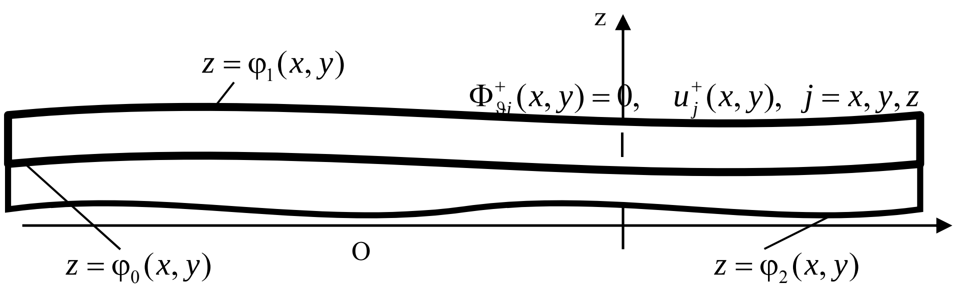
$$z = \varphi_1(x,y): \quad \theta = \theta^+, \quad \theta = T - T_0$$

$$-\frac{q_x}{\Lambda_1}\frac{\partial\varphi_1}{\partial x} - \frac{q_y}{\Lambda_1}\frac{\partial\varphi_1}{\partial x} + \frac{q_z}{\Lambda_1} = q^+_{\vartheta_1}, \quad q_x = -\lambda_{11}\frac{\partial\theta}{\partial x}\ (x,y,z;\ 1,2,3) \tag{1.1}$$

а также неклассические механические граничные условия одновременно и первой, и второй краевых задач несвязанной теории термоупругости

$$z = \varphi_1(x,y): \ \frac{\sigma_{jx}}{\Lambda_1}\frac{\partial\varphi_1}{\partial x} + \frac{\sigma_{jy}}{\Lambda_1}\frac{\partial\varphi_1}{\partial y} - \frac{\sigma_{jz}}{\Lambda_1} = \Phi^+_{\vartheta j},\ u_j(\varphi_1) = u_x^+,\ j = x,y,z$$

$$\Lambda_k = \sqrt{1 + \left(\frac{\partial\varphi_k}{\partial x}\right)^2 + \left(\frac{\partial\varphi_k}{\partial y}\right)^2}, \quad k = 0,1,2 \tag{1.2}$$



Фигура

а на противоположной лицевой поверхности $z = \varphi_2(x,y)$ пакета никакие условия не заданы. (Задачи с такими или с аналогичными граничными условиями считаются неклассическими краевыми задачами теории упругости). Требуется определить температурное поле и напряжённо-деформированное состояние пакета, когда между слоями выполняются условия полного теплового

$$z = \varphi_0(x,y): \ \theta^{(1)}(z = \varphi_0) = \theta^{(2)}(z = \varphi_0)$$

$$\left(q_x^{(1)} - q_x^{(2)}\right)\frac{\partial\varphi_0}{\partial x} + \left(q_y^{(1)} - q_y^{(2)}\right)\frac{\partial\varphi_0}{\partial y} - \left(q_z^{(1)} - q_z^{(2)}\right) = 0 \tag{1.3}$$

и полного механического

$$z = \varphi_0:$$

$$\left(\sigma_{jx}^{(1)} - \sigma_{jx}^{(2)}\right)\frac{\partial\varphi_0}{\partial x} + \left(\sigma_{jy}^{(1)} - \sigma_{jy}^{(2)}\right)\frac{\partial\varphi_0}{\partial y} - \left(\sigma_{jz}^{(1)} - \sigma_{jz}^{(2)}\right) = 0, \ j = x,y,z \tag{1.4}$$

$$U_j = u_j^{(1)} - u_j^{(2)} = 0,\ j = x,y, \quad u_z^{(1)} = u_z^{(2)}$$

контактов, а также неполного механического контака краевых задач теории упругости [8]

$$z=\varphi_0: U_z = u_z^{(1)}(\varphi_0)-u_z^{(2)}(\varphi_0)=0,\ U_x = u_x^{(1)}(\varphi_0)-u_x^{(2)}(\varphi_0)\neq 0,\ (x,y)$$

$$\Lambda_0 T_x = \sigma_{xz}^{(i)}\left(\varphi_0\right)-\sigma_{xx}^{(i)}(\varphi_0)\frac{\partial\varphi_0}{\partial x}-\sigma_{xy}^{(i)}(\varphi_0)\frac{\partial\varphi_0}{\partial y}\quad (x,y),\ \ i=1,2 \tag{1.5}$$

$$\Lambda_0 T_z^{(s)} = \sigma_{zz}^{(i,s)}\left(\varphi_0\right)-\sigma_{xz}^{(i,s)}(\varphi_0)\frac{\partial\varphi_0}{\partial x}-\sigma_{yz}^{(i,s)}(\varphi_0)\frac{\partial\varphi_0}{\partial y},\ \ i=1,2$$

а) по аналогии с гипотезой Винклера – Фусса

$$\sigma_{zz}^{(1)}(z=\varphi_0)=\sigma_{zz}^{(2)}(z=\varphi_0),\quad u_z^{(1)}(z=\varphi_0)=u_z^{(2)}(z=\varphi_0)$$

$$u_x^{(1)}(z=\varphi_0)-u_x^{(2)}(z=\varphi_0)=U_x\ \ (x,y) \tag{1.6}$$

$$\sigma_{xz}^{(1)}(z=\varphi_0)=\sigma_{xz}^{(2)}(z=\varphi_0)=T_x=\mu_x U_x\quad (x,y)$$

(эта модель предполагает в зоне контакта шероховатых поверхностей образование плотного слоя толщины h^{im}, который подвергается сдвигу вдоль поверхности $z=\varphi_0\left(x,y\right)$ контакта на величину $U=\left(+u_x\right)-\left(-u_x\right)$, где $\left(\pm u_x\right)$ (x,y) – перемещения точек верхнего и нижнего (краёв) берегов воображаемого уплотнённого слоя контакта шероховатостей и $\tau = G^{im}\,U/h^{im}$ – соответственно его касательное напряжение и модуль сдвига).

б) по аналогии закона сухого трения Кулона

$$\sigma_{zz}^{(1)}(z=\varphi_0)=\sigma_{zz}^{(2)}(z=\varphi_0),\quad u_z^{(1)}(z=\varphi_0)=u_z^{(2)}(z=\varphi_0)$$

$$u_x^{(1)}(z=\varphi_0)-u_x^{(2)}(z=\varphi_0)=U \tag{1.7}$$

$$\sigma_{xz}^{(1)}(z=\varphi_0)=\sigma_{xz}^{(2)}(z=\varphi_0)=\tau=f\sigma_{zz}^{(1)}(z=\varphi_0)=f\sigma_{zz}^{(2)}(z=\varphi_0).$$

Доказаны, что всегда существуют классические краевые задачи теории упругости и теории теплопроводности, решения которых совпадают с решениями сформулированных неклассических задач [3,7].

Для решения поставленных краевых задач приведём уравнения и соотношения теории термоупругости ортотропного тела с учётом объёмных сил $\vec{P}=\left\{P_x,P_y,P_z\right\}$ и изменения температурного поля $\theta=T-T_0$ по модели Дюгамеля – Неймана

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+P_x=0,\frac{\partial u_x}{\partial x}=e_1+\beta_{11}\theta,\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}=a_{44}\sigma_{yz} \tag{1.8}$$

$$e_m=a_{1m}\sigma_{xx}+a_{2m}\sigma_{yy}+a_{3m}\sigma_{zz},\quad m=1,2,3\ \ (y,z,x;4,5,6),$$

где σ_{ij} – компоненты тензора напряжений, u_x,u_y,u_z – компоненты вектора перемещения, a_{ij} – коэффициенты упругой податливости, β_{jj} – коэффициенты теплового линейного расширения.

Допускается возможное медленное изменение во времени заданных функций, при этом, не вызывая ощутимых динамических эффектов в пакете слоёв. Исходя из этого, здесь и в дальнейшем в формулах и соотношениях время t не будет фигурировано. Приведём также уравнение стационарной задачи теплопроводности ортотропного тела [9,10]

$$\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}=W,\quad q_x=-\lambda_{11}\frac{\partial\theta}{\partial x}\ \left(x,y,x;\,1,2,3\right), \tag{1.9}$$

где q_x, q_y, q_z – компоненты вектора плотности теплового потока, $\lambda_{11},\lambda_{22},\lambda_{33}$ – коэффициенты теплопроводности, W – заданная плотность источника тепла.

В уравнениях и соотношениях (1.8),(1.9) перейдём к безразмерным координатам и безразмерным перемещениям по формулам

$$\xi=\frac{x}{l},\ \ \eta=\frac{y}{l},\ \zeta=\frac{z}{h}=\varepsilon^{-1}\frac{z}{l},\ u=\frac{u_x}{l},\ v=\frac{u_y}{l},\ w=\frac{u_z}{l},\varepsilon=\frac{h}{l}, \tag{1.10}$$

где l – некоторый продольный характерный размер слоёв.

Подставив (1.10) в (1.8), (1.9), получаем сингулярно-возмущённую геометрическим малым параметром ε систему уравнений и соотношений, асимптотическое решение которых, согласно [4,5], складывается из двух решений. Первое из них, называемое внутренним (внешним) решением, удовлетворяет граничным условиям, заданным на лицевых поверхностях пакета. Второе решение, называемое решением задачи пограничного слоя, на лицевых поверхностях пластины удовлетворяет соответствующим однородным (нулевым) условиям, а в сумме с внутренним решением должно удовлетворить граничным условиям, заданным на торцах пакета. Поскольку рассматриваются ортотропные слои (пластины бесконечных размеров), следовательно, решается только внутренняя задача. Решение ищется в виде асимптотического разложения.

$$Q^{(i)}\left(x,y,z\right)=\sum_{s=0}^{S}\varepsilon^{s+\chi_Q}Q^{(i,s)}\left(\xi,\eta,\zeta\right),\ Q^{(i,m)}=0,\ m<0,\ i=1,2\ , \tag{1.11}$$

где $Q^{(i)}$ – любая из неизвестных компонент вектора перемещения u_j, тензора напряжений σ_{ij}, χ_Q характеризует асимптотический порядок соответствующей величины, $\chi_u=0$ – для всех перемещений, $\chi_\sigma=-1$ – для всех напряжений, а для температурной функции и компонент вектора плотности теплового потока должны быть соответственно $\chi_\theta=-1,\ \chi_{q_x}=\chi_{q_y}=-1,\ \chi_{q_z}=-2$.

Одновременно представим заданные объёмные силы и плотности источников тепла W в виде асимптотических разложений:

$$P_x=\sum_{s=0}^{S}\varepsilon^{-2+s}l^{-1}P_x^{(s)}\left(\xi,\eta,\zeta\right)\ \ (x,y,z),\ W=\sum_{s=0}^{S}\varepsilon^{-3+s}l^{-1}W^{(s)}\left(\xi,\eta,\zeta\right)$$
$$Q^{(0)}=Q,\ Q^{(s)}=0,\ s\neq 0,\ Q=\left\{P_j,W\right\}. \tag{1.12}$$

Это означает, что объёмные силы и источник тепла могут влиять на напряжённо-деформированное состояние пакета слоёв, начиная с первого шага итерационного процесса, если их асимптотические порядки будут соответственно ε^{-2} и ε^{-3}.

Подставив (1.11), (1.12) в систему сингулярных уравнений и приравняв коэффициенты при $\varepsilon^s\left(s=0,1,2,\ldots,S\right)$ в левых и правых частях уравнений, получим непротиворечивую систему дифференциальных уравнений относительно неизвестных коэффициентов разложения (1.11), что свидетельствует о правильности выбранной асимптотики. После интегрирования полученной системы разрешающих уравнений для температурной функции, компонент тензора напряжений и вектора перемещения получаются рекуррентные формулы, которым присвоим номер соответствующего слоя $(i=1,2)$ пакета и представим в размерных координатах и перемещениях. Для температурной функции имеем:

$$\theta^{(i,s)} = zA^{(i,s)} + B^{(i,s)} + \frac{1}{\lambda_{33}^{(i)}}\Psi^{(i,s)}$$
$$\Psi^{(i,s)} = -\int_0^z\left[\int_0^\beta\left(\lambda_{11}^{(i)}\frac{\partial^2\theta^{(i,s-2)}}{\partial x^2} + \lambda_{22}^{(i)}\frac{\partial^2\theta^{(i,s-2)}}{\partial y^2} + W^{(i,s)}\right)d\alpha\right]d\beta,\ i=1,2, \tag{1.13}$$

а для компонент тензора напряжений и вектора перемещения

$$\sigma_{jz}^{(i,s)} = \sigma_{jz0}^{(i,s)}(x,y) + \sigma_{jz*}^{(i,s)}(x,y,z), \quad j=x,y,z$$
$$\sigma_{xx}^{(i,s)} = A_{13}^{(i)}\sigma_{zz0}^{(i,s)}(x,y) + \sigma_{xx*}^{(i,s)}(x,y,z)\ (xx,yy;\,1,2)$$
$$u_x^{(i,s)} = u_{x0}^{(i,s)}\left(x,y\right) + zA_{55}^{(i)}\sigma_{xz0}^{(i,s)} + u_{x*}^{(i,s)}(x,y,z) \quad (x,y,z;5,4,3),\ i=1,2$$
$$\sigma_{xy}^{(i,s)} = \frac{1}{a_{66}^{(i)}}\left(\frac{\partial u_x^{(i,s-1)}}{\partial y} + \frac{\partial u_y^{(i,s-1)}}{\partial x}\right),\ \Delta^{(i)} = a_{11}^{(i)}a_{22}^{(i)} - a_{12}^{(i)2}$$
$$\sigma_{jz*}^{(i,s)} = -\int_0^z\left(\frac{\partial\sigma_{jx}^{(i,s-1)}}{\partial x} + \frac{\partial\sigma_{jy}^{(i,s-1)}}{\partial y} + P_j^{(i,s)}\right)dz, \quad j=x,y,z$$
$$\sigma_{xx*}^{(i,s)}(x,y,z) = A_{13}^{(i)}\sigma_{zz*}^{(i,s)} + B_{11}^{(i)}\frac{\partial u_x^{(i,s-1)}}{\partial x} + B_{12}^{(i)}\frac{\partial u_y^{(i,s-1)}}{\partial y} + \gamma_{11}^{(i)}\theta_i^{(s)}\ \left(x,y;1,2\right)$$
$$u_{x*}^{(i,s)} = \int_0^z\left(a_{55}^{(i)}\sigma_{xz*}^{(i,s)} - \frac{\partial u_z^{(i,s-1)}}{\partial x}\right)dz \quad (x,y;5,4) \tag{1.14}$$
$$u_{z*}^{(i,s)} = \int_0^z\left(A_{33}^{(i)}\sigma_{zz*}^{(i,s)} - A_{13}^{(i)}\frac{\partial u_x^{(i,s-1)}}{\partial x} - A_{23}^{(i)}\frac{\partial u_y^{(i,s-1)}}{\partial y} + \gamma_{33}^{(i)}\theta_i^{(s)}\right)dz$$
$$A_{33}^{(i)} = a_{13}^{(i)}A_{13}^{(i)} + a_{23}^{(i)}A_{23}^{(i)} + a_{33}^{(i)}, \qquad A_{kk}^{(i)} = a_{kk}^{(i)},\ k=4,5,6$$
$$A_{j3}^{(i)} = -a_{13}^{(i)}B_{j1}^{(i)} - a_{23}^{(i)}B_{j2}^{(i)}, \quad j=1,2;\ B_{11}^{(i)} = \frac{a_{22}^{(i)}}{\Delta^{(i)}},\ B_{12}^{(i)} = -\frac{a_{12}^{(i)}}{\Delta^{(i)}} \quad (1,2)$$
$$\gamma_{11}^{(i)} = \frac{a_{12}^{(i)}\beta_{22}^{(i)} - a_{22}^{(i)}\beta_{11}^{(i)}}{\Delta^{(i)}} \quad (1,2), \quad \gamma_{33}^{(i)} = A_{13}^{(i)}\beta_{11}^{(i)} + A_{23}^{(i)}\beta_{22}^{(i)} + \beta_{33}^{(i)}$$

Полученные общие интегралы (1.13),(1.14) системы уравнений (1.8),(1.9) содержат по восемь функций интегрирования для каждого слоя: $A^{(i,s)}, B^{(i,s)}$, $\sigma_{xz0}^{(i,s)}, \sigma_{yz0}^{(i,s)}, \sigma_{zz0}^{(i,s)}$, $u_{x0}^{(i,s)}$, $u_{y0}^{(i,s)}, u_{z0}^{(i,s)}$, $i = 1, 2$, которые однозначно определяются из неклассических тепловых и механических граничных условий (1.1),(1.2) и соответствующих условий контакта слоёв (1.3) – (1.7) под номерами $i = 1$ и $i = 2$.

2. Решения сформулированных краевых задач. Удовлетворив на лицевой поверхности $z = \varphi_1(x, y)$ неклассическим граничным условиям (1.1) задачи теплопроводности и условиям полного теплового контакта (1.3), получаем значения температурных функций первого и второго слоёв пакета:

$$\theta^{(1)} = \theta^{+} + \frac{\varphi_1 - z}{\lambda_{33}^{(1)}} q_{\vartheta z}^{+} - \frac{(\varphi_1 - z)^2}{2\lambda_{33}^{(1)}} W^{(1)}, \quad \theta^{(2)} = zA^{(2)} + B^{(2)} - \frac{z^2}{2\lambda_{33}^{(2)}} W^{(2)}$$

$$A^{(2)} = -\frac{1}{\lambda_{33}^{(2)}}\left(q_{\vartheta z}^{+} - \varphi_0 W^{(2)}\right) + \frac{\varphi_1 - \varphi_0}{2\lambda_{33}^{(2)}} W^{(1)} \tag{2.1}$$

$$B^{(2)} = B^{(1)} + \varphi_0 A^{(1)}\left(1 - \frac{\lambda_{33}^{(1)}}{\lambda_{33}^{(2)}}\right) + \varphi_0^2\left(\frac{1}{2\lambda_{33}^{(2)}} - \frac{1}{\lambda_{33}^{(1)}}\right) W^{(1)}$$

$$A^{(1)} = -\frac{1}{\lambda_{33}^{(1)}}\left(q_{\vartheta z}^{+} - \varphi_1 W^{(1)}\right), \qquad B^{(1)} = \theta^{+} + \frac{\varphi_1}{\lambda_{33}^{(1)}} q_{\vartheta z}^{+} - \frac{\varphi_1^2}{2\lambda_{33}^{(1)}} W^{(1)},$$

а из неклассических механических граничных условий (1.2) с учётом веса слоёв определяем все компоненты тензора напряжений и вектора перемещения первого слоя:

$$\sigma_{zz}^{(1,s)}(z) = \sigma_1^{(s)}(z) + \bar{\sigma}_{zz}^{(1,s-1)}$$

$$\sigma_{xz}^{(1,s)}(z) = A_{13}^{(1)} \sigma_1^{(s)}(z) \frac{\partial \varphi_1}{\partial x} + \gamma_{11}^{(1)} \theta^{(1,s)} \frac{\partial \varphi_1}{\partial x} + \bar{\sigma}_{xz}^{(1,s-1)}$$

$$\sigma_{xx}^{(1,s)}(z) = A_{13}^{(1)} \sigma_1^{(s)}(z) + \gamma_{11}^{(1)} \theta^{(1,s)} + \bar{\sigma}_{xx}^{(1,s-1)} \qquad (x, y;\ 13, 23;\ 11, 22)$$

$$\sigma_{xy}^{(i,s)}(z) = \frac{1}{A_{66}^{(i)}}\left(\frac{\partial u_y^{(i,s-1)}}{\partial x} + \frac{\partial u_x^{(i,s-1)}}{\partial y}\right), \quad i = 1, 2 \tag{2.2}$$

$$u_z^{(1,s)}(z)=A_{33}^{(1)}(\varphi_1-z)\left[\frac{z-\varphi_1}{2}g\rho^{(1)}+\frac{\theta^{(1,s)}}{\Delta_1^{(1)}}\left(\gamma_{11}^{(1)}\left(\frac{\partial\varphi_1}{\partial x}\right)^2+\gamma_{22}^{(1)}\left(\frac{\partial\varphi_1}{\partial y}\right)^2\right)\right]+$$

$$+(\varphi_1-z)\gamma_{33}^{(1)}\theta^{(1,s)}+u_z^{+(s)}+A_{33}^{(1)}\int_0^z\overline{\sigma}_{zz}^{(1,s-1)}dz$$

$$u_x^{(1,s)}(z)=u_x^{+(s)}+A_{55}^{(1)}(\varphi_1-z)\gamma_{11}^{(1)}\theta^{(1,s)}\frac{\partial\varphi_1}{\partial x}+A_{55}^{(1)}\int_0^z\overline{\sigma}_{xz}^{(1,s-1)}dz+$$

$$+A_{55}^{(1)}A_{13}^{(1)}(\varphi_1-z)\left[\frac{z-\varphi_1}{2}g\rho^{(1)}+\frac{\theta^{(1,s)}}{\Delta_1^{(1)}}\left(\gamma_{11}^{(1)}\left(\frac{\partial\varphi_1}{\partial x}\right)^2+\gamma_{22}^{(1)}\left(\frac{\partial\varphi_1}{\partial y}\right)^2\right)\right]\frac{\partial\varphi_1}{\partial x}$$

$$(x,y;\,5,4)\qquad u_x^{+(0)}=u_x^+,\ u_x^{+(s)}=0,\ s\neq 0\quad (x,y,z),$$

где обозначены

$$\sigma_1^{(s)}(z)=(z-\varphi_1)g\rho^{(1)}+\frac{\theta^{(1,s)}}{\Delta_1^{(1)}}\left(\gamma_{11}^{(1)}\left(\frac{\partial\varphi_1}{\partial x}\right)^2+\gamma_{22}^{(1)}\left(\frac{\partial\varphi_1}{\partial y}\right)^2\right)$$

$$\Lambda_k=\sqrt{1+\left(\frac{\partial\varphi_k}{\partial x}\right)^2+\left(\frac{\partial\varphi_k}{\partial y}\right)^2},\quad \Delta_k^{(i)}=1-A_{13}^{(i)}\left(\frac{\partial\varphi_k}{\partial x}\right)^2-A_{23}^{(i)}\left(\frac{\partial\varphi_k}{\partial y}\right)^2 \tag{2.3}$$

$$k=0,1,\quad i=1,2$$

$$\overline{\sigma}_{xz}^{(1,s-1)}=\overline{\sigma}_{xy}^{(1,s-1)}\frac{\partial\varphi_1}{\partial y}+\left(B_{11}^{(1)}\frac{\partial u_x^{(i,s-1)}}{\partial x}+B_{12}^{(1)}\frac{\partial u_y^{(1,s-1)}}{\partial y}\right)\left(\frac{\partial\varphi_1}{\partial x}\right)^2\quad (x,y;\,11,22)$$

$$\overline{\sigma}_{xx}^{(1,s-1)}=A_{13}^{(1)}\overline{\sigma}_{zz}^{(1,s-1)}+\left(B_{11}^{(1)}\frac{\partial u_x^{(1,s-1)}}{\partial x}+B_{12}^{(1)}\frac{\partial u_y^{(1,s-1)}}{\partial y}\right)\left(\frac{\partial\varphi_1}{\partial x}\right)^2\quad (x,y;13,23;11,22)$$

$$\overline{\sigma}_{zz}^{(1,s-1)}=\frac{2}{\Delta_1^{(1)}}\sigma_{xy}^{(1,s-1)}\frac{\partial\varphi_1}{\partial x}\frac{\partial\varphi_1}{\partial y}+\frac{1}{\Delta_1^{(1)}}\left(B_{11}^{(1)}\frac{\partial u_x^{(1,s-1)}}{\partial x}+B_{12}^{(1)}\frac{\partial u_y^{(1,s-1)}}{\partial y}\right)\left(\frac{\partial\varphi_1}{\partial x}\right)^2+$$

$$+\frac{1}{\Delta_1^{(1)}}\left(B_{12}^{(1)}\frac{\partial u_x^{(1,s-1)}}{\partial x}+B_{22}^{(1)}\frac{\partial u_y^{(1,s-1)}}{\partial y}\right)\left(\frac{\partial\varphi_1}{\partial y}\right)^2$$

Для определения функций интегрирования компонент тензора напряжений и вектора перемещения второго слоя на поверхности контакта слоёв $z=\varphi_0(x,y)$ целесообразно ввести функции $T_x^{(s)},T_y^{(s)},T_z^{(s)}$, которые представляют из себя составляющие по координатным осям $Oxyz$ компоненты тензора полного напряжения в точках поверхности контакта слоёв:

$$z=\varphi_0:\quad U_j=u_j^{(2)}-u_j^{(1)}=0,\;\; j=x,y,z$$

$$\Lambda_0 T_x^{(s)}=\sigma_{xz}^{(i,s)}\left(\varphi_0\right)-\sigma_{xx}^{(i,s)}(\varphi_0)\frac{\partial\varphi_0}{\partial x}-\sigma_{xy}^{(i,s)}(\varphi_0)\frac{\partial\varphi_0}{\partial y}\quad (x,y)$$

$$\Lambda_0 T_z^{(s)}=\sigma_{zz}^{(i,s)}\left(\varphi_0\right)-\sigma_{xz}^{(i,s)}(\varphi_0)\frac{\partial\varphi_0}{\partial x}-\sigma_{yz}^{(i,s)}(\varphi_0)\frac{\partial\varphi_0}{\partial y},\; i=1,2 \qquad (2.4)$$

Заметим, что уравнения (2.4) справедливы и для первого $(i=1)$ и для второго $(i=2)$ слоёв, что позволяет компоненты тензора полного напряжения $T_x^{(s)},T_y^{(s)},T_z^{(s)}$ выразить через уже известные компоненты тензора напряжений (2.2) первого слоя

$$\Lambda_0 T_x^{(s)}=\left(\frac{\partial\varphi_1}{\partial x}-\frac{\partial\varphi_0}{\partial x}\right)\left(A_{13}^{(1)}\sigma_1^{(s)}(\varphi_0)+\gamma_{11}^{(1)}\theta^{(1,s)}\right)+\bar\sigma_{xz}^{(1,s-1)}+\bar\sigma_{xx}^{(1,s-1)}\frac{\partial\varphi_0}{\partial x}$$

$$(x,y;13,23;11,22)$$

$$\Lambda_0 T_z^{(s)}=\sigma_1^{(s)}(\varphi_0)\left(1-A_{13}^{(1)}\frac{\partial\varphi_1}{\partial x}\frac{\partial\varphi_0}{\partial x}-A_{23}^{(1)}\frac{\partial\varphi_1}{\partial y}\frac{\partial\varphi_0}{\partial y}\right)- \qquad (2.5)$$

$$-\theta^{(1,s)}\left(1-\gamma_{11}^{(1)}\frac{\partial\varphi_1}{\partial x}\frac{\partial\varphi_0}{\partial x}-\gamma_{22}^{(1)}\frac{\partial\varphi_1}{\partial y}\frac{\partial\varphi_0}{\partial y}\right)+\bar\sigma_{zz}^{(1,s-1)}-\bar\sigma_{xz}^{(1,s-1)}\frac{\partial\varphi_0}{\partial x}-\bar\sigma_{yz}^{(1,s-1)}\frac{\partial\varphi_0}{\partial x},$$

после чего формулы компонент тензора напряжений и вектора перемещения второго слоя принимают вид:

$$\sigma_{zz}^{(2,s)}(z)=\sigma_{zz}^{(1,s)}(\varphi_0)+\sigma_2^{(s)}(z)+\bar\sigma_{zz}^{(2,s-1)},\;\Pi_0^{(s)}=\frac{\Lambda_0}{\Delta_0^{(2)}}\left(T_x^{(s)}\frac{\partial\varphi_0}{\partial x}+T_y^{(s)}\frac{\partial\varphi_0}{\partial y}+T_z^{(s)}\right)$$

$$\sigma_2^{(s)}(z)=\Pi_0^{(s)}-\left(\varphi_0-z\right)g\rho^{(2)}+\frac{\theta^{(2,s)}}{\Delta_0^{(2)}}\left(\gamma_{11}^{(2)}\left(\frac{\partial\varphi_0}{\partial x}\right)^2+\gamma_{22}^{(2)}\left(\frac{\partial\varphi_0}{\partial y}\right)^2\right) \qquad (2.6)$$

$$\sigma_{xz}^{(2,s)}(z)=\sigma_{xz}^{(1,s)}(\varphi_0)+A_{13}^{(1)}\sigma_2^{(s)}(z)\frac{\partial\varphi_0}{\partial x}+\Lambda_0 T_x^{(s)}+\gamma_{11}^{(2)}\theta^{(2,s)}\frac{\partial\varphi_0}{\partial x}+\bar\sigma_{xz}^{(2,s-1)}$$

$$\sigma_{xx}^{(2,s)}\left(z\right)=A_{13}^{(1)}\sigma_2^{(s)}(z)+\gamma_{11}^{(2)}\theta^{(2,s)}+\bar\sigma_{xx}^{(2,s-1)}\qquad (x,y;13,23;11,22)$$

$$u_z^{(2,s)}\left(z\right)=u_z^{(1,s)}\left(\varphi_0\right)+u_z^{+(s)}+A_{33}^{(2)}\left(\varphi_0-z\right)\left[\sigma_{zz}^{(1,s)}(\varphi_0)+\Pi_0-\frac{\varphi_0-z}{2}g\rho^{(2)}+\right.$$

$$\left.+\gamma_{33}^{(2)}\theta^{(2,s)}+\frac{\theta^{(2,s)}}{\Delta_0^{(2)}}\left(\gamma_{11}^{(2)}\left(\frac{\partial\varphi_0}{\partial x}\right)^2+\gamma_{22}^{(2)}\left(\frac{\partial\varphi_0}{\partial y}\right)^2\right)\right]+A_{33}^{(2)}\int_{\varphi_0}^{z}\bar\sigma_{zz}^{(2,s-1)}dz$$

$$u_x^{(2,s)}\left(z\right)=u_z^{(1,s)}\left(\varphi_0\right)+u_x^{+(s)}+A_{55}^{(2)}\left(\varphi_0-z\right)\left(\sigma_{xz}^{(1,s)}(\varphi_0)+\Lambda_0 T_x^{(s)}+\gamma_{11}^{(2)}\theta^{(2,s)}\frac{\partial\varphi_0}{\partial x}\right)+$$

$$+A_{55}^{(2)}A_{13}^{(2)}\left(\varphi_0-z\right)\left[\Pi_0-\frac{\varphi_0-z}{2}g\rho^{(2)}+\frac{\theta^{(2,s)}}{\Delta_0^{(2)}}\left(\gamma_{11}^{(2)}\left(\frac{\partial\varphi_0}{\partial x}\right)^2+\gamma_{22}^{(2)}\left(\frac{\partial\varphi_0}{\partial y}\right)^2\right)\right]\frac{\partial\varphi_0}{\partial x}$$

$$+A_{55}^{(2)}\int_{\varphi_0}^{z}\overline{\sigma}_{xz}^{(2,s-1)}dz\ (x,y;13,23;55,44),\quad u_x^{+(0)}=u_x^{+},\ u_x^{+(s)}=0,\ s\neq 0\quad (x,y,z).$$

Здесь обозначены:

$$\overline{\sigma}_{xz}^{(2,s-1)}=\overline{\sigma}_{xy}^{(2,s-1)}\frac{\partial\varphi_0}{\partial y}+\left(B_{11}^{(2)}\frac{\partial u_x^{(2,s-1)}}{\partial x}+B_{12}^{(2)}\frac{\partial u_y^{(2,s-1)}}{\partial y}\right)\left(\frac{\partial\varphi_0}{\partial x}\right)^2$$

$$\overline{\sigma}_{xx}^{(2,s-1)}=A_{13}^{(2)}\overline{\sigma}_{zz}^{(2,s-1)}+\left(B_{11}^{(2)}\frac{\partial u_x^{(2,s-1)}}{\partial x}+B_{12}^{(2)}\frac{\partial u_y^{(2,s-1)}}{\partial y}\right)\left(\frac{\partial\varphi_0}{\partial x}\right)^2 \tag{2.7}$$

$(x,y;13,23;11,22)$

$$\overline{\sigma}_{zz}^{(2,s-1)}=\frac{2}{\Delta_0^{(2)}}\sigma_{xy}^{(2,s-1)}\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_0}{\partial y}+\frac{1}{\Delta_0^{(2)}}\left(B_{11}^{(2)}\frac{\partial u_x^{(2,s-1)}}{\partial x}+B_{12}^{(2)}\frac{\partial u_y^{(2,s-1)}}{\partial y}\right)\left(\frac{\partial\varphi_2}{\partial x}\right)^2+$$

$$+\frac{1}{\Delta_0^{(2)}}\left(B_{12}^{(2)}\frac{\partial u_x^{(2,s-1)}}{\partial x}+B_{22}^{(2)}\frac{\partial u_y^{(2,s-1)}}{\partial y}\right)\left(\frac{\partial\varphi_2}{\partial y}\right)^2$$

Таким образом, асимптотическим методом выведены рекуррентные формулы (2.1)-(2-3), (2.5)-(2.7), позволяющие с любой заранее заданной асимптотической точностью $O(\varepsilon^S)$ вычислить температурную функцию и компоненты полей напряжений и перемещений двухслойного пакета из ортотропных пластин переменной толщины при полном тепловом и механическом контактах слоёв, если на лицевой поверхности пакета заданы неклассические краевые условия (1.1),(1.2).

Заметим, что условие полного контакта слоёв (1.4) предполагает отсутствие сдвиговой деформации слоёв $U_x^{(s)}=u_x^{(1,s)}(z=\varphi_0)-u_x^{(2,s)}(z=\varphi_0)=0\ (x,y)$. Если же механический контакт между слоями неполный, то происходит смещение (сдвиг) слоёв $U_x^{(s)}=u_x^{(1,s)}(z=\varphi_0)-u_x^{(2,s)}(z=\varphi_0)\neq 0\ (x,y)$. При этом, на поверхности $z=\varphi_0(x,y)$ раздела слоёв касательные составляющие $T_x^{(s)},T_y^{(s)}$ полного напряжения определяются одной из известных моделей неполного контакта (1.6) или (1.7). В таком случае по являющейся аналогом модели Винклера – Фусса (1.6) для каждого шага итерации S имеем:

$$T_x^{(s)}=\mu_x U_x^{(s)},\quad U_x^{(s)}=u_x^{(1,s)}(z=\varphi_0)-u_x^{(2,s)}(z=\varphi_0)\neq 0\quad (x,y), \tag{2.8}$$

а по закону сухого трения Кулона (1.7)

$$T_x^{(s)}=f_x T_z^{(s)}\qquad (x,y). \tag{2.9}$$

Следовательно, выведенные асимптотическим методом рекуррентные формулы (2.1)-(2.3), (2.5)-(2.7), с учётом значений $T_x^{(s)}, T_y^{(s)}$ (2.8) или (2.9)п, позволяют с любой асимптотической точностью $O(\varepsilon^S)$ вычислить температурную функцию и компоненты тензора напряжений и вектора перемещений двухслойного пакета из ортотропных пластин переменной толщины также согласно выбранной модели неполного механического контакта слоёв.

Таким образом, рекуррентные расчётные формулы (1.10),(1.11), (2.1)-(2-3), (2.5)-(2.7) позволяют вычислить температурную функцию, а также компоненты тензора напряжений и вектора перемещения в слоях пластины с любой асимптотической точностью $O(\varepsilon^S)$ в размерных координатах и перемещениях

$$Q^{(i)}\left(x,y,z\right)=\sum_{s=0}^{S}Q^{(i,s)}\left(x,y,z\right). \tag{2.10}$$

Они одновременно служат готовым алгоритмом компьютерной программы для аналитического (при необходимости) и численного решений поставленных краевых задач с заданной точностью. В качестве примера рассмотрим частный случай, когда двухслойная пластина состоит из слоёв постоянной толщины $\varphi_k\left(x,y\right)=h_k=\text{const},\ k=0,1,2$, постоянной плотности $\rho^{(i)}=\text{const},$ с источниками тепла постоянной интенсивности $W^{(i)}=\text{const},$ с постоянными граничными условиями

$$\theta^+=\text{const},\ \left(\theta^+,q_j^+,u_j^+,\sigma_{jz}^+\right),\ \ j=x,y,z\,. \tag{2.11}$$

Ограничившись исходным приближением, с учётом (2.1)–(2.4) по рекуррентным расчётным формулам (1.10)–(1.11), вычислив значения температурных функций, компонент векторов плотностей потоков теплоты, а также компонент тензоров напряжений и векторов перемещений, ля первого слоя получим:

$$\begin{gathered}
\theta^{(1)}=\theta^+ +\frac{\varphi_1-z}{\lambda_{33}^{(1)}}q_z^+ -\frac{\left(\varphi_1-z\right)^2}{2\lambda_{33}^{(1)}}W^{(1)},\qquad q_z=q_z^+ +(\varphi_1-z)W_1\\
u_x^{(1)}=u_x^+,\quad u_y^{(1)}=u_y^+,\quad \sigma_{xz}^{(1)}=\sigma_{yz}^{(1)}=\sigma_{xy}^{(1)}=0\\
\sigma_{zz}^{(1)}=\rho^{(1)}g\left(z-\varphi_1\right),\quad \sigma_{xx}^{(1)}=\rho^{(1)}gA_{13}^{(1)}\left(z-\varphi_1\right)+\gamma_{11}^{(1)}\theta^{(1)}\ \ (x,y;1,2)\\
u_z^{(1)}=u_z^+ +\frac{\left(z-\varphi_1\right)^2}{2}\left(\rho^{(1)}gA_{33}^{(1)}-\frac{\gamma_{33}^{(1)}}{\lambda_{33}^{(1)}}q_z^+\right)+\gamma_{33}^{(1)}\left(z-\varphi_1\right)\left(\theta^+ -\frac{\left(z-\varphi_1\right)^2}{6\lambda_{33}^{(1)}}W^{(1)}\right)
\end{gathered} \tag{2.12}$$

а для второго слоя имеем:

$$\begin{gathered}
\theta^{(2)}=\theta^+ +\left(\frac{\varphi_0-z}{\lambda_{33}^{(2)}}+\frac{\varphi_1-\varphi_0}{\lambda_{33}^{(1)}}\right)q^+ +\left[\frac{\left(\varphi_0-z\right)\left(\varphi_1-\varphi_0\right)}{\lambda_{33}^{(2)}}-\frac{\left(\varphi_1-\varphi_0\right)^2}{2\lambda_{33}^{(1)}}\right]W^{(1)}-\\
-\frac{\left(\varphi_0-z\right)^2}{2\lambda_{33}^{(2)}}W^{(2)},\qquad q^{(2)}=q_z^+ +\left(\varphi_1-\varphi_0\right)W^{(1)}+(\varphi_0-z)W^{(2)}
\end{gathered} \tag{2.13}$$

$$\sigma_{xz}^{(2)} = \sigma_{yz}^{(2)} = \sigma_{xy}^{(2)} = 0,\ \sigma_{zz}^{(2)} = \rho^{(1)} g\left(\varphi_0 - \varphi_1\right) + \rho^{(2)} g\left(z - \varphi_0\right)$$

$$\sigma_{xx}^{(2)} = \rho^{(1)} g A_{13}^{(2)} \left(\rho^{(1)} g\left(\varphi_0 - \varphi_1\right) + \rho^{(2)} g\left(z - \varphi_0\right)\right) + \gamma_{11}^{(2)} \theta^{(2)},\ (x, y; 1, 2)$$

$$u_y^{(2)} = u_y^+,\ \ u_x^{(2)} = u_x^+$$

$$u_z^{(2)} = \overline{u}_z + \frac{\left(z - \varphi_0\right)^2}{2}\left(\rho^{(2)} g A_{33}^{(2)} - \frac{\gamma_{33}^{(2)}}{\lambda_{33}^{(2)}}\overline{q}\right) + \gamma_{33}^{(2)}\left(z - \varphi_0\right)\left(\overline{\theta} - \frac{\left(z - \varphi_0\right)^2}{6\lambda_{33}^{(2)}} W^{(2)}\right)$$

$$\overline{u}_z = u_z^{(1)}(z = \varphi_0),\ \ \overline{\theta} = \theta^{(1)}(z = \varphi_0),\ \overline{q} = q^{(1)}(z = \varphi_0)$$

Заметим, что решение (2.5),(2.6) примера с неклассическими граничными условиями (2.5) – математически точное (замкнутое) для слоёв постоянной толщины, поскольку следующие шаги итерации дают нули. Учитывая это, на поверхности $z = \varphi_2(x, y) = h_2 = \mathrm{const}$, по формулам (2.7) вычислим

$$Q^{-cal} = Q(z = h_2),\ Q = \left\{\theta^{(2)}, q^{(2)} \sigma_{xz}^{(2)}, \sigma_{yz}^{(2)}, \sigma_{zz}^{(2)}, u_x^{(2)}, u_y^{(2)}, u_z^{(2)}\right\}. \qquad (2.14)$$

Непосредственной подстановкой можно убедиться [2,3,5,6], что решение (2.6),(2.7) краевой задачи с неклассическими граничными условиями (2.5) одновременно является решением задачи с классическими граничными условиями:

$$\begin{aligned} &z = \varphi_1 = h_1:\quad \theta^+ = \mathrm{const},\quad u_j^+(x, y) = \mathrm{const},\ \ j = x, y, z \\ &z = \varphi_2 = h_2:\ q_2^- = q_2^{-cal}, \sigma_{xz}^{-(2)} = \sigma_{xz}^{-cal} = 0\ \ (x, y),\ \sigma_{xz}^{-(2)} = \sigma_{zz}^{-cal} \end{aligned} \qquad (2.15)$$

Таким образом, асимптотическое решение каждой задачи стационарной теплопроводности и несвязанной теории термоупругости с асимптотической точностью $O(\varepsilon^S)$ совпадает с решением определённой краевой задачи с классическими граничными условиями [3].

Исследование выполнено при финансовой поддержке ГКНМОНРА «15T-2C343»

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Поступила в редакцию 14.11.2016

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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 70, №2, 2017 Механика

PROPAGATION OF ELASTIC WAVES IN A PLANE WAVEGUIDE LAYER ON THE BASIS OF A SIMPLIFIED MODEL OF THE COSSERAT CONTINUUM

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Keywords: micropolar material, Cosserat continuum, acoustic waves, elastic waveguide layer, wave energy localization, localized waves.
Բանալի բառեր. Միկրոպոլյար նյութ, Կոսերայի միջավայր, ակուստիկ ալիքներ, առաձգական ալիքատար շերտ, ալիքային էներգիայի տեղայնացում, տեղայնացված ալիքներ:
Ключевые слова: микрополярная среда, континуум Коссера, акустические волны, упругий слой-волновод, локализация волновой энергии, локализованные волны.

Համբարձումյան Ս.Ա.,Ավետիսյան Ա.Ս., Բելուբեկյան Մ.Վ.,
Առաձգական ալիքների տարածումը հարթ շերտ-ալիքատարում, Կոսերայի միջավայրի պարզեցված մոդելի հաշվառումով

Դիտարկվում է բարձր-հաճախականային և ցածր-հաճախականային ակուստիկ ալիքների տարածումը առաձգական հարթ շերտ-ալիքատարում, Կոսերայի միջավայրի պարզեցված մոդելի հաշվառմամբ: Հարթ և հակահարթ դեֆորմացիաների խնդիրներում, տարբեր եզրային պայմանների դեպքում, ձևակերպված են եզրային արժեքի խնդիրները հաշվի առնելով նյութի միկրոպոլյար հատկությունը: Երկար ալիքային և կարճ ալիքային մոտարկումների դեպքում ստացված արդյունքները համադրված են առաձգականության դասական տեսության արդյունքների հետ: Բացահայտված են ալիքատարի մակերևույթների մոտ ալիքային էներգիայի հնարավոր տեղայնացման պայմանները: Ցույց է տրված, որ հակահարթ դեֆորմացիայի խնդրում նյութի միկրոպոլյար հատկությունը սահքի ալիքների մակերևութային տեղայնացման չի բերում: Գտնվել են միկրոպոլյարության հաշվառման դեպքում հարմոնիկ ալիքների տարածման սեղմված հաճախականայի գոտիները: Ցույց է տրված, որ հարթ դեֆորմացիայի ալիքային ազդանշանի դեպքում միկրոպոլյար հատկությունը կարող է բերել նոր տեղայնացված ալիքի տարածման:

Амбарцумян С.А., Аветисян А.С, Белубекян М.В.
Распространение упругих волн в плоском слое-волноводе с учётом упрощённой модели континуума Коссера

Рассматривается распространение высокочастотных и низкочастотных акустических волн в плоском упругом слое-волноводе на основе упрощённой модели среды Коссера. Учитывая наличие микро-полярности среды, для различных комбинаций граничных условий на поверхности волновода сформулированы граничные задачи плоской и антиплоской деформаций. В длинноволновом и коротковолновом приближениях полученные результаты сравниваются с результатами классической теории упругости. Найдены условия для возможной локализации волновой энергии вблизи поверхности волновода. Показано, что учёт микрополярности материала в задаче антиплоской деформации не приводит к существованию высокочастотных локализованных форм. В задаче плоской деформации учёт микро-полярности материала, при различных граничных условиях на поверхности волновода может вызвать как искажение частотного диапазона существования локализованных волн Рэлея, так и привести к появлению нового частотного диапазона возможных локализованных волн плоской деформации. Найдены частотные полосы локализованных и гармонических форм колебаний.

The problem of propagation of high-frequency and low-frequency acoustic waves in a plane elastic waveguide layer on the basis of a simplified model of the Cosserat continuum is considered. In view, the presence of micro polarity of the medium, boundary value problems for plane and antiplane deformations for different combinations of boundary conditions on the waveguide surface are formulated. In a long-wave and a short-wave approximations the obtained results are compared with the results of the classical theory of elasticity. The conditions for a possible localization of the wave energy near the surface of the waveguide are found. It is shown that in the antiplane deformation problem, the considering of micropolarity of the material is not leads to the possibility of existence of high localized forms. The frequency bands of localized and harmonic waveforms are found. In the plane deformation problem, considering of micropolarity of the material under different surface conditions may cause as a distortion of the frequency band of the localized Rayleigh wave existence so as the emergence of a new frequency band of possible localized waves.

Introduction. In the classical theory of elasticity it is known that the reason of localization of high-frequency waves near the medium interface boundary is the disturbance of homogeneity of effective physical and mechanical characteristics of these fields. We first encounter the problem of wave energy localization in the primary sources [1÷4]. Within the framework of the classical theory of elasticity, more about localization of wave energy near the medium interface boundary can be found in [5÷7], and others. However, when changes in the microstructure of the body are essential (that is near the cracks and chipping, where the stress gradients are essential) there appears a discrepancy between the results of the classical theory of elasticity and the experiments' results. Such discrepancies also appear in the case of granular medium and multi molecular structures, such as polymers.

The influence of microstructure is particularly evident in the case of elastic oscillations of high frequency and short wavelength. W. Voigt [8] attempted to overcome the disadvantages of the classical theory of elasticity under the assumption that the interaction of two parts of the body through the area element is transmitted not only by the force vector, but also by the vector momentum. However, the complete theory of asymmetric elasticity was developed only in 1909 by Francois and Eugene Cosserat brothers [9]. Currently Cosserat theory is in rapid development.

There is an extensive literature on the study of mechanics problems based on the micropolar theory of elasticity (or based on the Cosserat continuum). General works of A. C. Eringen and others [10,11] and Vladimir Yerofeyev's work [12] should be noted.

In this article the problems of waves propagation in a flat elastic waveguide with due regard to the internal rotation of the medium particles are considered. The limiting cases of short and long waves (high and low frequency acoustic waves) on the basis of a simplified model of the Cosserat continuum are investigated.

1. Basic relations of a simplified model of the Cosserat continuum.

In general, the motion equations in the asymmetric elasticity theory are written as:

$$\begin{aligned} &\sigma_{ji,j} + X_i = \rho \ddot{u}_i \\ &\mu_{ji,j} + \in_{ijk} \sigma_{jk} + Y_i = J \ddot{\varphi}_i \end{aligned} \qquad (1.1)$$

where σ_{ij} and μ_{ik} are force and moment stresses, respectively, X_i and Y_i are mass forces, $\in_{ijk}$ is the Levi-Civita tensor, ρ is the material density, u_i are the displacement vector components, φ_i are the rotation vector components at medium unit point, J is the rotary inertia.

The material relations of isotropic material for σ_{ji} force and μ_{ji} moment stresses are:

$$\begin{aligned} &\sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ji} + \lambda\delta_{ji}\gamma_{kk} \\ &\mu_{ji} = (\gamma+\varepsilon)\omega_{ji} + (\gamma-\varepsilon)\omega_{ij} + \beta\delta_{ji}\omega_{kk} \end{aligned} \qquad (1.2)$$

These relations (1.2) involve material constants $\lambda;\ \mu;\ \alpha;\ \beta$, which are independent, and Kronecker delta δ_{ik}. And besides, these material constants and their combinations are positive definite ones

$$\mu > 0\ ;\quad \gamma > 0\ ;\quad \alpha > 0\ ;\quad \varepsilon > 0\ ;\quad 3\lambda + 2\mu > 0\ ;\quad 3\beta + 2\gamma > 0\ .$$

Now, if we exclude σ_{ij} and μ_{ik} stresses from the motion equations (1.1), using the constitutive equations and defining relations for tensors

$$\gamma_{ji} = u_{i,j} - \in_{kji} \varphi_k\ ,\qquad \omega_{ji} = \varphi_{i,j}$$

we will obtain a system in vector form of six equations in terms of displacements $\vec{u} = \{u_i\}$ and rotations $\vec{\varphi} = \{\varphi_i\}$:

$$\begin{aligned} &\square_2 \vec{u} + (\lambda + \mu - \alpha)\,\mathrm{graddiv}\vec{u} + 2\alpha\,\mathrm{rot}\,\vec{\varphi} + \vec{X} = 0 \\ &\square_4 \vec{\varphi} + (\beta + \gamma - \varepsilon)\,\mathrm{graddiv}\,\vec{\varphi} + 2\alpha\,\mathrm{rot}\vec{u} + \vec{Y} = 0 \end{aligned} \tag{1.3}$$

Where vector and operators $\square_2$ and $\square_4$ are given as

$$\square_2 = (\mu + \alpha)\Delta - \rho\partial_t^2\ ,\qquad \square_4 = (\gamma + \varepsilon)\Delta - 4\alpha J\partial_t^2\ .$$

Many authors have investigated the problem of distribution and localization of elastic waves by means of a system of general equations of the asymmetric elasticity theory. Ambartsumyan S.A. and Belubekyan M.V. [13] have also investigated the generalized Rayleigh waves in a micropolar continuous medium. V.R. Parfitt and A.C. Eringen [14], as well as J. Stefaniak [15] have investigated the reflection of a plane wave from a free boundary of the half space.

The same problem was discussed in the expanded paper of S. Kaliski, J. Kupelewski and C. Rymarz [16]. Propagation of waves in a plate and generalized Lamb waves have been considered in W.Nowacki and W.K. Nowacki articles [17-18].

In general, the equations and relations in the micropolar theory are quite complex, so far simple models [19 ÷ 21] are used often for solving some specific problems. On the other hand, the most significant effects, associated with moment stresses under consideration, occur in dynamic problems. For such problems, in particularly, where elastic wave propagation is studied, a simple model considering only the dynamics of the internal rotation of the particles, was proposed on the basis of the Cosserat model. Simplified Cosserat model for dynamic problems, apparently independently of one another, has been proposed in works [22 ÷ 24].

In the Cartesian coordinate system $\{x_i\}$ for a simplified Cosserat model the known linear motion equations of the classical theory of elasticity are applied

$$\frac{\partial\sigma_{ji}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2};\qquad i;\, j \in \{1;2;3\} \tag{1.4}$$

However, in (1.4) the shear stresses are not symmetrical, and are defined as a generalization of the classical Hooke's law for isotropic material

$$\sigma_{ij} = 2\mu\gamma_{ij} + \delta_{ij}\nu\gamma_{kk} + J\left(\partial^2\omega_{ij}/\partial t^2\right) \tag{1.5}$$

where strain tensor γ_{ij} is defined by the usual way

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{1.6}$$

and the transposed tensor of additional rotations ω_{ij} defines the asymmetry of shear stresses $\left(\omega_{ij} = -\omega_{ji}\right)$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right). \tag{1.7}$$

Consideration of two-dimensional dynamic problems, when all the physico-mechanical characteristics of the elastic field don't depend on the coordinate x_3, i.e. $\left(\partial/\partial x_3 \equiv 0\right)$, is much easier. As in the general micropolar elasticity theory, in the simplified theory of the Cosserat continuum the constitutive equations (1.5) and the equations of motion (1.4) allow the separation of the problems to plane and antiplane deformations.

This model was used to solve a number of problems on propagation of acoustic waves, the reviews are given in [20, 25].

In the next, for convenience instead of $\{x_i\}$ coordinates we will use $\{x; y; z\}$ coordinates, and instead of the displacement vector components $\{u_i\}$ we will use $\{u; \mathrm{v}; w\}$ notation.

Then, for plane strain problems from (1.4) in view of (1.5) ÷ (1.7) the following equations of motion are obtained:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}; \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho \frac{\partial^2 \mathrm{v}}{\partial t^2}. \tag{1.8}$$

The corresponding constitutive equations are

$$\begin{aligned} &\sigma_{xx} = (\lambda + 2\mu)\gamma_{xx} + \lambda\gamma_{yy}; \qquad \sigma_{yy} = (\lambda + 2\mu)\gamma_{yy} + \lambda\gamma_{xx}; \\ &\sigma_{xy} = 2\mu\gamma_{xy} + J\frac{\partial^2 \omega_{xy}}{\partial t^2}; \qquad \sigma_{yx} = 2\mu\gamma_{xy} + J\frac{\partial^2 \omega_{yx}}{\partial t^2}. \end{aligned} \tag{1.9}$$

The defining relations are

$$\begin{aligned} &\gamma_{xx} = \frac{\partial u}{\partial x}; \quad \gamma_{xx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right); \quad \gamma_{yy} = \frac{\partial \mathrm{v}}{\partial y}; \\ &\omega_{xy} = \left(\frac{\partial u}{\partial y} - \frac{\partial \mathrm{v}}{\partial x}\right) = -\omega_{yx}. \end{aligned} \tag{1.10}$$

For antiplane strain problems we obtain the equations of motion:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2} \tag{1.11}$$

the constitutive equations:

$$\sigma_{xz} = 2\mu\gamma_{xz} + J\frac{\partial^2\omega_{xz}}{\partial t^2}; \quad \sigma_{zx} = 2\mu\gamma_{zx} + J\frac{\partial^2\omega_{zx}}{\partial t^2}$$

$$\sigma_{yz} = 2\mu\gamma_{yz} + J\frac{\partial^2\omega_{yz}}{\partial t^2}; \quad \sigma_{zy} = 2\mu\gamma_{zy} + J\frac{\partial^2\omega_{zy}}{\partial t^2} \tag{112}$$

the defining relations:

$$\gamma_{xz} = \frac{1}{2}\frac{\partial w}{\partial x}; \quad \gamma_{yz} = \frac{1}{2}\frac{\partial w}{\partial y}; \quad \omega_{xz} = \frac{\partial w}{\partial x} = -\omega_{zx}; \quad \omega_{yz} = \frac{\partial w}{\partial y} = -\omega_{zy} \tag{1.13}$$

2. The antiplane deformation problem. Let us consider an elastic, homogeneous isotropic waveguide, which occupies the region $\Omega \triangleq \{-\infty < x < \infty; \ \ 0 \le y \le h; \ \ -\infty < z < \infty\}$ Equations of purely shear waves (1.11), with accounting the material equations (1.12) and defining relations (1.13) are reduced to the form:

$$\mu\Delta w + J\frac{\partial^2}{\partial t^2}\Delta w = \rho\frac{\partial^2 w}{\partial t^2}\,. \tag{2.1}$$

Presenting the solution of equation (2.1) in the form of normal harmonic waves

$$w(x,y,t) = w_0(y)\exp[i(\omega t - kx)] \tag{2.2}$$

where ω is the oscillation frequency, $k \triangleq 2\pi/\lambda$ is the wave number, λ is the length of the wave, $w_0(y)$ is amplitude function, which determines the distribution across the waveguide's thickness, we obtain the following ordinary differential equation:

$$w_0''(y) + k^2q^2w_0(y) = 0\,, \tag{2.3}$$

where

$$q^2 \triangleq \eta^2\big/\left(1-\beta_k\eta^2\right) - 1\ ;\ \eta^2 \triangleq \omega^2/(k^2c_t^2)\ ;\ c_t^2 \triangleq \mu/\rho\ ;\ \beta_k \triangleq (Jk^2)/\rho \tag{2.4}$$

From these notations one can see that for all wave numbers k values $\eta > 0$ and $\beta_k > 0$ are positive.

The condition of the existence of harmonic oscillations $q^2 > 0$ is easily obtained from the first notation (2.4) in the form of

$$1/(1+\beta_k) < \eta^2 < 1/\beta_k \quad \text{or} \quad \sqrt{\frac{\mu k^2}{Jk^2+\rho}} < \omega < \sqrt{\mu/J} \tag{2.5}$$

If the condition (2.5) is valid, the general solution of equation (2.3) can be represented by trigonometric functions as

$$w_0(y) = A\sin(kqy) + B\cos(kqy) \tag{2.6}$$

It should be noted, that when the micro rotation does not take into account (if $J \equiv 0$ then $\beta_k \to 0$), the condition (2.5) takes the known form $\eta > 1$. From (2.5) it is also obvious that

for normal short waves, when $\lambda \ll 2\pi\sqrt{J/\rho}$, the frequency range is very narrow $\sqrt{\mu/J} - o(\lambda^2\rho/4\pi^2 J) < \omega < \sqrt{\mu/J}$.
Consequently, the condition for the existence of harmonic waves is transformed into

$$\omega_n \in \left(\sqrt{\mu/J} - o(\lambda^2\rho/4\pi^2 J);\ \sqrt{\mu/J}\right) \tag{2.7}$$

Here ω_n is the oscillation frequency of corresponding harmonic. The resulting waveforms and the corresponding frequencies are determined by the boundary conditions on the waveguide walls.
In particularly, the problems for a waveguide with boundary conditions of clamped or traction free wall, according to (2.6) lead to the definition of the phase velocity satisfying (2.5). Herewith consideration of the internal rotation reduces the given phase velocity $\sqrt{\eta}$. If waveguide walls $y = 0$ and $y = h$ are clamped: $w_0(0) = 0$ and $w_0(h) = 0$, then from the dispersion equation the values of natural frequencies are obtained:

$$q_n \triangleq \frac{n\pi}{kh};\ \text{where } n = 0;\ 1;\ 2;\ \ldots \tag{2.8}$$

$$\eta^2 = \left(1 - q_n^2\right) \Big/ \left[1 + \beta_k\left(1 - q_n^2\right)\right];$$

The condition of existence (2.5) of the n[th] oscillations harmonic is transformed into

$$\sqrt{\frac{\mu k^2}{Jk^2 + \rho}} < kc_t \cdot \sqrt{\frac{1 + n^2\pi^2/k^2h^2}{1 + (Jk^2/\rho)\cdot\left(1 + n^2\pi^2/k^2h^2\right)}} < \sqrt{\mu/J} \tag{2.9}$$

and the value $n = 0$ corresponds to the limiting wave, for which the frequency is defined as $\omega_{01} = \sqrt{k^2\mu/\left(\rho + Jk^2\right)}$. For higher harmonics when $n \to \infty$, the limiting frequency is $\omega_{02} = \sqrt{\mu/J}$ (Fig.1a).
From (2.9) it also follows that in this frequency range there always exist harmonics with numbers $n \geq \left[2h/\lambda\right]$.

From (2.9), taking into account (2.4), it follows that the phase velocity of the n^{th} harmonic is represented as

$$\mathrm{v}_f(k) \triangleq \frac{\omega}{k} = \sqrt{\mu\left(1 + \left(n\pi/kh\right)^2\right) \Big/ \left[\rho + Jk^2\left(1 + \left(n\pi/kh\right)^2\right)\right]}$$

The behavior of harmonics phase velocity is shown on Fig. 1b.
In frequency intervals

$$0 < \eta < 1/\left(1 + \beta_k\right) \quad \text{or} \quad \eta > 1/\beta_k \tag{2.10}$$

according to (2.4) we get $q^2 < 0$. Then, non-harmonic solutions of the wave formation equation (2.3), with the notation $p \triangleq iq = \sqrt{1 - \rho\left(\omega^2/k^2\right)/\left(\mu - J\omega^2\right)}$, are represented by hyperbolic functions

$$w_0(y) = A \cdot \mathrm{sh}(kpy) + B \cdot \mathrm{ch}(kpy) \tag{2.11}$$

For the frequency and phase velocity of the normal wave, we obtain the ranges

$$0 < \omega < \sqrt{\frac{\mu k^2}{Jk^2 + \rho}} \quad \text{or} \quad \omega > \sqrt{\mu/J} \tag{2.12}$$

$$0 < \mathrm{v}_f(k) < \sqrt{\frac{\mu}{Jk^2 + \rho}} \quad \text{or} \quad \mathrm{v}_f(k) > k^{-1} \cdot \sqrt{\mu/J}$$

Here $\sqrt{\mu k^2 / \left(Jk^2 + \rho\right)}$ and $\sqrt{\mu/J}$ are the limiting frequencies for harmonic formations of normal waves across the waveguide thickness.

For a waveguide with camped walls or traction free walls according to (2.12) the problem under consideration leads to determination of an oscillation frequency (or phase velocity) which does not satisfy condition (2.11).

3. The plane deformation problem. In the elastic, isotropic homogeneous waveguide $\Omega \triangleq \{-\infty < x < \infty;\ \ 0 \le y \le h;\ \ \infty < z < \infty\}$ the plane strain equations (1.8) in view of the material equations (1.9) and the defining relations (1.10) are reduced to the system of equations for the components $u(x, y, t)$ and $\mathrm{v}(x, y, t)$ of the displacement vector

$$\begin{aligned} \mu\Delta u + (\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y}\right) - J\frac{\partial^3}{\partial y \partial t^2}\left(\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y}\right) = \rho\frac{\partial^2 u}{\partial t^2} \\ \mu\Delta \mathrm{v} + (\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y}\right) - J\frac{\partial^3}{\partial x \partial t^2}\left(\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y}\right) = \rho\frac{\partial^2 \mathrm{v}}{\partial t^2} \end{aligned} \tag{3.1}$$

By means of Lame's transformation for plane strain problems

$$u \triangleq \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\ ; \qquad \mathrm{v} \triangleq \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x} \tag{3.2}$$

the system of equations (3.1) gives a separate equations of longitudinal and transverse waves [28], for Lame's functions $\varphi(x, y, t)$ and $\psi(x, y, t)$

$$(\lambda + \mu)\Delta\varphi = \rho\frac{\partial^2 \varphi}{\partial t^2} \tag{3.3}$$

$$\mu\Delta\psi + J\frac{\partial^2}{\partial t^2}(\Delta\psi) = \rho\frac{\partial^2 \psi}{\partial t^2} \tag{3.4}$$

Equation (3.4) coincides with equation (2.1) of the antiplane problem. Representing the solutions of equations (3.3) and (3.4) in a form of normal wave

$$\varphi(x, y, t) = \varphi_0(y)\exp\left[i(\omega t - kx)\right];\ \psi(x, y, t) = \psi_0(y)\exp\left[i(\omega t - kx)\right] \tag{3.5}$$

we obtain ordinary differential equations for amplitude functions $\varphi_0(y)$ and $\psi_0(y)$, which general solutions are

$$\varphi_0(y) = A\sin(kpy) + C\cos(kpy)\ ; \quad \psi_0(y) = D\sin(kqy) + B\cos(kqy) \tag{3.6}$$

Where q has the same notation as in (2.4) and

$$p \triangleq \sqrt{\theta\eta - 1}\ ; \quad \theta \triangleq \mu/(\lambda + 2\mu) \tag{3.7}$$

Let us consider a wave process in a waveguide with different boundary conditions.

3.1. Navier Conditions on both walls of the waveguide. Suppose that Navier conditions are defined on the wall $y = 0$ of the waveguide

$$\varphi = 0; \quad \partial\psi/\partial y = 0 \tag{3.8}$$

Satisfying solution (3.6) to conditions (3.8), we obtain $C = D = 0$. Thus the solution forms for the required functions $\varphi_0(y)$ and $\psi_0(y)$ are simplified, by what problems with different conditions on the other surface of the waveguide can be investigated.
The simplest version of the boundary value problem is when Navier conditions (3.8) are also given at the wall $y = h$. In that case, as in the general classical theory of elasticity, the effect of micropolarity is included in the components of the elastic displacement, while the wave equations for Lame's function (3.3) and (3.4) are completely separated from each other.

$$u(x, y, t) = -k\left[A\mathrm{sh}(kp_1 y) + kqB\cos(kqy)\right]\exp\left[i(\omega t - kx)\right] \tag{3.9}$$

$$\mathrm{v}(x, y, t) = k\left[pA\sin(kpy) + B\mathrm{ch}(kq_1 y)\right]\exp\left[i(\omega t - kx)\right] \tag{3.10}$$

Here $p_1 = ip = \sqrt{1 - \theta\eta}$, and $q_1 = iq = \sqrt{1 - \eta^2/\left(1 - \beta_k \eta^2\right)}$ are the coefficients of formation in the plane strain problem.
Moreover, wave of (3.9) and (3.10) types will exist at the phase velocity $\mathrm{v}_{1f}(\lambda) \le (\lambda/2\pi)\sqrt{\mu/\left(J + \rho\left(\lambda^2/4\pi^2\right)\right)}$ for all permitted frequencies $\omega < \sqrt{\mu/J}$. For higher frequencies $\omega > \sqrt{\mu/J}$ the waves will exist with the phase velocity $(\lambda/2\pi)\sqrt{\mu/J} < \mathrm{v}_{2f}(\lambda) \le c_l$, which length is determined by the physical characteristics of the micropolar material. Phase zones of such waves' existence are shown in Fig. 2.
The second option of setting a boundary value problem assumes that with the boundary conditions (3.8) on $y = 0$, the clamped boundary conditions on the other wall of the waveguide must be satisfied

$$u(x, h, t) \equiv 0\ ; \quad \mathrm{v}(x, h, t) \equiv 0 \tag{3.11}$$

Conditions (3.11) with the use of required functions $\varphi_0(y), \psi_0(y)$ and in the view of (3.5), are represented as:

$$-ik\varphi_0(h) + \psi_0'(h) = 0\ ; \quad \varphi_0'(h) + ik\psi_0(h) = 0 \tag{3.12}$$

Satisfying solution (3.6) when $C = D = 0$ to satisfy conditions (3.12), we obtain a system of algebraic equations for arbitrary constants A and B. Condition for the existence of nontrivial solutions gives the dispersion equation

$$\mathrm{tg}(kph) = -pq \cdot \mathrm{tg}(kqh) \tag{3.13}$$

Equation (3.13) always has a solution satisfying to (2.5).

The question arise does the equation has any solution satisfying the condition $0 < \eta < 1/(1+\beta_k)$ of (2.10) Such decision would mean the existence of localized waves near the walls of the waveguide layer. To answer this question it's sufficient to consider equation (3.13) in a short-wave approximation ($kh \gg 1$). By substituting $p = ip_1$ and $q = iq_1$, equation (3.13) reduces to the equation with hyperbolic functions, where in a short-wave approximation we obtain

$$\text{th}(khp_1) \approx khp_1 \quad \text{and} \quad \text{th}(khq_1) \approx khq_1 \tag{3.14}$$

The condition of the existence of waves is obtained as

$$\eta = \frac{1+\theta}{(1+\beta_k)\theta} > \frac{1}{1+\beta_k}; \tag{3.15}$$

From (3.15) it follows that (3.13) cannot have roots, which satisfy to the first condition of (2.10).

On the other hand, when

$$\beta_k > \theta\ ; \quad \text{or} \quad k^2 > \rho\theta J^{-1} \tag{3.16}$$

the characteristic equation (3.13) has roots, which satisfy the second condition of (2.10)

$$\theta^{-1} > \eta > \beta_k^{-1} \quad \text{or} \quad (\lambda+2\mu)/\rho > \omega/k > \mu/(Jk^2) \tag{3.17}$$

The frequency of these waves will be limited for each length λ_0 in a way

$$2\pi(\lambda+2\mu)/(\lambda_0\rho) > \omega(\lambda) > \lambda_0\mu/(2\pi J) \tag{3.18}$$

3.2. Mixed boundary conditions on the surfaces of the waveguide. Suppose that in addition to Navier conditions (3.8) on the wall $y = 0$, on the other wall $y = h$ of the waveguide the conditions of mechanically free boundary are defined

$$\sigma_{yy}(x,h,t) = 0 \quad \text{and} \quad \sigma_{yx}(x,h,t) = 0 \tag{3.19}$$

Conditions (3.19) by means of functions $\varphi(x,y,t)$ and $\psi(x,y,t)$, in the view of (3.5) reduce to

$$\begin{aligned} &(\lambda+2\mu)\varphi_0'' - k^2\lambda\varphi_0 + 2ik\mu\psi_0' = 0 \\ &-2ik\varphi_0' + (1-\beta_k\eta)\psi_0'' + k^2(1+\alpha\eta)\psi_0 = 0 \end{aligned} \tag{3.20}$$

Assuming that, in addition to conditions (3.19) on the other wall of the waveguide $y = 0$ Navier conditions (3.8) are defined and using solutions (3.6) when $C = D = 0$, from (3.20) we obtain the equations for the arbitrary constants A and B. The dispersion equation

$$(2-\eta)[1+\beta_k\eta-(1-\beta_k\eta)q^2]\cdot\text{tg}(khp) + 4pq\cdot\text{tg}(khq) = 0 \tag{3.21}$$

is obtained from the condition of existence of nontrivial solutions.

To investigate the waves, localized at the free surface, which satisfy (2.5), it is more convenient to rewrite equation (3.21) as follows:

$$(2-\eta)[1+\beta_k\eta+(1-\beta_k\eta)q_1^2]\cdot\text{th}(khp_1) - 4p_1q_1\cdot\text{th}(khq_1) = 0 \tag{3.22}$$

where $p = ip_1$ and $q = iq_1$.

From equation (3.22) in a short-wave approximation, when $kh \gg 1$, we obtain the equation [26].

$$(2-\eta)[1+\beta_k\eta+(1-\beta_k\eta)q_1^2]-4p_1q_1=0 \tag{3.23}$$

If we ignore the internal rotation of the particles ($\beta_k \equiv 0$) from (3.23), we obtain the dispersion relation of Rayleigh waves [17].

In a long-wave approximation, when $k^2h^2 \ll 1$, assuming $\text{th}z \approx z - z^3/3$, a well-known dispersion equation of one-dimensional bending oscillations of a plate can be obtained [27]

$$(1+4\beta_k)\eta = \frac{4(1-\theta)k^2h^2}{3} \tag{3.24}$$

From dispersion equation (3.24), the equation of oscillations of a plate can be restored as

$$D\frac{\partial^4 w}{\partial x^4} - 8hJ\frac{\partial^4 w}{\partial x^2 \partial t^2} + 2\rho h\frac{\partial^2 w}{\partial t^2} = 0, \tag{3.25}$$

where $D \triangleq \left(2Eh^3\right)/3(1-\nu^2)$.

The same analytical result was obtained in [27] on the basis of Kirchhoff's hypothesis.

4. Numerical analysis of the wave process behavior.

In the case of wave propagation for antiplane deformation in an elastic micropolar waveguide, the band of permitted frequencies is constrained by micropolarity of the material.

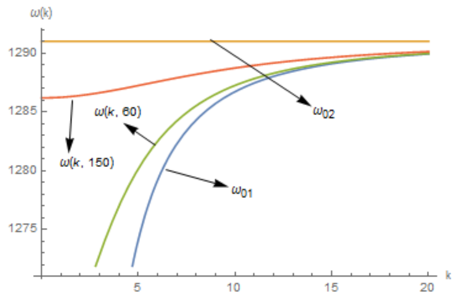

Fig. 1a. The region of frequencies of harmonic forms of shear wave's oscillations and the behavior of the natural frequencies

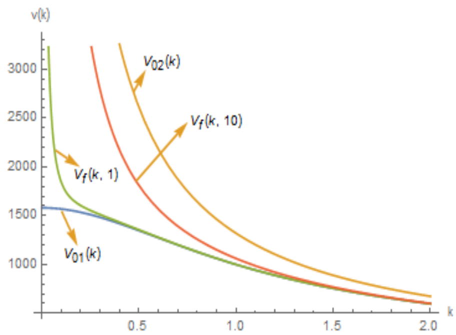

Fig. 1b. The zone and the behavior of the phase velocity of the shear wave's harmonic forms in a waveguide with mechanically free surfaces

In the case of ignoring microrotation (if $J \equiv 0$ then $\beta_k \to 0$) the condition of existence of harmonic waveforms takes the known form $\eta > 1$, while the microrotation account in the material narrows the band of the permitted frequencies and takes the form (2.9).

Figure 1a shows the frequency region of existence of waves with harmonic forms of oscillations for a waveguide of $h = 50 \cdot 10^{-3}$ m thickness from a material with physical-mechanical constants $\mu = 5 \times 10^9$ N/m², $J = 3 \times 10^3$ kg/m, , $\rho = 2 \times 10^3$ kg/m³, $\lambda = 12 \times 10^9$ N/ m².

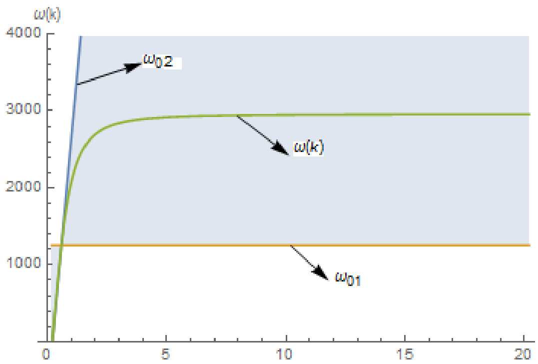

Fig. 2. The region of frequencies of plain deformation wave's localized form and the behavior of the frequency.

Here we see that the natural frequencies of oscillations forms for all the harmonics are enclosed in the region $\omega_{01} < \omega_n(k) < \omega_{02}$.

For low harmonics the dispersion form with a phase velocity $v_{01}(k) = \sqrt{\mu/\left(Jk^2 + \rho\right)}$ is the limiting one, and for the higher harmonics the limiting dispersion form is the one with a phase velocity $v_{02}(k) = k^{-1}\sqrt{\mu/J}$ (Fig. 1b).

The calculations also show that the accounting of micropolarity of the material results in appearance of forbidden frequency zones for a shear wave harmonics in a waveguide with rigidly clamped or mechanically free surfaces $0 < \omega_n(k) < \sqrt{\mu k^2/\left(Jk^2 + \rho\right)}$ or $\omega_n(k) > \sqrt{\mu/J}$.

In the case of plane strain wave propagation in an elastic micropolar waveguide the accounting of rotations leads to a possible localization of wave energy near the surface of the waveguide.

In frequency determination zone (3.17) the wave signal of plane deformation is localized near the surface of the waveguide and has a propagation frequency $\omega(k) = k\sqrt{\left(c_l^2 + c_t^2\right)/\left(1 + Jk^2/\rho\right)}$.

From Fig. 2 we can see that the localized wave signals of plain deformation have a wavelength $\lambda \le \pi\left(c_l/c_t\right)\sqrt{J/\rho}$.

Conclusion. On the basis of a simplified model of the Cosserat continuum, the conditions of possible localization of the wave energy with different boundary conditions on the surfaces of the elastic micropolar waveguide are obtained. The conditions for a possible localization of the wave energy near the surfaces of the waveguide are found. It is shown that in the antiplane deformation problem for a waveguide with clamped or mechanically free walls, the accoun of material micropolarity doesn't lead to the possibility of localized forms existence of high frequency. In the plain strain problem the micropolarity account of under different boundary conditions may cause a distortion of a frequency band of the existence of localized Rayleigh waves, and the emergence of a new frequency band of possible localized waves. Frequency bands of localized and harmonic waveforms are found.

In a long-wave and a short-wave approximation the obtained results are compared with the results of the classical theory of elasticity. Characteristic distribution of elastic displacement across the thickness of the waveguide with different combinations of boundary conditions is given.

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Received 16.12.2016

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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 70, №2, 2017 Механика

AMPLITUDE-PHASE DISTORTION OF THE NORMAL HIGH-FREQUENCY SHEAR WAVES IN HOMOGENEOUS ELASTIC WAVEGUIDE WITH WEAKLY ROUGH SURFACES

Avetisyan A.S., Hunanyan A.A.

Keywords: instability of normal waves, weakly rough surfaces, delay frequency, internal resonance.
Ключевые слова: неустойчивость нормальной волны, слабо-неоднородные поверхности, полоса задержки частот, внутренний резонанс.
Բանալի բառեր. նորմալ ալիքների անկայունություն; թույլ անհարթ մակերևույթներ; հաճախականության լռության գոտիներ; ներքին ռեզոնանս:

Аветисян А.С., Унанян А.А.
Амплитудно-фазовые искажения высокочастотной нормальной сдвиговой волны в однородном упругом волноводе с слабо-неоднородными поверхностями

Исследуется влияние слабой неоднородности поверхностей упругого слоя-волновода на распространение нормальной сдвиговой волны при механически свободных слабо-неоднородных поверхностях волновода. В приповерхностных зонах волновода виртуально выделяются тонкие прослойки переменной толщины. В выделенных упругих прослойках вводятся функции распределения упругих сдвигов (гипотезы MELS). Ввод гипотез MELS позволяет более подробно исследовать процесс искажения нормальной волны. Оно сделает более удобным исследования волновых процессов в волноводах с усложнёнными свойствами и сложными характерными неоднородностями материала волновода и его поверхностей. Показывается, что в отличие от идеально гладких поверхностей, слабая неоднородность механически свободных поверхностей приводит к искажению распространяющейся нормальной волны. Происходит частичная локализация волновой энергии в приповерхностных прослойках волновода. Появляются частотные зоны умолчания (а также зоны частотного пропускания) вновь формированной волны.

Ավետիսյան Ա.Ս., Հունանյան Ա.Ա.
Բարձր հաճախակայինության սահքի նորմալ ալիքի լայնույթա-փուլային աղավաղումը, մակերևութային թույլ անհամասեռությամբ, համասեռ առաձգական ալիքատարում

Դիտարկվում է մեխանիկորեն ազատ մակերևույթներով առաձգական ալիքատարի մակերևույթների թույլ անահարթության ազդեցությունը սահքի նորմալ ալիքի տարածման վրա: Վիրտուլ ընտրվում են փոփոխական հաստությամբ բարակ շերտեր մերձ-մակերևույթային շերտերում: Առաձգական սահքի բաշխման ֆունկցիաներ են ներմուծվում ընտրված առաձգական շերտերում (MELS վարկածներ): MELS վարկածների ներմուծումը ավելի կհեշտացնի բարդ հատկություններով նյութերց կազմված և անահարթ մակերևույթներով ալիքատարներում ալիքային պրոցեսների ուսումնասիրությունը: Ցույց է տրված, որ ի հակադրություն իդեալական հարթ մակերևույթների՝ մեխանիկորեն ազատ մակերևույթների թույլ անահարթությունը տանում է տարածվող նորմալ ալիքի աղավաղման: Տեղի է ունենում ալիքային էներգիայի մասնակի տեղայնացում ալիքատարի մերձ-մակերևույթային շերտերում: Հայտնվում են նոր ձևավորվող ալիքների հաճախականության լռության գոտիներ (ինչպես նաև հաճախականության թողունակության գոտիներ):

The influence of weak roughness of mechanically free elastic waveguide surfaces on propagation of normal shear wave is investigated. Thin layers of variable thickness are virtually separated in near-surface areas. Distribution functions of elastic shears are introduced in separated elastic layers (hypotheses MELS). Introduction of hypotheses

MELS will make the study of wave processes in waveguides with complicated properties and sophisticated characteristic roughness of the material of the waveguide and its surfaces more convenient. It is shown that in contrast to perfectly smooth surfaces, weak roughness of the mechanically free surfaces leads to distortion of propagating normal waves. Partial localization of wave energy occurs in near-surface layers of the waveguide. Frequency zones of silence of newly formed waves (as well as zones of frequency bandwidth) appear as well.

Introduction

The interaction of ultrasound wave with rough surface of waveguides currently is actively investigated from both theoretical and experimental points of view (see e.g. [1-3]). It is related to applications of elastic wave phenomena in modern technology: telecommunications (signal processing), medicine (ultrasound measurement), metallurgy (nondestructive control), etc.

In studies of propagation of high-frequency wave signals (high frequency, short waves) in layered waveguides it is especially important to take into account the real roughness (non-smoothness) of surfaces of the waveguide. It is especially important in cases where the length of the wave signal is of the same order with the amplitude or average step of the surface roughness. There is a huge body of references about wave propagation in layered waveguide with perfectly smooth surfaces of attachment of the layers. However, smooth surface is an idealized model for which it is not always possible to rigorously determine or estimate the characteristics of the wave field more accurately, especially in near-surface zones of the waveguide. The roughness of the waveguide layers definitely complicates the mathematical model, but provides opportunity to identify near-surface wave effects and more accurately calculate the quantitative characteristics of the formed wave field in the near-surface area.

There are different theoretical approaches and practical tools for investigating surface waves propagation on rough surfaces (see, for instance, [4-7]). Many papers (see e.g. [8-11]) are dedicated to different cases of normal high-frequency short monochromatic waves stability loses, such as localization of wave energy, internal resonance, occurrence of forbidden zones of frequency, etc.

Possible distortion of the amplitude and phase functions for normal distribution of the wave signal in a weakly rough elastic waveguide are investigated in [12-14]. The occurrence of internal resonance is studied, and conditions for existence of forbidden zones of frequency are revealed using the hypotheses of magneto-electro-elastic layered systems (MELS hypotheses).

In [15], wave propagation in inhomogeneous media with self-similar structure is studied using fractional calculus, along with the space-time discontinuous Galerkin methods. One and two dimensional problems are studied to demonstrate the capability of the proposed model in modeling inhomogeneous media.

In this paper, we propose a new approach for studying the influence of roughness of the surface of the layer-waveguide on the propagation of elastic, normal shear wave by so-called MELS hypotheses.

1. Problem Statement

Let us assume that pure shear normal wave signal

$$w(x,y,t)=W_0(y)\times\exp\left[i(k_0x-\omega_0t)\right], \tag{1.1}$$

$u(x,y,t)\equiv 0$, $v(x,y,t)\equiv 0$, (1.2)

is propagating in elastic, isotropic waveguide $\Omega := \left\{ |x| < \infty;\ h_{-}(x) \le y \le h_{+}(x);\ |z| < \infty \right\}$ with rough surfaces $y = h_{-}(x)$ and $y = h_{+}(x)$. Here ω_0 is the frequency of the source of wave signal, $k_0 \triangleq (2\pi/\lambda_0)$ is the wave number and λ_0 is the length of wave signal. Then, the equation of motion of the medium has the following form

$$\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2} = c_0^{-2}\frac{\partial^2 w(x,y,t)}{\partial t^2}, \tag{1.3}$$

where $c_0^2 = G_0/\rho_0$ is the speed of the shear normal wave in the waveguide, G_0 is the shear modulus and ρ_0 is the density of the waveguide material.

It is assumed, that the roughness of the waveguide surfaces $y = h_{\pm}(x)$ are represented by the following harmonic functions

$$\begin{cases} h_{+}(x) = h_0\left[1 + \varepsilon_{+}\cdot\sin(k_{+}\cdot x) + \delta_{+}\cdot\cos(k_{+}\cdot x)\right], \\ h_{-}(x) = -h_0\left[1 + \varepsilon_{-}\cdot\sin(k_{-}\cdot x) + \delta_{-}\cdot\cos(k_{-}\cdot x)\right], \end{cases} \tag{1.4}$$

where h_0 is the half-thickness of basic layer of the waveguide, $\varepsilon_{\pm}$ and $\delta_{\pm}$ are the relative amplitude coefficients of the heights of roughness profiles with $\{\varepsilon_{\pm};\ \delta_{\pm}\} \ll 1$, because the heights of the protrusions of roughness $h_0\cdot\varepsilon_{\pm}$ and $h_0\cdot\delta_{\pm}$ are always much less than the basic layer thickness: $\{h_0\cdot\varepsilon_{\pm};\ h_0\cdot\delta_{\pm}\} \ll h_0$, $k_{\pm} \triangleq 2\pi/\lambda_{\pm}$ is the number of the waviness of roughness profile and $\lambda_{\pm}$ is the step (wavelength) of the roughness profiles.

The boundary conditions on mechanically free non-smooth surfaces of the waveguide $\sigma_{ij}(x,y)\cdot n_j^{\pm}(x) = 0$ are written respectively in this form:

$$h_{\pm}'(x)\cdot\left.\frac{\partial w(x,y)}{\partial x}\right|_{y=h_{\pm}(x)} + \left.\frac{\partial w(x,y)}{\partial y}\right|_{y=h_{\pm}(x)} = 0. \tag{1.5}$$

It is evident from (1.3)-(1.5), that its solution must explicitly depend on the roughness of the surfaces. Since the roughness is weak $\{h_0\cdot\varepsilon_{\pm};\ h_0\cdot\delta_{\pm}\} \ll h_0$, the interaction of roughness will mainly be available in case of high-frequency (shortwave) wave signals, for which $\lambda_0 \sim \lambda_{\pm} \ll h_0$, or equivalently $k_0 h_0 \sim k_{\pm} h_0 \gg 1$. Then, one might be interested in investigation of the influence of surfaces roughness of the waveguide on the propagation of normal high-frequency shear waves.

2. Problem Solution

There are two methods to solve the problem: the method of successive approximations and the method of introduced hypotheses. Later in this article we will compare wave characteristics of the received wave fields.

2.1. First Approach

When high-frequency, normal shear signal (1.1) is propagated in elastic waveguide, interaction of the wave signal with the roughness of the surfaces in the near-surface areas occurs, which consequently leads to amplitude and phase distortion of the primary signal. New harmonics appear and a new amplitude-phase interaction is formed.

We use Fourier method of variables separation, and the solution of the boundary value problem (1.3)-(1.5) is represented in the following form:

$$w(x,y,t)=\sum_{n=1}^{\infty}W_n(y)\cdot X_n(x)\cdot\exp(-i\omega_{0n}t)\,. \tag{2.1}$$

Then the conditions of mechanically free surfaces of the waveguide, on rough surfaces $y=h_{\pm}(x)$ respectively, for each harmonic of propagating wave will have the following form

$$W_n'(h_{\pm}(x))=\mp h_0k_{\pm}\cdot\left[\varepsilon_{\pm}\cdot\cos(k_{\pm}\cdot x)-\delta_{\pm}\cdot\sin(k_{\pm}\cdot x)\right]\cdot\frac{X_n'(x)}{X_n(x)}W_n(h_{\pm}(x))\,. \tag{2.2}$$

It is suggested, that the equations for determining the desired functions $X_n(x)$ and $W_n(y)$ are shown in the form

$$\begin{cases}W_n''(y)+k_n^2\left[\eta_n^2-1\right]W_n(y)=0,\\ X_n''(x)+k_n^2X_n(x)=0,\end{cases} \tag{2.3}$$

where the following assignment for appropriate harmonics $\eta_n^2\triangleq\omega_n^2k_n^{-2}c_0^{-2}$ has been taken into account, k_n is the wave number (formation coefficient through the thickness of the waveguide), corresponding to the generated n-th harmonic.

From surface conditions (2.2) it follows that the undamped solutions of (2.3) in the directions of the propagation $\pm Ox$ (for $\mathrm{Im}\left[k_n\right]\equiv 0$) are shown in the following form

$$\begin{cases}W_n(y)=C_{1n}\exp\left(ik_n\alpha_n y\right)+C_{2n}\exp\left(-ik_n\alpha_n y\right),\\ X_n(x)=C_{\pm}\exp\left(\pm ik_n x\right),\end{cases} \tag{2.4}$$

which, for slow waves, i.e. when $\alpha_n^2\triangleq\eta_n^2-1<0$, corresponds to the damped harmonics from the surface up to the depth of the waveguide, and for fast waves, i.e. when $\alpha_n^2\triangleq\eta_n^2-1\geq 0$, corresponds to harmonic forms over the thickness of the waveguide.

From (2.3) it also follows that fast damped waves occur in the directions of wave propagation

$\pm Ox$ in the case of $\mathrm{Re}\left[k_n\right] \equiv 0$:

$$\begin{cases} W_n''(y) - k_n^2\left[\eta_n^2 - 1\right]W_n(y) = 0 \\ X_n''(x) - k_n^2 X_n(x) = 0 \end{cases}.$$

For slow wave, i.e. when $\alpha_n^2 \triangleq \eta_n^2 - 1 < 0$, the solution corresponds to harmonic forms over the thickness of the waveguide, and for fast wave, i.e. when $\alpha_n^2 \triangleq \eta_n^2 - 1 \geq 0$, it corresponds to damped harmonics from the surface up to the depth of the waveguide.

Taking into account that the roughness of the surface of the waveguide is weak and its impact on the propagating wave is described by boundary conditions (2.2), the solution of system (2.3) is represented in this form

$$X_n(x) = \sum_{m=0}^{\infty} \gamma^m \cdot A_{mn} \exp\left(ik_{*m}x\right). \tag{2.5}$$

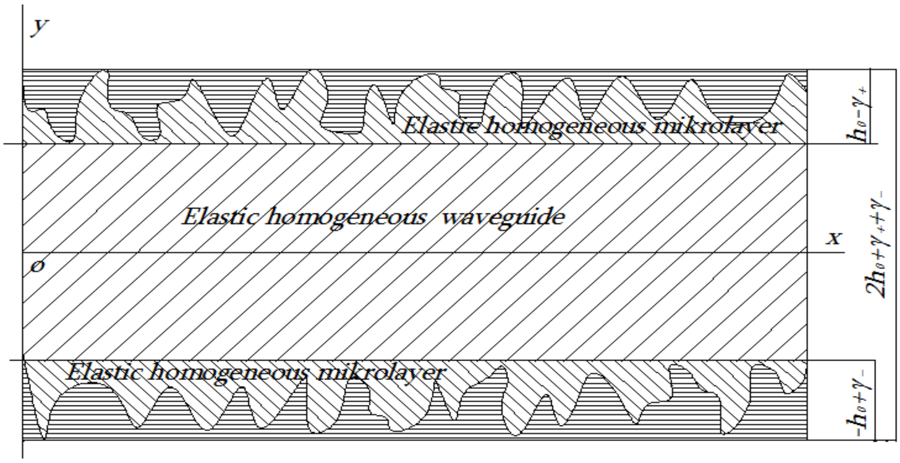

Fig.2.1. The model of elastic waveguide as a multilayer waveguide

Moreover, the value $m = 0$ corresponds to the case of homogeneous waveguide. Here, the introduced wave number k_{*m} should be formed by the impact of normal wave signal and roughness of the surfaces of the waveguide.

The roughness of the surfaces, in its turn, is characterized by the greatest common divisor of wave numbers $k_* \triangleq \min\left\{k_+/p;\ k_-/q\right\} = 2\pi/\lambda_*$ is the smallest common wave number of roughness on the surfaces corresponding to the generated m-th harmonic waves, and $\gamma \triangleq \max\left\{\sqrt{\varepsilon_\pm^2 + \delta_\pm^2}\right\} \ll 1$ is a small parameter characterizing the weak roughness of the surfaces of the waveguide.

$$k_n\alpha_n\left[C_{n1}\exp\left(ik_n\alpha_n h_\pm(x)\right)-C_{n2}\exp\left(-ik_n\alpha_n h_\pm(x)\right)\right]\sum_{m=0}^{\infty}\gamma^m\cdot A_{mn}\exp\left(ik_{*m}x\right)=$$
$$=\mp h_0 k_\pm\cdot\left[\varepsilon_\pm\cdot\cos(k_\pm\cdot x)-\delta_\pm\cdot\sin(k_\pm\cdot x)\right]\times \tag{2.6}$$
$$\times\left[C_{n1}\exp\left(ik_n\alpha_n h_\pm(x)\right)+C_{n2}\exp\left(-ik_n\alpha_n h_\pm(x)\right)\right]\sum_{m=0}^{\infty}k_{*m}\gamma^m\cdot A_{mn}\exp\left(ik_{*m}x\right)$$

Considering that the right hand sides of boundary conditions (2.6) are in small $m+1$ order in the $n=0$ approximation, for non-trivial solutions of (2.6) we obtain the dispersion equation with the following solution

$$\omega_{0n}=k_{0n}c_0=\frac{2\pi c_0}{\lambda_{0n}}=\frac{\pi n c_0}{h_0}. \tag{2.7}$$

Consequently, interaction of the normal wave (1.1) with surface roughness $h_\pm(x)$ is not occur in the $n=0$ approximation, and the propagating wave is still normal as in $n=0$ approximation of longitudinally weakly rough waveguide with mechanically free surfaces [13]

$$w_0(x,y,t)=\sum_{n=1}^{\infty}A_{0n}\cdot\exp\left[i\left(\frac{\pi n x}{h_0}-\omega_{0n}t\right)\right]. \tag{2.8}$$

From the conditions of synchronization of the surface distortions at the mid-plane of the waveguide $y=0$, we get

$$\exp\left[i\left(k_{+m}-k_{-m}\right)x\right]=-\frac{k_+\cdot\left[\varepsilon_+\cdot\cos(k_+\cdot x)-\delta_+\cdot\sin(k_+\cdot x)\right]}{k_-\cdot\left[\varepsilon_-\cdot\cos(k_-\cdot x)-\delta_-\cdot\sin(k_-\cdot x)\right]}. \tag{2.9}$$

Considering that the wave number is formed as $k_{1n}(x)=k_{+n}-k_{-n}$ and $k_*\triangleq\min\left\{pk_+;\ qk_-\right\}=2\pi/\lambda_*$, it is easy to get the allowed wavelengths from (2.9) for the first approximation:

$$\lambda_*(x)=\lambda_0\cdot 2\pi\arccos^{-1}\left\{-\frac{k_+\cdot\left[\varepsilon_+\cdot\cos(k_+\cdot x)-\delta_+\cdot\sin(k_+\cdot x)\right]}{k_-\cdot\left[\varepsilon_-\cdot\cos(k_-\cdot x)-\delta_-\cdot\sin(k_-\cdot x)\right]}\right\}. \tag{2.10}$$

Then from the boundary equations (2.6) for the first approximation we will have

$$\exp\left(ik_{0n}\alpha_n\left(h_+(x)-h_-(x)\right)\right)-\exp\left(-ik_{0n}\alpha_n\left(h_+(x)-h_-(x)\right)\right)=0,$$

therefore formation coefficient of generated distortions of waves is obtained as

$$k_{0n}\alpha_{1n}=\frac{\pi n}{h_+(x)-h_-(x)}. \tag{2.11}$$

The wave number of the first generated harmonic depends on the surfaces of the non-smooth waveguide

$$k_{1n}(x)=\left(\frac{\pi n}{h_0}\right)\left(h_+(x)-h_-(x)\right)\left[h_0^2+\left(h_+(x)-h_-(x)\right)^2\right]^{-1/2}. \tag{2.12}$$

In the first approximation, the interaction of the normal wave with surface non-smoothness affects to the propagating wave:

$$w_1(x,y,t)=\sum_{n=1}^{\infty}W_{1n}(y)\cdot X_{1n}(x)\cdot\exp(-i\omega_{0n}t), \tag{2.13}$$

where

$$\begin{cases} W_{1n}(y)=C_{1n}\exp\left(i\dfrac{h_0\cdot k_{1n}(x)y}{\left(h_+(x)-h_-(x)\right)}\right)+C_{2n}\exp\left(-i\dfrac{h_0\cdot k_{1n}(x)y}{\left(h_+(x)-h_-(x)\right)}\right), \\ X_{1n}(x)=\gamma A_{0n}\exp\left(ik_{1n}(x)\cdot x\right). \end{cases} \tag{2.14}$$

Note that if the rough surfaces are "symmetric" with respect to the mid-plane of the waveguide, i.e.

$$-h_-(x)=h_+(x)=h(x)=h_0\left[1+\varepsilon\cdot\sin(k\cdot x)+\delta\cdot\cos(k\cdot x)\right], \tag{2.15}$$

then from relations (2.11) and (2.12) for the wave number over the thickness of the waveguide and the coefficient of formation, respectively, are obtained as follows

$$k_{1n}^s(x)=\frac{\pi n}{h_0}\cdot\left[\frac{h_0^2}{4h^2(x)}+1\right]^{-1/2}; \tag{2.16}$$

$$k_{1n}^s(x)\cdot\alpha_{1n}^s(x)=\frac{\pi n}{h(x)}\cdot\left[\frac{h_0^2}{h^2(x)}+4\right]^{-1/2}. \tag{2.17}$$

The solution (2.14) will be correspondingly transformed into

$$\begin{cases} W_{1n}^s(y)=C_{1n}^s\exp\left(i\left(\dfrac{h_0\cdot k_{1n}^s(x)}{h(x)}\right)y\right)+C_{2n}^s\exp\left(-i\left(\dfrac{h_0\cdot k_{1n}^s(x)}{h(x)}\right)y\right), \\ X_{1n}^s(x)=\gamma A_{0n}\exp\left(ik_{1n}^s(x)\cdot x\right). \end{cases} \tag{2.18}$$

In the case of "synchronous" (parallel to each other) roughness on the surfaces of the waveguide, will have the following representations:

$$\begin{cases} h_+(x)=h_0\left[1+\varepsilon\cdot\sin(k\cdot x)+\delta\cdot\cos(k\cdot x)\right], \\ h_-(x)=-h_0\left[1-\varepsilon\cdot\sin(k\cdot x)-\delta\cdot\cos(k\cdot x)\right]. \end{cases} \tag{2.19}$$

Then from relations (2.11) and (2.12) for the wave number over the thickness of the waveguide and the coefficient of formation, respectively, are obtained as follows:

$$k_{1n}^*(x)=\frac{\sqrt{5}}{10}\cdot\frac{\pi n}{h_0};\ k_{1n}^*(x)\cdot\alpha_{1n}^*(x)=\frac{\sqrt{5}}{20}\cdot\frac{\pi n}{h_0}. \tag{2.20}$$

The solution (2.14) changes accordingly

$$\begin{cases} W_{1n}^{*}(y) = C_{1n}^{*} \exp\left(i\dfrac{\sqrt{5}}{20} \cdot \dfrac{\pi n}{h_0} y \right) + C_{2n}^{*} \exp\left(-i\dfrac{\sqrt{5}}{20} \cdot \dfrac{\pi n}{h_0} y \right), \\ X_{1n}(x) = \gamma A_{0n} \exp\left(i\dfrac{\sqrt{5}}{10} \cdot \dfrac{\pi n}{h_0} \cdot x \right). \end{cases} \tag{2.21}$$

2.2. Second Approach

To analyze the propagation of the normal, pure shear wave signal (1.1) and (1.2), taking into account that in the isotropic waveguide $\Omega := \{|x| < \infty;\ h_-(x) \le y \le h_+(x);\ |z| < \infty\}$ roughness of surfaces $y = h_-(x)$ and $y = h_+(x)$ are described by the functions (1.4), the near-surface thin layers with variable thickness (the waveguide is presented as three-layer, see Fig.2) are virtually selected $\Omega = \Omega_- \cup \Omega_0 \cup \Omega_+$, where

$$\begin{cases} \Omega_- \triangleq \{|x| < \infty;\ h_-(x) \le y \le -h_0 + \gamma_-;\ |z| < \infty\}, \\ \Omega_0 \triangleq \{|x| < \infty;\ -h_0 + \gamma_- \le y \le h_0 - \gamma_+;\ |z| < \infty\}, \\ \Omega_+ \triangleq \{|x| < \infty;\ h_0 - \gamma_+ \le y \le h_+(x);\ |z| < \infty\}. \end{cases} \tag{2.22}$$

We intend to solve the equation of medium motion (1.3) for all three layers separately with boundary conditions (1.5) on mechanically free, non-smooth surfaces $y = h_-(x)$ and $y = h_+(x)$ for elastic displacements $w_\pm(x, y, t)$ (respectively for layers $\Omega_\pm$), and the conditions of continuity on virtual cross-sections $y = -h_0 + \gamma_-$ and $y = h_0 - \gamma_+$

$$w_0(x,y,t)\big|_{y=-h_0+\gamma_-} = w_-(x,y,t)\big|_{y=-h_0+\gamma_-},$$

$$w_0(x,y,t)\big|_{y=h_0-\gamma_+} = w_+(x,y,t)\big|_{y=h_0-\gamma_+}, \tag{2.23}$$

$$\left.\frac{\partial w_0(x,y,t)}{\partial y}\right|_{y=-h_0+\gamma_-} = \left.\frac{\partial w_-(x,y,t)}{\partial y}\right|_{y=-h_0+\gamma_-},$$

$$\left.\frac{\partial w_0(x,y,t)}{\partial y}\right|_{y=h_0-\gamma_+} = \left.\frac{\partial w_+(x,y,t)}{\partial y}\right|_{y=h_0-\gamma_+}. \tag{2.24}$$

Considering the thinness of the surface layers $\Omega_\pm$, the solution in them are represented with the hypotheses of MELS [11, 13] taking into account the nature of the changes arising from surface roughness $y = h_-(x)$ and $y = h_+(x)$

$$w_+(x,y)=\frac{\mathrm{sh}\left(\mu_+\left[y-h_0+\gamma_+\right]\right)}{\mathrm{sh}\left(\mu_+\left[h_+(x)-h_0+\gamma_+\right]\right)}\cdot\left[w_+\left(x,h_+(x)\right)-w_0\left(x,h_0-\gamma_+\right)\right]+$$

$$+w_0\left(x,h_0-\gamma_+\right); \tag{2.25}$$

$$w_-(x,y)=\frac{\mathrm{sh}\left(\mu_-\left[y+h_0-\gamma_-\right]\right)}{\mathrm{sh}\left(\mu_-\left[h_-(x)+h_0-\gamma_-\right]\right)}\cdot\left[w_-\left(x,h_-(x)\right)-w_0\left(x,-h_0+\gamma_-\right)\right]+$$

$$+w_0\left(x,-h_0+\gamma_-\right), \tag{2.26}$$

where the values $w_+\left(x,h_+(x)\right)$ and $w_-\left(x,h_-(x)\right)$ are determined from the conditions on mechanically free surface (1.5) as follows:

$$w_+\left(x,h_+(x)\right)=\frac{\mu_+\cdot\mathrm{cth}\left(\mu_+\left[h_+(x)-h_0+\gamma_+\right]\right)\cdot\left[1-\{h_+'(x)\}^2\right]}{\mu_+\cdot\mathrm{cth}\left(\mu_+\left[h_+(x)-h_0+\gamma_+\right]\right)\cdot\left[1-\{h_+'(x)\}^2\right]+h_+'(x)}\cdot w_0\left(x,h_0-\gamma_+\right); \tag{2.27}$$

$$w_-\left(x,h_-(x)\right)=\frac{\mu_-\cdot\mathrm{cth}\left(\mu_-\left[h_-(x)+h_0-\gamma_-\right]\right)\cdot\left[1-\left(h_-'(x)\right)^2\right]}{\mu_-\cdot\mathrm{cth}\left(\mu_-\left[h_-(x)+h_0-\gamma_-\right]\right)\cdot\left[1-\left(h_-'(x)\right)^2\right]+h_-'(x)}\cdot w_0\left(x,-h_0+\gamma_-\right). \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.25) and (2.26), we reach the solution in the near-surface thin layers of the waveguide formed by the propagation of the normal wave $w_0(x,y,t)=W_0(y)\cdot\exp\left[i(k_0x-\omega_0t)\right]$ in the basic layer Ω_0:

$$w_+(x,y)=\left\{\begin{array}{l}1-\dfrac{\mathrm{sh}\left(\mu_+\left[y-h_0+\gamma_+\right]\right)}{\mathrm{sh}\left(\mu_+\left[h_+(x)-h_0+\gamma_+\right]\right)}\times\\ \times\dfrac{h_+'(x)}{\mu_+\cdot\mathrm{cth}\left(\mu_+\left[h_+(x)-h_0+\gamma_+\right]\right)\cdot\left[1-\{h_+'(x)\}^2\right]+h_+'(x)}\end{array}\right\}\cdot w_0\left(x,h_0-\gamma_+\right); \tag{2.29}$$

$$w_-(x,y)=\left\{\begin{array}{l}1-\dfrac{\mathrm{sh}\left(\mu_-\left[y+h_0-\gamma_-\right]\right)}{\mathrm{sh}\left(\mu_-\left[h_-(x)+h_0-\gamma_-\right]\right)}\times\\ \times\dfrac{h_-'(x)}{\mu_-\cdot\mathrm{cth}\left(\mu_-\left[h_-(x)+h_0-\gamma_-\right]\right)\cdot\left[1-(h_-'(x))^2\right]+h_-'(x)}\end{array}\right\}\cdot w_0\left(x,-h_0+\gamma_-\right). \tag{2.30}$$

Let us represent the normal wave in the basic layer Ω_0 in a common form

$$w_0(x,y,t)=\left[A\cos(\mu_*y)+B\sin(\mu_*y)\right]\cdot\exp\left[i(k_*x-\omega_0t)\right], \tag{2.31}$$

here $k_* \triangleq \min\{pk_+;\ qk_-\} = 2\pi/\lambda_*$ is the smallest common wave number of the roughness on the surfaces corresponding to the generated harmonic of the wave.

From the conditions of continuity of mechanical stresses (2.24), we obtain a dispersion equation to determine the formation coefficient μ_*:

$$\begin{aligned}&\mu_*^2 - \mu_* \cdot \operatorname{ctg}\left(\mu_*(2h_0 - (\gamma_+ + \gamma_-))\right) \cdot \left(f_+\left(\mu_+; h_+(x)\right) - f_-\left(\mu_-; h_-(x)\right)\right) = \\ &= - f_+\left(\mu_+; h_+(x)\right) \cdot f_-\left(\mu_-; h_-(x)\right),\end{aligned} \tag{2.32}$$

in which

$$f_+\left(\mu_+; h_+(x)\right) \triangleq \left[\begin{aligned}&\frac{1}{\operatorname{sh}\left(\mu_+\left[h_+(x) - h_0 + \gamma_+\right]\right)} \times \\ &\times \frac{1}{\mu_+ \cdot \operatorname{cth}\left(\mu_+\left[h_+(x) - h_0 + \gamma_+\right]\right) \cdot \left[1 - \left\{h_+'(x)\right\}^2\right] + h_+'(x)} \cdot \mu_+ h_+'(x)\end{aligned}\right], \tag{2.33}$$

$$f_-\left(\mu_-; h_-(x)\right) \triangleq \left[\begin{aligned}&\frac{1}{\operatorname{sh}\left(\mu_-\left[h_-(x) + h_0 - \gamma_-\right]\right)} \times \\ &\times \frac{1}{\mu_- \cdot \operatorname{cth}\left(\mu_-\left[h_-(x) + h_0 - \gamma_-\right]\right) \cdot \left[1 - \left\{h_-'(x)\right\}^2\right] + h_-'(x)} \cdot \mu_- h_-'(x)\end{aligned}\right]. \tag{2.34}$$

They characterize the influence of the rough surfaces on the formation coefficient.

It is obvious, that the solution of the dispersion equation (2.32) significantly depends on the surface roughness $h_\pm(x)$.

3. Numerical Analysis of Obtained Results

Considering the surface roughness, in the first approach, the solutions for formation coefficient $k_{0n}\alpha_{1n}$ and wave number $k_{1n}(x)$ are obtained in the forms (2.11) and (2.12) respectively. As expected, the variable thickness through the waveguide plays the main role in these expressions $\xi(x) \triangleq h_+(x) - h_-(x)$, by means of which the wave process can be controlled.

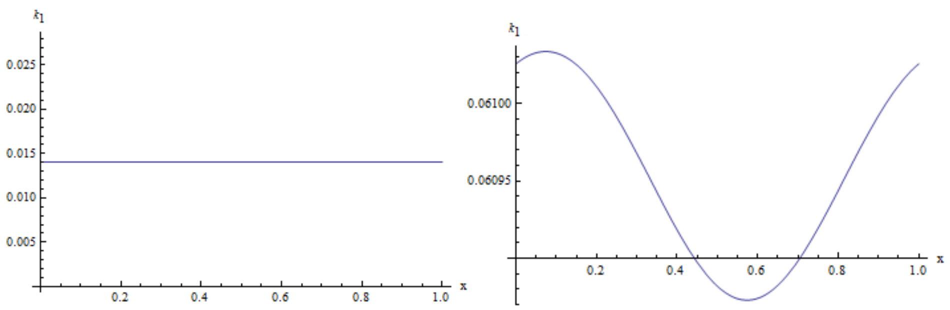

Fig. 3.1. The wave number for “synchronous” and “symmetric” surface roughness of the waveguide (the first approach)

Graphics of the formation coefficient and the wave number for different particular characteristic surfaces of roughness are given in Figs. 3.1 and 3.2 using the relations (2.15)-(2.21), respectively. From the figures of the wave number and formation coefficient it follows that for "symmetric" surface roughness of the waveguide (2.15) the changes of these values are characteristically different from the case of "synchronous" surface roughness (2.19).

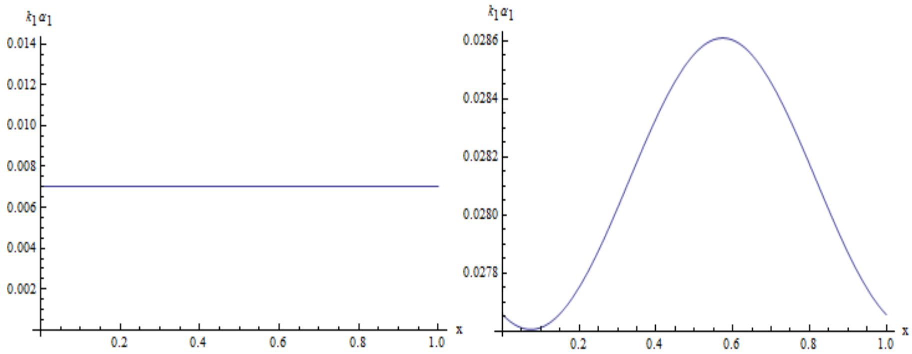

Fig. 3.2. The formation coefficient for "synchronous" and "symmetric" surface roughness of the waveguide (the first approach)

From Figs. 3.1 and 3.2 it is obvious that in the case of "symmetric" surface roughness of the waveguide (2.15), the wave number and the formation coefficient are periodically changed with respect to the half-thickness of the waveguide in the interval $x \in [0;\ \lambda_*]$.

In the case of "synchronous" surface roughness of the waveguide (2.19) the wave number and the formation coefficient are only changed by a constant value for each n-th harmonic.

In the general case of arbitrary surface roughness $y = h_-(x)$ and $y = h_+(x)$ from (2.9)-(2.14) it follows that due to the difference of surface roughness in the near-surface areas there occur qualitatively identical, but quantitatively different harmonics, a synchronization which occurs at the mid-plane $y = 0$. From (2.11) and (2.12) it is obvious that the wave number $k_{1n}(x)$ and the formation coefficient $k_{0n}\alpha_{1n}(x)$ for the propagation of the waves is always positive, since $h_+(x) - h_-(x) > 0$. From relations (2.18) and (2.21) we can easily get the nature of the changes of elastic shear through the thickness of the waveguide, according to the variable thickness of the waveguide (see Figs. 3.3 and 3.4). The picture of elastic shear $W_{1n}^s(y)$ over the thickness of the waveguide for the "symmetric" surface roughness is defined by relation (2.18) and is shown in Fig.3.3. Fig. 3.3 shows that over the thickness of the waveguide for the "symmetric" surface roughness (2.15), the normal waveform is periodically distorted depending on the law of variation of its thickness $\xi(x) \triangleq h_+(x) - h_-(x)$. Accordingly, the phase velocity of the generated harmonic is also changed. The elastic shear $W_{1n}^*(y)$ over the thickness of the waveguide for "synchronous" surface roughness is defined by relation (2.21) and is shown in Fig.3.4. From (2.20) it follows that in this case only short waves with lengths $\lambda_* = \sqrt{5} \cdot h_0 / n$ propagate for large numbers of harmonics n, such that $n\lambda_* \ll \sqrt{5} \cdot h_0$.

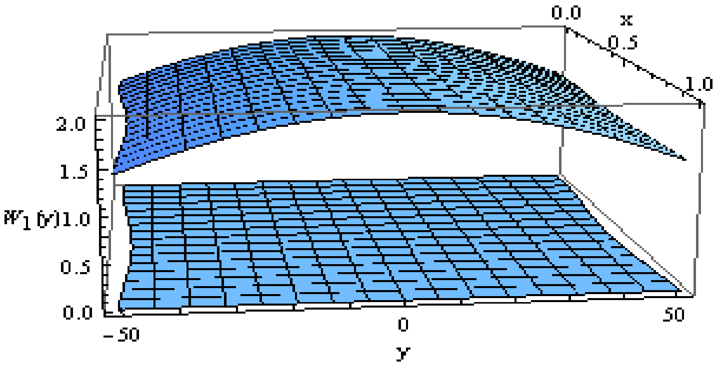

Fig. 3.3. The elastic shear through the thickness of the waveguide for "symmetric" surface roughness (the first approach)

Solving the problem with the method of hypotheses MELS, through the thickness of the waveguide we obtain the expression of elastic shear in the basic layer Ω_0 in the form of (2.31), which is analytically continued in both near-surface zones Ω_- and Ω_+, accordingly (2.30) and (2.29). The image over the thickness of the waveguide is constructed after determining the formation coefficient μ_* from the dispersion equation (2.32). From relations (2.29)-(2.34) it is obvious that the solutions, received in the near-surface zones Ω_- and Ω_+, are characteristically the same, but numerically different at different surface roughness $h_+(x)$ and $h_-(x)$.

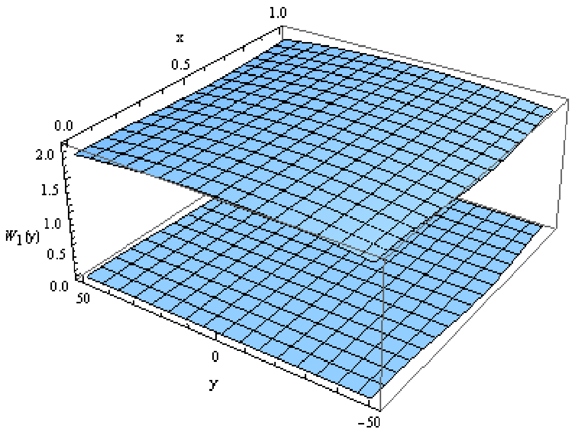

Fig. 3.4. The elastic shear through the thickness of the waveguide for "synchronous" surface roughness (the first approach)

The dispersion equation (2.32) is much simplified in the cases of “symmetric” (2.15) and “synchronous”
(2.19) surface roughness, considering the expressions of the coefficients of the dispersion equation $f_{+}\left(\mu_{+};h_{+}(x)\right)$ and $f_{-}\left(\mu_{-};h_{-}(x)\right)$, in relations (2.33) and (2.34) respectively. Fig. 3.5 shows the graphical dependence of the formation coefficient μ_{*} on x.

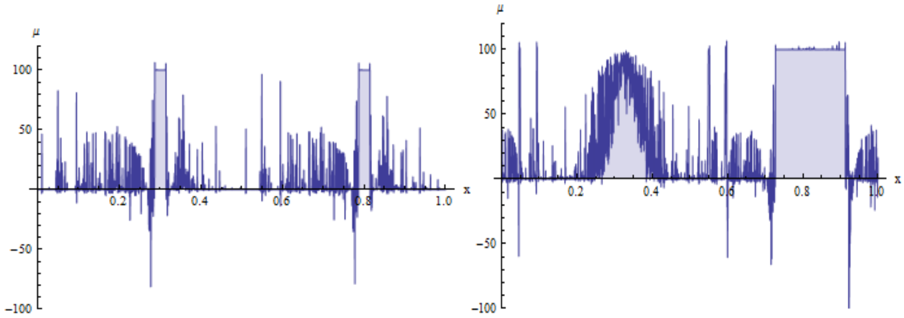

Fig. 3.5. The formation coefficient for “synchronous” and “symmetric” surface roughness of the waveguide (the second approach)

To each formation coefficient μ_{*n} naturally corresponds a wave number $k_{*n} = 2\pi/\lambda_{*n} = \sqrt{\omega_{0n}^{2}c_{0}^{-2} - \mu_{*n}^{2}}$.

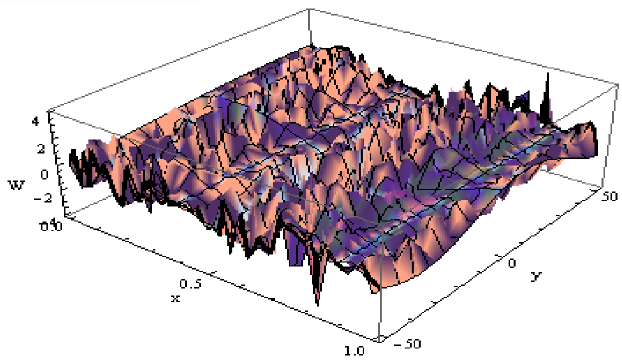

Fig. 3.6. The elastic shear through the thickness of the waveguide for “symmetric” surface roughness (the second approach)

From the dispersion equation (2.32) and the relations (2.33) and (2.34) it is evident that in the absence of roughness on the surfaces of the waveguide, i.e. when $h_{+}'(x) = h_{-}'(x) \equiv 0$, both introduced multipliers (2.33) and (2.34) become zero and from the dispersion equation we obtain the case of homogeneous waveguide $\mu_{*n} = \mu_{0n} = \pi n / h_{0}$.

From the obtained graphs it is also seen how the presence of “symmetric” (2.15) or “synchronous” (2.19) surface roughness of relatively homogeneous waveguide leads to distortion of forms (formation coefficient μ_{*n} and wave number k_{*n}).

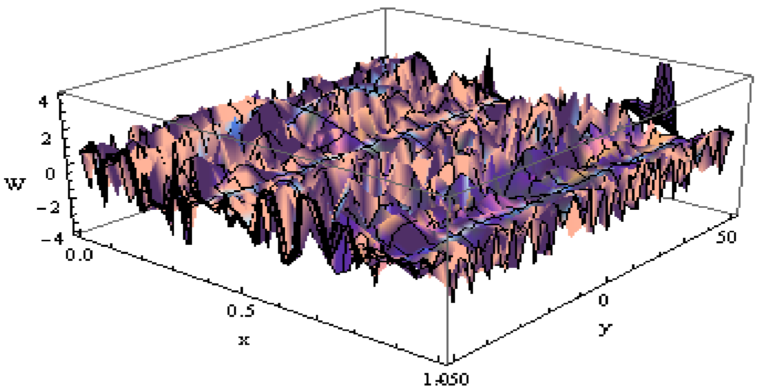

Fig. 3.7. The elastic shear through the thickness of the waveguide for "synchronous" surface roughness (the second approach)

From relations (2.32)-(2.34) and the received graphs it is also clear that weak surface roughness do not lead to appearance of damped propagating harmonics through the depth of the waveguide. Partial localization of the wave energy occurs only in the thin surface rough layers, which can be seen in the given figures of elastic shear over the thickness of the waveguide. The images of elastic shear throughout the thickness of the waveguide in particular "symmetric" (2.15) and "synchronous" (2.19) surface roughness cases are shown in Figs. 3.6 and 3.7.

4. Conclusion

It is shown that weak surface roughness lead to instability of a normal propagating wave in the waveguide. The presence of surface roughness can lead to prohibition of waves of certain lengths depending on the characteristic values of the functions of the roughness. Only partial localization of wave energy in thin near-surface areas of roughness occurs. The localized surface waves do not occur. The introduced method of hypotheses MELS allows to analyze the process of distortion of the normal waves, that will make it convenient for studies of wave processes in waveguides with complicated properties and sophisticated characteristic roughness of the material of the waveguide and its surfaces.

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Funding: This study was funded by RA MES State Committee of Science (grant number 15T-2C026).

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Received 12.04.2017

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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 70, №2, 2017 Механика

УДК 539.3

ПЕРВАЯ ДИНАМИЧЕСКАЯ КРАЕВАЯ ЗАДАЧА ТЕОРИИ УПРУГОСТИ ДЛЯ ТРЁХСЛОЙНОЙ ПЛАСТИНКИ

Закарян Т.В.[1]

Ключевые слова: слоистая пластинка, вынужденные колебания, резонанс, асимптотический метод.
Key words: laminated plate, forced oscillations, resonance, asymptotic method.
Բանալի բառեր: շերտավոր սալ, ստիպողական տատանումներ, ռեզոնանս, ասիմպտոտիկ մեթոդ

Zakaryan T.V.
First dynamic boundary problem of the elasticity theory for three-layered plate
The three-dimensional dynamic problem for orthotropic three-layered plate of symmetric shape is solved. It is assumed that the values of components of stress tensors, harmonically changing during the time, are given on front surfaces. There is a full contact between layers. The general asymptotical solution is obtained. It is shown that if external interactions are polynomials of tangential coordinates then the solution will be mathematically exact. The illustrativ eexample is discussed.

Զաքարյան Տ.Վ.
Առաձգականության տեսության առաջին դինամիկական եզրային խնդիրը եռաշերտ սալի համար
Լուծված է առաձգականության տեսության տարածական դինամիկական խնդիր սիմետրիկ շերտերից բաղկացած եռաշերտ օրթոտրոպ սալի համար: Փաթեթի դիմային մակերևույթների վրա տրված են լարման թենզորի համապատասխան բաղադրիչները, որոնք ըստ ժամանակի փոփոխվում են հարմոնիկորեն: Շերտերի միջև կոնտակտը լրիվ է: Ստացված է ընդհանուր ասիմպտոտիկական լուծումը: Ցույց է տրված, որ երբ արտաքին ազդեցությունները հանդիսանում են տանգենցիալ կոորդինատների նկատմամբ բազմանդամներ, ներքին խնդրի լուծումը դառնում է մաթեմատիկորեն ճշգրիտ: Արտածված են ռեզոնանսի առաջացման պայմանները: Բերված է բնութագրիչ օրինակ:

Решена трёхмерная динамическая задача для ортотропной трёхслойной пластинки симметричной структуры. Считается, что на лицевых поверхностях пакета заданы значения соответствующих компонент тензора напряжений, которые изменяются во времени гармонически. Контакт между слоями полный. Получено общее асимптотическое решение. Показано, что если внешние воздействия являются многочленами от тангенциальных координат, решение становится математически точным. Приведён иллюстрационный пример.

Введение. Для решения плоских и пространственных статических и динамических задач балок-полоси пластин оказался эффективным асимптотический метод решения сингулярно возмущённых дифференциальных уравнений. Решению статических плоских и пространственных задач однослойных и многослойных балок и пластин посвящены монографии [1,2]. Некоторые классы задач о вынужденных колебаниях однослойных и многослойных пластин решены в [3-5]. Первая динамическая краевая задача для изотропной полосы решена в [6], а для ортотропной полосы – в [7]. Первая динамическая пространственная краевая задача, для прямоугольной пластинки решена в [8]. В данной работе решена та же задача для трёхслойной пластинки симметричной структуры.

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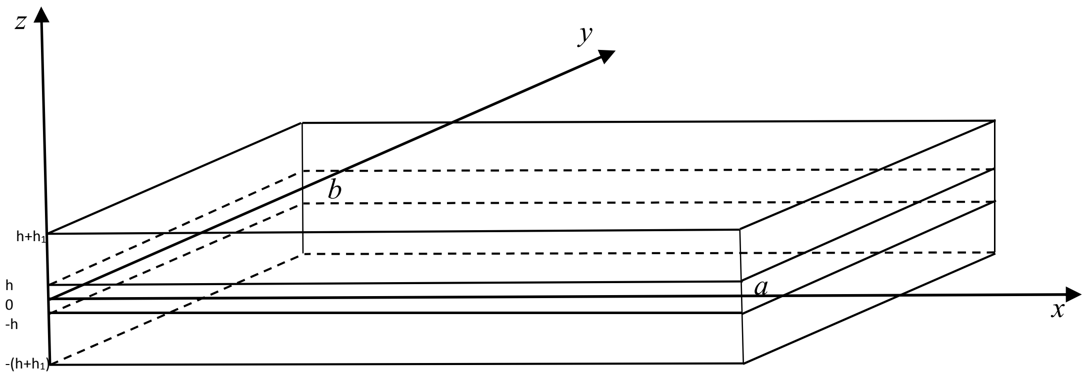
[1] Работа доложена на Международной школе-конференции молодых учёных, 3-7 октября, 2016, Цахкадзор, Армения.

1.Основные уравнения и постановка краевой задачи. Имеем трёхслойную пластинку, занимающую область $D=\{(x,y,z):\ 0\le x\le a,\ 0\le y\le b,$ $-(h+h_1)\le z\le h+h_1,\ 2(h+h_1)=H<<l,\ l=\min(a,b)\}$ (фиг.1).



Фиг. 1

Требуется найти решение уравнений движения

$$\frac{\partial\sigma_{xx}^k}{\partial x}+\frac{\partial\sigma_{xy}^k}{\partial y}+\frac{\partial\sigma_{xz}^k}{\partial z}=\rho^k\frac{\partial^2 u^k}{\partial t^2},\ \frac{\partial\sigma_{xy}^k}{\partial x}+\frac{\partial\sigma_{yy}^k}{\partial y}+\frac{\partial\sigma_{yz}^k}{\partial z}=\rho^k\frac{\partial^2 v}{\partial t^2},\qquad(1.1)$$

$$\frac{\partial\sigma_{xz}^k}{\partial x}+\frac{\partial\sigma_{yz}^k}{\partial y}+\frac{\partial\sigma_{zz}^k}{\partial z}=\rho^k\frac{\partial^2 w^k}{\partial t^2},\ \ k=I,II,III,$$

при соотношениях упругости (обобщённый закон Гука)

$$\frac{\partial u^k}{\partial x}=a_{11}^k\sigma_{xx}^k+a_{12}^k\sigma_{yy}^k+a_{13}^k\sigma_{zz}^k,\ \frac{\partial v^k}{\partial y}=a_{12}^k\sigma_{xx}^k+a_{22}^k\sigma_{yy}^k+a_{23}^k\sigma_{zz}^k,$$

$$\frac{\partial w^k}{\partial z}=a_{13}^k\sigma_{xx}^k+a_{23}^k\sigma_{yy}^k+a_{33}^k\sigma_{zz}^k,\ \frac{\partial u^k}{\partial y}+\frac{\partial v^k}{\partial x}=a_{66}^k\sigma_{xy}^k,\ (1.2)$$

$$\frac{\partial w^k}{\partial x}+\frac{\partial u^k}{\partial z}=a_{55}^k\sigma_{xz}^k,\ \frac{\partial w^k}{\partial y}+\frac{\partial v^k}{\partial z}=a_{44}^k\sigma_{yz}^k.$$

и следующих граничных условиях на лицевых поверхностях пластинки:

$$\sigma_{xz}^I(x,y,h+h_1,t)=\sigma_{xz}^+(\xi,\eta)\exp(i\Omega t),$$

$$\sigma_{yz}^I(x,y,h+h_1,t)=\sigma_{yz}^+(\xi,\eta)\exp(i\Omega t),$$

$$\sigma_{zz}^I(x,y,h+h_1,t)=\sigma_{zz}^+(\xi,\eta)\exp(i\Omega t),\qquad(1.3)$$

$$\xi=x/l,\quad \eta=y/l,\quad l=\min(a,b),$$

$$\sigma_{xz}^{III}\left(x,y,-(h+h_1),t\right)=-\sigma_{xz}^{-}\left(\xi,\eta\right)\exp\left(i\Omega t\right),$$
$$\sigma_{yz}^{III}\left(x,y,-(h+h_1),t\right)=-\sigma_{yz}^{-}\left(\xi,\eta\right)\exp\left(i\Omega t\right),\qquad(1.4)$$
$$\sigma_{zz}^{III}\left(x,y,-(h+h_1),t\right)=-\sigma_{zz}^{-}\left(\xi,\eta\right)\exp\left(i\Omega t\right),$$

где Ω – частота внешнего воздействия, и условиях полного контакта между слоями

$$\sigma_{xz}^{I}(x,y,h,t)=\sigma_{xz}^{II}(x,y,h,t),\quad \sigma_{yz}^{I}(x,y,h,t)=\sigma_{yz}^{II}(x,y,h,t),$$
$$\sigma_{zz}^{I}(x,y,h,t)=\sigma_{zz}^{II}(x,y,h,t),\quad u^{I}(x,y,h,t)=u^{II}(x,y,h,t),\qquad(1.5)$$
$$v^{I}(x,y,h,t)=v^{II}(x,y,h,t),\quad w^{I}(x,y,h,t)=w^{II}(x,y,h,t).$$
$$\sigma_{xz}^{II}(x,y,-h,t)=\sigma_{xz}^{III}(x,y,-h,t),\quad \sigma_{yz}^{II}(x,y,-h,t)=\sigma_{yz}^{III}(x,y,-h,t),$$
$$\sigma_{zz}^{II}(x,y,-h,t)=\sigma_{zz}^{III}(x,y,-h,t),\quad u^{II}(x,y,-h,t)=u^{III}(x,y,-h,t),\qquad(1.6)$$
$$v^{II}(x,y,-h,t)=v^{III}(x,y,-h,t),\quad w^{II}(x,y,-h,t)=w^{III}(x,y,-h,t).$$

2. Асимптотическое решение задачи. Решение сформулированной задачи будем искать в виде

$$\sigma_{\alpha\beta}^{k}\left(x,y,z,t\right)=\sigma_{ij}^{k}\left(x,y,z\right)\exp\left(i\Omega t\right),$$
$$\alpha,\beta=x,y,z,\quad i,j=1,2,3,\quad k=I,II,III,\qquad(2.1)$$
$$\left(u^{k}(x,y,z,t),v^{k}(x,y,z,t),w^{k}(x,y,z,t)\right)=$$
$$=\left(u_x^{k}(x,y,z),u_y^{k}(x,y,z),u_z^{k}(x,y,z)\right)\exp\left(i\Omega t\right).$$

Перейдя в динамических уравнениях и соотношениях упругости к безразмерным координатам и перемещениям:

$$\xi=x/l,\quad \eta=y/l,\quad \zeta=z/H,\ U=u_x/l,\quad V=u_y/l,\quad W=u_z/l,\qquad(2.2)$$

и подставив (2.2) в эти преобразованные уравнения, получим сингулярно возмущённую малым параметром $\varepsilon=h/l$ систему, решение которой складывается из решений внутренней задачи (I^{int}) и пограничного слоя (I_b). Решение внутренней задачи будем искать в виде

$$\sigma_{ij}^{k\,\text{int}}=\varepsilon^{-1+s}\sigma_{ij}^{k(s)}(\xi,\eta,\zeta),\quad i,j=1,2,3,\quad s=\overline{0,N}$$
$$(U^{k\,\text{int}},V^{k\,\text{int}},W^{k\,\text{int}})=\varepsilon^{s}(U^{k(s)},V^{k(s)},W^{k(s)}),\ k=I,II,III,\qquad(2.3)$$

Подставив (2.3) в эту систему и приравняв коэффициенты при одинаковых степенях ε, получим непротиворечивую систему для определения величин $\sigma_{ij}^{k(s)},U^{k(s)},V^{k(s)},W^{k(s)}$:

Из этой системы все напряжения можно выразить через перемещения по формулам:

$$\sigma_{13}^{k(s)}=\frac{1}{a_{55}^{k}}\left(\frac{\partial U^{k(s)}}{\partial\zeta}+\frac{\partial W^{k(s-1)}}{\partial\xi}\right),\quad \sigma_{23}^{k(s)}=\frac{1}{a_{44}^{k}}\left(\frac{\partial V^{k(s)}}{\partial\zeta}+\frac{\partial W^{k(s-1)}}{\partial\eta}\right),$$

$$\sigma_{12}^{k(s)} = \frac{1}{a_{66}^k}\left(\frac{\partial U^{k(s-1)}}{\partial \eta} + \frac{\partial V^{k(s-1)}}{\partial \xi}\right),$$

$$\sigma_{11}^{k(s)} = \frac{1}{\Delta^k}\left(-A_{23}^k \frac{\partial W^{k(s)}}{\partial \zeta} + A_{22}^k \frac{\partial U^{k(s-1)}}{\partial \xi} - A_{12}^k \frac{\partial V^{k(s-1)}}{\partial \eta}\right), \qquad (2.4)$$

$$\sigma_{22}^{k(s)} = \frac{1}{\Delta^k}\left(-A_{13}^k \frac{\partial W^{k(s)}}{\partial \zeta} - A_{12}^k \frac{\partial U^{k(s-1)}}{\partial \xi} + A_{33}^k \frac{\partial V^{k(s-1)}}{\partial \eta}\right),$$

$$\sigma_{33}^{k(s)} = \frac{1}{\Delta^k}\left(A_{11}^k \frac{\partial W^{k(s)}}{\partial \zeta} - A_{23}^k \frac{\partial U^{k(s-1)}}{\partial \xi} - A_{13}^k \frac{\partial V^{k(s-1)}}{\partial \eta}\right),$$

где

$$A_{11}^k = a_{11}^k a_{22}^k - \left(a_{12}^k\right)^2, \; A_{12}^k = a_{12}^k a_{33}^k - a_{23}^k a_{13}^k, \; A_{13}^k = a_{11}^k a_{23}^k - a_{13}^k a_{12}^k, \; k = I, II, III$$

$$A_{22}^k = a_{22}^k a_{33}^k - \left(a_{23}^k\right)^2, \; A_{23}^k = a_{13}^k a_{22}^k - a_{12}^k a_{23}^k, \; A_{33}^k = a_{11}^k a_{33}^k - \left(a_{13}^k\right)^2, \qquad (2.5)$$

$$\Delta^k = a_{11}^k A_{22}^k - a_{12}^k A_{12}^k - a_{13}^k A_{23}^k, \quad \mathrm{A}_{ij}^{III} = A_{ij}^I, \; \Delta^{III} = \Delta^I,$$

$Q^{k(m)} \equiv 0$ при $m < 0$,

для определения $U^{k(s)}$ получим уравнение

$$\frac{\partial^2 U^{k(s)}}{\partial \zeta^2} + a_{55}^k \rho^k \Omega_*^2 U^{k(s)} = R_U^{k(s)}, \qquad (2.6)$$

$$R_U^{k(s)} = -a_{55}^k \left(\frac{\partial \sigma_{11}^{k(s-1)}}{\partial \xi} + \frac{\partial \sigma_{12}^{k(s-1)}}{\partial \eta}\right) - \frac{\partial^2 W^{k(s-1)}}{\partial \xi \partial \zeta},$$

для определения $V^{k(s)}$ получим уравнение

$$\frac{\partial^2 V^{k(s)}}{\partial \zeta^2} + a_{44}^k \rho^k \Omega_*^2 V^{k(s)} = R_V^{k(s)}, \qquad (2.7)$$

$$R_V^{k(s)} = -a_{44}^k \left(\frac{\partial \sigma_{12}^{k(s-1)}}{\partial \xi} + \frac{\partial \sigma_{22}^{k(s-1)}}{\partial \eta}\right) - \frac{\partial^2 W^{k(s-1)}}{\partial \eta \partial \zeta},$$

для определения $W^{k(s)}$ имеем уравнение

$$A_{11}^k \frac{\partial^2 W^{k(s)}}{\partial \zeta^2} + \Delta^k \rho^k \Omega_*^2 W^{k(s)} = R_W^{k(s)}, \qquad (2.8)$$

$$R_W^{k(s)} = -\Delta^k \left(\frac{\partial \sigma_{13}^{k(s-1)}}{\partial \xi} + \frac{\partial \sigma_{23}^{k(s-1)}}{\partial \eta}\right) + A_{23}^k \frac{\partial^2 U^{k(s-1)}}{\partial \xi \partial \zeta} + A_{13}^k \frac{\partial^2 V^{k(s-1)}}{\partial \eta \partial \zeta},$$

Решениями уравнений (2.6)-(2.8) являются:

$$U^{k(s)} = C_1^{k(s)}(\xi,\eta)\sin\gamma_1^k\zeta + C_2^{k(s)}(\xi,\eta)\cos\gamma_1^k\zeta + U_\tau^{k(s)}(\xi,\eta,\zeta),$$
$$V^{k(s)} = C_3^{k(s)}(\xi,\eta)\sin\gamma_2^k\zeta + C_4^{k(s)}(\xi,\eta)\cos\gamma_2^k\zeta + V_\tau^{k(s)}(\xi,\eta,\zeta), \quad (2.9)$$
$$W^{k(s)} = C_5^{k(s)}(\xi,\eta)\sin\gamma_3^k\zeta + C_6^{k(s)}(\xi,\eta)\cos\gamma_3^k\zeta + W_\tau^{k(s)}(\xi,\eta,\zeta),$$

$$\gamma_1^k = \Omega_*\sqrt{\rho^k a_{55}^k},\ \gamma_2^k = \Omega_*\sqrt{\rho^k a_{44}^k},\ \gamma_3^k = \Omega_*\sqrt{\rho^k \Delta^k / A_{11}^k},\ \gamma_i^{III} = \gamma_i^I$$

где $U_\tau^{k(s)}, V_\tau^{k(s)}, W_\tau^{k(s)}$ – частные решения этих уравнений.

Подставив значения $U^{k(s)}, V^{k(s)}, W^{k(s)}$ в (2.4), для напряжений $\sigma_{13}^{k(s)}, \sigma_{23}^{k(s)}, \sigma_{33}^{k(s)}$ будем иметь:

$$\sigma_{13}^{k(s)} = \Omega_*\sqrt{\rho^k / a_{55}^k}\left(C_1^{k(s)}(\xi,\eta)\cos\gamma_1^k\zeta - C_2^{k(s)}(\xi,\eta)\sin\gamma_1^k\zeta\right) + f_{13}^{k(s)}(\xi,\eta,\zeta),$$

$$\sigma_{23}^{k(s)} = \Omega_*\sqrt{\rho^k / a_{44}^k}\left(C_3^{k(s)}(\xi,\eta)\cos\gamma_2^k\zeta - C_4^{k(s)}(\xi,\eta)\sin\gamma_2^k\zeta\right) + f_{23}^{k(s)}(\xi,\eta,\zeta), \quad (2.10)$$

$$\sigma_{33}^{k(s)} = \Omega_*\sqrt{\rho^k A_{11}^k / \Delta^k}\left(C_5^{k(s)}(\xi,\eta)\cos\gamma_3^k\zeta - C_6^{k(s)}(\xi,\eta)\sin\gamma_3^k\zeta\right) + f_{33}^{k(s)}(\xi,\eta,\zeta),$$

где

$$f_{13}^{k(s)} = \frac{1}{a_{55}^k}\left(\frac{\partial U_\tau^{k(s)}}{\partial\zeta} + \frac{\partial W^{k(s-1)}}{\partial\xi}\right),\quad f_{23}^{k(s)} = \frac{1}{a_{44}^k}\frac{\partial V_\tau^{k(s)}}{\partial\zeta} + \frac{1}{a_{55}^k}\frac{\partial W^{k(s-1)}}{\partial\eta},$$
$$f_{33}^{k(s)} = \frac{A_{11}^k}{\Delta^k}\frac{\partial W_\tau^{k(s)}}{\partial\zeta} - \frac{1}{\Delta^k}\left(A_{23}^k\frac{\partial U^{k(s-1)}}{\partial\xi} + A_{13}^k\frac{\partial V^{k(s-1)}}{\partial\eta}\right), \quad (2.11)$$

Удовлетворив условиям (1.3)-(1.6) и решив соответствующую алгебраическую систему, определим все функции $C_j^{k(s)}$:

$$C_1^{I(s)} = \frac{1}{B_1^I}(\mathrm{d}_1^{(s)} + B_2^I C_2^{I(s)}),\ C_1^{II(s)} = \frac{1}{2g_1}\left(m_1\left(d_{16}^{(s)} + d_{13}^{(s)}\right) + m_2\left(b_1 d_{10}^{(s)} - d_7^{(s)}\right)\right),$$

$$C_1^{III(s)} = \frac{1}{B_1^I}(\mathrm{d}_4^{(s)} - B_2^I C_2^{III(s)}),\ C_2^{I(s)} = \frac{1}{m_2}\left(d_{13}^{(s)} + B_2^{II0} C_1^{II(s)} + B_1^{II0} C_2^{II(s)}\right),$$

$$C_2^{II(s)} = \frac{1}{2g_2}\left(m_1\left(d_{16}^{(s)} - d_{13}^{(s)}\right) + m_2\left(b_1 d_{10}^{(s)} + d_7^{(s)}\right)\right), \quad (2.12)$$

$$C_2^{III(s)} = \frac{b_1}{m_1}\left(d_{10}^{(s)} - B_1^{II0} C_1^{II(s)} - B_2^{II0} C_2^{II(s)}\right),$$

$$C_3^{I(s)} = \frac{1}{B_3^I}(d_2^{(s)} + B_4^I C_4^{I(s)}),\quad C_3^{II(s)} = \frac{1}{2g_3}\left(m_3\left(d_{17}^{(s)} + d_{14}^{(s)}\right) + m_4\left(b_2 d_{11}^{(s)} - d_8^{(s)}\right)\right),$$

$$C_3^{III(s)} = \frac{1}{B_3^I}(d_5^{(s)} - B_4^I C_4^{III(s)}),\ C_4^{I(s)} = \frac{1}{m_4}\left(d_{14}^{(s)} + B_4^{II0} C_3^{II(s)} + B_3^{II0} C_4^{II(s)}\right),$$

$$C_4^{II(s)} = \frac{1}{2g_4}\left(m_3\left(d_{17}^{(s)} - d_{14}^{(s)}\right) + m_4\left(b_2 d_{11}^{(s)} + d_8^{(s)}\right)\right),$$

$$C_4^{III(s)} = \frac{b_2}{m_3}\left(d_{11}^{(s)} - B_3^{II0} C_3^{II(s)} - B_4^{II0} C_4^{II(s)}\right),$$

$$C_5^{I(s)} = \frac{1}{B_5^I}(\mathrm{d}_3^{(s)} + B_6^I C_6^{I(s)}), \quad C_5^{II(s)} = \frac{1}{2g_5}\left(m_5\left(d_{18}^{(s)} + d_{15}^{(s)}\right) + m_6\left(b_3 d_{12}^{(s)} - d_9^{(s)}\right)\right),$$

$$C_5^{III(s)} = \frac{1}{B_5^I}(\mathrm{d}_6^{(s)} - B_6^I C_6^{III(s)}),\ C_6^{I(s)} = \frac{1}{m_6}\left(d_{15}^{(s)} + B_6^{II0} C_5^{II(s)} + B_5^{II0} C_6^{II(s)}\right),$$

$$C_6^{II(s)} = \frac{1}{2g_6}\left(m_5\left(d_{18}^{(s)} - d_{15}^{(s)}\right) + m_6\left(b_3 d_{12}^{(s)} + d_9^{(s)}\right)\right),$$

$$C_6^{III(s)} = \frac{b_3}{m_5}\left(d_{12}^{(s)} - B_5^{II0} C_5^{II(s)} - B_6^{II0} C_6^{II(s)}\right),$$

где

$$m_1 = \frac{\sin\gamma_1^I(\zeta_1 - \zeta_0)}{\cos\gamma_1^I\zeta_1}, \quad m_2 = \frac{\cos\gamma_1^I(\zeta_1 - \zeta_0)}{\cos\gamma_1^I\zeta_1},$$

$$b_1 = \sqrt{\frac{a_{55}^I\rho^{II}}{a_{55}^{II}\rho^I}},\ b_2 = \sqrt{\frac{a_{44}^I\rho^{II}}{a_{44}^{II}\rho^I}},\ b_3 = \sqrt{\frac{A_{11}^{II}\rho^{II}\Delta^I}{A_{11}^I\rho^I\Delta^{II}}},$$

$$g_1 = \frac{1}{2\cos\gamma_1^I\zeta_1}\left[(\mathrm{b}_1 - 1)\cos(\gamma_1^I\zeta_1 - (\gamma_1^I + \gamma_1^{II})\zeta_0) + (\mathrm{b}_1 + 1)\cos(\gamma_1^I\zeta_1 + (\gamma_1^{II} - \gamma_1^I)\zeta_0)\right],$$

$$g_2 = \frac{1}{2\cos\gamma_1^I\zeta_1}\left[(\mathrm{b}_1 - 1)\sin(-\gamma_1^I\zeta_1 + (\gamma_1^I + \gamma_1^{II})\zeta_0) + (\mathrm{b}_1 + 1)\sin(\gamma_1^I\zeta_1 + (\gamma_1^{II} - \gamma_1^I)\zeta_0)\right],$$

$$(\mathrm{g}_1, \mathrm{g}_3; \gamma_1^i, \gamma_2^i; b_1, b_2),\ (\mathrm{g}_2, \mathrm{g}_4; \gamma_1^i, \gamma_2^i; b_1, b_2),\ (m_1, m_3; \gamma_1^i, \gamma_2^i),\ (m_2, m_4; \gamma_1^i, \gamma_2^i),$$

$$(\mathrm{g}_1, \mathrm{g}_5; \gamma_1^i, \gamma_3^i; b_1, b_3),\ (\mathrm{g}_2, \mathrm{g}_6; \gamma_1^i, \gamma_3^i; b_1, b_3),\ (m_1, m_5; \gamma_1^i, \gamma_3^i),\ (m_2, m_6; \gamma_1^i, \gamma_3^i),$$

$$B_1^I = \cos\gamma_1^I\zeta_1,\ B_2^I = \sin\gamma_1^I\zeta_1, \quad B_1^{i0} = \cos\gamma_1^i\zeta_0,\ B_2^{i0} = \sin\gamma_1^i\zeta_0,$$
$$B_3^I = \cos\gamma_2^I\zeta_1,\ B_4^I = \sin\gamma_2^I\zeta_1, \quad B_3^{i0} = \cos\gamma_2^i\zeta_0,\ B_4^{i0} = \sin\gamma_2^i\zeta_0,\ i = \mathrm{I}, \mathrm{II}, \qquad (2.13)$$
$$B_5^I = \cos\gamma_3^I\zeta_1,\ B_6^I = \sin\gamma_3^I\zeta_1, \quad B_5^{i0} = \cos\gamma_3^i\zeta_0,\ B_6^{i0} = \sin\gamma_3^i\zeta_0,$$

$$d_1^{(s)} = \frac{1}{\Omega_*}\sqrt{a_{55}^I/\rho^I}\left(\sigma_{xz}^+ - f_{13}^{I(s)}(\xi, \eta, \zeta_1)\right), d_2^{(s)} = \frac{1}{\Omega_*}\sqrt{a_{44}^I/\rho^I}\left(\sigma_{yz}^+ - f_{23}^{I(s)}(\xi, \eta, \zeta_1)\right),$$

$$d_3^{(s)} = \frac{1}{\Omega_*}\sqrt{\frac{\Delta^I}{A_{11}^I\rho^I}}\left(\sigma_{zz}^+ - f_{33}^{I(s)}(\xi, \eta, \zeta_1)\right),\ d_4^{(s)} = -\frac{1}{\Omega_*}\sqrt{a_{55}^I/\rho^I}\left(\sigma_{xz}^- + f_{13}^{III(s)}(\xi, \eta, -\zeta_1)\right),$$

$$d_5^{(s)} = -\frac{1}{\Omega_*}\sqrt{a_{44}^I/\rho^I}\left(\sigma_{yz}^- + f_{23}^{III(s)}(\xi,\eta,-\zeta_1)\right),\ d_6^{(s)} = -\frac{1}{\Omega_*}\sqrt{\frac{\Delta^I}{A_{11}^I\rho^I}}\left(\sigma_{zz}^- + f_{33}^{III(s)}(\xi,\eta,-\zeta_1)\right),$$

$$d_7^{(s)} = \frac{1}{\Omega_*}\sqrt{a_{55}^I/\rho^I}\left(f_{13}^{II(s)}(\xi,\eta,\zeta_0) - f_{13}^{I(s)}(\xi,\eta,\zeta_0)\right) - \frac{B_1^{I0}}{B_1^I}d_1^{(s)},$$

$$d_8^{(s)} = \frac{1}{\Omega_*}\sqrt{a_{44}^I/\rho^I}\left(f_{23}^{II(s)}(\xi,\eta,\zeta_0) - f_{23}^{I(s)}(\xi,\eta,\zeta_0)\right) - \frac{B_3^{I0}}{B_3^I}d_2^{(s)},$$

$$d_9^{(s)} = \frac{1}{\Omega_*}\sqrt{\frac{\Delta^I}{A_{11}^I\rho^I}}\left(f_{33}^{II(s)}(\xi,\eta,\zeta_0) - f_{33}^{I(s)}(\xi,\eta,\zeta_0)\right) - \frac{B_5^{I0}}{B_5^I}d_3^{(s)},$$

$$d_{10}^{(s)} = \frac{1}{\Omega_*}\sqrt{a_{55}^{II}/\rho^{II}}\left(f_{13}^{III(s)}(\xi,\eta,-\zeta_0) - f_{13}^{II(s)}(\xi,\eta,-\zeta_0)\right) + \frac{B_1^{I0}}{b_1B_1^I}d_4^{(s)},$$

$$d_{11}^{(s)} = \frac{1}{\Omega_*}\sqrt{a_{44}^{II}/\rho^{II}}\left(f_{23}^{III(s)}(\xi,\eta,-\zeta_0) - f_{23}^{II(s)}(\xi,\eta,-\zeta_0)\right) + \frac{B_3^{I0}}{b_2B_3^I}d_5^{(s)},$$

$$d_{12}^{(s)} = \frac{1}{\Omega_*}\sqrt{\frac{\Delta^{II}}{A_{11}^{II}\rho^{II}}}\left(f_{33}^{III(s)}(\xi,\eta,-\zeta_0) - f_{33}^{II(s)}(\xi,\eta,-\zeta_0)\right) + \frac{B_5^{I0}}{b_3B_5^I}d_6^{(s)},$$

$$d_{13}^{(s)} = U_\tau^{II(s)} - U_\tau^{I(s)} - \frac{B_2^{I0}}{B_1^I}d_1^{(s)},\quad d_{14}^{(s)} = V_\tau^{II(s)} - V_\tau^{I(s)} - \frac{B_4^{I0}}{B_3^I}d_2^{(s)},$$

$$d_{15}^{(s)} = W_\tau^{II(s)} - W_\tau^{I(s)} - \frac{B_6^{I0}}{B_5^I}d_3^{(s)},\quad d_{16}^{(s)} = U_\tau^{III(s)} - U_\tau^{II(s)} - \frac{B_2^{I0}}{B_1^I}d_4^{(s)},$$

$$d_{17}^{(s)} = V_\tau^{III(s)} - V_\tau^{II(s)} - \frac{B_4^{I0}}{B_3^I}d_5^{(s)},\quad d_{18}^{(s)} = W_\tau^{III(s)} - W_\tau^{II(s)} - \frac{B_6^{I0}}{B_5^I}d_6^{(s)},$$

Решение будет конечным, если $B_i \neq 0,\ m_i \neq 0,\ g_i \neq 0,\ i = 1,2,3,...,6$. Если же какое-либо из этих величин обращается в ноль, то будет возникать резонанс.

Решение внутренней задачи становится математически точным, если входящие в граничные условия (1.3), (1.4) функции являются многочленами. В качестве иллюстрации приведём решение, соответствующее условиям:

$$\begin{gathered}\sigma_{zz}^+(\zeta_1) = -Z_1^+ = \text{const},\quad \sigma_{zz}^-(\zeta_1) = -Z_2^+ = \text{const},\\ \sigma_{xz}^+ = 0,\quad \sigma_{yz}^+ = 0,\ \sigma_{xz}^- = 0,\quad \sigma_{yz}^- = 0,\end{gathered} \tag{2.14}$$

$$C_5^{I(0)} = \frac{1}{B_5^I}(\mathrm{d}_3^{(0)} + B_6^I C_6^{I(0)}),\quad C_5^{II(0)} = -\frac{1}{2g_5B_5^I\Omega_*}\sqrt{\frac{\Delta^I}{A_{11}^I\rho^I}}(Z_1^+ + Z_2^-),$$

$$C_5^{III(0)} = \frac{1}{B_5^I}(\mathrm{d}_6^{(0)} - B_6^I C_6^{III(0)}),$$

$$C_6^{II(0)}=\frac{1}{\Omega_*}\sqrt{\frac{\Delta^I}{A_{11}^I\rho^I}}\Big[(\mathrm{b}_1-1)\sin(-\gamma_1^I\zeta_1+(\gamma_1^I+\gamma_1^{II})\zeta_0)+$$

$$+(\mathrm{b}_1+1)\sin(\gamma_1^I\zeta_1+(\gamma_1^{II}-\gamma_1^I)\zeta_0)\Big]^{-1}(\mathrm{Z}_1^+-\mathrm{Z}_2^-),$$

$$C_6^{I(0)}=\frac{1}{m_6}\left(d_{15}^{(0)}+B_6^{II0}C_5^{II(0)}+B_5^{II0}C_6^{II(0)}\right),\qquad(2.15)$$

$$C_6^{III(0)}=\frac{b_3}{m_5}\left(d_{12}^{(0)}-B_5^{II0}C_5^{II(0)}-B_6^{II0}C_6^{II(0)}\right),\qquad C_{1,2,3,4}^{I,II,III}=0,$$

$$W^{k(0)}=C_5^{k(0)}(\xi,\eta)\sin\gamma_3^k\zeta+C_6^{k(0)}(\xi,\eta)\cos\gamma_3^k\zeta,$$

$$\sigma_{11}^{k(0)}=-\frac{A_{23}^k}{\Delta^k}\frac{\partial W^{k(0)}}{\partial\zeta},\ \ \sigma_{22}^{k(0)}=-\frac{A_{13}^k}{\Delta^k}\frac{\partial W^{k(0)}}{\partial\zeta},\ \ \sigma_{33}^{k(0)}=\frac{A_{11}^k}{\Delta^k}\frac{\partial W^{k(0)}}{\partial\zeta},$$

$$u^k=0,\ v^k=0,\ \sigma_{12}^k=0,\ \ \sigma_{13}^k=0,\ \ \sigma_{23}^k=0,\ w^k=lW^{k(0)}\exp(i\omega\mathrm{t}),$$

$$\sigma_{xx}^k=\varepsilon^{-1}\sigma_{11}^{k(0)}\exp(i\omega\mathrm{t}),\ \sigma_{yy}^k=\varepsilon^{-1}\sigma_{22}^{k(0)}\exp(i\omega\mathrm{t}),\ \sigma_{zz}^k=\varepsilon^{-1}\sigma_{33}^{k(0)}\exp(i\omega\mathrm{t}),$$

Заключение. Решена пространственная первая динамическая краевая задача теории упругости для трёхслойной ортотропной пластинки симметричной структуры. Асимптотическим методом построен итерационный процесс для определения компонент тензора напряжений и вектора перемещения во внутренней задаче. Показано, что если внешнее воздействие есть многочлен по тангенциальным координатам, то итерационный процесс обрывается и получается математически точное решение внутренней задачи. Приведён иллюстрационный пример.

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Поступила в редакцию 12.10.2016

ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 70, №2, 2017 Механика

RESONANCE AND LOCALIZED SHEAR VIBRATION OF BI-MATERIAL ELASTIC RESONATOR

Ghazaryan K.B., Papyan A.A.

Key words: Shear waves, localized waves, resonator, internal resonance.
Ключевые слова: сдвиговые волны, локализованные волны, внутренний резонанс.
Բանալի բառեր. սահքի ալիքներ, տեղայնացված ալիքներ, ներքին ռեզոնանս

Ղազարյան Կ.Բ., Պապյան Ա.Ա.

Ռեզոնանսային և տեղայնացված սահքի տատանումները բաղադրյալ առաձգական ռեզոնատորում

Աշխատանքը նվիրված է բաղադրյալ ռեզոնատորում սահքի տատանումների խնդրին, երբ ռեզոնատորը բաղկացած է երկու տարբեր առաձգական նյութերից և ունի ուղղանկյուն հատույթ: Դիտարկվել է այն դեպքը, երբ ռեզոնատորի մի կողմը ազատ է լարումներից, իսկ մնացած երեք կողմերը կոշտ ամրակցված են: Ցույց է տրված, որ գոյություն ունեն երկու տիպի տատանումներ՝տեղայանացված և սեփական: Ցույց է տրված երկու տարբեր տիրույթներում տեղայանցված և սեփական տատանումների հաճախությունների համընկնման հնարավորությունը:

Казарян К.Б., Папян А.А.

Резонансные и локализованные сдвиговые колебания в упругом составном резонаторе

В работе рассмотрена задача сдвиговых колебаний составного резонатора прямоугольного сечения, состоящего из двух различных упругих материалов, когда одна из сторон резонатора свободна от напряжений, а остальные три жёстко закреплены. Установлено существование двух различных типов колебаний: локализованных и собственных. Показана возможность совпадения локализованных и собственных частот колебаний различных форм, приводяшее к эффекту внутреннего резонанса.

The paper is dedicated to the problem of shear vibration of compound resonator, made from two different elastic materials, with rectangular cross section, when one side of the resonator is traction free, three other sides are clamped. The existence of two different types of vibration, namely localized and natural types are established. Possibility of coinciding of localized and natural frequencies from two different spectrums are shown, resulting in the internal resonance occurrence that does not exist in one phase material resonator, with ordinary boundary conditions.

Introduction.

A number of studies and reviews devoted to specific cases of localized waves edge resonance in elastic systems are presented in [1].The correlation between effects of resonance and localisation of shear waves in elasic resonator were have been firstly reported in a modal problem [2], where was shown that due to vibration localisation frequencies the internal resonance can occur. In [3] classical compound systems are analyzed formed by the pairs of coupled resonators, including a system of elastically coupled masses, a system of rigid rods separated by a notch, and an optical system made by a pair of dielectric films separated by a thin metallic layer. Non linear effects in elastic resonators are considered in [4].

Statement of the problem.

In Cartesian system (x, y, z) a two phase bi-material elastic resonator is considered, occupying a region $-b \le x \le a;\ 0 \le y \le d;\ -\infty < z < \infty$. The resonator consists of the two different elastic materials: (1) of length b, bulk density $\rho^{(1)}$, shear modules $G^{(1)}$ and (2) of length a, bulk density $\rho^{(2)}$, shear modules $G^{(2)}$ (Fig.1).

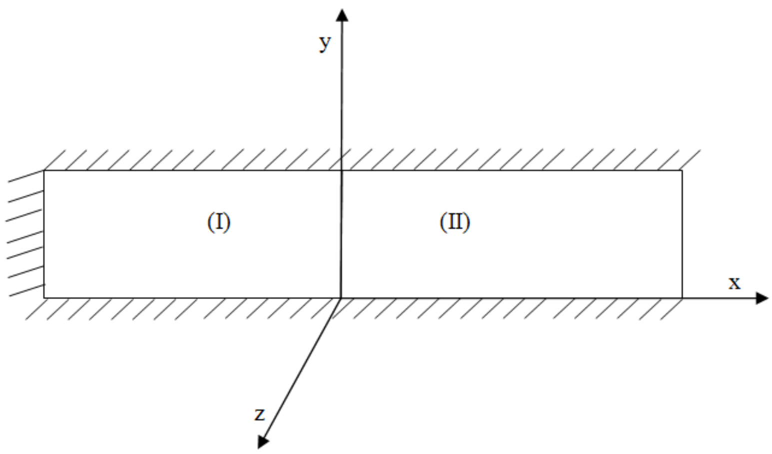

Fig.1. Resonator's cross-section

$s = 1,\ \ s = 2$ stand for first and second materials, correspondingly.

We take the following boundary conditions at the resonators walls

$$U^{(1)} = 0;\ \ U^{(2)} = 0 \ \ y = 0, \quad y = d, \tag{1}$$

$$U^{(1)} = 0 \ \ x = -b, \quad \frac{\partial U^{(2)}}{\partial x} = 0; \quad x = a. \tag{2}$$

We also take the ideal contact conditions of continuity for the displacements and the stresses of two different materials at the interface $x = 0$

$$U^{(1)} = U^{(2)}\ ; \quad G^{(1)} \frac{\partial U^{(1)}}{\partial x} = G^{(2)} \frac{\partial U^{(2)}}{\partial x}. \tag{3}$$

Solutions of the problem

The solutions of Eq.(1) satisfying boundary conditions at $y = 0, y = d$ we present in the form

$$U^{(s)}(x,y,t)=\sum_{n=1}^{\infty}U_{0n}^{(s)}(x)\sin(\lambda_n y)\exp(i\omega t)$$

$$\lambda_n=\frac{\pi n}{d};\ n=1,2,3..... \tag{4}$$

Functions $U_{0n}^{(s)}(x)$ satisfy the equations

$$\frac{d^2U_{0n}^{(1)}}{dx^2}+\lambda_1^2\left(\eta^2-n^2\right)U_0^{(1)}=0, \tag{5}$$

$$\frac{d^2U_{0n}^{(2)}}{dx^2}+\lambda_1^2\left(\alpha^2\eta^2-n^2\right)U_0^{(2)}=0,$$

Here $\eta=\dfrac{\omega}{\lambda_1 c_1};\quad c_1^2=\dfrac{G^{(1)}}{\rho^{(1)}};\ c_2^2=\dfrac{G^{(2)}}{\rho^{(2)}};\ \alpha=\dfrac{c_1}{c_2}$.

Solutions of Eq. (5) satisfying boundary condition $x=-b$ and contact conditions at $x=0$ can be written as

$$U_{0n}^{(1)}(x)=C\left(\sinh\left(x\lambda_1\sqrt{n^2-\eta^2}\right)+\tanh\left(b\lambda_1\sqrt{n^2-\eta^2}\right)\cosh\left(x\lambda_1\sqrt{n^2-\eta^2}\right)\right),$$

$$U_{0n}^{(2)}(x)=C\left(\frac{\gamma\sqrt{n^2-\eta^2}}{\sqrt{n^2-\alpha^2\eta^2}}\sinh\left(x\lambda_1\sqrt{n^2-\alpha^2\eta^2}\right)+\right.$$

$$\left.+\tanh\left(b\lambda_1\sqrt{n^2-\eta^2}\right)\cosh\left(x\lambda_1\sqrt{n^2-\alpha^2\eta^2}\right)\right) \tag{6}$$

Here C is an arbitrary constant, $\gamma=G^{(2)}/G^{(1)}$.

Satisfying solutions $U_{0n}^{(2)}(x)$ to the boundary condition at $x=a$ we get the dispersion equation defining dimensionless frequencies η

$$\frac{\gamma\sqrt{n^2-\eta^2}}{\sqrt{n^2-\alpha^2\eta^2}}+\tanh\left(b\lambda_1\sqrt{n^2-\eta^2}\right)\tanh\left(a\lambda_1\sqrt{n^2-\alpha^2\eta^2}\right)=0 \tag{7}$$

Analysis of dispersion equation

In the frequency regions $\eta<n$ if $\alpha\le 1$; $\eta<n\alpha^{-1}$ if $\alpha\ge 1$ the dispersion equation (7) has not solutions. In other regions of η the dispersion equation defines spectral correlations

for the resonator frequencies and may have two types of solution as it occurs in the problems of shear waves propogation in layered waveguides [5,6], and in the modal problem, considered in [2] .

The first type of solution gives a series of modes corresponding to natural vibration in the frequency regions $\eta > n\alpha^{-1}(\alpha < 1)$ or $\eta > n(\alpha > 1)$, $n = 1, 2.....$

The second type gives a series of modes corresponding to localized vibration in the frequency regions $m < \eta < m\alpha^{-1}$ if $\alpha < 1$ or $m\alpha^{-1} < \eta < m$ if $\alpha > 1$ $(m = 1, 2.....)$

In the frequency regions, $\eta > n\alpha^{-1}(\alpha < 1)$ or $\eta > n(\alpha > 1)$ we have the dispersion equation defining the spectrum of the resonator natural frequencies

$$\frac{\gamma\sqrt{\eta^2 - n^2}}{\sqrt{\alpha^2\eta^2 - n^2}} - \tan\left(b\lambda_1\sqrt{\eta^2 - n^2}\right)\tan\left(a\lambda_1\sqrt{\alpha^2\eta^2 - n^2}\right) = 0 \quad (8)$$

In the frequency region $m < \eta < m\alpha^{-1}$, $m = 1, 2....$ when $\alpha < 1$ me have the following dispersion equations defining the spectrum of the resonator localized frequencies

$$\frac{\gamma\sqrt{\eta^2 - m^2}}{\sqrt{m^2 - \alpha^2\eta^2}} + \tan\left(b\lambda_1\sqrt{\eta^2 - m^2}\right)\tanh\left(a\lambda_1\sqrt{m^2 - \alpha^2\eta^2}\right) = 0 \quad (9)$$

When $\alpha > 1$ in region $m\alpha^{-1} < \eta < m$ $m = 1, 2.....$ the dispersion equation defining the spectrum of the resonator localized frequencies can be written as

$$\frac{\gamma\sqrt{m^2 - \eta^2}}{\sqrt{\eta^2 - \alpha^2 m^2}} - \tanh\left(b\lambda_1\sqrt{m^2 - \eta^2}\right)\tan\left(a\lambda_1\sqrt{\eta^2 - \alpha^2 m^2}\right) = 0 \quad (10)$$

When $\alpha = 1$ the dispersion equation of natural frequencies as the form

$$\gamma - \tan\left(b\lambda_1\sqrt{\eta^2 - n^2}\right)\tan\left(a\lambda_1\sqrt{\eta^2 - n^2}\right) = 0 \quad (11)$$

while the dispersion equation of localized frequencies

$$\gamma + \tanh\left(b\lambda_1\sqrt{m^2 - \eta^2}\right)\tanh\left(a\lambda_1\sqrt{m^2 - \eta^2}\right) = 0\sqrt{b^2 - 4ac} \quad (12)$$

has no solutions.

α		η_* $a\lambda_1 = 1$ $b\lambda_1 = 0.5$	η_* $a\lambda_1 = 0.5$ $b\lambda_1 = 1$
0.1		22.70	17.17
0.3		17.51	12.27
0.5		11.10	11.38
0.7		9.22	10.42
0.9		7.77	9.69
1.0		7.19	9.48
1.1		6.79	9.30
1.3		6.22	8.89
1.5		5.77	8.25
1.7		5.31	7.55
1.9		4.85	6.94
2.0		4.63	6.79

Table 1. Minimal natural frequencies of the resonator first mode

Based on the numerical analysis of the dispersion equations (8,11) defining the natural frequencies η_*, in the Table 1 the data for the minimal natural frequencies η_* related to dependence from geometrical parameter α are presented for first mode $n = 1$ of the resonator oscillation. The numerical calculations have been carried out for resonators with parameters γ=0.5 $a\lambda_1 = 1$, $b\lambda_1 = 0.5$. Data of the Table 1 shows that the minimal frequencies decreasing with increase of α. On the other hand, the localized vibration frequencies increasing with increase of mode number m and in some cases the frequency of m mode of localized vibration may coincide with minimal frequency of natural vibration of $n = 1$ mode. The coincidence of these frequencies results in the effect of an internal resonance.

$a\lambda_1$	$b\lambda_1$	α	$\eta_{nat} \approx \eta_*$	m
0.1	0.1	0.5	31.46	24
0.6	0.2	0.7	17.24	14
0.09	0.03	0.3	68.52	33
0.5	0.1	0.5	31.47	24
2.5	5.0	2.0	1.35	2
0.6	2.0	1.7	1.83	2
0.5	5.0	1.2	1.65	2
2.5	5.0	1.2	1.62	2

Table 2. Resonance frequencies data

On the Table 3 the resonance frequencies of localized and natural frequencies are presented for different cases where the internal resonance occur. The number m corresponds to localized vibration frequencies modes, the minimal frequencies of natural vibration correspond to $n = 1$. The numerical calculations have been carried out for resonators with γ=0.5, for different cases of $a\lambda_1, b\lambda_1$.

Conclusion

Shear vibration of bi-material elastic resonator with rectangular cross section is considered, when one side of the resonator is traction free, three other sides are clamped. The corresponding dispersion equations are obtained defining spectral correlations for resonator frequencies. It is shown that dispersion equation may have two different kinds of frequency spectrums, namely natural frequency spectrum and localized frequency spectrum. The equation of frequency spectrums are analyzed numerically in detail. Possibility of coinciding (internal resonance) of frequencies from two spectrums are shown. Based on numerical analysis the resonance frequencies of localized and natural frequencies are presented for different cases where the internal resonance occur.

Acknowledgements

The authors gratefully acknowledge professor M.V.Belubekyan for helpful discussions of results of the paper.

This research was supported by the State Committee of Science of Armenia, Grant No. SCS 115T-2C031.

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Received 22.09.2016

ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 70, №2, 2017 Механика

UDC 539.3

ON THE SOLUTION TO INTEGRAL EQUATIONS OF ONE CLASS OF MIXED AND CONTACT PROBLEMS BY THE DEGENERATE KERNEL METHOD

Mkhitaryan S.M.

Key words: integral equations, degenerate kernels, contact problems, Shtaerman generalized problem, regular infinite system of equations.
Ключевые слова: интегральные уравнения, вырожденные ядра, контактные задачи, обобщённая задача Штаермана, регулярные бесконечные системы уравнений.
Բանալի բառեր՝ ինտեգրալ հավասարումներ, վերածվող կորիզներ, կոնտակտային խնդիրներ, Շտաերմանի կոնտակտային խնդիր

Մխիթարյան Ս.Մ.
Վերածվող կորիզների մեթոդով խառը և կոնտակտային խնդիրների մի դասի ինտեգրալ հավասարումների լուծման մասին

Դեֆորմացվող պինդ մարմնի մեխանիկայի խառը և կոնտակտային խնդիրների բավականաչափ լայն դաս նկարագրվում է սիմետրիկ կորիզներով Ֆրեդհոլմի երկրորդ սեռի ինտեգրալ հավասարումներով: Հոդվածում զարգացվում է այդ հավասարումների լուծման վերածվող կորիզների հայտնի մեթոդը: Շարադրված մեթոդիկան լուսաբանվում է մակերևութային կառուցվածքի հաշվառումով առաձգական կիսահարթությանը դրոշմի սեղմման Ի.Յա. Շտաերմանի կոնտակտային խնդրի ինտեգրալ հավասարման օրինակի վրա:

Мхитарян С.М.
О решении интегральных уравнений одного класса смешанных и контактных задач методом вырожденных ядер

Довольно широкий класс смешанных и контактных задач механики деформируемого твёрдого тела описывается интегральными уравнениями Фредгольма второго рода с симметрическими ядрами. Для решения таких уравнений в статье развивается известный метод вырожденных ядер.

Изложенная методика иллюстрируется на примере интегрального уравнения обобщённой контактной задачи И.Я. Штаермана о вдавливании штампа в упругую полуплоскость с учётом поверхностной структуры основания.

A fairly wide class of mixed and contact problems of mechanics of deformable solids is described by Fredholm integral equations of the second kind with symmetric kernels. For solving such equations, a well-known method of degenerate kernels is developed in the paper. The stated methodology is illustrated on the example of an integral equation of the E.Ja. Shtaerman generalized contact problem on indentation of a punch into an elastic half-plane taking into account the surface structure of the base.

Introduction. The method of integral equations being one of the effective methods of solution of mixed and contact problems of mechanics of deformable solids was widely applied in numerous investigations [1-9]. By the method of Green function, such problems are directly reduced to Fredholm integral equations (IE) of the first kind as well, but most of them can be transformed into Fredholm equations of the second kind. The latter equations can also directly arise in contact problems. This is the case in the problem of contact interaction between the elastic bodies taking into account the factor of the surface structure of the bodies contacting between each other, usually the factor of roughness by Shtaerman model of contact [1]. According to this model, because of the local deformations, the arising local displacements in each point of the contact zone are proportional to the contact stress at the very point. In such formulation in [10] an axially symmetric contact problem on indentation of a punch, circular in the plan, into a rough elastic half-space, also described by Fredholm IE of the second kind, is considered.

Numerous effective methods of solving the Fredholm IE of the second kind [11-13] are developed and among them the Fredholm method of reducing the original IE to the system of linear algebraic equations (SLAE) holds a special place. The procedure of reducing to SLAE is greatly simplified in case of degenerate kernels of IE. That is why the method of the degenerate kernels of IE solution, when the original kernel is approximated by the degenerate kernel with great exactness, has got an intensive development [11-13].

In the present paper, the method of degenerate kernels is applied to solving the Fredholm IE of the second kind with symmetrical kernels, by which integral operators with discrete spectra are generated and for these operators corresponding spectral relationship are well-known. The idea of the paper lies in the fact that based on the spectral relationship bilinear expansions of the kernels in the form of infinite series are written, then these infinite series are replaced by the finite series and, by that, the original kernels are approximated by degenerate kernels.

There is a list of symmetric kernels, for which the spectral relationship [7, 8, 14, 15] of Fredholm IE of the second kind are well known; with such kernels in the framework of the above mentioned E.Ja. Shtaerman contact model a wide class of contact problems is described. The method of degenerated kernels is concretely illustrated here on the example of E.Ja. Shtaerman generalized problem [1] on indentation of a punch of the general configuration into an elastic half-plane. It is proved that the approximate solution by the method of degenerate kernels, as the number of summands of the finite series increases infinitely, tends to the exact solution of the problem. For this purpose the issue of regularity of the corresponding infinite SLAE is investigated. In particular cases the numerical analysis of the problem is conducted.

1. General preconditions of the method of degenerate kernels.

Let us have Fredholm IE of the second kind

$$\varphi(x)+\lambda\int_L K(x,s)\varphi(s)\,ds = f(x) \qquad (x\in L) \tag{1.1}$$

with symmetrical quadratically summarized on $L\times L$ by kernel $K(x,s)$, where L is a finite or infinite interval of the numerical axis. The integral operator K, originated by kernel $K(x,s)(x,s\in L\times L)$, has discrete specter and for it let spectral relationships take place

$$\int_L K(x,s)\varphi_n(s)w(s)\,ds = \lambda_n\varphi_n(x) \quad (n=0,1,2,...). \tag{1.2}$$

Here λ_n are eigen values, $\varphi_n(x)$ are eigen functions, composing full orthogonal systems in space $L_2(L)$, and $w(x)$ is non-negative weight function by which functions $\varphi_n(x)$ are orthogonal:

$$\int\limits_L \varphi_n(x)\varphi_m(x)w(x)dx = \begin{cases} 0 & (m \neq n); \\ h_n & (m = n). \end{cases} \tag{1.3}$$

As kernels $K(x,s)$ with above properties the following kernels can be taken

1) $\ln\frac{1}{|x-s|}$ $\quad L=(-a,a)$; 2) $\ln\frac{x+s}{|x-s|}$ $\quad L=(b,a)\quad (b>0)$;

3) $\ln\frac{1}{2\sin(|x-s|/2)}$ $\quad L=(-\alpha,\alpha)$; 4) $\frac{1}{|x-s|^{\mu}}$ $\quad (0<\mu<1/2)$; $\quad L=(-a,a)$;

5) $K_{\mu}(|x-s|)/|x-s|^{\mu}$ $(|\mu|<1/2)$; $\quad L=(-a,a)$; $\quad L=(0,\infty)$;

6) $\int\limits_0^{\infty} J_{\mu}(\lambda x)J_{\mu}(\lambda s)\lambda^{2\gamma}d\lambda\,(\chi\geq 0; |\gamma|<1/2)$; $\qquad L=(0,a)$;

7) $2\int\limits_0^{\pi}\frac{e^{-\chi_0\sqrt{x^2+s^2-2xs\cos u}}}{\sqrt{x^2+s^2-2xs\cos u}}\cos(mn)du$ $\;(\chi_0>0,\ m=0,1,2,...)$; $\;L=(0,a)$.

Here $K_{\mu}(x)$ – Macdonald's function of index μ and $J_{\mu}(x)$ – Bessel's function of first kind of index μ.

Fredholm IEs of the first kind with these kernels describe numerous mixed and contact problems of mechanics of deformable solids. In [7, 8, 14, 15], as well as in papers cited in [14, 15] for such kernels spectral relationships of type (1.2) and related to them integral relationships are established.

Now for the function $f(x)$ from $L_2(L)$ we write the formulas of Fourier generalized transformations in the system of functions $\varphi_n(x)$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n\varphi_n(x) \quad (x\in L) \\ a_n &= \frac{1}{h_n}\int\limits_L f(x)\varphi_n(x)w(x)dx \quad (n=0,1,2,...). \end{aligned} \tag{1.4}$$

Using formulas (1.4) for kernel $K(x,s)$ at fixed x, the following bilinear expansion of the kernel in the system of functions $\varphi_n(x)$ will be obtained:

$$K(x,s) = \sum_{m=0}^{\infty}\frac{\lambda_m}{h_m}\varphi_m(x)\varphi_m(s) \quad ((x,s)\in L\times L). \tag{1.5}$$

If in expansion (1.5) we replace the infinite sum with the finite sum restricting the number of terms by n, then thereby kernel $K(x,s)$ will be approximated by the degenerate kernel $K_n(x,s)$:

$$K(x,s) \approx K_n(x,s) = \sum_{m=0}^{n}\frac{\lambda_m}{h_m}\varphi_m(x)\varphi_m(s). \tag{1.6}$$

Further, in IE (1.1) kernel $K(x,s)$ is replaced by $K_n(x,s)$ from (1.6). After the simple transformations we shall have

$$\varphi(x)+\lambda\sum_{m=0}^{n}\frac{\lambda_m}{h_m}X_m\varphi_m(x)=f(x) \quad (x\in L)$$
$$X_m=\int_L \varphi(s)\varphi_m(s)\,ds \qquad \left(m=\overline{0,n}\right). \tag{1.7}$$

From here the approximate solution of the original IE (1.1) will be in the form of

$$\varphi(x)\approx f(x)-\lambda\sum_{m=0}^{n}\frac{\lambda_m}{h_m}X_m\varphi_m(x) \quad (x\in L), \tag{1.8}$$

of course, if the coefficients X_m are already determined. For the determination of these coefficients we multiply both parts of (1.7) by $\varphi_k(x)$ $\left(k=\overline{0,n}\right)$ and integrate the obtained equality over the interval L. As a result, we come to the following SLAE:

$$X_k+\lambda\sum_{m=0}^{n}\frac{\lambda_m}{h_m}R_{km}X_m=f_k \quad \left(k=\overline{0,n}\right) \tag{1.9}$$
$$R_{km}=\int_L \varphi_m(x)\varphi_k(x)\,dx \quad \left(k,m=\overline{0,n}\right); \quad f_k=\int_L f(x)\varphi_k(x)\,dx.$$

Thus, the method of degenerate kernels in the above described form reduces the solution of the original IE to the solution of SLAE (1.9).

Note, that in paper [10] with the help of bilinear expansion (1.5) for a symmetric kernel in the form of Veber-Sonin integral the solution of corresponding Fredholm IE of the second kind is reduced to the solution of the regular infinite SLAE. In paper [16] the method of reduction of the general class of integral equations with the symmetric quadratically summable difference or summation or difference-summation kernels to regular infinite SLAE is suggested. Moreover, by means of expanding the kernel function in Fourier cosine-series or in the series of other complete orthogonal systems of functions bilinear expansion of (1.5) type is applied. However, for the noted above class of kernels the application of expansions (1.5) in eigenvalue functions of kernels is more convenient and the use of degenerate kernels technique based on above expansions turns to be more simple.

This method is applicable to the solution of IE of I.Ja. Shtaerman generalized contact problem [1].

2. The formulation of the contact problem and derivation of basic equations. Generalizing the I.Ja. Shtaerman contact problem [1], we assume that the absolutely rigid punch, the base of which in the cross-section cut by the plane Oxy is described by the equation $y=f(x)$, is indented under the influence of the central vertical force P and overturning moment M into the elastic half-plane with Young module E and Poisson coefficients ν (Fig.1).

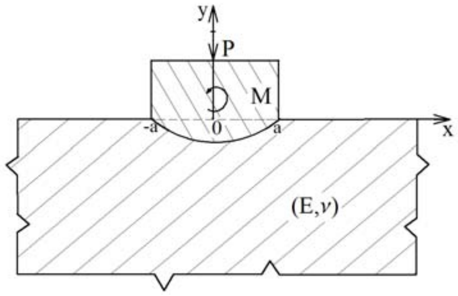

Fig.1

Here, instead of the Hertz smooth contact model we take the I.Ja. Shtaerman contact model [1] which takes into account the factor of the surface structure of deformable bodies contacting between themselves. According to this model the vertical displacements of the boundary points of the elastic half-plane are consisted of two summands. The first summand [1]

$$\mathrm{v}(x) = -\vartheta\int_{-a}^{a} \ln\frac{a}{|x-s|}\,p(s)\,ds + C \qquad \left(-\infty < x < \infty;\ \ \vartheta = \frac{2\left(1-\nu^2\right)}{\pi E}\right)$$

arises in consequence of global deformation of the elastic body caused by the applied in the contact area $-a < x < a$ pressure $p(x)$ of the punch on the foundation in accordance with the differential equations of linear elasticity theory. The second summand $\mathrm{v}_0(x)$ arises in consequence of local deformations, conditioned by roughness (non-smoothness) of the contact surface, and it is considered, that at each point of the contact area it is proportional to the pressure $p(x)$ at the same point: $\mathrm{v}_0(x) = -\chi\, p(x)$, where χ is some coefficient, depending on the surface structure of the elastic body. Eventually, for the vertical displacements $\mathrm{v}_2(x)$ of the boundary points of the elastic half-plane we shall have

$$\mathrm{v}_2(x) = \mathrm{v}(x) + \mathrm{v}_0(x) = -\chi p(x) - \vartheta\int_{-a}^{a} \ln\frac{a}{|x-s|}p(s)\,ds + C \quad (-a < x < a). \tag{2.1}$$

On the other hand, the vertical displacements $\mathrm{v}_1(x)$ of the punch, as an absolutely rigid body, have the form of

$$\mathrm{v}_1(x) = \Delta + \alpha x \quad (-a < x < a), \tag{2.2}$$

where α is the angle of the rigid rotation of the punch, and Δ is its settling.

Now, substituting (2.1) and (2.2) into the contact condition [1]

$$\mathrm{v}_1(x) - \mathrm{v}_2(x) = \delta - f(x) \quad (-a < x < a),$$

for the determination of the unknown contact pressure, we obtain the following Fredholm IE of the second kind:

$$\chi p(x) + \vartheta\int_{-a}^{a} \ln\frac{a}{|x-s|}p(s)\,ds = \delta - \alpha x - f(x) \quad (-a < x < a). \tag{2.3}$$

Here, $\delta - \Delta + C$ is denoted by δ.

The governing IE (GIE) (2.3) should be considered under the conditions of the punch equilibrium

$$\int_{-a}^{a} p(x)dx = P; \quad \int_{-a}^{a} xp(x)dx = M. \tag{2.4}$$

Equations (2.3)–(2.4) will be the basic equations of the considered contact problem. In them we pass to dimensionless coordinates and values, assuming

$\xi = x/a$, $\eta = s/a$; $\vartheta_0 = a\vartheta/\chi$; $\alpha_0 = a\alpha/\chi E$; $\delta_0 = \delta/\chi E$;

$p_0(\xi) = p(a\xi)/E$; $f_0(\xi) = f(a\xi)/E\chi$ $(-1 < \xi, \eta < 1)$.

As a result, GIE (2.3) is transformed into the following GIE:

$$p_0(\xi) + \vartheta_0 \int_{-1}^{1} \ln \frac{1}{|\xi - \eta|} p_0(\eta) d\eta = \delta_0 - \alpha_0 \xi - f_0(\xi) \quad (-1 < \xi < 1), \tag{2.5}$$

and the conditions (2.4) – into the following conditions:

$$\int_{-1}^{1} p_0(\xi)d\xi = P_0 \ (P_0 = P/aE); \quad \int_{-1}^{1} \xi p_0(\xi)d\xi = M_0 \ (M_0 = M/a^2 E). \tag{2.6}$$

3. The solution of GIE (2.5)-(2.6) by the method of degenerate kernels. The method described in section 1 will be applied to the equations (2.5)-(2.6). In the given case $L = (-1,1)$ and the spectral relationships (1.2) have the form of [7, 8, 14]

$$\frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|\xi - \eta|} \frac{T_n(\eta) d\eta}{\sqrt{1-\eta^2}} = \begin{cases} \frac{1}{n} T_n(\xi) & (n = 1, 2, ...); \\ \ln 2 & (n = 0); \quad (-1 < \xi < 1); \end{cases}$$

where $T_n(\xi)$ are Chebishev polynomials of the first kind, the conditions of orthogonality (1.3) has the form

$$\int_{-1}^{1} T_m(\xi) T_n(\xi) \frac{d\xi}{\sqrt{1-\xi^2}} = \begin{cases} 0 & (m \neq n); \\ \pi & (m = n = 0); \\ \pi/2 & (m = n \neq 0) \end{cases} \qquad \left(w(\xi) = 1\Big/\sqrt{1-\xi^2} \right)$$

and the formulas of Fourier generalized transformation (1.4) have the form of

$$f(\xi) = \sum_{m=0}^{\infty} f_m T_m(\xi) \quad (-1 < \xi < 1)$$

$$f_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(\xi) d\xi}{\sqrt{1-\xi^2}}; \quad f_m = \frac{2}{\pi} \int_{-1}^{1} \frac{f(\xi) T_m(\xi) d\xi}{\sqrt{1-\xi^2}} \quad (m = 1, 2, ...).$$

As a result, the bilinear expansion of the kernel (1.5) in the given case for the symmetrical logarithmic kernel is written in the form of

$$\ln \frac{1}{|\xi - \eta|} = \sum_{m=0}^{\infty} a_m T_m(\xi) T_m(\eta) \quad (-1 < \xi, \eta < 1) \qquad a_m = \begin{cases} \ln 2 & (m = 0); \\ \frac{2}{m} & (m = 1, 2, ...). \end{cases}$$

Later in accordance with (1.6), (1.8) and (1.9) in the given case the degenerate kernel is represented by the formula

$$\ln \frac{1}{|\xi - \eta|} \approx K_n(\xi, \eta) = \sum_{m=0}^{n} a_m T_m(\xi) T_n(\eta) \quad (-1 < \xi, \eta < 1),$$

the approximate solution of GIE (2.5) is represented by the formula

$$p_0(\xi) \approx \delta_0 - \alpha_0 \xi - f_0(\xi) - \vartheta_0 \sum_{m=0}^{n} a_m X_m T_m(\xi) \qquad (-1 \le \xi \le 1), \tag{3.1}$$

and the unknown coefficients X_m are determined from SLAE

$$X_k + \vartheta_0 \sum_{m=0}^{n} a_m R_{km} X_m = g_k \qquad \left(k = \overline{0,n}\right)$$

$$R_{km} = \int_{-1}^{1} T_k(\xi) T_m(\xi) d\xi \quad \left(k,m = \overline{0,n}\right); \quad g_k = \int_{-1}^{1} g(\xi) T_k(\xi) d\xi; \tag{3.2}$$

$$g(\xi) = \delta_0 - \alpha_0 \xi - f_0(\xi) \qquad (-1 \le \xi \le 1).$$

By the substitution $\xi = \cos t$ integrals R_{km} and g_k are transformed into the integrals

$$R_{km} = \int_0^{\pi} \cos kt \cos mt \sin t dt \quad \left(k,m = \overline{0,n}\right);$$

$$g_k = \delta_0 R_{k0} - \alpha_0 R_{k1} - f_k \quad \left(k = \overline{0,n}\right); \quad f_k = \int_0^{\pi} f_0(\cos t) \cos kt \sin t dt \tag{3.3}$$

and are easily calculated. Upon that

$$R_{km} = \begin{cases} -\dfrac{1+(-1)^{m+k}}{2}\left[\dfrac{1}{(m+k)^2-1} + \dfrac{1}{(m-k)^2-1}\right] & (m \ne k-1;\; m \ne k+1); \\ 0 & (m = k-1;\; m = k+1). \end{cases} \tag{3.4}$$

From here, particularly,

$$R_{00} = 2; \quad R_{k0} = \begin{cases} -\dfrac{1+(-1)^k}{k^2-1} & (k \ne 1); \\ 0 & (k = 1); \end{cases} \quad R_{k1} = \begin{cases} \dfrac{(-1)^k - 1}{k^2-4} & (k \ne 2); \\ 0 & (k = 2); \end{cases} \quad \left(k = \overline{0,n}\right).$$

Now, taking into account the expression of the coefficients g_k from (3.3), SLAE (3.2) is represented in the form of

$$X_k + \vartheta_0 \sum_{m=0}^{n} L_{km} X_m = \delta_0 R_{k0} - \alpha_0 R_{k1} - f_k \qquad \left(k = \overline{0,n}\right)$$

$$L_{km} = a_m R_{km} \qquad \left(k,m = \overline{0,n}\right). \tag{3.5}$$

Let us the solution of SLAE (3.5) for the right - hand side equal to R_{k0} denote by $X_k^{(1)}$, for the right - hand side R_{k1} – by $X_k^{(2)}$, and for the right –hand part f_k – by $X_k^{(3)}$.

Then solution (3.5) is represented in the form of

$$X_k = \delta_0 X_k^{(1)} - \alpha_0 X_k^{(2)} - X_k^{(3)} \quad \left(k = \overline{0,n}\right). \tag{3.6}$$

Then, referring to the conditions of punch equilibrium (2.6), with the help of (3.1) and (3.6) for the determination of parameters δ_0 and α_0 we obtain the following SLAE:

$$\begin{cases} a_{11}\delta_0 + a_{12}\alpha_0 = b_1 \\ a_{21}\delta_0 + a_{22}\alpha_0 = b_2 \end{cases} \tag{3.7}$$

$$a_{11}=2-\vartheta_0\sum_{m=0}^{n}a_mR_{m0}X_m^{(1)};\quad a_{12}=\vartheta_0\sum_{m=0}^{n}a_mR_{m0}X_m^{(2)};$$

$$a_{21}=\vartheta_0\sum_{m=0}^{n}a_mR_{m1}X_m^{(1)};\quad a_{22}=\frac{2}{3}-\vartheta_0\sum_{m=0}^{n}a_mR_{m1}X_m^{(2)};$$

$$b_1=P_0+f_0-\vartheta_0\sum_{m=0}^{n}a_mR_{m0}X_m^{(3)};\quad b_2=-M_0-f_1+\vartheta_0\sum_{m=0}^{n}a_mR_{m1}X_m^{(3)}.$$

By equations (3.7) the dependence between the geometrical parameters of the problem δ_0 and α_0, corresponding to the reduced settlement of the punch and its reduced angle of rotation, respectively, with power parameters P_0 and M_0 is established.

Note, that taking into account (3.6) the solution (3.1) may be written in the form of

$$p_0(\xi)=\left[1-\vartheta_0\sum_{m=0}^{n}a_mX_m^{(1)}T_m(\xi)\right]\delta_0-\left[\xi-\alpha_0\sum_{m=0}^{n}a_mX_m^{(2)}T_m(\xi)\right]\alpha_0+$$
$$+\vartheta_0\sum_{m=0}^{n}a_mX_m^{(3)}T_m(\xi)\qquad(-1\le\xi\le 1).\tag{3.8}$$

In order to investigate the convergence of the approximate solution (3.8) to the exact solution of GIE (2.5)–(2.6), it is necessary to pass from the final SLAE (3.5) to the infinite SLAE:

$$X_k+\vartheta_0\sum_{m=0}^{\infty}L_{km}X_m=\delta_0R_{k0}-\alpha_0R_{k1}-f_k\quad(k=1,2,...).\tag{3.9}$$

Coming out from (3.4), it is easy to observe that at different parities of k and m we have $R_{km}=0$. That is why in (3.4) and (3.9) k and m should be considered simultaneously even or odd numbers. Then the infinite system (3.9) splits up into the following two separate infinite SLAE, corresponding to the symmetric and skew-symmetric parts of the considered contact problem

$$X_{2p}+\vartheta_0\sum_{q=0}^{\infty}L_{2p,2q}X_{2q}=\delta_0R_{2p,0}-f_{2p}\quad(p=0,1,2,...)$$
$$L_{2p,2q}=a_{2q}R_{2p,2q};\quad a_{2q}=\begin{cases}\ln 2\ (q=0) & (k=2p,\ m=2q);\\ \dfrac{1}{q} & (q=1,2,...);\end{cases}\tag{3.10}$$

$$X_{2p-1}+\vartheta_0\sum_{q=1}^{\infty}L_{2p-1,2q-1}X_{2q-1}=-\alpha_0R_{2p-1,1}-f_{2p-1}\quad(p=1,2,...)$$
$$(k=2p-1,m=2q-1)\tag{3.11}$$
$$L_{2p-1,2q-1}=a_{2q-1}R_{2p-1,2q-1};\quad a_{2q-1}=\frac{2}{2q-1}\qquad(p,q=1,2,...).$$

Here, according to (3.4)

$$R_{2p,2q} = -\left[\frac{1}{4(p+q)^2-1} + \frac{1}{4(p-q)^2-1}\right] \qquad (p,q = 0,1,2,...);$$

$$R_{2p-1,2q-1} = -\left[\frac{1}{4(p+q-1)^2-1} + \frac{1}{4(p-q)^2-1}\right] \quad (p,q = 1,2,...). \tag{3.12}$$

4. The investigation of the infinite systems (3.10)–(3.11). These infinite systems will be investigated on regularity. With this aim referring to the infinite system (3.10) we estimate the sums [12]

$$S_{2p} = \vartheta_0 \sum_{q=0}^{\infty} a_{2q} \left|R_{2p,2q}\right| \quad (p = 0,1,2...).$$

Taking into account the expression $R_{2p,2q}$ from (3.12), it may be written

$$S_{2p} \le \vartheta_0 \sum_{q=0}^{\infty} a_{2q} \left[\frac{1}{\left|4(p+q)^2-1\right|} + \frac{1}{\left|4(p-q)^2-1\right|}\right] = \frac{2\ln 2}{\left|4p^2-1\right|}\vartheta_0 +$$

$$+\vartheta_0\left[S_{2p}^{(1)} + S_{2p}^{(2)}\right] \qquad (p = 0,1,2...);$$

$$S_{2p}^{(1)} = \sum_{q=1}^{\infty}\frac{1}{q}\frac{1}{4(p+q)^2-1}; \quad S_{2p}^{(2)} = \sum_{q=1}^{\infty}\frac{1}{q}\frac{1}{\left|4(p-q)^2-1\right|}. \tag{4.1}$$

It is evident, that

$$\frac{1}{4(p+q)^2-1} \le \frac{1}{4q^2-1} \quad (p = 0,1,2,...).$$

Therefore

$$S_{2p}^{(1)} = \sum_{q=1}^{\infty}\frac{1}{q\left[4(p+q)^2-1\right]} \le \sum_{q=1}^{\infty}\frac{1}{q\left(4q^2-1\right)} = \sum_{q=1}^{\infty}\frac{1}{q(2q-1)(2q+1)} = \frac{1}{2}\ln\frac{2}{\sqrt{e}}.$$

Here the well-known formula from [17] (p. 22, form. 0.238.1) was applied. Thus,

$$S_{2p}^{(1)} \le \frac{1}{2}\ln\frac{2}{\sqrt{e}} \quad (p = 0,1,2,...). \tag{4.2}$$

The sums $S_{2p}^{(1)}$ will also be estimated with the help of Cauchy - Buniakovsky inequality:

$$S_{2p}^{(1)} \le \sqrt{\sum_{q=1}^{\infty}\frac{1}{q^2}}\sqrt{\sum_{q=1}^{\infty}\frac{1}{\left[4(p+q)^2-1\right]^2}} \le \frac{\pi}{\sqrt{6}}\sqrt{\sum_{q=1}^{\infty}\frac{1}{\left(4q^2-1\right)^2}}.$$

Calculate the sum

$$\sum_{q=1}^{\infty}\frac{1}{\left(4q^2-1\right)^2} = \frac{1}{4}\sum_{q=1}^{\infty}\left(\frac{1}{2q-1} - \frac{1}{2q+1}\right)^2 =$$

$$= \frac{1}{4}\left[\sum_{q=1}^{\infty}\frac{1}{(2q-1)^2} - 2\sum_{q=1}^{\infty}\frac{1}{(2q-1)(2q+1)} + \sum_{q=1}^{\infty}\frac{1}{(2q+1)^2}\right] = \frac{\pi^2-8}{16}.$$

Here the expressions of these sums from [17] (p.53. form.1.444.6 and 1.444.7 for $x=0$) were applied. As a result,

$$S_{2p}^{(1)} \le \frac{\pi}{4\sqrt{6}}\sqrt{\pi^2-8} \quad (p=0,1,2,...). \tag{4.3}$$

Based on the estimations (4.2)–(4.3), we shall have

$$S_{2p}^{(1)} \le \min\left\{\frac{1}{2}\ln\frac{2}{\sqrt{e}}, \frac{\pi}{4\sqrt{6}}\sqrt{\pi^2-8}\right\} = \frac{1}{2}\ln\frac{2}{\sqrt{e}} = 0.0965736 (p=0,1,2,...). \tag{4.4}$$

We pass to the estimation of the sums $S_{2p}^{(2)}$. At first note, that as above

$$S_0^{(2)} = \sum_{q=1}^{\infty}\frac{1}{q\left(4q^2-1\right)} = \frac{1}{2}\ln\frac{2}{\sqrt{e}}. \tag{4.5}$$

Then, again using Cauchy-Buniakovsky inequality, we can write $(p=1,2,...)$:

$$S_{2p}^{(2)} = \sum_{q=1}^{\infty}\frac{1}{q}\frac{1}{\left|4(p-q)^2-1\right|} \le \sqrt{\sum_{q=1}^{\infty}\frac{1}{q^2}}\sqrt{\sum_{q=1}^{\infty}\frac{1}{\left[4(p-q)^2-1\right]^2}} = \frac{\pi}{\sqrt{6}}\sqrt{\sum_{q=1}^{\infty}\frac{1}{\left[4(p-q)^2-1\right]^2}}$$

Separately, estimate the sums,

$$\sum_{q=1}^{\infty}\frac{1}{\left[4(p-q)^2-1\right]^2} = \underset{(p-q=r)}{\sum_{q=1}^{\infty}\frac{1}{\left[4(p-q)^2-1\right]^2}} + \underset{(q-p=r)}{\sum_{q=p+1}^{\infty}\frac{1}{\left[4(q-p)^2-1\right]^2}} =$$

$$= \sum_{r=0}^{p-1}\frac{1}{\left(4r^2-1\right)^2} + \sum_{r=1}^{\infty}\frac{1}{\left(4r^2-1\right)^2} \le 1 + 2\sum_{r=1}^{\infty}\frac{1}{\left(4r^2-1\right)^2}.$$

Again using the obtained above value for the sum of the last series, we find

$$S_{2p}^{(2)} \le \sqrt{1+\frac{\pi^2-8}{8}}\frac{\pi}{\sqrt{6}} = \frac{\pi^2}{4\sqrt{3}} \quad (p=1,2,...). \tag{4.6}$$

As

$$\frac{1}{\left|4p^2-1\right|} \le 1 \quad (p=0,1,2,...), \quad \frac{1}{2}\ln\frac{2}{\sqrt{e}} < \frac{\pi^2}{4\sqrt{3}},$$

then from (4.1) with the help of the estimations (4.4)–(4.6) we shall have

$$S_{2p} \le \left(2\ln 2 + \frac{1}{2}\ln\frac{2}{\sqrt{e}} + \frac{\pi^2}{4\sqrt{3}}\right)\vartheta_0 = \left(2\ln 2 + \frac{1}{2}\ln\frac{2}{\sqrt{e}} + \frac{\pi^2}{4\sqrt{3}}\right)\vartheta_0 \quad (p=0,1,2,...).$$

We require that the following condition will be fulfilled

$$\left(2\ln 2 + \frac{1}{2}\ln\frac{2}{\sqrt{e}} + \frac{\pi^2}{4\sqrt{3}}\right)\vartheta_0 \le q_0 < 1.$$

Whence we obtain the following condition of complete regularity [12] of the infinite system (3.10):

$$\vartheta_0 \le \frac{q_0}{2\ln 2 + \frac{1}{2}\ln\left(2/\sqrt{e}\right) + \pi^2/4\sqrt{3}} \quad (0 < q_0 < 1). \tag{4.7}$$

Now we shall show that for those ϑ_0, for which the condition of complete regularity (4.7) is not fulfilled, the infinite system (3.10) is quasi-completely regular, i.e. a complete regularity in (3.10) begins with some number. For this it is sufficient to show, that $\lim_{p\to\infty} S_{2p} = 0.$ Turning to the estimation of sums $S_{2p}^{(1)}$, consider the function

$$f(x) = 1\Big/ x\left[4(x+p)^2 - 1\right] \quad (x \ge 1, \quad p = 0,1,2,...).$$

It is evident, that the function $f(x)$ at $x \ge 1$ monotonously decreases and its value at the point $x = q\ (q = 1,2,...)$ coincides with the corresponding member of the series $S_{2p}^{(1)}$. Herewith the sum of the series $S_{2p}^{(1)}$, beginning from the second member, is equal to the area of the figure, consisted of the elementary rectangles with the bases of unique lengths.

Therefore,

$$S_{2p}^{(1)} < \frac{1}{4(p+1)^2 - 1} + \int_1^\infty \frac{dx}{x\left[4(x+p)^2 - 1\right]} \quad (p = 0,1,2,...).$$

For the calculation of this integral we use the well-known expression of the corresponding indefinite integral from [17] (p. 84, form. 2.18.4 at $m = n = 1$). As a result,

$$S_{2p}^{(1)} < \frac{1}{4p^2 + 8p + 3} + \frac{1}{2(4p^2 - 1)} \times$$
$$\times\left[2p\ln\left(\frac{2p+1}{2p+3}\right) - 2\ln 2 + \ln\left(4p^2 + 8p + 3\right)\right] \qquad (p = 1,2,...). \tag{4.8}$$

From (4.8) it follows that

$$S_{2p}^{(1)} = 0\left(1/p^2\right) \text{ at } p \to \infty.$$

We pass to the estimation of the sum $S_{2p}^{(2)}$, representing them in the form of

$$S_{2p}^{(2)} = U_p + \mathrm{V}_p;\ U_p = \sum_{q=1}^{p} \frac{1}{q} \frac{1}{\left|4(p-q)^2 - 1\right|} = \frac{1}{p} + W_p$$
$$W_p = \sum_{q=1}^{p-1} \frac{1}{q} \frac{1}{4(p-q)^2 - 1};\ \ \mathrm{V}_p = \sum_{q=p+1}^{\infty} \frac{1}{q} \frac{1}{4(p-q)^2 - 1} \quad (p = 2,3,...). \tag{4.9}$$

For the estimation of the sum W_p we introduce to consideration the function

$$h_1(x) = 1\Big/ x\left[4(p-x)^2 - 1\right] \qquad (1 \le x \le p-1;\ \ p = 3,4,...).$$

It is easy to show, that this function decreases on the line segment $1 \le x \le p_*$ and increases on the line segment $p_* \le x \le p-1$, wherein

$$p_* = \frac{1}{6}\left(4p - \sqrt{4p^2 + 3}\right)\left(1 < p_* < p - 1;\ p \ge 3\right).$$

Let the number p_* be between two consecutive natural numbers p_0 and $p_0 + 1$. Then

$$W_p \le \frac{1}{4p^2-8p+3}+\frac{1}{3(p-1)}+\int_1^{p-1} h_1(x)\,dx=\frac{1}{4p^2-8p+3}+$$
$$+\frac{1}{3(p-1)}+\int_1^{p-1}\frac{dx}{x\left[4(p-x)^2-1\right]} \qquad (p\ge 3). \tag{4.10}$$

Passing to the estimation of the sum V_p, represent them in the form of

$$V_p=\sum_{r=1}^{\infty}\frac{1}{p+r}\frac{1}{4r^2-1}$$

and introduce to consideration the decreasing function

$$h_2(x)=\frac{1}{(p+x)(4x^2-1)} \quad (x\ge 1).$$

It is evident that

$$V_p \le \frac{1}{3(p+1)}+\int_1^{\infty} h_2(x)\,dx=\frac{1}{3(p+1)}+\int_1^{\infty}\frac{dx}{(p+x)(4x^2-1)}. \tag{4.11}$$

Further, by the above-mentioned formula from [17] we calculate the integrals from (4.10)-(4.11). After the simple transformations we find

$$U_p \le \frac{1}{p}+\frac{1}{4p^2-8p+3}+\frac{1}{3(p-1)}+\frac{1}{2(4p^2-1)}\left\{\ln\left[(2p-3)\times\right.\right.$$
$$\left.\left.\times(2p-1)(p-1)^2\right]-\ln 3\right\}-\frac{p}{4p^2-1}\ln\frac{2p-3}{3(2p-1)} \qquad (p\ge 4);$$
$$V_p \le \frac{1}{3(p+1)}+\frac{1}{4p^2-1}\left[\frac{4p+1}{2}\ln 3-\ln 2-\ln(1+p)\right]. \tag{4.12}$$

Now from (4.1), (4.8), (4.9) and (4.12) it follows that

$$S_{2p}=0(1/p) \text{ as } p\to\infty$$

and, therefore quasi-complete regularity of the infinite system (3.10) is proved. By the pretty analogous way it is possible to conduct the investigation on regularity of the infinite system (3.11).

On the base of the foregoing, the method of reduction [11] is applicable to the infinite systems (3.10)–(3.11), i.e. the solutions of the corresponding (3.10)–(3.11) of the finite SLAE as $n\to\infty$ tends to the solutions of the infinite systems.

5. Numerical results. For Poisson numerical coefficients of an elastic half-plane material we take $\nu=0{,}25$. Then the dimensionless parameter $\nu=0{,}25$ may be represented in the form of

$$\vartheta_0=a\vartheta/\chi=\frac{15}{8\pi}\lambda_0;\ \lambda_0=a/E\chi.$$

Now for the particular configurations of the punch base, when $f_0(\xi)=1$ or $f_0(\xi)=\xi^2$ we solve SLAE (3.5) at various values of the parameter λ_0. As a result, the coefficients $X_k^{(j)}$ $\left(j=1,2,3;\ k=\overline{0,n}\right)$ are determined and by formula (3.6) the coefficients

X_k are obtained. Later on from system (3.7) the parameters δ_0 and α_0 are determined, and besides it was accepted here that $P_0 = 0{,}001$, $M_0 = 0{,}00001$. Then using the results of these calculations, the values of the dimensionless contact pressure under the punch, $p_0(\xi)$, as well as the values $p_0(\pm 1)$ are calculated by the formula (3.8).

In case of $f_0(\xi) = 1$ the graphs of $p_0(\xi)$ are practically rectilinear segments, parallel to the axis of the abscissa, which in the process of increase of λ_0 are removing from the axis of the abscissa. And in the case of $f_0(\xi) = \xi^2$ the graphs of $p_0(\xi)$ at small λ_0, corresponding to the big values of the local displacements in the contact zone, practically represent rectilinear segments near the axis of the abscissa. But with the increase of the parameter λ_0, when the local displacements become small values, the graphs of $p_0(\xi)$ gradually take the form of the parabola with branches going to infinity (Fig. 2).

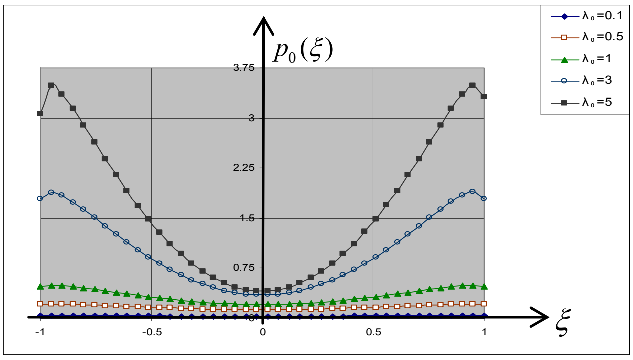

Fig. 2

Such parabolas are characteristic for the classical contact problems, when the contact pressure at the ends of the contact zone becomes infinite.

Values δ_0, α_0 and $p_0(\pm 1)$ when $f_0(\xi) = 1$ are given in Table 1 for different λ_0.

Table 1

λ_0	δ_0	α_0	$p_0(-1)$	$p_0(1)$
0.001	0.000834	-0.00015	0.000684	0.000984
0.005	0.002171	-0.00015	0.002022	0.002324
0.1	0.034514	-0.00016	0.035603	0.03593
0.3	0.105856	-0.00019	0.116166	0.116546
0.5	0.180771	-0.00022	0.207713	0.208146
0.8	0.298264	-0.00026	0.361064	0.361576
1	0.379268	-0.00028	0.471833	0.472396
5	2.17879	-0.00079	3.30652	3.30809
10	4.57025	-0.0014	7.4174	7.4202
20	9.45821	-0.00261	16.1845	16.1898
50	24.3141	-0.00623	43.8278	43.8403
100	49.2167	-0.01222	91.3401	91.3645

In Table 2 the same parameters when $f_0(\xi) = \xi^2$ are represented.

Table 2

λ_0	δ_0	α_0	$p_0(-1)$	$p_0(1)$
0.001	0.0015	-0.0015	0.00135	0.00165
0.005	0.005502	-0.00015	0.005351	0.005652
0.1	0.100548	-0.00016	0.100366	0.100693
0.3	0.300642	-0.00019	0.300393	0.300773
0.5	0.500736	-0.00022	0.500417	0.50085
0.8	0.800874	-0.00026	0.800446	0.800958
1	1.00096	-0.00028	1.00046	1.00103
5	5.00271	-0.00079	5.00055	5.00212
10	10.0048	-0.0014	10.0004	10.0032
20	20.009	-0.00261	19.9999	20.0051
50	50.0215	-0.00623	49.9976	50.0101
100	100.042	-0.01222	99.9931	100.018

With the increase of the parameter λ_0, which corresponds to the gradual transition into the smooth contact model, quantities δ_0 and $p_0(\pm 1)$ greatly increase, while values of δ_0 all the time remain very small.

Now we shall find out the conditions of the absence of the punch rotation when a given system of forces acts on the punch. Setting $\alpha_0 = 0$ into system (3.7), we obtain the following necessary values of M_0 and δ_0, providing the absence of the punch rotation:

$$M_0 = -\left\{\left[f_1 - \vartheta_0 Z_n^{(3)}\right]\left[2 - \vartheta_0 Y_n^{(1)}\right] + \vartheta_0 Z_n^{(1)}\left(P_0 + f_0 - \vartheta_0 Y_n^{(3)}\right)\right\} \Big/ \left(2 - \vartheta_0 Y_n^{(1)}\right);$$

$$\delta_0 = \left[P_0 + f_0 - \vartheta_0 Y_n^{(3)}\right] \Big/ \left(2 - \vartheta_0 Y_n^{(1)}\right); \qquad (5.1)$$

$$Y_n^{(j)} = \sum_{m=0}^{n} a_m R_{m_0} \mathrm{X}_m^{(j)}; \quad Z_n^{(j)} = \sum_{m=0}^{n} a_m R_{m_1} \mathrm{X}_m^{(j)}; \qquad (j = 1,2,3).$$

The calculated by the formulas (5.1) values of δ_0 and M_0 for different λ_0 when $f_0(\xi) = 1 + \xi$ or $f_0(\xi) = \xi + \xi^2$ for the same value of P_0 are given in Table 3.

Table 3

λ_0	$f_0(\xi)=1+\xi$		$f_0(\xi)=\xi+\xi^2$	
	δ_0	M_0	δ_0	M_0
0.001	0.0015	0.000666	0.000834	0.000666
0.005	0.005502	0.003318	0.002171	0.003318
0.1	0.100548	0.061202	0.034514	0.061202
0.3	0.300642	0.157898	0.105856	0.157898
0.5	0.500736	0.231072	0.180772	0.231072
0.8	0.800874	0.312955	0.298264	0.312956
1	1.00096	0.35511	0.379268	0.355111
5	5.00271	0.636414	2.17881	0.636418
10	10.0048	0.714155	4.57033	0.714167
20	20.009	0.764888	9.45859	0.764915
50	50.0215	0.802989	24.3169	0.803067
100	100.042	0.818321	49.2273	0.818474

From this table it is seen that at small λ_0, when the local displacements are significant, δ_0 and M_0 are enough small. However, they also increase with the increase of λ_0.

Conclusion. Rather a wide class of contact and mixed problems of mechanics of deformable solids is described by Fredholm integral equations of the second kind with symmetric kernels, for which the corresponding integral spectral relationships are well-known. In the paper for solving such equations, the well-known method of degenerate kernels is developed which reduces their solution to the solution of SLAE. The described method is illustrated on the example of the I.Ja. Shtaerman generalized contact problem on the punch indentation into an elastic half-plane taking into account the surface structure of the foundation.

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Поступила в редакцию 20.09.2016

ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

Մեխանիկա 70, №2, 2017 Механика

UDC 539.3

THREE-DIMENSIONAL PROBLEM OF WAVES PROPAGATION IN HALF-SPACE WITH AN ELASTICALLY RESTRAINED BOUNDARY

Sarkisyan S.V.

Key words: elastic half-space, surface wave, elastically restrained boundary

Ключевые слова: упругое полупространство, поверхностная волна, упруго-стеснённая граница

Բանալի բառեր. առաձգական կիսատարածություն, մակերևութային ալիքներ, առաձգականորեն կաշկանդված եզր

Саркисян С.В.

Трёхмерная задача о распространении волн в полупространстве с упруго-стеснённой границей

В работе получены дисперсионные уравнения трёхмерной задачи о распространении волн в полупространстве с упруго-стеснённой границей. Исследование задачи упрощается введением потенциальных функций. Показано, что при плоской деформации упругое стеснение границы полупространства приводит к уменьшению степени локализации поверхностной волны. Трёхмерная поверхностная волна существует лишь для двух видов граничных условий, когда поверхность полупространства свободна от напряжений и стеснённая свободная поверхность. В случае стеснённой свободной поверхности трёхмерная поверхностная волна обладает свойством дисперсии.

Սարգսյան Ս.Վ.

Եռաչափ դրվածքով առաձգականորեն կաշկանդված եզրով կիսատարածությունում ալիքների տարածման խնդիրը

Եռաչափ դրվածքով պոտենցիալ ֆունկցիաների ներմուծմամբ դիտարկված է առաձգականորեն կաշկանդված եզրով կիսատարածությունում մակերևութային ալիքների գոյության հարցը: Ստացված են մակերևութային ալիքների փուլային արագության նկատմամբ բնութագրիչ հավասարումներ: Ցույց է տրված, որ հարթ դեֆորմացիայի դեպքում եզրային պայմանը բերում է մակերևութային ալիքի տեղայնացման աստիճանի նվազմանը: Եռաչափ մակերևութային ալիքները գոյություն ունեն միայն երկու դեպքում՝ կիսատարածության եզրը ազատ է լարումներից կամ կաշկանդված ազատ եզրով կիսատարածություն: Կաշկանդված եզրով կիսատարածությունում եռաչափ մակերևութային ալիքները օժտված են դիսպերսիայով:

In this paper we obtain the dispersion equation of three-dimensional wave propagation problem in half-space with an elastic- restrained border. Research of the problem is simplified by the introduction of potential functions. It is shown that by plane strain the elastic half-space constraint of boundary leads to decreasing of the surface wave localization degree. Three-dimensional surface wave exists only for two kinds of boundary conditions, when the surface of half-space is free from the stresses and the free surface is restrained. In the case of constrained free surface the three-dimensional surface wave has a dispersion property.

Introduction. Surface waves propagation study represents a separate research in science. In the study of surface waves the plane and antiplane deformation was generally considered. For the first time the existence of surface waves was indicated by Rayleigh [1], where he examined the plane problem for half-space with stress free from the boundary. Solution of the three-dimensional problem was obtained by Knowles [2], who generalized the Rayleigh problem. These results are mentioned in monograph [3]. Another option of space problem was investigated in [4]. In work [5] the three-dimensional problems for elastic space waves propagation in isotropic half-space with two options of half-space boundary conditions was researched: free boundary and when we have one shear displacement at the border of half-space, one of the tangential stress and normal stress is equal to zero. In monograph [6] the summary of elastic waves propagation space problems is given. Study of three-dimensional surface waves for various types of mixed boundary conditions on the surface of the half-space is given in work [7]. It is shown that dispersion equation has a root for two types of boundary conditions: free surface and the surface, where displacements in one tangential direction are forbidden.

Unlike the classical Rayleigh' problem, M.V. Belubekyan [8] considers two types of complex boundary conditions instead of free surface boundary conditions for an isotropic elastic half-space. It is assumed that either normal stress is constricted in the perpendicular direction to the surface normal and shear is equal to zero, or the normal stress is equal to zero and tangent is restrainedly.The conditions are set, at which the surface wave cannot exists. The problem of periodic waves propagation in an elastic layer when in the layer boundaries the normal and shear stresses restrained were investigated in works [9,10]. Here the influence of restraint factor to the phase velocity of the symmetric and asymmetric vibration layer is shown.

As a result of the integral Radon' transformation [11] the space problems of the dynamic theory of elasticity are reduced to the plane problem regarding the Radon' transformation images. In the work [12] the introduction problem of dynamic potential for solving three-dimensional problems of the dynamic theory of elasticity is investigated, in which the antiplane displacementis not used (for example, in the problem of the dynamics of the surface of an elastic half-space, where the contribution of the surface wave is dominant). Applying the Radon' transformation, the solution of three-dimensional elasticity problem is comes to solving the corresponding plane problem. Development of asymptotic models of Rayleigh' surface waves, Stoneley' and Scholte-Gogoladze' interface waves were studied in work [13]. In work [14] the wave propagation problem in an elastic half-space is studied, when the half-restrained free edge conditions on the half-space boundary are given. Using the Radon' integral transformation, a dispersion equation for determining the velocity of surface wave propagation is obtained and the numerical experiment for the different physical and mechanical parameters characterizing the media is made.

In this paper we obtain the dispersion equation of three-dimensional wave propagation problem in half-space with an elastic- restrained border. Research of the problem is simplified by the introduction of potential functions like plane strain problems [3,5]. It is shown that by plane strain the elastic half-space constraint of boundary leads to decreasing of the surface wave localization degree. Three-dimensional surface wave exists only for two kinds of boundary conditions, when the surface of half-space is free from the stresses and the free surface is restrained. In the case of constrained free surface the three-dimensional surface wave has a dispersion property. By mixed boundary conditions at the surface, the propagation angle affects the phase velocity to the three-dimensional surface wave.

Statement of the Problem. Consider the harmonic vibrations of an isotropic elastic half-space $-\infty < x < \infty$, $-\infty < z < \infty$, $0 \le y < \infty$. Vibrations described by three-dimensional motion equations [3]:

$$(\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} + \mu\Delta\vec{u} = \rho\ddot{\vec{u}} \tag{1}$$

where $\vec{u}$ – displacement vector, λ, μ – Lame' parameter, ρ – density.

Suppose that the following boundary conditions are given [8] on the boundary of the half-space $y = 0$:

$$\sigma_{yx} = \alpha_* u,\ \ \sigma_{yy} = \beta_* \mathrm{v},\ \ \sigma_{yz} = \gamma_* w\left(\alpha_*, \beta_*, \gamma_* > 0\right) \tag{2}$$

These conditions were proposed by Mindlin [16] for study the elastic wave reflections problem from the boundary of the half-space. In work [8] the conditions for the existence of Rayleigh waves in the case of elastic- restrained boundary (plane strain) were researched by M. Belubekyan. Periodic waves propagation in elastic layer is studied in works [9,10].

In particular case by $\alpha_* = \beta_* = \gamma_* = 0$ we get conditions of free boundary.

To solve the problem of surface waves propagation, the potential functions $\varphi(x, y, z, t)$ and $\psi(x, y, z, t)$ [5] are introduced like in problems of plane strain:

$$u = \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial z}, \qquad w = \frac{\partial\varphi}{\partial z} + \frac{\partial\psi}{\partial x}. \qquad (3)$$

By means of (1) and (3) with taking into accounts damping conditions

$$\lim_{y\to\infty} \vec{u} = 0, \lim_{y\to\infty} \varphi = 0, \lim_{y\to\infty} \psi = 0$$

the displacements u, v, w are determined in the form [5]:

$$u(x, y, z, t) = -i\begin{bmatrix} Ak\cos\gamma e^{-\upsilon_1 ky} + \\ +(Bk\cos\gamma + Ck\sin\gamma)e^{-\upsilon_2 ky}\end{bmatrix}\exp i(\omega t - xk\cos\gamma - zk\sin\gamma),$$

$$\mathrm{v}(x, y, z, t) = -k\left[A\upsilon_1 e^{-\upsilon_1 ky} + B\upsilon^{-1}{}_2 e^{-\upsilon_2 ky}\right]\exp i(\omega t - xk\cos\gamma - zk\sin\gamma), \qquad (4)$$

$$w(x, y, z, t) = -i\begin{bmatrix} Ak\sin\gamma e^{-\upsilon_1 ky} + \\ +(Bk\sin\gamma - Ck\cos\gamma)e^{-\upsilon_2 ky}\end{bmatrix}\exp i(\omega t - xk\cos\gamma - zk\sin\gamma),$$

where k – wave number, $\nu_1^2 = 1 - \theta\eta$, $\nu_2^2 = 1 - \eta$, $\theta = \frac{c_t^2}{c_l^2} < 1$, $\eta = \frac{\omega^2}{k^2 c_t^2} < 1$ – dimensionless phase velocity of the three-dimensional surface wave, γ – sharp angle of wave propagation in plane $0xz$, A, B and C – arbitrary constants.

Applying Hooke's law the boundary conditions (2) by y=0 comes to the form:

$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right) - \alpha_* u = 0, \mu\left(\frac{\partial \mathrm{v}}{\partial z} + \frac{\partial w}{\partial y}\right) - \gamma_* w = 0,$$

$$\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial \mathrm{v}}{\partial y} - \beta_* \mathrm{v} = 0 \qquad (5)$$

Dispersion equations and numerical results. Satisfying solution (4) to the boundary conditions (5) we get the dispersion equation:

$$\begin{aligned}&4\nu_1\nu_2-(2-\eta)^2+\nu_2\left[\eta\left(\alpha_0+\beta_0\nu_1\nu_2^{-1}\right)+\alpha_0\beta_0\left(\nu_2^{-1}-\nu_1\right)\right]+\\&+\gamma_0\left[4\nu_1+\eta\left(\alpha_0+\beta_0\nu_1\nu_2^{-1}\right)+\alpha_0\beta_0\left(\nu_2^{-1}-\nu_1\right)-(2-\eta)^2\nu_2^{-1}\right]+\\&+\mathrm{tg}^2\gamma\Big[4\nu_1\nu_2-(2-\eta)^2+\nu_2\eta\left(\gamma_0+\beta_0\nu_1\nu_2^{-1}\right)+\nu_2\gamma_0\beta_0\left(\nu_2^{-1}-\nu_1\right)+\\&+\alpha_0\left(4\nu_1-(2-\eta)^2\nu_2^{-1}+\eta\left(\gamma_0+\beta_0\nu_1\nu_2^{-1}\right)+\gamma_0\beta_0\left(\nu_2^{-1}-\nu_1\right)\right)\Big]=0,\end{aligned}\tag{6}$$

where $\alpha_0=\dfrac{\alpha_*}{\mu k},\ \beta_0=\dfrac{\beta_*}{\mu k},\ \gamma_0=\dfrac{\gamma_*}{\mu k}.$

From equation (6) is follows that three-dimensional surface wave possesses dispersion feature. For given equation let's consider the following particular cases.

- Dispersion equation (6) in case of plane strain comes to the following $(\gamma=\gamma_0=0)$

$$\begin{aligned}&(2-\eta)^2-4\sqrt{(1-\eta)(1-\eta\theta)}-\alpha_0\eta\sqrt{1-\eta}-\beta_0\eta\sqrt{1-\eta\theta}-\\&-\alpha_0\beta_0\left(1-\sqrt{(1-\eta)(1-\eta\theta)}\right)=0.\end{aligned}\tag{7}$$

Equation (7) by $\alpha_0=\beta_0=0$ is coincides with Rayleigh' classical equation. Compared with the Rayleigh's equation, the equation (7) is dispersion, since solution depends on α_0,β_0. Dispersion equation (7) by $\alpha_0=0$ either $\beta_0=0$ has been received in work [8], where the conditions of existence of surface waves were set, depending on the coefficient characterizing the elastic restraint and the wave length.

Equation (7) has a root $\eta=0$, to which the trivial solution is corresponds. Following to work [15], eliminating root $\eta=0$, the equation (7) comes to the following:

$$\begin{aligned}&D(\eta)\equiv\eta-\frac{(1-\theta)\sqrt{1-\eta}}{\sqrt{1-\eta}+\sqrt{1-\theta\eta}}\left(4-\alpha_0\beta_0\right)-\\&-\alpha_0\sqrt{1-\eta}-\beta_0\sqrt{1-\theta\eta}-\alpha_0\beta_0=0.\end{aligned}\tag{8}$$

Function $D(\eta)$ takes the following values:

$$D(0) = -0.5(1-\theta)(4-\alpha_0\beta_0) - \alpha_0 - \beta_0 - \alpha_0\beta_0,$$
$$D(1) = 1 - \beta_0\sqrt{1-\theta} - \alpha_0\beta_0.$$

Equation (8) will have solution in the interval $\eta \in (0,1)$, if $D(0) < 0, \quad D(1) > 0$ and this solution will be unique if $\frac{dD}{d\eta} > 0.$ Choosing values α_0 and β_0, which satisfy to the given conditions, we can get the values for surface wave phase velocity depending on the degree of restraint surface of the half-space.

In table 1 the numerical results are given, which calculated by equation (8) for η parameter, characterizing the square of phase velocity of space wave depending on parameters, characterizing degree of restraint surface of the half-space by $\theta = 0.33$. The table shows that the restraint boundaries either at the direction of the normal or tangential direction leads to an increase in the dimensionless phase velocity of the surface wave. By the oppression of the border at the same time in both directions of the surface wave phase velocity at first increases, reaching a maximum value, then decreases. Thus, the boundary elastic half-space constraint reduces the degree of surface wave of localization (slow decay of the amplitude).

- Dispersion equation (6) for three-dimensional problem in case of free boundary $(\alpha_0 = \beta_0 = \gamma_0 = 0)$ comes to the following:

$$\left(1+\mathrm{tg}^2\gamma\right)\left((2-\eta)^2 - 4\sqrt{(1-\eta)(1-\eta\theta)}\right) = 0 \qquad (9)$$

By different mixed boundary conditions [7] we get corresponding equations from dispersion equation (6):

a) displacement is prohibited in tangent direction

$$\sigma_{yy} = 0, \sigma_{yz} = 0, u = 0$$

$$(2-\eta)^2 - 4\sqrt{(1-\eta)(1-\eta\theta)} - \eta(1-\eta)\,\mathrm{ctg}^2\gamma = 0 \qquad (10)$$

b) displacement is prohibited in one of tangent directions

$$\sigma_{yy} = 0,\ \sigma_{yx} = 0,\ w = 0,\ (2-\eta)^2 - 4\sqrt{(1-\eta)(1-\eta\theta)} - \eta(1-\eta)\,\mathrm{tg}^2\gamma = 0 \qquad (11)$$

Table 1

α_0	β_0	η
0	0	0.8464
0	0.2	0.9005
0	0.4	0.9405
0	0.6	0.9686
0	0.8	0.9867
0	1	0.9966
0.2	0	0.8712
0.4	0	0.8903
0.6	0	0.9052
0.8	0	0.9172
1	0	0.9269
0.2	0.2	0.9045
0.4	0.4	0.9238
0.6	0.6	0.9216
0.8	0.8	0.9001
1	1	0.8540

c) displacement is prohibited in both tangent directions:

$$\sigma_{yy}=0,\ \ u=0,\ \ w=0,\ \eta\left(1+\mathrm{t}\,g^2\gamma\right)=0$$

d) displacement is prohibited in normal direction:

$$\mathrm{v}=0,\ \ \sigma_{yx}=0,\ \ \sigma_{yz}=0,\ \eta\sqrt{1-\theta\eta}\left(1+\mathrm{tg}^2\gamma\right)=0$$

e) displacement is prohibited in one of tangent directions and in normal direction

$$\mathrm{v}=0,\ w=0,\ \sigma_{yx}=0,\ \eta\sqrt{1-\theta\eta}+\mathrm{tg}^2\gamma\sqrt{1-\eta}\left(1-\sqrt{(1-\eta)(1-\theta\eta)}\right)=0$$

f) displacement is prohibited in one of tangent directions and in normal direction

$$\mathrm{v}=0,\ u=0,\ \sigma_{yz}=0,\ \eta\sqrt{1-\theta\eta}+\mathrm{ctg}^2\gamma\sqrt{1-\eta}\left(1-\sqrt{(1-\eta)(1-\theta\eta)}\right)=0$$

g) displacement is prohibited in all directions

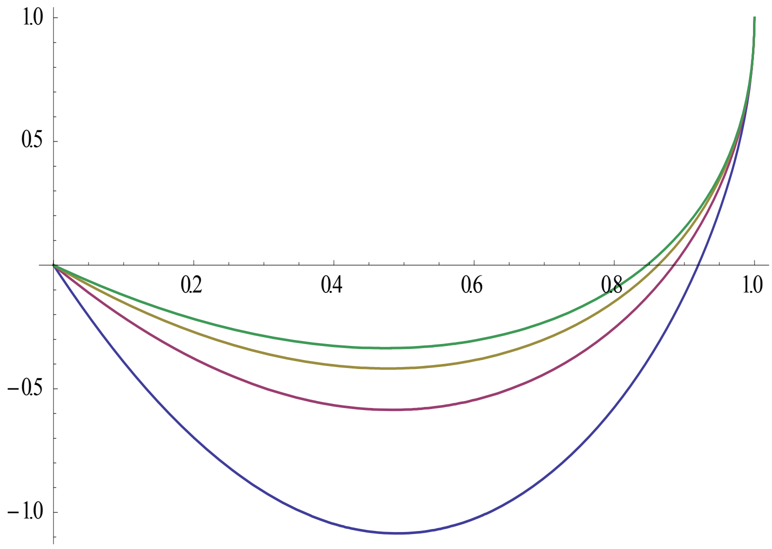
$$u=0,\ \mathrm{v}=0,\ w=0,\ \left(1+\mathrm{tg}^2\gamma\right)\left(1-\sqrt{(1-\eta)(1-\theta\eta)}\right)=0$$

Studies have shown that three-dimensional surface wave exist only for two kinds of boundary conditions. In the case where the half-surface is free from stresses, the known Rayleigh equation is obtained (9). If we have restrained free surface (displacement if forbidden in tangential direction) dispersion equations are comes to the equation (10) or (11). In case of free surface constrained these equations have a single root $\eta<1$ and three-dimensional surface wave possesses the feature of dispersion. The figure shows dependence of the phase velocity of the three-dimensional surface wave from the propagation angle. In table 2 the values of η parameter are shown, which characterized the square of phase velocity of three-dimensional surface wave depends on propagation angle by θ =0.33.

Table 2 and graph show that by mixed boundary conditions on the surface, and propagation angle affects to the phase velocity of the three-dimensional surface wave. By the displacement prohibition in one of tangential direction, with an increase of the angle value, the three-dimensional surface wave phase velocity decreases (increases). By $\gamma=\frac{\pi}{2}(0)$ the value of the phase velocity of the three-dimensional surface wave is exactly coincides to the value of the phase velocity of the Rayleigh surface waves.

Table 2

γ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
η(10)	-	0.9198	0.8850	0.8624	0.8464
(11)	0.8464	0.8624	0.8850	0.9198	-



Conclusion. Dispersion equations for the space problem of wave propagation in half-space with an elastic-restrained border are obtained. The elastic half-space restraint of boundary in plane strain leads to decreasing of localization degree of the surface wave (to the slow decay of the amplitude). Three-dimensional surface wave exists only for two kinds of boundary conditions - when the surface of half-space is free from the stresses and in case of cramped free surface (displacement is prohibitedin one of tangential direction). In case of cramped free surface three-dimensional surface wave possesses the feature of dispersion. By mixed boundary conditions on the surface, the angle of propagation has the influence to the phase velocity of three-dimensional surface wave.Increasing this angle the values of three-dimensional surface wave phase velocity decreases (increases), tends to the value of the phase velocity of the Rayleigh surface waves.

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Received 09.12.2016

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СОДЕРЖАНИЕ

2-го номера журнала «Механика», 2017 г., том 70

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