Известия НАН Армении, Математика, том 59, н. 1, 2024, стр. 86 – 100.

UNICITY OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENTIAL-DIFFERENCE POLYNOMIALS

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Abstract. In this paper, we study unicity of meromorphic functions concerning differential-difference polynomials and mainly prove: Let $k_1, k_2, \cdots, k_n$ be non-negative integers and $k = \max\{k_1, k_2, \cdots, k_n\}$, let l be the number of distinct values of $\{0, c_1, c_2, \cdots, c_n\}$, let s be the number of distinct values of $\{c_1, c_2, \cdots, c_n\}$, let $f(z)$ be a non-constant meromorphic function of finite order satisfying $N(r,f) \le \frac{1}{8(lk+l+2s-1)+1}T(r,f)$, let $m_1(z), m_2(z), \cdots, m_n(z)$, $a(z), b(z)$ be small functions of $f(z)$ such that $a(z) \not\equiv b(z)$, let $(c_1, k_1), (c_2, k_2), \cdots, (c_n, k_n)$ be distinct and let $F(z) = m_1(z)f^{(k_1)}(z+c_1)+m_2(z)f^{(k_2)}(z+c_2)+\cdots+m_n(z)f^{(k_n)}(z+c_n)$. If $f(z)$ and $F(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv F(z)$. Our results improve and extend some results due to [1, 18, 20].

MSC2020 numbers: 30D35.
Keywords: meromorphic function; small function; difference polynomial.

1. Introduction and main results

In this paper, a meromorphic function always means meromorphic in the whole complex plane. We use the following standard notations in value distribution theory, see [7, 15, 16]: $T(r,f), N(r,f), m(r,f), \cdots$.

We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function $\alpha(z)$ is said to be a small function of $f(z)$ if it satisfies $T(r,\alpha) = S(r,f)$.

Let $\alpha(z)$ be a small function of both $f(z)$ and $g(z)$. If $f(z)-\alpha(z)$ and $g(z)-\alpha(z)$ have the same zeros counting multiplicities (ignoring multiplicities), then we call that $f(z)$ and $g(z)$ share $\alpha(z)$ CM (IM). Let $N(r,\alpha)$ be the counting function of common zeros of both $f(z) - \alpha(z)$ and $g(z) - \alpha(z)$ with counting multiplicities. If

$$N\left(r, \frac{1}{f-\alpha}\right) + N\left(r, \frac{1}{g-\alpha}\right) - 2N(r,\alpha) \le S(r,f) + S(r,g),$$

then we call that $f(z)$ and $g(z)$ share $\alpha(z)$ CM almost.

[1]This paper is supported by the NNSF of China(Grant No 12171127) and the NSF of Zhejiang Province (LY21A010012).

Let $f(z)$ be a non-constant meromorphic function. Define

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{\log^+ T(r,f)}{\log r}$$

by the order of $f(z)$.

For a nonzero complex constant $\eta\in\mathbb{C}$, we define the difference operators of $f(z)$ as $\triangle_\eta f(z)=f(z+\eta)-f(z)$ and $\triangle_\eta^k f(z)=\triangle_\eta(\triangle_\eta^{k-1}f(z)), k\in\mathbb{N}, k\geq 2$.

Let $f(z)$ be a non-constant meromorphic function, let $n_0, n_1, \cdots, n_k$ be non-negative integers, let $c_0, c_1, \cdots, c_k$ be finite values, we call that $M(f)=f^{n_0}(z+c_0)(f')^{n_1}(z+c_1)\cdots(f^{(k)})^{n_k}(z+c_k)$ is a differential-difference monomial, and its degree $\gamma_M=n_0+n_1+\cdots+n_k$. Let $H=a_1M_1(f)+a_2M_2(f)+\cdots+a_nM_n(f)$ be a homogeneous differential-difference polynomial, where $a_1(z), a_2(z), \cdots, a_n(z)$ are small functions of $f(z)$ and $\gamma_{M_1}=\gamma_{M_2}=\cdots=\gamma_{M_n}$.

Let $N_{k)}(r,f)$ be the counting function for poles of $f(z)$ with multiplicity $\leq k$ and let $N_{(k}(r,f)$ be the counting function for poles of $f(z)$ with multiplicity $\geq k$.

Nevanlinna [7, 15, 16] proved the famous five-value theorem.

Theorem A. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a_j (j=1,2,\cdots,5)$ be five distinct values on extend complex plane. If $f(z)$ and $g(z)$ share $a_j (j=1,2,\cdots,5)$ IM, then $f(z)\equiv g(z)$.*

Li and Qiao[11] proved the five-small function theorem.

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a_j(z) (j=1,2,\cdots,5)$ be five distinct small functions of both $f(z)$ and $g(z)$ (one may be ∞). If $f(z)$ and $g(z)$ share $a_j(z) (j=1,2,\cdots,5)$ IM, then $f(z)\equiv g(z)$.*

In 1976, Rubel and Yang[14] proved the following result.

Theorem C. *Let $f(z)$ be a non-constant entire function, and let a, b be two distinct finite values. If $f(z)$ and $f'(z)$ share a, b CM, then $f(z)\equiv f'(z)$.*

In 1992, Zheng and Wang[19] proved:

Theorem D. *Let $f(z)$ be a non-constant entire function, and let $a(z), b(z)$ be two distinct small functions of $f(z)$. If $f(z)$ and $f'(z)$ share $a(z), b(z)$ CM, then $f(z)\equiv f'(z)$.*

In 1995, Fang[5] proved the following theorem.

Theorem E. *Let $f(z)$ be a non-constant meromorphic function such that $N(r,f)=S(r,f)$, let n be a positive integer, let a, b be two distinct finite complex values, and let $F(z)=f^{(n)}(z)+a_1(z)f^{(n-1)}(z)+\cdots+a_n(z)f(z)$, where $a_1(z), a_2(z), \cdots, a_n(z)$ are small functions of $f(z)$. If $f(z)$ and $F(z)$ share a, b CM almost, then $f(z)\equiv F(z)$.*

In 2006, Chen[1] studied the case of meromorphic function satisfying $N(r,f)\leq \frac{1}{8n+17}T(r,f)$, and proved the following result.

Theorem F. *Let n be a positive integer, let $f(z)$ be a non-constant meromorphic function satisfying $N(r,f) \le \frac{1}{8n+17}T(r,f)$, and let $a(z), b(z)$ be two distinct small functions of $f(z)$, and let $F(z) = f^{(n)}(z) + a_1(z)f^{(n-1)}(z) + \cdots + a_n(z)f(z)$, where $a_1(z), a_2(z), \cdots, a_n(z)$ are small functions of $f(z)$. If $f(z)$ and $F(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv F(z)$.*

Recently, a number of articles focused on value distribution in shifts or difference operators of meromorphic functions. In particular, some papers studied the unicity of meromorphic functions sharing values with their shifts or difference operators (see [3, 4, 6, 9, 12, 13, 20]).

In 2011, Heittokangas et al.[9] proved the following result.

Theorem G. *Let $f(z)$ be a non-constant entire function of finite order, let η be a nonzero constant, and let a, b be two distinct finite values. If $f(z)$ and $f(z+\eta)$ share a, b CM, then $f(z) \equiv f(z+\eta)$.*

In 2014, Zhang and Liao[20] proved the following result.

Theorem H. *Let $f(z)$ be an entire function of finite order, let η be a nonzero constant, and let a, b be two distinct finite values. If $f(z)$ and $\triangle_\eta f(z)$ share a, b CM, then $f(z) \equiv \triangle_\eta f(z)$.*

Liu et al.[12] replaced $\triangle_\eta f(z)$ by the general difference polynomial and proved the following result:

Theorem I. *Let $f(z)$ be a non-constant entire function of finite order, let n be a positive integer, let $a(z)$, $b(z)$ be be two distinct small functions of $f(z)$, and let $F(z) = m_1 f(z+c_1) + m_2 f(z+c_2) + \cdots + m_n f(z+c_n)$, where $m_1, m_2, \cdots, m_n$ are nonzero complex numbers and $c_1, c_2, \cdots, c_n$ are distinct finite values. If $f(z)$ and $F(z)$ share $a(z), b(z)$ CM, then $f(z) \equiv F(z)$.*

In 2017, Yang and Liu[18] extended Theorem J and proved the following theorem.

Theorem J. *Let $f(z)$ be a non-constant meromorphic function of finite order, let n be a positive integer, let $a(z), b(z)$ be two distinct small functions of $f(z)$, let $m_1, m_2, \cdots, m_n$ be nonzero complex numbers, let $c_1, c_2, \cdots, c_n$ be distinct finite complex numbers, and let*

$$F(z) = m_1 f(z+c_1) + m_2 f(z+c_2) + \cdots + m_n f(z+c_n).$$

If $f(z)$ and $F(z)$ share $a(z)$, $b(z)$ CM almost and $N(r,f) \le \frac{1}{27n}T(r,f)$, then $f(z) \equiv F(z)$.

In this paper, we extend and improve the above results.

Theorem 1.1. *Let $k_1, k_2, \cdots, k_n$ be non-negative integers and $k = \max\{k_1, k_2, \cdots, k_n\}$, let l be the number of distinct values of $\{0, c_1, c_2, \cdots, c_n\}$, let s be the number of distinct values of $\{c_1, c_2, \cdots, c_n\}$, let $f(z)$ be a non-constant meromorphic function* *of finite order satisfying* $N(r,f) \leq \frac{1}{8(lk+l+2s-1)+1}T(r,f)$*, let* $m_1(z)$*,* $m_2(z)$*,* $\cdots$*,* $m_n(z)$*,* $a(z), b(z)$ *be small functions of* $f(z)$ *such that* $a(z) \not\equiv b(z)$*, let* (c_1,k_1)*,*(c_2,k_2)*,* $\cdots$*,* (c_n,k_n) *be distinct and let*

(1.1) $F(z) = m_1(z)f^{(k_1)}(z+c_1) + m_2(z)f^{(k_2)}(z+c_2) + \cdots + m_n(z)f^{(k_n)}(z+c_n).$

If $f(z)$ *and* $F(z)$ *share* $a(z), b(z)$ *CM, then* $f(z) \equiv F(z)$*.*

Remark 1.1. Let $l = s = 1, k = n, 0 = c_1 = c_2 = \cdots = c_n$, then Theorem 1 is also valid. If $F(z) = f^{(n)}(z)+a_1(z)f^{(n-1)}(z)+\cdots+a_n(z)f(z)$, where $a_1(z), a_2(z)\cdots, a_n(z)$ are small functions of $f(z)$. Then by Theorem 1.1 we get Theorem F.

Corollary 1.1. *Let* $k = 0$*, let* $f(z)$ *be a non-constant meromorphic function of finite order, let* n *be a positive integer, let* $a(z)$*,* $b(z)$ *be be two distinct small functions of* $f(z)$*, let* $m_1, m_2, \cdots, m_n$ *be nonzero complex numbers, let* $c_1, c_2, \cdots, c_n$ *be distinct finite complex numbers, and let*

$$F(z) = m_1 f(z+c_1) + m_2 f(z+c_2) + \cdots + m_n f(z+c_n).$$

If $f(z)$ *and* $F(z)$ *share* $a(z)$*,* $b(z)$ *CM almost and* $N(r,f) \leq \frac{1}{24n+1}T(r,f)$*, then* $f(z) \equiv F(z)$*.*

By Corollary 1.1, we get Theorem J.

The following example illustrates that the condition

$$N(r,f) \leq \frac{1}{8(lk+l+2s-1)+1}T(r,f)$$

is necessary in Theorem 1.1.

Example 1.1. Let $f(z) = \frac{e^z+1}{e^z-1}$ and $F(z) = f(z) - f(z+c) - f(z+2c) = -\frac{e^{z+c}+1}{e^{z+c}-1}$, where $c = \pi i$. It is easy to see that $f(z)$ and $F(z)$ share $1, -1$ CM. But $f(z) \not\equiv F(z)$.

Theorem 1.2. *Let* $F(z), l, k, s$ *be the same as Theorem 1, let* $f(z)$ *be a non-constant meromorphic function of finite order satisfying* $N_{1)}(r,f) \leq \frac{1}{5(lk+l+2s-1)}T(r,f)$*, and let* $a(z), b(z)$ *be distinct small functions of* $f(z)$*. If* $f(z)$ *and* $F(z)$ *share* $a(z), b(z), \infty$ *CM, then* $f(z) \equiv F(z)$*.*

The following example illustrates that the condition

$$N_{1)}(r,f) \leq \frac{1}{5(lk+l+2s-1)}T(r,f)$$

is necessary in Theorem 1.2.

Example 1.2. Let $f(z) = \frac{e^z+1}{e^z-1}$ and $F(z) = f(z) + f(z+c) - f(z+2c) - f(z+3c) - f(z+4c) = -\frac{e^z+1}{e^z-1}$, where $c = 2\pi i$. It is easy to see that $f(z)$ and $F(z)$ share $1, -1, \infty$ CM. But $f(z) \not\equiv F(z)$.

Theorem 1.3. *Let $f(z)$ be a non-constant meromorphic function of finite order, let $a(z), b(z)$ be two distinct small functions of $f(z)$, and let $H(f)$ be a homogeneous differential-difference polynomial of f with* $\deg H = m$*. If $f^m(z)(m \geq 2)$ and $H(f)$ share $a(z), b(z), \infty$ CM, then $f^m(z) \equiv H(f)$.*

2. Some Lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1. [7, 15, 16]. *Let $f(z)$ be a non-constant meromorphic function, and let $a_i(z)(i = 1, 2)$ be two distinct small functions of $f(z)$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

Lemma 2.2. [17]. *Let $f(z)$ be a non-constant meromorphic function, and let $a_i(z)(i = 1, 2, 3)$ be three distinct small functions of $f(z)$. Then for any $0 < \varepsilon < 1$,*

$$2T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{f - a_i}\right) + \varepsilon T(r, f) + S(r, f).$$

Lemma 2.3. [2]. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let η be a non-zero finite complex number. Then*

$$N(r, f(z + \eta)) = N(r, f(z)) + S(r, f).$$

Lemma 2.4. [2, 8, 10]. *Let $f(z)$ be a non-constant meromorphic function of finite order, let k be a positive integer and let η be a non-zero finite complex number. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \quad m\left(r, \frac{f(z + \eta)}{f(z)}\right) = S(r, f).$$

Lemma 2.5. [5]. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions satisfying*

$$N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f), \quad N(r, g) + N\left(r, \frac{1}{g}\right) = S(r, g).$$

If $f(z)$ and $g(z)$ share 1 CM almost, then either $f(z)g(z) \equiv 1$ or $f(z) \equiv g(z)$.

Lemma 2.6. [7, 15, 16]. *Let $f(z)$ be a non-constant meromorphic function, let $n(\geq 2)$ be a positive integer, and let $a_1(z), a_2(z) \cdots a_n(z)$ be distinct small functions of $f(z)$. Then*

$$m\left(r, \frac{1}{f - a_1}\right) + \cdots + m\left(r, \frac{1}{f - a_n}\right) \leq m\left(r, \frac{1}{f - a_1} + \cdots + \frac{1}{f - a_n}\right) + S(r, f).$$

Lemma 2.7. [16]. *Let k be a positive integer and let $f(z)$ be a meromorphic function such that $f^{(k)}(z) \not\equiv 0$. Then*

$$\begin{aligned} T(r, f^{(k)}) \leq & T(r, f) + k\overline{N}(r, f) + S(r, f), \\ N\left(r, \frac{1}{f^{(k)}}\right) \leq & N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \end{aligned}$$

Lemma 2.8. [1]. *Let $0 \leq \lambda \leq \frac{1}{4}$ and let $f(z)$ and $g(z)$ be two meromorphic functions satisfying*

$$\overline{N}(r, f) \leq \lambda T(r, f), \ \overline{N}(r, g) \leq \lambda T(r, g).$$

If $f(z)$ and $g(z)$ share 0, 1 CM almost, and

$$\overline{\lim_{\substack{r\to\infty \\ r\in I}}} \frac{N(r,0) + N(r,1)}{T(r,f) + T(r,g)} < \frac{2-8\lambda}{3},$$

where $I \subset [0, \infty)$ is a set of infinite linear measure, then $\frac{1}{f-1} - \frac{c}{g-1} = d$, where $c(\neq 0), d$ are two constants.

By imitating the proof of Lemma 2.8, we can prove the following lemma.

Lemma 2.9. *Let $0 \leq \lambda < 1$ and let $f(z)$ and $g(z)$ be two meromorphic functions satisfying*

$$\overline{N}(r, f) \leq \lambda T(r, f), \ \overline{N}(r, g) \leq \lambda T(r, g).$$

If $f(z)$ and $g(z)$ share 0, 1, ∞ CM almost, and

$$\overline{\lim_{\substack{r\to\infty \\ r\in I}}} \frac{N(r,0) + N(r,1)}{T(r,f) + T(r,g)} < \frac{2-2\lambda}{3},$$

where $I \subset [0, \infty)$ is a set of infinite linear measure, then $\frac{1}{f-1} - \frac{c}{g-1} = d$, where $c(\neq 0), d$ are two constants.

Lemma 2.10. *Let k, l be non-negative integers. Then*

$$\begin{aligned}
&(1) \quad \frac{9lk+25l-8}{15lk+45l-13} < \frac{16lk+48l-22}{24lk+72l-21} (k \geq 1, l \geq 1), \\
&(2) \quad \frac{9lk+25l-24}{15lk+45l-43} < \frac{16lk+48l-54}{24lk+72l-69} (k \geq 0, l \geq 2), \\
&(3) \quad \frac{6lk+16l-6}{9lk+27l-9} < \frac{10lk+30l-12}{15lk+45l-15} (k \geq 1, l \geq 1), \\
&(4) \quad \frac{6lk+16l-16}{9lk+27l-27} < \frac{10lk+30l-32}{15lk+45l-45} (k \geq 0, l \geq 2).
\end{aligned}$$

3. Proof of Theorems

Proof of Theorem 1.1. Set

$$g(z)=\frac{f(z)-a(z)}{b(z)-a(z)}, \tag{3.1}$$

$$G(z)=\frac{F(z)-a(z)}{b(z)-a(z)}. \tag{3.2}$$

Since $f(z)$ and $F(z)$ share $a(z), b(z)$ CM, we know that $g(z)$ and $G(z)$ share 0, 1 CM almost.

It follows from (3.1) and (3.2) that

$$T(r,g)=T(r,f)+S(r,f), \tag{3.3}$$

$$T(r,G)=T(r,F)+S(r,f), \tag{3.4}$$

$$N(r,g)=N(r,f)+S(r,f). \tag{3.5}$$

Hence, by (1.1), (3.2) and Lemma 2.3, we get

$$N(r,G)=N(r,F)+S(r,f)\le s\left(N(r,f)+k\overline{N}(r,f)\right)+S(r,f). \tag{3.6}$$

It follows that

$$T(r,F)\le (s+sk)T(r,f)+S(r,f). \tag{3.7}$$

Hence, we obtain

$$S(r,g)=S(r,f), S(r,f)=S(r,g),$$

$$S(r,F)=S(r,f), S(r,G)=S(r,f).$$

Since $g(z)$ and $G(z)$ share 0, 1 CM almost, we have

$$\begin{aligned} N(r,0)+N(r,1) &\le N\left(r,\frac{1}{G-g}\right)+S(r,f)\\ &\le N\left(r,\frac{1}{F-f}\right)+S(r,f)\\ &\le T(r,F-f)+S(r,f)\\ &= m(r,F-f)+N(r,F-f)+S(r,f). \end{aligned} \tag{3.8}$$

It follows from Lemmas 2.3 and 2.4 that

$$m(r,F-f)\le m\left(r,\frac{F-f}{f}\right)+m(r,f)+S(r,f)\le m(r,f)+S(r,f), \tag{3.9}$$

$$N(r,F-f)\le lN(r,f^{(k)})+S(r,f)\le l\left(N(r,f)+k\overline{N}(r,f)\right)+S(r,f). \tag{3.10}$$

By (3.8)-(3.10) and $N(r,f) \leq \frac{1}{8(lk+l+2s-1)+1}T(r,f)$, we obtain

$$\begin{aligned} N(r,0)+N(r,1) \leq & m(r,f)+l\left(N(r,f)+k\overline{N}(r,f)\right)+S(r,f) \\ \leq & T(r,f)+(l-1+lk)N(r,f)+S(r,f) \\ \leq & \frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+S(r,f). \end{aligned} \tag{3.11}$$

By Nevanlinna's first fundamental theorem and (3.11), we have

$$\begin{aligned} 2T(r,f) = & 2T(r,g)+S(r,f) = T\left(r,\frac{1}{g}\right)+T\left(r,\frac{1}{g-1}\right)+S(r,f) \\ \leq & N(r,0)+N(r,1)+m\left(r,\frac{1}{g}\right)+m\left(r,\frac{1}{g-1}\right)+S(r,f) \\ \leq & \frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right)+S(r,f). \end{aligned} \tag{3.12}$$

Set

$$\begin{aligned} a_1(z) = & m_1 a^{(k_1)}(z+c_1)+m_2 a^{(k_2)}(z+c_2)+\cdots+m_n a^{(k_n)}(z+c_n), \\ b_1(z) = & m_1 b^{(k_1)}(z+c_1)+m_2 b^{(k_2)}(z+c_2)+\cdots+m_n b^{(k_n)}(z+c_n). \end{aligned} \tag{3.13}$$

By Lemma 2.4, we obtain

$$m\left(r,\frac{F-a_1}{f-a}\right)=S(r,f),\ m\left(r,\frac{F-b_1}{f-b}\right)=S(r,f).$$

Set

$$W(F,a_1,b_1)=\left|\begin{array}{ccc} F & a_1 & b_1 \\ F' & a_1' & b_1' \\ F'' & a_1'' & b_1'' \end{array}\right|.$$

By Lemma 2.4, we have

$$m\left(r,\frac{W(F,a_1,b_1)}{f-a_1}\right)=S(r,f),\ m\left(r,\frac{W(F,a_1,b_1)}{f-b_1}\right)=S(r,f). \tag{3.14}$$

If $W(F,a_1,b_1)\equiv 0$, then $b_1\equiv ka_1$, where k is a nonzero constant. Obviously, $W(F,a_1)\not\equiv 0$, where

$$W(F,a_1)=\left|\begin{array}{cc} F & a_1 \\ F' & a_1' \end{array}\right|.$$

Then by Lemma 2.4, we have

$$m\left(r,\frac{W(F,a_1)}{f-a_1}\right)=S(r,f),\ m\left(r,\frac{W(F,a_1)}{f-ka_1}\right)=S(r,f). \tag{3.15}$$

By (3.3), (3.15), Lemmas 2.3, 2.4 and 2.6, we obtain

$$
\begin{aligned}
& m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right)\\
\le & m\left(r,\frac{F-a_1}{f-a}\right)+m\left(r,\frac{F-b_1}{f-b}\right)+m\left(r,\frac{1}{F-a_1}\right)+m\left(r,\frac{1}{F-b_1}\right)+S(r,f)\\
\le & m\left(r,\frac{1}{F-a_1}+\frac{1}{F-ka_1}\right)+S(r,f)\\
\le & m\left(r,\frac{W(F,a_1)}{F-a_1}+\frac{W(F,a_1)}{F-ka_1}\right)+m\left(r,\frac{1}{W(F,a_1)}\right)+S(r,f)\\
\le & T\left(r,W(F,a_1)\right)+S(r,f)\\
\le & T(r,c_1F'+c_2F)+S(r,f)\le T(r,F)+\overline{N}(r,F)+S(r,f)\\
\le & T(r,F)+\frac{s}{8(lk+l+2s-1)+1}T(r,f)+S(r,f),
\end{aligned}
\tag{3.16}
$$

where c_1 and c_2 are small functions of f.

If $W(F,a_1,b_1)\not\equiv 0$, then by (3.14), Lemmas 2.4 and 2.6, we have

$$
\begin{aligned}
& m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right)\\
\le & m\left(r,\frac{F-a_1}{f-a}\right)+m\left(r,\frac{F-b_1}{f-b}\right)+m\left(r,\frac{1}{F-a_1}\right)+m\left(r,\frac{1}{F-b_1}\right)+S(r,f)\\
\le & m\left(r,\frac{1}{F-a_1}+\frac{1}{F-b_1}\right)+S(r,f)\\
\le & m\left(r,\frac{W(F,a_1,b_1)}{F-a_1}+\frac{W(F,a_1,b_1)}{F-b_1}\right)+m\left(r,\frac{1}{W(F,a_1,b_1)}\right)+S(r,f)\\
\le & T\left(r,W(F,a_1,b_1)\right)+S(r,f)\\
\le & T(r,d_1F''+d_2F'+d_3F)+S(r,f)\le T(r,F)+2\overline{N}(r,F)+S(r,f)\\
\le & T(r,F)+\frac{2s}{8(lk+l+2s-1)+1}T(r,f)+S(r,f),
\end{aligned}
\tag{3.17}
$$

where d_1, d_2 and d_3 are small functions of f.

It follows from (3.16) and (3.17), we deduce that

$$
\begin{aligned}
& m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right)\\
\le & T(r,F)+\frac{2s}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).
\end{aligned}
\tag{3.18}
$$

By (3.12) and (3.18), we have

$$
\begin{aligned}
2T(r,f)\le & \frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+T(r,F)\\
& +\frac{2s}{8(lk+l+2s-1)+1}T(r,f)+S(r,f),
\end{aligned}
$$

that is

$$\frac{7lk+7l+14s-6}{8(lk+l+2s-1)+1}T(r,f)\le T(r,F)+S(r,f). \tag{3.19}$$

Taking $\lambda=\frac{1}{8(lk+l+2s-1)+1}$. Then by (1), (2) of Lemma 2.10, (3.11), (3.19) and $N(r,f)\le\frac{1}{8(lk+l+2s-1)+1}T(r,f)$, we get

$$\begin{aligned}\overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{N(r,0)+N(r,1)}{T(r,g)+T(r,G)}&=\overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{N(r,0)+N(r,1)}{T(r,f)+T(r,F)+S(r,f)}\\ &\le\overline{\lim_{\substack{r\to\infty\\ r\in I}}}\frac{\frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+S(r,f)}{T(r,f)+\frac{7lk+7l+14s-6}{8(lk+l+2s-1)+1}T(r,f)+S(r,f)}\\ &\le\frac{9lk+9l+16s-8}{15lk+15l+30s-13}<\frac{16lk+16l+32s-22}{24(lk+l+2s-1)+3}=\frac{2-8\lambda}{3}.\end{aligned} \tag{3.20}$$

Hence, by Lemma 2.8, we have

$$\frac{1}{G-1}-\frac{c}{g-1}=d, \tag{3.21}$$

where $c(\neq 0), d$ are two constants. Now we consider two cases.

Case 1. $d=0$. Hence

$$G=\frac{g-1}{c}+1. \tag{3.22}$$

Next, we consider three subcases.

Case 1.1. $N(r,0)\neq S(r,f)$.

Thus there exists z_0 such that $g(z_0)=G(z_0)=0$. It follows from (3.22) that $g(z)\equiv G(z)$.

Case 1.2. $N(r,0)=S(r,f)$, $N(r,1)\neq S(r,f)$.

Obviously,

$$N\left(r,\frac{1}{G}\right)=N\left(r,\frac{1}{g-1+c}\right). \tag{3.23}$$

Suppose that $c\neq 1$. Then by (3.3), (3.5) and (3.23), we obtain

$$\begin{aligned}T(r,f)=&T(r,g)+S(r,f)\\ \le&N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-1+c}\right)+N(r,g)+S(r,f)\\ \le&N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{G}\right)+N(r,f)+S(r,f)\\ \le&\frac{1}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).\end{aligned}$$

It follows $T(r,f)\le S(r,f)$, a contradiction. So $c=1$, that is $g(z)\equiv G(z)$.

Case 1.3. $N(r,0)=S(r,f)$, $N(r,1)=S(r,f)$.

By (3.3), (3.5) and Lemma 2.1, we have

$$\begin{aligned}T(r,f) =&T(r,g)+S(r,f)\\ \leq& N\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g}\right)+N(r,g)+S(r,f)\\ \leq& N(r,1)+N(r,0)+N(r,f)+S(r,f)\\ \leq& \frac{1}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).\end{aligned}$$

It follows $T(r,f)\leq S(r,f)$, a contradiction.

Case 2. $d\neq 0$. In the following, we consider two subcases.

Case 2.1. $\frac{c}{d}\neq 1,0$.

By (3.3), (3.5), (3.6), (3.11) and Lemma 2.2, we have

$$\begin{aligned}&2T(r,f)=2T(r,g)+S(r,f)\\ \leq& N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g-\left(1-\frac{c}{d}\right)}\right)+N(r,g)+S(r,f)\\ \leq& N(r,0)+N(r,1)+N(r,g)+N(r,G)+S(r,f)\\ \leq& \frac{9lk+9l+16s-8}{8(lk+l+2s-1)+1}T(r,f)+(s+1)N(r,f)+sk\overline{N}(r,f)+S(r,f)\\ \leq& \frac{9lk+9l+17s+sk-7}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).\end{aligned}$$

It follows $T(r,f)\leq S(r,f)$, a contradiction.

Case 2.2. $\frac{c}{d}=1$. Hence $c=d(d\neq 0)$,

$$\frac{1}{G(z)-1}=\frac{dg(z)}{g(z)-1}. \tag{3.24}$$

Obviously $N(r,0)=S(r,f)$. Otherwise, there exists z_0 such that $g(z_0)=G(z_0)=0$. Thus by (3.24) $G(z_0)=\infty$, a contradiction. If $d\neq -1$, then we have

$$\begin{aligned}&T(r,f)=T(r,g)+S(r,f)\\ \leq& N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-\frac{1}{d+1}}\right)+N(r,g)+S(r,f)\leq N(r,f)+S(r,f)\\ \leq& \frac{1}{8(lk+l+2s-1)+1}T(r,f)+S(r,f).\end{aligned}$$

It follows $T(r,f)\leq S(r,f)$, a contradiction.

If $d=-1$, then by (3.24), we obtain $g(z)G(z)\equiv 1$. Thus, we have

$$(f-a)^2=\frac{(b-a)^2(f-a)}{F-a}. \tag{3.25}$$

By Nevanlinna's first fundamental theorem and (3.25), we have

$$\begin{aligned}2T(r,f) \le& T\left(r,(f-a)^2\right)+S(r,f)\\ =&T\left(r,\frac{(b-a)^2(f-a)}{F-a}\right)+S(r,f)\\ \le&T\left(r,\frac{F-a}{f-a}\right)+S(r,f)\\ =&N\left(r,\frac{F-a}{f-a}\right)+m\left(\frac{F-a}{f-a}\right)+S(r,f)\\ \le&m\left(r,\frac{F-a_1}{f-a}\right)+m\left(r,\frac{a_1-a}{f-a}\right)+N(r,F)+S(r,f)\\ \le&m\left(r,\frac{1}{f-a}\right)+N(r,F)+S(r,f)\\ \le&T(r,f)+N(r,F)+S(r,f).\end{aligned}$$

It follows that

$$\begin{aligned}T(r,f) \le& N(r,F)+S(r,f)\\ \le& s\left(N(r,f)+k\overline{N}(r,f)\right)+S(r,f)\\ \le&\frac{s+sk}{8(lk+l+2s-1)+1}T(r,f)+S(r,f),\end{aligned}$$

that is $T(r,f)\le S(r,f)$, a contradiction.

Combining Case 1 with Case 2, we deduce that $g(z)\equiv G(z)$. It follows that $f(z)\equiv F(z)$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Set

$$g(z)=\frac{f(z)-a(z)}{b(z)-a(z)},\ G(z)=\frac{F(z)-a(z)}{b(z)-a(z)}.$$

By $f(z)$ and $F(z)$ share $a(z), b(z), \infty$ CM, we know that $g(z)$ and $G(z)$ share 0, 1, ∞ CM almost.

We prove Theorem 1.2 by contradiction, suppose that $f(z)\not\equiv F(z)$, that is $g(z)\not\equiv G(z)$. Let

$$\phi=\frac{G(g-1)}{g(G-1)}. \tag{3.26}$$

Obviously, we know that $\phi(z)\not\equiv 0,\infty$, and

$$N(r,\phi)+N\left(r,\frac{1}{\phi}\right)=S(r,g)+S(r,G)=S(r,f).$$

By (3.26), we have

$$g-G=(\phi-1)g(G-1). \tag{3.27}$$

Let z_0 be a common pole of both $g(z)$ and $G(z)$ with multiplicity $m \geq 2$. Since $g(z)$ and $G(z)$ share ∞ CM almost, then by (3.27), we know that z_0 is the zero of $\phi(z) - 1$ with multiplicity at least m.

Next, we consider two cases.

Case 1. $\phi'(z) \not\equiv 0$, by (3.27), Lemma 2.7 and $N_{1)}(r, f) \leq \frac{1}{5(lk+l+2s-1)} T(r, f)$, we obtain

$$\begin{aligned} N(r,f) = N(r,g) &= N_{1)}(r,g) + N_{(2}(r,g) \\ &\leq \frac{1}{5(lk+l+2s-1)} T(r,f) + 2N\left(r, \frac{1}{\phi'}\right) + S(r,g) \\ &\leq \frac{1}{5(lk+l+2s-1)} T(r,f) + 2N\left(r, \frac{1}{\phi}\right) + 2\overline{N}(r,\phi) + S(r,g) \\ &\leq \frac{1}{5(lk+l+2s-1)} T(r,f) + S(r,g). \end{aligned}$$

Thus, we have

$$N(r,f) \leq \frac{1}{5(lk+l+2s-1)} T(r,f) + S(r,f). \tag{3.28}$$

Case 2. $\phi'(z) \equiv 0$, that is $\phi(z) \equiv c$. If $c = 1$, then by (3.26), we get $g(z) \equiv G(z)$, a contradiction. If $c \neq 1$, then by $\frac{G(g-1)}{g(G-1)} \equiv c$, we know that (3.28) is valid also.

By means of (3), (4) of Lemma 2.10 and Lemma 2.9, it is easy to prove Theorem 1.2 by imitating the proof of Theorem 1.1 and replacing (3.11), (3.19) and (3.20) respectively with the following three formulas:

$$\begin{aligned} N(r,0) + N(r,1) &\leq m(r,f) + l\left(N(r,f) + k\overline{N}(r,f)\right) + S(r,f) \\ &\leq T(r,f) + (l-1+lk)N(r,f) + S(r,f) \\ &\leq \frac{6lk+6l+10s-6}{5(lk+l+2s-1)} T(r,f) + S(r,f). \end{aligned}$$

$$\frac{4lk+4l+8s-4}{5(lk+l+2s-1)} T(r,f) \leq T(r,F) + S(r,f).$$

$$\begin{aligned} \overline{\lim_{\substack{r\to\infty \\ r\in I}}} \frac{N(r,0)+N(r,1)}{T(r,g)+T(r,G)} &= \overline{\lim_{\substack{r\to\infty \\ r\in I}}} \frac{N(r,0)+N(r,1)}{T(r,f)+T(r,F)+S(r,f)} \\ &\leq \overline{\lim_{\substack{r\to\infty \\ r\in I}}} \frac{\frac{6lk+6l+10s-6}{5(lk+l+2s-1)} T(r,f) + S(r,f)}{T(r,f) + \frac{4lk+4l+8s-4}{5(lk+l+2s-1)} T(r,f) + S(r,f)} \\ &\leq \frac{6lk+6l+10s-6}{9lk+19l+18s-9} < \frac{10lk+10l+20s-12}{15lk+15l+30s-15} = \frac{2-2\lambda}{3}. \end{aligned}$$

Proof of Theorem 1.3. Set

$$g(z) = \frac{f^m - a(z)}{b(z) - a(z)},$$

$$G(z)=\frac{H-a(z)}{b(z)-a(z)}.$$

By f^m and H share $a(z), b(z), \infty$ CM, we know that $g(z)$ and $G(z)$ share 0, 1, ∞ CM almost. Next, we consider two cases.

Case 1. $a(z)\equiv 0$, $b(z)\not\equiv 0$. In the following, we consider two subcases.

Case 1.1. $N(r,\frac{1}{G})\neq S(r,G)$. From the conditions of Theorem 1.3, we have

$$N(r,0)-\overline{N}(r,0)\neq S(r,G)+S(r,g). \tag{3.29}$$

Set $\psi(z)=\frac{G'(z)}{1-G(z)}-\frac{g'(z)}{1-g(z)}$. If $\psi(z)\not\equiv 0$, then by Nevanlinna's first fundamental theorem, we get

$$\begin{aligned}N(r,0)-\overline{N}(r,0)\leq &N(r,\frac{1}{\psi})\leq T(r,\psi)+O(1)\\ &=m(r,\psi)+N(r,\psi)+O(1)=S(r,G)+S(r,g),\end{aligned}$$

which contradicts with(3.29). Hence $\psi(z)\equiv 0$, we get $G(z)-1=c(g(z)-1)$. By (3.29), we know that there exists z_0 satisfying $g(z_0)=G(z_0)=0$. Hence $c=1$, that is $g(z)\equiv G(z)$. It follows $f^m(z)\equiv F(z)$.

Similarly, $N(r,G)=S(r,G)$ and $N(r,g)=S(r,g)$.

Case 1.2. $N(r,\frac{1}{G})=S(r,G)$.

Obviously, $N(r,\frac{1}{g})=S(r,g)$, by Lemma 2.5, $N(r,\frac{1}{G})+N(r,G)=S(r,G)$, $N(r,\frac{1}{g})+N(r,g)=S(r,g)$. It follows from $g(z)$ and $G(z)$ share 1 CM almost, that $g(z)G(z)\equiv 1$, we have $f^mF\equiv b^2$, that is $\frac{F}{f^m}=\frac{b^2}{f^{2m}}$. Hence, we get

$$m\left(r,\frac{F}{f^m}\right)=m\left(r,\frac{b^2}{f^{2m}}\right)=2mT(r,f)+S(r,f), \tag{3.30}$$

it follows from $m\left(r,\frac{F}{f^m}\right)\leq S(r,f)$ and (3.30) that $T(r,f)=S(r,f)$, a contradiction.

Case 2. $a(z)\not\equiv 0$.

In the following, we consider two subcases.

Case 2.1. $a(z)\not\equiv 0, b(z)\not\equiv 0$.

Let $f^m(z)\not\equiv F(z)$, by Lemma 2.2, we have

$$\begin{aligned}2T(r,f^m)\leq&\overline{N}\left(r,\frac{1}{f^m}\right)+\overline{N}(r,f^m)+\overline{N}\left(r,\frac{1}{f^m-a}\right)+\overline{N}\left(r,\frac{1}{f^m-b}\right)\\&+\varepsilon T(r,f^m)+S(r,f^m)\\ \leq&\frac{1}{m}N\left(r,\frac{1}{f^m}\right)+N\left(r,\frac{1}{f^m-F}\right)+\varepsilon T(r,f^m)+S(r,f^m)\\ \leq&\frac{1}{m}T(r,f^m)+T(r,f^m-F)+\varepsilon T(r,f^m)+S(r,f^m)\\ \leq&\frac{1}{m}T(r,f^m)+T(r,f^m)+\varepsilon T(r,f^m)+S(r,f^m)\\ \leq&\left(\frac{1}{m}+1+\varepsilon\right)T(r,f^m)+S(r,f^m).\end{aligned}$$

Let $\varepsilon = \frac{1}{4} < \frac{1}{2}$, and $m \geq 2$. It follows that $T(r, f^m) \leq S(r, f^m)$, a contradiction.

Case 2.2. $a(z) \not\equiv 0, b(z) \equiv 0$.

By using the same argument as used in Case 1, we obtain a contradiction. So $f^m(z) \equiv H\left(f(z)\right)$. This completes the proof of Theorem 1.3.

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Поступила 22 декабря 2022

После доработки 09 мая 2023

Принята к публикации 23 мая 2023