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CROSSING MALMQUIST SYSTEMS WITH CERTAIN TYPES

F. N. WANG, K. LIU

Nanchang University, Nanchang, Jiangxi, P. R. China[1]
E-mails: *2671059355@qq.com; liukai418@126.com; liukai@ncu.edu.cn*

Abstract. In this paper, we will present the expression of meromorphic solutions on the crossing differential or difference Malmquist systems of certain types using Nevanlinna theory. For instance, we consider the admissible meromorphic solutions of the crossing differential Malmquist system

$$\begin{cases} f_1'(z) = \dfrac{a_1(z)f_2(z) + a_0(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) = \dfrac{a_2(z)f_1(z) + b_0(z)}{f_1(z) + d_2(z)}, \end{cases}$$

where $a_1(z)d_1(z) \not\equiv a_0(z)$ and $a_2(z)d_2(z) \not\equiv b_0(z)$.

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1. Introduction

The Malmquist theorem, originally published in [6], states that the Malmquist type differential equation

$$f'(z) = R(z, f(z)), \tag{1.1}$$

where $R(z, f(z))$ is a rational function in z and f, admits a transcendental meromorphic solution, then (1.1) reduces to a differential Riccati equation

$$f'(z) = a_0(z) + a_1(z)f(z) + a_2(z)f(z)^2, \tag{1.2}$$

where $a_i(z)(i = 0, 1, 2)$ are rational functions. The original proof in [6] was independent of Nevanlinna theory, however, Nevanlinna theory is an efficient method to prove and generalize the above result, some details can be found in [4, Chapter 10]. We assume that the reader is familiar with the basic notations of Nevanlinna theory, see [3, 4, 5, 12].

To generalize the Riccati or Malmquist equations, as far as we know, Tu and Xiao [7] firstly considered the meromorphic solutions of system of higher-order algebraic differential equations, which will be called the crossing Malmquist systems in the paper. Recently, there are some results for the meromorphic solutions of

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several systems, see [9, 10]. We give the following presentation for our proceeding consideration, which is a corollary of [7, Theorem 2], where the admissible meromorphic solutions imply that the coefficients of the system are rational functions or small functions with respect to $f_1(z)$ and $f_2(z)$.

Theorem A. If the following system

$$\begin{cases} f_2'(z) = \dfrac{a_{p_1}(z)f_1(z)^{p_1} + \cdots + a_1(z)f_1(z) + a_0(z)}{b_{q_1}(z)f_1(z)^{q_1} + \cdots + b_1(z)f_1(z) + b_0(z)}, \\ f_1'(z) = \dfrac{c_{p_2}(z)f_2(z)^{p_2} + \cdots + c_1(z)f_2(z) + c_0(z)}{d_{q_2}(z)f_2(z)^{q_2} + \cdots + d_1(z)f_2(z) + d_0(z)} \end{cases} \tag{1.3}$$

has a paired admissible meromorphic solution (f_1, f_2), then $d_1 d_2 \leq 4$, where $d_i := \max\{p_i, q_i\}, i = 1, 2$.

Obviously, Theorem A can be viewed as the generalization of Malmquist theorem. Moreover, the case $q_1 \geq 1$, $q_2 \geq 1$ can occur. See the example below given by Tu and Xiao [7].

Example 1.1. $(f_1, f_2) = (e^z, e^{-z})$ *is a paired entire solution of the crossing Malmquist system*

$$\begin{cases} f_1'(z) = \dfrac{1}{f_2(z)}, \\ f_2'(z) = -\dfrac{1}{f_1(z)}. \end{cases} \tag{1.4}$$

Actually, all meromorphic solutions of (1.4) can be expressed by $(f_1, f_2) = (e^{\frac{1}{c}z+d_1}, e^{-\frac{1}{c}z+d_2})$, where $e^{d_1+d_2} = c$. From the two equations in (1.4), then $f_1 f_2 = c$ follows immediately, where c is a non-zero constant. Thus, by $\frac{f_1'}{f_1} = \frac{1}{c}$, we get $f_1 = e^{\frac{1}{c}z+d_1}$, then $f_2 = e^{-\frac{1}{c}z+d_2}$, where $e^{d_1+d_2} = c$.

We proceed to consider the admissible meromorphic solutions of the generalization of the system (1.4) as follows

$$\begin{cases} f_1'(z) = \dfrac{a_1(z)f_2(z) + a_0(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) = \dfrac{a_2(z)f_1(z) + b_0(z)}{f_1(z) + d_2(z)}, \end{cases} \tag{1.5}$$

where $a_1(z)d_1(z) \not\equiv a_0(z)$ and $a_2(z)d_2(z) \not\equiv b_0(z)$. We obtain the following theorem.

Theorem 1.1. *The admissible entire solutions* (f_1, f_2) *of* (1.5) *satisfy one of the following cases:*

(*i*) *If* $a_1(z) = 0$, $d_2(z) = 0$, *then* $f_1(z) = \frac{\int (a_0(z)+b_0(z))dz}{f_2(z)+d_1(z)}$, *where* $d_1'(z) = -a_2(z)$.

(*ii*) *If* $a_1(z) = 0$, $d_2(z) \neq 0$, *then* $f_1(z) + d_2 = \frac{\int (a_0(z)+b_0(z)-a_2(z)d_2)dz}{f_2(z)+d_1(z)}$, *where* $d_2(z) = d_2$ *and* $d_1'(z) = -a_2(z)$.

(iii) *If* $a_1(z) \neq 0$, $d_2(z) \neq 0$, *then*

$$f_1(z) + d_2(z) = \frac{\int (a_0(z) + b_0(z) - a_1(z)d_1(z) - a_2(z)d_2(z))dz}{f_2(z) + d_1(z)},$$

where $d_1'(z) = -a_2(z)$ *and* $d_2'(z) = -a_1(z)$.

The another example below, given by Gao [2], shows that the case $d_1 d_2 = 4$ can occur in Theorem A.

Example 1.2. $(f_1, f_2) = (\frac{1}{e^z+1}, \frac{1}{e^z-1})$ *is a paired meromorphic solution of the crossing Malmquist system*

$$\begin{cases} f_1'(z) = \dfrac{-f_2(z)^2 - f_2(z)}{(2f_2(z)+1)^2}, \\ f_2'(z) = \dfrac{f_1(z)^2 - f_1(z)}{(2f_1(z)-1)^2}. \end{cases} \tag{1.6}$$

The system (1.6) has no any transcendental entire solutions. Otherwise, assume that (f_1, f_2) are transcendental entire functions, using the Valiron-Mohon'ko theorem [4, Theorem 2.2.5] and a basic formula $T(r, f') \leq T(r, f) + S(r, f)$ for an entire function f, then

$$\begin{aligned} 2T(r, f_2) &= T(r, f_2^2) + O(1) = T(r, f_1') + O(1) \leq T(r, f_1) + O(1) \\ &= \frac{1}{2}T(r, f_1^2) + O(1) = \frac{1}{2}T(r, f_2') + O(1) \leq \frac{1}{2}T(r, f_2) + O(1), \end{aligned}$$

thus $T(r, f_2) = O(1)$, which is impossible. We find that $(f_1, f_2) = (\frac{1}{1-e^z}, -\frac{1}{e^z+1})$ is also a paired meromorphic solution of (1.6). However, we have not obtained all meromorphic solutions satisfying the system (1.6). Remark that all the above two solutions (f_1, f_2) of (1.6) are meromorphic functions with no zeros. We obtain the following theorem to describe the partial meromorphic solutions of (1.6).

Theorem 1.2. *If* $f_1(z)$ *and* $f_2(z)$ *are two finite order meromorphic solutions of* (1.6) *with no zeros and simple poles only, then* $f_1(z) = \frac{1}{\alpha e^z+1}$ *and* $f_2(z) = \frac{1}{\alpha e^z-1}$, *where* α *is a non-zero constant.*

Without loss of generalization, we rewrite (1.3) as follows

$$\begin{cases} f_1'(z) = \dfrac{a_2(z)f_2(z)^2 + a_1(z)f_2(z) + a_0(z)}{b_2(z)f_2(z)^2 + b_1(z)f_2(z) + b_0(z)}, \\ f_2'(z) = \dfrac{c_2(z)f_1(z)^2 + c_1(z)f_1(z) + c_0(z)}{d_2(z)f_1(z)^2 + d_1(z)f_1(z) + d_0(z)}, \end{cases} \tag{1.7}$$

where $a_i(z), b_i(z), c_i(z), d_i(z) (i = 0, 1, 2)$ are small functions with respect to $f_1(z)$ and $f_2(z)$. From Theorem A, we see that there are four cases for d_1 and d_2 as follows

(i) (d_1, d_2): (4,1), (1,4);

(ii) (d_1, d_2): (3,1), (1,3);

(*iii*) (d_1, d_2): (2,2), (2,1), (1,2);

(*iv*) (d_1, d_2): (1,1).

Three new examples in the following remark with Example 1.1 and Example 1.2 show that there exist meromorphic solutions for all cases $(i)-(iv)$ indeed.

Remark 1.1. *For the case* $(d_1, d_2) = (4, 1)$*, we see that*

$$(f_1(z), f_2(z)) = (\sec z, \tan\frac{z}{2})$$

solves the following system

$$\begin{cases} f_1'(z) = \dfrac{2f_2(z)(1+f_2(z)^2)}{(1-f_2(z)^2)^2}, \\ f_2'(z) = \dfrac{f_1(z)}{f_1(z)+1}. \end{cases} \tag{1.8}$$

For the case $(d_1, d_2) = (3, 1)$*, we see that* $(f_1(z), f_2(z)) = \left(\frac{e^z}{(e^z-1)^2}, \frac{1}{e^z-1}\right)$ *solves the following system*

$$\begin{cases} f_1'(z) = -f_2(z) - 3f_2(z)^2 - 2f_2(z)^3, \\ f_2'(z) = -f_1(z). \end{cases} \tag{1.9}$$

For the case $(d_1, d_2) = (2, 1)$*, we see that* $(f_1(z), f_2(z)) = (\frac{1}{e^z-1}, e^z)$ *solves the following system*

$$\begin{cases} f_1'(z) = \dfrac{-f_2(z)}{(f_2(z)-1)^2}, \\ f_2'(z) = \dfrac{1+f_1(z)}{f_1(z)}. \end{cases} \tag{1.10}$$

The examples on $(d_1, d_2) = (1, 4), (1, 3), (1, 2)$ *can be constructed easily by the above.*

Gao [1, Theorem 1.2] obtained a difference version of Theorem A as follows.

Theorem B. If the following system

$$\begin{cases} f_2(z+c_1)\cdots f_2(z+c_n) = \dfrac{a_{p_1}(z)f_1(z)^{p_1} + \cdots + a_1(z)f_1(z) + a_0(z)}{b_{q_1}(z)f_1(z)^{q_1} + \cdots + b_1(z)f_1(z) + b_0(z)}, \\ f_1(z+d_1)\cdots f_1(z+d_m) = \dfrac{c_{p_2}(z)f_2(z)^{p_2} + \cdots + c_1(z)f_2(z) + c_0(z)}{d_{q_2}(z)f_2(z)^{q_2} + \cdots + d_1(z)f_2(z) + d_0(z)} \end{cases} \tag{1.11}$$

has a paired admissible meromorphic solution (f_1, f_2), where f_1 and f_2 are all meromorphic functions with hyper-order less than one. Then $d_1 d_2 \leq nm$, where $d_i := \max\{p_i, q_i\}$.

Gao [1] also obtained that (e^z, e^{-z}) is a paired transcendental meromorphic solution of the crossing difference Malmquist system

$$\begin{cases} f_1(z+1)f_1(z-1) = \dfrac{1}{f_2(z)^2}, \\ f_2(z+1)f_2(z-1) = \dfrac{1}{f_1(z)^2}. \end{cases} \tag{1.12}$$

Our proceeding theorem shows that all transcendental entire solutions with finite order of (1.12).

Theorem 1.3. *The transcendental entire solutions with finite order of* (1.12) *should satisfy one of the following two cases:*

(*i*) $(f_1(z), f_2(z)) = \left(e^{\alpha z+\beta}, e^{-\alpha z+\nu}\right)$, *where* $\nu + \beta = ki\pi$ *and* k *is an integer;*
(*ii*) $(f_1(z), f_2(z)) = \left(e^{\frac{B}{4}z^2+\frac{A+B}{2}z+D}, e^{-\frac{B}{4}z^2-\frac{A+B}{2}z+H}\right)$, *where* $\frac{B}{2} + 2D + 2H = 2ki\pi$ *and* k *is an integer.*

2. Lemmas

To prove Theorem 1.1, we need the following modification of Hayman inequality which relates to the zeros of f and $f^{(n)} - b$, where b is a non-zero small function with respect to f.

Lemma 2.1. [11] *Let* $f(z)$ *be a transcendental meromorphic function satisfying*

$$N\left(r, \frac{1}{f}\right) = S(r, f).$$

For any small functions $b(z)(\not\equiv 0)$ *of* f, *then*

$$N\left(r, \frac{1}{f^{(n)} - b}\right) \neq S(r, f).$$

In order to prove Theorem 1.2, we need the following lemma, which can be found in [8, Theorem 1.1].

Lemma 2.2. *Let* f *and* g *be transcendental entire functions with finite order, such that* f *and* g' *share* 0 *CM,* g *and* f' *share* 0 *CM. Then* f *and* g *satisfy one of the following three cases:*

(1) $f = \gamma g$, *where* γ *is a non-zero constant;*
(2) $f = \lambda \sin(az + b)$ *and* $g = \gamma \cos(az + b)$, *where* a, b, λ, γ *are constants with* $a\lambda\gamma \neq 0$ *and* $\lambda = i\gamma^2$;
(3) $fg = \beta f'g'$, *where* β *is a non-zero constant.*

3. Proofs of Theorems

Proof of Theorem 1.1. Firstly, rewrite (1.5) into

$$\begin{cases} f_1'(z) - a_1(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) - a_2(z) = \dfrac{b_2(z)}{f_1(z) + d_2(z)}, \end{cases} \tag{3.1}$$

where $b_1(z) = a_0(z) - a_1(z)d_1(z)$ and $b_2(z) = b_0(z) - a_2(z)d_2(z)$.

Using Valiron-Mohon'ko theorem [4, Theorem 2.2.5], we have

$$\begin{aligned} T(r, f_2(z)) + S(r, f_2(z)) &= T(r, f_1'(z)) \leq 2T(r, f_1(z)) + S(r, f_1(z)) \\ &\leq 2T(r, f_2'(z)) \leq 4T(r, f_2(z)) + S(r, f_2(z)). \end{aligned}$$

Hence, we assume that $S(r) := S(r, f_1(z)) = S(r, f_2(z))$. We will discuss four cases for the entire functions $f_1(z)$ and $f_2(z)$ below.

Case 1. If $a_1(z) = 0$, $d_2(z) = 0$, then

$$N\left(r, \frac{1}{f_2(z) + d_1(z)}\right) = N(r, f_1'(z)) + S(r) = S(r), \tag{3.2}$$

$$N\left(r, \frac{1}{f_2'(z) - a_2(z)}\right) = N(r, f_1(z)) + S(r) = S(r),$$

which can be written as

$$N\left(r, \frac{1}{(f_2(z) + d_1(z))' - d_1'(z) - a_2(z)}\right) = S(r). \tag{3.3}$$

By Lemma 2.1, (3.2) and (3.3), for avoiding a contradiction, then $d_1'(z) = -a_2(z)$ holds. In this case, from (3.1), we have

$$\begin{cases} f_1'(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) + d_1'(z) = \dfrac{b_2(z)}{f_1(z)}. \end{cases} \tag{3.4}$$

It follows from (3.4),

$$\begin{cases} f_1'(z)f_2(z) + f_1'(z)d_1(z) = b_1(z), \\ f_2'(z)f_1(z) + d_1'(z)f_1(z) = b_2(z). \end{cases} \tag{3.5}$$

Summing the two equations in (3.5), we get

$$(f_1(z)(f_2(z) + d_1(z)))' = b_1(z) + b_2(z),$$

thus

$$f_1(z) = \frac{\int (b_1(z) + b_2(z))dz}{f_2(z) + d_1(z)} = \frac{\int (a_0(z) + b_0(z))dz}{f_2(z) + d_1(z)}.$$

Case 2. If $a_1(z) = 0$, $d_2(z) \neq 0$, then we affirm that $d_2(z)$ must be a constant. From the second equation of (3.1), we have

$$N\left(r, \frac{1}{f_1(z) + d_2(z)}\right) = S(r).$$

From the first equation of (3.1), we have

$$N\left(r, \frac{1}{f_1'(z)}\right) = N\left(r, \frac{1}{(f_1(z) + d_2(z))' - d_2'(z)}\right) = S(r),$$

for avoiding a contradiction, we have $d_2(z)$ must be a constant d_2. Furthermore, the second equation of (3.1) shows that

$$N\left(r, \frac{1}{f_2'(z) - a_2(z)}\right) = N\left(r, \frac{f_1(z) + d_2(z)}{b_2(z)}\right) = S(r),$$

which implies that

$$N\left(r, \frac{1}{(f_2(z) + d_1(z))' - d_1'(z) - a_2(z)}\right) = S(r). \tag{3.6}$$

The first equation of (3.1) shows also that

$$N\left(r, \frac{1}{f_2(z) + d_1(z)}\right) = S(r). \tag{3.7}$$

By Lemma 2.1, (3.6) and (3.7), $-d_1'(z) - a_2(z) = 0$ holds for avoiding a contradiction, that is $d_1'(z) = -a_2(z)$, so we have

$$\begin{cases} f_1'(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) + d_1'(z) = \dfrac{b_2(z)}{f_1(z) + d_2}. \end{cases} \tag{3.8}$$

It follows from (3.8), we get

$$\Big((f_1(z) + d_2)(f_2(z) + d_1(z))\Big)' = b_1(z) + b_2(z),$$

thus

$$f_1(z) + d_2 = \frac{\int (b_1(z) + b_2(z))dz}{f_2(z) + d_1(z)} = \frac{\int (a_0(z) + b_0(z) - a_2(z)d_2)dz}{f_2(z) + d_1(z)}.$$

Case 3. If $a_1(z) \neq 0$, $d_2(z) = 0$, then (3.1) changes into

$$\begin{cases} f_1'(z) - a_1(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) - a_2(z) = \dfrac{b_2(z)}{f_1(z)}, \end{cases} \tag{3.9}$$

where $b_1(z) = a_0(z) - a_1(z)d_1(z)$ and $b_2(z) = b_0(z)$. The first equation of (3.9) implies that

$$N\left(r, \frac{1}{f_1'(z) - a_1(z)}\right) = S(r),$$

the second equation of (3.9) implies that

$$N\left(r, \frac{1}{f_1(z)}\right) = S(r).$$

By Lemma 2.1 and the above two equations, we get a contradiction.

Case 4. If $a_1(z) \neq 0$, $d_2(z) \neq 0$, then

$$N\left(r, \frac{1}{f_2'(z) - a_2(z)}\right) = N\left(r, \frac{f_1(z) + d_2(z)}{b_2(z)}\right) = S(r),$$
$$N\left(r, \frac{1}{f_2(z) + d_1(z)}\right) = N\left(r, \frac{f_1'(z) - a_1(z)}{b_1(z)}\right) = S(r).$$

Since

$$N\left(r, \frac{1}{(f_2(z) + d_1(z))' - d_1'(z) - a_2(z)}\right) = S(r),$$

by Lemma 2.1, we obtain $-d_1'(z) - a_2(z) = 0$ for avoiding a contradiction, that is $d_1'(z) = -a_2(z)$. In addition,

$$N\left(r, \frac{1}{f_1'(z) - a_1(z)}\right) = S(r),$$
$$N\left(r, \frac{1}{f_1(z) + d_2(z)}\right) = S(r),$$

we can have $d_2'(z) = -a_1(z)$. Thus, we have

$$\begin{cases} f_1'(z) + d_2'(z) = \dfrac{b_1(z)}{f_2(z) + d_1(z)}, \\ f_2'(z) + d_1'(z) = \dfrac{b_2(z)}{f_1(z) + d_2(z)}. \end{cases} \tag{3.10}$$

From (3.10), we get

$$\Big((f_1(z) + d_2(z))(f_2(z) + d_1(z))\Big)' = b_1(z) + b_2(z),$$

thus

$$\begin{aligned} f_1(z) + d_2(z) &= \frac{\int (b_1(z) + b_2(z))dz}{f_2(z) + d_1(z)} \\ &= \frac{\int (a_0(z) + b_0(z) - a_1(z)d_1(z) - a_2(z)d_2(z))dz}{f_2(z) + d_1(z)}. \end{aligned}$$

The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. Since $f_1(z)$ and $f_2(z)$ are meromorphic solutions with finite order of (1.6) with no zeros and simple poles only, then we assume that $f_1(z) = \frac{1}{g_1(z)}$ and $f_2(z) = \frac{1}{g_2(z)}$, where $g_1(z)$ and $g_2(z)$ are entire functions with finite order. Thus, the system (1.6) means that

$$\begin{cases} g_1'(z) = \dfrac{1 + g_2(z)}{(2 + g_2(z))^2} g_1^2(z), \\ g_2'(z) = \dfrac{g_1(z) - 1}{(2 - g_1(z))^2} g_2^2(z). \end{cases} \tag{3.11}$$

From the above system and g_1, g_2 are entire functions with simple zeros only, then we see that g_1 and $2+g_2$ have the same zeros and same multiplicities, g_2 and $2-g_1$ have the same zeros and same multiplicities. Hence, we assume

$$\begin{cases} g_1(z) = e^{P(z)}(2+g_2(z)), \\ g_2(z) = e^{Q(z)}(2-g_1(z)), \end{cases} \tag{3.12}$$

and

$$\begin{cases} g_1'(z) = e^{2P(z)}(1+g_2(z)), \\ g_2'(z) = e^{2Q(z)}(g_1(z)-1), \end{cases} \tag{3.13}$$

where $P(z)$ and $Q(z)$ are polynomials. Then, we rewrite (3.13) as

$$\begin{cases} \dfrac{(g_1(z)-1)'}{1+g_2(z)} = e^{2P(z)}, \\ \dfrac{(1+g_2(z))'}{g_1(z)-1} = e^{2Q(z)}. \end{cases} \tag{3.14}$$

From (3.14), we can get that $(g_1-1)'$ and $1+g_2$ share 0 CM, $(1+g_2)'$ and g_1-1 share 0 CM. By Lemma 2.2, then we discuss three cases for g_1 and g_2 below.

Case 1. $g_1 - 1 = \gamma(1+g_2)$, where γ is a non-zero constant, $e^{2P(z)} = \gamma^2 e^{2Q(z)}$. Substitute $g_1 - 1 = \gamma(1+g_2)$ into (3.12), we have

$$\begin{cases} \gamma(1+g_2)+1 = e^{P(z)}(2+g_2), \\ g_2 = e^{Q(z)}[2-\gamma(1+g_2)-1], \end{cases} \tag{3.15}$$

we see that (3.15) is represented by

$$\begin{cases} e^{P(z)} = \dfrac{\gamma(1+g_2)+1}{2+g_2}, \\ e^{Q(z)} = \dfrac{g_2}{1-\gamma(1+g_2)}. \end{cases} \tag{3.16}$$

(i) If $e^{P(z)} = \gamma e^{Q(z)}$, then we have

$$\frac{\gamma(1+g_2)+1}{2+g_2} = \frac{\gamma g_2}{1-\gamma(1+g_2)}. \tag{3.17}$$

Then

$$\gamma g_2^2 + 2\gamma g_2 = -\gamma^2 g_2^2 - 2\gamma^2 g_2 - \gamma^2 + 1. \tag{3.18}$$

So we obtain $\gamma = -1$, then $g_1 = -g_2$ follows. However, in this case, the first equation of (3.11) reduces into

$$g_1'(z) = \frac{1-g_1}{(2-g_1)^2} g_1^2,$$

which has no any transcendental entire solutions by Malmquist theorem.

(ii) If $e^{P(z)} = -\gamma e^{Q(z)}$, then

$$\frac{\gamma(1+g_2)+1}{2+g_2} = \frac{-\gamma g_2}{1-\gamma(1+g_2)}. \tag{3.19}$$

Then

$$-\gamma g_2^2 - 2\gamma g_2 = -\gamma^2 g_2^2 - 2\gamma^2 g_2 - \gamma^2 + 1. \tag{3.20}$$

So we obtain $\gamma = 1$, then $g_1 - g_2 = 2$ follows. From the first equation of (3.12), we see $e^{P(z)} \equiv 1$. Thus, the first equation of (3.14) also implies that

$$\frac{(1+g_2)'}{1+g_2} = 1,$$

so $g_2 = \alpha e^z - 1$, where α is a non-zero constant. Then $g_1 = \alpha e^z + 1$.

Case 2. If $g_1 - 1 = \lambda \sin(az+b)$ and $1 + g_2 = \gamma \cos(az+b)$, where a, b, λ, γ are constants with $a\lambda\gamma \neq 0$ and $\lambda = i\gamma^2$, then $e^{2P(z)} = \gamma^2 e^{2Q(z)}$ follows by (3.13). From (3.12), we have

$$\begin{cases} e^{P(z)} = \dfrac{1 + i\gamma^2 \sin(az+b)}{1 + \gamma\cos(az+b)}, \\ e^{Q(z)} = \dfrac{\gamma\cos(az+b) - 1}{1 - i\gamma^2 \sin(az+b)}. \end{cases} \tag{3.21}$$

(i) If $e^{P(z)} = \gamma e^{Q(z)}$, then

$$\frac{1 + i\gamma^2 \sin(az+b)}{1 + \gamma\cos(az+b)} = \frac{\gamma(\gamma\cos(az+b) - 1)}{1 - i\gamma^2 \sin(az+b)}.$$

Thus

$$\gamma^4 \sin^2(az+b) + 1 = -\gamma^3 \sin^2(az+b) + \gamma^3 - \gamma,$$

which is impossible for the reason that there is no γ satisfying

$$\begin{cases} \gamma^4 = -\gamma^3, \\ \gamma^3 - \gamma = 1. \end{cases}$$

(ii) If $e^{P(z)} = -\gamma e^{Q(z)}$, we have

$$\frac{1 + i\gamma^2 \sin(az+b)}{1 + \gamma\cos(az+b)} = \frac{-\gamma(\gamma\cos(az+b) - 1)}{1 - i\gamma^2 \sin(az+b)}.$$

Then

$$\gamma^4 \sin^2(az+b) + 1 = \gamma^3 \sin^2(az+b) - \gamma^3 + \gamma, \tag{3.22}$$

which is also impossible for the reason that there is no γ satisfying

$$\begin{cases} \gamma^4 = \gamma^3, \\ -\gamma^3 + \gamma = 1. \end{cases} \tag{3.23}$$

Case 3. If $(g_1 - 1)(1 + g_2) = \beta(g_1 - 1)'(1 + g_2)' = \beta g_1' g_2'$, where β is a non-zero constant, we have $e^{2P(z)+2Q(z)} = \frac{1}{\beta} := \tau^2$. From (3.12), we have

$$\begin{cases} g_1 = -e^{P(z)+Q(z)} g_1 + 2e^{P(z)+Q(z)} + 2e^{P(z)}, \\ g_2 = -e^{P(z)+Q(z)} g_2 - 2e^{P(z)+Q(z)} + 2e^{Q(z)}. \end{cases} \tag{3.24}$$

If $\tau \neq -1$, we have

$$\begin{cases} g_1 = \dfrac{2\tau + 2e^{P(z)}}{1+\tau}, \\ g_2 = \dfrac{-2\tau + 2e^{Q(z)}}{1+\tau}. \end{cases} \tag{3.25}$$

Substitute (3.25) into the first equation of (3.13), we have

$$\frac{2P'(z)}{1+\tau} = e^{P(z)}\left(1 + \frac{-2\tau + 2e^{Q(z)}}{1+\tau}\right).$$

The above equation implies that $1 + \frac{-2\tau}{1+\tau} = 0$, that is $\tau = 1$ and $e^{P(z)+Q(z)} = 1$, thus $P(z) = z + b$, where b is a constant. In the same way, substitute (3.25) into the second equation of (3.13), we have

$$\frac{2Q'(z)}{1+\tau} = e^{Q(z)}\left(\frac{2\tau + 2e^{P(z)}}{1+\tau} - 1\right).$$

The above equation implies that $\frac{2\tau}{1+\tau} - 1 = 0$, that is $\tau = 1$ and $e^{P(z)+Q(z)} = 1$, thus $Q(z) = z + a$, where a is a constant. However, this is in contradiction with $e^{P(z)+Q(z)} = 1$, so this case is omitted.

If $\tau = -1$, from the two equations in (3.24), we have $e^{P(z)+Q(z)} = -1$, $e^{P(z)} = 1$ and $e^{Q(z)} = -1$. From the first equation of (3.12), we have $g_1 = 2 + g_2$. Thus, the first equation of (3.14) also implies that

$$\frac{(1+g_2)'}{1+g_2} = 1,$$

so $g_2 = \alpha e^z - 1$, where α is a non-zero constant. Then $g_1 = \alpha e^z + 1$. The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. If $(f_1(z), f_2(z))$ is the paired transcendental entire solutions of the complex difference system (1.12), then we have $f_1(z)$ and $f_2(z)$ must have no zeros, thus we assume that $f_1(z) = e^{h_1(z)}$ and $f_2(z) = e^{h_2(z)}$, where $h_1(z)$ and $h_2(z)$ are non-constant polynomials. So

$$\begin{cases} e^{h_1(z+1)}e^{h_1(z-1)} = e^{-2h_2(z)}, \\ e^{h_2(z+1)}e^{h_2(z-1)} = e^{-2h_1(z)}, \end{cases} \tag{3.26}$$

it follows

$$\begin{cases} h_1(z+1) + h_1(z-1) + 2h_2(z) = 2ki\pi, \\ h_2(z+1) + h_2(z-1) + 2h_1(z) = 2mi\pi, \end{cases} \tag{3.27}$$

where k, m are integers. Shifting forward and backward on (3.27), we have

$$\begin{cases} h_1(z+2) + h_1(z) + 2h_2(z+1) = 2ki\pi, \\ h_2(z+2) + h_2(z) + 2h_1(z+1) = 2mi\pi, \end{cases} \tag{3.28}$$

and

$$\begin{cases} h_1(z) + h_1(z-2) + 2h_2(z-1) = 2ki\pi, \\ h_2(z) + h_2(z-2) + 2h_1(z-1) = 2mi\pi. \end{cases} \tag{3.29}$$

The first equation of (3.28) and the first equation of (3.29) can be rewritten as follows

$$\begin{cases} 2h_2(z+1) = -h_1(z+2) - h_1(z) + 2ki\pi, \\ 2h_2(z-1) = -h_1(z-2) - h_1(z) + 2ki\pi. \end{cases} \tag{3.30}$$

Combining the above system (3.30) and the second equation of (3.27), we have

$$2(2mi\pi - 2h_1(z)) = 4ki\pi - 2h_1(z) - h_1(z+2) - h_1(z-2),$$

thus, we have

$$h_1(z+2) + h_1(z-2) - 2h_1(z) = 4ki\pi - 4mi\pi. \tag{3.31}$$

From (3.31), we also have

$$h_1(z+2) - h_1(z) = h_1(z) - h_1(z-2) + 4ki\pi - 4mi\pi,$$

which implies that

$$F(z+2) = F(z) + 4ki\pi - 4mi\pi$$

by letting $F(z) = h_1(z) - h_1(z-2)$. We discuss two cases below.

Case 1. If $m = k$, then $F(z)$ must be a periodic function with period 2, thus $F(z)$ is a non-zero constant 2α for the reason that $h_1(z)$ is a non-constant polynomial. Thus $h_1(z) - h_1(z-2) = 2\alpha$, it follows $h_1(z) = \alpha z + \beta$.

Case 2. If $m \neq k$, then $F(z)$ must be a non-constant linear polynomial, that is $F(z) = Bz + A$. Thus, $h_1(z) - h_1(z-2) = Bz + A$, $B \neq 0$. In this case, we have $h_1(z)$ is a linear polynomial when $B = 0$ and is a polynomial with degree two when $B \neq 0$, we assume that $h_1(z) = \frac{B}{4}z^2 + \frac{A+B}{2}z + D$, where D is any constant.

Using the similar method as above, we also obtain

$$h_2(z+2) + h_2(z-2) - 2h_2(z) = 4mi\pi - 4ki\pi, \tag{3.32}$$

which implies that

$$h_2(z+2) - h_2(z) = h_2(z) - h_2(z-2) + 4mi\pi - 4ki\pi,$$

it follows

$$G(z+2) = G(z) + 4mi\pi - 4ki\pi$$

by letting $G(z) = h_2(z) - h_2(z-2)$. There are two cases to be discussed as follows.

Case 1. If $m = k$, then $G(z)$ must be a periodic function with period 2, thus $G(z)$ is also a non-zero constant 2μ. Then $h_2(z) - h_2(z-2) = 2\mu$, that is $h_2(z) = \mu z + \nu$.

Case 2. If $m \neq k$, then $G(z)$ must be a non-constant linear polynomial, that is $G(z) = Ez + F$. Thus, $h_2(z) - h_2(z-2) = Ez + F$, $E \neq 0$. In this case, we have $h_2(z)$ is a linear polynomial when $E = 0$ and is a polynomial with degree two when $E \neq 0$, we assume that $h_2(z) = \frac{E}{4}z^2 + \frac{E+F}{2}z + H$, where H is any constant.

We also remark that the degree of $h_1(z)$ and $h_2(z)$ are equal. Substitute $h_1(z) = \alpha z + \beta$ and $h_2(z) = \mu z + \nu$ into the first equation of (3.27), we have $\mu = -\alpha$ and $\nu + \beta = ki\pi$. Substitute $h_1(z) = \frac{B}{4}z^2 + \frac{A+B}{2}z + D$ and $h_2(z) = \frac{E}{4}z^2 + \frac{E+F}{2}z + H$ into the system of (3.27), we have $E = -B$, $F = -A$, $\frac{B}{2} + 2D + 2H = 2ki\pi$, $\frac{E}{2} + 2D + 2H = 2mi\pi$ and $B = 2ki\pi - 2mi\pi$. The proof of Theorem 1.3 is thus completed.

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