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POWER OF AN ENTIRE FUNCTION SHARING ONE VALUE PARTIALLY WITH ITS DERIVATIVE

S. MAJUMDER, J. SARKAR

Raiganj University, Raiganj, West Bengal, India
E-mails: *sm05math@gmail.com*, *sjm@raiganjuniversity.ac.in*
jeetsarkar.math@gmail.com

Abstract. In the paper, we investigate the uniqueness problem of a power of an entire function that share one value partially with its derivatives and obtain a result, which improve several previous results. Also in the paper we include some applications of our main result.

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1. Introduction and main result

In this paper, a meromorphic function f always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with standard notation and main results of Nevanlinna Theory (see, e.g., [3, 8]). By $S(r,f)$ we denote any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside of an exceptional set of finite linear measure. A meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$. Moreover, we use notation $\rho(f)$ for the order of a meromorphic function f. As usual, the abbreviation CM means counting multiplicities, while IM means ignoring multiplicities. Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C}$. If $g-a=0$ whenever $f-a=0$, we write $f=a \Rightarrow g=a$.

In 1996, Brück [1] discussed the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture A. *Let f be a non-constant entire function such that*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} \notin \mathbb{N} \cup \{\infty\}.$$

If f and f' share one finite value a CM, then $f'-a = c(f-a)$, where $c \in \mathbb{C}\setminus\{0\}$.

The conjecture for the special cases (1) $a=0$ and (2) $N(r,0;f') = S(r,f)$ had been confirmed by Brück [1].

Though the conjecture is not settled in its full generality, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with its derivatives. Specially, it was observed by Yang and Zhang [9] that Brück's conjecture holds if instead of an entire function one considers its suitable power. They proved the following theorem.

Theorem A. [9] *Let f be a non-constant entire function and $n \in \mathbb{N}$ such that $n \geq 7$. If f^n and $(f^n)'$ share 1 CM, then $f^n \equiv (f^n)'$ and $f(z) = c\exp(\frac{z}{n})$, where $c \in \mathbb{C} \setminus \{0\}$.*

In 2010, Zhang and Yang [12] improved and generalised Theorem A by considering higher order derivatives and by lowering the power of the entire function. In one of their results they also considered IM sharing of values. We now state two results of Zhang and Yang [12].

Theorem B. [12] *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k+1$. If f^n and $(f^n)^{(k)}$ share 1 CM, then $f^n \equiv (f^n)^{(k)}$ and $f(z) = c\exp(\frac{\lambda}{n}z)$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$.*

Theorem C. [12] *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k+2$. If f^n and $(f^n)^{(k)}$ share 1 IM, then the conclusion of Theorem B holds.*

In connection to Theorem C, Zhang and Yang [12] posed the problem of investigating the validity of the result for $n \geq k+1$. They could prove Theorem C for $n \geq k+1$ but only in the case when $k = 1$. We now recall the result.

Theorem D. [12] *Let f be a non-constant entire function and $n \in \mathbb{N} \setminus \{1\}$. If f^n and $(f^n)'$ share 1 IM, then $f^n \equiv (f^n)'$ and $f(z) = c\exp(z)$, where $c \in \mathbb{C} \setminus \{0\}$.*

In the paper, we have extended and improved above Theorems in the following directions:

(1) We relax the nature of sharing with the idea of “partially"sharing value.
(2) We replace the first derivative $(f^n)'$ in Theorem D by the general derivative $(f^n)^{(k)}$.

We now state our main result as follows.

Theorem 1.1. *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k+1$. If $f^n = 1 \Rightarrow (f^n)^{(k)} = 1$, then only one of the following cases holds:*

(1) *$f^n \equiv (f^n)^{(k)}$ and $f(z) = c\exp\left(\frac{\lambda}{n}z\right)$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$,*
(2) *$n = 2$ and $f(z) = c_0\exp\left(\frac{1}{4}z\right) + c_1$, where $c_0, c_1 \in \mathbb{C} \setminus \{0\}$ such that $c_1^2 = 1$.*

If $k \geq 2$, then from Theorem 1.1, we have the following corollary.

Corollary 1.1. *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $k \geq 2$ and $n \geq k+1$. If $f^n = 1 \Rightarrow (f^n)^{(k)} = 1$, then $f^n \equiv (f^n)^{(k)}$ and $f(z) = c\exp\left(\frac{\lambda}{n}z\right)$, where $c, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^k = 1$.*

Clearly Corollary 1.1 improves Theorems A-D for the case when $k \geq 2$.

We now make the following observation on the conclusions of Theorem 1.1:

From the conclusion (2), we see that $k = 1$ and $n = 2$. Note that

$$f^2 - 1 = c_0^2 \exp\left(\frac{1}{2}z\right) + 2c_0c_1 \exp\left(\frac{1}{4}z\right)$$

and

$$(f^2)' - 1 = \frac{1}{2}\Big(c_0^2 \exp\left(\frac{1}{2}z\right) + c_0c_1 \exp\left(\frac{1}{4}z\right) - 2\Big).$$

It is easy to conclude that $(f^2)' = 1 \not\Rightarrow f^2 = 1$. Therefore if we add the condition that $(f^2)' = 1 \Rightarrow f^2 = 1$ in Theorem 1.1, then the conclusion (2) will be automatically ruled out.

As a result, from Theorem 1.1, we immediately have the following corollary.

Corollary 1.2. *Let f be a non-constant entire function and $k, n \in \mathbb{N}$ such that $n \geq k+1$. If f^n and $(f^n)^{(k)}$ share 1 IM, then the conclusion of Theorem B holds.*

Obviously Corollary 1.2 improves Theorem D.

Now we exhibit the following example to show that the condition "$n \geq k+1$" in Theorem 1.1 and Corollary 1.2 is sharp.

Example 1.1. *Let $f(z) = \exp(\frac{z}{2}) + 2\exp(\frac{z}{4}) + 1$ and $k = n = 1$. It is easy to verify that $f(z) = 1 \Rightarrow f'(z) = 1$, but $f(z)$ does not satisfy any case of Theorem 1.1.*

Example 1.2. *Let $f(z) = 2\exp(\frac{z}{2}) - 1$ and $k = n = 1$. It is easy to verify that f and f' share 1 IM, but $f \not\equiv f'$.*

2. Auxiliary lemmas

Lemma 2.1. *([5], [4], Theorem 4.1]) Let f be a non-constant entire function such that $\rho(f) \leq 1$ and $k \in \mathbb{N}$. Then $m(r, \frac{f^{(k)}}{f}) = o(\log r)$ as $r \to \infty$.*

Lemma 2.2. [7] *Let f be a non-constant meromorphic function and let $a_n(\not\equiv 0), a_{n-1}, \ldots, a_0$ be small functions of f. Then $T(r, \sum_{i=0}^{n} a_i f^i) = nT(r,f) + S(r,f)$.*

Now we introduce some basic ideas about normal families.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [6]).

Now we introduce the notation of the spherical derivative. Let h be a non-constant meromorphic function. The spherical derivative of h at $z \in \mathbb{C}$ is given as

$$h^{\#}(z) = \frac{|h'(z)|}{1+|h(z)|^2}.$$

We remember that h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M \ \forall\ z \in \mathbb{C}$.

Here we introduce some other results related to Zalcman's lemma. We also use Zalcman's lemma to prove our Lemma 2.5 which plays an important role in the proof of the main result of the paper.

The following lemma is the famous Marty's Criterion.

Lemma 2.3. [6] *A family $\mathcal{F}$ of meromorphic functions on a domain D is normal and only if for each compact subset $K \subseteq D$, there exists a constant M such that $f^{\#}(z) \leq M \ \forall\ f \in \mathcal{F}$ and $z \in K$.*

Zalcman's lemma.[[11]] *Let $\mathcal{F}$ be a family of functions holomorphic in a domain D. If $\mathcal{F}$ is not normal at $z_0 \in D$, then there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers ρ_n, $\rho_n \to 0$ and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta) \to g(\zeta)$$

converges locally uniformly in $\mathbb{C}$, where g is a non-constant entire function. The function g may be taken to satisfy the normalization $g^{\#}(\zeta) \leq g^{\#}(0) = 1 \ \forall\ \zeta \in \mathbb{C}$.

Lemma 2.4. [2] *Let f be a non-constant entire function such that $N(r,f) = O(\log r)$ as $r \to \infty$. If f has bounded spherical derivative on $\mathbb{C}$, then $\rho(f) \leq 1$.*

It does not seem that Theorem 1.1 can be proved by using the methods in [12]. In order to prove Theorem 1.1, we need the following result related to normal families.

Lemma 2.5. *Let f be a non-constant entire function such that*

$$(f^{k+1})'(f^{k+1} - (f^{k+1})^{(k)}) = \varphi f^{k+1}(f^{k+1} - 1),$$

where $\varphi(\not\equiv 0)$ is an entire function and $k \in \mathbb{N}$. If

$$f = 0 \Rightarrow (f^{k+1})^{(k)} = 0 \ \text{ and } \ f^{k+1} = 1 \Rightarrow (f^{k+1})^{(k)} = 1,$$

then $\rho(f) \leq 1$.

Proof. Let $\mathcal{F} = \{F_\omega\}$, where $F_\omega(z) = F(\omega + z) = f^{k+1}(\omega + z)$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ is a family of entire functions defined on $\mathbb{C}$. By assumption, we have $F(\omega + z) = 0 \Rightarrow F^{(k)}(\omega + z) = 0$ and $F(\omega + z) = 1 \Rightarrow F^{(k)}(\omega + z) = 1$. If $k = 1$, then by Theorem 1.3 [**?**], $\mathcal{F}$ is normal in $\mathbb{C}$. Henceforth we assume that $k \geq 2$.

Since normality is a local property, it is enough to show that $\mathcal{F}$ is normal at each point $z_0 \in \mathbb{C}$. Suppose on the contrary that $\mathcal{F}$ is not normal at z_0. Again since normality is a local property, we may assume that $\mathcal{F}$ is a family of holomorphic functions in a domain $D = \{z : 0 < |z - z_0| < R\}$, where $R > 0$. Then by Zalcman's lemma, there exist a sequence of functions $F_n \in \mathcal{F}$, where $F_n(z) = f^{k+1}(\omega_n + z)$, a sequence of complex numbers, z_n, $z_n \to z_0$ and a sequence of positive numbers ρ_n, $\rho_n \to 0$ such that

$$H_n(\zeta) = F_n(z_n + \rho_n\zeta) \to H(\zeta) \tag{2.1}$$

locally uniformly in $\mathbb{C}$, where H is a non-constant entire function such that $H^{\#}(\zeta) \leq 1$, $\forall\, \zeta \in \mathbb{C}$. Then by Lemma 2.4, we deduce that $\rho(H) \leq 1$.

Also by Hurwitz's theorem we conclude that all the zeros of H have multiplicity at least $k+1$. Clearly $H^{(k)} \not\equiv 0$. It is easy to deduce from (2.1) that

$$H_n^{(i)}(\zeta) = \rho_n^i F_n^{(i)}(z_n + \rho_n\zeta) \to H^{(i)}(\zeta) \tag{2.2}$$

locally uniformly in $\mathbb{C}$ for all $i \in \mathbb{N}$.

Now we claim that 1 is not a Picard exceptional value of H. If not, suppose 1 is a Picard exceptional value of H. Then by the second fundamental theorem, we have

$$\begin{aligned} T(r,H) \leq \overline{N}(r,0;H) + \overline{N}(r,1;H) + S(r,H) &\leq \frac{1}{k+1}\,N(r,0;H) + S(r,H) \\ &\leq \frac{1}{k+1}T(r,H) + S(r,H), \end{aligned}$$

which is impossible. Hence 1 is not a Picard exceptional value of H.

Suppose $H(\zeta_0) = 1$. Hurwitz's theorem implies the existence of a sequence $\zeta_n \to \zeta_0$ with

$$H_n(\zeta_n) = F_n(z_n + \rho_n\zeta_n) = 1.$$

Since $F = 1 \Rightarrow F^{(k)} = 1$, we have $H_n^{(k)}(z_n + \rho_n\zeta_n) = 1$. Then from (2.2), we have

$$H^{(k)}(\zeta_0) = \lim_{n\to\infty} H_n^{(k)}(\zeta_n) = \lim_{n\to\infty} \rho_n^k = 0.$$

Hence $H = 1 \Rightarrow H^{(k)} = 0$. First we suppose 0 is a Picard exceptional value of H. Since H is a non-constant entire function of order at most one and H has no zeros, then by Hadamard's Factorization theorem, we get $H(\zeta) = A\exp(\lambda\zeta)$, where $A, \lambda \in \mathbb{C} \setminus \{0\}$. Since $H = 1 \Rightarrow H^{(k)} = 0$, we get a contradiction.

Next we suppose that 0 is not a Picard exceptional value of H. Since all the zeros of H have multiplicity at least $k+1$, one can easily conclude that $H = 0 \Rightarrow H^{(k)} = 0$.

Also by the given condition, we have

$$\varphi_n(z_n+\rho_n\zeta)F_n(z_n+\rho_n\zeta)(F_n(z_n+\rho_n\zeta)-1) = F_n'(z_n+\rho_n\zeta)\big(F_n(z_n+\rho_n\zeta)-F_n^{(k)}(z_n+\rho_n\zeta)\big)$$

and so

$$\begin{aligned}(2.3)\qquad & \rho_n^{k+1}\varphi_n(z_n+\rho_n\zeta)F_n(z_n+\rho_n\zeta)(F_n(z_n+\rho_n\zeta)-1)\\ = \;& \rho_n F_n'(z_n+\rho_n\zeta)\big(\rho_n^k F_n(z_n+\rho_n\zeta)-\rho_n^k F_n^{(k)}(z_n+\rho_n\zeta)\big).\end{aligned}$$

Then from (2.1), (2.2) and (2.3), we conclude that

$$\rho_n^{k+1}\varphi_n(z_n+\rho_n\zeta)\to\psi_1(\zeta) \tag{2.4}$$

locally uniformly in $\mathbb{C}$, where ψ_1 is an entire function. Again using (2.1), (2.2) and (2.4), we deduce from (2.3) that

$$\psi_1(\zeta)H(\zeta)(H(\zeta)-1)=-H'(\zeta)H^{(k)}(\zeta). \tag{2.5}$$

Since $\rho(H)\leq 1$, it follows from (2.5) that $\rho(\psi_1)\leq 1$. Therefore applying Lemma 2.1, we deduce from (2.5) that $m(r,\psi_1)=o(\log r)$ as $r\to\infty$. Since $N(r,\psi_1)=0$, we have $T(r,\psi_1)=o(\log r)$ as $r\to\infty$, which implies that ψ_1 is a constant. We can write $\psi_1=c_1$, where $c_1\in\mathbb{C}\setminus\{0\}$. Consequently from (2.5), we have

$$c_1H(\zeta)(H(\zeta)-1)=-H'(\zeta)H^{(k)}(\zeta). \tag{2.6}$$

Let ζ_0 be a zero of H of multiplicity $m(\geq k+1)$. Then from (2.6), we conclude that $m=k+1$ and so all the zeros of H have multiplicity exactly $k+1$.

We claim that H is a transcendental entire function. If not, suppose that H is a polynomial. Since zeros of H are of multiplicity exactly $k+1$, H is a polynomial of degree $k+1$. Consequently we may assume that $H(\zeta)=a\,(\zeta-\zeta_0)^{k+1}$, where $a\in\mathbb{C}\setminus\{0\}$. Therefore $H^{(k)}(\zeta)=(k+1)!a(\zeta-\zeta_0)$. Note that $H(\zeta)-1=a\,(\zeta-\zeta_0)^{k+1}-1$. Since $H=1\Rightarrow H^{(k)}=0$, we obtain a contradiction. Hence H is a transcendental entire function.

Therefore we may assume that

$$H=h^{k+1}, \tag{2.7}$$

where h is a transcendental entire function having only simple zeros. Now (2.7) yields

$$\begin{aligned}(2.8)\qquad & H^{(k)}=(h^{k+1})^{(k)}=((k+1)h^kh')^{(k-1)}\\ =\;& (k+1)\big(kg^{k-1}(h')^2+h^kh''\big)^{(k-2)}\\ =\;& k(k+1)\big((k-1)h^{k-2}(h')^3\big)^{(k-3)}+k(k+1)\big(2h^{k-1}h'h''\big)^{(k-3)}\\ & +(k+1)\big(kh^{k-1}h'h''\big)^{(k-3)}+(k+1)\big(h^kh'''\big)^{(k-3)}\\ =\;& \cdots\cdots\cdots\cdots\\ =\;& (k+1)!h(h')^k+\frac{k(k-1)}{4}(k+1)!h^2(h')^{k-2}h''+\cdots+(k+1)h^kh^{(k)}\\ =\;& (k+1)!h(h')^k+\frac{k(k-1)}{4}(k+1)!h^2(h')^{k-2}h''+R_1(h),\end{aligned}$$

where $R_1(h)$ is a differential polynomial in h with constant coefficients and each term of $R_1(h)$ contains $h^m(3 \leq m \leq k)$ as a factor.

Denote by $N(r,1;H \mid\geq 2)$ the counting function of multiple 1-points of H.

Now we divide the following two cases.

Case 1. Suppose $N(r,1;H \mid\geq 2) = 0$. Then from (2.6), we conclude that h' has no zeros and so $\frac{h}{h'}$ is an entire function. Again from (2.6), we have $\frac{h}{h'} = -\frac{k+1}{c_1}\frac{H^{(k)}}{H-1}$ and so by Lemma 2.1, we deduce that $m(r,\frac{h}{h'}) = o(\log r)$ as $r \to \infty$. Since $N(r,\frac{h}{h'}) = 0$, we have $T(r,\frac{h}{h'}) = o(\log r)$ as $r \to \infty$, which implies that $\frac{h}{h'}$ is a constant. We can write $\frac{h}{h'} = c_2$, where $c_2 \in \mathbb{C}\setminus\{0\}$. On integration, we have $h(\zeta) = c_3\exp(\frac{1}{c_2}\zeta)$, where $c_3 \in \mathbb{C}\setminus\{0\}$. This shows that H has no zeros, which is impossible.

Case 2. Suppose $N(r,1;H \mid\geq 2) \neq 0$. Now from (2.6), (2.7) and (2.8), we have

$$c_1h^{k+1}(h^{k+1}-1) = -(k+1)h^kh'\big((k+1)!h(h')^k + \frac{k(k-1)}{4}(k+1)!h^2(h')^{k-2}h'' + R_1(h)\big),$$

i.e.,

$$c_1(h^{k+1}-1) = -(k+1)(k+1)!(h')^{k+1} - \frac{k(k-1)(k+1)}{4}(k+1)!h(h')^{k-1}h'' + R_1(h). \tag{2.9}$$

Differentiating (2.9) once, we get

$$c_1(k+1)h^kh' = -(k+1)!(k+1)^2(h')^kh'' - \frac{k(k-1)(k+1)(k+1)!}{4}\big((h')^kh'' + (k-1)h(h')^{k-2}(h'')^2 + h(h')^{k-1}h'''\big) + R_2(h), \tag{2.10}$$

where $R_2(h)$ is a differential polynomial in h.

Let ζ_0 be a zero of h. Now from (2.9) and (2.10), we have respectively

$$c_1 = (k+1)(k+1)!(h'(\zeta_0))^{k+1} \tag{2.11}$$

and

$$\big(k+1+\frac{k(k-1)}{4}\big)(h'(\zeta_0))^kh''(\zeta_0) = 0. \tag{2.12}$$

If $h''(\zeta_0) \neq 0$, then from (2.11) and (2.12) we arrive at a contradiction. Hence $h''(\zeta_0) = 0$ and so $h = 0 \Rightarrow h'' = 0$. Let $H_1 = \frac{h''}{h}$. Clearly $H_1 \not\equiv 0$. One can easily prove that H_1 is a non-zero constant, say $\lambda \in \mathbb{C}$. Therefore

$$h'' = \lambda_1 h. \tag{2.13}$$

Solving (2.13), we get

$$h(\zeta) = A_1\exp\big(\sqrt{\lambda_1}\zeta\big) + B_1\exp\big(-\sqrt{\lambda_1}\zeta\big),$$

where $A_1, B_1 \in \mathbb{C}\setminus\{0\}$. Note that

$$h'(\zeta) = A_1\sqrt{\lambda_1}\exp\big(\sqrt{\lambda_1}\zeta\big) - \sqrt{\lambda_1}B_1\exp\big(-\sqrt{\lambda_1}\zeta\big).$$

Again differentiating (2.13) and using it repeatedly, we have

$$h^{(2i)} = {\lambda_1}^i h \;\text{ and }\; h^{(2i+1)} = {\lambda_1}^i h', \;\text{ where }\; i = 1, 2, \ldots. \tag{2.14}$$

Then from (2.7) and (2.14), one can easily deduce that

$$(h^{k+1})^{(k)} = \tilde{c}_1 h(h')^k + \tilde{c}_2 h^2 (h')^{k-1} + \tilde{c}_3 h^3 (h')^{k-2} + \ldots + \tilde{c}_k h^k h' + \tilde{c}_{k+1} h^{k+1}, \tag{2.15}$$

where $\tilde{c}_1 = (k+1)!$ and $\tilde{c}_i \in \mathbb{C}$ for $i \geq 2$.

First we suppose $\tilde{c}_{k+1} \neq 0$. Let ζ_1 be a multiple zero of $H - 1$. Then obviously $H(\zeta_1) = 1$, $H'(\zeta_1) = 0$ and $H^{(k)}(\zeta_1) = 0$. Note that $H' = (k+1)h^k h'$. Since $H'(\zeta_1) = 0$, it follows that $h'(\zeta_1) = 0$ and $h(\zeta_1) \neq 0$. Therefore from (2.15), we get $\tilde{c}_{k+1} = 0$, which is impossible.

Next we suppose $\tilde{c}_{k+1} = 0$. Let $g = \frac{h'}{h}$. Obviously both $g - \sqrt{\lambda_1}$ and $g + \sqrt{\lambda_1}$ have no zeros. Now from (2.6) and (2.15), we deduce that

$$\begin{aligned} &c_1 h^{k+1} + (k+1)\big(\tilde{c}_1 (h')^{k+1} + \tilde{c}_2 h (h')^k + \\ &\quad \tilde{c}_3 h^2 (h')^{k-1} + \ldots + \tilde{c}_k h^{k-1} (h')^2\big) = c_1. \end{aligned} \tag{2.16}$$

Putting $h' = gh$ into (2.16), we get

$$(k+1)\big(\tilde{c}_1 g^{k+1} + \tilde{c}_2 g^k + \tilde{c}_3 g^{k-1} + \ldots + \tilde{c}_k g^2\big) + c_1 = \frac{c_1}{h^{k+1}}. \tag{2.17}$$

Note that the right hand side of (2.17) has no zeros. Consequently the left hand side may not have no zeros. Since both $g - \sqrt{\lambda_1}$ and $g + \sqrt{\lambda_1}$ have no zeros, we conclude that the left hand side of (2.17) must be one of the forms (i) $(k+1)\tilde{c}_1 (g - \sqrt{\lambda_1})^{k+1}$, (ii) $(k+1)\tilde{c}_1 (g + \sqrt{\lambda_1})^{k+1}$ and (iii) $(k+1)\tilde{c}_1 (g - \sqrt{\lambda_1})^m (g + \sqrt{\lambda_1})^n$, where $m + n = k + 1$. Note that

$$\begin{aligned} (k+1)\tilde{c}_1 \big(g - \sqrt{\lambda_1}\big)^{k+1} &= (k+1)\tilde{c}_1 g^{k+1} - (k+1)^2 \tilde{c}_1 \sqrt{\lambda_1} g^k + \cdots \\ &\quad + (-1)^k (k+1)^2 \tilde{c}_1 (\sqrt{\lambda_1})^k g + (-1)^{k+1} (k+1)\tilde{c}_1 (\sqrt{\lambda_1})^{k+1}. \end{aligned} \tag{2.18}$$

If the left hand side of (2.17) $= (k+1)\tilde{c}_1 (g - \sqrt{\lambda_1})^{k+1}$, then the left hand side of (2.17) and the right hand side of (2.18) must be identical. Note that the coefficient of g of the right hand side of (2.18) is non-vanishing. Therefore from (2.17) and (2.18), we arrive at a contradiction. Similarly if the left hand side of (2.17) $=$ $(k+1)\tilde{c}_1 (g + \sqrt{\lambda_1})^{k+1}$, then we get a contradiction. Again if the left hand side of (2.17) $= (k+1)\tilde{c}_1 (g - \sqrt{\lambda_1})^m (g + \sqrt{\lambda_1})^n$, where $m + n = k + 1$, then by a simple calculation we deduce that $m = n$ and $(-1)^m (k+1)\tilde{c}_1 (\sqrt{\lambda_1})^{m+n} = c_1$, i.e., $2m = k + 1$ and $(-1)^m (k+1)(k+1)!(\sqrt{\lambda_1})^{k+1} = c_1$. Clearly k is odd. Note that

$$\begin{aligned} H(\zeta) = (h(\zeta))^{k+1} &= A_1^{k+1} \exp\big((k+1)\sqrt{\lambda}\zeta\big) + \\ &\quad \ldots + B_1^{k+1} \exp\big(-(k+1)\sqrt{\lambda}\zeta\big). \end{aligned} \tag{2.19}$$

Now from (2.6) we get $c_1H^2(\zeta) + H'(\zeta)H^{(k)}(\zeta) = c_1H(\zeta)$ and so from (2.19) we can easily conclude that $c_1 + (k+1)^{k+1}(\sqrt{\lambda})^{k+1} = 0$. Since $(-1)^m(k+1)(k+1)!(\sqrt{\lambda_1})^{k+1} = c_1$, we get $(-1)^{\frac{k+1}{2}}(k+1)! + (k+1)^k = 0$, which is impossible for $k \geq 2$.

Hence all the foregoing discussion shows that $\mathcal{F}$ is normal at z_0. Consequently $\mathcal{F}$ is normal in $\mathbb{C}$. Hence by Lemma 2.3, there exists $M > 0$ satisfying $F^{\#}(\omega) = F_{\omega}^{\#}(0) < M$ for all $\omega \in \mathbb{C}$. Consequently by Lemma 2.4, we conclude that $\rho(F) \leq 1$ and so $\rho(f) \leq 1$. This completes the proof. □

3. Proof of the main result

Proof of Theorem 1.1. Let $F = f^n$. We put

$$\varphi = \frac{F'(F - F^{(k)})}{F(F-1)}. \tag{3.1}$$

Differentiating (3.1) once, we get

$$F''(F - F^{(k)}) + F'(F' - F^{(k+1)}) = \varphi' F(F-1) + \varphi F'(2F-1). \tag{3.2}$$

Now we divide the following two cases.

Case 1. Suppose $\varphi \not\equiv 0$. Then $F \not\equiv F^{(k)}$ and from (3.1), we get

$$\varphi = \frac{F'}{F-1}\Big(1 - \frac{F^{(k)}}{F}\Big). \tag{3.3}$$

Let z_0 be a zero of f with multiplicity p_0. Then z_0 is a zero of F and $F^{(k)}$ of multiplicities np_0 and $np_0 - k$ respectively and so from (3.1), we get in some neighbourhood of z_0

$$\varphi(z) = O\big((z - z_0)^{np_0 - k - 1}\big). \tag{3.4}$$

Since $n \geq k+1$, it follows from (3.4) that φ is analytic at z_0. Let z_1 be a zero of $F - 1$ of multiplicity p_1. Since $F = 1 \Rightarrow F^{(k)} = 1$, it follows that z_1 is a zero of $F^{(k)} - 1$ with multiplicity q_1, say. By Taylor's theorem we get in some neighbourhood of z_1

$$F(z) = 1 + a_{p_1}(z - z_1)^{p_1} + O(z - z_1)^{p_1+1},$$

$$F^{(k)}(z) = 1 + b_{q_1}(z - z_1)^{q_1} + O(z - z_1)^{q_1+1}$$

and

$$F'(z) = p_1 a_{p_1}(z - z_1)^{p_1 - 1} + O(z - z_1)^{p_1},$$

where $a_{p_1} \neq 0$ and $b_{q_1} \neq 0$. Consequently in some neighbourhood of z_1

$$\begin{aligned} F(z) - F^{(k)}(z) &= a_{p_1}(z-z_1)^{p_1} + O(z-z_1)^{p_1+1} \text{ if } p_1 < q_1 \\ &= -b_{q_1}(z-z_1)^{q_1} + O(z-z_1)^{q_1+1} \text{ if } q_1 < p_1 \\ &= (a_{p_1} - b_{q_1})(z-z_1)^{p_1} + O(z-z_1)^{p_1+1} \text{ if } p_1 = q_1. \end{aligned}$$

Then in some neighbourhood of z_1, we get from (3.1) that $\varphi(z) = O\big((z-z_1)^{t-1}\big)$, where $t = \min\{p_1, q_1\} \geq 1$ if $p_1 \neq q_1$ and $t \geq p_1 = q_1 \geq 1$ otherwise. Therefore we conclude that φ is analytic at z_1.

Since f is an entire function, from the above discussion, we deduce that φ is an entire function. Also (3.3) gives $m(r,\varphi) = S(r,f)$ and so $T(r,\varphi) = S(r,f)$. Again from (3.1), we get

$$\frac{1}{F} = \frac{1}{\varphi}\Big(\frac{F'}{F-1} - \frac{F'}{F}\Big)\Big(1 - \frac{F^{(k)}}{F}\Big). \tag{3.5}$$

Therefore we have $m(r, \frac{1}{F}) = S(r,f)$ and so $m(r, \frac{1}{f}) = S(r,f)$.

First we suppose $n > k+1$. Then from (3.4) we get $N(r,0;f) \leq N(r,0;\varphi) = S(r,f)$. Since $m(r, \frac{1}{f}) = S(r,f)$, we conclude that $T(r,f) = T(r, \frac{1}{f}) + O(1) = S(r,f)$, which is impossible.

Next we suppose $n = k+1$. Let z_0 be a zero of f with multiplicity p_0. So z_0 is a zero of F and $F^{(k)}$ of multiplicities $(k+1)p_0$ and $(k+1)p_0 - k$ respectively. If $p_0 \geq 2$, then from (3.1), we see that z_0 is a zero of φ, i.e., $\varphi(z_0) = 0$. Next we suppose that $p_0 = 1$. Clearly from (3.1), we get $\varphi(z_0) \neq 0$. Then in some neighbourhood of z_0, we get by Taylor's expansion

$$F(z) = \tilde{a}_{k+1}(z-z_0)^{k+1} + \tilde{a}_{k+2}(z-z_0)^{k+2} + \ldots \; (\tilde{a}_{k+1} \neq 0). \tag{3.6}$$

Clearly

$$\begin{cases} F'(z) = (k+1)\tilde{a}_{k+1}(z-z_0)^{k} + (k+2)\tilde{a}_{k+2}(z-z_0)^{k+1} + \ldots, \\ F''(z) = (k+1)k\tilde{a}_{k+1}(z-z_0)^{k-1} + (k+2)(k+1)\tilde{a}_{k+2}(z-z_0)^{k} + \ldots, \\ \ldots\ldots, \\ F^{(k)}(z) = (k+1)!\tilde{a}_{k+1}(z-z_0) + \ldots, \\ F^{(k+1)}(z) = (k+1)!\tilde{a}_{k+1} + \ldots \end{cases} \tag{3.7}$$

Now from (3.2), (3.6) and (3.7), we deduce that

$$\left(\varphi(z_0) - (k+1)(k+1)!\tilde{a}_{k+1}\right)(z-z_0)^{k} + O\big((z-z_0)^{k+1}\big) \equiv 0,$$

which implies that $(k+1)!\tilde{a}_{k+1} = \frac{\varphi(z_0)}{k+1}$, i.e., $F^{(k+1)}(z_0) = \frac{\varphi(z_0)}{k+1}$. Consequently we get

$$F = 0 \Rightarrow F^{(k+1)} = \frac{\varphi}{k+1}. \tag{3.8}$$

Now from Lemma 2.5, we conclude that $\rho(F) \leq 1$. Consequently using Lemma 2.1, we deduce from (3.3) that $m(r,\varphi) = o(\log r)$ as $r \to \infty$.

Since $N(r,\varphi)=0$, we have $T(r,\varphi)=o(\log r)$ as $r\to\infty$, which implies that φ is a constant. We can write $\varphi=c$, where $c\in\mathbb{C}\setminus\{0\}$. Then from (3.1), we have

$$F'(F-F^{(k)})=cF(F-1). \tag{3.9}$$

Also from (3.9), one can easily conclude that f has only simple zeros, i.e., all the zeros of F have multiplicity exactly $k+1$.

We claim that F is a transcendental entire function. If not, suppose that F is a polynomial. Since zeros of F are of multiplicity exactly $k+1$, F is a polynomial of degree $k+1$. Consequently we may assume that $F(z)=a\left(z-\hat{z}_0\right)^{k+1}$, where $a\in\mathbb{C}\setminus\{0\}$. Therefore $F^{(k)}(z)=(k+1)!a(z-\hat{z}_0)$. Note that $F(z)-1=a\left(z-\hat{z}_0\right)^{k+1}-1$ and $F^{(k)}(z)-1=(k+1)!a(z-\hat{z}_0)-1$. It is clear that $F-1$ has $k+1$ distinct zeros. Since $F=1\Rightarrow F^{(k)}=1$, we obtain a contradiction. Hence F is a transcendental entire function.

Now applying Lemma 2.1, we deduce from (3.5) that $m\left(r,\frac{1}{F}\right)=o(\log r)$ as $r\to\infty$.

Let 0 be a Picard exceptional value of F. Then $T(r,F)=T\left(r,\frac{1}{F}\right)+O(1)=m\left(r,\frac{1}{F}\right)+O(1)=o(\log r)$ as $r\to\infty$, which implies that F is a constant. Therefore we arrive at a contradiction. Hence 0 is not a Picard exceptional value of F.
If 1 is a Picard exceptional value of F, by the second fundamental theorem, we get

$$\begin{aligned}
&(k+1)T(r,f)=T(r,F)+O(1)\leq \overline{N}(r,0;F)+\overline{N}(r,1;F)+S(r,F)\\
\leq\ &\frac{1}{k+1}\,N(r,0;f)+S(r,f)\leq\frac{1}{k+1}T(r,f)+S(r,f),
\end{aligned}$$

which is impossible. Hence 1 is not a Picard exceptional value of F. Also we have

$$F=f^{k+1}. \tag{3.10}$$

Therefore from (3.10), we deduce that

$$\begin{aligned}
F^{(k)}&=(k+1)!f(f')^k+\frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''+\cdots+(k+1)f^kf^{(k)} \qquad (3.11)\\
&=(k+1)!f(f')^k+\frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''+R_1(f),
\end{aligned}$$

where $R_1(f)$ is a differential polynomial in f with constant coefficients and each term of $R_1(f)$ contains $f^m(3\leq m\leq k)$ as a factor. Differentiating (3.11) once, we get

$$\begin{aligned}
F^{(k+1)}&=(f^{k+1})^{(k+1)} \qquad (3.12)\\
&=(k+1)!(f')^{k+1}+\frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''+(k+1)f^kf^{(k+1)}\\
&=(k+1)!(f')^{k+1}+\frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''+S_1(f),
\end{aligned}$$

where $S_1(f)$ is a differential polynomial in f and each term of $S_1(f)$ contains f and its higher powers as a factor. Again differentiating (3.12) once, we get

$$F^{(k+2)} = (f^{k+1})^{(k+2)} = \frac{(k+1)!(k+1)(k+2)}{2}(f')^k f'' + S_2(f), \tag{3.13}$$

where $S_2(f)$ is a differential polynomial in f and each term of $S_2(f)$ contains f and its higher powers as a factor.

Now from (3.9), we deduce that

$$\frac{(k+1)f'\left(f^{k+1} - (f^{k+1})^{(k)}\right)}{f\left(f^{k+1} - 1\right)} = c, \tag{3.14}$$

i.e.,

$$(k+1)f^{k+1}f' - (k+1)f'(f^{k+1})^{(k)} - cf^{k+2} = -cf. \tag{3.15}$$

Denote by $N(r, 1; F \mid\geq 2)$ the counting function of multiple 1-points of F.

Now we divide the following two sub-cases.

Sub-case 1.1. Suppose $N(r, 1; F \mid\geq 2) = 0$. Then from (3.14), we conclude that $f' \neq 0$. Since f is a transcendental entire function and $\rho(f) \leq 1$, it follows that

$$f'(z) = d_0 \exp(\lambda z), \tag{3.16}$$

where $d_0, \lambda \in \mathbb{C} \setminus \{0\}$. On integration, we have

$$f(z) = \frac{d_0}{\lambda} \exp(\lambda z) + d_1, \tag{3.17}$$

where $d_1 \in \mathbb{C}$. Since 0 is not a Picard exceptional value of $F = f^{k+1}$, it follows that $d_1 \neq 0$.

Let z_0 be a zero of f. Then $f(z_0) = 0$ and $F(z_0) = 0$. Also (3.8) gives $F^{(k+1)}(z_0) = \frac{c}{k+1}$. Now from (3.12), we conclude that

$$\left(f'(z_0)\right)^{k+1} = \frac{c}{(k+1)(k+1)!}. \tag{3.18}$$

Also from (3.17), we have

$$d_0 \exp(\lambda z_0) = -\lambda d_1, \text{ i.e., } d_0^{k+1} \exp((k+1)\lambda z_0) = (-\lambda d_1)^{k+1}. \tag{3.19}$$

Again from (3.16) and (3.18), we deduce that

$$d_0^{k+1} \exp((k+1)\lambda z_0) = \frac{c}{(k+1)(k+1)!}. \tag{3.20}$$

Therefore from (3.19) and (3.20), we deduce that

$$(-\lambda d_1)^{k+1} = \frac{c}{(k+1)(k+1)!}. \tag{3.21}$$

Now from (3.17), we have

$$\begin{aligned} f^{k+1}(z) &= \left(\frac{d_0}{\lambda}\right)^{k+1}\exp((k+1)\lambda z)+\binom{k+1}{1}\left(\frac{d_0}{\lambda}\right)^{k} d_1\exp(k\lambda z)+\dots \\ &\quad +\binom{k+1}{k-1}\left(\frac{d_0}{\lambda}\right)^{2} d_1^{k-1}\exp(2\lambda z)+\binom{k+1}{k}\frac{d_0}{\lambda}d_1^{k}\exp(\lambda z)+d_1^{k+1}. \end{aligned}$$

and

$$\begin{aligned} f^{k+2}(z) &= \left(\frac{d_0}{\lambda}\right)^{k+2}\exp((k+2)\lambda z)+\binom{k+2}{1}\left(\frac{d_0}{\lambda}\right)^{k+1} d_1\exp((k+1)\lambda z)+\dots \\ &\quad +\binom{k+2}{k}\left(\frac{d_0}{\lambda}\right)^{2} d_1^{k}\exp(2\lambda z)+\binom{k+2}{k+1}\frac{d_0}{\lambda}d_1^{k+1}\exp(\lambda z)+d_1^{k+2}. \end{aligned}$$

Therefore

$$\begin{aligned} &(f^{k+1}(z))^{(k)} \\ &= \left(\frac{d_0}{\lambda}\right)^{k+1}((k+1)\lambda)^k\exp((k+1)\lambda z)+\binom{k+1}{1}\left(\frac{d_0}{\lambda}\right)^{k}(k\lambda)^k d_1\exp(k\lambda z)+\dots \\ &\quad +\binom{k+1}{k-1}\left(\frac{d_0}{\lambda}\right)^{2}(2\lambda)^k d_1^{k-1}\exp(2\lambda z)+\binom{k+1}{k}\frac{d_0}{\lambda}\lambda^k d_1^{k}\exp(\lambda z). \end{aligned}$$

Now from (3.15), we deduce that

$$\begin{aligned} (3.22)\quad &\left\{\frac{k+1}{\lambda^k}-(k+1)^{k+1}-\frac{c}{\lambda^{k+1}}\right\}\frac{d_0^{k+2}}{\lambda}\exp((k+2)\lambda z) \\ &+\left\{\frac{(k+1)^2}{\lambda^k}-(k+1)^2k^k-\frac{c(k+2)}{\lambda^{k+1}}\right\}d_0^{k+1}d_1\exp((k+1)\lambda z) \\ &+\dots+\left\{\frac{(k+1)^2}{\lambda}-(k+1)^2\lambda-c\binom{k+2}{k}\frac{1}{\lambda^2}\right\}d_0^2d_1^k\exp(2\lambda z) \\ &+\left\{(k+1)-\frac{c(k+2)}{\lambda}\right\}d_0d_1^{k+1}\exp(\lambda z)-cd_1^{k+2}=-\frac{cd_0}{\lambda}\exp(\lambda z)-cd_1. \end{aligned}$$

Clearly from (3.22), we have

$$(3.23)\quad d_1^{k+1}=1 \ \text{ and } \ \left\{(k+1)-\frac{c(k+2)}{\lambda}\right\}d_0d_1^{k+1}=-\frac{cd_0}{\lambda},\ \text{ i.e., }\ c=\lambda.$$

Now from (3.21) and (3.23), we have

$$(3.24)\qquad \lambda^k=(-1)^{k+1}\frac{1}{(k+1)(k+1)!}.$$

First we suppose $k=1$. Then from (3.23) and (3.24), we have respectively $d_1^2=1$ and $c=\lambda=\frac{1}{4}$. Also from (3.17), we have $f(z)=c_0\exp\left(\frac{1}{4}z\right)+d_1$, where $c_0=4d_0$.

Next we suppose $k\geq 2$. Now from (3.23) and (3.24), we calculate that

$$(3.25)\quad \frac{k+1}{\lambda^k}-(k+1)^{k+1}-\frac{c}{\lambda^{k+1}}=(k+1)\big((-1)^{k+1}k(k+1)!-(k+1)^k\big)\neq 0$$

for $k\geq 2$. Therefore one can easily arrive at a contradiction from Lemma 2.2 and (3.22).

Sub-case 1.2. Suppose $N(r,1;F\mid\geq 2)\neq 0$. Now differentiating (3.15) once, we have

$$(k+1)^2f^k(f')^2+(k+1)f^{k+1}f''-(k+1)f''(f^{k+1})^{(k)}- \quad (3.26)$$
$$(k+1)f'(f^{k+1})^{(k+1)}-c(k+2)f^{k+1}f'=-cf'.$$

Again differentiating (3.26) once, we have

$$(k+1)^2kf^{k-1}(f')^3+3(k+1)^2f^kf'f''+(k+1)f^{k+1}f'''-(k+1)f'''(f^{k+1})^{(k)}$$
$$-2(k+1)f''(f^{k+1})^{(k+1)}-(k+1)f'(f^{k+1})^{(k+2)}-c(k+2)(k+1)f^k(f')^2 \quad (3.27)$$
$$-c(k+2)f^{k+1}f''=-cf''.$$

Now from (3.11) (3.12), (3.13) and (3.27), we get

$$-(k+1)(k+1)!\frac{k^2+3k+6}{2}(f')^{k+1}f''+S_3(f)=-cf'', \quad (3.28)$$

where $S_3(f)$ is a differential polynomial in f and each term of $S_3(f)$ contains f and its higher powers as a factor. Let z_0 be a zero of f. Now from (3.18) and (3.28), we have respectively

$$(f'(z_0))^{k+1}=\frac{c}{(k+1)(k+1)!} \quad (3.29)$$

and

$$(k+1)(k+1)!\frac{k^2+3k+6}{2}(f'(z_0))^{k+1}f''(z_0)=cf''(z_0). \quad (3.30)$$

If $f''(z_0)\neq 0$, then from (3.29) and (3.30) we arrive at a contradiction. Hence $f''(z_0)=0$ and so $f=0\Rightarrow f''=0$. Let

$$H_1=\frac{f''}{f}. \quad (3.31)$$

Clearly $H_1\not\equiv 0$. One can easily prove that H_1 is a non-zero constant. Let us suppose that $H_1=\tilde{\lambda}\in\mathbb{C}\setminus\{0\}$. Now from (3.31), we deduce that

$$f''=\tilde{\lambda}f. \quad (3.32)$$

Differentiating (3.32) and using it repeatedly, we have

$$f^{(2i)}=\tilde{\lambda}^if \ \text{ and } \ f^{(2i+1)}=\tilde{\lambda}^if', \ \text{ where } \ i=1,2,\ldots. \quad (3.33)$$

First we suppose k is odd. Then from (3.11) and (3.33), one can easily deduce that

$$(f^{k+1})^{(k)}=c_1f(f')^k+c_3f^3(f')^{k-2}+\ldots+c_kf^kf', \quad (3.34)$$

where $c_1=(k+1)!$ and $c_i\in\mathbb{C}$ for $i\geq 3$.

Denote by $\overline{N}(r,1;F,F^{(k)}\mid\geq 2)$ the reduced counting function of common multiple 1-points of F and $F^{(k)}$.

If z_1 is a zero of $F-1$ with multiplicity $p_1 \geq 2$ and a zero of $F^{(k)}-1$ with multiplicity $q_1 \geq 2$, then from (3.9), we deduce that

$$\overline{N}(r,1;F,F^{(k)} \mid\geq 2)=0. \tag{3.35}$$

Let z_1 be a zero of $F-1$ of multiplicity p_1. Then from (3.35), we conclude that z_1 is a simple zero of $F^{(k)}-1$. Obviously $F(z_1)=1$ and $F^{(k)}(z_1)=1$, i.e., $(f^{k+1})^{(k)}(z_1)=1$. Note that $F'=(k+1)f^kf'$. Since $F'(z_1)=0$, it follows that $f'(z_1)=0$ and $f(z_1)\neq 0$. Therefore from (3.34), we conclude that $1=0$, which is impossible.

Next we suppose k is even. Solving (3.32), we get

$$f(z)=A_1\exp\big(\sqrt{\tilde{\lambda}}z\big)+B_1\exp\big(-\sqrt{\tilde{\lambda}}z\big), \tag{3.36}$$

where $A_1,B_1\in\mathbb{C}\setminus\{0\}$. Note that

$$f'(z)=A_1\sqrt{\tilde{\lambda}}\exp\big(\sqrt{\tilde{\lambda}}z\big)-\sqrt{\tilde{\lambda}}B_1\exp\big(-\sqrt{\tilde{\lambda}}z\big) \tag{3.37}$$

$$(f(z))^{k+1}=A_1^{k+1}\exp\big((k+1)\sqrt{\tilde{\lambda}}z\big)+\ldots+B_1^{k+1}\exp\big(-(k+1)\sqrt{\tilde{\lambda}}z\big) \tag{3.38}$$

$$(f(z))^{k+2}=A_1^{k+2}\exp\big((k+2)\sqrt{\tilde{\lambda}}z\big)+\ldots+B_1^{k+2}\exp\big(-(k+2)\sqrt{\tilde{\lambda}}z\big) \tag{3.39}$$

and so

$$\begin{aligned}\big((f(z))^{k+1}\big)^{(k)} &= A_1^{k+1}\big((k+1)\sqrt{\tilde{\lambda}}\big)^k\exp\big((k+1)\sqrt{\tilde{\lambda}}z\big)+\ldots\\ &\quad +B_1^{k+1}(-1)^k\big((k+1)\sqrt{\tilde{\lambda}}\big)^k\exp\big(-(k+1)\sqrt{\tilde{\lambda}}z\big).\end{aligned} \tag{3.40}$$

Therefore from (3.15) and (3.36)-(3.40), we deduce that

$$\begin{aligned}&A_1^{k+2}\Big(\big((k+1)\sqrt{\tilde{\lambda}}\big)^{k+1}-(k+1)\sqrt{\tilde{\lambda}}+c\Big)\exp\big((k+2)\sqrt{\tilde{\lambda}}z\big)+\ldots\\ &+B_1^{k+2}\Big((-1)^{k+1}\big((k+1)\sqrt{\tilde{\lambda}}\big)^{k+1}+(k+1)\sqrt{\tilde{\lambda}}+c\Big)\exp\big(-(k+2)\sqrt{\tilde{\lambda}}z\big)\\ =\ &A_1c\exp\big(\sqrt{\tilde{\lambda}}z\big)+B_1c\exp\big(-\sqrt{\tilde{\lambda}}z\big).\end{aligned} \tag{3.41}$$

Now from (3.41), one can easily conclude that

$$\big((k+1)\sqrt{\tilde{\lambda}}\big)^{k+1}-(k+1)\sqrt{\tilde{\lambda}}+c=0 \text{ and } -\big((k+1)\sqrt{\tilde{\lambda}}\big)^{k+1}+(k+1)\sqrt{\tilde{\lambda}}+c=0.$$

Solving we get $c=0$, which is impossible.

Case 2. Suppose $\varphi\equiv 0$. Since $F'\not\equiv 0$, we get $F\equiv F^{(k)}$. Now $F\equiv F^{(k)}$ implies that $\rho(F)=1$, i.e., $\rho(f)=1$ and f has no zeros. Therefore we conclude that $f(z)=d\exp\big(\frac{\lambda}{n}z\big)$, where $d,\lambda\in\mathbb{C}\backslash\{0\}$ such that $\lambda^k=1$. This completes the proof. □

4. Some applications

From Theorem 1.1, we see that the problem of the entire function g and its k-th derivative sharing one value a is related to the problem of the non-linear differential equation $g'(g-g^{(k)})-\varphi g(g-a)=0$ having a non-constant entire solution, where φ is an entire function. In general, it is difficult to judge whether the differential equation has a non-constant solution. However for the very special case $g=f^n$, where $n\in\mathbb{N}$, we can solve the equation completely.

As the applications of Theorem 1.1, we now present the following results.

Theorem 4.1. *Let φ be an entire function and $k,n\in\mathbb{N}$. Suppose F is a non-constant meromorphic solution of the differential equation $F'(F-F^{(k)})-\varphi F(F-1)=0$, where $F=f^n$ and $n\geq k+1$. Then only one of the following cases holds:*

1. *$F\equiv F^{(k)}$ and $f(z)=c\exp\left(\frac{\lambda}{n}z\right)$, where $c,\lambda\in\mathbb{C}\setminus\{0\}$ such that $\lambda^k=1$,*
2. *$n=2$ and $f(z)=c_0\exp\left(\frac{1}{4}z\right)+c_1$, where $c_0,c_1\in\mathbb{C}\setminus\{0\}$ such that $c_1^2=1$.*

Theorem 4.2. *Let φ be a non-constant entire function and $k,n\in\mathbb{N}$. Suppose F is a non-constant meromorphic solution of the differential equation $F'\left(F-F^{(k)}\right)-\varphi F(F-1)=0$, where $F=f^n$ and $n\geq k+1$. Then $k=1$, $n=2$ and $f(z)=c_0\exp\left(\frac{1}{4}z\right)+c_1$, where $c_0,c_1\in\mathbb{C}\setminus\{0\}$ such that $c_1^2=1$.*

Proof of Theorem 4.1. We have

$$F'(F-F^{(k)})=\varphi F(F-1), \tag{4.1}$$

where $F=f^n$ and φ is a non-constant entire function. Let F be a non-constant meromorphic solution of the equation (4.1). Now we divide the following two cases.

Case 1. Suppose $\varphi\not\equiv 0$. Since φ is a non-constant entire function, from (4.1), one can easily conclude that F is a non-constant entire function. Now we prove that $F=1\Rightarrow F^{(k)}=1$. If 1 is a Picard exceptional value of F, then obviously $F=1\Rightarrow F^{(k)}=1$. Next we suppose that 1 is not a Picard exceptional value of F. Let z_0 be a zero of $F-1$ of multiplicity p_0. Clearly z_0 is a zero of F' of multiplicity p_0-1. Then from (4.1), we deduce that z_0 must be a zero of $F-F^{(k)}$. Since $F-F^{(k)}=(F-1)-(F^{(k)}-1)$, it follows that z_0 is a zero of $F^{(k)}-1$. So $F=1\Rightarrow F^{(k)}=1$. Since $\varphi\not\equiv 0$, we have $F\not\equiv F^{(k)}$. Now proceeding in the same way as done in the proof of Case 1 of Theorem 1.1, one can easily conclude that $k=1$, $n=2$ and $f(z)=c_0\exp\left(\frac{1}{4}z\right)+c_1$, where $c_0,c_1\in\mathbb{C}\setminus\{0\}$ such that $c_1^2=1$.

Case 2. Suppose $\varphi\equiv 0$. Since $F'\not\equiv 0$, it follows that $F\equiv F^{(k)}$. We now want to prove that F is an entire function. For this let z_1 be a pole of F of multiplicity p_1. Then z_1 is also a pole $F^{(k)}$ of multiplicity p_1+k. Since $F\equiv F^{(k)}$, we arrive at a contradiction. Hence F is an entire function. The fact $F \equiv F^{(k)}$ implies that $\rho(F) = 1$, i.e., $\rho(f) = 1$. Also $F \equiv F^{(k)}$ implies that f has no zeros. Therefore we conclude that $f(z) = d\exp\left(\frac{\lambda}{n}z\right)$, where $d, \lambda \in \mathbb{C}\backslash\{0\}$ such that $\lambda^k = 1$. This completes the proof. □

Proof of Theorem 4.2. Since φ is a non-constant entire function, it follows that $\varphi \not\equiv 0$. Now the proof of Theorem 4.2 follows directly from the proof of Theorem 4.1. So we omit the proof. □

Now from Theorems 4.1 and 4.2, we immediately obtain the following corollary.

Corollary 4.1. *Let φ be a non-constant entire function and $k, n \in \mathbb{N}$ such that $k \geq 2$. Then the differential equation $F'(F - F^{(k)}) - \varphi F(F - 1) = 0$, where $F = f^n$ and $n \geq k + 1$ has no solutions.*

Following example shows that the condition "$n \geq k+1$" in Theorem 4.1 is sharp.

Example 4.1. *Let $f(z) = \exp(z) + \exp(-z)$, $k = 2$, $n = 1$ and $\varphi = 0$. Clearly f satisfies the differential equation (4.1), but f does not satisfy any case of Theorem 4.1.*

Список литературы

[1] R. Brück, "On entire functions which share one value CM with their first derivative", Results Math., **30**, 21 – 24 (1996).
[2] J. M. Chang, L. Zalcman, "Meromorphic functions that share a set with their derivatives", J. Math. Anal. Appl., **338**, 1191 – 1205 (2008).
[3] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford (1964).
[4] J. Heittokangas, R. Korhonen, J. Rättyä, "Generalized logarithmic derivative estimates of Gol'dberg-Grinshtein type", Bull. Londan Math. Soc., **36**, 105 – 114 (2004).
[5] V. Ngoan, I. V. Ostrovskii, "The logarithmic derivative of a meromorphic function" [in Russian], Akad. Nauk. Armyan, SSR Dokl., **41**, 272 – 277 (1965).
[6] J. Schiff, Normal Families, Berlin (1993).
[7] C. C. Yang, "On deficiencies of differential polynomials II", Math. Z., **125**, 107 – 112 (1972).
[8] C. C. Yang, H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, Dordrecht/Boston/London (2003).
[9] L. Z. Yang, J. L. Zhang, "Non-existence of meromorphic solutions of Fermat type functional equation", Aequations Math., **76** (1-2), 140 – 150 (2008).
[10] L. Zalcman, "A heuristic principle in complex theory", Amer. Math. Monthly, **82**, 813 – 817 (1975).
[11] L. Zalcman, "Normal families, new perspectives", Bull. Amer. Math. Soc., **35**, 215 – 230 (1998).
[12] J. L. Zhang, L. Z. Yang, "A power of an entire function sharing one value with its derivative", Comput. Math. Appl., **60**, 2153 – 2160 (2010).

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