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UNIQUENESS OF L-FUNCTIONS AND GENERAL MEROMORPHIC FUNCTIONS IN LIGHT OF TWO SHARED SETS

R. SAHA, S. MALLICK

Cooch Behar Panchanan Barma University, West Bengal, India[1]
E-mails: *ripan.tfg.saha@gmail.com; sanjay.mallick1986@gmail.com; smallick.ku@gmail.com*

Abstract. In this paper, we have dealt with the uniqueness problem of a general meromorphic function with $\mathcal{L}$ function in terms of two shared sets. In our main theorem, we deal with general meromorphic functions instead of meromorphic functions having finitely many poles. As a corollary of our main theorem, we have shown that our result not only fills the gap of some theorems of [3] and [1] for $m = n-1$ but also reduces the cardinality of the main range set and hence our result significantly improves all the results in this direction.

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1. Introduction

Let $\mathbb{N}$ be the set of natural numbers and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Throughout the paper by meromorphic functions we shall mean it is meromorphic in the complex plane and by L-functions we mean it is L-functions in the Selberg class which is defined [7, 8] to be a Dirichelet series

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{-s}} \tag{1.1}$$

satisfying the following axioms:

- (i) *Ramanujan hypothesis :* $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$;
- (ii) *Analytic continuation :* There is a non-negative integer m such that $(s-1)^m \mathcal{L}(s)$ is an entire function of finite order;
- (iii) *Functional equation:* $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega \overline{\Lambda_{\mathcal{L}}(1-\overline{s})}, \tag{1.2}$$

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where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \nu_j), \tag{1.3}$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $\text{Re}\nu_j \geq 0$ and $|\omega| = 1$;

- (iv) *Euler product hypothesis :* $\log \mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

This class includes many of the known entire Dirichlet series with Euler product, including the Riemann zeta function and the Dirichlet L-functions. Since an L-function can be analytically continued to a meromorphic function, the study of uniquely determining an L-function, gradually moved towards uniquely determining the L-functions with respect to the meromorphic functions having finitely many poles. A lot of research has already been pursued by various researchers [7, 8, 6, 9, 3] in this direction. Below we recall some of these results and the gradual development. But before that, we recall some basic definitions. For standard notations of Nevanlinna theory, we suggest our reader to follow [2].

Definition 1.1. [4, 5] *Let k be a non-negative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [4] *For $A \subset \overline{\mathbb{C}}$ we define $E_f(A, k) = \cup_{a \in A} E_k(a; f)$, where k is a non-negative integer or infinity. If $E_f(A, k) = E_g(A, k)$, then we say that f and g share the set A with weight k.*

We write f, g share (A, k) to mean that f, g share the set A with weight k. We say that f, g share a set A IM or CM if and only if f, g share $(A, 0)$ or (A, ∞) respectively.

2. Gradual development and motivation

In 2010, B. Q. Li proved the following theorem.

Theorem A. [6] *Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a non-constant L-function $\mathcal{L}$ share (a,∞) and $(b,0)$, then $\mathcal{L}=f$.*

In 2018, Yuan, Li and Yi [9] considered the uniqueness of L-functions with meromorphic functions having finitely many poles under set sharing and proved the following theorem.

Theorem B. [9] *Let $S=\{\omega_1,\omega_2,...,\omega_l\}$, where $\omega_1,\omega_2,...,\omega_l$ are all distinct zeros of the polynomial $P(\omega)=\omega^n+a\omega^m+b$. Here l is a positive integer satisfying $1\leq l\leq n$, n and m are relatively prime positive integers with $n\geq 5$ and $n>m$, and a,b,c are nonzero finite constants, where $c\neq\omega_j$ for $1\leq j\leq l$. Let f be a non-constant meromorphic function such that f has finitely many poles in the complex plane, and let $\mathcal{L}$ be a non-constant L-function. If f and $\mathcal{L}$ share S CM and c IM, then $\mathcal{L}=f$.*

In 2020, Kundu and Banerjee [3] considered the case $c=0$ of Theorem B and provided the following theorem.

Theorem C. [3] *Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles and S be as defined in* Theorem B. *Here a,b are two non-zero constants and n,m are relatively prime positive integers such that $n>2m$. If f and a non-constant L -function $\mathcal{L}$ share (S,∞) and $(0,0)$, then $\mathcal{L}=f$.*

Very recently, Banerjee and Kundu [1] proved the following result.

Theorem D. [1] *Let S be defined as in* Theorem B*, f be a meromorphic function having finitely many poles in $\mathbb{C}$ and let $\mathcal{L}$ be a non-constant L -function. Suppose $E_f(S,s)=E_{\mathcal{L}}(S,s)$ and for some finite $c\notin S, f$ and $\mathcal{L}$ share $(c,0)$. Also let $a_i(i=1,2,\dots,n-m)$ be the zeros of nz^k+ma, where $k=n-m(\geq 1)$ and denote $S'=\{a_1,a_2,\dots,a_{n-m}\}$.*

I. *Suppose $c=0$.*

When (i) $s\geq 2, n>2m+2$ or (ii) $s=1, n>2m+3$ or (iii) $s=0, n>2m+8$; then $f\equiv\mathcal{L}$.

II. *Suppose $c\neq 0$.*

(A) *Let $c\in S'$. When $l=n$ and (i) $s=1, n>2k+2$ or (ii) $s=0, n>2k+5$; or when $l=n-1$ and (i) $s\geq 2, n>2k+2$ or (ii) $s=1, n>2k+3$ or (iii) $s=0, n>2k+8$; then $f\equiv\mathcal{L}$.*

(B) *Next let $c\notin S'$. When (i) $s\geq 2, n>2k+4$ or (ii) $s=1, n>2k+5$ or (iii) $s=0, n>2k+10$; then $f\equiv\mathcal{L}$.*

From the above discussion, one would naturally observe that in *Theorem B-D* all the authors always considered the set S to be the zeros of the polynomial $P(z) = z^n + az^m + b$. Now if we take $m = n-1$, then the condition of *Theorem C* and condition **I** of *Theorem D* become absurd; i.e., for $m = n-1$ *Theorem C* and **I** of *Theorem D* is not applicable. Also one can notice that all the authors always considered a special class of meromorphic functions; i.e., they always considered meromorphic functions having finitely many poles. Therefore the uniqueness of L-functions with general meromorphic functions is yet to be dealt with. At this moment naturally, the following two questions come into mind.

Question 2.1. *For $m = n-1$, if a non-constant meromorphic function having finitely many poles and an L-function share (S,t) and $(c,0)$, are they equal?*

Question 2.2. *If a general non-constant meromorphic function and an L-function share (S,t) and $(c,0)$, then are they equal?*

In this paper, we have answered the above two questions affirmatively. Not only that by considering the polynomial $P(z) = z^n + az^{n-1} + b$, we have shown the uniqueness of a general non-constant meromorphic function with a non-constant L-function when they share the set (S,t) and $(\eta,0)$, where η is the zero of $P'(z)$. As a corollary of our main theorem, we have shown that our result not only fills the gap of *Theorem C* and I of *Theorem D* for $m = n-1$ but also significantly improves *Theorem B-C* and **I** of *Theorem D*.

3. Main result

Now we state the following theorem which is the main result of the paper.

Theorem 3.1. *Let $P(z) = z^n + az^{n-1} + b$, with $n \geq 3$ and a, b are non-zero constants such that the polynomial has no multiple zero. Suppose that f, $\mathcal{L}$ share (S,t) and $(\eta,0)$, where $t \in \mathbb{N} \cup \{0\}$, S be the set of zeros of $P(z)$, η be the zero of $P'(z)$, f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function.*

(I) *Suppose $\eta = 0$. If*
   (i) *$t \geq 5$, with*
   - $n > 2 + (2 + \frac{3}{n-2})(1 - \Theta(\infty; f))$;

   (ii) *$t = 4$, with*
   - $n > \max\{2 + (2 + \frac{3}{n-2})(1 - \Theta(\infty; f)), 3\}$;
   - $n = 3$ *and* $\Theta(\infty; f) > \frac{5}{6}$;

(iii) $t=3$, *with*
- $n > \max\{2+(2+\frac{3}{n-2})\,(1-\Theta(\infty;f))\,,\ 4\}$;
- $n=4$ *and* $\Theta(\infty;f) > \frac{11}{23}$;
- $n=3$ *and* $\Theta(\infty;f) > \frac{7}{8}$;

(iv) $t=2$, *with*
- $n > 2+\left(2+\frac{12}{3n-8}\right)(1-\Theta(\infty;f))$;

(v) $t=1$, *with*
- $n > \max\{2+\frac{5}{2}+(\frac{6}{n-3})(1-\Theta(\infty;f)),\ 4\}$;
- $n=4$ *and* $\Theta(\infty;f) > \frac{13}{17}$.

(vi) $t=0$, *with*
- $n > \max\{2+\left(4+\frac{14}{n-4}\right)(1-\Theta(\infty;f)),\ 4\}$;

then we get $f \equiv \mathcal{L}$.

(II) *Suppose* $\eta \neq 0$. *If*

(i) $t \geq 2$, *with*
- $n > \max\{4+2(1-\Theta(\infty;f)),\ 4\}$;

(ii) $t=1$, *with*
- $n > \max\{5+\frac{5}{2}(1-\Theta(\infty;f)),\ 4\}$;

(iii) $t=0$, *with*
- $n > \max\{8+4(1-\Theta(\infty;f)),\ 4\}$;

then we get $f \equiv \mathcal{L}$.

Corollary 3.1. *Let* $P(z) = z^n + az^{n-1} + b$, *with* $n \geq 3$ *and* a, b *are non-zero constants such that the polynomial has no multiple zero. Suppose that* f, $\mathcal{L}$ *share* (S,t) *and* $(\eta,0)$, *where* $t \in \mathbb{N}\cup\{0\}$, S *be the set of zeros of* $P(z)$, η *be the zero of* $P'(z)$, f *be a non-constant meromorphic function having finitely many poles and* $\mathcal{L}$ *be a non-constant* L*-function.*

(I) *Suppose* $\eta = 0$. *If (i)* $n \geq 3$ *when* $t \geq 2$, *(ii)* $n \geq 4$ *when* $t=1$, *(iii)* $n \geq 5$ *when* $t=0$; *then we get* $f \equiv \mathcal{L}$.

(II) *Suppose* $\eta \neq 0$. *If (i)* $n \geq 5$ *when* $t \geq 2$, *(ii)* $n \geq 6$ *when* $t=1$, *(iii)* $n \geq 9$ *when* $t=0$; *then we get* $f \equiv \mathcal{L}$.

Remark 3.1. *In Corollary 3.1 one can observe that for* $\eta = 0$ *the least cardinality of the set* S *is* $3,\ 4$ *and* 5 *when* $t \geq 2,\ t=1$ *and* $t=0$ *respectively, whereas in Theorem D it was* $5,\ 6$ *and* 11. *Again in Theorem C the cardinality* 3 *was achieved in the case of CM sharing but from Corollary 3.1 the same is achieved for weight* 2 *only. Also in Theorem 3.1 we deal with general meromorphic functions instead of meromorphic functions having finitely many poles. Therefore our result is not only improved but also an extended version of Theorem B-C and I of Theorem D.*

4. Lemmas

In this section, we discuss some lemmas which will play key role to prove our main result. For the convenience of the reader, let us shortly recall some definitions and notations which will be required to prove the lemmas.

Definition 4.1. *Let f be a meromorphic function. We denote the order of f by $\rho(f)$, where*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log(T(r,f))}{\log r}. \tag{4.1}$$

By $S(r,f)$ we mean any quantity satisfying $S(r,f) = O(\log(rT(r,f)))$ for all r possibly outside a set of finite linear measure. If f is a function of finite order, then $S(r,f) = O(\log r)$ for all r.

Definition 4.2. *Let f and g be two non-constant meromorphic functions such that f and g share $(a,0)$, where $a \in \overline{\mathbb{C}}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. Then*

- *$\overline{N}_d(r,a;f)$ denotes the reduced counting function of those a-points of f and g where $p > q$.*
- *$N_E^{1)}(r,a;f)$ denotes the counting function of those a-points of f and g where $p = q = 1$.*

  In the same way we can define $\overline{N}_d(r,a;g)$ and $N_E^{1)}(r,a;g)$.
- *$\overline{N}(r,a;f| = 1)$ denotes the reduced counting function of simple a-points of f.*
- *$\overline{N}_*(r,a;f,g)$ denotes the reduced counting function of those a-points of f and g where $p \neq q$. Clearly $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f) = \overline{N}_d(r,a;f) + \overline{N}_d(r,a;g)$.*
- *$\overline{N}(r,a;f\mid\geq m)$ denotes the reduced counting function of those a points of f whose multiplicities are not less than m.*
- *$N(r,a;f \mid \quad g \neq b_1,b_2,\ldots,b_q)$ denotes the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1,2,\ldots,q$; where $a,b_1,b_2,\ldots,b_q \in \overline{\mathbb{C}}$.*

Definition 4.3. *Let $f(z)$ be a non-constant meromorphic function in the complex plane and $a \in \overline{\mathbb{C}}$. Then*

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

Observe that $0 \leq \Theta(a,f) \leq 1$.

For two non-constant meromorphic functions F and G, set

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right), \tag{4.2}$$

and

$$\Phi=\frac{F'}{F-1}-\frac{G'}{G-1}. \tag{4.3}$$

Lemma 4.1. [10] *Let F, G share $(1,0)$ and $H\not\equiv 0$. Then*

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 4.2. *Let f be a non-constant meromorphic function and $\mathcal{L}$ be an non-constant L-function sharing a set S IM, where $|S|\geq 3$. Then $\rho(f)=\rho(\mathcal{L})=1$. Furthermore, $S(r,f)=O(\log r)=S(r,\mathcal{L})$.*

Proof. Proceeding in a similar method as done in the proof of Theorem 5, [9, see p. 6], we can obtain $\rho(f)=\rho(\mathcal{L})=1$. So we omit it.
Since $\rho(f)=\rho(\mathcal{L})=1$, so from the definition of $S(r,f)$ we get $S(r,f)=O(\log r)=S(r,\mathcal{L})$. □

Lemma 4.3. *Let f, g be two non-constant meromorphic functions sharing (1,t), where $t\in\mathbb{N}\cup\{0\}$. Then*

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f).$$

Proof. Since f and g share $(1,t)$, we observe that

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)=2\overline{N}(r,1;f).$$

Case-I : Suppose $t\geq 2$.
Let z_0 be a 1 point of f with multiplicity p and a 1 point of g of multiplicity q. Since f and g share $(1,t)$, therefore $p\leq t$ implies $p=q$.

Subcase - I : When $p\leq t$. If $p=1$, then z_0 is counted once in both $N_E^{1)}(r,1;f)$ and $N(r,1;f)$. On the other hand z_0 is not counted in $\overline{N}_*(r,1;f,g)$. Again if $p\neq 1$, then z_0 is counted p times (i.e., at least 2 times) in $N(r,1;f)$ and in this case z_0 is not counted in $N_E^{1)}(r,1;f)$ and $\overline{N}_*(r,1;f,g)$. Therefore z_0 is counted at least 2 times in $N_E^{1)}(r,1;f)+(1-t)\,\overline{N}_*(r,1;f,g)+N(r,1;f)$.

Subcase - II : When $p\geq(t{+}1)$. If $p=q$, then z_0 is counted p time (i.e., at least 3 times) in $N(r,1;f)$ and z_0 is not counted in $N_E^{1)}(r,1;f)$ and $\overline{N}_*(r,1;f,g)$. When $p\neq q$, then z_0 is counted $(1-t)$ times in $(1-t)\overline{N}_*(r,1;f,g)$ and counted p times (i.e., at least $t{+}1$ times) in $N(r,1;f)$ and z_0 is not counted in $N_E^{1)}(r,1;f)$; i.e., z_0 is counted at least $(1-t)+(t+1)=2$ times in $N_{E}^{1)}(r,1;f)+(1-t)\,\overline{N}_{*}(r,1;f,g)+N(r,1;f)$. Now since z_0 is counted two times in $\overline{N}(r,1;f)+\overline{N}(r,1;g)$. Therefore in any sub case we have

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_{E}^{1)}(r,1;f)+(1-t)\,\overline{N}_{*}(r,1;f,g)+N(r,1;f).$$

Case-II : Suppose $t=1$.
Then clearly

$$\overline{N}(r,1;f)\leq N(r,1;f|=1)+N(r,1;f)=N_{E}^{1)}(r,1;f)+N(r,1;f).$$

Therefore, for $t=1$

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_{E}^{1)}(r,1;f)+(1-t)\,\overline{N}_{*}(r,1;f,g)+N(r,1;f).$$

Case-III : Suppose that $t=0$. Let z_0 be a 1 point of f with multiplicity p and a 1 point of g of multiplicity q. If $p=q=1$, then z_0 is counted 2 times in both $2\overline{N}(r,1;f)$ and $N_{E}^{1)}(r,1;f)+(1-t)\,\overline{N}_{*}(r,1;f,g)+N(r,1;f)$, as $\overline{N}_{*}(r,1;f,g)$ does not count z_0. If $p=1,\ q\neq 1$, then also z_0 is counted 2 times in both $2\overline{N}(r,1;f)$ and $N_{E}^{1)}(r,1;f)+(1-t)\,\overline{N}_{*}(r,1;f,g)+N(r,1;f)$, as in this case $N_{E}^{1)}(r,1;f)$ does not count z_0. Finally if $p\neq 1$, then z_0 is counted at least 2 times in $(1-t)\,\overline{N}_{*}(r,1;f,g)+$ $N(r,1;f)$ and z_0 is not counted in $N_{E}^{1)}(r,1;f)$.
Therefore, for $t=0$,

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_{E}^{1)}(r,1;f)+(1-t)\,\overline{N}_{*}(r,1;f,g)+N(r,1;f).$$

And hence in any case,

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)\leq N_{E}^{1)}(r,1;f)+(1-t)\,\overline{N}_{*}(r,1;f,g)+N(r,1;f). \qquad \square$$

Lemma 4.4. *Let $P(z)=z^n+az^{n-1}+b$, with $n\geq 3$ and a, b are non-zero constants such that the polynomial has no multiple zero and S be the set of all zeros of $P(z)$. Define*

$$F=\frac{f^n+af^{n-1}}{-b}\ \text{and}\ G=\frac{\mathcal{L}+a\mathcal{L}^{n-1}}{-b} \tag{4.4}$$

and

$$P^{'}(z)=n\prod_{i=1}^{2}(z-\eta_i)^{q_i}, \tag{4.5}$$

where $\eta_1=0$, $\eta_2=\frac{a(n-1)}{n}$, $q_1=n-2$ and $q_2=1$.

Let f be a non-constant meromorphic functions and $\mathcal{L}$ be a non-constant L-function such that $f,\mathcal{L}$ share (S,t) and η_j IM and if $\Phi\not\equiv 0$ then

$$\overline{N}(r,\eta_j;f)\leq\frac{1}{q_j}[\overline{N}_{*}(r,1;F,G)+\overline{N}(r,\infty;f)]+O(\log r).$$

Proof. By the given condition clearly F and G share $(1,t)$. Also by (4.5) we have

$$F' = -\frac{n}{b}\prod_{i=1}^{2}(f-\eta_i)^{q_i}\, f' \quad \text{and} \quad G' = -\frac{n}{b}\prod_{i=1}^{2}(\mathcal{L}-\eta_i)^{q_i}\, \mathcal{L}'.$$

Thus we see that

$$\Phi = \frac{-n\prod_{i=1}^{2}(f-\eta_i)^{q_i}\, f'}{b(F-1)} - \frac{-n\prod_{i=1}^{2}(\mathcal{L}-\eta_i)^{q_i}\, \mathcal{L}'}{b(G-1)}. \tag{4.6}$$

Let z_0 be a zero of $f-\eta_j$ with multiplicity r and a zero of $\mathcal{L}-\eta_j$ with multiplicity v. Then that would be a zero of Φ of multiplicity $\mu = \min\{q_j r + r - 1, q_j v + v - 1\} \geq q_j$. So by a simple calculation we can write

$$\begin{aligned}
&\overline{N}\,(r,\eta_j;f) = \overline{N}\,(r,\eta_j;\mathcal{L}) \leq \frac{1}{\mu}N(r,0;\Phi) \leq \frac{1}{\mu}T(r,\Phi) \\
&\leq \frac{1}{\mu}[N(r,\Phi) + S(r,F) + S(r,G)] \\
&\leq \frac{1}{\mu}\left[\overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\right] + S(r,F) + S(r,G) \\
&\leq \frac{1}{q_j}\left[\overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\mathcal{L})\right] + S(r,f) + S(r,\mathcal{L}).
\end{aligned}$$

Now using Lemma (4.2) and the fact that $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$, we have

$$\overline{N}\,(r,\eta_j;f) = \overline{N}\,(r,\eta_j;\mathcal{L}) \leq \frac{1}{q_j}\left[\overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f)\right] + O(\log r). \qquad \square$$

Lemma 4.5. *Let $F^* - 1 = a_n\prod_{i=1}^{n}(f-w_i)$ and $G^* - 1 = a_n\prod_{i=1}^{n}(\mathcal{L}-w_i)$, where f be a non-constant meromorphic function, $\mathcal{L}$ be an non-constant L-function, $a_n, w_i \in \mathbb{C}-\{0\}$ for all $i \in \{1,2,\ldots,n\}$. Further suppose that F^* and G^* share $(1,t)$, where $t \in \mathbb{N}\cup\{0\}$ and $\eta_j \neq w_i$ for $i = 1,2,\cdots,n$. Then*

$$\overline{N}_d(r,1;F^*) \leq \frac{1}{t+1}\left[\overline{N}(r,\eta_j;f) + \overline{N}(r,\infty;f) - N_1(r,0;f')\right] + O(\log r),$$

where $N_1(r,0;f') = N(r,0;f'|f \neq 0,\eta_1,w_1,w_2,...,w_n)$. Similar expression also holds for $\overline{N}_d(r,1;G^*)$.

Proof. Since F^* and G^* share $(1,t)$, in view of *Lemma (4.2)* and $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$, we find by using first fundamental theorem that

$$\begin{aligned}
\overline{N}_d(r,1;F^*) &\leq \overline{N}(r,1;F^*| \geq t+2) \leq \frac{1}{t+1}\left[N(r,1;F^*) - \overline{N}(r,1;F^*)\right] \\
&\leq \frac{1}{t+1}\left[\sum_{i=1}^{n}\left(N(r,w_i;f) - \overline{N}(r,w_i;f)\right)\right] \\
&\leq \frac{1}{t+1}\left[N(r,0;f'|f-\eta_j \neq 0) - N_1(r,0;f')\right] \\
&\leq \frac{1}{t+1}\left[N(r,0;\frac{f'}{f-\eta_j}) - N_1(r,0;f')\right] \\
&\leq \frac{1}{t+1}\left[T(r,\frac{f'}{f-\eta_j}) - N_1(r,0;f')\right] + O(1) \\
&\leq \frac{1}{t+1}\left[N(r,\infty;\frac{f'}{f-\eta_j}) - N_1(r,0;f')\right] + S(r,f) \\
&\leq \frac{1}{t+1}\left[\overline{N}(r,\infty;f) + \overline{N}(r,\eta_j;f) - N_1(r,0;f')\right] + O(\log r).
\end{aligned}$$

This proves the lemma. □

Remark 4.1. *Let F and G be defined by* (4.4)*. If F, G share $(1,t)$ and f, $\mathcal{L}$ share η_j IM, then using* Lemma (4.4) *and* Lemma (4.5)*, in view of $\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$, we get*

$$\begin{aligned}
\overline{N}_*(r,1;F,G) &= \overline{N}_d(r,1;F) + \overline{N}_d(r,1;G) \\
&\leq \frac{1}{t+1}\left[\overline{N}\left(r,\eta_j;f\right) + \overline{N}(r,\infty;f) + \overline{N}\left(r,\eta_j;\mathcal{L}\right)\right] + O(\log r) \\
&\leq \frac{2}{t+1}\overline{N}\left(r,\eta_j;f\right) + \frac{1}{t+1}\overline{N}(r,\infty;f) + O(\log r) \qquad (4.7) \\
&\leq \frac{2}{q_j(t+1)}\left[\overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f)\right] + \frac{1}{t+1}\overline{N}(r,\infty;f) + O(\log r).
\end{aligned}$$

This implies that

$$\begin{aligned}
\left(1 - \frac{2}{q_j(t+1)}\right)\overline{N}_*(r,1;F,G) &\leq \frac{(2+q_j)}{q_j(t+1)}\overline{N}(r,\infty;f) + O(\log r) \\
((t+1)q_j - 2)\,\overline{N}_*(r,1;F,G) &\leq (2+q_j)\overline{N}(r,\infty;f) + O(\log r). \qquad (4.8)
\end{aligned}$$

Lemma 4.6. *Let $P(z) = z^n + az^{n-1} + b$, with $n \geq 3$ and a, b are non-zero constants such that the polynomial has no multiple zero. Suppose that f, $\mathcal{L}$ share (S,t) and $(\eta,0)$, where $t \in \mathbb{N}\cup\{0\}$, η be the zero of $P'(z)$, f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function. Further suppose that*

(4.9)
$$\mathcal{F} = \frac{f^n + af^{n-1}}{-b} = -\frac{1}{b}f^{n-1}(f+a) \ \text{ and } \ \mathcal{G} = \frac{\mathcal{L}^n + a\mathcal{L}^{n-1}}{-b} = -\frac{1}{b}\mathcal{L}^{n-1}(\mathcal{L}+a).$$

When $\eta = 0$ and

(i) $t \geq 5$, *with*
- $n > 2 + (2 + \frac{3}{n-2})(1 - \Theta(\infty; f))$;

(ii) $t = 4$, *with*
- $n > \max\{2 + (2 + \frac{3}{n-2})(1 - \Theta(\infty; f)), 3\}$;
- $n = 3$ *and* $\Theta(\infty; f) > \frac{5}{6}$;

(iii) $t = 3$, *with*
- $n > \max\{2 + (2 + \frac{3}{n-2})\,(1 - \Theta(\infty; f))\,,\ 4\}$;
- $n = 4$ *and* $\Theta(\infty; f) > \frac{11}{23}$;
- $n = 3$ *and* $\Theta(\infty; f) > \frac{7}{8}$;

(iv) $t = 2$, *with*
- $n > 2 + \left(2 + \frac{12}{3n-8}\right)(1 - \Theta(\infty; f))$;

(v) $t = 1$, *with*
- $n > \max\{2 + \frac{5}{2} + (\frac{6}{n-3})(1 - \Theta(\infty; f)),\ 4\}$;
- $n = 4$ *and* $\Theta(\infty; f) > \frac{13}{17}$.

(vi) $t = 0$, *with*
- $n > \max\{2 + \left(4 + \frac{14}{n-4}\right)(1 - \Theta(\infty; f)),\ 4\}$;

or, $\eta \neq 0$ *and*

(i) $t \geq 2$, *with*
- $n > 4 + 2(1 - \Theta(\infty; f))$;

(ii) $t = 1$, *with*
- $n > 5 + \frac{5}{2}(1 - \Theta(\infty; f))$;

(iii) $t = 0$, *with*
- $n > 8 + 4(1 - \Theta(\infty; f))$;

we get $\frac{1}{\mathcal{F}-1} = \frac{A}{\mathcal{G}-1} + B$, *where* $A(\neq 0),\ B \in \mathbb{C}$.

Proof. According to the assumptions of the lemma, we clearly have $\mathcal{F}$, $\mathcal{G}$ share $(1, t)$ and f, $\mathcal{L}$ share $(\eta, 0)$. Here

$$\mathcal{F}' = -\frac{1}{b} f^{n-2}(nf + a(n-1))f' = -\frac{n}{b} f^{n-2}\left(f - \frac{a(1-n)}{n}\right) f'$$

and

$$\mathcal{G}' = -\frac{1}{b} \mathcal{L}^{n-2}(n\mathcal{L} + a(n-1))\mathcal{L}' = -\frac{n}{b} \mathcal{L}^{n-2}\left(\mathcal{L} - \frac{a(1-n)}{n}\right) \mathcal{L}'.$$

Now consider H as given by (4.2) for $\mathcal{F}$ and $\mathcal{G}$. Firstly we suppose that $H \not\equiv 0$. Now we distinguish the following cases.

Case 1. $\Phi \equiv 0$.

Then by integrating we get, $(F - 1) = A(G - 1)$, where $A(\neq 0) \in \mathbb{C}$. Therefore, $F' = AG'$ and $F'' = AG''$. Which implies that $H \equiv 0$. Which is a contradiction.

Case 2. $\Phi \not\equiv 0$.

First let us assume that $\eta=\eta_1=0$. Then clearly $q_1=n-2$. Also let $\eta_2=\frac{a(1-n)}{n}$. Since $H\not\equiv 0$, it can be easily verified that H has only simple poles and these poles come from the following points.

(i) η_2 -points of f and $\mathcal{L}$.

(ii) η_1-points of f and $\mathcal{L}$ having different multiplicity.

(iii) Poles of f and $\mathcal{L}$.

(iv) 1 -points of $\mathcal{F}$ and $\mathcal{G}$ having different multiplicities.

(v) Those zeros of f' and $\mathcal{L}'$, which are not zeros of $\prod_{i=1}^{2}(f-\eta_i)(\mathcal{F}-1)$ and $\prod_{i=1}^{2}(\mathcal{L}-\eta_i)(\mathcal{G}-1)$ respectively. Therefore we obtain

$$
\begin{aligned}
N(r,H) &\leq \overline{N}\,(r,\eta_2;f)+\overline{N}\,(r,\eta_2;\mathcal{L})+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) \\
&+\overline{N}_*\,(r,\eta_1;f,\mathcal{L})+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}_0\,(r,0;f')+\overline{N}_0\,(r,0;\mathcal{L}')\,,
\end{aligned} \tag{4.10}
$$

where $\overline{N}_0\,(r,0;f')$ and $\overline{N}_0\,(r,0;\mathcal{L}')$ denotes the reduced counting functions of those zeros of f' and $\mathcal{L}'$, which are not zeros of $\prod_{i=1}^{2}(f-\eta_i)(\mathcal{F}-1)$ and $\prod_{i=1}^{2}(\mathcal{L}-\eta_i)(\mathcal{G}-1)$ respectively.

Using the second fundamental theorem, we get,

$$
\begin{aligned}
(n+1)T(r,f) &\leq \overline{N}(r,1;\mathcal{F})+\textstyle\sum_{i=1}^{2}\overline{N}\,(r,\eta_i;f)+ \\
&+\overline{N}(r,\infty;f)-\overline{N}_0\,(r,0;f')+S(r,f),
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
(n+1)T(r,\mathcal{L}) &\leq \overline{N}(r,1;\mathcal{G})+\textstyle\sum_{i=1}^{2}\overline{N}\,(r,\eta_i;\mathcal{L})+ \\
&+\overline{N}(r,\infty;\mathcal{L})-\overline{N}_0\,(r,0;\mathcal{L}')+S(r,\mathcal{L}).
\end{aligned} \tag{4.12}
$$

Now combining (4.11) and (4.12) with the help of Lemmas (4.1) – (4.4) and then using $\overline{N}(r,\infty;\mathcal{L})=O(\log r)$, and (4.10) we get,

$$
\begin{aligned}
&(n+1)\{T(r,f)+T(r,\mathcal{L})\}\leq \overline{N}(r,1;\mathcal{F})+\overline{N}(r,1;\mathcal{G})+\sum_{i=1}^{2}\left[\overline{N}\,(r,\eta_i;f)+\overline{N}\,(r,\eta_i;\mathcal{L})\right] \\
&+[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})]-\overline{N}_0\,(r,0;f')-\overline{N}_0\,(r,0;\mathcal{L}')+S(r,f)+S(r,\mathcal{L}) \\
\leq\quad & N_E^{1)}(r,1;\mathcal{F})+(1-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+N(r,1;F)+ \\
&\sum_{i=1}^{2}[\overline{N}(r,n_i;f)+\overline{N}(r,\eta_i;\mathcal{L})]+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')-\overline{N}_0(r,0;\mathcal{L})+\ O(\log r)
\end{aligned}
$$

$$
\begin{aligned}
(4.13)\leq \quad & \left[\overline{N}\left(r,\eta_2;f\right)+\overline{N}\left(r,\eta_2;\mathcal{L}\right)\right]+\overline{N}(r,\infty;f)+\overline{N}_*(r,\eta_1;f,\mathcal{L})\\
& +\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;\mathcal{L}')+(1-t)\\
& \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+N(r,1;\mathcal{F})+\sum_{i=1}^{2}[\overline{N}(r,n_i;f)+\overline{N}(r,\eta_i;\mathcal{L})]\\
& +\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')-\overline{N}_0(r,0;\mathcal{L}')+O(\log r)\\
\leq \quad & 2\left[\overline{N}\left(r,\eta_2;f\right)+\overline{N}\left(r,\eta_2;\mathcal{L}\right)\right]+2\overline{N}(r,\infty;f)+3\overline{N}(r,\eta_1;f)\\
& +N(r,1;\mathcal{F})+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)\\
& \leq 2\{T(r,f)+T(r,\mathcal{L})\}+2\overline{N}(r,\infty;f)+\frac{3}{n-2}[\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
& +\overline{N}(r,\infty;f)]+nT(r,f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)\\
& \leq 2\{T(r,f)+T(r,\mathcal{L})\}+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+\frac{3}{n-2}\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
& +nT(r,f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)\\
& \leq (2+n)T(r,f)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+2T(r,\mathcal{L})\\
& +(2+\frac{3}{n-2}-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& nT(r,\mathcal{L})\leq T(r,f)+T(r,\mathcal{L})+(2+\tfrac{3}{n-2})\overline{N}(r,\infty;f)+\\
& \qquad +(2+\tfrac{3}{n-2}-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\ O(\log r).
\end{aligned}
\tag{4.14}
$$

In a similar manner, we get

$$
\begin{aligned}
& nT(r,f)\leq T(r,f)+T(r,\mathcal{L})+(2+\tfrac{3}{n-2})\overline{N}(r,\infty;f)+\\
& \qquad +(2+\tfrac{3}{n-2}-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\ O(\log r).
\end{aligned}
\tag{4.15}
$$

Let $T(r)=\max\{T(r,f)\ ,\ T(r,\mathcal{L})\}$. Then from (4.14) and (4.15) we get,

$$
\begin{aligned}
nT(r)\ \leq\ & 2T(r)+(2+\tfrac{3}{n-2})\overline{N}(r,\infty;f)+\\
& +(2+\tfrac{3}{n-2}-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).
\end{aligned}
\tag{4.16}
$$

Again from (4.13) we get

$$
(n+1)\{T(r,f)+T(r,\mathcal{L})\}\leq\left[\overline{N}\left(r,\eta_2;f\right)+\overline{N}\left(r,\eta_2;\mathcal{L}\right)\right]+\overline{N}(r,\infty;f)
$$

$$
\begin{aligned}
&+\overline{N}_*(r,\eta_1;f,\mathcal{L})+\ \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+(1-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+N(r,1;\mathcal{F})\\
&+\sum_{i=1}^{2}[\overline{N}(r,n_i;f)+\overline{N}(r,\eta_i;\mathcal{L})]+\overline{N}(r,\infty;f)+O(\log r)\\
\leq\ & \left[\overline{N}\,(r,\eta_2;f)+\overline{N}\,(r,\eta_2;\mathcal{L})\right]+\overline{N}(r,\infty;f)+\overline{N}_d(r,\eta_1;f)\\
&+\overline{N}_d(r,\eta_1;\mathcal{L})+\ \overline{N}_*(r,1;\mathcal{F},\mathcal{G})+(1-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&+N(r,1;\mathcal{F})+[\overline{N}(r,n_2;f)+\overline{N}(r,\eta_2;\mathcal{L})]+\overline{N}(r,n_1;f)\\
&+\overline{N}(r,\eta_1;\mathcal{L})+\overline{N}(r,\infty;f)+O(\log r)\\
\leq\ & 2\{T(r,f)+T(r,\mathcal{L})\}+2\overline{N}(r,\infty;f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
&+nT(r,f)+N_2(r,\eta_1;f)+N_2(r,\eta_1;\mathcal{L})+O(\log r)\\
\leq\ & (n+3)T(r,f)+3T(r,\mathcal{L})+2\overline{N}(r,\infty;f)\\
&+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)
\end{aligned}
$$

$$
\begin{aligned}
nT(r,\mathcal{L})\leq 2T(r,f)+2T(r,\mathcal{L})+2\overline{N}(r,\infty;f)+&\\
+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).&
\end{aligned}
\tag{4.17}
$$

In a similar manner, we get

$$
\begin{aligned}
nT(r,f)\leq 2T(r,f)+2T(r,\mathcal{L})+2\overline{N}(r,\infty;f)+&\\
+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).&
\end{aligned}
\tag{4.18}
$$

Then combining (4.17) and (4.18) we get

$$
nT(r)\ \leq\ 4T(r)+2\overline{N}(r,\infty;f)+(2-t)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).
\tag{4.19}
$$

Subcase 2.1. When $t\geq 5$ or $t=4$ with $n\geq 4$, we get from (4.16) that

$$
nT(r)\ \leq\ 2T(r)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+O(\log r);
$$

i.e.,

$$
nT(r)\ \leq\ (2+(2+\frac{3}{n-2})(1-\Theta(\infty;f))+\epsilon)T(r)+O(\log r),
$$

which is a contradiction for $n>2+(2+\frac{3}{n-2})(1-\Theta(\infty;f))$.

When $t=4$ and $n=3$, we get from (4.16) that

$$
3T(r)\ \leq\ 2T(r)+5\overline{N}(r,\infty;f)+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).
\tag{4.20}
$$

Now from (4.8) we get, $\overline{N}_*(r,1;F,G)\leq\overline{N}(r,\infty;f)+O(\log r)$. Hence (4.20) gives $T(r)\ \leq\ 6(1-\Theta(\infty;f)+\epsilon)T(r)+O(\log r)$, which is a contradiction as $\Theta(\infty;f)>\frac{5}{6}$.

Subcase 2.2. When $t=3$ and $n\geq 5$, we get from (4.16) that

$$
nT(r)\ \leq\ 2T(r)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+O(\log r);
$$

i.e.,

$$nT(r) \ \leq \ (2+(2+\frac{3}{n-2})\,(1-\Theta(\infty;f)+\epsilon))T(r)+O(\log r),$$

which is a contradiction for $n>2+(2+\frac{3}{n-2})(1-\Theta(\infty;f))$.

When $n=4$, we get from (4.16) that

$$4T(r) \ \leq \ 2T(r)+\frac{7}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r). \tag{4.21}$$

Now using (4.8) in (4.21) we get,

$$4T(r) \ \leq \ 2T(r)+\left(\frac{7}{2}+\frac{1}{3}\right)\overline{N}(r,\infty;f)+O(\log r);$$

i.e.,

$$4(r) \ \leq \ 2T(r)+\frac{23}{6}(1-\Theta(\infty;f)+\epsilon)T(r)+O(\log r),$$

which is a contradiction for $\Theta(\infty;f)>\frac{11}{23}$.
When $t=3$ and $n=3$ from (4.16) we get

$$3T(r) \ \leq \ 2T(r)+5\overline{N}(r,\infty;f)+2\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r).$$

Now using (4.8) we get $3T(r)\leq 2T(r)+8\overline{N}(r,\infty;f)+O(\log r)$; i.e.,

$$3T(r)\leq 2T(r)+8(1-\Theta(\infty;f)+\epsilon)T(r)+O(\log r);$$

which is a contradiction for $\Theta(\infty;f)>\frac{7}{8}$.

Subcase 2.3. When $t=2$, from (4.8) we get

$$(3n-8)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leq n\overline{N}(r,\infty;f)+O(\log r).$$

Since $(3n-8)>0$, we get

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leq\frac{n}{3n-8}\overline{N}(r,\infty;f)+O(\log r). \tag{4.22}$$

Now from (4.16) using (4.22) we get

$$\begin{aligned} nT(r) &\leq 2T(r)+(2+\frac{3}{n-2})\overline{N}(r,\infty;f)+\frac{3}{n-2}\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+O(\log r)\\ &\leq \left(2+\left(2+\frac{12}{3n-8}\right)(1-\Theta(\infty;f)+\epsilon)\right)T(r)+O(\log r), \end{aligned}$$

which is a contradiction for $n>2+\left(2+\frac{12}{3n-8}\right)(1-\Theta(\infty;f)$.

Subcase 2.4. When $t=1$, $n\geq 5$ then $(2n-6)>0$.
Therefore from (4.8) we get

$$(2n-6)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leq n\overline{N}(r,\infty;f)+O(\log r). \tag{4.23}$$

Now from (4.16) using (4.23), we get

$$\begin{aligned} nT(r) &\leq 2T(r) + (2 + \frac{3}{n-2})\overline{N}(r, \infty; f) + (1 + \frac{3}{n-2})\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r), \\ &\leq 2T(r) + \left(2 + \frac{1}{2} + \frac{6}{n-3}\right)(1 - \Theta(\infty; f) + \epsilon)\, T(r) + O(\log r) \end{aligned}$$

which is a contradiction for $n > 2 + \frac{5}{2} + (\frac{6}{n-3})(1 - \Theta(\infty; f))$.

When $t = 1$ and $n = 4$, we get from (4.8) that

$$\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) \leq 2\overline{N}(r, \infty; f) + O(\log r). \tag{4.24}$$

Now from (4.16) using (4.24), we get

$$\begin{aligned} 4T(r) &\leq 2T(r) + \frac{7}{2}\overline{N}(r, \infty; f) + \frac{5}{2}\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r) \\ &\leq 2T(r) + \frac{17}{2}(1 - \Theta(\infty; f) + \epsilon)T(r) + O(\log r), \end{aligned}$$

which is a contradiction for $\Theta(\infty; f) > \frac{13}{17}$.

Subcase 2.5. When $t = 0$, $n \geq 5$ then $(n-4) > 0$.

Therefore from (4.8), we get

$$(n-4)\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) \leq n\overline{N}(r, \infty; f) + O(\log r). \tag{4.25}$$

Now from (4.16) using (4.25), we get

$$\begin{aligned} nT(r) &\leq 2T(r) + (2 + \frac{3}{n-2})\overline{N}(r, \infty; f) + (2 + \frac{3}{n-2})\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r) \\ &\leq 2T(r) + \left(4 + \frac{14}{n-4}\right)(1 - \Theta(\infty; f) + \epsilon)T(r) + O(\log r), \end{aligned}$$

which is a contradiction for $n > 2 + \left(4 + \frac{14}{n-4}\right)(1 - \Theta(\infty; f))$.

Next suppose that $\eta = \eta_2 (\neq 0)$. Hence $q_2 = 1$. Then proceeding similarly as we have done above for (4.16) and (4.19) we can easily get the following

$$nT(r) \leq 2T(r) + 5\overline{N}(r, \infty; f) + (5 - t)\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r) \tag{4.26}$$

and

$$nT(r) \leq 4T(r) + 2\overline{N}(r, \infty; f) + (2 - t)\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r). \tag{4.27}$$

Subcase 2.6. When $t \geq 2$, we get from (4.27) that

$$nT(r) \leq 4T(r) + 2\overline{N}(r, \infty; f) + O(\log r);$$

i.e.,

$$nT(r) \leq (4 + 2(1 - \Theta(\infty; f)) + \epsilon)T(r) + O(\log r),$$

which is a contradiction for $n > 4 + 2(1 - \Theta(\infty; f))$.

Subcase 2.7. When $t = 1$, then from (4.7) we get

$$\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) \leq \overline{N}(r, \eta; f) + \frac{1}{2}\overline{N}(r, \infty; f) + O(\log r). \tag{4.28}$$

Now from (4.27) using (4.28), we get

$$\begin{aligned} nT(r) &\leq 4T(r) + 2\overline{N}(r, \infty; f) + \overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r) \\ &\leq 5T(r) + \frac{5}{2}\left(1 - \Theta(\infty; f) + \epsilon\right) T(r) + O(\log r), \end{aligned}$$

which is a contradiction for $n > 5 + \frac{5}{2}(1 - \Theta(\infty; f))$.

Subcase 2.8. When $t = 0$, then from (4.7) we get

$$\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) \leq 2\overline{N}(r, \eta; f) + \overline{N}(r, \infty; f) + O(\log r). \tag{4.29}$$

Now from (4.27) using (4.29), we get

$$\begin{aligned} nT(r) &\leq 4T(r) + 2\overline{N}(r, \infty; f) + 2\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + O(\log r) \\ &\leq 8T(r) + 4(1 - \Theta(\infty; f) + \epsilon)T(r) + O(\log r), \end{aligned}$$

which is a contradiction for $n > 8 + 4(1 - \Theta(\infty; f))$.
Thus we see from above that $H \equiv 0$. Hence on integration, we obtain

$$\frac{1}{\mathcal{F} - 1} = \frac{A}{\mathcal{G} - 1} + B,$$

where $A(\neq 0), B \in \mathbb{C}$. □

Lemma 4.7. *Let $\mathcal{F}$ and $\mathcal{G}$ be defined by* (4.9), *then $\mathcal{F}\mathcal{G} \neq a$, where a is non-zero complex constant.*

Proof. On the contrary, suppose that $\mathcal{F}\mathcal{G} = \zeta \neq 0$. Then

$$f^{n-1}(f + a)\mathcal{L}^{n-1}(\mathcal{L} + a) = \zeta b^2 = \zeta_1(say) \neq 0. \tag{4.30}$$

Let $\alpha_1 = 0$ and $\alpha_2 = -a$. Then it is clear from (4.30) that each α_i-point of f is a pole of $\mathcal{L}$ and vice-versa.

Let z_0 be a α_2 point of $\mathcal{L}$ of multiplicity r, then it will be a pole of f of multiplicity ν, such that $r = \nu n$. Since $\nu \geq 1$, so $r \geq n$; i.e., $\frac{1}{r} \leq \frac{1}{n}$. Similar argument can be made for α_1 point of $\mathcal{L}$. Now using the second fundamental theorem in view of

$\overline{N}(r,\infty;\mathcal{L}) = O(\log r)$ we get

$$\begin{aligned} T(r,\mathcal{L}) &\leq \overline{N}(r,\alpha_1;\mathcal{L}) + \overline{N}(r,\alpha_2;\mathcal{L}) + \overline{N}(r,\infty;\mathcal{L}) + S(r,\mathcal{L}) \\ &\leq \frac{2}{n}T(r,\mathcal{L}) + O(\log r), \end{aligned}$$

which is a contradiction as $n \geq 3$. □

5. Proof Of the theorems

Proof of Theorem 3.1 Let f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function. Suppose $E_f(S,t) = E_{\mathcal{L}}(S,t)$ and $E_f(\eta,0) = E_{\mathcal{L}}(\eta,0)$ where $t \in \mathbb{N} \cup \{0\}$ and η is the zero of $P'(z)$. Consider $\mathcal{F}$ and $\mathcal{G}$ as defined by (4.9).

Therefore in view of the Lemma 4.6 we get

$$\frac{1}{\mathcal{F}-1} = \frac{A}{\mathcal{G}-1} + B, \tag{5.1}$$

where $A(\neq 0), B \in \mathbb{C}$. Hence we have

$$T(r,\mathcal{F}) = T(r,\mathcal{G}) + O(1). \tag{5.2}$$

Since

$$T(r,\mathcal{F}) = nT(r,f) + O(1) \;\; and \;\; T(r,\mathcal{G}) = nT(r,\mathcal{L}) + O(1). \tag{5.3}$$

So (5.2) implies that

$$T(r,f) = T(r,\mathcal{L}) + O(1). \tag{5.4}$$

Case 1. If $B \neq 0$. Then from (5.1) we get,

$$\mathcal{F} = \frac{(B+1)\mathcal{G} + (A-B-1)}{B\mathcal{G} + (A-B)}. \tag{5.5}$$

Subcase 1.1. If $B \neq -1$. Then from (5.5) we get,

$$\mathcal{F} = \frac{(B+1)\left(\mathcal{G} - \frac{B-A+1}{B+1}\right)}{B\left(\mathcal{G} - \frac{B-A}{B}\right)}. \tag{5.6}$$

Now clearly, $\frac{B-A+1}{B+1} \neq \frac{B-A}{B}$, as if $\frac{B-A+1}{B+1} = \frac{B-A}{B}$ then $A = 0$, which is absurd.

Subcase 1.1.1. If $B - A \neq 0$. Then from (5.6) it is clear that $\overline{N}\left(r, \frac{B-A}{B}; \mathcal{G}\right) = \overline{N}(r,\infty;\mathcal{F})$.

Now using second fundamental theorem, (5.4) and $\overline{N}(r,\infty;\mathcal{L})=O(\log r)$ we get,

$$\begin{aligned}
T(r,\mathcal{G}) &\leq \overline{N}(r,0;\mathcal{G})+\overline{N}\left(r,\frac{B-A}{B};\mathcal{G}\right)+\overline{N}(r,\infty;\mathcal{G})+S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{L})+\overline{N}(r,-a;\mathcal{L})+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})+S(r,\mathcal{G})\\
&\leq 2T(r,\mathcal{L})+(1-\Theta(\infty;f)+\epsilon)T(r,f)+S(r,\mathcal{G})\\
&\leq (3-\Theta(\infty;f)+\epsilon)T(r,\mathcal{L})+S(r,\mathcal{G})\\
&\leq \left(\frac{3-\Theta(\infty;f)+\epsilon}{n}\right)T(r,\mathcal{G})+S(r,\mathcal{G}),
\end{aligned}$$

which is a contradiction.

Subcase 1.1.2. If $B-A=0$. Then from (5.6) we get,

$$\mathcal{F}=\frac{(B+1)\left(\mathcal{G}-\frac{1}{B+1}\right)}{B\mathcal{G}}. \tag{5.7}$$

Now it is clear from (5.7) that $\overline{N}\left(r,\frac{1}{B+1};\mathcal{G}\right)=\overline{N}(r,0;\mathcal{F})$ and $\overline{N}(r,0;\mathcal{G})=\overline{N}(r,\infty;\mathcal{F})$.
Now using second fundamental theorem, (5.4) and $\overline{N}(r,\infty;\mathcal{L})=O(\log r)$ we get,

$$\begin{aligned}
T(r,\mathcal{G}) &\leq \overline{N}(r,0;\mathcal{G})+\overline{N}\left(r,\frac{1}{B+1};\mathcal{G}\right)+\overline{N}(r,\infty;\mathcal{G})+S(r,\mathcal{G})\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,\infty;\mathcal{L})+S(r,\mathcal{G})\\
&\leq 2T(r,f)+(1-\Theta(\infty;f)+\epsilon)T(r,f)+S(r,\mathcal{G})\\
&\leq (3-\Theta(\infty;f)+\epsilon)T(r,\mathcal{L})+S(r,\mathcal{G})\\
&\leq \left(\frac{3-\Theta(\infty;f)+\epsilon}{n}\right)T(r,\mathcal{G})+S(r,\mathcal{G}),
\end{aligned}$$

which is a contradiction.

Subcase 1.2. If $B=-1$. Then from (5.5) we get,

$$\mathcal{F}=\frac{A}{-\mathcal{G}+A+1}. \tag{5.8}$$

Subcase 1.2.1. If $A\neq -1$. Then from (5.8) it is clear that $\overline{N}\left(r,(A+1);\mathcal{G}\right)=\overline{N}(r,\infty;\mathcal{F})$.
Now using second fundamental theorem, (5.4) and $\overline{N}(r,\infty;\mathcal{L})=O(\log r)$ we get,

$$\begin{aligned}
T(r,\mathcal{G}) &\leq \overline{N}(r,0;\mathcal{G})+\overline{N}\left(r,(A+1);\mathcal{G}\right)+\overline{N}(r,\infty;\mathcal{G})+S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{L})+\overline{N}(r,-a;\mathcal{L})+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})+S(r,\mathcal{G})\\
&\leq (3-\Theta(\infty;f)+\epsilon)T(r,\mathcal{L})+S(r,\mathcal{G})\\
&\leq \left(\frac{3-\Theta(\infty;f)+\epsilon}{n}\right)T(r,\mathcal{G})+S(r,\mathcal{G}),
\end{aligned}$$

which is a contradiction.

Subcase 1.2.2. If $A = -1$. Then from (5.8) we get,

$$\mathcal{F}\mathcal{G} = 1,$$

which is a contradiction in view of *Lemma (4.7)*.

Case 2. If $B = 0$. Then from (5.1) we get,

$$\mathcal{G} - 1 = A(\mathcal{F} - 1). \tag{5.9}$$

Subcase 2.1. $A \neq 1$. Then from (5.9) we get,

$$A\mathcal{F} = \mathcal{G} - (1 - A). \tag{5.10}$$

Suppose η is not an e.v.P. of f and $\mathcal{L}$. Then there exists z_0 such that $f(z_0) = \mathcal{L}(z_0) = \eta$.
Let $\xi = -\frac{1}{b}\eta^{n-1}(\eta + a)$. Then clearly $F(z_0) = G(z_0) = \xi$ and since $P(z)$ has only simple zeros, we have $\xi \neq 1$.
Now from (5.10) we get,

$$(\xi - 1)(A - 1) = 0,$$

which is a contradiction.
Now let η be an e.v.P of $\mathcal{L}$ and hence it will be an e.v.P of f also.

Subcase 2.1.1. Suppose that $\eta = 0$. Then 0 is an e.v.P of f and $\mathcal{L}$.
Again, it is clear from (5.10) that $\overline{N}\left(r, (1-A); \mathcal{G}\right) = \overline{N}(r, 0; \mathcal{F})$.
Now using second fundamental theorem, (5.4) and $\overline{N}(r, \infty; \mathcal{L}) = O(\log r)$ we get,

$$\begin{aligned} T(r, \mathcal{G}) &\leq \overline{N}(r, 0; \mathcal{G}) + \overline{N}\left(r, (1-A); \mathcal{G}\right) + \overline{N}(r, \infty; \mathcal{G}) + S(r, \mathcal{G}) \\ &\leq \overline{N}(r, 0; \mathcal{L}) + \overline{N}(r, -a; \mathcal{L}) + \overline{N}(r, 0; f) + \overline{N}(r, -a; f) + \overline{N}(r, \infty; \mathcal{L}) + S(r, \mathcal{G}) \\ &\leq T(r, \mathcal{L}) + T(r, f) + S(r, \mathcal{G}) \leq \left(\frac{2}{n}\right) T(r, \mathcal{G}) + S(r, \mathcal{G}), \end{aligned}$$

which is a contradiction as $n \geq 3$.

Subcase 2.1.2. let $\eta \neq 0$, then using second fundamental theorem, (5.4) and $\overline{N}(r, \infty; \mathcal{L}) = O(\log r)$ we get,

$$\begin{aligned} T(r, \mathcal{G}) &\leq \overline{N}(r, 0; \mathcal{G}) + \overline{N}\left(r, (1-A); \mathcal{G}\right) + \overline{N}(r, \infty; \mathcal{G}) + S(r, \mathcal{G}) \\ &\leq \overline{N}(r, 0; \mathcal{L}) + \overline{N}(r, -a; \mathcal{L}) + \overline{N}(r, 0; f) + \overline{N}(r, -a; f) + \overline{N}(r, \infty; \mathcal{L}) + S(r, \mathcal{G}) \\ &\leq 2T(r, \mathcal{L}) + 2T(r, f) + S(r, \mathcal{G}) \leq \left(\frac{4}{n}\right) T(r, \mathcal{G}) + S(r, \mathcal{G}), \end{aligned}$$

which is a contradiction as $n \geq 5$.

Subcase 2.2. $A = 1$ and hence $\mathcal{F} = \mathcal{G}$. That is, we get

$$-\frac{1}{b}f^{n-1}(f+a) = -\frac{1}{b}\mathcal{L}^{n-1}(\mathcal{L}+a) \tag{5.11}$$

$$\implies (f^n - \mathcal{L}^n) + a(f^{n-1} - \mathcal{L}^{n-1}) = 0. \tag{5.12}$$

Let $h = \frac{f}{\mathcal{L}}$. Then from (5.11) we get,

$$\mathcal{L}(h^n - 1) + a(h^{n-1} - 1) = 0. \tag{5.13}$$

If $h \neq 1$, then we can write (5.13) as

$$\mathcal{L} = -a\frac{(h-v)(h-v^2)...(h-v^{n-2})}{(h-u)(h-u^2)...(h-u^{n-1})}, \tag{5.14}$$

where $u = \exp(2\pi i/n)$ and $v = \exp(2\pi i/(n-1))$. Noting that n and $(n-1)$ are relatively prime positive integers, then the numerator and denominator of (5.14) have no common factors. Since $\mathcal{L}$ can have atmost one pole in the complex plane, hence whenever $n \geq 3$ we can see that there exists at least one distinct roots of $h^n = 1$ such that they are Picard exceptional values of h.

Subcase 2.2.1. When $\eta = 0$; i.e., f and $\mathcal{L}$ share $(0, 0)$, then from (5.11) it is clear that f and $\mathcal{L}$ have same zeros and poles with counting multiplicity. Therefore, h is an entire function with no zeros; i.e., when $n \geq 3$ there are at least two Picard exceptional value of h, and so it follows by (5.14) that h and thus $\mathcal{L}$ are constants, which is impossible.

Therefore, we must have $h = 1$; i.e., $f = \mathcal{L}$.

Subcase 2.2.2. When $\eta \neq 0$, for $n \geq 5$ there are at least three Picard exceptional value of h, and so it follows by (5.14) that h and thus $\mathcal{L}$ are constants, which is impossible.

Therefore, we must have $h = 1$; i.e., $f = \mathcal{L}$.

Proof of Corollary 3.1 If f be a meromorphic function having finitely many poles, then we have

$$\Theta(\infty; f) = 1. \tag{5.15}$$

Therefore using (5.15), the desired results follow from the proofs of Theorem 3.1.

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