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UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH RESPECT TO THEIR SHIFTS CONCERNING DERIVATIVES

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Abstract. An example in the article shows that the first derivative of $f(z) = \frac{2}{1-e^{-2z}}$ sharing 0 CM and $1, \infty$ IM with its shift πi cannot obtain they are equal. In this paper, we study the uniqueness of meromorphic function sharing small functions with their shifts concerning its $k-th$ derivatives. We use a different method from Qi and Yang [18] to improves entire function to meromorphic function, the first derivative to the $k-th$ derivatives, and also finite values to small functions. As for $k = 0$, we obtain: Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$, let c be a nonzero finite value, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty \in \hat{S}(f)$ be two distinct small functions of $f(z)$ such that $a(z)$ is a periodic function with period c and $b(z)$ is any small function of $f(z)$. If $f(z)$ and $f(z+c)$ share $a(z), \infty$ CM, and share $b(z)$ IM, then either $f(z) \equiv f(z+c)$ or

$$e^{p(z)} \equiv \frac{f(z+c) - a(z+c)}{f(z) - a(z)} \equiv \frac{b(z+c) - a(z+c)}{b(z) - a(z)},$$

where $p(z)$ is a non-constant entire function of $\rho(p) < 1$ such that $e^{p(z+c)} \equiv e^{p(z)}$.

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1. Introduction and main results

Throughout this paper, we assume that the reader have a knowledge of the fundamental results and the standard notations of the Nevanlinna value distribution theory. See([6, 20, 21]). In the following, a meromorphic function f means meromorphic in the whole complex plane. Define

$$\rho(f) = \varlimsup_{r\to\infty} \frac{log^{+}T(r,f)}{logr},$$

$$\rho_2(f) = \varlimsup_{r\to\infty} \frac{log^{+}log^{+}T(r,f)}{logr}$$

by the order and the hyper-order of f, respectively. When $\rho(f) < \infty$, we say f is of finite order.

By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic function $a(z)$ satisfying $T(r, a) = S(r, f)$ is called a small function of f. We denote $S(f)$ as the family of all small meromorphic functions of f which includes the constants in $\mathbb{C}$. Moreover, we define $\hat{S}(f) = S(f) \cup \{\infty\}$. We say that two non-constant meromorphic functions f and g share small function a CM(IM) if $f - a$ and

$g-a$ have the same zeros counting multiplicities (ignoring multiplicities). Moreover, we introduce the following notation: $S_{(m,n)}(a) = \{z|z$ is a common zero of $f(z+c)-a(z)$ and $f(z)-a(z)$ with multiplicities m and n respectively$\}$. $\overline{N}_{(m,n)}(r, \frac{1}{f-a})$ denotes the counting function of f with respect to the set $S_{(m,n)}(a)$. $\overline{N}_{n)}(r, \frac{1}{f-a})$ denotes the counting function of all distinct zeros of $f-a$ with multiplicities at most n. $\overline{N}_{(n}(r, \frac{1}{f-a})$ denotes the counting function of all zeros of $f-a$ with multiplicities at least n.

We say that two non-constant meromorphic functions f and g share small function a CM(IM)almost if

$$N(r, \frac{1}{f-a}) + N(r, \frac{1}{g-a}) - 2N(r, f=a=g) = S(r,f) + S(r,g),$$

or

$$\overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{g-a}) - 2\overline{N}(r, f=a=g) = S(r,f) + S(r,g),$$

respectively.

For a meromorphic function $f(z)$, we denote its shift by $f_c(z) = f(z+c)$.

Rubel and Yang [19] studied the uniqueness of an entire function concerning its first order derivative, and proved the following result.

Theorem A. Let $f(z)$ be a non-constant entire function, and let a, b be two finite distinct complex values. If $f(z)$ and $f'(z)$ share a, b CM, then $f(z) \equiv f'(z)$.

Zheng and Wang [23] improved Theorem A and proved

Theorem B. Let $f(z)$ be a non-constant entire function, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty$ be two distinct small functions of $f(z)$. If $f(z)$ and $f^{(k)}(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv f^{(k)}(z)$.

Li and Yang [15] improved Theorem B and proved

Theorem C. Let $f(z)$ be a non-constant entire function, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty$ be two distinct small functions of $f(z)$. If $f(z)$ and $f^{(k)}(z)$ share $a(z)$ CM, and share $b(z)$ IM. Then $f(z) \equiv f^{(k)}(z)$.

Recently, the value distribution of meromorphic functions concerning difference analogue has become a popular research, see [1, 2, 4 – 9, 12 – 14, 16 – 18]. Heittokangas et al [7] obtained a similar result analogue of Theorem A concerning shifts.

Theorem D. Let $f(z)$ be a non-constant entire function of finite order, let c be a nonzero finite complex value, and let a, b be two finite distinct complex values. If $f(z)$ and $f(z+c)$ share a, b CM, then $f(z) \equiv f(z+c)$.

In [17], Qi-Li-Yang investigated the value sharing problem with respect to $f'(z)$ and $f(z+c)$. They proved

Theorem E. Let $f(z)$ be a non-constant entire function of finite order, and let a, c be two nonzero finite complex values. If $f'(z)$ and $f(z+c)$ share $0, a$ CM, then $f'(z) \equiv f(z+c)$.

Recently, Qi and Yang [18] improved Theorem E and proved

Theorem F. Let $f(z)$ be a non-constant entire function of finite order, and let a, c be two nonzero finite complex value. If $f'(z)$ and $f(z+c)$ share 0 CM and a IM, then $f'(z) \equiv f(z+c)$.

Of above theorem, it's naturally to ask whether the condition $0, a$ can be replaced by two distinct small functions, and f' can be replaced by $f^{(k)}$?

In this article, we give a positive answer. In fact, we prove the following more general result.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$, let c be a nonzero finite value, k be a positive integer, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty \in \hat{S}(f)$ be two distinct small functions. If $f^{(k)}(z)$ and $f(z+c)$ share $a(z), \infty$ CM, and share $b(z)$ IM, then $f^{(k)}(z) \equiv f(z+c)$.*

Example 1.1. [9] *Let $f(z) = \frac{2}{1-e^{-2z}}$, and let $c = \pi i$. Then $f'(z)$ and $f(z+c)$ share 0 CM and share $1, \infty$ IM, but $f'(z) \not\equiv f(z+c)$.*

This example shows that for meromorphic functions, the conclusion of Theorem 1 doesn't hold even when sharing ∞ CM is replaced by sharing ∞ IM when $k = 1$. We believe there are examples for any k, but we can not construct them.

As for $k = 0$, Li and Yi [13] obtained

Theorem G. Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$, let c be a nonzero finite value, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty \in \hat{S}(f)$ be two distinct small functions. If $f(z)$ and $f(z+c)$ share $a(z)$ CM, and share $b(z)$ IM, then $f(z) \equiv f(z+c)$.

Remark 1.1. *Theorem G holds when $f(z)$ is a non-constant meromorphic function of $\rho_2(f) < 1$ such that $N(r, f) = S(r, f)$.*

Theorem H. [8] Let $f(z)$ be a non-constant meromorphic function of finite order, let c be a nonzero finite value, and let $a(z) \not\equiv \infty$, $b(z) \not\equiv \infty$ and $d(z) \not\equiv \infty \in \hat{S}(f)$ be three distinct small functions such that $a(z)$, $b(z)$ and $d(z)$ are periodic functions with period c. If $f(z)$ and $f(z+c)$ share $a(z), b(z)$ CM, and $d(z)$ IM, then $f(z) \equiv f(z+c)$.

We can ask a question that whether the small periodic function $d(z)$ of $f(z)$ can be replaced by any small function of $f(z)$?

In this paper, we obtain our second result.

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$, let c be a nonzero finite value, and let $a(z) \not\equiv \infty, b(z) \not\equiv \infty \in \hat{S}(f)$ be two distinct small functions of $f(z)$ such that $a(z)$ is a periodic function with period c and $b(z)$ is a small function of $f(z)$. If $f(z)$ and $f(z+c)$ share $a(z), \infty$ CM, and share $b(z)$ IM, then either $f(z) \equiv f(z+c)$ or*

$$e^{p(z)} \equiv \frac{f(z+c) - a(z+c)}{f(z) - a(z)} \equiv \frac{b(z+c) - a(z+c)}{b(z) - a(z)},$$

where $p(z)$ is a non-constant entire function of $\rho(p) < 1$ such that $e^{p(z+c)} \equiv e^{p(z)}$.

We can obtain the following corollary from the proof of Theorem 1.2.

Corollary 1.1. *Under the same condition as in Theorem 2, then $f(z) \equiv f(z+c)$ holds if one of conditions satisfies*
(i) $b(z)$ is a periodic function with period nc ;
(ii) $\rho(b(z)) < \rho(e^{p(z)})$;
(iii) $\rho(b(z)) < 1$.

Example 1.2. *Let $f(z) = \frac{e^z}{1-e^{-2z}}$, and let $c = \pi i$. Then $f(z+c) = \frac{-e^z}{1-e^{-2z}}$, and $f(z)$ and $f(z+c)$ share $0, \infty$ CM, but $f(z) \not\equiv f(z+c)$.*

Example 1.3. *Let $f(z) = e^z$, and let $c = \pi i$. Then $f(z+c) = -e^z$, and $f(z)$ and $f(z+c)$ share $0, \infty$ CM, $f(z)$ and $f(z+c)$ attain different values everywhere in the complex plane, but $f(z) \not\equiv f(z+c)$.*

Above two examples of show that "2CM+1IM"is necessary.

Example 1.4. *Let $f(z) = e^{e^z}$, then $f(z+\pi i) = \frac{1}{e^{e^z}}$. It is easy to verify that $f(z)$ and $f(z+\pi i)$ share $0, 1, \infty$ CM, but $f(z) = \frac{1}{f(z+\pi i)}$. On the other hand, we obtain $f(z) = f(z+2\pi i)$.*

Example 1.4 tells us that if we drop the assumption $\rho_2(f) < 1$, we can get another relation.

By Theorem 1.1 and Theorem 1.2, we still believe the latter situation of Theorem 2 can be removed, that is to say, only the case $f(z) \equiv f(z+c)$ occurs. So we raise a conjecture here.

Conjecture. Under the same condition as in Theorem 1.2, is $f(z) \equiv f(z+c)$?

2. Some lemmas

Lemma 2.1. [6] *Let f be a non-constant meromorphic function of $\rho_2(f) < 1$, and let c be a non-zero complex number. Then*

$$m(r, \frac{f(z+c)}{f(z)}) = S(r, f),$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2.2. [10, 20, 21] *Let f_1 and f_2 be two non-constant meromorphic functions in $|z| < \infty$, then*

$$N(r, f_1 f_2) - N(r, \frac{1}{f_1 f_2}) = N(r, f_1) + N(r, f_2) - N(r, \frac{1}{f_1}) - N(r, \frac{1}{f_2}),$$

where $0 < r < \infty$.

Lemma 2.3. [6] *Let f be a non-constant meromorphic function of $\rho_2(f) < 1$, and let c be a non-zero complex number. Then*

$$T(r, f(z)) = T(r, f(z+c)) + S(r, f),$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2.4. *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$ such that $\overline{N}(r, f) = S(r, f)$, let c be a nonzero constant, k be a positive integer, and let $a(z)$ be a small function of $f(z+c)$ and $f^{(k)}(z)$. If $f(z+c)$ and $f^{(k)}(z)$ share $a(z), \infty$ CM, and $N(r, \frac{1}{f^{(k)}(z+c) - a^{(k)}(z)}) = S(r, f)$, then $T(r, e^p) = S(r, f)$, where p is an entire function of order less than 1.*

Proof. Since f is a transcendental meromorphic function of $\rho_2(f) < 1$, $\overline{N}(r, f) = S(r, f)$, and f_c and $f^{(k)}$ share a and ∞ CM, then there is an entire function p of order less than 1 such that

$$f_c - a = e^p(f^{(k)} - a_{-c}^{(k)}) + e^p(a_{-c}^{(k)} - a). \tag{2.1}$$

Suppose on the contrary that $T(r, e^p) \not\equiv S(r, f)$.
Set $g = f_c^{(k)} - a^{(k)}$. Differentiating (2.1) k times we have

$$g = (e^p)^{(k)} g_{-c} + k(e^p)^{(k-1)} g'_{-c} + \cdots + k(e^p)' g_{-c}^{(k-1)} + e^p g_{-c}^{(k)} + B^{(k)}, \tag{2.2}$$

where $B = e^p(a_{-c}^{(k)} - a)$.
It is easy to see that $g \not\equiv 0$. Then we rewrite (2.2) as

$$1 - \frac{B^{(k)}}{g} = De^p, \tag{2.3}$$

where

$$D = e^{-p}[(e^p)^{(k)} \frac{g_{-c}}{g} + k(e^p)^{(k-1)} \frac{g'_{-c}}{g} + \cdots$$
$$+ k(e^p)' \frac{g_{-c}^{(k-1)}}{g} + (e^p) \frac{g_{-c}^{(k)}}{g}]. \tag{2.4}$$

Since f is a transcendental meromorphic function with $\rho_2(f) < 1$ and $f^{(k)}$ and f_c share ∞ CM, we can see from $\overline{N}(r, f) = S(r, f)$, Lemma 2.1 and Lemma 2.3 that

$$(1 + o(1))N(r, f) + S(r, f) = N(r, f_c) = N(r, f^{(k)}),$$

and on the other hand

$$k\overline{N}(r,f_c)+N(r,f_c)=N(r,f_c^{(k)}),\overline{N}(r,f_c)=\overline{N}(r,f^{(k)})=\overline{N}(r,f),$$

which follows from above equalities that $N(r,f^{(k)})=N(r,f_c^{(k)})+S(r,f)$, and thus we can know that g and g_{-c} share ∞ CM almost. It is easy to see from the assumption f_c and $f^{(k)}$ share ∞ CM that there exists no simple pole point of f_c. Now we estimate $N(r,\frac{g_{-c}^{(i)}}{g})$. Let z_0 be a pole of f with multiplicity n, than z_0 is a pole of g with multiplicity $n+2k$, and also z_0 is a pole of $g_{-c}^{(i)}$ with multiplicity $n+k+i$. Then we can see that z_0 is a zero point of $\frac{g_{-c}^{(i)}}{g}$ with $k-i$. Let z_1 be a pole of f_c with multiplicity m, then z_1 is a pole of g with multiplicity $m+k$, and also z_1 is a pole of $g_{-c}^{(i)}$ with multiplicity $m+i$. Then we can see that z_1 is a zero point of $\frac{g_{-c}^{(i)}}{g}$ with $k-i$. Note that $N(r,\frac{1}{f_c^{(k)}-a^{(k)}})=N(r,\frac{1}{g})=S(r,f)$, then $N(r,\frac{g_{-c}^{(i)}}{g})=S(r,f)$, and hence

$$\begin{aligned} T(r,D) &\le \sum_{i=0}^{k}(T(r,\frac{(e^p)^{(i)}}{e^p})+T(r,\frac{C_k^i g_{-c}^{(k-i)}}{g}))+S(r,f) \\ &\le \sum_{i=0}^{k}(S(r,e^p)+m(r,\frac{g_{-c}^{(i)}}{g_{-c}})+N(r,\frac{g_{-c}^{(i)}}{g}))+S(r,f) \\ &= S(r,e^p)+S(r,f), \end{aligned} \tag{2.5}$$

where C_k^i is a combinatorial number. By (2.1) and Lemma 2.1, we get

$$T(r,e^p)\le T(r,f_c)+T(r,f^{(k)})+S(r,f)\le 2T(r,f)+S(r,f). \tag{2.6}$$

Then it follows from (2.5) that $T(r,D)=S(r,f)$. Next we discuss two cases.

Case 1. $e^{-p}-D\not\equiv 0$. Rewrite (2.3) as

$$ge^p(e^{-p}-D)=B^{(k)}. \tag{2.7}$$

We claim that $D\equiv 0$. Otherwise, using the Lemma 2.8 to e^{-p}, we get

$$\begin{aligned} &m(r,\frac{1}{e^{-p}-D})+N(r,\frac{1}{e^{-p}-D})=T(r,e^{-p}) \\ &\le \overline{N}(r,e^{-p})+\overline{N}(r,\frac{1}{e^{-p}})+\overline{N}(r,\frac{1}{e^{-p}-D}) \\ &+S(r,e^p)=\overline{N}(r,\frac{1}{e^{-p}-D})+S(r,f)\le T(r,e^{-p})+S(r,f), \end{aligned}$$

that is to say

$$T(r,e^p)=T(r,e^{-p})+O(1)=\overline{N}(r,\frac{1}{e^{-p}-D})+S(r,f)$$

and

$$N(r,\frac{1}{e^{-p}-D})=N_1(r,\frac{1}{e^{-p}-D})+S(r,f).$$

It follows form above two equalities that

$$T(r,e^p)=N_1(r,\frac{1}{e^{-p}-D})+S(r,f).$$

Because the numbers of zeros and poles of $B^{(k)}$ are $S(r,f)$, we can see from (2.7) and $\overline{N}(r,f)=S(r,f)$ that the multiplicities of poles of g are almost 1. And then

$$\begin{aligned}N(r,f)+k\overline{N}(r,f)&=N(r,g)+S(r,f)=N(r,\frac{1}{e^{-p}-D})+S(r,f)\\ &=N_1(r,f)+S(r,f)\le\overline{N}(r,f)+S(r,f)=S(r,f).\end{aligned}$$

it follows from above that $\overline{N}(r,\frac{1}{e^{-p}-D})=S(r,f)$. Then by Lemma 2.8 in the following we can obtain

$$\begin{aligned}T(r,e^p)&=T(r,e^{-p})+O(1)\\ &\le\overline{N}(r,e^{-p})+\overline{N}(r,\frac{1}{e^{-p}})+\overline{N}(r,\frac{1}{e^{-p}-D})\\ &+S(r,e^p)=S(r,f),\end{aligned}\tag{2.8}$$

which contradicts with present assumption. Thus $D\equiv 0$. Then by (2.7) we get

$$g=B^{(k)}.\tag{2.9}$$

Integrating (2.9), we get

$$f_c=e^p(a_{-c}^{(k)}-a)+P+a,\tag{2.10}$$

where P is a polynomial of degree at most $k-1$. (2.10) implies

$$T(r,f_c)=T(r,e^p)+S(r,f).\tag{2.11}$$

Substituting (2.9) and (2.10) into (2.1) we can obtain

$$e^p(a_{-c}^{(k)}-a)+P=e^{p+p_{-c}}L_{-c},\tag{2.12}$$

where L_{-c} is the differential polynomial in

$$p'_{-c},\ldots,p_{-c}^{(k)},a_{-2c}-a_{-c},(a_{-2c}-a_{-c})',\ldots,(a_{-2c}-a_{-c})^{(k)},$$

and it is a small function of $f(z+c)$. On the one hand

$$2T(r,e^p)=T(r,e^{2p})=m(r,e^{2p})\le m(r,e^{p+p_{-c}})+m(r,\frac{e^p}{e^{p_{-c}}})\le T(r,e^{p+p_{-c}})+S(r,f).\tag{2.13}$$

On the other hand, we can prove similarly that

$$T(r,e^{p+p_{-c}})\le 2T(r,e^p)+S(r,f).\tag{2.14}$$

So

$$T(r,e^{p+p_{-c}})=2T(r,e^p)+S(r,f).\tag{2.15}$$

By (2.11), (2.12) and (2.15) we can get $T(r,e^p)=2T(r,e^p)+S(r,f)$, which is $T(r,e^p)=S(r,f)$, a contradiction.

Case 2. $e^{-p} - D \equiv 0$. Immediately, we get $T(r, e^p) = S(r, f)$, but it's impossible.

Of above discussions, we conclude that $T(r, e^p) = S(r, f)$. □

Lemma 2.5. *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$ such that $\overline{N}(r, f) = S(r, f)$, let k be a positive integer and $c \neq 0$ a complex value, and let $a \not\equiv \infty$ and $b \not\equiv \infty$ be two distinct small functions of f. Suppose*

$$L(f_c) = \begin{vmatrix} f_c - a & a - b \\ f_c' - a' & a' - b' \end{vmatrix}$$

and

$$L(f^{(k)}) = \begin{vmatrix} f^{(k)} - a & a - b \\ f^{(k+1)} - a' & a' - b' \end{vmatrix},$$

and f_c and $f^{(k)}$ share a, ∞ CM, and share b IM, then $L(f_c) \not\equiv 0$ and $L(f^{(k)}) \not\equiv 0$.

Proof. Suppose that $L(f_c) \equiv 0$, then we can get $\frac{f_c' - a'}{f_c - a} \equiv \frac{a' - b'}{a - b}$. Integrating both side of above we can obtain $f_c - a = C_1(a - b)$, where C_1 is a nonzero constant. So by Lemma 2.3, we have $T(r, f) = T(r, f_c) + S(r, f) = T(r, C(a - b) + a) = S(r, f)$, a contradiction. Hence $L(f_c) \not\equiv 0$.

Since $f^{(k)}$ and f_c share a CM and b IM, and f is a transcendental meromorphic function of $\rho_2(f) < 1$ such that $\overline{N}(r, f) = S(r, f)$, then by the Lemma 2.8, we get

$$\begin{aligned} T(r, f_c) &\leq \overline{N}(r, \frac{1}{f_c - a}) + \overline{N}(r, \frac{1}{f_c - b}) + \overline{N}(r, f_c) + S(r, f) \\ &= \overline{N}(r, \frac{1}{f^{(k)} - a}) + \overline{N}(r, \frac{1}{f^{(k)} - b}) + S(r, f) \\ &\leq 2T(r, f^{(k)}) + S(r, f). \end{aligned} \tag{2.16}$$

Hence a and b are small functions of $f^{(k)}$. If $L(f^{(k)}) \equiv 0$, then we can get $f^{(k)} - a = C_2(a - b)$, where C_2 is a nonzero constant. And we get $T(r, f^{(k)}) = S(r, f^{(k)})$. Combing (2.16) we obtain $T(r, f) = T(r, f_c) + S(r, f) = T(r, C(a - b) + a) = S(r, f)$, a contradiction. □

Lemma 2.6. *Let f be a transcendental meromorphic function, let $k_j (j = 1, 2, \ldots, q)$ be distinct constants, and let $a \not\equiv \infty$ and $b \not\equiv \infty$ be two distinct small functions of f. Again let $d_j = a - k_j(a - b)$ $(j = 1, 2, \ldots, q)$. Then*

$$m(r, \frac{L(f_c)}{f_c - a}) = S(r, f), \quad m(r, \frac{L(f_c)}{f_c - d_j}) = S(r, f).$$

for $1 \leq i \leq q$ and

$$m(r, \frac{L(f_c) f_c}{(f_c - d_1)(f_c - d_2) \cdots (f_c - d_m)}) = S(r, f),$$

where $L(f_c)$ is defined as in Lemma 2.5, and $2 \leq m \leq q$.

Proof. Obviously, we have

$$m(r, \frac{L(f_c)}{f_c - a}) \leq m(r, \frac{(a' - b')(f_c - a)}{f_c - a}) + m(r, \frac{(a - b)(f'_c - a')}{f_c - a}) = S(r, f),$$

and

$$\frac{L(f_c)f_c}{(f_c - d_1)(f_c - d_2)\cdots(f_c - d_q)} = \sum_{i=1}^{q} \frac{C_i L(f_c)}{f_c - d_i},$$

where $C_i = \frac{d_j}{\prod\limits_{j\neq i}(d_i - d_j)}$ are small functions of f. By Lemma 2.1 and above, we have

$$m(r, \frac{L(f_c)f_c}{(f_c - d_1)(f_c - d_2)\cdots(f_c - d_q)}) = m(r, \sum_{i=1}^{q} \frac{C_i L(f_c)}{f_c - d_i})$$

$$\leq \sum_{i=1}^{q} m(r, \frac{L(f_c)}{f_c - d_i}) + S(r, f) = S(r, f). \tag{2.17}$$

□

Lemma 2.7. *Let f and g be are two non-constant meromorphic functions such that $\overline{N}(r, f) = S(r, f)$, and let $a \not\equiv \infty$ and $b \not\equiv \infty$ be two distinct small functions of f and g. If*

$$H = \frac{L(f)}{(f - a)(f - b)} - \frac{L(g)}{(g - a)(g - b)} \equiv 0,$$

where

$$L(f) = (a' - b')(f - a) - (a - b)(f' - a')$$

and

$$L(g) = (a' - b')(g - a) - (a - b)(g' - a').$$

And if f and g share a, ∞ CM, and share b IM, then either $2T(r, f) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-b}) + S(r, f)$, or $f = g$.

Proof. Integrating H which leads to

$$\frac{g - b}{g - a} = C\frac{f - b}{f - a},$$

where C is a nonzero constant.

If $C = 1$, then $f = g$. If $C \neq 1$, then from above, we have

$$\frac{a - b}{g - a} \equiv \frac{(C - 1)f - Cb + a}{f - a},$$

and

$$T(r, f) = T(r, g) + S(r, f) + S(r, g).$$

It follows that $N(r, \frac{1}{f - \frac{Cb - a}{C - 1}}) = N(r, \frac{1}{a - b}) = S(r, f)$. Then by Lemma 2.8 in the following,

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f - a}) + \overline{N}(r, \frac{1}{f - \frac{Cb - a}{C - 1}}) + S(r, f)$$

$$\leq \overline{N}(r, \frac{1}{f - a}) + S(r, f) \leq T(r, f) + S(r, f),$$

and

$$T(r,f) \le \overline{N}(r,f) + \overline{N}(r,\frac{1}{f-b}) + \overline{N}(r,\frac{1}{f-\frac{Cb-a}{C-1}}) + S(r,f)$$
$$\le \overline{N}(r,\frac{1}{f-b}) + S(r,f) \le T(r,f) + S(r,f),$$

that is $T(r,f) = \overline{N}(r,\frac{1}{f-a}) + S(r,f)$ and $T(r,f) = \overline{N}(r,\frac{1}{f-b}) + S(r,f)$, and hence $2T(r,f) = \overline{N}(r,\frac{1}{f-a}) + \overline{N}(r,\frac{1}{f-b}) + S(r,f)$. □

Lemma 2.8. [22] *Let $f(z)$ be a non-constant meromorphic function, and let $a_j \in \hat{S}(f)$ be q distinct small functions for all $j = 1, 2, \ldots, q$. Then*

$$(q-2-\epsilon)T(r,f) \le \sum_{j=1}^{q} \overline{N}(r,\frac{1}{f-a_j}) + S(r,f), r \notin E,$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Remark 2.1. *Lemma 2.8 is true when $\infty, a_1, a_2, \cdots, a_q \in \hat{S}(f)$ with $S(r,f)$ in our notation, in other words, even if exceptional sets are of infinite linear measure. But they are not of infinite logarithmic measure.*

Lemma 2.9. [11] *Let f and g be two non-constant meromorphic functions. If f and g share $0, 1, \infty$ IM, and f is a bilinear transformation of g, then f and g assume one of the following six relations: (i) $fg = 1$; (ii) $(f-1)(g-1) = 1$; (iii) $f+g = 1$; (iv) $f = cg$; (v) $f-1 = c(g-1)$; (vi) $[(c-1)f+1][(c-1)g-c] = -c$, where $c \ne 0, 1$ is a complex number.*

Lemma 2.10. [3] *Let f, F and g be three non-constant meromorphic functions, where $g = F(f)$. Then f and g share three values IM if and only if there exist an entire function h such that, by a suitable linear fractional transformation, one of the following cases holds:*
(i) $f \equiv g$;
(ii) $f = e^h$ and $g = a(1 + 4ae^{-h} - 4a^2e^{-2h})$ have three IM shared values $a \ne 0$, $b = 2a$ and ∞;
(iii) $f = e^h$ and $g = \frac{1}{2}(e^h + a^2e^{-h})$ have three IM shared values $a \ne 0$, $b = -a$ and ∞;
(iv) $f = e^h$ and $g = a + b - abe^{-h}$ have three IM shared values $ab \ne 0$ and ∞;
(v) $f = e^h$ and $g = \frac{1}{b}e^{2h} - 2e^h + 2b$ have three IM shared values $b \ne 0$, $a = 2b$ and ∞;
(vi) $f = e^h$ and $g = b^2e^{-h}$ have three IM shared values $a \ne 0$, 0 and ∞.

Lemma 2.11. [10, 20, 21] *Let f and g be two non-constant meromorphic functions, and let $\rho(f)$ and $\rho(g)$ be the order of f and g, respectively. Then $\rho(fg) \le \max\{\rho(f), \rho(g)\}$.*

Remark 2.2. *We can see from the proof that Lemma 2.9* [11] *and Lemma 2,10* [20] *are still true when f and g share three value IM almost.*

3. The proof of Theorem 1.1

If $f_c \equiv f^{(k)}$, there is nothing to prove. Suppose $f_c \not\equiv f^{(k)}$. Since f is a non-constant meromorphic function of $\rho_2(f) < 1$, f_c and $f^{(k)}$ share a, ∞ CM, then we get

$$\frac{f^{(k)} - a}{f_c - a} = e^h, \tag{3.1}$$

where h is an entire function, and it is easy to know from (2.1) that $h = -p$.

Since f is a transcendental meromorphic function of $\rho_2(f) < 1$ and $f^{(k)}$ and f_c share ∞ CM, we can see from Lemma 2.1 and Lemma 2.3 that

$$(1 + o(1))N(r, f) + S(r, f) = N(r, f_c) = N(r, f^{(k)}),$$

which implies

$$\overline{N}(r, f) = S(r, f).$$

Furthermore, from the assumption that $f^{(k)}$ and f_c share a and ∞ CM and b IM, then by Lemma 2.1, Lemma 2.8 and above equality, we get

$$\begin{aligned}
T(r, f_c) &\le \overline{N}(r, \frac{1}{f_c - a}) + \overline{N}(r, \frac{1}{f_c - b}) + \overline{N}(r, f_c) + S(r, f) \\
&= \overline{N}(r, \frac{1}{f^{(k)} - a}) + \overline{N}(r, \frac{1}{f^{(k)} - b}) + S(r, f) \\
&\le N(r, \frac{1}{f_c - f^{(k)}}) + S(r, f) \le T(r, f_c - f^{(k)}) + S(r, f) \\
&\le m(r, f_c - f^{(k)}) + N(r, f_c - f^{(k)}) + S(r, f) \\
&\le m(r, f_c) + m(r, 1 - \frac{f^{(k)}}{f_c}) + N(r, f_c) + S(r, f) \le T(r, f_c) + S(r, f).
\end{aligned}$$

That is

$$T(r, f_c) = \overline{N}(r, \frac{1}{f_c - a}) + \overline{N}(r, \frac{1}{f_c - b}) + S(r, f). \tag{3.2}$$

By (3.1) and (3.2) we have

$$T(r, f_c) = T(r, f_c - f^{(k)}) + S(r, f) = N(r, \frac{1}{f_c - f^{(k)}}) + S(r, f). \tag{3.3}$$

and by Lemma 2.1,

$$\begin{aligned}
T(r, e^h) = m(r, e^h) = m(r, \frac{f^{(k)} - a_{-c}^{(k)} + a_{-c}^{(k)} - a}{f_c - a}) \le m(r, \frac{a_{-c}^{(k)} - a}{f_c - a}) \\
+ m(r, \frac{f^{(k)} - a_{-c}^{(k)}}{f_c^{(k)} - a^{(k)}}) + m(r, \frac{f_c^{(k)} - a^{(k)}}{f_c - a}) \le m(r, \frac{1}{f_c - a}) + S(r, f).
\end{aligned} \tag{3.4}$$

Then it follows from (3.1) and (3.3) that

(3.5)
$$m(r,\frac{1}{f_c-a})=m(r,\frac{e^h-1}{f^{(k)}-f_c})\le m(r,\frac{1}{f^{(k)}-f_c})+m(r,e^h-1)\le T(r,e^h)+S(r,f).$$

Then by (3.4) and (3.5)

$$T(r,e^h)=m(r,\frac{1}{f_c-a})+S(r,f). \tag{3.6}$$

On the other hand, (3.1) can be rewritten as

$$\frac{f^{(k)}-f_c}{f_c-a}=e^h-1, \tag{3.7}$$

which implies

$$\overline{N}(r,\frac{1}{f_c-b})\le\overline{N}(r,\frac{1}{e^h-1})+S(r,f)=T(r,e^h)+S(r,f). \tag{3.8}$$

Thus, by (3.2), (3.6) and (3.8)

$$\begin{aligned}m(r,\frac{1}{f_c-a})+N(r,\frac{1}{f_c-a})&=\overline{N}(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{f_c-b})+S(r,f)\\&\le\overline{N}(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{e^h-1})+S(r,f)\\&\le\overline{N}(r,\frac{1}{f_c-a})+m(r,\frac{1}{f_c-a})+S(r,f),\end{aligned}$$

which implies

$$N(r,\frac{1}{f_c-a})=\overline{N}(r,\frac{1}{f_c-a})+S(r,f). \tag{3.9}$$

And then

$$\overline{N}(r,\frac{1}{f_c-b})=T(r,e^h)+S(r,f). \tag{3.10}$$

Set

$$\varphi=\frac{L(f_c)(f_c-f^{(k)})}{(f_c-a)(f_c-b)}, \tag{3.11}$$

and

$$\psi=\frac{L(f^{(k)})(f_c-f^{(k)})}{(f^{(k)}-a)(f^{(k)}-b)}. \tag{3.12}$$

It is easy to know that $\varphi\not\equiv 0$ because of Lemma 2.5 and $f\not\equiv f^{(k)}$. We know that $N(r,\varphi)\le\overline{N}(r,f)=S(r,f)$ by (3.11). By Lemma 2.1 and Lemma 2.6 we have

$$\begin{aligned}T(r,\varphi)=m(r,\varphi)+N(r,\varphi)&=m(r,\frac{L(f_c)(f_c-f^{(k)})}{(f_c-a)(f_c-b)})+S(r,f)\\&\le m(r,\frac{L(f_c)f_c}{(f_c-a)(f_c-b)})+m(r,1-\frac{f^{(k)}}{f_c})+S(r,f)=S(r,f),\end{aligned}$$

that is

$$T(r,\varphi)=S(r,f). \tag{3.13}$$

Let $d = a - j(a-b)(j \neq 0, 1)$. Obviously, by Lemma 2.1 and Lemma 2.6, we obtain

$$\begin{aligned} m(r, \frac{1}{f_c}) &= m(r, \frac{1}{(b-a)\varphi}(\frac{L(f_c)}{f_c - a} - \frac{L(f_c)}{f_c - b})(1 - \frac{f^{(k)}}{f_c})) \\ &\leq m(r, \frac{1}{\varphi}) + m(r, \frac{L(f_c)}{f_c - a} - \frac{L(f_c)}{f_c - b}) \\ &+ m(r, 1 - \frac{f^{(k)}}{f_c}) + S(r,f) = S(r,f). \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} m(r, \frac{1}{f_c - d}) &= m(r, \frac{L(f_c)(f_c - f^{(k)})}{\varphi(f_c - a)(f_c - b)(f_c - d)}) \\ &\leq m(r, 1 - \frac{f^{(k)}}{f_c}) + m(r, \frac{L(f_c)f_c}{(f_c - a)(f_c - b)(f_c - d)}) \\ &+ S(r,f) = S(r,f). \end{aligned} \tag{3.15}$$

Set

$$\phi = \frac{L(f_c)}{(f_c - a)(f_c - b)} - \frac{L(f^{(k)})}{(f^{(k)} - a)(f^{(k)} - b)}. \tag{3.16}$$

We discuss two cases.

Case 1 $\phi \equiv 0$. Integrating the both sides of (3.16) which leads to

$$\frac{f_c - a}{f_c - b} = C\frac{f^{(k)} - a}{f^{(k)} - b}, \tag{3.17}$$

where C is a nonzero constant. Then by Lemma 2.7 we get

$$2T(r, f_c) = \overline{N}(r, \frac{1}{f_c - a}) + \overline{N}(r, \frac{1}{f_c - b}) + S(r,f), \tag{3.18}$$

which contradicts with (3.2).

Case 2 $\phi \not\equiv 0$. By (3.3), (3.13) and (3.16) we can obtain

$$\begin{aligned} T(r, f_c) &= T(r, f_c - f^{(k)}) + S(r,f) = T(r, \frac{\phi(f_c - f^{(k)})}{\phi}) + S(r,f) \\ &= T(r, \frac{\varphi - \psi}{\phi}) + S(r,f) \leq T(r, \varphi - \psi) + T(r, \phi) + S(r,f) \\ &\leq T(r, \psi) + T(r, \phi) + S(r,f) \leq T(r, \psi) + \overline{N}(r, \frac{1}{f_c - b}) + S(r,f). \end{aligned} \tag{3.19}$$

On the other hand,

$$\begin{aligned} T(r, \psi) &= T(r, \frac{L(f^{(k)})(f_c - f^{(k)})}{(f^{(k)} - a)(f^{(k)} - b)}) \\ &= m(r, \frac{L(f^{(k)})(f_c - f^{(k)})}{(f^{(k)} - a)(f^{(k)} - b)}) + N(r, \psi) \\ &\leq m(r, \frac{L(f^{(k)})}{f^{(k)} - b}) + m(r, \frac{f_c - f^{(k)}}{f^{(k)} - a}) + \overline{N}(r,f) + S(r,f) \\ &\leq m(r, \frac{1}{f_c - a}) + S(r,f) = \overline{N}(r, \frac{1}{f_c - b}) + S(r,f). \end{aligned} \tag{3.20}$$

Hence combining (3.19) and (3.20), we obtain

$$T(r,f_c)\le 2\overline{N}(r,\frac{1}{f_c-b})+S(r,f). \tag{3.21}$$

If $a_{-c}^{(k)}\equiv a$, then by (3.1) and Lemma 2.1 we can get

$$\begin{aligned}T(r,e^h)&=m(r,e^h)=m(r,\frac{f^{(k)}-a_{-c}^{(k)}}{f_c-a})\\ &\le m(r,\frac{f^{(k)}-a_{-c}^{(k)}}{f_c^{(k)}-a^{(k)}})+m(r,\frac{f_c^{(k)}-a^{(k)}}{f_c-a})=S(r,f).\end{aligned} \tag{3.22}$$

It follows from (3.10), (3.21), (3.22) and Lemma 2.3 that $T(r,f)=T(r,f_c)+S(r,f)=S(r,f)$. It's impossible.

If $a_{-c}^{(k)}\equiv b$, then by (3.10), (3.21) and and Lemma 2.1,

$$\begin{aligned}T(r,f_c)&\le m(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+S(r,f)\\ &\le m(r,\frac{f^{(k)}-a_{-c}^{(k)}}{f_c^{(k)}-a^{(k)}})+m(r,\frac{f_c^{(k)}-a^{(k)}}{f_c-a})+m(r,\frac{1}{f^{(k)}-b})\\ &+\overline{N}(r,\frac{1}{f^{(k)}-b})+S(r,f)\le T(r,f^{(k)})+S(r,f),\end{aligned}$$

which implies

$$T(r,f_c)\le T(r,f^{(k)})+S(r,f). \tag{3.23}$$

Lemma 2.3 implies

$$T(r,f^{(k)})\le T(r,f)+k\overline{N}(r,f)+S(r,f)=T(r,f_c)+S(r,f), \tag{3.24}$$

and it follows from the fact f_c and $f^{(k)}$ share a CM and b IM, (3.2) and (3.23) that

$$\begin{aligned}T(r,f^{(k)})&=T(r,f_c)+S(r,f)\\ &=\overline{N}(r,\frac{1}{f_c-a})+\overline{N}(r,\frac{1}{f_c-b})+S(r,f)\\ &=\overline{N}(r,\frac{1}{f^{(k)}-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+S(r,f).\end{aligned} \tag{3.25}$$

By Lemma 2.1, Lemma 2.8, (3.2) and (3.25), we have

$$\begin{aligned}2T(r,f^{(k)})&\le\overline{N}(r,\frac{1}{f^{(k)}-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+\overline{N}(r,\frac{1}{f^{(k)}-d})+\overline{N}(r,f^{(k)})\\ &+S(r,f)\le 2T(r,f^{(k)})-m(r,\frac{1}{f^{(k)}-d})+S(r,f)\end{aligned}$$

Immediately,

$$m(r,\frac{1}{f^{(k)}-d})=S(r,f). \tag{3.26}$$

By the First Fundamental Theorem, Lemma 2.1, Lemma 2.2, (3.14), (3.25), (3.26) and f is a transcendental meromorphic function of $\rho_2(f) < 1$, we obtain

$$\begin{aligned} m(r, \frac{f_c-d}{f^{(k)}-d}) &\leq m(r, \frac{f_c}{f^{(k)}-d}) + m(r, \frac{d}{f^{(k)}-d}) + O(1) \\ &\leq T(r, \frac{f_c}{f^{(k)}-d}) - N(r, \frac{f_c}{f^{(k)}-d}) + S(r,f) \\ &= m(r, \frac{f^{(k)}-d}{f_c}) + N(r, \frac{f^{(k)}-d}{f_c}) - N(r, \frac{f_c}{f^{(k)}-d}) + S(r,f) \\ &\leq N(r, \frac{1}{f_c}) - N(r, \frac{1}{f^{(k)}-d}) + N(r, f^{(k)}) - N(r,f) + S(r,f) \\ &= T(r, \frac{1}{f_c}) - T(r, \frac{1}{f^{(k)}-d}) + S(r,f) \\ &= T(r, f_c) - T(r, f^{(k)}) + S(r,f) = S(r,f). \end{aligned}$$

Thus

$$m(r, \frac{f_c-d}{f^{(k)}-d}) = S(r,f). \tag{3.27}$$

It's easy to see that $N(r,\psi) = S(r,f)$ and (3.12) can be rewritten as

$$\psi = [\frac{a-d}{a-b}\frac{L(f^{(k)})}{f^{(k)}-a} - \frac{b-d}{a-b}\frac{L(f^{(k)})}{f^{(k)}-b}][\frac{f_c-d}{f^{(k)}-d} - 1]. \tag{3.28}$$

Then by Lemma 2.6, (3.27) and (3.28) we can get

$$T(r,\psi) = m(r,\psi) + N(r,\psi) = S(r,f). \tag{3.29}$$

By (3.2), (3.19) and (3.29) we get

$$\overline{N}(r, \frac{1}{f_c-a}) = S(r,f). \tag{3.30}$$

Moreover, by Lemma 2.1, (3.2), (3.25) and (3.30), we have

$$m(r, \frac{1}{(f_c-a)^{(k)}}) = m(r, \frac{1}{f_c^{(k)}-b_c}) = m(r, \frac{1}{f^{(k)}-b}) + S(r,f) = S(r,f), \tag{3.31}$$

and it follows from above, (3.6) and (3.10) that

$$\begin{aligned} \overline{N}(r, \frac{1}{f_c-b}) &= m(r, \frac{1}{f_c-a}) + S(r,f) \\ &\leq m(r, \frac{1}{(f_c-a)^{(k)}}) + m(r, \frac{(f_c-a)^{(k)}}{f_c-a}) + S(r,f) = S(r,f). \end{aligned} \tag{3.32}$$

Then by (3.2), (3.30), (3.32) and Lemma 2.3, we obtain

$$\begin{aligned} T(r,f) = T(r,f_c) + S(r,f) &= \overline{N}(r, \frac{1}{f_c-a}) \\ &+ \overline{N}(r, \frac{1}{f_c-b}) + S(r,f) = S(r,f), \end{aligned} \tag{3.33}$$

which implies $T(r,f) = S(r,f)$, a contradiction.

So by (3.6), (3.10), (3.21), the First Fundamental Theorem, Lemma 2.8 and Remark 2.1 we can get

$$\begin{aligned}T(r,f_c) &\leq 2m(r,\frac{1}{f_c-a})+S(r,f)\leq 2m(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})\\ &+S(r,f)=2T(r,f^{(k)})-2N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})+S(r,f)\\ &\leq \overline{N}(r,\frac{1}{f^{(k)}-a})+\overline{N}(r,\frac{1}{f^{(k)}-b})+\overline{N}(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})\\ &+\overline{N}(r,f^{(k)})-2N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})+S(r,f)\\ &\leq T(r,f_c)-N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})+S(r,f),\end{aligned}$$

which implies that

$$N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})=S(r,f). \tag{3.34}$$

Consequently, Lemma 2.1 and Lemma 2.3 can deduce

$$N(r,\frac{1}{f^{(k)}-a_{-c}^{(k)}})=N(r,\frac{1}{f_c^{(k)}-a^{(k)}})=S(r,f).$$

Then applying Lemma 2.4, we have $T(r,e^h)=T(r,e^p)+O(1)=S(r,f)$, and it follows from (3.10) and (3.21) we can get $T(r,f)=T(r,f_c)+S(r,f)=S(r,f)$, a contradiction. This completes the proof of Theorem 1.

4. The Proof of Theorem 1.2

If $f(z)\equiv f(z+c)$, there is nothing to do. Assume that $f(z)\not\equiv f(z+c)$. Since $f(z)$ is a transcendental meromorphic function of $\rho_2(f)<1$, f and $f(z+c)$ share $a(z),\infty$ CM, then there is a nonzero entire function $p(z)$ of order less than 1 such that

$$\frac{f(z+c)-a(z)}{f(z)-a(z)}=e^{p(z)}, \tag{4.1}$$

then by Lemma 2.1 and $a(z)$ is a periodic function with period c,

$$T(r,e^p)=m(r,e^p)=m(r,\frac{f(z+c)-a(z+c)}{f(z)-a(z)})=S(r,f). \tag{4.2}$$

On the other hand, (4.1) can be rewritten as

$$\frac{f(z+c)-f(z)}{f(z)-a(z)}=e^{p(z)}-1, \tag{4.3}$$

and then we get

$$\overline{N}(r,\frac{1}{f(z)-b(z)})\leq N(r,\frac{1}{e^{p(z)}-1})=S(r,f). \tag{4.4}$$

Denote $N_{(m,n)}(r, \frac{1}{f(z)-b(z)})$ by the zeros of $f(z)-b(z)$ with multiplicities m and the zeros of $f_c(z)-b(z)$ with multiplicities n, where m, n are two positive integers. Thus, we can obtain

$$N(r, \frac{1}{f(z)-b(z)}) = \sum_{k=2}^{n} N_{(1,k)}(r, \frac{1}{f(z)-b(z)}) + \sum_{l=2}^{m} N_{(l,1)}(r, \frac{1}{f(z)-b(z)})$$
$$+ \sum_{l=2}^{m}\sum_{k=2}^{n} N_{(l,k)}(r, \frac{1}{f(z)-b(z)}) \le \overline{N}(r, \frac{1}{f(z)-b(z)}) + m\overline{N}(r, \frac{1}{f(z+c)-b(z)})$$
(4.5)
$$+ N(r, \frac{1}{e^{p(z)}-1}) \le (m+1)\overline{N}(r, \frac{1}{f(z)-b(z)}) + S(r,f) = S(r,f),$$

that is

$$N(r, \frac{1}{f(z+c)-b(z+c)}) = N(r, \frac{1}{f(z)-b(z)}) = S(r,f). \tag{4.6}$$

Similarly, we also have

$$N(r, \frac{1}{f(z+c)-b(z)}) = S(r,f). \tag{4.7}$$

Set

$$\psi(z) = \frac{f(z+c)-b(z+c)}{f(z)-b(z)}. \tag{4.8}$$

It is easy to see that

$$N(r, \frac{1}{\psi(z)}) \le N(r, \frac{1}{f(z+c)-b(z+c)}) + N(r, b(z)) = S(r,f), \tag{4.9}$$

$$N(r, \psi(z)) \le N(r, \frac{1}{f(z)-b(z)}) + N(r, b(z)) = S(r,f). \tag{4.10}$$

Hence by Lemma 2.1 and above,

$$T(r, \psi(z)) = m(r, \psi(z)) + N(r, \psi(z)) = S(r,f) \tag{4.11}$$

According to (4.1) and (4.8),we have

$$(e^{p(z)} - \psi(z))f(z) + \psi(z)b(z) + a(z) - b(z+c) - a(z)e^{p(z)} \equiv 0. \tag{4.12}$$

We discuss following two cases.

Case 1 $e^{p(z)} \not\equiv \psi(z)$. Then by (4.2), (4.11) and (4.12) we obtain $T(r,f) = S(r,f)$, a contradiction.

Case 2 $e^{p(z)} \equiv \psi(z)$. Then by (4.1) we have

$$f(z+c) = e^{p(z)}(f(z)-a(z)) + a(z), \tag{4.13}$$

and

$$N(r, \frac{1}{f(z+c)-b(z)}) = N(r, \frac{1}{f(z)-a(z)+\frac{a(z)-b(z)}{e^{p(z)}}}) = S(r,f). \tag{4.14}$$

If $b(z)$ is a periodic function of period c, then by (4.12) we can get $e^{p(z)} \equiv 1$, which implies $f(z) \equiv f(z+c)$, a contradiction. Obviously, $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv a(z)$. Otherwise, we can deduce $a(z) \equiv b(z)$, a contradiction.

Next, we discuss three subcases.

Subcase 2.1 $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv b(z)$ and $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv b(z-c)$. Then according to (4.6), (4.7),(4.14) and Lemma 2.8, we can get

$$T(r, f(z)) \leq \overline{N}(r, \frac{1}{f(z) - a(z) - \frac{a(z)-b(z)}{e^{p(z)}}}) + \overline{N}(r, \frac{1}{f(z) - b(z)})$$

$$+ \overline{N}(r, \frac{1}{f(z) - b(z-c)}) + S(r, f) = S(r, f), \tag{4.15}$$

that is $T(r, f(z)) = S(r, f)$, a contradiction.

Subcase 2.2 $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \equiv b(z)$, but $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv b(z-c)$. It follows that $e^{p(z)} \equiv 1$. Therefore by (4.1) we have $f(z) \equiv f(z+c)$, a contradiction.

Subcase 2.3 $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \equiv b(z)$, $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \equiv b(z-c)$. It follows that $e^{p(z)} \equiv 1$. Therefore by (4.1) we have $f(z) \equiv f(z+c)$, a contradiction.

Subcase 2.4 $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \not\equiv b(z)$ and $a(z) - \frac{a(z)-b(z)}{e^{p(z)}} \equiv b(z-c)$. It is easy to see that

$$\frac{a(z) - b(z)}{a(z-c) - b(z-c)} = e^{p(z)}. \tag{4.16}$$

Furthermore, (4.12) implies

$$\frac{a(z+c) - b(z+c)}{a(z) - b(z)} = e^{p(z)}, \tag{4.17}$$

$$\frac{a(z) - b(z)}{a(z-c) - b(z-c)} = e^{p(z-c)}. \tag{4.18}$$

It follows from (4.16) and (4.18) that

$$e^{p(z)} = e^{p(z+c)}. \tag{4.19}$$

By (4.1), (4.8) and (4.19), we know that $f(z)$ and $f(z+nc)$ share $a(z)$ and ∞ CM, so we set

$$F(z) = \frac{f(z) - a(z)}{b(z) - a(z)}, \quad G(z) = \frac{f(z+nc) - a(z)}{b(z+nc) - a(z+nc)}. \tag{4.20}$$

Since $f(z)$ and $f(z+nc)$ share $a(z)$ and ∞ CM, and $(b(z), b(z+nc)$ CM, so $F(z)$ and $G(z)$ share $0, \infty$ CM almost, and 1 CM almost. We claim that F is not a bilinear transform of G. Otherwise, we can see from Lemma 2.9 that if (i) occurs, we have $N(r, f(z)) = N(r, F(z)) + S(r, f) = S(r, f)$, then by Remark 1 and Theorem G, we get $f(z) \equiv f(z+c)$, a contradiction.

If (ii) occurs, we have $N(r, f(z)) = N(r, F(z)) + S(r, f) = S(r, f)$, then by Remark 1 and Theorem G, we get $f(z) \equiv f(z+c)$, a contradiction.

If (iii) occurs, we have

$$N(r,\frac{1}{f(z)-a(z)})=S(r,f),\quad N(r,\frac{1}{f(z)-b(z)})=S(r,f). \tag{4.21}$$

Then it follows from above, $a(z)-\frac{a(z)-b(z)}{e^{p(z)}}\not\equiv a(z)$, $a(z)-\frac{a(z)-b(z)}{e^{p(z)}}\not\equiv b(z)$ and Lemma 2.8 that $T(r,f)=S(r,f)$, a contradiction.

If (iv) occurs, we have $F(z)\equiv jG(z)$, that is

$$\frac{b(z+nc)-a(z+nc)}{b(z)-a(z)}=j(\frac{f(z+nc)-a(z)}{f(z)-a(z)}), \tag{4.22}$$

where $j\neq 0,1$ is a finite constant. Then it follows from above, (4.17) and (4.19) that $e^{np(z)}=je^{np(z)}$, therefore we have $j=1$, a contradiction.

If (v) occurs, we have

$$N(r,\frac{1}{f(z)-a(z)})=S(r,f). \tag{4.23}$$

Then by Lemma 2.8, (4.7), (4.14) and $b(z-c)\not\equiv a(z)$, we obtain $T(r,f)=S(r,f)$, a contradiction.

If (vi) occurs, we have

$$N(r,f(z))=N(r,F(z))+S(r,f)=S(r,f), \tag{4.24}$$

and hence we can see from Theorem G and Remark 1 that $f(z)\equiv f(z+c)$, a contradiction.

Therefore, $F(z)$ is not a linear fraction transformation of $G(z)$. If $b(z)$ is a small function with period nc, that is $b(z+(n-1)c)\equiv b(z-c)$, we can set

$$\begin{aligned}D(z)&=(f(z)-b(z))(b(z+nc)-b(z+(n-1)c))\\&\quad-(f(z+nc)-b(z+nc))(b(z)-b(z-c))\\&=(f(z)-b(z-c))(b(z+nc)-b(z+(n-1)c))\\&\quad-(f(z+nc)-b(z+(n-1)c))(b(z)-b(z-c))\end{aligned}$$

If $D(z)\equiv 0$, then we have $f(z+nc)-b(z-c)\equiv -(f(z)-b(z-c))$. And thus we know that $f(z)$ and $f(z+nc)$ share $a(z),b(z-c)$ and ∞ CM. We suppose

$$F_1(z)=\frac{f(z)-a(z)}{b(z-c)-a(z)},G_1(z)=\frac{f(z+nc)-a(z)}{b(z-c)-a(z)}. \tag{4.25}$$

Then we know that $F_1(z)$ and $G_1(z)$ share $0,1,\infty$ CM almost and $G_1(z)=-F_1(z)$. So by Lemma 2.10, we will obtain either $N(r,f(z))=N(r,F_1)+S(r,f)=S(r,f)$, but in this case, according to Theorem G and Remark 1, we can deduce a contradiction. Or $F_1(z)=G_1(z)$, that is $f(z)\equiv f(z+nc)$. Therefore, we obtain $f(z)\equiv b(z-c)$, that is $T(r,f(z))=S(r,f)$, a contradiction.

Hence $D(z) \not\equiv 0$, and by (4.7)-(4.8), (4.14) and Lemma 2.1, we have

$$\begin{aligned}
2T(r, f(z)) &= m(r, \frac{1}{f(z) - b(z)}) + m(r, \frac{1}{f(z) - b(z-c)}) + S(r, f) \\
&= m(r, \frac{1}{f(z) - b(z)} + \frac{1}{f(z) - b(z-c)}) + S(r, f) \\
&\leq m(r, \frac{D(z)}{f(z) - b(z)} + \frac{D(z)}{f(z) - b(z-c)}) + m(r, \frac{1}{D(z)}) + S(r, f) \\
&\leq m(r, D) + N(r, D) \leq m(r, f(z)) + N(r, f(z)) + S(r, f) \\
&= T(r, f) + S(r, f),
\end{aligned} \tag{4.26}$$

which implies $T(r, f) = S(r, f)$, a contradiction.

By (4.16) we have

$$\frac{\Delta_c b(z)}{1 - e^{p(z)}} + b(z) = a(z). \tag{4.27}$$

Combining (4.18) and the fact that $a(z)$ is a small function with period c, we can get

$$\frac{\Delta_c b(z+c)}{1 - e^{p(z)}} + b(z+c) = a(z). \tag{4.28}$$

According to (4.27) and (4.28), we obtain

$$e^{p(z)} = \frac{b_{2c}(z) - b_c(z)}{\Delta_c b(z)}. \tag{4.29}$$

So if $\rho(b(z)) < \rho(e^{p(z)})$, we can follows from (4.28) and Lemma 2.11 that

$$\rho(e^{p(z)}) = \rho(\frac{b_{2c}(z) - b_c(z)}{\Delta_c^2 b(z)}) \leq \rho(b(z)) < \rho(e^{p(z)}), \tag{4.30}$$

which is a contradiction.

If $\rho(b(z)) < 1$, we claim that $p(z) \equiv B$ is a non-zero constant. Otherwise, the order of right hand side of (4.28) is 0, but the left hand side is 1, which is impossible. Therefore, by (4.1) we know that $f(z+c) - a(z) = B(f(z) - a(z))$, and then by Lemma 2.10 we will get $N(r, f) = S(r, f)$, so by Theorem G and Remark 1 we can obtain $f(z) \equiv f(z+c)$, a contradiction.

This completes Theorem 1.2.

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