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MOVABILITY OF MORPHISMS IN AN ENRICHED PRO-CATEGORY AND IN A J-SHAPE CATEGORY

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Abstract. Various types of movability for abstract classical pro-morphisms or coherent mappings, and for abstract classical or strong shape morphisms was given by the same authors in some previous paper [10], [11], [12]. In the present paper we introduce and study the notions of (uniform) movability, and (uniform) co-movability for a new type of pro-morphisms and shape morphisms belonging to the so called *enriched pro-category* pro^J-$\mathcal{C}$ and to the corresponding shape category $Sh^J_{(\mathcal{C},\mathcal{D})}$, which were introduced by N. Uglešić [27].

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1. Introduction

The notion of movability for metric compacta was introduced by K. Borsuk [2] as an important shape invariant. The movable spaces are a generalization of spaces having the shape of ANR's. The movability assumption allows a series of important results in algebraic topology (like the Whitehead and Hurewicz theorems) to remain valid when the homotopy pro-groups are replaced by the corresponding shape groups. The term "movability"comes from the geometric interpretation of the definition in the compact case: if X is a compactum lying in a space $M \in AR$, one says that X is movable if for every neighborhood U of X in M there exists a neighborhood $V \subset U$ of X such that for every neighborhood $W \subset U$ of X there is a homotopy $H : V \times [0,1] \to U$ such that $H(x,0) = x$ and $H(x,1) \in W$ for every $x \in V$. One shows that the choice of $M \in AR$ is irrelevant [2]. After the notion of movability had been expressed in terms of ANR-systems for arbitrary topological spaces [16], [17], it became clear that one could define it in arbitrary pro-categories. The definition of a movable object in an arbitrary pro-category and that of uniform movability were both given by Maria Moszyńska [20]. Uniform movability is important in the study of mono- and epi-morphisms in pro-categories and in the study of the shape of pointed spaces. In the book of Sibe Mardešić and

Jack Segal [17] all these approaches and applications of various types of movability are discussed.

Besides the classic case of movability pro-objects and shape objects, some notions of movability for some morphisms appear in the papers of T. Yagasaki [28] and [29], Z.Čerin [3], and D. A. Edwards and P. Tulley McAuley [6]. Unfortunately, these approaches are just particular cases and they do not deal with the movability of shape morphisms in the general case of an abstract shape theory.

Some categorical approaches to movability in shape theory were given by P.S. Gevorgyan [7], [8], P.S. Gevorgyan and I. Pop [9], Avakyan and Gevorgyan [1], and I. Pop [21], [23].

The idea of considering the notions of movability for abstract pro-morphisms and shape morphisms came from the article [22] of the second author, in which the notion of movability is defined for a covariant functor and for a natural transformation (functorial morphism). Then, considering the inverse systems as functors and the pro-morphisms as natural transformations, various types of movability can be obtained, for pro-morphisms and shape morphisms, which is done in the papers of P.S. Gevorgyan and I. Pop [10], [11], [12]. But what is achieved by introducing this property? In short: if $m : X \to Y$ is a pro-morphism or a shape morphism and if X or Y is a movable pro-object or a shape-object then m is a movable morphism. And if $Y = X$ and $m = 1_X$, then X is movable if and only if the morphism 1_X is movable. We see that the movability of morphisms (pro- or shape-) is a generalization of the movability of objects in that category. And then, to obtain a theorem on the morphism m, assuming that X or Y is movable, it may happen that the same result should be obtained with the weaker condition that m be movable.

In the present paper we introduce and study the notions of movability for a new type of pro-morphisms and shape morphisms associated with a category $\mathcal{C}$ and a pair $(\mathcal{C}, \mathcal{D})$ respectively, namely belonging to a so called *enriched pro-category* pro^J-$\mathcal{C}$, and respectively to the corresponding shape category $Sh^J_{(\mathcal{C},\mathcal{D})}$ having as the realizing subcategory the category pro^J-$\mathcal{D}$ for $(J, \leq)$ a directed partially ordered set, according to the article [27] by N. Uglešić.

Because by particularization of the set $(J, \leq)$ one can obtain the classical abstract shape theory and the so-called coarse shape theory, the results of this article can be considered as generalizations of the corresponding results from the papers [9], [10], and [12].

2. Enriched pro-category and J-shape category

In this section are given the notions and results from [27] necessary for the approach of our paper. Other notions and necessary results from shape theory can be found in the books [17] and [4].

Definition 2.1. Let $\mathcal{C}$ be a category, let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in $\mathcal{C}$ and let $J = (J, \leq)$ be a directed partially ordered set. A *J-morphism* (of $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{C}$) is every triple $(\mathbf{X}, ((f^j_\mu), \phi), \mathbf{Y})$, denoted $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$, where $((f^j_\mu), \phi)$ is an ordered pair consisting of a function $\phi : M \to \Lambda$, called the *index function*, and, for each $\mu \in M$, of a family (f^j_μ) of $\mathcal{C}$-morphisms $f^j_\mu : X_{\phi(\mu)} \to Y_\mu, j \in J$, such that, for every related pair $\mu' \geq \mu$ in M, there exists a $\lambda \in \Lambda, \lambda \geq \phi(\mu), \phi(\mu')$, and there exists a $j \in J$ so that for every $j' \geq j$,

$$f^{j'}_\mu p_{\phi(\mu)\lambda} = q_{\mu\mu'} f^{j'}_{\mu'} p_{\phi(\mu')\lambda}, \tag{2.1}$$

i.e., makes the following diagram commutative

$$\begin{tikzcd}
& X_\lambda \arrow[dl, "p_{\phi(\mu)\lambda}"'] \arrow[dr, "p_{\phi(\mu')\lambda}"] & \\
X_{\phi(\mu)} \arrow[d, "f^{j'}_\mu"'] & & X_{\phi(\mu')} \arrow[d, "f^{j'}_{\mu'}"] \\
Y_\mu & & Y_{\mu'} \arrow[ll, "q_{\mu\mu'}"']
\end{tikzcd}$$

If the index function ϕ is increasing and, for every pair $\mu \leq \mu'$, one may put $\lambda = \phi(\mu')$, then (f^j_μ, ϕ) is said to be a *simple* J-morphism.

If, in addition, $M = \Lambda$ and $\phi = 1_\Lambda$, then $(f^j_\lambda, 1_\Lambda)$ is said to be a *level* J-morphism.

Further, if the equality (2.1) holds for every $j \in J$, then $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ is said to be a *commutative* J-morphism.

Remark 2.1. a) The *composition* of two J-morphisms $(f^j_\mu, \phi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ and $(g^j_\nu, \psi) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ is defined as $(h^j_\nu, \chi) : \mathbf{X} \to \mathbf{Z}$, with $\chi = \phi \circ \psi$ and $h^j_\nu = g^j_\nu \circ f^j_{\psi(\nu)}$, $j \in J$, $\nu \in N$. This composition is associative.

b) The *identity J-morphism* of the inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is $(1^j_{X_\lambda}, 1_\Lambda) : \mathbf{X} \to \mathbf{X}$ with $1^j_{X_\lambda} = 1_{X_\lambda}$ for any $j \in J$, where 1_{X_λ} is the identity morphism of X_λ in the category $\mathcal{C}$.

c) For a category $\mathcal{C}$ and a directed partially ordered set J there exists a category inv^J-$\mathcal{C}$ having the object class $Ob(inv^J\text{-}\mathcal{C}) = Ob(inv\text{-}\mathcal{C})$ and the morphism class $Mor(inv^J\text{-}\mathcal{C})$ of all sets $(inv^J\text{-}\mathcal{C})(\mathbf{X}, \mathbf{Y})$ of all J-morphisms (f^j_μ, ϕ) of $\mathbf{X}$ to $\mathbf{Y}$, endowed with the composition and identities described in a) and b).

Definition 2.2. A J-morphism $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ of inverse systems in $\mathcal{C}$ is said to be *equivalent to* a J-morphism $({f'}_\mu^j), \phi') : \mathbf{X} \to \mathbf{Y}$, denoted by $(f_\mu^j, \phi) \sim ({f'}_\mu^j, \phi')$, if every $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geq \phi(\mu), \phi'(\mu)$, and a $j \in J$ such that, for every $j' \geq j$,

$$f_\mu^{j'} p_{\phi(\mu)\lambda} = {f'}_\mu^{j'} p_{\phi'(\mu)\lambda}, \tag{2.2}$$

i.e., makes the following diagram commutative

$$\begin{tikzcd}
X_\lambda \arrow[r, "p_{\phi(\mu)\lambda}"] \arrow[d, "p_{\phi'(\mu)\lambda}"'] & X_{\phi(\mu)} \arrow[d, "f_\mu^{j'}"] \\
X_{\phi'(\mu)} \arrow[r, "{f'}_\mu^{j'}"'] & Y_\mu
\end{tikzcd}$$

Remark 2.2. a) The defining equality (2.2) holds for every $\lambda' \geq \lambda$;

b) The relation $\sim$ is an equivalence relation on each set $(inv^J\text{-}\mathcal{C})(\mathbf{X}, \mathbf{Y})$;

c) The equivalence class $[(f_\mu^j, \phi)]$ of a J-morphism is denoted by $\mathbf{f}$;

d) Let $(f^j, \phi), ({f'}_\mu^j, \phi') : \mathbf{X} \to \mathbf{Y}$ and $(g_\nu^j, \psi), ({g'}_\nu^j, \psi') : \mathbf{Y} \to \mathbf{Z}$ be J-morphisms of inverse systems in $\mathcal{C}$. If $(f_\mu^j, \phi) \sim ({f'}_\mu^j, \phi')$ and $(g_\nu^j, \psi) \sim ({g'}_\nu^j, \psi')$, then $(g_\nu^j, \psi)(f_\mu^j, \phi) \sim ({g'}_\nu^j, \phi')({f'}_\mu^j, \phi')$;

e) By the above remarks one may compose the equivalence classes of J-morphisms of inverse systems in $\mathcal{C}$ by means of any pair of their representatives, i.e., $\mathbf{gf} = \mathbf{h}$, where $\mathbf{h}$ is the equivalence class of $(h_\nu^j, h) = (g_\nu^j, \psi)(f_\mu^j, \phi) = (g_\nu^j f_{g(\nu)}^j, \phi\psi)$. The corresponding quotient category $(inv^J\text{-}\mathcal{C})/\sim$ is denoted by $pro^J\text{-}\mathcal{C}$. The morphisms of this category are called *J-pro-morphisms.* There exists a subcategory $(pro^J\text{-}\mathcal{C})_c \subseteq pro^J\text{-}\mathcal{C}$ determined by all equivalence classes having commutative representatives. This category is isomorphic to the quotient category $(inv^J\text{-}C)_c/\sim$. Also $pro\text{-}\mathcal{C} = (inv\text{-}\mathcal{C})/\sim$ can be considered as a subcategory of $(pro^J\text{-}\mathcal{C})_c$ and, consequently as a subcategory of $pro^J\text{-}\mathcal{C}$

f) Now using the fact that if $(\Lambda, \leq)$ is a directed set and $(M, \leq)$ is a cofinite directed set, then every function $\phi : M \to \Lambda$ admits an increasing function $\phi' : M \to \Lambda$ such that $\phi \leq \phi'$ (see [17], Ch.I, §1.2, Lemma 1), it can be proved that: if $\mathbf{f} : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to Y = (Y_\mu, q_{\mu\mu'}, M)$ is a morphism in $pro^J\text{-}\mathcal{C}$, with $(M, \leq)$ cofinite, then $\mathbf{f}$ admits a simple representative $({f'}_\mu^j, \phi') : \mathbf{X} \to \mathbf{Y}$ ([27], Lemma 6).

g) There exists a covariant functor $\underline{I} \equiv \underline{I}_\mathcal{C}^J : pro\text{-}\mathcal{C} \to pro^J\text{-}\mathcal{C}$, by: $\underline{I}(\mathbf{X}) = \mathbf{X}$, for every inverse system $\mathbf{X}$ in $\mathcal{C}$, and if $\mathbf{f} \in pro\text{-}\mathcal{C}(\mathbf{X}, \mathbf{Y})$ and $\mathbf{f} = [(f_\mu, \phi)]$, then $\underline{I}(\mathbf{f}) = [(f_\mu^j, \phi)] \in (pro^J\text{-}\mathcal{C})(\mathbf{X}, \mathbf{Y})$, where for each $\mu \in M$, $f_\mu^j = f_\mu$ for all $j \in J$. Thus, every induced J-morphism is commutative, and therefore $\underline{I}_\mathcal{C}^J : pro\text{-}\mathcal{C} \to (pro^J\text{-}\mathcal{C})_c \subseteq$

pro^J-$\mathcal{C}$. It is easy to see that this functor is faithful ([27], Theorem 1), but it is not full ([27], Remark 2).

h) Every inverse system $\mathbf{X}$ in $\mathcal{C}$ is isomorphic in pro^J-$\mathcal{C}$ to a cofinite inverse system $\mathbf{X}'$.

An important theorem is the following ([27], Theorem 2; [17], Ch.I, §1.3, Theorem 3):

Theorem 2.1. *Let* $\mathbf{f}:\mathbf{X}\to\mathbf{Y}\in(pro^J\text{-}\mathcal{C})(\mathbf{X},\mathbf{Y})$. *Then there exist inverse systems* $\mathbf{X}'$ *and* $\mathbf{Y}'$ *in* $\mathcal{C}$ *having the same cofinite index set* $(N,\leq)$, *there exists a morphism* $\mathbf{f}':\mathbf{X}'\to\mathbf{Y}'$ *having a level representative* $({f'}^j_\nu, 1_N)$ *and there exists isomorphisms* $\mathbf{i}:\mathbf{X}\to\mathbf{X}'$ *and* $j:\mathbf{Y}\to\mathbf{Y}'$ *of* pro^J-$\mathcal{C}$ *such that the following diagram in* pro^J-$\mathcal{C}$ *commutes*

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\mathbf{f}"] \arrow[d, "\mathbf{i}"'] & \mathbf{Y} \arrow[d, "\mathbf{j}"] \\
\mathbf{X}' \arrow[r, "\mathbf{f}'"'] & \mathbf{Y}'
\end{tikzcd}$$

Remark 2.3. a) If $J=\{1\}$, then $pro^{(1)}$-$\mathcal{C}=pro$-$\mathcal{C}$;

b) If $(J,\leq)=(\mathbb{N},\leq)$, then $pro^{\mathbb{N}}$-$\mathcal{C}=pro^*$-$\mathcal{C}$ is the pro-category obtained from the category $(inv^*$-$\mathcal{C})$ with so-called, $*$-*morphisms* [14];

c) If J is a directed partially ordered set having $maxJ$, then pro^J-$\mathcal{C}\cong pro$-$\mathcal{C}$. The "inclusion"functor $I: pro$-$C\to pro^J$-C is a category isomorphism;

d) If J and K are finite directed partially ordered sets, then there exist the isomorphisms: pro^J-$\mathcal{C}\cong pro^K$-$\mathcal{C}\cong pro$-$\mathcal{C}$;

e) If there exists $maxJ$, then for every L there exists the canonical inclusion functor $\underline{I}: pro^J$-$\mathcal{C}\to pro^L$-$\mathcal{C}$ keeping the objects fixed;

f) Let J be a well ordered set and let K be a directed partially ordered set, both without maximal elements, such that there exists an increasing function $\phi: J\to K$ such that $\phi[J]$ is cofinal in K. Then there exists a functor $\underline{T}: pro^J$-$\mathcal{C}\to pro^K$-$\mathcal{C}$ which keeps the objects fixed and does not depend on ϕ. Furthermore, for every pair $\mathbf{X}$ and $\mathbf{Y}$ of inverse systems in $\mathcal{C}$, $\mathbf{X}\cong\mathbf{Y}$ in pro^J-$\mathcal{C}$ iff $\mathbf{X}\cong\mathbf{Y}$ in pro^K-$\mathcal{C}$.

Remark 2.4. A pro^J-C category is called an *enriched pro-category.* An enriched pro-category is interesting and useful by itself because, in general, it divides (classifies) the objects into larger classes (isomorphisms types) than the underling pro-category pro-$\mathcal{C}$. In addition, with the help of such an enriched pro-category one can construct in the usual way a corresponding J-*shape theory.*

Definition 2.3. A J-pro-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is said to be pro^J-$\mathcal{D}$ *equivalent* to a J-pro-morphism $\mathbf{f}':\mathbf{X}'\to\mathbf{Y}'$, denoted by $\mathbf{f}\sim\mathbf{f}'$, if there exist two canonical isomorphisms $\mathbf{i}:\mathbf{X}\to\mathbf{X}'$ and $\mathbf{j}:\mathbf{Y}\to\mathbf{Y}'$ of *pro*-$\mathcal{D}$ such that the following diagram in *pro*-$\mathcal{D}$ commutes:

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\mathbf{i}"] \arrow[d, "\mathbf{f}"'] & \mathbf{X}' \arrow[d, "\mathbf{f}'"] \\
\mathbf{Y} \arrow[r, "\mathbf{j}"'] & \mathbf{Y}'
\end{tikzcd}$$

The equivalence class of a J-pro-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is denoted by $\langle\mathbf{f}\rangle$.

Remark 2.5. If $\mathbf{f}\sim\mathbf{f}'$ and $\mathbf{g}\sim\mathbf{g}'$, then $\mathbf{gf}\sim\mathbf{g}'\mathbf{f}'$, so the composition $\langle\mathbf{g}\rangle\langle\mathbf{f}\rangle=\langle\mathbf{gf}\rangle$ is well defined.

Definition 2.4. For a pair of categories $(\mathcal{C},\mathcal{D})$ with $\mathcal{D}$ a dense full (equivalent, full and pro-reflective, [26]) subcategory of $\mathcal{C}$, the (*abstract*) *J-shape category* $Sh^J_{(\mathcal{C},\mathcal{D})}$ is defined as follows. The objects of this category are all the objects of $\mathcal{C}$. A morphism $F\in Sh^J_{(C,D)}(X,Y)$ is the (*pro*^{J}-$\mathcal{D}$)-equivalence class $\langle\mathbf{f}\rangle$ of a J-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ of *pro*^{J}-$\mathcal{D}$ for an arbitrary choice of $\mathcal{D}$-expansions $\mathbf{p}:X\to\mathbf{X}$, $\mathbf{q}:Y\to\mathbf{Y}$. In other words, a J-shape morphism $F:X\to Y$ is given by a diagram

$$\begin{tikzcd}
\mathbf{X} \arrow[d, "\mathbf{f}"'] & X \arrow[l, "\mathbf{p}"'] \arrow[d, "F"] \\
\mathbf{Y} & Y \arrow[l, "\mathbf{q}"]
\end{tikzcd}$$

The composition of two J-shape morphisms $F:X\to Y$, $F=\langle\mathbf{f}\rangle$ and $G:Y\to Z$, $G=\langle\mathbf{g}\rangle$, is defined by representatives, i.e., $GF:X\to Z$, $GF=\langle\mathbf{gf}\rangle$.

The identity J-shape morphism on an object X, $1_X:X\to X$, is the (*pro*^{J}-D)-equivalence class $\langle 1_{\mathbf{X}}\rangle$ of the identity morphism $1_{\mathbf{X}}$ of $\mathbf{X}$ in *pro*^{J}-$\mathcal{D}$.

Since $Sh^J_{(C,D)}(X,Y)\approx$ *pro*^{J}-$\mathcal{D}(\mathbf{X},\mathbf{Y})$ is a set, the J-shape category $Sh^J_{(C,D)}$ is well defined, and that its realizing category is *pro*^{J}-$\mathcal{D}$.

An interesting particular case of J-shape morphism is the following: If $f:X\to Y$ is a morphism in the category $\mathcal{C}$ and $\mathbf{p}:X\to\mathbf{X}$, $\mathbf{q}:Y\to\mathbf{Y}$ are $\mathcal{D}$-expansions, then there exists a morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in *pro*^{J}-$\mathcal{D}$, such that the following diagram in *pro*^{J}-$\mathcal{C}$ commutes:

$$\begin{tikzcd}
\mathbf{X} \arrow[d, "\mathbf{f}"'] & X \arrow[l, "\mathbf{p}"'] \arrow[d, "f"] \\
\mathbf{Y} & Y \arrow[l, "\mathbf{q}"]
\end{tikzcd}$$

This is a result of the definition of an expansion, [17] (Ch. I, §2.1). If we take other $\mathcal{D}$-expansions $\mathbf{p}':X\to\mathbf{X}'$, $\mathbf{q}':Y\to\mathbf{Y}'$, we obtain another morphism $\mathbf{f}':\mathbf{X}'\to\mathbf{Y}'$ in *pro*^{J}-$\mathcal{D}$, such that $\mathbf{f}'\mathbf{p}'=\mathbf{q}'f$. And because $(\mathbf{f}'\mathbf{i})\mathbf{p}=\mathbf{f}'\mathbf{p}'=\mathbf{q}'f=\mathbf{jq}f=(\mathbf{jf})\mathbf{p}$, which implies $\mathbf{f}'\mathbf{i} = \mathbf{j}\mathbf{f}$, such that $\mathbf{f} \sim \mathbf{f}'$ in pro^J-$\mathcal{D}$, and in this way we can associate with every $f \in \mathcal{C}(X,Y)$ a pro^J-$\mathcal{D}$-equivalence class $\langle \mathbf{f} \rangle$, i.e., a J-shape morphism $F \in Sh^J_{(\mathcal{C},\mathcal{D})}(X,Y)$.

If one defines $S^J(X) = X$, $X \in Ob\mathcal{C}$, and $S^J(f) = F = \langle \mathbf{f} \rangle$, $f \in \mathcal{C}(X,Y)$, we obtain a covariant functor $S^J \equiv S^J_{(\mathcal{C},\mathcal{D})} : \mathcal{C} \to Sh^J_{(\mathcal{C},\mathcal{D})}$, called (*abstract*) *J-shape functor*.

Theorem 2.2 ([27], Theorem 5)**.** *Let $\mathcal{D}$ be a full and pro-reflective subcategory of $\mathcal{C}$ and J a directed partially ordered set. Then, for every pair $P, Q \in Ob\mathcal{D}$, the following statements are equivalent:*

(i) P and Q are isomorphic objects of $\mathcal{D}$, $P \cong Q$ in $\mathcal{D} \subseteq \mathcal{C}$;

(ii) P and Q have the same shape, $Sh(P) = Sh(Q)$, i.e., $P \cong Q$ in $Sh_{(\mathcal{C},\mathcal{D})}$;

(iii) P and Q have the same J-shape, $Sh^J_{(\mathcal{C},\mathcal{D})}(P) = Sh^J_{(\mathcal{C},\mathcal{D})}(Q)$, i.e., $P \cong Q$ in $Sh^J_{(\mathcal{C},\mathcal{D})}$.

Theorem 2.3 ([27], Corollary 2)**.** *Let $\mathcal{C}$ a category and $\mathcal{D}$ a full and pro-reflective subcategory. Then*

(i) $Sh_{(\mathcal{C},\mathcal{D})} = Sh^{\{1\}}_{(\mathcal{C},\mathcal{D})}$;

(ii) $Sh^*_{(\mathcal{C},\mathcal{D})} = Sh^{\mathbb{N}}_{(\mathcal{C},\mathcal{D})}$, where $Sh^*_{(\mathcal{C},\mathcal{D})}$ is the coarse shape category [14]*;*

(iii) If J is a directed partially ordered set having $\max J$, then $Sh^J_{(\mathcal{C},\mathcal{D})} \cong Sh_{(\mathcal{C},\mathcal{D})}$.

3. Movability and uniform movability properties for J-morphisms

All sets of indices of inverse systems are supposed to be cofinite directed sets. This condition is not restrictive (cf. [17], Ch.I, §1.2).

First we recall from [17] the notions of movable and uniform movable inverse systems.

An object $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-$\mathcal{C}$ is *movable* provided every $\lambda \in \Lambda$ admits a $\lambda' \geq \lambda$ (called a *movability index* of λ) such that each $\lambda'' \geq \lambda$ admits a morphism $r : X_{\lambda'} \to X_{\lambda''}$ of $\mathcal{C}$ which satisfies

$$p_{\lambda\lambda''} \circ r = p_{\lambda\lambda'}, \tag{3.1}$$

i.e., makes the following diagram commutative

$$\begin{tikzcd} X_{\lambda'} \arrow[rr, "p_{\lambda\lambda'}"] \arrow[dr, dashed, "r"'] & & X_\lambda \\ & X_{\lambda''} \arrow[ur, "p_{\lambda\lambda''}"'] & \end{tikzcd}$$

An object $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-$\mathcal{C}$ is called a *uniform movable* if every $\lambda \in \Lambda$ admits a $\lambda' \geq \lambda$ (called a *uniform movability index* of λ) such that there is a morphism $\mathbf{r} : X_{\lambda'} \to \mathbf{X}$ in pro-$\mathcal{C}$ satisfying

$$\mathbf{p}_\lambda \circ \mathbf{r} = p_{\lambda\lambda'}, \tag{3.2}$$

where $\mathbf{p}_\lambda : \mathbf{X} \to X_\lambda$ is the morphism of pro-$\mathcal{C}$ given by $1_\lambda : X_\lambda \to X_\lambda$.

Definition 3.1. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$ and $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism of inverse systems. We say that the J-morphism (f^j_μ, ϕ) is *movable* (J*-movable*) if every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $j \in J$, such that each $\mu' \in M$, $\mu' \geq \mu$, and $j' \geq j$ admit a morphism $u^{j'} : X_\lambda \to Y_{\mu'}$ in the category $\mathcal{C}$, which satisfies

$$f^{j'}_\mu \circ p_{\phi(\mu)\lambda} = q_{\mu\mu'} \circ u^{j'}, \tag{3.3}$$

i.e., makes the following diagram commutative

$$\begin{tikzcd} X_{\phi(\mu)} \arrow[r, "f^{j'}_\mu"] & Y_\mu \\ X_\lambda \arrow[u, "p_{\phi(\mu)\lambda}"] \arrow[r, dashed, "u^{j'}"'] & Y_{\mu'} \arrow[u, "q_{\mu\mu'}"'] \end{tikzcd}$$

The pair of indices (λ, j) is called a *J-movability pair* of μ with respect to the J-morphism (f^j_μ, ϕ).

The composition $f^j_\mu \circ p_{\phi(\mu)\lambda}$ for $\lambda \geq \phi(\mu)$ is denoted by $f^j_{\mu\lambda}$ (cf. [17], Ch.II, §2.1). With this notation the relation (3.3) becomes

$$f^{j'}_{\mu\lambda} = q_{\mu\mu'} \circ u^{j'}.$$

Note that if (λ, j) is a J-movability pair of μ with respect to (f^j_μ, ϕ), then so is any pair $(\widetilde{\lambda}, \widetilde{j})$, with $\widetilde{\lambda} \geq \lambda$ and $\widetilde{j} \geq j$.

Example 3.1. Let (X) be a rudimentary system in the category $\mathcal{C}$ and (f^j_μ, ϕ) : $(X) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\phi(\mu) = 1, \forall \mu \in M$, a J-morphism of inverse systems. It is not hard to verify that (f^j_μ, ϕ) is movable.

More generally, if we consider a morphism $(f^j_\mu, \phi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ such that there exists $\lambda_M \in \Lambda$ satisfying $\lambda_M \geq \phi(\mu)$ for any $\mu \in M$, then (f^j_μ, ϕ) is J-movable. Indeed, for an arbitrary index $\mu \in M$ and $\mu' \geq \mu$ there exists $j \in J$ such that for $j' \in J$, $j' \geq j$, we have $f^{j'}_\mu \circ p_{\phi(\mu)\lambda_M} = q_{\mu\mu'} \circ f^{j'}_{\mu'} \circ p_{\phi(\mu')\lambda_M} = q_{\mu\mu'} \circ u^{j'}$, where $u^{j'} = f^{j'}_{\mu'} \circ p_{\phi(\mu')\lambda_M}$ is a morphism from X_{λ_M} to $Y_{\mu'}$. So, (λ_M, j) is a J-movability pair for $\mu \in M$.

Remark 3.1. a) If $J = \{1\}$, that is, inv^J-$\mathcal{C} = inv$-$\mathcal{C}$, then the condition of movability for a morphism of inverse systems $(f_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ is written as $f_\mu \circ p_{\phi(\mu)\lambda} = q_{\mu\mu'} u$, for a $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$, $\mu' \geq \mu$, and $u : X_\lambda \to Y_{\mu'}$ a morphism in $\mathcal{C}$. And this is the definition of movability for an usual morphism of inverse systems (cf. [10], Definition 2.2).

b) If $(J, \leq) = (\mathbb{N}, \leq)$, i.e., inv^J-$\mathcal{C} = inv^*$-$\mathcal{C}$, the condition of movability for a $*$-morphism $(f_\mu^n, \phi) : \mathbf{X} \to \mathbf{Y}$ is the following: every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $n \in \mathbb{N}$, such that each $\mu' \in M$, $\mu' \geq \mu$, and $m \geq n$, admit a morphism $u^m : X_\lambda \to Y_{\mu'}$ in the category $\mathcal{C}$, which satisfies

$$f_\mu^m \circ p_{\phi(\mu)\lambda} = q_{\mu\mu'} \circ u^m. \tag{3.4}$$

Proposition 3.1. *An inverse system* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *is movable if and only if the identity* J*-morphism* $(1^j_{X_\lambda}, 1_\Lambda)$ *is movable.*

Proof. If λ' is a movability index of λ with respect to $\mathbf{X}$, then a pair (λ', j), $j \in J$, is a J-movability pair for λ with respect to the identity J-morphism, and conversely. □

Theorem 3.1. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ *be inverse systems in the category* $\mathcal{C}$ *and let* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$, $(g_\nu^j, \psi) : \mathbf{Y} \to \mathbf{Z}$ *be* J*-morphisms of inverse systems. If* (g_ν^j, ψ) *is movable, then the composition* $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ *is also a movable* J*-morphism.*

Proof. Recall that by definition of the composition of J-morphisms we have $\chi = \phi \circ \psi$ and $h_\nu^j = g_\nu^j \circ f^j_{\psi(\nu)}$. If (g_ν^j, ψ) is movable, and if (μ, j) is a J-movability pair of an index $\nu \in N$, then for any index $\nu' \in N$, $\nu' \geq \nu$, there is an index $j \in J$ and a morphism $u^{j'} : Y_\mu \to Z_{\nu'}$, $j' \geq j$, such that $g_\nu^{j'} \circ q_{\psi(\nu)\mu} = r_{\nu\nu'} \circ u^{j'}$ or the next diagram is commutative

$$\begin{tikzcd}
Y_{\psi(\nu)} \arrow[r, "g_\nu^{j'}"] & Z_\nu \\
Y_\mu \arrow[u, "q_{\psi(\nu)\mu}"] \arrow[r, dashed, "u^{j'}"'] & Z_{\nu'} \arrow[u, "r_{\nu\nu'}"']
\end{tikzcd}$$

Now consider $\lambda \in \Lambda$ such that $\lambda \geq \phi(\mu)$, $\lambda \geq \phi(\psi(\nu))$, and $f^{j'}_{\psi(\nu)} \circ p_{\phi(\psi(\nu))\lambda} = q_{\psi(\nu)\mu} \circ f_\mu^{j'} \circ p_{\phi(\mu)\lambda}$. Consider the morphism $u'^{j'} = u^{j'} \circ f_\mu^{j'} \circ p_{\phi(\mu)\lambda} : X_\lambda \to Z_{\nu'}$. For this morphism we obtain: $r_{\nu\nu'} \circ u'^{j'} = (r_{\nu\nu'} \circ u^{j'}) \circ f_\mu \circ p_{\phi(\mu)\lambda} = g_\nu \circ q_{\psi(\nu)} \circ f_\mu^{j'} \circ p_{\phi(\mu)\lambda} = g_\nu \circ f^{j'}_{\psi(\nu)} \circ p_{\phi(\psi(\nu))} = h_\nu^{j'} \circ p_{\chi(\nu)\lambda}$, i.e., the following diagram is commutative

$$\begin{tikzcd} X_{\chi(\nu)} \arrow[r, "h_\nu^{j'}"] & Z_\nu \\ X_\lambda \arrow[u, "p_{\chi(\nu)\lambda}"] \arrow[r, dashed, "u'^{j'}"'] & Z_{\nu'} \arrow[u, "r_{\nu\nu'}"'] \end{tikzcd}$$

Thus, (h_ν^j, χ) is a movable J-morphism. □

Corollary 3.1. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be an arbitrary inverse system and* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *be a movable inverse system. Then any* J*-morphism* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ *is movable.*

Proof. Since $(f_\mu^j, \phi) = (1_{Y_\mu}^j, 1_M) \circ (f_\mu^j, \phi)$ and $(1_{Y_\mu}^j, 1_M) : \mathbf{Y} \to \mathbf{Y}$ is a movable J-morphism by Proposition 3.1, then (f_μ^j, ϕ) is also J-movable according to Theorem 3.1. □

Theorem 3.2. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ *be inverse systems in the category* $\mathcal{C}$, *and let* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$, $(g_\nu^j, \psi) : \mathbf{Y} \to \mathbf{Z}$ *be* J*-morphisms. If* (f_μ^j, ϕ) *is movable, then the composition* $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ *is also a movable* J*-morphism.*

Proof. For a given index $\nu \in N$, consider a movability pair (λ, j) of $\psi(\nu)$, $\lambda \geq \phi(\psi(\nu))$, with respect to (f_μ^j, ϕ). Let us prove that (λ, j) is a movability pair of ν with respect to the J-morphism (h_ν^j, χ).

Let $\nu' \in N$, $\nu' \geq \nu$, be any index and let $\mu' \geq \psi(\nu'), \psi(\nu)$ be an index such that for $j' \geq j$

$$r_{\nu\nu'}^{j'} \circ g_{\nu'}^{j'} \circ q_{\psi(\nu')\mu'} = g_\nu^{j'} \circ q_{\psi(\nu)\mu'}.$$

By the J-movability of $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$, for $q_{\psi(\nu)\mu'}$ there exists a morphism $u^{j'} : X_\lambda \to Y_{\mu'}$ such that

$$f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\lambda} = q_{\psi(\nu)\mu'} \circ u^{j'}.$$

Define $U^{j'} : X_\lambda \to Z_{\nu'}$ by

$$U^{j'} = g_{\nu'}^{j'} \circ q_{\psi(\nu')\mu'} \circ u^{j'}.$$

Now we have: $r_{\nu\nu'}^{j'} \circ U^{j'} = r_{\nu\nu'}^{j'} \circ g_{\nu'}^{j'} \circ q_{\psi(\nu')\mu'} \circ u^{j'} = g_\nu^{j'} \circ q_{\psi(\nu)\mu'} \circ u^{j'} = g_\nu^{j'} \circ q_{\psi(\nu)\mu} \circ f_\mu^{j'} \circ p_{\phi(\mu)\lambda} = g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\chi(\nu)\lambda} = h_\nu^{j'} \circ p_{\chi(\nu)\lambda}$. □

Corollary 3.2. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be a movable inverse system and let* $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *be an arbitrary inverse system. Then any* J*-morphism* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ *is movable.*

Proof. Since $(f^j_\mu, \phi) = (f^j_\mu, \phi) \circ (1^j_{X_\lambda}, 1_\Lambda)$ and the identity J-morphism $(1^j_{X_\lambda}, 1_\Lambda) : \mathbf{X} \to \mathbf{X}$ is movable by Proposition 3.1, then (f^j_μ, ϕ) is also J-movable according to Theorem 3.2. □

Corollary 3.3. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a movable inverse system in the category $\mathcal{C}$. If an inverse system $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is J-dominated by $\mathbf{X}$, i.e., there exist two J-morphisms $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ and $(g^j_\lambda, \psi) : \mathbf{Y} \to \mathbf{X}$ such that $(f^j_\mu, \phi) \circ (g^j_\lambda, \psi) = (1^j_{Y_\mu}, 1_M)$, then $\mathbf{Y}$ is movable.*

Proof. By hypothesis and Proposition 3.1, $(1^j_{X_\lambda}, 1_\Lambda)$ is J-movable. Then by the equality $(1^j_{X_\lambda}, 1_\Lambda) \circ (g^j_\lambda, \psi) = (g^j_\lambda, \psi)$ and by Theorem 3.1 it follows that (g^j_ν, ψ) is J-movable. Hence, the composition $(f^j_\mu, \phi) \circ (g^j_\lambda, \psi) = (1^j_{Y_\mu}, 1_M)$ is also J-movable by Theorem 3.2. Therefore, Proposition 3.1 implies that $\mathbf{Y}$ is a movable inverse system. □

Remark 3.2. Corollary 3.3 is a generalization of a classical result for the movability of inverse systems [17] (Ch. II, §6.1, Theorem 1) here with a proof based on the J-movability property of J-morphisms.

Definition 3.2. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism. We say that the J-morphism (f_μ, ϕ) is *uniformly movable*(*J-uniformly movable*) if every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $j \in J$ such that for $j' \geq j$ there is a J-morphism of inverse systems $\mathbf{u}^{\mathbf{j}'} : X_\lambda \to \mathbf{Y}$ satisfying

$$f^{j'}_{\mu\lambda} = \mathbf{q}_\mu \circ \mathbf{u}^{j'} \tag{3.5}$$

i.e., the following diagram commutes

$$\begin{tikzcd}
X_\lambda \arrow[rr, "f^{j'}_{\mu\lambda}"] \arrow[dr, dashed, "\mathbf{u}^{j'}"'] & & Y_\mu \\
& \mathbf{Y} \arrow[ur, "\mathbf{q}_\mu"'] &
\end{tikzcd}$$

where $f^{j'}_{\mu\lambda} = f^{j'}_\mu \circ p_{\phi(\mu)\lambda}$ and $\mathbf{q}_\mu : \mathbf{Y} \to Y_\mu$ is the J-morphism of inverse systems given by $1^j_{Y_\mu} : Y_\mu \to Y_\mu$.

The pair (λ, j) is called a *J-uniform movability pair of μ with respect to* (f^j_μ, ϕ).

Remark 3.3. If (λ, j) is a J-uniform movability pair, then so is any pair $(\widetilde{\lambda}, \widetilde{j})$, $\widetilde{\lambda} \geq \lambda$, $\widetilde{j} \geq j$.

Remark 3.4. Note that the J-morphism $\mathbf{u}^{j'} : X_\lambda \to \mathbf{Y}$ determines for every $\mu_1 \in M$ a morphism $u^{j'}_{\mu_1} : X_\lambda \to Y_{\mu_1}$ in $\mathcal{C}$ such that for $\mu_1 \leq \mu_2$ we have $q_{\mu_1\mu_2} \circ u^{j'}_{\mu_2} = u^{j'}_{\mu_1}$

and $u^{j'}_{\mu} = f^{j'}_{\mu\lambda}$. In particular, for $\mu' \in M$, $\mu' \geq \mu$, we have $q_{\mu\mu'} \circ u^{j'}_{\mu'} = u^{j'}_{\mu} = f^{j'}_{\mu\lambda}$, so that J-uniform movability of J-morphisms implies J-movability.

Proposition 3.2. *An inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is uniformly movable if and only if the identity J-morphism $1_{\mathbf{X}} = (1^j_{X_\lambda}, 1_\Lambda)$ is J-uniformly movable.*

Proof. Suppose $\mathbf{X}$ is uniformly movable. Let $\lambda \in \Lambda$. Note that a uniform movability index $\lambda' \geq \lambda$ together with $j \in J$ arbitrary constitutes a pair (λ', j) of J-uniform movability of λ with respect to the identity $1_{\mathbf{X}} = (1^j_{X_\lambda}, 1_\Lambda)$. Conversely, suppose $1_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$ is a uniformly movable J-morphism. Note that for any $\lambda \in \Lambda$ if (λ', j) is a J-uniform movability pair of λ with respect to $1_{\mathbf{X}}$, then λ' is a uniform movability index of λ for $\mathbf{X}$. $\square$

Example 3.2. Let (X) be a rudimentary system in the category $\mathcal{C}$. It is easy to see that any J-morphism of inverse systems $(f^j_\mu) : (X) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is J-uniformly movable.

More generally, if we consider a J-morphism $(f^j_\mu, \phi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ such that there exists $\lambda_M \in \Lambda$ satisfying $\lambda_M \geq \phi(\mu)$ for any $\mu \in M$, then (f^j_μ, ϕ) is J-uniformly movable. Indeed, it is not difficult to verify that for any index $\mu \in M$, a J-uniformly movable pair is (λ_M, j), for an arbitrary $j \in J$.

Theorem 3.3. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ be inverse systems in the category $\mathcal{C}$ and $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$, $(g^j_\nu, \psi) : \mathbf{Y} \to \mathbf{Z}$ morphisms of inverse systems. If (g^j_ν, ψ) is J-uniformly movable, then the composition $(h^j_\nu, \chi) = (g^j_\nu, \psi) \circ (f^j_\mu, \phi)$ is also a J-uniformly movable morphism.*

Proof. We use the notations from the proof of Theorem 3.1 replacing $r_{\nu\nu'} : Z_{\nu'} \to Z_\nu$ by $\mathbf{r}_\nu : \mathbf{Z} \to Z_\nu$ and $u^{j'} : Y_\mu \to Z_{\nu'}$ by $\mathbf{u}^{j'} : Y_\mu \to \mathbf{Z}$. Then we have $g^{j'}_\nu \circ q_{\psi(\nu)\mu} = \mathbf{r}^{j'}_\nu \circ \mathbf{u}^{j'}$. And by defining $\mathbf{u'}^{j'} = \mathbf{u}^{j'} \circ f^{j'}_{\mu\lambda} : X_\lambda \to \mathbf{Z}$, we obtain $\mathbf{r}_\nu \circ \mathbf{u'}^{j'} = h^{j'}_{\nu\lambda}$. $\square$

Corollary 3.4. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an arbitrary inverse system and let $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be a uniformly movable inverse system. Then any J-morphism $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ is J-uniformly movable.*

Proof. Since $(f^j_\mu, \phi) = 1^J_{\mathbf{Y}} \circ (f^j_\mu, \phi)$ and $1^J_{\mathbf{Y}} : \mathbf{Y} \to \mathbf{Y}$ is J-uniformly movable by Proposition 3.2, then (f^j_μ, ϕ) is also uniformly movable according to Theorem 3.3.

Theorem 3.4. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ be inverse systems in the category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$, $(g^j_\nu, \psi) : \mathbf{Y} \to \mathbf{Z}$ be morphisms of inverse systems. Suppose that (f^j_μ, ϕ) is J-uniformly movable. Then the composition $(h^j_\nu, \chi) = (g^j_\nu, \psi) \circ (f^j_\mu, \phi)$ is also a uniformly movable morphism.*

Proof. Using the notations from the proof of Theorem 3.2, there exists $\mathbf{u}^{j'}: X_\lambda \to \mathbf{Y}$, such that $f^{j'}_{\psi(\nu)\lambda} = \mathbf{q}_{\psi(\nu)} \circ \mathbf{u}^{j'}$. Then for $\mathbf{U}^{j'} : X_\lambda \to \mathbf{Z}$, $\mathbf{U}^{j'} = g^{j'}_\nu \circ \mathbf{u}^{j'}$, we have $h^{j'}_{\nu\lambda} = g^{j'}_\nu \circ f^{j'}_{\psi(\nu)\lambda} = \mathbf{r}_\nu \circ \mathbf{U}^{j'}$. $\square$

Corollary 3.5. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be uniformly movable inverse system and let $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be an arbitrary inverse system. Then any J-morphism $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ is uniformly movable.*

Proof. Since $(f^j_\mu, \phi) = (f^j_\mu, \phi) \circ (1^j_{X_\lambda}, 1_\Lambda)$ and the identity J-morphism $(1^j_{X_\lambda}, 1_\Lambda)$ is uniformly movable by Proposition 3.2, then (f^j_μ, ϕ) is also J-uniformly movable according to Theorem 3.4. $\square$

Corollary 3.6. *Let $\mathbf{X}$ and $\mathbf{Y}$ be inverse systems in the category $\mathcal{C}$. Suppose that $\mathbf{X}$ is uniformly movable and $\mathbf{Y}$ is J-dominated by $\mathbf{X}$. Then $\mathbf{Y}$ is uniformly movable.*

Proof. We use the notations from Corollary 3.3. By hypothesis and Proposition 3.2, $(1^j_{X_\lambda}, 1_\Lambda)$ is J-uniformly movable. Then by the equality $(1^j_{X_\lambda}, 1_\Lambda) \circ (g^j_\nu, \psi) = (g^j_\nu, \psi)$ and by Theorem 3.3 we have that (g^j_ν, ψ) is J-uniformly movable. Hence, by Theorem 3.4 the composition $(f^j_\mu, \phi) \circ (g^j_\lambda, \psi) = (1^j_{Y_\mu}, 1_M)$ is also J-uniformly movable. Finally, using Proposition 3.2 we conclude that $\mathbf{Y}$ is a uniformly movable inverse system. $\square$

4. Co-movability properties for J-morphisms

Definition 4.1. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism in $\mathcal{C}$. We say that the (f_μ, ϕ) is a *co-movable J-morphism* provided every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $j \in J$ (the pair (λ, j) being called a *co-movability pair* of μ relative to (f^j_μ, ϕ)) such that each $\lambda' \geq \phi(\mu)$ and $j' \in J$, $j' \geq j$ admit a morphism $r^{j'} : X_\lambda \to X_{\lambda'}$ of $\mathcal{C}$ which satisfies

$$f^{j'}_{\mu\lambda} = f^{j'}_{\mu\lambda'} \circ r^{j'}, \tag{4.1}$$

i.e., makes the following outside diagram commutative

$$\begin{tikzcd}
 & Y_\mu & \\
 & X_{\phi(\mu)} \arrow[u, "f^{j'}_\mu"'] & \\
X_\lambda \arrow[uur, "f^{j'}_{\mu\lambda}"] \arrow[ur, "p_{\phi(\mu)\lambda}"'] \arrow[rr, dashed, "r^{j'}"'] & & X_{\lambda'} \arrow[uul, "f^{j'}_{\mu\lambda'}"'] \arrow[ul, "p_{\phi(\mu)\lambda'}"]
\end{tikzcd}$$

Definition 4.2. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism of inverse systems. We say that the (f^j_μ, ϕ) is a *uniformly co-movable J-morphism* provided every $\mu \in M$ admits $\lambda \in \Lambda$, $\lambda \geq \phi(\mu)$ and $j \in J$ (the pair (λ, j) being called a *uniform co-movability pair of* μ relative to (f^j_μ, ϕ)) such that, for $j' \in J$, $j' \geq j$, there is a morphism $\mathbf{r}^{j'} : X_\lambda \to \mathbf{X}$ of inverse systems satisfying

$$f^{j'}_{\mu\lambda} = \mathbf{f}^{j'}_\mu \circ \mathbf{r}^{j'}, \tag{4.2}$$

i.e., makes the following outside diagram commutative

$$\begin{tikzcd}
 & Y_\mu & \\
 & X_{\phi(\mu)} \arrow[u, "f^{j'}_\mu"'] & \\
X_\lambda \arrow[uur, "f^{j'}_{\mu\lambda}"] \arrow[ur, "p_{\phi(\mu)\lambda}"'] \arrow[rr, dashed, "\mathbf{r}^{j'}"'] & & \mathbf{X} \arrow[uul, "\mathbf{f}^{j'}_\mu"'] \arrow[ul, "\mathbf{p}_{\phi(\mu)}"]
\end{tikzcd}$$

where $\mathbf{f}^{j'}_\mu = f^{j'}_\mu \circ \mathbf{p}_{\phi(\mu)}$.

Remark 4.1. Note that the morphism $\mathbf{r}^{j'} : X_\lambda \to \mathbf{X}$ is given by some morphisms $r^{j'}_{\lambda'} : X_\lambda \to X_{\lambda'}$ such that if $\lambda'_1 \leq \lambda'_2$ then $r^{j'}_{\lambda'_1} = p_{\lambda'_1\lambda'_2} \circ r^{j'}_{\lambda'_2}$. The relation $f^{j'}_{\mu\lambda} = \mathbf{f}^{j'}_\mu \circ \mathbf{r}^{j'}$ means $f^{j'}_{\mu\lambda} = f^{j'}_\mu \circ r^{j'}_{\phi(\mu)}$. Therefore, $\lambda' \geq \phi(\mu)$ implies $f^{j'}_{\mu\lambda} = f^{j'}_\mu \circ p_{\phi(\mu)\lambda'} \circ r^{j'}_{\lambda'} = f^{j'}_{\mu\lambda'} \circ r^{j'}_{\lambda'}$. In this way we have that uniform J-co-movability implies J-co-movability.

Remark 4.2. If (λ, j) is a co-movability (uniform co-movability) pair of μ relative to the J-morphism (f^j_μ, ϕ) then so is any pair $(\widetilde{\lambda}, \widetilde{j})$, with $\widetilde{\lambda} \geq \lambda$ and $j \geq j$.

Definition 4.3. A J-morphism $(f^j_\mu, \phi) : ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda) \to ((Y_\mu, *), q_{\mu\mu'}, M)$ of pointed sets is said to have the *Mittag-Leffler property* provided every $\mu \in M$ admits a pair (λ, j), $\lambda \geq \phi(\mu)$, $j \in J$, (an ML pair for μ with respect to (f^j_μ, ϕ)), such that for any $\lambda' \in \Lambda$, with $\lambda' \geq \lambda$, and $j' \geq j$ one has

$$f^{j'}_{\mu\lambda'}(X_{\lambda'}) = f^{j'}_{\mu\lambda}(X_\lambda). \tag{4.3}$$

Note that if $J = \{1\}$ and (f^j_μ, ϕ) is replaced by $1_{(X,*)}$ we obtain the Mittag-Leffler property for an inverse sistems in the category Set_* (cf. [17], Ch. II, §6.2).

Theorem 4.1. *A J-morphism of inverse systems of pointed sets is co-movable if and only if it has the Mittag-Lefler property.*

Proof. Let $(f^j_\mu, \phi) : ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda) \to ((Y_\mu, *), q_{\mu\mu'}, M)$ be a J-morphism with the Mittag-Leffler property. Then for $\mu \in M$ there is a ML pair (λ, j), $\lambda \geq \phi(\mu)$

such that (4.3) holds for each $\lambda' \geq \lambda$ and $j' \geq j$. We can prove that (λ, j) is a co-movability pair of μ with respect to (f^j_μ, ϕ). If $\lambda' \geq \phi(\mu)$ and $\lambda' \geq \lambda$, the relation (4.3) defines a map of pointed sets $r^{j'} : (X_\lambda, *) \to (X_{\lambda'}, *)$ such that $f^{j'}_{\mu\lambda'} \circ r^{j'} = f^{j'}_{\mu\lambda}$. For any other $\lambda'' \geq \phi(\mu)$, one choose $\lambda''' \geq \lambda''$, $\phi(\mu)$ and consider $r'^{j'} : X_\lambda \to X_{\lambda'''}$ such that $f^{j'}_{\mu\lambda'''} \circ r'^{j'} = f^{j'}_{\mu\lambda}$. Then the composition $r^{j'} := p_{\lambda''\lambda'''} \circ r'^{j'}$ satisfies the relation $f^{j'}_{\mu\lambda''} \circ r^{j'} = f^{j'}_{\mu\lambda''} \circ p_{\lambda''\lambda'''} \circ r'^{j'} = f^{j'}_{\mu\lambda'''} \circ r'^{j'} = f^{j'}_{\mu\lambda}$.

Conversely, let (f^j_μ, ϕ) be a co-movable J-morphism. Let $\mu \in M$ and $\lambda \in \Lambda$ with $\lambda \geq \phi(\mu)$ and $j \in J$ a co-movability pair of μ with respect to (f^j_μ, ϕ). Then, for $\lambda' \geq \lambda$ and $j' \geq j$ there exists $r^{j'} : (X_\lambda, *) \to (X_{\lambda'}, *)$ such that $f^{j'}_{\mu\lambda'} \circ r^{j'} = f^{j'}_{\mu\lambda}$. This implies the inclusion $f^{j'}_{\mu\lambda}(X_\lambda) \subseteq f^{j'}_{\mu\lambda'}(X_{\lambda'})$. The converse inclusion follows from the relation $f^{j'}_{\mu\lambda} \circ p_{\lambda\lambda'} = f^{j'}_{\mu\lambda'}$. □

Proposition 4.1. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in a the category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-morphism of inverse systems. If $\mathbf{X}$ is a movable (uniformly movable) inverse system and $\mathbf{Y}$ is an arbitrary inverse system, then (f^j_μ, ϕ) is a co-movable (uniformly co-movable) J-morphism.*

Proof. It is easy to prove that if $\mu \in M$ and $\lambda \in \Lambda$ is a movability (uniform movability) index for $\phi(\mu)$, then a pair (λ, j) with an arbitrary $j \in J$ is a co-movability (uniform co-movability) pair for μ with respect to the J-morphism (f^j_μ, ϕ). □

Theorem 4.2. *An inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is movable (uniformly movable) if and only if the identity J-morphism $1^J_{\mathbf{X}}$ is co-movable (uniformly co-movable) for an arbitrary directed partially ordered set J.*

Proof. If X is movable (uniformly movable), then by Proposition 4.1 the morphism $1^J_{\mathbf{X}}$ is co-movable (uniformly co-movable). Conversely, let $1^J_{\mathbf{X}}$ be a co-movable (uniformly co-movable) J-morphism and let (λ', j) be a co-movability (uniform co-movability) pair of a given $\lambda \in \Lambda$ with respect to $1^J_{\mathbf{X}} = (1_{X_\lambda}, 1_\Lambda)$. It is easy to verify that λ' is a movability (uniform movability) index of λ for the inverse system $\mathbf{X}$. □

Using Theorems 4.1, 4.2 and Proposition 4.1, we obtain the following corollary (see [17], Ch. II, §6.2, Corollary 4).

Corollary 4.1. *An inverse system of pointed set $(\mathbf{X}, *)$ is movable if and only if it has the Mittag-Leffler property, in particular, if all bonding functions are surjective.*

The following theorem is a generalization of Proposition 4.1.

Theorem 4.3. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ be inverse systems in the category $\mathcal{C}$ and let $(f^j_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$, $(g^j_\nu, \psi) : \mathbf{Y} \to \mathbf{Z}$ be*

J-morphisms. Suppose that (f^j_μ,ϕ) is co-movable (uniformly co-movable). Then the composition $(h^j_\nu,\chi)=(g^j_\nu,\psi)\circ(f^j_\mu,\phi)$ is also a co-movable (uniformly co-movable) J-morphism.

Proof. At first we note that $h^j_{\nu\lambda}=g^j_\nu\circ f^j_{\psi(\nu)\lambda}$. Then if (λ,j) is a co-movability pair for $\psi(\nu)$, we have $f^{j'}_{\psi(\nu)\lambda}=f^{j'}_{\psi(\nu)\lambda'}\circ r^{j'}$, for $\lambda'\geq\lambda$ and $j'\geq j$. By this we have $h^{j'}_{\nu\lambda}=g^{j'}_\nu\circ f^{j'}_{\psi(\nu)\lambda'}\circ r^{j'}=h^{j'}_{\nu\lambda'}\circ r^{j'}$, which is the condition of co-movability for J-morphism (h^j_ν,χ). For the property of uniform co-movability the proof is similar.
□

Remark 4.3. The assertion of Corollary 3.1 in the case of co-movability of J-morphisms is false even if $J=\{1\}$. To show this, consider the following inverse sequences of groups:

$$\mathbf{G}=(G_n,p_{nn'}),\text{where } G_n=\mathbb{Z} \text{ and } p_{nn'}(m)=2^{n'-n}m;$$

$$\mathbf{H}=(H_n,q_{nn'}),\text{where } H_n=\oplus^n\mathbb{Z}=\mathbb{Z}\oplus\ldots\oplus\mathbb{Z} \text{ and}$$
$$q_{nn+1}(m_1,\ldots,m_{n'})=(m_1,\ldots,m_{n'-n})$$

The pro-group $\mathbf{H}$ is movable (see [17], Ch.II, §6.1, Example 2).

Now consider the following morphism $(f_n,1_{\mathbb{N}}):\mathbf{G}\to\mathbf{H}$ with

$$f_n:G_n\to H_n,\quad f_n(m)=(2^{n-1}m,2^{n-2}m,\ldots,2m,m).$$

We can verify that in this way we obtain a level morphism of pro-groups. Indeed,

$$(f_n\circ p_{nn'})(m)=f_n(2^{n'-n}m)=(2^{n-1}2^{n'-n}m,2^{n-2}2^{n'-n}m,\ldots,2\cdot2^{n'-n}m,2^{n'-n}m)=$$
$$=(2^{n'-1}m,2^{n'-2}m,\ldots,2^{n'-n+1}m,2^{n'-n}m)$$

and

$$(q_{nn'}\circ f_{n'})(m)=q_{nn'}(2^{n'-1}m,2^{n'-2}m,\ldots,2m,m)=(2^{n'-1}m,2^{n'-2}m,\ldots,2^{n'-n}m).$$

So $f_n\circ p_{nn'}=q_{nn'}\circ f_{n'}$ and hence, $(f_n,1_{\mathbb{N}}):\mathbf{G}\to\mathbf{H}$ is a level morhism of pro-groups.

Now the condition of co-movability (4.1) for the morphism $(f_n,1_{\mathbb{N}})$ becomes

$$f_{nn'}=f_{nn''}\circ r\quad\text{or}\quad f_n\circ p_{nn'}=f_n\circ p_{nn''}\circ r.$$

Consider the last relation written for $n=1$ and $n''=n'+1$:

$$(f_1\circ p_{1n'})(m)=(f_1\circ p_{1n'+1}\circ r)(m)\Leftrightarrow 2^{n'-1}m=2^{n'}r(m)\Leftrightarrow m=2\cdot r(m)\Leftrightarrow r(m)=\frac{m}{2}$$

for any $m\in\mathbb{Z}$, which is impossible because r is an endomorphism of $\mathbf{Z}$. Thus, the morphism $(f_n,1_{\mathbb{N}})$ is not co-movable although $\mathbf{H}$ is movable.

In addition, by Proposition 4.1, we conclude that $\mathbf{G}$ is not movable (the result also proved in [17], Ch.II, §6.1).

Remark 4.4. Since Corollary 3.1 is a consequence of Theorem 3.1, Remark 4.3 suggests that a result for the properties of co-movability and uniform co-movability analogous to that from Theorem 3.1 for movability and uniform movability is false. But imposing for (f_μ^j, ϕ) to be a J-isomorphism, we obtain a positive result.

Theorem 4.4. *Let* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, $\mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ *be inverse systems in a category* $\mathcal{C}$. *Let* $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ *be a* J*-isomorphism and let* $(g_\nu^j, \psi) : \mathbf{Y} \to \mathbf{Z}$ *be a (uniformly) co-movable* J*-morphism. Then the composition* $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ *is also a (uniformly) co-movable* J*-morphism.*

Proof. Without loss of generality, we can assume that $\phi : \Lambda \to M$ is an increasing function [17] (Ch.I, §1.2, Lemma 2). Since $(f_\mu^j, \phi) : \mathbf{X} \to \mathbf{Y}$ be a J-isomorphism we can also assume that ϕ is a bijection. Let $({f'}_\lambda^j, \phi') : \mathbf{Y} \to \mathbf{X}$, where $\phi' = \phi^{-1}$, be an inverse J-morphism of (f_μ^j, ϕ).
Now suppose that (g_ν^j, ψ) is a co-movable J-morhiasm. To prove that the composition $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ is also a co-movable J-morphism, consider an arbitrary $\nu \in N$ and take a co-movability pair (μ, j) of ν with respect to J-morphism (g_ν^j, ψ).

Let's prove that $(\phi(\mu), j)$ is a co-movability pair of ν with respect to J-morphism (h_ν^j, χ). Consider any $\lambda' \geq \chi(\nu)$, $\chi(\nu) = \phi(\psi(\nu))$. Note that $\phi'(\lambda') \geq \psi(\nu)$ because ϕ' is an increasing function. Hence, for any $j' \geq j$ there exists a morphism $r^{j'} : Y_\mu \to Y_{\phi'(\lambda')}$ in the category $\mathcal{C}$ satisfying the relation

$$g_{\nu\mu}^{j'} = g_{\nu\phi'(\lambda')}^{j'} \circ r^{j'}, \text{ i.e., } g_\nu^{j'} \circ q_{\psi(\nu)\mu} = g_\nu^{j'} \circ q_{\psi(\nu)\phi'(\lambda')} \circ r^{j'}. \tag{4.4}$$

Now define the morphism $R^{j'} : X_{\phi(\mu)} \to X_{\lambda'}$ by

$$R^{j'} = {f'}_{\lambda'}^{j'} \circ r^{j'} \circ f_\mu^{j'} \tag{4.5}$$

and prove that $h_{\nu\phi(\mu)}^{j'} = h_{\nu\lambda'}^{j'} \circ R^{j'}$, i.e.,

$$g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\phi(\mu)} = g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\lambda'} \circ R^{j'}. \tag{4.6}$$

Indeed, by (4.4) and (4.5), one has

$$\begin{aligned} g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\lambda'} \circ R^{j'} &= g_\nu^{j'} \circ \left(f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\lambda'}\right) \circ {f'}_{\lambda'}^{j'} \circ r^{j'} \circ f_\mu^{j'} = \\ = g_\nu^{j'} \circ \left(q_{\psi(\nu)\phi'(\lambda')} \circ f_{\phi'(\lambda')}^{j'}\right) \circ {f'}_{\lambda'}^{j'} \circ r^{j'} \circ f_\mu^{j'} &= g_\nu^{j'} \circ q_{\psi(\nu)\phi'(\lambda')} \circ 1_{Y_{\phi'(\lambda')}} \circ r^{j'} \circ f_\mu^{j'} = \\ &= g_\nu^{j'} \circ q_{\psi(\nu)\mu} \circ f_\mu^{j'} = g_\nu^{j'} \circ f_{\psi(\nu)}^{j'} \circ p_{\phi(\psi(\nu))\phi(\mu)}. \end{aligned}$$

So, the composition $(h_\nu^j, \chi) = (g_\nu^j, \psi) \circ (f_\mu^j, \phi)$ is co-movable. In the same way one can prove the case of uniform co-movability. □

5. Properties of movability and co-movability for J-pro-morphisms

Proposition 5.1. *Let* $(f_\mu^j,\phi),({f'}_\mu^j,\phi'):\mathbf{X}=(X_\lambda,p_{\lambda\lambda'},\Lambda)\to\mathbf{Y}=(Y_\mu,q_{\mu\mu'},M)$ *be two equivalent* J*-morphisms of inverse systems.*

(i) If the J*-morphism* (f_μ^j,ϕ) *is movable (uniformly movable) then the* J*-morphism* $({f'}_\mu^j,\phi')$ *is also movable (uniformly movable).*

(ii) If the J*-morphism* (f_μ^j,ϕ) *is co-movable (uniformly co-movable) then the* J*-morphism* $({f'}_\mu^j,\phi')$ *is also co-movable (uniformly co-movable).*

Proof. (i) Suppose that (f_μ^j,ϕ) is J-movable and $(f_\mu^j,\phi)\sim({f'}_\mu^j,\phi')$. We need to prove that (f'^j_μ,ϕ') is also J-movable.

Let $\mu\in M$ be any index. Consider a movability pair (λ,j) of μ with respect to J-morphism (f_μ^j,ϕ). There is no loss of generality in assuming that $\lambda\geq\phi(\mu),\phi'(\mu)$ and

$$f_\mu^j\circ p_{\phi(\mu)\lambda}=f'^j_\mu\circ p_{\phi'(\mu)\lambda}. \tag{5.1}$$

Consider any $\mu'\geq\mu$. By assumption for any $j'\geq j$ there is a morphism $u^{j'}:X_\lambda\to Y_{\mu'}$ such that

$$f_{\mu\lambda}^{j'}=q_{\mu\mu'}\circ u^{j'}. \tag{5.2}$$

Then by (5.1) and (5.2) we have

$$f'^{j'}_{\mu\lambda}=f'^{j'}_\mu\circ p_{\phi'(\mu)\lambda}=f^{j'}_\mu\circ p_{\phi(\mu)\lambda}=f^{j'}_{\mu\lambda}=q_{\mu\mu'}\circ u^{j'}$$

which means that (λ,j) is also movability pair of μ with respect to J-morphism (f'^j_μ,ϕ').

The case of uniform movability is proved similarly.

(ii) Let $\mu\in M$ be any index and let (λ,j) be a co-movability pair for μ with respect to J-morphism (f_μ^j,ϕ). We can assume that $\lambda\geq\phi(\mu),\phi'(\mu)$ and (5.1) holds.

Now we prove that (λ,j) is also a co-movability pair of μ with respect to (f'^j_μ,ϕ'). Let $\lambda'\geq\phi'(\mu)$ be any index and let λ'' be an index with $\lambda''\geq\lambda',\phi(\mu)$ which satisfies

$$f^{j'}_\mu\circ p_{\phi(\mu)\lambda''}=f'^{j'}_\mu\circ p_{\phi'(\mu)\lambda''} \tag{5.3}$$

for given $j'\geq j$. By assumption there is a morphism $r^{j'}:X_\lambda\to X_{\lambda''}$ such that

$$f^{j'}_{\mu\lambda}=f^{j'}_{\mu\lambda''}\circ r^{j'}. \tag{5.4}$$

Define the morphism $R^{j'}:X_\lambda\to X_{\lambda'}$ by

$$R^{j'}=p_{\lambda'\lambda''}\circ r^{j'}. \tag{5.5}$$

By (5.1), (5.3), (5.4), and (5.5) one has

$$f'^{j'}_{\mu\lambda'} \circ R^{j'} = f'^{j'}_{\mu\lambda'} \circ p_{\lambda'\lambda''} \circ r^{j'} = f'^{j'}_{\mu} \circ p_{\phi'(\mu)\lambda'} \circ p_{\lambda'\lambda''} \circ r^{j'} =$$
$$= f^{j'}_{\mu} \circ p_{\phi(\mu)\lambda''} \circ r^{j'} = f^{j'}_{\mu} \circ p_{\phi(\mu)\lambda} = f'^{j'}_{\mu} \circ p_{\phi'(\mu)\lambda} = f'^{j'}_{\mu\lambda},$$

which is the condition for co-movability for the J-morphism (f'^{j}_{μ}, ϕ'). The case of uniform movability can be proved similarly. □

Thanks to Proposition 5.1, we can give the following definition.

Definition 5.1. (i) A J-pro-morphism $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$ is called movable (uniformly movable) if $\mathbf{f}^J$ admits a representation (f^j_μ, ϕ) which is J-movable (uniformly J-movable).

(ii) A J-pro-morphism $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$ is called co- movable (uniformly co-movable) if $\mathbf{f}^J$ admits a representation (f^j_μ, ϕ) which is J-co-movable (uniformly J-co-movable).

The next theorem follows from Theorems 3.1, 3.2, 3.3, 3.4 and Corollaries 3.1, 3.2, 3.4, 3.5.

Theorem 5.1. *A (pre- or post-) composition of an arbitrary J-pro-morphism with a movable (uniformly movable) J-pro-morphism is a movable (uniformly movable) J-pro-morphism. In particular, if $\mathbf{X}$ or $\mathbf{Y}$ is a movable (uniformly movable) inverse system, then $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$ is a movable (uniformly movable) J-pro-morphism.*

Taking into account Corollaries 3.3 and 3.6, we obtain

Proposition 5.2. *Let $\mathbf{X}$ and $\mathbf{Y}$ be J-pro-morphisms. If $\mathbf{Y}$ is a movable (uniformly movable) and $\mathbf{X}$ is dominated by $\mathbf{Y}$ in pro^J-$\mathcal{C}$, then $\mathbf{X}$ is also movable (uniformly movable).*

The following theorem is an immediate consequence of Theorem 4.3

Theorem 5.2. *Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be inverse systems in the category $\mathcal{C}$ and let $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$, $\mathbf{g}^J : \mathbf{Y} \to \mathbf{Z}$ be J-pro-morphisms in pro^J-$\mathcal{C}$. If $\mathbf{f}^J$ is a co-movable (uniformly co-movable) J-pro-morphism, then the composition $\mathbf{h}^J = \mathbf{g}^J \circ \mathbf{f}^J$ is also a co-movable (uniformly co-movable) J-pro-morphism.*

Remark 5.1. It follows from the example from Remark 4.3 that if $\mathbf{g}^J$ is a co-movable J-pro-morphism and $\mathbf{f}^J$ is an arbitrary J-pro-morphism, then the composition $\mathbf{h}^J = \mathbf{g}^J \circ \mathbf{f}^J$ is not necessarily a co-movable J-pro-morphism.

However, the following theorem is true (follows from Theorem 4.4).

Theorem 5.3. *Let* $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ *be inverse systems in the category* $\mathcal{C}$ *and let* $\mathbf{f}^J : \mathbf{X} \to \mathbf{Y}$, $\mathbf{g}^J : \mathbf{Y} \to \mathbf{Z}$ *be* J*-pro-morphisms in* pro^J*-*$\mathcal{C}$*. If* $\mathbf{g}^J$ *is a co-movable (uniformly co-movable)* J*-pro-morphism and* $\mathbf{f}^J$ *is a* J*-pro-isomorphism, then the composition* $\mathbf{h}^J = \mathbf{g}^J \circ \mathbf{f}^J$ *is a co-movable (uniformly co-movable)* J*-pro-morphism.*

6. Properties of movability and co-movability for J-shape morphisms

Consider $(\mathcal{C}, \mathcal{D})$ a pair of categories with $\mathcal{D}$ a dense subcategory of $\mathcal{C}$. If $X, Y \in Ob\mathcal{C}$ and $p : X \to \mathbf{X}$, $q : Y \to \mathbf{Y}$ are $\mathcal{D}$-expansions, then by Remark 2.4 and Definitions 2.3 and 2.4, a J-shape morphism from X to Y is an equivalence class $\langle \mathbf{f} \rangle$ of a J-pro-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$.

Theorem 6.1. *In the above conditions, let* $p' : X \to \mathbf{X}'$ *and* $q' : Y \to \mathbf{Y}'$ *be other* $\mathcal{D}$*-expansions of* X *and* Y*, respectively. If the* J*-pro-morphisms* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ *define the same* J*-shape morphism* $F : X \to Y$ *and if* $\mathbf{f}$ *is a movable (uniformly movable)* J*-pro-morphism, then* $\mathbf{f}'$ *is the same.*

Proof. By Definition 2.3 there exists a commutative diagram

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\mathbf{i}"] \arrow[d, "\mathbf{f}"'] & \mathbf{X}' \arrow[d, "\mathbf{f}'"] \\ \mathbf{Y} \arrow[r, "\mathbf{j}"'] & \mathbf{Y}' \end{tikzcd}$$

where $\mathbf{i}$ and $\mathbf{j}$ are J-pro-isomorphisms. If $\mathbf{f}$ is a movable (uniformly movable), then by Theorem 5.1 the composition $\mathbf{j} \circ \mathbf{f}$ is J-movable (uniformly J-movable). Therefore, by the same theorem, $\mathbf{f}' = (\mathbf{j} \circ \mathbf{f}) \circ \mathbf{i}'^{-1}$ is J-movable (uniformly J-movable). □

Definition 6.1. A J-shape morphism $F : X \to Y$ is called movable (uniformly movable) if it can be represented by a movable (uniformly movable) J-pro-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $F = \langle \mathbf{f} \rangle$.

Theorem 6.2. *With the notation from Theorem 6.1, if* $\mathbf{f}$ *is a co-movable (uniformly co-movable)* J*-pro-morphism, then* $\mathbf{f}'$ *is the same.*

Proof. As above we have $\mathbf{j} \circ \mathbf{f} = \mathbf{f}' \circ \mathbf{i}$. If $\mathbf{f}$ is co-movable (uniformly co-movable), then by Theorem 5.2, the composition $\mathbf{j} \circ \mathbf{f}$ is co-movable (uniformly co-movable). Then $\mathbf{f}' = (\mathbf{j} \circ \mathbf{f}) \circ \mathbf{i}^{-1}$ is co-movable (uniformly co-movable) by Theorem 5.3. □

Definition 6.2. A J-shape morphism $F : X \to Y$ is called co-movable (uniformly co-movable) if it can be represented by a co-movable (uniformly co-movable) J-pro-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $F = \langle \mathbf{f} \rangle$.

Remark 6.1. All properties of movability (uniform movability) and co-movability (uniform co-movability) of J-morphisms and J-pro-morphisms of inverse systems can be transferred to appropriate properties for J-shape morphisms and for morphisms in the category $\mathcal{C}$ of a shape theory $Sh^J_{(\mathcal{C},\mathcal{D})}$. For example, by Theorems 5.1, 5.2, and 5.3 we obtain the following theorem.

Theorem 6.3. *(i) A (pre-or post-) composition of an arbitrary J-shape morphism with a movable (uniformly movable) J-shape morphism is a movable (uniformly movable) J-shape morphism. In particular, if X or Y is a movable (uniformly movable) object, then any J-shape morphism $F : X \to Y$ is movable (uniformly movable);*

(ii) Let $F : X \to Y$, $G : Y \to Z$ be J-shape morphisms in the J-shape category $Sh^J_{(\mathcal{C},\mathcal{D})}$. If F is co-movable (uniformly co-movable), then the composition $H = G\circ F$ also is co-moavble (uniformly co-movable).

(iii) If $F : X \to Y$ is a J-shape isomorphism and $G : Y \to Z$ is a co-movable (uniformly co-movable) J-shape morphism, then $H = G\circ F$ is a co-movable (uniformly co-movable) J-shape morphism.

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