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ON FRACTIONAL KIRCHHOFF PROBLEMS WITH LIOUVILLE-WEYL FRACTIONAL DERIVATIVES

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Abstract. In this paper, we study the following fractional Kirchhoff-type problem with Liouville-Weyl fractional derivatives:

$$\begin{cases} \left[a + b\Big(\int_{\mathbb{R}}(|u|^2 + |_{-\infty}D_x^{\beta}u|^2)dx\Big)^{\varrho-1}\right]({}_xD_{\infty}^{\beta}({}_{-\infty}D_x^{\beta}u) + u) = |u|^{2_{\beta}^*-2}u, \ in\ \mathbb{R}, \\ u \in \mathbb{I}_{-}^{\beta}(\mathbb{R}), \end{cases}$$

where $\beta \in (0, \frac{1}{2})$, ${}_{-\infty}D_x^{\beta}u(\cdot), {}_xD_{\infty}^{\beta}u(\cdot)$ denote the left and right Liouville-Weyl fractional derivatives, $2_{\beta}^* = \frac{2}{1-2\beta}$ is fractional critical Sobolev exponent $a \geq 0$ and $b > 0$. Under suitable values of the parameters ϱ, a and b, we obtain a non-existence result of nontrivial solutions of infinitely many nontrivial solutions for the above problem.

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Keywords: Liouville-Weyl fractional derivatives; Kirchhoff-type problem; non-existence result; infinitely many nontrivial solutions.

1. Introduction

The purpose of this article is to study the non-existence results for the following fractional Kirchhoff-type equation with Liouville-Weyl fractional derivatives:

$$(1.1)\quad \begin{cases} \left[a + b\Big(\int_{\mathbb{R}}(|u|^2 + |_{-\infty}D_x^{\beta}u|^2)dx\Big)^{\varrho-1}\right]({}_xD_{\infty}^{\beta}({}_{-\infty}D_x^{\beta}u) + u) = |u|^{2_{\beta}^*-2}u, \ in\ \mathbb{R}, \\ u \in \mathbb{I}_{-}^{\beta}(\mathbb{R}), \end{cases}$$

where $\beta \in (0, \frac{1}{2})$, ${}_{-\infty}D_x^{\beta}u(\cdot), {}_xD_{\infty}^{\beta}u(\cdot)$ denote the left and right Liouville-Weyl fractional derivatives, $2_{\beta}^* = \frac{2}{1-2\beta}$ is fractional critical Sobolev exponent, $a \geq 0$ and $b > 0$.

The theory of fractional operators for a long time remained hidden from the scientific community, with its pioneering works involving the integrals and fractional derivatives of Liouville, Riemann, Grunwald-Letnikov and Riemann-Liouville [6, 10, 30]. Then, around 1974, at a conference at the University of New Haven, in the United States, the first international conference on fractional calculus took place [24]. From that moment on, fractional calculus began to be disseminated and disseminated and countless fractional derivatives have been introduced, each one with its importance and relevance in the field of fractional operators [1, 8, 9, 12, 14, 17, 18, 19, 22]. We highlight in a special way, when it comes to applications in: medicine, engineering, physics, biology among other areas [6, 10, 11, 13, 20, 23].

We note that when $a = 1$, $b = 0$, problem (1.1) boils down to a fractional differential equation of the type

$$ {}_xD^{\beta}_{+\infty}({}_{-\infty}D^{\beta}_x u) = g(u), \text{ in } \mathbb{R}, $$

which is a special case of the fractional advection-dispersion equation proposed by Benson et. all. [3, 4, 5]. When $\beta \in (\frac{1}{2}, 1)$ several existence and multiplicity results can be found in [25, 26] and the reference therein. Recently, the case $\beta \in (0, \frac{1}{2})$ was considered in [28, 29].

On the other hand, in these last years, the study of Kirchhoff problems with fractional derivatives have been attracted the attention from many mathematicians. For instance, Nyamoradi and Zhou [15] dealt with the existence of nontrivial solutions for a Kirchhoff type problem with Liouville-Weyl fractional derivatives by using minimal principle and Morse theory. Nyamoradi et. all. [16] studied a class of Schrödinger-Kirchhoff equation with Liouville-Weyl fractional derivatives and obtained the existence and multiplicity of solutions by using mountain pass theorem and the symmetric mountain pass theorem. Tayyebi and Nyamoradi [21] established the existence and multiplicity of nontrivial solutions for a Kirchhoff equation with Liouville-Weyl fractional derivatives by using symmetric mountain pass theorem, Morse theory combined with local linking arguments and the Clark's theorem. The authors in [2] by using local linking arguments and Morse theory studied the existence and multiplicity of solutions for a fractional Kirchhoff equation with Liouville-Weyl fractional derivatives.

Since we did not find in the literature any paper dealing with problems involving fractional derivatives and critical exponent, motivated by the previous works, in the present paper we intend to show the non-existence results for problem (1.1) by applying suitable variational arguments.

2. Preliminaries and main results

In this section, we recall some useful preliminaries which will play an important role to solve the problem (1.1), and we state the main results of this work.

Definition 2.1. *The left and right Liouville-Weyl fractional integrals of order* $0 < \beta < 1$ *on the whole axis* $\mathbb{R}$ *are defined by*

$$ {}_{-\infty}I^{\beta}_x \phi(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} (x-\xi)^{\beta-1}\phi(\xi)d\xi, \tag{2.1} $$

$$ {}_xI^{\beta}_{\infty} \phi(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\infty} (\xi-x)^{\beta-1}\phi(\xi)d\xi. \tag{2.2} $$

respectively, where $x \in \mathbb{R}$.

The left and right Liouville-Weyl fractional derivatives of order $0 < \beta < 1$ *on the whole axis* $\mathbb{R}$ *are defined by*

$$ {}_{-\infty}D_x^{\beta}\phi(x) = \frac{d}{dx}{}_{-\infty}I_x^{1-\beta}\phi(x), \tag{2.3}$$

$$ {}_{x}D_{\infty}^{\beta}\phi(x) = -\frac{d}{dx}{}_{x}I_{\infty}^{1-\beta}\phi(x). \tag{2.4}$$

respectively, where $x \in \mathbb{R}$.

2.1. **Fractional space of Sobolev type.** By argument in [29], we will look for weak solutions of the problem (1.1) hence the natural setting involves the fractional space of Sobolev type $\mathbb{I}_{-}^{\beta}(\mathbb{R})$ defined as

$$\mathbb{I}_{-}^{\beta}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : {}_{-\infty}D_x^{\beta}u \in L^2(\mathbb{R})\}$$

endowed with the scalar product

$$\langle u, v\rangle_{\beta} = \int_{\mathbb{R}} u(x)v(x)dx + \int_{\mathbb{R}} {}_{-\infty}D_x^{\beta}u(x) \cdot {}_{-\infty}D_x^{\beta}v(x)dx$$

and norm

$$\|u\|_{\beta} = \Big(\int_{\mathbb{R}} u^2dx + \int_{\mathbb{R}} |{}_{-\infty}D_x^{\beta}u(x)|^2dx\Big)^{1/2}.$$

It is well known that $(\mathbb{I}_{-}^{\beta}(\mathbb{R}), \langle .,.\rangle_{\beta})$ is a Hilbert space. Moreover, for $\beta \in (0, \frac{1}{2})$ we have the continuous embedding

$$\mathbb{I}_{-}^{\beta}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}) \text{ for every } p \in [2, 2_{\beta}^{*}], \tag{2.5}$$

where $2_{\beta}^{*} = \frac{2}{1-2\beta}$ is the fractional critical Sobolev exponent.

In the case $a = 1$, $b = 0$, the problem (1.1) will be transformed into the following critical problem with Liouville-Weyl fractional derivatives:

$$ {}_{x}D_{\infty}^{\beta}({}_{-\infty}D_x^{\beta}u) + u = |u|^{2_{\beta}^{*}-2}u, \ \textit{in } \mathbb{R}. \tag{2.6}$$

Set

$$S_{\beta} := \inf_{u\in\mathbb{I}_{-}^{\beta}(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}}(|u|^2 + |{}_{-\infty}D_x^{\beta}u|^2)dx}{\Big(\int_{\mathbb{R}} |u(x)|^{2_{\beta}^{*}}dx\Big)^{\frac{2}{2_{\beta}^{*}}}}. \tag{2.7}$$

For any $\varepsilon > 0$, we can define $\widetilde{u}(x)$ as $u_{\varepsilon}(x) = \sqrt{\varepsilon}\widetilde{u}\left(\frac{x}{\varepsilon}\right)$, where $\widetilde{u}(x)$ is a minimizer for S_{β}. Clearly, $u_{\varepsilon}(x)$ is also a minimizer for S_{β}, satisfying (2.6) and

$$\int_{\mathbb{R}}(|u_{\varepsilon}|^2 + |{}_{-\infty}D_x^{\beta}u_{\varepsilon}|^2)dx = \int_{\mathbb{R}} |u_{\varepsilon}(x)|^{2_{\beta}^{*}}dx = S_{\beta}^{\frac{2_{\beta}^{*}}{2_{\beta}^{*}-2}}. \tag{2.8}$$

Now, under suitable values of the parameters a, b and ϱ, we state the main results of this paper as follow:

Theorem 2.1. *Suppose that* $\varrho > 1$ *and* $\beta \in (0, \frac{1}{2})$. *Then, problem (1.1) has no nontrivial solution under one of the following conditions:*

(i) $\varrho = \frac{2_{\beta}^{*}}{2}$, $a = 0$ *and* $b > S_{\beta}^{-\varrho}$;

(ii) $\varrho = \frac{2^*_\beta}{2}$, *$a > 0$ and $b \geq S_\beta^{-\varrho}$;*

(iii) $\varrho > \frac{2^*_\beta}{2}$, *$a, b > 0$ satisfy*

$$\frac{2a(\varrho-1)}{2\varrho - 2^*_\beta}\left(\frac{(2\varrho-2^*_\beta)bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}}{a(2^*_\beta-2)}\right)^{\frac{2^*_s-2}{2(\varrho-1)}} > 1;$$

(iv) $\varrho = \frac{1+2\beta}{1-2\beta}$, *$a, b > 0$ satisfy $1 < 4abS_\beta^{\varrho+1}$.*

Theorem 2.2. *Suppose that $\varrho > 1$ and $\beta \in (0, \frac{1}{2})$. Then the following properties hold:*

(i) $\varrho \neq \frac{2^*_\beta}{2}$, *$a = 0$ and $b > 0$, then problem (1.1) has infinitely many positive solutions and these solutions are*

$$b^{\frac{1}{2^*_\beta - 2\varrho}} S_\beta^{\frac{2^*_\beta(\varrho-1)}{(2^*_\beta-2\varrho)(2^*_\beta-2)}} u_\varepsilon \quad \textit{for any } \varepsilon > 0.$$

(ii) $\varrho = \frac{2^*_\beta}{2}$, *$a > 0$ and $b < S_\beta^{-\varrho}$, then problem (1.1) has infinitely many positive solutions and these solutions are given by*

$$\left(\frac{a}{1 - bS_\beta^{\varrho}}\right) u_\varepsilon \quad \textit{for any } \varepsilon > 0.$$

(iii) $\varrho > \frac{2^*_\beta}{2}$, *$a, b > 0$ satisfy*

$$\frac{2a(\varrho-1)}{2\varrho - 2^*_\beta}\left(\frac{(2\varrho-2^*_\beta)bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}}{a(2^*_\beta-2)}\right)^{\frac{2^*_\beta-2}{2(\varrho-1)}} = 1, \tag{2.9}$$

then problem (1.1) has infinitely many positive solutions and these solutions are

$$\left(\frac{a(2^*_\beta-2)}{(2\varrho-2^*_\beta)bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}}\right)^{\frac{1}{2(\varrho-1)}} u_\varepsilon \quad \textit{for any } \varepsilon > 0.$$

3. Proof of the main results

In this section, we deal with the proof of Theorems 2.1 and 2.2. Let us introduce the energy functional associated with problem (1.1):

$$J(u) = \frac{a}{2}\|u\|_\beta^2 + \frac{b}{2\varrho}\|u\|_\beta^{2\varrho} - \frac{1}{2^*_\beta}\int_{\mathbb{R}} |u(x)|^{2^*_\beta} dx, \tag{3.1}$$

which is well-defined for each $u \in \mathbb{I}^\beta_-(\mathbb{R})$. We know that $J \in C^1(\mathbb{I}^\beta_-(\mathbb{R})$. Moreover, it is easy to see that a weak solution of problem (1.1) is a critical point of the functional J.

Firstly, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose that $u \in \mathbb{I}_{-}^{\beta}(\mathbb{R})\setminus\{0\}$ is a solution of (1.1). Hence,

(i) from (2.7), we have

$$S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2\varrho} = S_\beta^{-\varrho}\|u\|_\beta^{2\varrho} < b\|u\|_\beta^{2\varrho} = \int_{\mathbb{R}^N}|u(x)|^{2_\beta^*}dx \le S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2_\beta^*} = S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2\varrho}.$$

which gives a contradiction. Then, (i) holds true.

(ii) In view of (2.7), one can get

$$S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2\varrho} = S_\beta^{-\varrho}\|u\|_\beta^{2\varrho} \le b\|u\|_\beta^{2\varrho} < a\|u\|_\beta^2 + b\|u\|_\beta^{2\varrho} = \int_{\mathbb{R}^N}|u(x)|^{2_\beta^*}dx \le S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2\varrho},$$

which is impossible. Then, (ii) is satisfied.

(iii) Using the Young's inequality and (2.7), we can get

$$\begin{aligned} S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2_\beta^*} &= S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{\frac{2\varrho-2_\beta^*}{\varrho-1}}\|u\|_\beta^{\frac{\varrho 2_\beta^*-2\varrho}{\varrho-1}} \\ &\le a\|u\|_\beta^2 + \frac{2_\beta^*-2}{2(\varrho-1)}\left(\frac{2a(\varrho-1)}{2\varrho-2_\beta^*}\right)^{-\frac{2\varrho-2_\beta^*}{2_\beta^*-2}} S_\beta^{-\frac{(\varrho-1)2_\beta^*}{2_\beta^*-2}}\|u\|_\beta^{2\varrho} \\ &< a\|u\|_\beta^2 + b\|u\|_\beta^{2\varrho} \\ &= \int_{\mathbb{R}^N}|u(x)|^{2_\beta^*}dx \le S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2_\beta^*}, \end{aligned}$$

which leads to a contradiction. So, (iii) is verified.

(iv) From geometric-arithmetic inequality and (2.7) one can get

$$\begin{aligned} \|u\|_\beta^{\varrho+1} &< 2\sqrt{ab}S_\beta^{\frac{\varrho+1}{2}}\|u\|_\beta^{\varrho+1} \le (a\|u\|^2 + b\|u\|^{2\varrho})S_\beta^{\frac{\varrho+1}{2}} \\ &\le S_\beta^{\frac{\varrho+1}{2}}\int_{\mathbb{R}}|u(x)|^{2_\beta^*}dx \le S_\beta^{\frac{\varrho+1}{2}}S_\beta^{-\frac{2_\beta^*}{2}}\|u\|_\beta^{2_\beta^*} = \|u\|_\beta^{\varrho+1} \end{aligned}$$

a contradiction. Hence, we get the result (iv). □

Secondly, we give the proof of Theorem 2.2. To this end, for any $\varepsilon > 0$, we set

$$v_{\varepsilon,\beta}(x) = \vartheta^{\frac{1}{2_\beta^*-2}}u_\varepsilon(x), \tag{3.2}$$

and it is a positive solution of (2.6). So, $v_{\varepsilon,s}$ satisfies

$$\vartheta({}_xD_\infty^\beta({}_{-\infty}D_x^\beta v_{\varepsilon,\beta}) + v_{\varepsilon,\beta}) = |v_{\varepsilon,\beta}|^{2_\beta^*-2}v_{\varepsilon,\beta},\ in\ \mathbb{R}. \tag{3.3}$$

Then, if

$$\vartheta = a + b\Big(\int_{\mathbb{R}}(|v_{\varepsilon,\beta}|^2 + |{}_{-\infty}D_x^\beta v_{\varepsilon,\beta}|^2)dx\Big)^{\varrho-1}, \tag{3.4}$$

we can deduce that $v_{\varepsilon,\beta}$ is a solution of (1.1). Since u_ε satisfies (2.8), then by inserting (3.2) into (3.4) we can infer that

$$\vartheta = a + bS_\beta^{\frac{2_\beta^*(\varrho-1)}{2_\beta^*-2}}\vartheta^{\frac{2(\varrho-1)}{2_\beta^*-2}}. \tag{3.5}$$

Furthermore, if $\vartheta \in (0,+\infty)$ is a solution of (3.5), then $v_{\varepsilon,\beta}$ is a solution of problem (1.1).

Proof of Theorem 2.2. (i) If $\varrho \neq \frac{2^*_\beta}{2}$, then $\frac{2(\varrho-1)}{2^*_\beta-2} \neq 1$. So, if $a = 0$, (3.5) has solution

$$\vartheta = b^{\frac{2^*_\beta-2}{2^*_\beta-2\varrho}} S_\beta^{\frac{2^*_\beta(\varrho-1)(2^*_\beta-2)}{(2^*_\beta-2)(2^*_\beta-2\varrho)}}.$$

Hence, in view of (3.2) we get the result (i).

(ii) If $\varrho = \frac{2^*_\beta}{2}$, then $\frac{2(\varrho-1)}{2^*_\beta-2} = 1$. So, (3.5) is equivalent to

$$\vartheta = a + bS_\beta^\varrho \vartheta, \tag{3.6}$$

and then $\vartheta = \frac{1}{1-bS_\beta^\varrho} > 0$. Hence, by (3.2) it follows that (ii) holds true.

(iii) If $\varrho > \frac{2^*_\beta}{2}$, then $\frac{2(\varrho-1)}{2^*_\beta-2} > 1$. Define

$$\varphi(\vartheta) := a\vartheta^{-1} + bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}} \vartheta^{\frac{2\varrho-2^*_\beta}{2^*_\beta-2}}$$

which implies that

$$\varphi(\vartheta) = 1 \quad \text{iff } \ \vartheta \text{ solves (3.5)}. \tag{3.7}$$

We can easily see that $\varphi(\vartheta)$ achieves its minimum at

$$\vartheta_0 = \left(\frac{a(2^*_\beta - 2)}{(2\varrho - 2^*_\beta) bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}} \right)^{\frac{2^*_\beta-2}{2(\varrho-1)}}$$

and

$$\min_{\vartheta>0} \varphi(\vartheta) = \varphi(\vartheta_0) = \frac{2a(\varrho-1)}{2\varrho - 2^*_\beta} \left(\frac{(2\varrho - 2^*_\beta) bS_\beta^{\frac{2^*_\beta(\varrho-1)}{2^*_\beta-2}}}{a(2^*_\beta - 2)} \right)^{\frac{2^*_\beta-2}{2(\varrho-1)}}.$$

By condition (2.9) we have $\varphi(\vartheta_0) = 1$, and from (3.7) we get that ϱ_0 is a solution of (3.5). From (3.2), we have the result (iii).

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Список литературы

[1] R. Almeida, "Caputo fractional derivative of a function with respect to another function", Commun. Nonlinear Sci. Numer. Simul. **44**, 460 – 481 (2017).

[2] A. A. Nori, N. Nyamoradi and N. Eghbali, "Multiplicity of Solutions for Kirchhoff Fractional Differential Equations Involving the Liouville-Weyl Fractional Derivatives", Journal of Contemporary Mathematical Analysis, **55**, **1**, 13 – 31 (2020).

[3] D. Benson, R. Schumer and M. Meerschaert, Fractional dispersion, Lévy motion, and the MADE tracer test, Trans Porous Med, **42**, 211 – 240 (2001).

[4] D. Benson, S. Wheatcraft and M. Meerschaert, Application of a fractional advection-dispersion equation, Water Resour Res, **36**, 1403 – 1412 (2000).

[5] D. Benson, S. Wheatcraft and M. Meerschaert, "The fractional-order governing equation of Lévy motion", Water Resour Res, **36**, 1413 – 1423 (2000).

[6] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag Berlin Heidelberg (2010).
[7] A. Jajarmi, D. Baleanu, S. Sadat Sajjadi and J. Nieto, Analysis and some applications of a regularized ψ-Hilfer fractional derivative, J. Comput. Appl. Math. **415**, 114476 (2022).
[8] F. Jarad and T. Abdeljawad, "Generalized fractional derivatives and Laplace transform", Disc. Cont. Dyn. Sys.- S **13**, no. 3, 709 – 722 (2020).
[9] F. Jarad, T. Abdeljawad, and D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, Adv. Difference Equ. 2012.1, 1 – 8 (2012).
[10] A. Kilbas, H. Srivastava and J. Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam (2006).
[11] D. Kumar and J. Singh, Fractional calculus in medical and health science, CRC Press (2020).
[12] Y. Luchko and J. Trujillo, "Caputo-type modification of the Erdélyi-Kober fractional derivative", Frac. Cal. Appl. Anal. **10.3**, 249 – 267 (2007).
[13] R. Magin, Fractional Calculus in Bioengineering, part 3, Critical Reviews in Biomedical Engineering 32.34 (2004).
[14] A. A. Nori, N. Nyamoradi, N. Eghbal, "Multiplicity of solutions for Kirchhoff fractional differential equations involving the Liouville-Weyl fractional derivatives", J. Contemp. Math. Anal. **55**, no. 1, 13 – 31 (2020).
[15] N. Nyamoradi and Y. Zhou, "Existence of solution for a Kirchhoff type fractional differential equations via minimal principle and morse theory", Topological Methods in Nonlinear Analysis, **46**, no. 2, 617 – 630 (2015).
[16] N. Nyamoradi, Y. Zhou, E. Tayyebi, D. Ahmad and A. Alsaedi, Nontrivial Solutions for Time Fractional Nonlinear Schrödinger-Kirchhoff Type Equations, Discrete Dynamics in Nature and Society, **2017**, 9281049 (2017).
[17] D. Oliveira, J. Vanterler da C. Sousa and G. Frederico, "Pseudo-fractional operators of variable order and applications", Soft Computing **26.10**, 4587 – 4605 (2022).
[18] D. Oliveira and E. Capelas de Oliveira. *On a Caputo-type fractional derivative.* Adv. Pure Appl. Math. 10.2 (2019): 81-91.
[19] D. Oliveira and E. Capelas De Oliveira, "Hilfer–Katugampola fractional derivatives", Comput. Appl. Math. **37.3**, 3672 – 3690 (2018).
[20] V. Tarasov, Handbook of Fractional Calculus with Applications, **5**, Berlin, de Gruyter (2019).
[21] E. Tayyebi and N. Nyamoradi, Existence of Nontrivial Solutions for Kirchhoff Type Fractional Differential Equations with Liouville-Weyl Fractional Derivatives, J. Nonlinear Funct. Anal. **2018**, Article ID 19 (2018).
[22] D. Tavares, R. Almeida, and D. M. Torres, "Caputo derivatives of fractional variable order: numerical approximations", Commun. Nonlinear Sci. Numer. Simul. **35**, 69 – 87 (2016).
[23] J. Tenreiro Machado, F. Mainardi, and V. Kiryakova, "Fractional calculus: Quo vadimus? (Where are we going?)", Frac. Cal. Appl. Anal. **18.2**, 495 – 526 (2015).
[24] J. Tenreiro Machado, V. Kiryakova and F. Mainardi, "Recent history of fractional calculus", Commun. Nonlinear Sci. Numer. Simul. **16.3**, 1140 – 1153 (2011).
[25] C. Torres, "Ground state solution for differential equations with left and right fractional derivatives", Math. Meth. Appl. Sci. **38**, 5063 – 5073 (2015).
[26] C. Torres Ledesma, "Existence and symmetric result for Liouville-Weyl fractional nonlinear Schrödinger equation", Commun Nonlinear Sci Numer Simulat **27**, 314 – 327 (2015).
[27] C. Torres Ledesma and J. Vanterler da C. Sousa, "Fractional integration by parts and Sobolev-type inequalities for ψ-fractional operators", Math. Meth. Appl. Sci. **45**, 9945 – 9966 (2022).
[28] C. E. Torres Ledesma, "Fractional Hamiltonian systems with vanishing potentials", Progr. Fract. Differ. Appl. **8**, no. 3, 1 – 19 (2022).
[29] C. E. Torres Ledesma, H. C. Gutierrez, J. A. Rodriguez, Z. Zhang, "Evennon-increasing solution for a Schrödinger type problem with Liouville-Weyl fractional derivative", Computat. Appl. Math. **41** (404) (2022) doi:10.1007/s40314-022-02124-6.
[30] J. Vanterler da C. Sousa and E. Capelas de Oliveira, "On the ψ-Hilfer fractional derivative", Commun Nonlinear Sci Numer Simulat **60**, 72 – 91 (2018).

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