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ON BENIGN SUBGROUPS CONSTRUCTED BY HIGMAN'S SEQUENCE BUILDING OPERATION

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Abstract. For Higman's sequence building operation ω_m and for any integer sequences set $\mathcal{B}$ the subgroup $A_{\omega_m \mathcal{B}}$ is benign in a free group G as soon as $A_{\mathcal{B}}$ is benign in G. Higman used this property as a key step to prove that a finitely generated group is embeddable into a finitely presented group if and only if it is recursively presented. We build the explicit analog of this fact, i.e., we explicitly give a finitely presented overgroup $K_{\omega_m \mathcal{B}}$ of G and its finitely generated subgroup $L_{\omega_m \mathcal{B}} \le K_{\omega_m \mathcal{B}}$ such that $G \cap L_{\omega_m \mathcal{B}} = A_{\omega_m \mathcal{B}}$ holds. Our construction can be used in explicit embeddings of finitely generated groups into finitely presented groups, which are theoretically possible by Higman's theorem. To build our construction we suggest some auxiliary "nested" free constructions based on free products with amalgamation and HNN-extensions.

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1. Introduction

Higman's fundamental result establishing connection between group theory and computability theory states: *a finitely generated group G can be embedded in a finitely presented group if and only if it is recursively presented* [9]. The requirement that G is finitely generated is not critical, and it can be replaced by the condition that G has an effectively enumerable countable set of generators, see the remark on p. 456 in [9].

Despite importance of this theorem, possibility of *explicit* embedding of any recursively presented group into some finitely presented group is a less intelligible issue, and it is open problem even for some well known groups. In particular, construction of an *explicit* embedding of the additive group $\mathbb{Q}$ of rationals into a finitely presented group was an open question mentioned by Bridson and de la Harpe as "Well-known problem" 14.10 (a) in Kourovka notebook [12] and also announced in [8]. Recently a direct solution to that problem was found by Belk, Hyde and Matucci in [6]; and an algorithm how to build such an explicit embedding was

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given in [17], *without* an explicit finitely presented group containing $\mathbb{Q}$ though. Also, based on recent work [2]–[5] it is possible to embed $\mathbb{Q}$ as a center for a continuum of non-isomorphic 2-generator groups. These along with some other remarks in the literature [19, 1] motivate research on explicit embeddings of recursively presented groups into finitely presented groups.

The key group-theoretic concept introduced in [9] is that of *benign subgroup*: a subgroup H is benign in a finitely generated group G, if there is a finitely presented overgroup K of G, and a finitely generated subgroup L of K such that $G \cap L = H$. Actually, the most part of [9] is dedicated to showing that if a subgroup H of a specific type is benign in the free group $G = \langle a, b, c\rangle$ of rank 3, then applying some specific kinds of operations to H (such as, the sequence building operation ω_m, see below) we again get a benign subgroup in G.

Denote by $\mathcal{E}$ the set of all functions $f : \mathbb{Z} \to \mathbb{Z}$ with finite supports. If $f(i) = 0$ for all $i < 0$ and $i \geq m$ (for a fixed $m = 1, 2, \ldots$), then f can be recorded as a sequence $f = (j_0, \ldots, j_{m-1})$ assuming $f(i) = j_i$ for $i = 0, \ldots, m-1$ [9]. Then the following words are defined in the free group $G = \langle a, b, c\rangle$ with respect to f:

$$b_f = b_0^{j_0} \cdots b_{m-1}^{j_{m-1}} \quad \text{and} \quad a_f = a^{b_f} = b_f^{-1} a\, b_f \tag{1.1}$$

where $b_i = b^{c^i}$ for $i = 1, \ldots, m-1$. Let $\mathcal{E}_m$ be the subset of all functions f of the above type. For any subset $\mathcal{B}$ of $\mathcal{E}$ denote $A_{\mathcal{B}} = \langle a_f \mid f \in \mathcal{B}\rangle$, in particular, $A_{\mathcal{E}_m} = \langle a_f \mid f \in \mathcal{E}_m\rangle$. See details and examples in [17].

For m and for any subset $\mathcal{B} \subseteq \mathcal{E}$ the *sequence building* operation ω_m is defined on $\mathcal{B}$ as follows: $\omega_m(\mathcal{B})$ consists of all $f \in \mathcal{E}$ for which for every $i \in \mathbb{Z}$ there exists a sequence $\big(f(mi+0), \ldots, f(mi+m-1)\big) \in \mathcal{B}$ [9]. In other words, this operation just constructs new sequences f by concatenation of some sequences of length m picked from $\mathcal{B}$. For details see [17], and also check Section 3 below where the new sequence (3.4) is built from the sequences $(6, 4, 5, 3), (7, 2, 4, 9) \in \mathcal{B}$ and from the zero sequence using ω_4. Having the subgroup $A_{\mathcal{B}} = \langle a_f \mid f \in \mathcal{B}\rangle$ of G one may construct the subgroup $A_{\omega_m\mathcal{B}} = \langle a_{\omega_m\mathcal{B}} \mid f \in \mathcal{B}\rangle$. And if $\mathcal{B} \subseteq \mathcal{E}_m$, then $A_{\mathcal{B}} \leq A_{\omega_m\mathcal{B}}$, see subsection 3.2 where samples of $A_{\mathcal{B}}$ and $A_{\omega_4\mathcal{B}}$ are given.

If for some $\mathcal{B} \subseteq \mathcal{E}$ the group $A_{\mathcal{B}}$ is benign in G for a given finitely presented overgroup K holding G, and for the finitely generated subgroup $L \leq K$, we stress that by denoting $K = K_{\mathcal{B}}$ and $L = L_{\mathcal{B}}$, and writing $G \cap L_{\mathcal{B}} = A_{\mathcal{B}}$ in $K_{\mathcal{B}}$. Clearly, $K_{\mathcal{B}}$ and $L_{\mathcal{B}}$ may not be unique for a given $\mathcal{B}$.

A main strategy of [9] is to start from a set $\mathcal{B} \subseteq \mathcal{E}$ for which the subgroup $A_{\mathcal{B}}$ is benign in G, and to show that if a new set $\mathcal{B}'$ is obtained from $\mathcal{B}$ by means of certain operations, then $A_{\mathcal{B}'}$ also is benign in G. In this terms [9, Lemma 4.10] states that if for the given $\mathcal{B} \subseteq \mathcal{E}$ the subgroup $A_{\mathcal{B}}$ is benign in G, then $A_{\omega_m \mathcal{B}}$ also is benign in G for any m.

The objective of this note is to additionally show that if the respective groups $K_{\mathcal{B}}$ and $L_{\mathcal{B}}$ can be constructed *explicitly*, then $K_{\omega_m \mathcal{B}}$ and $L_{\omega_m \mathcal{B}}$ can also be constructed *explicitly*:

Theorem 1.1. *Let $\mathcal{B} \subseteq \mathcal{E}$ be a sequences set such that $A_{\mathcal{B}}$ is benign in G and, moreover, the respective finitely presented group $K_{\mathcal{B}}$ and its finitely generated subgroup $L_{\mathcal{B}}$ are given explicitly. Then for any $m = 1, 2, \ldots$ the subgroup $A_{\omega_m \mathcal{B}}$ also is benign in G, and the finitely presented group $K_{\omega_m \mathcal{B}}$ and its finitely generated subgroup $L_{\omega_m \mathcal{B}}$ can also be given explicitly.*

The promised explicit group $K_{\omega_m \mathcal{B}}$ is given in (5.9), $L_{\omega_m \mathcal{B}}$ is given in (5.10), while the components $\Psi, \bar{\Delta}, L'$, etc., used in those formulas all are defined in Section 5 using some free constructions. And under explicitly given $K_{\mathcal{B}}$ and $L_{\mathcal{B}}$ one may understand, say, their presentations with generators and defining relations.

The proof of this theorem occupies sections 3–5 below. In particular, in Section 3 we build an initial embedding construction in which $A_{\omega_m \mathcal{B}}$ is an intersection of G with certain subgroup $W_{\mathcal{B}}$. As this construction is not yet finitely presented, we in Section 4 suggest some auxiliary "nested" free constructions (such as (4.3)), and using them we obtain the finitely presented $K_{\omega_m \mathcal{B}}$ in Section 5.

In order to avoid any repetition of material already published in [16, 17] or elsewhere, we below often adopt constructions from other work. This makes parts of the current text dependant on other articles, but the provided exact references, we hope, alleviate any inconvenience.

2. Preliminary information

2.1. **Free constructions.** For background information on free products with amalgamation and on HNN-extensions we refer to [7] and [10]. Notations vary in the literature, and to maintain uniformity we are going to adopt notations we used in [16].

If any groups G and H have subgroups, respectively, A and B isomorphic under $\varphi : A \to B$, then the (generalized) free product of G and H with amalgamated subgroups A and B is denoted by $G *_\varphi H$ (an alternative notation in the literature being $G *_{A=B} H$). When G and H are overgroups of the same subgroup A, and φ is just the *identical* isomorphism on A, we write $\Gamma = G *_A H$.

If G has subgroups A and B isomorphic under $\varphi : A \to B$, then the HNN-extension of the base G by some stable letter t with respect to the isomorphism φ

is denoted by $G *_\varphi t$. In case when $A = B$ and φ is *identity* on A, we may write $\Gamma = G *_A t$. We also use HNN-extensions $G *_{\varphi_1,\varphi_2,\ldots} (t_1, t_2, \ldots)$ with more than one stable letters, see [16] for details.

Our usage of the *normal forms* in free constructions is close to [7].

2.2. **Benign subgroups and Higman operations.** For detailed informaton on bening subgroups we refer to Sections 3, 4 in [9], see also Section 3 in [16]. Higman operations and their basic properties can be found in Section 2 in [9], see also Section 3 in [16] and Section 2 in [17].

From definition of bening subgroup it is very easy to see that arbitrary finitely generated subgroup H in any finitely presented group G is benign in G, for, the group G itself acts as a finitely presented overgroup of G with a finitely generated subgroup H, such that $H \cap H = H$. We are going to often use this remark in the sequel.

2.3. **Subgroups in free constructions.** The following two auxiliary facts are adopted from [16], and they follow from more general Lemma 2.2 and Lemma 2.4 in [16].

Corollary 2.1 (Corollary 2.3 in [16])**.** *Let $\Gamma = G *_A H$, and let $G' \le G$, $H' \le H$ be subgroups such that $G' \cap A = H' \cap A$. Then for $\Gamma' = \langle G', H' \rangle$ and $A' = G' \cap A$ we have:*

1. *$\Gamma' = G' *_{A'} H'$, in particular, if $A \le G', H'$, then $\Gamma' = G' *_A H'$;*
2. *$\Gamma' \cap A = A'$, in particular, if $A \le G', H'$, then $\Gamma' \cap A = A$;*
3. *$\Gamma' \cap G = G'$ and $\Gamma' \cap H = H'$.*

Corollary 2.2 (Corollary 2.5 in [16])**.** *Let $\Gamma = G *_A t$, and let $G' \le G$ be a subgroup. Then for $\Gamma' = \langle G', t \rangle$ and $A' = G' \cap A$ we have:*

1. *$\Gamma' = G' *_{A'} t$, in particular, if $A \le G'$, then $\Gamma' = G' *_A t$;*
2. *$\Gamma' \cap A = A'$, in particular, if $A \le G'$, then $\Gamma' \cap A = A$;*
3. *$\Gamma' \cap G = G'$.*

Remark 2.1. It is easy to adapt Corollary 2.2 for the case of multiple stable letters $t_1, \ldots, t_k$ which fix the same subgroup A in G. In such a case point (1) in Corollary 2.2 will read: $\Gamma' = G' *_{A'} (t_1, \ldots, t_k)$ for $\Gamma' = \langle G', t_1, \ldots, t_k \rangle$ and $A' = G' \cap A$. We are going to use this fact only once, in the proof of Lemma 5.4.

2.4. **The "conjugates collecting" process.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be some disjoint subsets in any group G. Then any element $w \in \langle \mathfrak{X}, \mathfrak{Y} \rangle$ can be written as:

$$w = u \cdot v = x_1^{\pm v_1} x_2^{\pm v_2} \cdots x_k^{\pm v_k} \cdot v$$

with some $v_1, v_2, \ldots, v_k,\ v \in \langle \mathfrak{Y} \rangle$, and $x_1, x_2, \ldots, x_k \in \mathfrak{X}$. The proof, examples and variations of this fact can be found in Subsection 2.6 in [16]. We use the name "conjugates collecting" just because we heavily used it in [16], and we need a name to refer to (we were unable to find a conventional name to this in the literature).

3. The initial embedding construction

3.1. **Construction of Δ.** The free group $\langle b, c\rangle$ contains a free subgroup $\langle b_i \mid i \in \mathbb{Z}\rangle$ of infinite rank, which for any $m = 1, 2, \ldots$ decomposes into a free product $B_m * \bar{B}_m$ with $B_m = \langle \ldots b_{-2}, b_{-1};\ b_m, b_{m+1}, \ldots\rangle$ and $\bar{B}_m = \langle b_0, \ldots, b_{m-1}\rangle$.

Introducing three stable letters g, h, k, all fixing B_m, build the HNN-extension:

$$\Gamma = \langle b, c\rangle *_{B_m} (g, h, k). \tag{3.1}$$

Denote $\bar{G} = \langle g, h, k\rangle$, and in analogy with b_i, b_f, a_f of (1.1) introduce $h_i = h^{k^i},\ h_f$, and $g_f = g^{h_f}$ in the free group $\bar{G}$. Fixing the subgroup $R = \langle g_f b_f^{-1} \mid f \in \mathcal{E}_m\rangle$ of Γ by means of a new stable letter a build the HNN-extension $\Gamma *_R a$.

The intersection $\langle b, c\rangle \cap R$ is trivial because the non-trivial words of type $g_f b_f^{-1}$ generate R freely, and so any non-trivial word they generate must involve at least one g, and hence it need to be outside $\langle b, c\rangle$. Then by (1) in Corollary 2.2 the subgroup generated in $\Gamma *_R a$ by $\langle b, c\rangle$ together with a is equal to $\langle b, c\rangle *_{\langle b,c\rangle \cap R} a = \langle b, c\rangle *_{\{1\}} a = \langle b, c\rangle * a$ which is the free group $G = \langle a, b, c\rangle$. So a, b, c generate a free subgroup in Γ, and hence the map sending a, b, c to $a, b^{c^m}\!, c$ can be continued to an isomorphism $\rho : G \to \langle a, b^{c^m}\!, c\rangle$. Identifying this ρ to a further stable letter r we arrive to the final HNN-extension of this section:

$$\Delta = \big(\Gamma *_R a\big) *_\rho r = \Big(\big(\langle b, c\rangle *_{B_m} (g, h, k)\big) *_R a\Big) *_\rho r. \tag{3.2}$$

3.2. **Obtaining $G \cap W_{\mathcal{B}} = A_{\omega_m \mathcal{B}}$ in Δ.** For any subset $\mathcal{B}$ of $\mathcal{E}$ denote $W_{\mathcal{B}} = \langle g_f,\ a,\ r \mid f \in \mathcal{B}\rangle$, and show that in Δ we have

$$G \cap W_{\mathcal{B}} = A_{\omega_m \mathcal{B}}. \tag{3.3}$$

Firstly notice that if $\mathcal{B}_m = \mathcal{B} \cap \mathcal{E}_m$, then $\omega_m(\mathcal{B}) = \omega_m(\mathcal{B}_m)$. Hence we may without loss of generality suppose $\mathcal{B} \subseteq \mathcal{E}_m$ below (if a *short* sequence contains less than m integers, we can without loss of generality extend its length to m by adding some extra 0's at the end).

For arbitrary sequence $f \in \omega_m \mathcal{B}$ the element $a_f = a^{b_f}$ is inside $W_{\mathcal{B}}$. Let us display this uncomplicated fact by a routine step-by-step construction example. Let $m = 4$ and let $(6, 4, 5, 3),\ (7, 2, 4, 9) \in \mathcal{B}$. Then by sequence building operations $\omega_4 \mathcal{B}$ contains the sequence, say,

$$f = (0,0,0,0,\ \ 7,2,4,9,\ \ 0,0,0,0,\ \ 0,0,0,0,\ \ 6,4,5,3,\ \ 7,2,4,9). \tag{3.4}$$

To show that $a^{b_f} \in W_{\mathcal{B}}$ start by the initial functions $l_1 = (7,2,4,9)$ and $l_2 = (6,4,5,3)$ in $\mathcal{B}$, and then use them by a few steps to arrive to the function f above. We are going to use the evident fact that the relation $(g_f b_f^{-1})^{\,a} = g_f b_f^{-1}$ is equivalent to $a^{g_f} = a^{b_f}$.

Step 1. Since $l_1 = (7,2,4,9)$ is in $\mathcal{B}$, then $g_{l_1} \in W_{\mathcal{B}}$, and so $a^{g_{l_1}} = a^{b_{l_1}} = a^{b_0^7\, b_1^2 b_2^4 b_3^9} \in W_{\mathcal{B}}$.

Step 2. Since $b_i^r = b_i^\rho = (b^\rho)^{(c^i)^\rho} = (b^{c^4})^{c^i} = b^{c^{i+4}} = b_{i+4}$, then conjugating the above obtained element $a^{b_{l_1}}$ by r we get:

$$\left(a^{b_{l_1}}\right)^r = \left(a^r\right)^{(b_0^7\, b_1^2 b_2^4 b_3^9)^{\,r}} = a^{b_4^7\, b_5^2 b_6^4 b_7^9} = a^{b_0^0\, b_1^0 b_2^0 b_3^0 \cdot b_4^7\, b_5^2 b_6^4 b_7^9} = a^{b_{l_3}} \in W_{\mathcal{B}}$$

for the sequence $l_3 = (0,0,0,0,\ 7,2,4,9)$.

Next, conjugating $a^{b_{l_3}}$ by g_{l_2} we have:

$$\left(a^{b_{l_3}}\right)^{g_{l_2}} = a^{b_{l_3} \cdot g_{l_2}} = a^{b_4^7\, b_5^2 b_6^4 b_7^9 \cdot g_{l_2}}.$$

Step 3. Each of stable letters g, h, k commutes with any b_i for $i < 0$ or $i \geq m = 4$, and so g_{l_2} commutes with $b_4^7\, b_5^2 b_6^4 b_7^9$ and so:

$$a^{b_4^7\, b_5^2 b_6^4 b_7^9 \cdot g_{l_2}} = a^{g_{l_2} \cdot b_4^7\, b_5^2 b_6^4 b_7^9}.$$

Then once more applying step 1 to $a^{g_{l_2}}$ we transform the above to:

$$\left(a^{g_{l_2}}\right)^{b_4^7\, b_5^2 b_6^4 b_7^9} = a^{b_0^6\, b_1^4 b_2^5 b_3^3 \cdot b_4^7\, b_5^2 b_6^4 b_7^9} = a^{b_{l_4}}$$

for the sequence $l_4 = (6,4,5,3,\quad 7,2,4,9)$. Then we repeat the above step 2 for *three times* i.e., conjugate the above by r^3 to get the element $a^{b_{l_5}}$ for the sequence:

$$l_5 = (0,0,0,0,\ \ 0,0,0,0,\ \ 0,0,0,0,\ \ 6,4,5,3,\ \ 7,2,4,9).$$

Next apply step 3 and step 1 again to conjugate $a^{b_{l_5}}$ by g_{l_1}. We get the element $a^{b_{l_6}}$ for the sequence:

$$l_6 = (7,2,4,9,\ \ 0,0,0,0,\ \ 0,0,0,0,\ \ 6,4,5,3,\ \ 7,2,4,9).$$

Then we again apply step 2, i.e., conjugate $a^{b_{l_6}}$ by r to discover in $W_{\mathcal{B}}$ the element $a^{b_f} = a_f$ with the sequence f promised in (3.4) above.

Since such a procedure can easily be performed for an *arbitrary* $f \in \omega_m\mathcal{B}$, we get that $A_{\omega_m\mathcal{B}} \leq W_{\mathcal{B}}$. And since also $A_{\omega_m\mathcal{B}} \leq G$, we have $A_{\omega_m\mathcal{B}} \leq G \cap W_{\mathcal{B}}$.

Next assume some word w from $W_{\mathcal{B}} = \langle g_f,\, a,\, r \mid f \in \mathcal{B}\rangle$ is in G. Since w also is in Δ, it can be brought to its normal form involving stable letter r and some elements from $\Gamma *_L a$. The latter elements, in turn, can be brought to normal forms involving stable letter a and some elements from Γ. Then the latters can further be brought to normal forms involving stable letters g, h, k and some elements from $\langle b, c\rangle$. That is,

w can be brought to a *unique "nested" normal form* reflecting three "nested" HNN-extensions in the right-hand side of (3.2). Let us detect the cases when it involves nothing but the letters a, b, c. The only relations of Γ involve g, h, k, and they are equivalent to $a^{g_f} = a^{b_f}$. Thus, the only way by which g, h, k may be eliminated in the normal form is to have in w subwords of type $g_f^{-1} a\, g_f = a^{g_f}$ which can be replaced by respective subwords $a^{b_f} \in G$. If after this procedure some subwords g_f still remain, then three scenario cases are possible:

Case 1. The word w may contain a subword of type $w' = g_f^{-1} a^{b_l} g_f$ for such an l that $l(i) = 0$ for $i = 0, \ldots, m{-}1$. Check the example of step 1, when this is achieved for $l = l_3 = (0, 0, 0, 0,\ \ 7, 2, 4, 9)$ and $f = l_2 = (6, 4, 5, 3)$. Then just replace w' by $a^{b_{l'}}$ for an $l' \in \omega_m \mathcal{B}$ (such as $l' = l_4 = (6, 4, 5, 3,\ \ 7, 2, 4, 9)$ in our example).

Case 2. If $w' = g_f^{-1} a^{b_l} g_f$, but the condition $l(i) = 0$ fails for an $i = 0, \ldots, m-1$, then g_f does *not* commute with b_l, so we cannot apply the relation $a^{g_f} = a^{b_f}$, and so $w \notin G$. Turning to example in steps 1–3, notice that for, say, $f = (7, 2, 4, 9) \in \mathcal{B}$ we may *never* get something like $a^{(g_f)^2} = \big(a^{b_0^7\, b_1^2\, b_2^4\, b_3^9}\big)^{g_f} = a^{(b_0^7\, b_1^2\, b_2^4\, b_3^9)^2}$ because g_f does not commute with b_0, b_1, b_2, b_3. That is, all the *new* functions l we get are from $\omega_m \mathcal{B}$ *only*.

Case 3. If g_f is in w, but is not in a subword $g_f^{-1} a^{b_l} g_f$, we again have $w \notin G$, unless all such g_f trivially cancel each other.

This means, if $w \in G$, then elimination of g, h, k turns w to a product of elements from $\langle r \rangle$ and of some a^{b_f} for some $f \in \omega_m \mathcal{B}$ (a also is of that type, as $(0) \in \mathcal{B}$). Now apply 2.4 for $\mathfrak{X} = \{a^{b_f} \mid f \in \omega_m \mathcal{B}\}$ and $\mathfrak{Y} = \{r\}$ to state that w is a product of some power r^i and of some elements each of which is an a^{b_f} conjugated by a power r^{n_i} of r. These conjugates certainly are in $\omega_m \mathcal{B}$ (see step 2 above), and so $w \in G$ if and only if $i = 0$, i.e., if $w \in A_{\omega_m \mathcal{B}}$.

Hence, equality (3.3) is established for any subset $\mathcal{B}$ of $\mathcal{E}$.

Remark 3.1. However, (3.3) cannot yet guarantee that $A_{\omega_m \mathcal{B}}$ is benign in G as soon as $A_{\mathcal{B}}$ is benign in G, because the group Δ in (3.2) is not finitely presented, and its subgroup $W_{\mathcal{B}} = \langle g_f,\, a,\, r \mid f \in \mathcal{B} \rangle$ may not be finitely generated, when $\mathcal{B}$ is infinite. The sections below will add these missing features replacing Δ by a much bulkier construction.

4. Auxiliary free constructions

In this section we generalize some of the results in Section 3 in [9]. Hence, the lemmas below may be of some independent interest also.

For any subgroup A of an arbitrary group G the well known equality $G \cap G^t = A$ holds in the HNN-extension $G *_A t$. It trivially follows, say, from uniqueness of the normal form in $G *_A t$. We need the following generalization of this fact:

Lemma 4.1. *Let $A_1, \dots, A_r$ be arbitrary subgroups in a group G. Then the following equality holds in the HNN-extension $G *_{A_1,\dots,A_r} (t_1, \dots, t_r)$:*

$$G \cap G^{t_1 \cdots t_r} = \bigcap_{i=1}^r A_i. \tag{4.1}$$

Proof. Choose a transversal T_{A_i} to A_i in G, $i = 1, \dots, r$. Take any $g \in G$, and show that if $g^{t_1 \cdots t_r} \in G$, then g is inside each of A_i. Write $g = a_1 l_1$ where $a_1 \in A_1$ and $l_1 \in T_{A_1}$. In turn, a_1 can be written as $a_1 = a_2 l_2$ where $a_2 \in A_2$ and $l_2 \in T_{A_1}$. This process can be continued for $A_3, \dots, A_r$. (the case when some of a_i or l_i, $i = 1, \dots, r$, are trivial is *not* ruled out). Since the inverse t_i^{-1} of the stable letter t_i also fixes A_i, calculation of the normal form for $g^{t_1 \cdots t_r}$ can be started via the following steps:

$$\begin{aligned} g^{t_1 \cdots t_r} &= t_r^{-1} \cdots t_1^{-1} a_1 l_1 \, t_1 \cdots t_r \\ &= t_r^{-1} \cdots t_2^{-1} a_1 t_1^{-1} l_1 \, t_1 \cdots t_r \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= a_r \, t_r^{-1} l_r \, t_{r-1}^{-1} l_{r-1} \cdots \, l_3 \, t_2^{-1} l_2 \, t_1^{-1} l_1 \, t_1 \cdots t_r \end{aligned} \tag{4.2}$$

The above belongs to G only if it contains no stable letters t_i. But the last line of (4.2) does not contain t_1 only when $l_1 = 1$, hence $t_1^{-1} l_1 \, t_1 = 1$, and $t_2^{-1} l_2 t_1^{-1} l_1 \, t_1 \, t_2 = t_2^{-1} l_2 \, t_2$. Then to exclude t_2 we must have $l_2 = 1$, hence $t_2^{-1} l_2 \, t_2 = 1$. At the end we get (4.2) reduced to $a_r t_r^{-1} l_r t_r = a_r$ where $l_r = 1$, and therefore $a_r \in \bigcap_{i=1}^r A_i$.

On the other hand, any $g \in \bigcap_{i=1}^r A_i$ is fixed by each of t_i, and so $g^{t_1 \cdots t_r} = g \in G$, and thus, $\bigcap_{i=1}^r A_i \subseteq G \cap G^{t_1 \cdots t_r}$. □

Another proof of this lemma could be deduced from Corollary 2.1 and Corollary 2.2 (in a manner rather similar to the proof of Lemma 4.3 below), but we prefer this version as it follows from more basic properties already.

Later we are going to use a specific free construction built for a system of groups via HNN-extensions and free products with amalgamation. Namely, let $G \le K_1, \dots, K_r$ be arbitrary groups such that $K_i \cap K_j = G$ for any distinct indices $i, j = 1, \dots, r$. If in each K_i we pick a subgroup L_i, and denote $G \cap L_i = A_i$, $i = 1, \dots, r$, we can build the following "nested" free construction:

$$\Theta = \Big(\cdots \Big(\big((K_1 *_{L_1} t_1) *_G (K_2 *_{L_2} t_2) \big) *_G (K_3 *_{L_3} t_3) \Big) \cdots \Big) *_G (K_r *_{L_r} t_r). \tag{4.3}$$

By these notations:

Lemma 4.2. *In the free construction* Θ *the following equality holds:*

$$\langle G, t_1, \ldots, t_r\rangle = G *_{A_1,\ldots,\, A_r} (t_1, \ldots, t_r).$$

Proof. Applying induction over r we for $r = 2$ have to display $\langle G, t_1, t_2\rangle = G *_{A_1,\, A_2} (t_1, t_2)$ in $\Theta = (K_1 *_{L_1} t_1) *_G (K_2 *_{L_2} t_2)$.

In $K_1 *_{L_1} t_1$ we by (1) in Corollary 2.2 have $\langle G, t_1\rangle = G *_{G \cap L_1} t_1 = G *_{A_1} t_1$. Similarly, $\langle G, t_2\rangle = G *_{A_2} t_2$ in $K_2 *_{L_2} t_2$. And since in Θ the intersection of both $\langle G, t_1\rangle$ and $\langle G, t_2\rangle$ with G clearly is G, we apply (1) in Corollary 2.1 to get:

$$\langle G, t_1, t_2\rangle = \big\langle \langle G, t_1\rangle, \langle G, t_2\rangle \big\rangle = (G *_{A_1} t_1) *_G (G *_{A_2} t_2).$$

But the above amalgamated free prodcut is noting but $G *_{A_1,\, A_2} (t_1, t_2)$, which is trivial to see by listing all the defining relations of both constructions: relations of G followed by relations stating that t_1 fixes the A_1 and t_2 fixes A_2 (plus the relations identifying both copies of G, if we initially assume them to be disjoint).

Next assume the proof is done for $r-1$, i.e.,

$$\langle G, t_1, \ldots, t_{r-1}\rangle = G *_{A_1,\ldots,\, A_{r-1}} (t_1, \ldots, t_{r-1}).$$

Again by (1) in Corollary 2.2 write $\langle G, t_r\rangle = G *_{G \cap L_r} t_r = G *_{A_r} t_r$. We have $\langle G, t_1, \ldots, t_{r-1}\rangle$ and $\langle G, t_r\rangle$ both intersect with G in G, and we by (1) in Corollary 2.1 get:

$$\langle G, t_1, \ldots, t_r\rangle = \big(G *_{A_1,\ldots,\, A_{r-1}} (t_1, \ldots, t_{r-1})\big) *_G (G *_{A_r} t_r) = G *_{A_1,\ldots,\, A_r} (t_1, \ldots, t_r).$$

□

Remark 4.1. The reader familiar with more general interpretations of free products with amalgamation (see Neumann's fundamental survey [18]) would notice that Θ in (4.3) is noting but the free product of the HNN-extensions $K_i *_{L_i} t_i$ with an amalgamated subgroup G. Indeed, the defining relations of this product can well be listed in an order matching the sintaxis of (4.3). Using the terms of [18] would allow us to avoid the bulky formula of (4.3), but it would require to involve here some new elements from [18] which would make the construction more complicated.

An immediate consequence of the above lemmas is:

Corollary 4.1. *If the subgroups* $A_1, \ldots, A_r$ *are benign in a finitely generated group* G*, then their intersection* $\bigcap_{i=1}^r A_i$ *also is benign in* G*. Moreover, if the finitely presented groups* K_i *with their finitely generated subgroups* L_i *can be given for each* A_i *explicitly, then the finitely presented* K *with its finitely generated subgroup* L *can be given for this intersection explicitly.*

Proof. By hypothesis we have some finitely presented overgroups $K_1,\dots,K_r$ of G and finitely generated $L_1,\dots,L_r$ such that $L_i \le K_i$ and $G\cap L_i = A_i$ for each $i=1,\dots,r$. Then the free construction Θ of (4.3) is finitely presented, since to the finitely many relations of K_i we only add the relations stating that t_i fixes the finitely many generators of L_i, plus (if needed) relations identifying the finitely many generators of all copies of G in $K_i *_{L_i} t_i$, $i=1,\dots,r$.

By Lemma 4.2 Θ contains the finitely generated subgroup $\langle G, t_1,\dots,t_r\rangle = G *_{A_1,\dots,A_r} (t_1,\dots,t_r)$, and by Lemma 4.1 we in that group have $G\cap L = \bigcap_{i=1}^r A_i$ for the finitely generated subgroup $L = G^{t_1\cdots t_r}$. $\square$

Lemma 4.3. *Let $A_1,\dots, A_r$ be arbitrary subgroups in a group G. Then in the HNN-extension $G *_{A_1,\dots,A_r} (t_1,\dots,t_r)$ the following equality holds:*

$$G\cap \Big\langle \bigcup_{i=1}^r G^{t_i}\Big\rangle = \Big\langle \bigcup_{i=1}^r A_i\Big\rangle. \tag{4.4}$$

Proof. For simplicity write the proof for the case $r=3$. Set $T=\langle A_1,A_2,A_3\rangle$. By Lemma 4.2:

$$G *_{A_1,A_2,A_3} (t_1,t_2,t_3) = \big((G *_{A_1} t_1) *_G (G *_{A_2} t_2)\big) *_G (G *_{A_3} t_3).$$

$G *_{A_1} t_1$ contains $G *_{A_1} G^{t_1}$, and in this subgroup we by (3) in Corollary 2.1 have $\langle T, G^{t_1}\rangle \cap G = T$. For the same reason $\langle T, G^{t_2}\rangle \cap G = T$. Noticing $\langle T, G^{t_1}, G^{t_2}\rangle = \big\langle \langle T, G^{t_1}\rangle, \langle T, G^{t_2}\rangle\big\rangle$ and applying to it (2) of Corollary 2.1 inside the group $(G *_{A_1} t_1) *_G (G *_{A_2} t_2)$ we have $\langle T, G^{t_1}, G^{t_2}\rangle \cap G = T$. Since also $\langle T, G^{t_3}\rangle \cap G = T$, we again by (2) have

$$\big\langle \langle T, G^{t_1}, G^{t_2}\rangle, \langle T,\, G^{t_3}\rangle\big\rangle \cap G = T.$$

But since

$$T \le \langle G^{t_1}, G^{t_2}, G^{t_3}\rangle,$$

it remains to notice

$$\big\langle \langle T, G^{t_1}, G^{t_2}\rangle, \langle T,\, G^{t_3}\rangle\big\rangle = \langle G^{t_1}, G^{t_2}, G^{t_3}\rangle. \qquad \square$$

Corollary 4.2. *If the subgroups $A_1,\dots, A_r$ are benign in a finitely generated group G, then their join $\big\langle \bigcup_{i=1}^r A_i\big\rangle$ also is benign in G. Moreover, if the finitely presented groups K_i with their finitely generated subgroups L_i can be given for each A_i explicitly, then the finitely presented K with its finitely generated subgroup L can be given for this join explicitly.*

Proof. Using the same constructions and notations as in the proof of Corollary 4.1 just notice that Θ is finitely presented, the join $L = \big\langle \bigcup_{i=1}^r G^{t_i}\big\rangle$ is finitely generated, and $G\cap L = \big\langle \bigcup_{i=1}^r A_i\big\rangle$ by Lemma 4.3. $\square$

5. Adding finite presentation to the construction

5.1. **The HNN-extension Ξ_m.** In free group $\langle b, c\rangle$ we for any integer m can define a pair of isomorphisms ξ_m and ξ'_m via:

$$\xi_m(b) = b_{-m+1}, \quad \xi'_m(b) = b_{-m} \quad \text{and} \quad \xi_m(c) = \xi'_m(c) = c^2. \tag{5.1}$$

It is easy to verify that $\xi_m(b_i) = b_{2i-m+1}$ and $\xi'_m(b_i) = b_{2i-m}$. The pair ξ_m, ξ'_m can be used to define the HNN-extension

$$\Xi_m = \langle b, c\rangle *_{\xi_m,\xi'_m} (t_m, t'_m).$$

Here t_m, t'_m are any stable letters, and the subscript m is used to stress the correlation with ξ_m, ξ'_m, as below we are going to use this construction for multiple values of m.

Lemma 5.1. *In the above notations we for any m have:*

$$\langle b, c\rangle \cap \langle b_m, t_m, t'_m\rangle = \langle b_m, b_{m+1}, \ldots\rangle,$$

$$\langle b, c\rangle \cap \langle b_{m-1}, t_m, t'_m\rangle = \langle b_{m-1}, b_{m-2}, \ldots\rangle.$$

Proof. For any integer m and i we have $b_i^{t_m} = \xi_m(b_i) = b_{2i-m+1}$ and $b_i^{t'_m} = \xi'_m(b_i) = b_{2i-m}$ from where we collect:

$$\begin{gathered}\ldots b_{m-2}^{t_m}=b_{m-3},\quad b_{m-1}^{t_m}=b_{m-1},\quad b_m^{t_m}=b_{m+1},\quad b_{m+1}^{t_m}=b_{m+3},\quad b_{m+2}^{t_m}=b_{m+5},\ldots\\ \ldots b_{m-2}^{t'_m}=b_{m-4},\quad b_{m-1}^{t'_m}=b_{m-2},\quad b_m^{t'_m}=b_m,\quad b_{m+1}^{t'_m}=b_{m+2},\quad b_{m+2}^{t'_m}=b_{m+4},\ldots\end{gathered} \tag{5.2}$$

The action of t_m^{-1} and ${t'_m}^{-1}$ can be deduced from the list above. From (5.2) it is straightforward that each of $b_m, b_{m+1}, \ldots$ indeed is in $\langle b_m, t_m, t'_m\rangle$. Say, $b_{m+8} = b_{m+4}^{t'_m} = b_{m+2}^{{t'_m}^2} = b_{m+1}^{{t'_m}^3} = b_m^{t_m \cdot {t'_m}^3} \in \langle b_m, t_m, t'_m\rangle$.

And on the other hand, bringing any word w on letters b_m, t_m, t'_m to the normal form in HNN-extension Ξ_m we first have to do cancellations like $t_m^{-1} b_m t_m = b_{m+1}$, and ${t'_m}^{-1} b_m t'_m = b_m$. Repeated applications of such steps may create in w some new letters $b_m, b_{m+1}, \ldots$ so that we may also have to do "reverse" cancellations like $t_m b_{m+1} t_m^{-1} = b_m$, $\;t_m b_{m+3} t_m^{-1} = b_{m+1}$, etc... or $\;t'_m b_m {t'}_m^{-1} = b_m$, $\;t'_m b_{m+2} {t'}_m^{-1} = b_{m+1}$, etc... That is, bringing w to normal form we *never* get a b_i outside $\langle b_m, b_{m+1}, \ldots\rangle$. If, in addition, w is in $\langle b, c\rangle$, then the normal form we obtained should contain no letters $t_m^{\pm 1}$ or ${t'_m}^{\pm 1}$. That is, if w is in $\langle b, c\rangle$, it in fact is in $\langle b_m, b_{m+1}, \ldots\rangle$, and we have $\langle b, c\rangle \cap \langle b_m, t_m, t'_m\rangle = \langle b_m, b_{m+1}, \ldots\rangle$.

The second equality stated by the lemma is proved analogously. □

Rules (5.1) define isomorphisms inside the free group $G = \langle a, b, c\rangle$ of rank 3 also, and we can define the HNN-extension $G *_{\xi_m,\xi'_m} (t_m, t'_m)$ which is noting but the ordinary free product $\langle a\rangle * \Xi_m$. Since $G = \langle a\rangle * \langle b, c\rangle$ and the subgroup $\langle b, c\rangle \cap \langle b_m, t_m, t'_m\rangle = \langle b_m, b_{m+1}, \dots\rangle$ involves *no* occurrence of the letter a, we from Lemma 5.1 deduce:

Lemma 5.2. *In the above notations we for any m have:*

$$G \cap \langle b_m, t_m, t'_m\rangle = \langle b_m, b_{m+1}, \dots\rangle,$$
$$G \cap \langle b_{m-1}, t_m, t'_m\rangle = \langle b_{m-1}, b_{m-2}, \dots\rangle.$$

5.2. **Some special benign subgroups.** With above information we obtain three types of benign subgroups:

Corollary 5.1. *In the above notations for any integer m:*

(1) $\langle b_m, b_{m+1}, \dots\rangle$ *is benign in* $\langle b, c\rangle$ *for the finitely presented group* Ξ_m *and its* 3*-generator subgroup* $\langle b_m, t_m, t'_m\rangle$,
(2) $\langle b_{m-1}, b_{m-2}, \dots\rangle$ *is benign in* $\langle b, c\rangle$ *for the finitely presented group* Ξ_m *and its* 3*-generator subgroup* $\langle b_{m-1}, t_m, t'_m\rangle$,
(3) $B_m = \langle \dots b_{-2}, b_{-1};\ b_m, b_{m+1}, \dots\rangle$ *is benign in* $\langle b, c\rangle$ *for the finitely presented group*

$$\Theta_m = \big(\Xi_m *_{\langle b_m, t_m, t'_m\rangle} x\big) *_{\langle b,c\rangle} \big(\Xi_0 *_{\langle b_{-1}, t_0, t'_0\rangle} x'\big)$$

and its 4*-generator subgroup* $P_m = \Big\langle \langle b, c\rangle^x,\ \langle b, c\rangle^{x'} \Big\rangle$.

Proof. Points (1) and (2) directly follow Lemma 5.1.

B_m is the (free) product of $\langle b_m, b_{m+1}, \dots\rangle$ and $\langle b_{-1}, b_{-2}, \dots\rangle$. Hence, point (3) follows from Lemma 4.3 and Corollary 4.2 for $r = 2$, $G = \langle b, c\rangle$, $K_1 = \Xi_m$, $K_2 = \Xi_0$, $L_1 = \langle b_m, t_m, t'_m\rangle$, $L_2 = \langle b_{-1}, t_0, t'_0\rangle$, $A_1 = \langle b_m, b_{m+1}, \dots\rangle$, $A_2 = \langle b_{-1}, b_{-2}, \dots\rangle$. Then Θ_m is noting but the group Θ from (4.3). □

Now we are able to replace our initial Γ from (3.1) by a *finitely presented* alternative:

$$\bar{\Gamma} = \Theta_m *_{P_m} (g, h, k) \tag{5.3}$$

with fixing action for g, h, k on the finitely generated subgroup P_m.

Lemma 5.3. *In above notations the following equalities hold in* $\bar{\Gamma}$:

(1) $\langle b, c\rangle \cap P_m = B_m$,
(2) $\langle b, c, g, h, k\rangle = \Gamma$.

Proof. The first point follows fom (3) in Corollary 5.1 (it holds in Θ_m, as it holds in $\bar{\Gamma}$). Next $\langle b, c, g, h, k\rangle = \langle b, c\rangle *_{\langle b,c\rangle \cap P_m} (g, h, k) = \Gamma$ by (1) in Corollary 2.2 and Remark 2.1. □

5.3. **Presenting R as a join of benign subgroups.** Following the original steps in subsection 3.1 we now would have to build the HNN-extension $\bar\Gamma *_R a$ by fixing the subgroup $R = \langle g_f b_f^{-1} \mid f \in \mathcal{E}_m\rangle$ of $\bar\Gamma$ by some stable letter a. But since R is *not finitely generated*, that HNN-extension would *not* be finitely presented, and we need some extra complications to arrive to a finitely presented free construction.

Denote the subgroup $\Phi_m = \langle b_0, \ldots, b_{m-1}, g, h_0, \ldots, h_{m-1}\rangle$ in $\bar\Gamma$, and notice that:

Lemma 5.4. *Φ_m is freely generated by $2m{+}1$ elements $b_0, \ldots, b_{m-1}, g, h_0, \ldots, h_{m-1}$ in $\bar\Gamma$.*

Proof. Firstly, $\bar B_m = \langle b_0, \ldots, b_{m-1}\rangle$ has trivial intersection with P_m because (1) in Lemma 5.3 implies $\bar B_m \cap P_m \le \big(\bar B_m \cap \langle b, c\rangle\big) \cap P_m = \bar B_m \cap \big(\langle b, c\rangle \cap P_m\big) = \bar B_m \cap B_m = frm[o]--$ due to $\langle B_m,\, \bar B_m\rangle = B_m * \bar B_m$. Therefore, we in $\bar\Gamma$ by (1) in Corollary 2.2 and by Remark 2.1 have:

$$\langle b_0, \ldots, b_{m-1},\; g, h, k\rangle = \bar B_m *_{\bar B_m \,\cap\, P_m} \langle g, h, k\rangle = \bar B_m *_{frm[o]--} \langle g, h, k\rangle = \bar B_m * \langle g, h, k\rangle$$

which simply is a free group of rank $m+3$. Since $h_0, \ldots, h_{m-1}$ generate a free subgroup inside $\langle g, h, k\rangle$, they together with $b_0, \ldots, b_{m-1}$ generate a free subgroup (of rank $2m+1$) inside $\langle b_0, \ldots, b_{m-1},\; g, h, k\rangle$. $\square$

Next we need a series of auxiliary benign subgroups inside $\bar\Gamma$. For an $s = 1, \ldots, m$ and for a sequence $f = (j_0, \ldots, j_{s-2}, j_{s-1}) \in \mathcal{E}_s$ denote $f^+ = (j_0, \ldots, j_{s-2},\, j_{s-1}{+}1)$ in $\mathcal{E}_s$, i.e., to get f^+ we just add 1 to the last coordinate of f. In these notations for any f the group $\bar\Gamma$ contains the elements $g_{f^+} \cdot b_{s-1}^{-1} \cdot g_f^{-1}$, such as, $g^{h_0^2 h_1^5 h_2^3 h_3^{\mathbf{8}}} \cdot b_3^{-1} \cdot g^{-\,h_0^2 h_1^5 h_2^3 h_3^7}$ for $f = (2,5,3,7)$ with $s = 4$. Denote:

$$V_{\mathcal{E}_s} = \Big\langle g_{f^+} \cdot b_{s-1}^{-1} \cdot g_f^{-1} \mid f \in \mathcal{E}_s \Big\rangle =$$

$$= \Big\langle g^{h_0^{i_0} \cdots\, h_{s-2}^{i_{s-2}} h_{s-1}^{i_{s-1}\mathbf{+1}}} \cdot b_{s-1}^{-1} \cdot g^{-h_0^{i_0} \cdots\, h_{s-2}^{i_{s-2}} h_{s-1}^{i_{s-1}}} \mid i_0 \ldots, i_{s-2}, i_{s-1} \in \mathbb{Z} \Big\rangle,$$

and establish a property for $V_{\mathcal{E}_s}$:

Lemma 5.5. *$V_{\mathcal{E}_s}$ is a benign subgroup in $\bar\Gamma$ for the some explicitly given finitely presented group and its finitely generated subgroup.*

Proof. By Lemma 5.4 for any $s = 1, \ldots, m$ the elements $b_{s-1}, g, h_0, \ldots, h_{s-1}$ are *free* generators for the $(s+2)$-generator subgroup $\langle b_{s-1}, g, h_0, \ldots, h_{s-1}\rangle$ of Φ_m. Hence, each of the following maps $\lambda_{i,\,j}$ can be continued to some isomorphism on

$\langle b_{s-1}, g, h_0, \dots, h_{s-1}\rangle$:

(5.4)

$$\begin{array}{llll}
\lambda_{s-1,\,0} & \text{sends} \;\; b_{s-1}, g, h_0, \dots, h_{s-2}, h_{s-1} & \text{to} & b_{s-1}, g^{h_0}, h_0, \dots, h_{s-2}, h_{s-1};\\
\lambda_{s-1,\,1} & \text{sends} \;\; b_{s-1}, g, h_0, \dots, h_{s-2}, h_{s-1} & \text{to} & b_{s-1}, g^{h_1}, h_0^{h_1}, \dots, h_{s-2}, h_{s-1};\\
\vdots & \qquad\vdots & & \qquad\vdots\\
\lambda_{s-1,\,s-1} & \text{sends} \;\; b_{s-1}, g, h_0, \dots, h_{s-2}, h_{s-1} & \text{to} & b_{s-1}, g^{h_{s-1}}, h_0^{h_{s-1}}, \dots, h_{s-2}^{h_{s-1}},\; h_{s-1}.
\end{array}$$

In particular, for $m = 1$ the map $\lambda_{0,0}$ sends b_0, g, h_0 to b_0, g^{h_0}, h_0; for $m = 2$ the map $\lambda_{1,0}$ sends b_1, g, h_0, h_1 to b_1, g^{h_0}, h_0, h_1 and $\lambda_{1,1}$ sends b_1, g, h_0, h_1 to $b_1, g^{h_1}, h_0^{h_1}, h_1$, etc...

Introducing for each isomorphism $\lambda_{i,\,j}$ a respective stable letter $l_{i,\,j}$ we construct the HNN-extension:

$$\Lambda_s = \bar\Gamma *_{\lambda_{s-1,\,0},\, \dots\, ,\lambda_{s-1,\,s-1}} (l_{s-1,\,0}, \dots, l_{s-1,\,s-1})$$

for each of $s = 1, \dots, m$.

The effects of conjugation by elements $l_{s-1,\,0}, \dots, l_{s-1,\,s-1}$ on the products $g_{f^+} \cdot b_{s-1}^{-1} \cdot g_f^{-1}$ is very easy to understand: $l_{s-1,\,i}$ just adds 1 to the i'th coordinate of f, say, for $s = 4$, $f = (2, 5, 3, 7)$ and $l_{3,\,2} = l_{4-1,\,3-1}$ we have:

(5.5)

$$\begin{aligned}
\Big(g_{f^+} \cdot b_3^{-1} \cdot g_f^{-1}\Big)^{l_{3,\,2}} &= \big(g^{h_2}\big)^{\left(h_0^{h_2}\right)^2 \left(h_1^{h_2}\right)^5 h_2^3\, h_3^{\mathbf{8}}} \cdot b_3^{-1} \cdot \big(g^{h_2}\big)^{-\,\left(h_0^{h_2}\right)^2 \left(h_1^{h_2}\right)^5 h_2^3\, h_3^7}\\
&= g^{h_2 \cdot\, h_2^{-1} h_0^2 h_2 h_2^{-1} h_1^5 h_2 h_2^3 h_3^{\mathbf{8}}} \cdot b_3^{-1} \cdot g^{-\, h_2 \cdot\, h_2^{-1} h_0^2 h_2 h_2^{-1} h_1^5 h_2 h_2^3 h_3^7}\\
&= g^{h_0^2 h_1^5 h_2^{\mathbf{4}} h_3^{\mathbf{8}}} \cdot b_3^{-1} \cdot g^{-\, h_0^2 h_1^5 h_2^{\mathbf{4}} h_3^7} = g_{f'\,+} \cdot b_3^{-1} \cdot g_{f'}^{-1} \in V_{\mathcal{E}_4}
\end{aligned}$$

where $f' = (2, 5, 3{+}1, 7) = (2, 5, \mathbf{4}, 7)$. In particular, actions of the above letters $l_{i,\,j}$ keep the elements from $V_{\mathcal{E}_s}$ inside $V_{\mathcal{E}_s}$.

For the sequence $f_0 = (0, \dots, 0) \in \mathcal{E}_s$ we have $g_{f_0^+} \cdot b_{s-1}^{-1} \cdot g_{f_0}^{-1} = g^{h_{s-1}} \cdot b_{s-1}^{-1} \cdot g^{-1}$. Applying the conjugate collection process of subsection 2.4 for $\mathfrak{X} = \{g^{h_{s-1}} \cdot b_{s-1}^{-1} \cdot g^{-1}\}$ and for $\mathfrak{Y} = \{l_{s-1,\,0}, \dots, l_{s-1,\,s-1}\}$ we see that any element w from $\langle \mathfrak{X}, \mathfrak{Y}\rangle \le \Lambda_s$ is a product of elements of $g_{f^+} \cdot b_{s-1}^{-1} \cdot g_f^{-1}$ (for certain sequences $f \in \mathcal{E}_s$) and of certain powers of the stable letters $l_{s-1,\,0}, \dots, l_{s-1,\,s-1}$. And w is inside $\bar\Gamma$ if and only if all those powers are cancelled out in the normal form, and w in fact is in $V_{\mathcal{E}_s}$, that is, denoting $L_s = \langle g^{h_{s-1}} \cdot b_{s-1}^{-1} \cdot g^{-1};\;\; l_{s-1,\,0}, \dots, l_{s-1,\,s-1}, \rangle$ we have:

$$\bar\Gamma \cap L_s = V_{\mathcal{E}_s},$$

i.e., $V_{\mathcal{E}_s}$ is benign in $\bar\Gamma$ for the above finitely presented group Λ_s and for its $(s+1)$-generator subgroup L_s. □

Lemma 5.6. *$R = \langle g_f b_f^{-1} \mid f \in \mathcal{E}_m\rangle$ is a benign subgroup in $\bar\Gamma$ for some explicitly given finitely presented group and its finitely generated subgroup.*

Proof. First show that R is generated by its $m+1$ subgroups $\langle g\rangle, V_{\mathcal{E}_1}, \ldots, V_{\mathcal{E}_m}$. For each $s=1,\ldots,m$ denote $Z_{\mathcal{E}_s} = \langle g_f b_f^{-1} \mid f \in \mathcal{E}_s\rangle$. In these notation R is noting but $Z_{\mathcal{E}_m}$ for $s=m$. It is easy to see that $\langle Z_{\mathcal{E}_{s-1}}, V_{\mathcal{E}_s}\rangle = Z_{\mathcal{E}_s}$ for each s (when $s=1$, then take $\langle g\rangle$ as $Z_{\mathcal{E}_0}$), see details in [17] based on an original idea from [9]. Then:

$$Z_{\mathcal{E}_m} = \langle Z_{\mathcal{E}_{m-1}}, V_{\mathcal{E}_m}\rangle = \langle Z_{\mathcal{E}_{m-2}}, V_{\mathcal{E}_{m-1}}, V_{\mathcal{E}_m}\rangle = \cdots = \big\langle \langle g\rangle, V_{\mathcal{E}_1}, \ldots V_{\mathcal{E}_{m-1}}, V_{\mathcal{E}_m}\big\rangle.$$

By Lemma 5.5 each $V_{\mathcal{E}_s}$ is benign $\bar\Gamma$ for an explicitly given finitely presented group Λ_s and its finitely generated subgroup L_s. And the finitely generated $\langle g\rangle$ is clearly benign in $\bar\Gamma$ for the finitely presented group $\Lambda_0 = \bar\Gamma$ and its finitely generated subgroup $L_0 = \langle g\rangle$.

It remains to load these components into Corollary 4.2 and into (4.3) to get the following finitely presented overgroup holding $\bar\Gamma$:

$$\bar\Theta = \Big(\cdots\Big(\big((\Lambda_0 *_{L_0} t_0) *_{\bar\Gamma} (\Lambda_1 *_{L_1} t_1)\big) *_{\bar\Gamma} (\Lambda_2 *_{L_2} t_2)\Big)\cdots\Big) *_{\bar\Gamma} (\Lambda_m *_{L_m} t_m), \tag{5.6}$$

and its finitely generated subgroup $Q = \big\langle \bar\Gamma^{\,t_0}, \ldots, \bar\Gamma^{\,t_m}\big\rangle$. □

5.4. **Proof for Theorem 1.1.** Now we can use the above constructions to finish the main proof. The last two steps of the construction in Section 3 are effortless to mimic. As $\bar\Theta$ of (5.6) is finitely presented, and Q is finitely generated, the HNN-extension $\bar\Theta *_Q a$ is finitely presented. Inside $\bar\Theta *_Q a$ the elements a, b, c generate the same free subgroup discussed in Section 3, and we can again define an isomorphism ρ sending a, b, c to a, b^{c^m}, c together with the *finitely presented* analog of Δ from (3.2):

$$\bar\Delta = \big(\bar\Theta *_Q a\big) *_\rho r. \tag{5.7}$$

For any $\mathcal{B} \subseteq \mathcal{E}_m$ we in analogy with Section 3 can denote $W_{\mathcal{B}} = \langle g_f,\, a,\, r \mid f \in \mathcal{B}\rangle$ in $\bar\Delta$. Since each g_f, a, r from $\bar\Delta$, in fact, is from Δ already, we in $\bar\Delta$ have the analog of (3.3) also:

$$G \cap W_{\mathcal{B}} = A_{\omega_m \mathcal{B}}. \tag{5.8}$$

Since $A_{\mathcal{B}}$ is benign in G, by Theorem 1.1 hypothesis there is a finitely presented (explicitly given) overgroup $K_{\mathcal{B}}$ of G with a finitely generated subgroup $L_{\mathcal{B}}$ so that $G \cap L_{\mathcal{B}} = A_{\mathcal{B}}$ in $K_{\mathcal{B}}$.

As $\bar\Delta$ was built purely via free constructions in which we are in position to control which new elements (such as, stable letters) to adjoin, we can make sure no element of $\bar\Delta$ outside G is contained in $K_{\mathcal{B}}$, and hence, we can construct the finitely presented amalgamated product $\bar\Delta *_G K_{\mathcal{B}}$.

The subgroup $A_{\mathcal{B}}$ is benign also in $\bar{\Delta}$. Indeed, the group $\bar{\Delta} *_G K_{\mathcal{B}}$ is finitely presented, and to its subgroup $L_{\mathcal{B}} = \langle A_{\mathcal{B}}, L_{\mathcal{B}} \rangle$ we may apply (3) in Corollary 2.1 to get $\bar{\Delta} \cap L_{\mathcal{B}} = A_{\mathcal{B}}$, because $A_{\mathcal{B}} \cap G = A_{\mathcal{B}}$ and $L_{\mathcal{B}} \cap G = A_{\mathcal{B}}$.

Next, being finitely generated $\langle b, c \rangle$ is benign in $\bar{\Delta}$ for the finitely presented $\bar{\Delta}$ and for the finitely generated $\langle b, c \rangle$, see remark in subsection 2.2.

Hence by Corollary 4.2 the join $\big\langle A_{\mathcal{B}}, \langle b, c \rangle \big\rangle = W_{\mathcal{B}}$ is benign in $\bar{\Delta}$. As its finitely presented overgroup we may take:

$$\Psi = \big((\bar{\Delta} *_G K_{\mathcal{B}}) *_{L_{\mathcal{B}}} y \big) \; *_{\bar{\Delta}} \; \big(\bar{\Delta} *_{\langle b,c \rangle} y' \big)$$

(see (4.3)), and as a finitely generated subgroup we may take $L' = \big\langle \bar{\Delta}^{y},\ \bar{\Delta}^{y'} \big\rangle$.

G clearly is benign in $\bar{\Delta}$. Hence by (5.8) and by Corollary 4.1 the intersection $G \cap W_{\mathcal{B}} = A_{\omega_m \mathcal{B}}$ is benign in $\bar{\Delta}$ for the finitely presented group:

$$K_{\omega_m \mathcal{B}} = (\Psi *_{L'} z) \; *_{\bar{\Delta}} \; (\bar{\Delta} *_G z'), \tag{5.9}$$

and its finitely generated subgroup:

$$L_{\omega_m \mathcal{B}} = \bar{\Delta}^{z\, z'}, \tag{5.10}$$

i.e., $\bar{\Delta} \cap L_{\omega_m \mathcal{B}} = A_{\omega_m \mathcal{B}}$ in $K_{\omega_m \mathcal{B}}$. But since $G \le \bar{\Delta}$ and $A_{\omega_m \mathcal{B}} \le G$, we conclude that $G \cap L_{\omega_m \mathcal{B}} = A_{\omega_m \mathcal{B}}$ also holds in $K_{\omega_m \mathcal{B}}$.

This completes the proof of Theorem 1.1.

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