Известия НАН Армении, Математика, том 59, н. 2, 2024, стр. 56 – 66.

PORTFOLIO VALUE-AT-RISK APPROXIMATION FOR GEOMETRIC BROWNIAN MOTION

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Abstract. Value-at-risk (VaR) serves as a measure for assessing the risk associated with individual securities and portfolios. When calculating VaR for portfolios, the dimension of the covariance matrix increases as more securities are included. In this study, we present a solution to address the issue of dimensionality by directly computing the VaR of a portfolio using a single security, therefore requiring only one variance and one mean. Our results demonstrate that, under the assumption of Gaussian distribution, the deviation between the computed VaR and actual values is relatively small.

MSC2020 numbers: 60J65; 65C30; 60H35; 97M30; 91B24.
Keywords: Portfolio value-at-risk; sum of lognormal distribution; Fenton-Wilkinson approximation.

1. Introduction

The computation of portfolio value-at-risk (VaR) typically involves a strict algorithm that assumes a distribution close to normal and incorporates a correlation matrix of security returns.

However, a challenge arises due to the increasing dimensionality of the covariance matrix, resulting in exponential computational burden with the inclusion of each new security (see [1]). In this paper, we propose an alternative calculation algorithm that assumes a Gaussian distribution of returns, while significantly reducing the computational burden. We examine this straightforward method and explore variations, focusing primarily on the maximum absolute difference between the two approaches. Initially, we investigate the maximum deviation in the case of positively correlated securities, followed by a general analysis encompassing various scenarios.

Existing literature primarily emphasizes the reduction of computational burden through simplification of matrix-based calculations. As our primary concern is a single quantity, it is more feasible to approximate the VaR itself. Other authors

[1]The research of the second author is partially supported by the Mathematical Studies Center at Yerevan State University

concentrate on improved methods of estimating VaR when the underlying distribution deviates from the Gaussian assumption (as seen in [2]). While we do not delve into these alternative approaches for computing VaR in portfolios with a sum of lognormal distributions, it may prove effective to incorporate third and fourth moments (see [3]). Given that we primarily deal with the sum of lognormal distributions, we employ existing approximation methods. Specifically, we utilize the Fenton-Wilkinson approximation ([4, 5]) due to its simplicity, although more accurate approximations exist (see [6, 7]). Remarkably, our findings indicate that the simple Fenton-Wilkinson approximation sufficiently approximates VaR under the assumption of normality.

The paper is organized as follows. In Section 2, we introduce the general framework and the proposed method. Section 3 analyzes the maximum potential difference, and finally, we conclude with a discussion of our results.

2. VaR computation and approximation

We deal with only two subsequent periods of time. Here we consider mixture of n securities represented by geometric Brownian motions

$$S(t) = \sum_{i=1}^{n} w_i S_i(t) \tag{2.1}$$

with $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$, and $S_i(t)$, $i = \overline{1,n}$ are processes satisfying the following stochastic differential equations (SDE).

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t) dW_i(t) \tag{2.2}$$

with $W_i(t) \sim N(0,t)$ not necessarily independent Brownian motions, i.e. for each i, $W_i(t+1) - W_i(t) \sim N(0,1)$ iid for each $t \in \{1, 2, ...\}$ [2] with correlation coefficients

$$\rho_{i,j}(t) = corr(W_i(t), W_j(t)) \tag{2.3}$$

As we deal with only two periods ($t = 0, 1$) and having no randomness in period 0, we take

$$\rho_{i,j}(t) = \rho_{ij} \tag{2.4}$$

Thus $S_i(t)$, $i = \overline{1,n}$ have log-normal distribution

$$\begin{aligned} S_i(t) &= S_i(0) e^{(\mu_i - \sigma_i^2/2)t + \sigma_i W_i(t)} \\ S_i(t) &\sim LogN\left(\ln S_i(0) + \left(\mu_i - \frac{\sigma_i^2}{2}\right)t, \sigma_i^2 t\right) \end{aligned} \tag{2.5}$$

[2]Note that we consider only discrete points of time. However, initially the Brownian motion should be defined on continuous domain. We are only interested in two periods $t = 0, 1$

The log-returns we denote by $X_i(t)$ for individual stock, and $X(t)$, for portfolio.

$$X_i(t) = \ln\left(\frac{S_i(t+1)}{S_i(t)}\right), \quad X_i(t) \sim N\left(\mu_i - \frac{\sigma_i^2}{2}, \sigma_i^2\right) \tag{2.6}$$

Note that correlation of log-returns is also $\rho(X_i(t), X_j(t)) =: \rho_{X_i,X_j} = \rho_{ij}$, thus yielding the following vector-distribution.

$$(X_1(t), ..., X_n(t)) \sim N\left(\mu = \begin{pmatrix} \mu_1 - \frac{\sigma_1^2}{2} \\ \vdots \\ \mu_n - \frac{\sigma_n^2}{2} \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_n\rho_{1n} & \dots & \sigma_n^2 \end{pmatrix}\right), \tag{2.7}$$

where μ is vector of means and Σ is covariance matrix. From (2.1) and (2.7), the portfolio VaR (the quantile of portfolio return $X(t)$), have the following form

$$VaR_X = \sum_{i=1}^{n} w_i S_i(0)\left(\mu_i - \frac{1}{2}\sigma_i^2\right) + z_{\alpha/2}\sqrt{(w_1S_1(0), ..., w_nS_n(0)) \cdot \Sigma \cdot \begin{pmatrix} w_1S_1(0) \\ \vdots \\ w_nS_n(0), \end{pmatrix}}$$

where VaR_X is value at risk for given portfolio, and $z_{\alpha/2}$ is quantile of standard normal distribution (with probability $P(X \le z_{\alpha/2}) = 1 - \frac{\alpha}{2}$) (see [1, 2]). Note that S(t) in (2.1), is distributed as sum of lognormal distributions, i.e.

$$S(t) \sim \sum_{i=1}^{n} w_i LogN\left(\ln S_i(0) + \left(\mu_i - \frac{\sigma_i^2}{2}\right)t, \sigma_i^2 t\right) \tag{2.8}$$

By Fenton - Wilkinson ([3, 4]) approximation we have

$$S(t) \sim_{approx} LogN(\mu_z(t); \sigma_z^2(t)) \sim: \widetilde{S(t)}, \tag{2.9}$$

where

$$\sigma_z^2(t) = \frac{1}{\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right) e^{\frac{\sigma_i^2 t}{2}}\right)^2} \cdot$$
$$\left[t\sum_{i,j=1}^{n} \rho_{ij}^S \sigma_i\sigma_j w_i w_j \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\left(S_j(0) + (\mu_j - \frac{1}{2}\sigma_j^2)\right) e^{\frac{\sigma_i^2 t + \sigma_j^2 t}{2}}\right]$$
$$\mu_z(t) = \ln\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right) e^{\frac{\sigma_i^2 t}{2}}\right) - \frac{\sigma_z^2(t)}{2} \tag{2.10}$$

Let's approximate $X(t)$ with return of $\widetilde{S(t)}$, $\widetilde{X_p(t)} := \ln\left(\frac{\widetilde{S(t+1)}}{\widetilde{S(t)}}\right)$, with

$$\widetilde{X_p(t)} \sim N(\mu_z(t+1), \sigma_z^2(t+1)) - N(\mu_z(t), \sigma_z^2(t)) \tag{2.11}$$

To completely determine the distribution, we additionally need covariance

$$C(t) = Cov\left(\ln\widetilde{S(t+1)}; \ln\widetilde{S(t)}\right) = E\left(\ln\widetilde{S(t+1)} \cdot \ln\widetilde{S(t)}\right) - E\left(\ln\widetilde{S(t+1)}\right) \cdot E\left(\ln\widetilde{S(t)}\right)$$

with $E(\ln \widetilde{S(t)}) = \mu_z(t)$. To find covariance $C(t)$ we use formula for exponential terms. So let's consider $E(e^{\ln \widetilde{S(t+1)}} \cdot e^{\ln \widetilde{S(t)}})$.

$$E\left(e^{\ln \widetilde{S(t+1)}} \cdot e^{\ln \widetilde{S(t)}}\right) = e^{\mu_z(t)+\mu_z(t+1)+\frac{\sigma_z^2(t)+\sigma_z^2(t)+2C(t)}{2}} \tag{2.12}$$

On the other hand

$$E\left(e^{\ln S(t+1)} \cdot e^{\ln S(t)}\right) = E\left(\sum_{i=1}^n w_i S_i(t) \cdot \sum_{i=1}^n w_i S_i(t)\right) \tag{2.13}$$

The covariance may be approximated by [3]

$$C(t) \approx \ln\left(E\left(\sum_{i=1}^n w_i S_i(t) \cdot \sum_{i=1}^n w_i S_i(t)\right)\right) - \mu_z(t) - \mu_z(t+1) - \frac{1}{2}\sigma_z^2(t) - \frac{1}{2}\sigma_z^2(t+1).$$

Hence we have the following approximate distribution

$$\widetilde{X_p(t)} \sim_{approx} N\left(\mu_z(t+1) - \mu_z(t), \sigma_z^2(t+1) + \sigma_z^2(t) - 2C(t)\right) \tag{2.14}$$

The idea is to approximate portfolio VaR using quantile of approximate distribution of $X_p(t)$ (for 2 consecutive days $t = 0; t+1 = 1$).

$$VaR_{X_p} = \mu_z(t+1) - \mu_z(t) + S(0) z_{\alpha/2} \sqrt{\sigma_z^2(t+1) + \sigma_z^2(t) - 2C(t)} \tag{2.15}$$

For $t = 0$, we have the following (by (2.10))

$$\begin{aligned} \sigma_z^2(0) &= \frac{1}{\left(\sum_{i=1}^n w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\right)^2} \cdot 0 = 0 \\ \mu_z(0) &= \ln\left(\sum_{i=1}^n w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\right) \end{aligned} \tag{2.16}$$

So for t=0 we have random variable with 0 variance, which is obvious as nothing is random in that period. For t+1=1 we have:

$$\begin{aligned} \sigma_z^2(1) &= \frac{1}{\left(\sum_{i=1}^n w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right) e^{\frac{\sigma_i^2}{2}}\right)^2} \cdot \\ &\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_i w_j \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\left(S_j(0) + (\mu_j - \frac{1}{2}\sigma_j^2)\right) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}\right] \\ \mu_z(1) &= \ln\left(\sum_{i=1}^n w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right) e^{\frac{\sigma_i^2}{2}}\right) - \frac{\sigma_z^2(1)}{2} \end{aligned} \tag{2.17}$$

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[3]It is indeed approximation, as in (2.12), $\widetilde{S(t)}$ is used, while in (2.13) $S(t)$.

Note that as one of our random variables has 0 variance the covariance can be taken to be 0. So we have:

$$VaR_{X_p} = \ln\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right) e^{\frac{\sigma_i^2}{2}}\right) - \ln\left(\sum_{i=1}^{n} w_i \left(S_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)\right)\right) - $$
$$- \frac{1}{2}\sigma_z^2(1) + z_{\alpha/2}\left(\sum_{i=1}^{n} w_i S_i(0)\right)\sqrt{\sigma_z^2(1)}.$$

Hereafter, we will consider only risk neutral pricing, i.e. $\mu_i = \frac{1}{2}\sigma_i^2$.

3. Difference in methods

We claim that difference between VaR_{X_p} and VaR_X is not big, in sense that there exist C such that

$$\left|\frac{VaR_{X_p} - VaR_X}{VaR_X}\right| < C$$

for any correlation coefficients ρ_{ij} with $i \neq j$; $i, j = \overline{1, n}$ and any weights w_i in risk neutral setting. Or at least we attempt to prove similar result[4].

Remark 3.1. *We don't yet know if C depends on general structure of ρ-s and σ-s, or is there any absolute constant. At least we will try to show the existence of some bounds. Also note that while we may not come to theoretically small C, in practice C is quite small.*

For the risk-neutral pricing (i.e. taking $\mu_i = \frac{1}{2}\sigma_i^2$) we obtain

(3.1)

$$VaR_X = z_{\alpha/2}\sqrt{(w_1 S_1(0), ..., w_n S_n(0)) \cdot \Sigma \cdot \begin{pmatrix} w_1 S_1(0) \\ \vdots \\ w_n S_n(0) \end{pmatrix}}$$

$$VaR_{X_p} = \ln\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right) - \ln\left(\sum_{i=1}^{n} w_i S_i(0)\right) -$$
$$- \frac{1}{2}\frac{1}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}\right] +$$
$$z_{\alpha/2}\left(\sum_{i=1}^{n} w_i S_i(0)\right)\sqrt{\frac{1}{\left(\sum_{i=1}^{n} w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}\right]}$$

Let's first consider the case where we deal only with non-negative correlations, i.e. $\rho_{ij} \geq 0$.

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[4]No formal derivations of approximation were given originally for log-normal approximation with Fenton-Wilkinson. So some computational comparisons had been done later, see [6].

3.1. **First bounds.** We have the following obvious (quite loose) bound for the fourth term of VaR_{X_p} in (3.1)

$$z_{\alpha/2}\frac{\sum_{i=1}^n w_iS_i(0)}{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}\sqrt{\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]} \le \tag{3.2}$$
$$z_{\alpha/2}\frac{\sum_{i=1}^n w_iS_i(0)}{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}e^{\frac{\sigma_{max}^2}{2}}VaR_X \le VaR_X e^{\frac{\sigma_{max}^2}{2}-\frac{\sigma_{min}^2}{2}}$$

And similarly

(3.3)

$$z_{\alpha/2}\frac{\sum_{i=1}^n w_iS_i(0)}{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}\sqrt{\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]} \ge$$
$$z_{\alpha/2}e^{\frac{-\sigma_{max}^2}{2}}\sqrt{\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]} \ge VaR_X e^{\frac{\sigma_{min}^2}{2}-\frac{\sigma_{max}^2}{2}}$$

Considering the first three terms of VaR_{X_p} in (3.1), and using the same argumentation we have the following bounds

$$\ln\left(\frac{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}{\sum_{i=1}^n w_iS_i(0)}\right)-\frac{1}{2}\frac{1}{\left(\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right] \ge$$
$$\frac{1}{2}\sigma_{min}^2-\frac{1}{2}e^{\sigma_{max}^2-\sigma_{min}^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_j\right]$$

which in turn, using the following $\sigma_{max}^2 \ge \sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_j \ge \sigma_{min}^2$, we give

(3.4)

$$\ln\left(\frac{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}{\sum_{i=1}^n w_iS_i(0)}\right)-\frac{1}{2}\frac{1}{\left(\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right] \ge$$
$$\frac{1}{2}\sigma_{min}^2-\frac{1}{2}\sigma_{max}^2e^{\sigma_{max}^2-\sigma_{min}^2}$$

Similarly one can derive

(3.5)

$$\ln\left(\frac{\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}}{\sum_{i=1}^n w_iS_i(0)}\right)-\frac{1}{2}\frac{1}{\left(\sum_{i=1}^n w_iS_i(0)e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_iw_jS_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right] \le$$
$$\frac{1}{2}\sigma_{max}^2-\frac{1}{2}\sigma_{min}^2e^{\sigma_{min}^2-\sigma_{max}^2}$$

Combining (3.2), (3.3), (3.4) and (3.5), we obtain the following bound

$$\begin{aligned}\frac{1}{2}\sigma^2_{min} - \frac{1}{2}\sigma^2_{max}e^{\sigma^2_{max}-\sigma^2_{min}} + VaR_X e^{\frac{\sigma^2_{min}}{2}-\frac{\sigma^2_{max}}{2}} \leq VaR_{X_p}\\ \leq \frac{1}{2}\sigma^2_{max} - \frac{1}{2}\sigma^2_{min}e^{\sigma^2_{min}-\sigma^2_{max}} + VaR_X e^{\frac{\sigma^2_{max}}{2}-\frac{\sigma^2_{min}}{2}}\end{aligned} \tag{3.6}$$

Note that this bound is indeed loose, as right side can get quite big thanks to exponent, while the left side can be quite small. Also note that, if $\sigma_i = \sigma_{max} = \sigma_{min}$, we retrieve $VaR_{X_p} = VaR_X$.

3.2. **Bounds for positive correlations.** The better bound stated in the following proposition can be obtained. First let's make some notations value of portfolio $VP := \sum_{i=1}^n w_i S_i(0)$ and

$$\sigma^2_{wS} := \frac{\sum_{i,j=1}^n w_i w_j \sigma_i \sigma_j S_i(0) S_j(0)}{\sum_{i,j=1}^n w_i w_j S_i(0) S_j(0)} \tag{3.7}$$

Proposition 3.1. *The following inequality holds if we assume non-negative correlations:*

$$\begin{aligned}\frac{1}{2}\sigma^2_{min} - \frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2 \frac{2\sigma^2_{max}-\sigma^2_{min}}{\sigma^2_{wS}} + VaR_X \leq VaR_{X_p} \leq\\ \frac{1}{2}\sigma^2_{max} - \frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2 + VaR_X\sqrt{\frac{2\sigma^2_{max}-\sigma^2_{min}}{\sigma^2_{wS}}}\end{aligned} \tag{3.8}$$

Proof. We make use of Holder's inequality for left part, and Abel's inequality for right part (3.8).

Lemma 3.1. *For positive values of x_i and w_i, the following inequality is true.*

$$\frac{\sum_{i=1}^n x_i w_i e^{x_i}}{\sum_{i=1}^n w_i e^{x_i}} \geq \frac{\sum_{i=1}^n x_i w_i}{\sum_{i=1}^n w_i}. \tag{3.9}$$

Proof. We define

$$H(a) = \frac{\sum_{i=1}^n x_i w_i e^{a x_i}}{\sum_{i=1}^n w_i e^{a x_i}} \tag{3.10}$$

and consider its derivative with respect to a.

$$H'(a) = \frac{\left(\sum_{i=1}^n x_i^2 w_i e^{a x_i}\right)\cdot\left(\sum_{i=1}^n w_i e^{a x_i}\right) - \left(\sum_{i=1}^n x_i w_i e^{a x_i}\right)^2}{\left(\sum_{i=1}^n w_i e^{a x_i}\right)^2} \tag{3.11}$$

Due to Holder's inequality the numerator is non-negative. Indeed, denote

$$a_k = x_k e^{\frac{a x_k}{2}}\sqrt{w_k}; \quad b_k = e^{\frac{a x_k}{2}}\sqrt{w_k} \tag{3.12}$$

Then the numerator is exactly

$$\left(\sum_{k=1}^n a_k\right)^2\left(\sum_{k=1}^n b_k\right)^2 - \left(\sum_{h=1}^n a_h b_h\right)^2 \geq 0.$$

By exactly the same technique, one can show that the following lemma is also true.

Lemma 3.2. *For positive values of x_i and w_i and for some strictly increasing function $f(x)$, the following inequality holds.*

$$\frac{\sum_{i=1}^n x_i w_i e^{f(x_i)}}{\sum_{i=1}^n w_i e^{f(x_i)}} \geq \frac{\sum_{i=1}^n x_i w_i}{\sum_{i=1}^n w_i} \tag{3.13}$$

This inequality is enough to show the first part of (3.8).

Proof. Consider only the last part of Var_{X_p} with $z_{\alpha/2}$, in (3.1).

$$A := z_{\alpha/2}\left(\sum_{i=1}^n w_i S_i(0)\right)\sqrt{\frac{1}{\left(\sum_{i=1}^n w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\left[\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}\right]}$$

$$= z_{\alpha/2}\left(\sum_{i=1}^n w_i S_i(0)\right)\left(\frac{\sum_{i,j=1}^n \rho_{ij} w_i w_j S_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^n w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\right.$$
$$\left.\left[\frac{\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}}{\sum_{i,j=1}^n \rho_{ij} w_i w_j S_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}}\right]\right)^{1/2}$$

For which using the inequality once and as soon as ρ-s are positive, we have

$$A \geq z_{\alpha/2}\left(\sum_{i=1}^n w_i S_i(0)\right)\left(\frac{\sum_{i,j=1}^n \rho_{ij} w_i w_j S_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^n w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\right.$$
$$\left.\left[\frac{\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0)S_j(0)}{\sum_{i,j=1}^n \rho_{ij} w_i w_j S_i(0)S_j(0)}\right]\right)^{1/2}$$

Note that here we have used the (3.13) twice [5]. Let's do it once more

$$A \geq z_{\alpha/2}\left(\sum_{i=1}^n w_i S_i(0)\right)\left(\frac{\sum_{i,j=1}^n w_i w_j S_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^n w_i S_i(0) e^{\frac{\sigma_i^2}{2}}\right)^2}\cdot\right.$$
$$\left.\frac{\sum_{i,j=1}^n \rho_{ij} w_i w_j S_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}}{\sum_{i,j=1}^n w_i w_j S_i(0)S_j(0)e^{\frac{\sigma_i^2}{2}}e^{\frac{\sigma_j^2}{2}}}\left[\frac{\sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0)S_j(0)}{\sum_{i,j=1}^n \rho_{ij} w_i w_j S_i(0)S_j(0)}\right]\right)^{1/2}$$

[5] We used it once for sum with i-s and once for sum with j-s.

and using Lemma 3.2 once more (again two times)[6], for the first term in square root we have

$$A \geq VaR_X \left(\frac{\sum_{i,j=1}^n \rho_{ij} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\sum_{i,j=1}^n w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}} \cdot \frac{\sum_{i,j=1}^n \sigma_i \sigma_j w_i w_j S_i(0) S_j(0)}{\sum_{i,j=1}^n \rho_{ij} w_i w_j S_i(0) S_j(0)} \right)^{1/2} = VaR_X \tag{3.14}$$

where the equality can be easily checked, just by multiplying sums.

For the next part of inequality we will make use of Abel's inequality. Denoting $\overline{x}_w = \sum w_i x_i$, the following lemma holds.

Lemma 3.3. *For positive values w_i, the following inequality is true*

$$\frac{\sum_{i=1}^n x_i w_i e^{x_i}}{\sum_{i=1}^n w_i e^{x_i}} \cdot \frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n x_i w_i} \leq \frac{|\max x_i| + R}{\overline{x}_w} \tag{3.15}$$

with $R = \max x_i - \min x_i$.

Without loss of generality, we can assume that x_i are in increasing order. Hence, by Abel's inequality (see [8]), we have

$$\sum_{i=1}^n x_i w_i e^{x_i} \leq (|x_n| + x_n - x_1) \max_j \sum_{i=1}^j w_i e^{x_i} = (|x_n| + x_n - x_1) \sum_{i=1}^n w_i e^{x_i} \tag{3.16}$$

Thus it will yield

$$\begin{aligned} \frac{\sum_{i=1}^n x_i w_i e^{x_i} \cdot \sum_{i=1}^n w_i}{\sum_{i=1}^n w_i e^{x_i} \cdot \sum_{i=1}^n x_i w_i} &\leq \frac{(|x_n| + x_n - x_1) \sum_{i=1}^n w_i e^{x_i} \cdot \sum_{i=1}^n w_i}{\sum_{i=1}^n w_i e^{x_i} \cdot \sum_{i=1}^n x_i w_i} \\ &= \frac{(|x_n| + x_n - x_1) \sum_{i=1}^n w_i}{\sum_{i=1}^n x_i w_i} = \frac{|\max x_i| + R}{\overline{x}_w} \end{aligned} \tag{3.17}$$

Using this inequality and considering A again, we obtain

$$\begin{aligned} A = z_{\alpha/2} \left(\sum_{i=1}^n w_i S_i(0) \right) & \left(\frac{\sum_{i,j=1}^n \rho_{ij} \sigma_i \sigma_j w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^n w_i S_i(0) e^{\frac{\sigma_i^2}{2}} \right)^2} \right)^{1/2} \\ \leq z_{\alpha/2} & \left(\frac{\sum_{i,j=1}^n \rho_{ij} \frac{\sigma_i \sigma_j}{2} w_i w_j S_i(0) S_j(0) e^{\frac{\sigma_i^2}{2}} e^{\frac{\sigma_j^2}{2}}}{\left(\sum_{i=1}^n w_i S_i(0) e^{\frac{\sigma_i^2}{2}} \right)^2} \cdot \frac{(\sum_{i=1}^n w_i S_i(0))^2}{\sum_{i,j=1}^n \rho_{ij} \frac{\sigma_i \sigma_j}{2} w_i w_j S_i(0) S_j(0)} \right. \\ & \left. \left[\sum_{i,j=1}^n \rho_{ij} \frac{\sigma_i \sigma_j}{2} w_i w_j S_i(0) S_j(0) \right] \right)^{1/2} \end{aligned}$$

[6]Note that numerator of expression in square brackets with $z_{\alpha/2}$ is VaR_X itself.

Using Lemma 3.3 for first two fractions, we come to the following result:

$$A \le z_{\alpha/2}\sqrt{\left[\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j w_i w_j S_i(0)S_j(0)\right]\frac{\max(\sigma_i\sigma_j)+(\sigma_{max}^2-\sigma_{min}^2)}{2\sigma_{wS}^2}} \tag{3.18}$$

or

$$A \le VaR_X\sqrt{\frac{\max(\sigma_i\sigma_j)+(\sigma_{max}^2-\sigma_{min}^2)}{2\sigma_{wS}^2}} \tag{3.19}$$

For the third term VaR_{X_p} in (3.1), note that expression in square brackets is bigger than $(\frac{1}{z_{\alpha/2}}VaR_X)^2$ for positive correlations, we obtain

$$\begin{aligned} VaR_{X_p} \ge \ln&\left(\frac{\sum_{i=1}^{n} w_i S_i(0)e^{\frac{\sigma_i^2}{2}}}{\sum_{i=1}^{n} w_i S_i(0)}\right) - \\ &\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2\frac{\max(\sigma_i\sigma_j)+(\sigma_{max}^2-\sigma_{min}^2)}{2\sigma_{wS}^2}+VaR_X \\ \ge&\frac{1}{2}\sigma_{min}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2+VaR_X \end{aligned} \tag{3.20}$$

and lastly the main formula can be derived.

$$\frac{1}{2}\sigma_{min}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}+VaR_X \le VaR_{X_p} \le$$
$$\frac{1}{2}\sigma_{max}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2+VaR_X\sqrt{\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}}$$

3.3. **Bounds for general case.** The following result is immediate consequence of above results. One can prove it by separating the stock considered into two groups: positively correlated and negatively, in the following sense. Taking $\rho_+ = \{ij|\rho_{ij} = \rho_{ji} > 0\}$ and $\rho_- = \{ij|\rho_{ij} = \rho_{ji} < 0\}$, other indices does not contribute to sum and using this grouping, we get the following result.

Proposition 3.2. *The following inequality holds:*

$$\begin{aligned} &\frac{1}{2}\sigma_{min}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 VaR_X^2\max\left(1,\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}\right) \\ &+VaR_X\min\left(1,\sqrt{\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}}\right) \le VaR_{X_p} \le \frac{1}{2}\sigma_{max}^2-\frac{1}{2}\left(\frac{1}{z_{\alpha/2}}\right)^2\left(\frac{1}{VP}\right)^2 \cdot \\ &\quad \cdot VaR_X^2\min\left(1,\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}\right)+VaR_X\max\left(1,\sqrt{\frac{2\sigma_{max}^2-\sigma_{min}^2}{\sigma_{wS}^2}}\right) \end{aligned}$$

4. Discussion and conclusion

Our calculations have revealed tighter bounds, in scenarios involving either solely positive correlation or both positive and negative correlations. This could be attributed to the relatively small magnitudes of the volatilities themselves, suggesting the potential for the derivation of improved bounds. Nevertheless, for portfolios with relatively confined volatility values, the current bounds prove sufficiently tight. It is worth noting that enhancing these bounds is primarily contingent on the theoretical justification of the proposed methodology. Computations indicate significantly tighter real bounds. For a three-stock portfolio, this translates to approximately $0.5 - 0.9\%$ of the Gaussian-VaR value, or roughly $0.1 - 0.2\%$ of the portfolio value. As the number of stocks increases, the disparity diminishes gradually. It is crucial to emphasize that the pursuit of better bounds is rooted in the theoretical validation of the proposed procedure.

For the above case compared to Gaussian-VaR, our lower bound deviates by no more than 0.00025%, showcasing its robustness. However, the upper bound exhibits a substantial discrepancy of up to 7.5%, a noteworthy disparity. From an empirical standpoint, particularly in domains where non-gaussian behavior may dominate, our methodology could yield significant differences. However, as of now, such disparities have not been observed in the context of stock portfolios.

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Поступила 03 июля 2023

После доработки 26 октября 2023

Принята к публикации 15 ноября 2023