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# DERIVATIVES OF MEROMORPHIC FUNCTIONS SHARING POLYNOMIALS WITH THEIR DIFFERENCE OPERATORS

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Abstract. In this paper, we investigate the uniqueness of meromorphic functions of finite order $f(z)$ concerning their difference operators $\Delta_c f(z)$ and derivatives $f'(z)$ and prove that if $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, $\infty$ CM, where $a(z)$ and $b(z)$ are two distinct polynomials, then they assume one of following cases: (1) $f'(z) \equiv \Delta_c f(z)$; (2) $f(z)$ reduces to a polynomial and $f'(z) - A\Delta_c f(z) \equiv (1-A)(c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0)$, where $A(\neq 1)$ is a nonzero constant and $c_n, c_{n-1}, \cdots, c_1, c_0$ are all constants. This generalizes the corresponding results due to Qi et al. and Deng et al.

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## 1. Introduction and main results

As we know, Nevanlinna theory plays a significant role in the study of the uniqueness theory of meromorphic functions. Recent years, the research about difference analogue of meromorphic functions has become a subject of some interests and there are extensive results on them. For the related results, the readers can refer to[1, 2, 6, 7, 10, 12, 16, 17, 20]. Throughout this paper, $c$ always means a nonzero complex constant. Given a meromorphic function $f(z)$, we recall that a difference operator $\Delta_c f(z)$ is defined by $\Delta_c f(z) = f(z+c) - f(z)$. Suppose that $f(z)$ and $g(z)$ are two meromorphic functions and $a$ is a finite complex constant. If $f(z) - a$ and $g(z) - a$ have the same zeros, then we say that they share $a$ IM(ignoring multiplicities). If $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities, then we say that they share $a$ CM(counting multiplicities). And the above definition also applies when a is a polynomial. Furthermore we use $\rho(f)$ to denote the order of $f(z)$.

In 2013, Chen and Yi[3] studied the unicity of $\Delta_c f(z)$ and $f(z)$ sharing three values CM and proved the following result.

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**Theorem 1.1.** [3] *Let $f(z)$ be a transcendental meromorphic function such that $\rho(f)$ is not an integer or infinite, and let $a$ and $b$ be two distinct constants. If $\Delta_c f(z)$ and $f(z)$ share $a$, $b$, $\infty$ CM, where $\Delta_c f(z) \not\equiv 0$, then $f(z) \equiv \Delta_c f(z)$.*

**Remark 1.1.** *In* [3]*, Chen and Yi conjectured that in Theorem 1.1, the condition that "$\rho(f)$ is not an integer" can be omitted.*

In 2014, Zhang et al.[20], Liu et al.[13] respectively confirmed this conjecture and proved the following result.

**Theorem 1.2.** [20, 13] *Let $f(z)$ be a transcendental entire function of finite order, and let $a$ and $b$ be two distinct constants. If $\Delta_c f(z)$ and $f(z)$ share $a$, $b$ CM, where $\Delta_c f(z) \not\equiv 0$, then $f(z) \equiv \Delta_c f(z)$.*

Later, Li et al.[11], Cui et al.[5], Lü et al.[14] successively considered a meromorphic function rather than a transcendental meromorphic function in Theorem 1.2 and obtained the following result.

**Theorem 1.3.** [11, 5, 14] *Let $f(z)$ be a meromorphic function of finite order, and let $a$ and $b$ be two distinct constants. If $\Delta_c f(z)$ and $f(z)$ share $a$, $b$, $\infty$ CM, where $\Delta_c f(z) \not\equiv 0$, then $f(z) \equiv \Delta_c f(z)$.*

In 2019, Li[12] continued the study of the unicity of $\Delta_c f(z)$ and $f(z)$ sharing polynomials CM rather than values CM, which generalizes Theorem 1.3.

**Theorem 1.4.** [12] *Let $f(z)$ be a transcendental meromorphic function of finite order. If $\Delta_c f(z)$ and $f(z)$ share $P_1$, $P_2$, $\infty$ CM where $P_1$ and $P_2$ are two distinct polynomials, then $f(z) \equiv \Delta_c f(z)$.*

During the study of the uniqueness of $\Delta_c f(z)$ and $f(z)$, many researchers may be inspired to think about the following question.

**Question 1.1.** *Do the theorems above still hold if it is $\Delta_c f(z)$ and $f'(z)$ that share values CM since there are certain similarities between derivatives and difference operators of meromorphic functions?*

In 2018, Qi et al.[15] gave a positive answer to this question and proved the following result.

**Theorem 1.5.** [15] *Let $f(z)$ be a transcendental meromorphic function such that $\rho(f)$ is not an integer or infinite, and let $a$ and $b$ be two distinct constants. If $\Delta_c f(z)$ and $f'(z)$ share $a$, $b$, $\infty$ CM, then $f'(z) \equiv \Delta_c f(z)$.*

**Remark 1.2.** *In* [15], *Qi et al. conjectured that Theorem 1.5 is still valid without the condition that "$\rho(f)$ is not an integer."*

In 2019, Deng et al.[6] not only confirmed this conjecture, but also showed that the condition "$f(z)$ is a transcendental meroporhic function"in Theorem 1.5 can be extended to "$f(z)$ is a meromorphic function."

**Theorem 1.6.** [6] *Let $f(z)$ be a meromorphic function of finite order, and let $a$ and $b$ be two distinct constants. If $\Delta_c f(z)$ and $f'(z)$ share $a$, $b$, $\infty$ CM, then $f'(z) \equiv \Delta_c f(z)$ or $f(z) = Az + B$, where $A$, $B$ are all constants and $A \neq a, b$, $Ac \neq a, b$.*

To further generalize and improve Theorem 1.6, a natural problem can be posed as follows.

**Question 1.2.** *Does Theorem 1.6 still hold if $\Delta_c f(z)$ and $f'(z)$ share polynomials CM?*

In this paper, we study this problem and obtain the following main result.

**Theorem 1.7.** *Let $f(z)$ be a meromorphic function of finite order, and let $a(z)$ and $b(z)$ be two distinct polynomials. If $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, $\infty$ CM, then they assume one of following cases.*

(1) $f'(z) \equiv \Delta_c f(z)$;

(2) *$f(z)$ reduces to a polynomial and $f'(z) - A\Delta_c f(z) \equiv (1 - A)(c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0)$, where $A (\neq 1)$ is a nonzero constant and $c_n$, $c_{n-1}$, $\cdots$, $c_1, c_0$ are all constants.*

**Remark 1.3.** *Theorem 1.6 is a special case of Theorem 1.7, which implies that Theorem 1.7 generalizes the result of Theorem 1.6.*

**Example 1.1.** *Let $f(z) = 2z + 1$, $c = 2$, $a(z) = 1$, $b(z) = 0$. Then $f'(z) = 2$, $\Delta_c f(z) = 4$. Obviously, $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, $\infty$ CM and $f'(z) - \frac{1}{2}\Delta_c f(z) = 0$. This example illustrates that the case* (2) in Theorem 1.7 may occur.

**Example 1.2.** *Let $f(z) = z^2$, $c = 1$, $a(z) = 2z + 3$, $b(z) = 2z + 2$. Then $f'(z) = 2z$, $\Delta_c f(z) = 2z + 1$. Obviously, $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, $\infty$ CM and $f'(z) - 2\Delta_c f(z) = -(2z + 2)$. This example illustrates that the case* (2) in Theorem 1.7 may occur.

**Example 1.3.** *Let $f(z) = z^3$, $c = 1$, $a(z) = 3z^2 + \frac{3}{2}z + \frac{1}{2}$, $b(z) = 3z^2 + 6z + 2$. Then $f'(z) = 3z^2$, $\Delta_c f(z) = 3z^2 + 3z + 1$. Obviously, $\Delta_c f(z)$ and $f'(z)$ share $a(z)$, $b(z)$, $\infty$ CM and $f'(z) - 2\Delta_c f(z) = -(3z^2 + 6z + 2)$. This example illustrates that the case* (2) in Theorem 1.7 may occur.

## 2. Some lemmas

We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna theory, as founded in[9, 18, 19]. Next, we give some lemmas, which play a key role in proving Theorem 1.7.

**Lemma 2.1.** [4, 8] *Suppose that $f(z)$ is a meromorphic function of finite order, and $c$ is a nonzero complex constant. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

**Lemma 2.2.** [4, 8] *Let $f(z)$ be a meromorphic function of finite order, let $c$ be a nonzero complex constant, and let $k$ be a positive integer. Then*

$$m\left(r, \frac{\Delta_c^k f(z)}{f(z)}\right) = S(r, f).$$

**Lemma 2.3.** [18, 19] *Suppose that $f_i(z)$ $(i = 1, \cdots, n)$ $(n \geq 2)$ are meromorphic functions and $g_i(z)$ $(i = 1, \cdots, n)$ $(n \geq 2)$ are entire functions satisfying*

(1) $\sum_{i=1}^{n} f_i(z)e^{g_i(z)} \equiv 0$;
(2) *when* $1 \leq k < l \leq n$, $g_k(z) - g_l(z)$ *are not constants;*
(3) *when* $1 \leq i \leq n,\ 1 \leq k < l \leq n$,

$$T(r, f_i) = o\{T\left(r, e^{g_k - g_l}\right)\}, \quad (r \to \infty,\ r \notin E),$$

*where $E \subset (1, \infty)$ is of finite linear measure or logarithmic measure.*

*Then $f_i \equiv 0$ for any $i = 1, \cdots, n$.*

**Lemma 2.4.** *Suppose that $a(z)$ is a polynomial satisfying $a(z+c) - a(z) = \frac{a'(z)}{R}$, where $R$ is a nonzero constant. Then*

(1) *when* $c = \frac{1}{R}$, $a(z)$ *is a polynomial of degree one or a constant;*
(2) *when* $c \neq \frac{1}{R}$, $a(z)$ *is a constant.*

**Proof.** Suppose that $a(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_1 z + a_0$, where $a_n, a_{n-1}, \cdots, a_0$ are all constants. Then

$$\begin{aligned}
a'(z) =& n a_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + (n-2)a_{n-2}z^{n-3} + \cdots + a_1,\\
a(z+c) =& a_n(z+c)^n + a_{n-1}(z+c)^{n-1} + a_{n-2}(z+c)^{n-2} + \cdots + a_1(z+c) + a_0\\
=& a_n(z^n + C_n^1 z^{n-1}c^1 + C_n^2 z^{n-2}c^2 + \cdots + C_n^{n-1}z^1 c^{n-1} + c^n)\\
& + a_{n-1}(z^{n-1} + C_{n-1}^1 z^{n-2}c^1 + C_{n-1}^2 z^{n-3}c^2 + \cdots + C_{n-1}^{n-2}z^1 c^{n-2} + c^{n-1}) + \cdots\\
& + a_1(z+c) + a_0\\
=& a_n z^n + (a_n C_n^1 c^1 + a_{n-1})z^{n-1} + (a_n C_n^2 c^2 + a_{n-1}C_{n-1}^1 c^1 + a_{n-2})z^{n-2} + \cdots\\
& + (a_n C_n^{n-1}c^{n-1} + a_{n-1}C_{n-1}^{n-2}c^{n-2} + \cdots + a_2 C_2^1 c^1 + a_1)z
\end{aligned}$$

$$+ (a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c^1 + a_0).$$

Thus,

$$\begin{aligned} a(z+c) - a(z) =& a_n C_n^1 c^1 z^{n-1} + (a_n C_n^2 c^2 + a_{n-1} C_{n-1}^1 c^1) z^{n-2} \\ &+ (a_n C_n^3 c^3 + a_{n-1} C_{n-1}^2 c^2 + a_{n-2} C_{n-2}^1 c^1) z^{n-3} \\ &+ \cdots + (a_n C_n^{n-1} c^{n-1} + a_{n-1} C_{n-1}^{n-2} c^{n-2} + \cdots + a_2 C_2^1 c^1) z \\ &+ (a_n c^n + a_{n-1} c^{n-1} + \cdots + a_2 c^2 + a_1 c^1). \end{aligned}$$

When $c = \frac{1}{R}$, by $a(z+c) - a(z) = \frac{a'(z)}{R}$, we can get $a_n = a_{n-1} = a_{n-2} = \cdots = a_2 = 0$. Hence, $a(z)$ is a polynomial of degree one or a constant.

When $c \neq \frac{1}{R}$, by $a(z+c) - a(z) = \frac{a'(z)}{R}$, we can get $a_n = a_{n-1} = a_{n-2} = \cdots = a_1 = 0$. Hence, $a(z)$ is a constant.

## 3. Proof of theorem 1.7

If $\Delta_c f(z) \equiv a(z)$, then by the condition that $\Delta_c f(z)$ and $f'(z)$ share $a(z)$ CM, we can get $f'(z) \equiv a(z)$. Thus $f'(z) \equiv \Delta_c f(z)$. If $\Delta_c f(z) \equiv b(z)$, then we can also get $f'(z) \equiv \Delta_c f(z)$ in the same way. Next, we consider the case of $\Delta_c f(z) \not\equiv a(z), b(z)$.

Note that $\Delta_c f(z)$ and $f'(z)$ share $a(z), b(z), \infty$ CM and $f(z)$ is a meromorphic function of finite order. Then by Lemma 2.1, we have

$$\frac{f'(z) - a(z)}{\Delta_c f(z) - a(z)} = e^{\alpha(z)}, \quad \frac{f'(z) - b(z)}{\Delta_c f(z) - b(z)} = e^{\beta(z)}, \tag{3.1}$$

where $\alpha(z)$ and $\beta(z)$ are two polynomials such that $\max\{\deg \alpha(z), \deg \beta(z)\} \leq \rho(f)$.

It follows from (3.1) that

$$(e^{\alpha(z)} - e^{\beta(z)}) \Delta_c f(z) = a(z) e^{\alpha(z)} - b(z) e^{\beta(z)} - a(z) + b(z). \tag{3.2}$$

If $e^{\alpha(z)} \equiv e^{\beta(z)}$, then from (3.2) we can obtain

$$[a(z) - b(z)](e^{\alpha(z)} - 1) = 0.$$

Since $a(z) \not\equiv b(z)$, we have $e^{\alpha(z)} \equiv 1$. Hence by (3.1), we can get $f'(z) \equiv \Delta_c f(z)$.

Next we consider the case of $e^{\alpha(z)} \not\equiv e^{\beta(z)}$.

It follows from (3.2), (3.1) that

$$\Delta_c f(z) = \frac{a(z) e^{\alpha(z)} - b(z) e^{\beta(z)} - a(z) + b(z)}{e^{\alpha(z)} - e^{\beta(z)}}, \tag{3.3}$$

$$f'(z) = \frac{e^{\alpha(z)} [a(z) e^{\alpha(z)} - b(z) e^{\beta(z)} - a(z) + b(z)]}{e^{\alpha(z)} - e^{\beta(z)}} - a(z) e^{\alpha(z)} + a(z). \tag{3.4}$$

Differentiating (3.3) yields

$$[\Delta_c f(z)]' = \frac{a'(z)e^{2\alpha(z)} + b'(z)e^{2\beta(z)} + [(a(z)-b(z))(\beta'(z)-\alpha'(z)) - a'(z) - b'(z)]e^{\alpha(z)+\beta(z)}}{(e^{\alpha(z)}-e^{\beta(z)})^2} + \frac{[(a(z)-b(z))\alpha'(z) - a'(z) + b'(z)]e^{\alpha(z)} - [(a(z)-b(z))\beta'(z) - a'(z) + b'(z)]e^{\beta(z)}}{(e^{\alpha(z)}-e^{\beta(z)})^2}. \tag{3.5}$$

It follows from (3.4) that

$$\begin{aligned}\Delta_c f'(z) =& \frac{e^{\alpha(z+c)}[a(z+c)e^{\alpha(z+c)} - b(z+c)e^{\beta(z+c)} - a(z+c) + b(z+c)]}{e^{\alpha(z+c)} - e^{\beta(z+c)}}\\ &- \frac{e^{\alpha(z)}[a(z)e^{\alpha(z)} - b(z)e^{\beta(z)} - a(z) + b(z)]}{e^{\alpha(z)} - e^{\beta(z)}}\\ &- a(z+c)e^{\alpha(z+c)} + a(z)e^{\alpha(z)} + a(z+c) - a(z).\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we obtain

$$\begin{aligned}&[a(z+c)-b(z+c)]e^{2\alpha(z)+\alpha(z+c)+\beta(z+c)} + [a(z+c)-b(z+c)]e^{\alpha(z+c)+2\beta(z)+\beta(z+c)}\\ &- 2[a(z+c)-b(z+c)]e^{\alpha(z)+\alpha(z+c)+\beta(z)+\beta(z+c)} - Q_1(z)e^{2\alpha(z)+\alpha(z+c)}\\ &+ [a(z)-b(z)]e^{2\alpha(z)+\beta(z)+\beta(z+c)} + [a(z)-b(z)]e^{\alpha(z)+\alpha(z+c)+2\beta(z)}\\ &- [a(z)-b(z)]e^{\alpha(z)+2\beta(z)+\beta(z+c)} - [a(z)-b(z)]e^{2\alpha(z)+\alpha(z+c)+\beta(z)}\\ &- Q_2(z)e^{\alpha(z+c)+2\beta(z)} - Q_3(z)e^{\alpha(z)+\alpha(z+c)+\beta(z)} + Q_4(z)e^{2\alpha(z)+\beta(z+c)}\\ &+ Q_5(z)e^{2\beta(z)+\beta(z+c)} + Q_6(z)e^{\alpha(z)+\beta(z)+\beta(z+c)} - Q_7(z)e^{\alpha(z)+\alpha(z+c)}\\ &+ Q_7(z)e^{\alpha(z)+\beta(z+c)} + Q_8(z)e^{\alpha(z+c)+\beta(z)} - Q_8(z)e^{\beta(z)+\beta(z+c)} \equiv 0,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}Q_1(z) &= a'(z) + b(z) - b(z+c),\\ Q_2(z) &= b'(z) + a(z) - b(z+c),\\ Q_3(z) &= [a(z)-b(z)][\beta'(z)-\alpha'(z)] - a'(z) - b'(z) - a(z) - b(z) + 2b(z+c),\\ Q_4(z) &= a'(z) + b(z) - a(z+c),\\ Q_5(z) &= b'(z) + a(z) - a(z+c),\\ Q_6(z) &= [a(z)-b(z)][\beta'(z)-\alpha'(z)] - a'(z) - b'(z) - a(z) - b(z) + 2a(z+c),\\ Q_7(z) &= [a(z)-b(z)]\alpha'(z) - a'(z) + b'(z),\\ Q_8(z) &= [a(z)-b(z)]\beta'(z) - a'(z) + b'(z).\end{aligned} \tag{3.8}$$

Next we consider three cases about deg $\alpha(z)$ and deg $\beta(z)$.

**Case 1.** deg $\alpha(z) >$ deg $\beta(z)$. Then (3.7) can be rewritten as

$$A_3(z)e^{3\alpha(z)} + A_2(z)e^{2\alpha(z)} + A_1(z)e^{\alpha(z)} + A_0(z) \equiv 0, \tag{3.9}$$

where

$$A_3(z) = [a(z+c) - b(z+c)]e^{\Delta_c\alpha(z)+\beta(z+c)} - [a(z) - b(z)]e^{\Delta_c\alpha(z)+\beta(z)} - Q_1(z)e^{\Delta_c\alpha(z)},$$
$$A_2(z) = -2[a(z+c) - b(z+c)]e^{\Delta_c\alpha(z)+\beta(z)+\beta(z+c)} + [a(z) - b(z)]e^{\beta(z)+\beta(z+c)}$$
$$+ [a(z) - b(z)]e^{\Delta_c\alpha(z)+2\beta(z)} - Q_3(z)e^{\Delta_c\alpha(z)+\beta(z)} + Q_4(z)e^{\beta(z+c)} - Q_7(z)e^{\Delta_c\alpha(z)},$$
$$A_1(z) = [a(z+c) - b(z+c)]e^{\Delta_c\alpha(z)+2\beta(z)+\beta(z+c)} - [a(z) - b(z)]e^{2\beta(z)+\beta(z+c)}$$
$$- Q_2(z)e^{\Delta_c\alpha(z)+2\beta(z)} + Q_6(z)e^{\beta(z)+\beta(z+c)} + Q_7(z)e^{\beta(z+c)} + Q_8(z)e^{\Delta_c\alpha(z)+\beta(z)},$$
(3.10)
$$A_0(z) = Q_5(z)e^{2\beta(z)+\beta(z+c)} - Q_8(z)e^{\beta(z)+\beta(z+c)}.$$

Obviously, for any $i = 0, 1, 2, 3$, we have

$$\rho(A_i(z)) < \text{deg } \alpha(z).$$

Hence, it follows from Lemma 2.3 that

$$A_i(z) \equiv 0 \ (i = 0, 1, 2, 3).$$

Next we discuss two subcases as follows.

**Subcase 1.1.** deg $\beta(z) = 0$. Then $\beta(z)$ is a constant.

It follows from (3.8), (3.10) and $A_3(z) \equiv A_0(z) \equiv 0$ that

$$\Delta_c a(z)e^{\beta} + (1 - e^{\beta})\Delta_c b(z) \equiv a'(z), \tag{3.11}$$

$$\Delta_c a(z)e^{\beta} \equiv a'(z) + b'(z)(e^{\beta} - 1). \tag{3.12}$$

Combining (3.11) with (3.12) yields

$$(e^{\beta} - 1)[b'(z) - \Delta_c b(z)] \equiv 0.$$

If $e^{\beta} = 1$, then by (3.1), we can get $f'(z) \equiv \Delta_c f(z)$ and $e^{\alpha(z)} \equiv e^{\beta(z)} \equiv 1$, which contradicts deg $\alpha(z) >$ deg $\beta(z)$.

If $b'(z) - \Delta_c b(z) \equiv 0$, then it follows from Lemma 2.4 that when $c = 1$, $b(z)$ is a polynomial of degree one or a constant; when $c \neq 1$, $b(z)$ is a constant.

**Subcase 1.1.1.** $b(z)$ is a constant. We let $b(z) \equiv b$. Then by (3.12), we can get $a'(z) \equiv e^{\beta}\Delta_c a(z)$. It follows from Lemma 2.4 that when $c = e^{-\beta}$, $a(z)$ is a polynomial of degree one or a constant; when $c \neq e^{-\beta}$, $a(z)$ is a constant.

If $a(z)$ is a constant, then we let $a(z) \equiv a$. By (3.8), (3.10) and $A_2(z) \equiv A_1(z) \equiv 0$, we have

$$\begin{cases} \left[e^{2\beta} - e^{\beta} - \alpha'(z)(e^{\beta} - 1)\right] e^{\Delta_c\alpha(z)} + e^{\beta}(1 - e^{\beta}) \equiv 0, \\ e^{\beta}(1 - e^{\beta})e^{\Delta_c\alpha(z)} + e^{2\beta} - e^{\beta} + \alpha'(z)(e^{\beta} - 1) \equiv 0. \end{cases}$$

Thus, we get $e^{\beta}=1$. Similarly, we can get a contradiction.

If $a(z)$ is a polynomial of degree one, then we can get $c=e^{-\beta}$ at once. Next we let $a(z)=Az+B$, where $A(\neq 0), B$ are all constants. By (3.8), (3.10) and $A_2(z)\equiv A_1(z)\equiv 0$, we have

$$\begin{cases} e^{\Delta_c\alpha(z)}\left[\alpha'(z)(Az+B-b)(e^{\beta}-1)-(Az+B-b)e^{2\beta}-(A-Az-B+b)e^{\beta}+A\right] \\ \qquad +(Az+B-b)e^{2\beta}+(A-Az-B+b)e^{\beta}-A\equiv 0, \\ \alpha'(z)(Az+B-b)(e^{\beta}-1)+(Az+B-b)e^{2\beta}+(A-Az-B+b)e^{\beta}-A \\ \qquad -e^{\Delta_c\alpha(z)}\left[(Az+B-b)e^{2\beta}+(A-Az-B+b)e^{\beta}-A\right]\equiv 0. \end{cases}$$

Thus, we get $e^{\beta}=1$. Similarly, we can get a contradiction.

**Subcase 1.1.2.** $b(z)$ is a polynomial of degree one. Firstly, we can get $c=1$. We let $b(z)=Dz+E$, where $D(\neq 0), E$ are all constants. By (3.12), we can get $a(z)$ is a polynomial of degree one or a constant.

If $a(z)$ is a constant, then by (3.12), we can get $e^{\beta}=1$. Similarly, we can get a contradiction.

If $a(z)$ is a polynomial of degree one, then by (3.12), we can get $e^{\beta}=1$ or $a(z)=Dz+F$, where $F(\neq E)$ is a constant. When $a(z)=Dz+F$, by (3.8), (3.10) and $A_2(z)\equiv A_1(z)\equiv 0$, we have

$$\begin{cases} \left[e^{2\beta}-e^{\beta}-\alpha'(z)(e^{\beta}-1)\right]e^{\Delta_c\alpha(z)}+e^{\beta}(1-e^{\beta})\equiv 0, \\ e^{\beta}(1-e^{\beta})e^{\Delta_c\alpha(z)}+e^{2\beta}-e^{\beta}+\alpha'(z)(e^{\beta}-1)\equiv 0. \end{cases}$$

Thus, we get $e^{\beta}=1$, which implies that we can only get $e^{\beta}=1$ in this case. Similarly, we can get a contradiction.

**Subcase 1.2.** $\deg\beta(z)\geq 1$. It follows from (3.10), $A_0(z)\equiv 0$ that $Q_8(z)\equiv 0$. By (3.8), we have

$$\beta'(z)\equiv\frac{a'(z)-b'(z)}{a(z)-b(z)}. \tag{3.13}$$

Since $\deg\beta(z)\geq 1$, we can get $\beta'(z)\not\equiv 0$. Then it follows from (3.13) that $a(z)-b(z)\equiv\tilde{A}e^{\beta(z)}$, where $\tilde{A}$ is a nonzero constant. But this is a contradiction.

**Case 2.** $\deg\beta(z)>\deg\alpha(z)$. Then (3.7) can be rewritten as

$$B_3(z)e^{3\beta(z)}+B_2(z)e^{2\beta(z)}+B_1(z)e^{\beta(z)}+B_0(z)\equiv 0, \tag{3.14}$$

where

$$B_3(z)=[a(z+c)-b(z+c)]e^{\alpha(z+c)+\Delta_c\beta(z)}-[a(z)-b(z)]e^{\alpha(z)+\Delta_c\beta(z)}+Q_5(z)e^{\Delta_c\beta(z)},$$

$$B_2(z)=-2[a(z+c)-b(z+c)]e^{\alpha(z)+\alpha(z+c)+\Delta_c\beta(z)}+[a(z)-b(z)]e^{2\alpha(z)+\Delta_c\beta(z)}$$

$$+[a(z)-b(z)]e^{\alpha(z)+\alpha(z+c)}-Q_2(z)e^{\alpha(z+c)}+Q_6(z)e^{\alpha(z)+\Delta_c\beta(z)}-Q_8(z)e^{\Delta_c\beta(z)},$$

$$B_1(z)=[a(z+c)-b(z+c)]e^{2\alpha(z)+\alpha(z+c)+\Delta_c\beta(z)}-[a(z)-b(z)]e^{2\alpha(z)+\alpha(z+c)}$$

$- Q_3(z)e^{\alpha(z)+\alpha(z+c)} + Q_4(z)e^{2\alpha(z)+\Delta_c\beta(z)} + Q_7(z)e^{\alpha(z)+\Delta_c\beta(z)} + Q_8(z)e^{\alpha(z+c)},$

(3.15)

$B_0(z) = -Q_1(z)e^{2\alpha(z)+\alpha(z+c)} - Q_7(z)e^{\alpha(z)+\alpha(z+c)}.$

Obviously, for any $i = 0, 1, 2, 3$, we have

$$\rho(B_i(z)) < \deg\ \beta(z).$$

Hence, it follows from Lemma 2.3 that

$$B_i(z) \equiv 0\ (i = 0, 1, 2, 3).$$

Next we discuss two subcases as follows.

**Subcase 2.1.** $\deg\alpha(z) = 0$. Then $\alpha(z)$ is a constant.

It follows from (3.8), (3.15) and $B_3(z) \equiv B_0(z) \equiv 0$ that

$$\Delta_c b(z)e^{\alpha} + (1 - e^{\alpha})\Delta_c a(z) \equiv b'(z), \tag{3.16}$$

$$\Delta_c b(z)e^{\alpha} \equiv b'(z) + a'(z)(e^{\alpha} - 1). \tag{3.17}$$

Combining (3.16) with (3.17) yields

$$(e^{\alpha} - 1)[a'(z) - \Delta_c a(z)] \equiv 0.$$

If $e^{\alpha} = 1$, then by (3.1), we can get $f'(z) \equiv \Delta_c f(z)$ and $e^{\alpha(z)} \equiv e^{\beta(z)} \equiv 1$, which contradicts $\deg\beta(z) > \deg\alpha(z)$.

If $a'(z) - \Delta_c a(z) \equiv 0$, then it follows from Lemma 2.4 that when $c = 1$, $a(z)$ is a polynomial of degree one or a constant; when $c \neq 1$, $a(z)$ is a constant.

**Subcase 2.1.1.** $a(z)$ is a constant. We let $a(z) \equiv a$. Then by (3.17), we can get $b'(z) \equiv e^{\alpha}\Delta_c a(z)$. It follows from Lemma 2.4 that when $c = e^{-\alpha}$, $b(z)$ is a polynomial of degree one or a constant; when $c \neq e^{-\alpha}$, $b(z)$ is a constant.

If $b(z)$ is a constant, then we let $b(z) \equiv b$. By (3.8), (3.15) and $B_2(z) \equiv B_1(z) \equiv 0$, we have

$$\begin{cases} \left[e^{2\alpha} - e^{\alpha} - \beta'(z)(e^{\alpha} - 1)\right] e^{\Delta_c\beta(z)} + e^{\alpha}(1 - e^{\alpha}) \equiv 0, \\ e^{\alpha}(1 - e^{\alpha})e^{\Delta_c\beta(z)} + e^{2\alpha} - e^{\alpha} + \beta'(z)(e^{\alpha} - 1) \equiv 0. \end{cases}$$

Thus, we get $e^{\alpha} = 1$. Similarly, we can get a contradiction.

If $b(z)$ is a polynomial of degree one, then we can get $c = e^{-\alpha}$ at once. Next we let $b(z) = Dz + E$, where $D(\neq 0), E$ are all constants. By (3.8), (3.15) and $B_2(z) \equiv B_1(z) \equiv 0$, we have

$$\begin{cases} e^{\Delta_c\beta(z)}\left[\beta'(z)(a - Dz - E)(e^{\alpha} - 1) - (a - Dz - E)e^{2\alpha} + (D + a - Dz - E)e^{\alpha} - D\right] \\ \qquad + (a - Dz - E)e^{2\alpha} - (D + a - Dz - E)e^{\alpha} + D \equiv 0, \\ \beta'(z)(a - Dz - E)(e^{\alpha} - 1) + (a - Dz - E)e^{2\alpha} - (D + a - Dz - E)e^{\alpha} + D \\ \qquad - e^{\Delta_c\beta(z)}\left[(a - Dz - E)e^{2\alpha} - (D + a - Dz - E)e^{\alpha} + D\right] \equiv 0. \end{cases}$$

Thus, we get $e^{\alpha}=1$. Similarly, we can get a contradiction.

**Subcase 2.1.2.** $a(z)$ is a polynomial of degree one. Firstly, we can get $c=1$. We let $a(z)=Az+B$, where $A(\neq 0), B$ are all constants. By (3.17), we can get $b(z)$ is a polynomial of degree one or a constant.

If $b(z)$ is a constant, then by (3.17), we can get $e^{\alpha}=1$. Similarly, we can get a contradiction.

If $b(z)$ is a polynomial of degree one, then by (3.17), we can get $e^{\alpha}=1$ or $b(z)=Az+F$, where $F(\neq B)$ is a constant. When $b(z)=Az+F$, by (3.8), (3.15) and $B_2(z)\equiv B_1(z)\equiv 0$, we have

$$\begin{cases} \left[e^{2\alpha}-e^{\alpha}-\beta'(z)(e^{\alpha}-1)\right]e^{\Delta_c\beta(z)}+e^{\alpha}(1-e^{\alpha})\equiv 0, \\ e^{\alpha}(1-e^{\alpha})e^{\Delta_c\beta(z)}+e^{2\alpha}-e^{\alpha}+\beta'(z)(e^{\alpha}-1)\equiv 0. \end{cases}$$

Thus, we get $e^{\alpha}=1$, which implies that we can only get $e^{\alpha}=1$ in this case. Similarly, we can get a contradiction.

**Subcase 2.2.** $\deg\alpha(z)\geq 1$. It follows from (3.15), $B_0(z)\equiv 0$ that $Q_7(z)\equiv 0$. By (3.8), we have

$$\alpha'(z)\equiv\frac{a'(z)-b'(z)}{a(z)-b(z)}. \tag{3.18}$$

Since $\deg\alpha(z)\geq 1$, we can get $\alpha'(z)\not\equiv 0$. Then it follows from (3.18) that $a(z)-b(z)\equiv\tilde{B}e^{\alpha(z)}$, where $\tilde{B}$ is a nonzero constant. But this is a contradiction.

**Case 3.** $\deg\alpha(z)=\deg\beta(z)$.

**Subcase 3.1.** $\deg\alpha(z)=\deg\beta(z)=0$. Then, $\alpha(z)$ and $\beta(z)$ are constants, which implies that $e^{\alpha(z)}$ and $e^{\beta(z)}$ are constants, too. It follows from (3.4) that $f'(z)$ can be represented as a linear representation of $a(z)$ and $b(z)$. Thus, $f'(z)$ is a polynomial. Then, $f(z)$ is a polynomial, too. By (3.1) we can deduce that $f'(z)-A\Delta_cf(z)\equiv(1-A)(c_nz^n+c_{n-1}z^{n-1}+\cdots+c_1z+c_0)$, where $A(\neq 0), c_n, c_{n-1},\cdots,c_1,c_0$ are all constants. And when $A=1$, we have $f'(z)\equiv\Delta_cf(z)$. Then from (3.1), we have $e^{\alpha(z)}\equiv e^{\beta(z)}\equiv 1$. But this contradicts the hypothesis $e^{\alpha(z)}\not\equiv e^{\beta(z)}$. Hence, $A\neq 1$.

**Subcase 3.2.** $\deg\alpha(z)=\deg\beta(z)\geq 1$. Then (3.7) can be rewritten as

$$\begin{aligned} &C_1(z)e^{3\alpha(z)+\beta(z)}+C_2(z)e^{\alpha(z)+3\beta(z)}+C_3(z)e^{2\alpha(z)+2\beta(z)}+C_4(z)e^{3\alpha(z)} \\ &+C_5(z)e^{\alpha(z)+2\beta(z)}+C_6(z)e^{2\alpha(z)+\beta(z)}+C_7(z)e^{3\beta(z)}+C_8(z)e^{2\alpha(z)} \\ &+C_9(z)e^{\alpha(z)+\beta(z)}+C_{10}(z)e^{2\beta(z)}\equiv 0, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} C_1(z)&=[a(z+c)-b(z+c)]e^{\Delta_c\alpha(z)+\Delta_c\beta(z)}-[a(z)-b(z)]e^{\Delta_c\alpha(z)}, \\ C_2(z)&=[a(z+c)-b(z+c)]e^{\Delta_c\alpha(z)+\Delta_c\beta(z)}-[a(z)-b(z)]e^{\Delta_c\beta(z)}, \\ C_3(z)&=-2[a(z+c)-b(z+c)]e^{\Delta_c\alpha(z)+\Delta_c\beta(z)}+[a(z)-b(z)]e^{\Delta_c\beta(z)}+ \end{aligned}$$

$$+ [a(z)-b(z)]e^{\Delta_c\alpha(z)},$$
$$C_4(z) = -Q_1(z)e^{\Delta_c\alpha(z)}, \quad C_5(z) = -Q_2(z)e^{\Delta_c\alpha(z)} + Q_6(z)e^{\Delta_c\beta(z)},$$
$$C_6(z) = -Q_3(z)e^{\Delta_c\alpha(z)} + Q_4(z)e^{\Delta_c\beta(z)}, \quad C_7(z) = Q_5(z)e^{\Delta_c\beta(z)},$$
$$C_8(z) = -Q_7(z)e^{\Delta_c\alpha(z)}, \quad C_9(z) = Q_7(z)e^{\Delta_c\beta(z)} + Q_8(z)e^{\Delta_c\alpha(z)},$$
$$C_{10}(z) = -Q_8(z)e^{\Delta_c\beta(z)}. \tag{3.20}$$

If $\deg(\alpha(z)-\beta(z)) = \deg(2\alpha(z)-\beta(z)) = \deg(\alpha(z)+\beta(z)) = \deg(3\alpha(z)-\beta(z)) = \deg(3\beta(z)-\alpha(z)) = \deg(\alpha(z)-2\beta(z)) = \deg(3\alpha(z)-2\beta(z)) = \deg(3\beta(z)-2\alpha(z)) = \deg\ \alpha(z) = \deg\ \beta(z)$, then for any $1 \le i < j \le 10, 1 \le n \le 10$, we can get

$$\rho(C_n(z)) < \rho(e^{g_i(z)-g_j(z)}) = \deg\ \alpha(z).$$

It follows from Lemma 2.3 that $C_n(z) \equiv 0 (n = 1, 2, \cdots, 10)$. Then

$$C_{10}(z) = -Q_8(z)e^{\Delta_c\beta(z)} \equiv 0,$$

which implies that $Q_8(z) \equiv 0$. Thus by (3.8), we have

$$\beta'(z) \equiv \frac{a'(z)-b'(z)}{a(z)-b(z)}.$$

Therefore, using the same method as in the proof of Subcase 1.2, we can get a contradiction.

Hence, we can only need to discuss the cases that some of $\deg(\alpha(z)-\beta(z))$, $\deg(2\alpha(z)-\beta(z))$, $\deg(\alpha(z)+\beta(z))$, $\deg(3\alpha(z)-\beta(z))$, $\deg(3\beta(z)-\alpha(z))$, $\deg(\alpha(z)-2\beta(z))$, $\deg(3\alpha(z)-2\beta(z))$, $\deg(3\beta(z)-2\alpha(z))$ are less than $\deg\ \alpha(z)$.

**Subcase 3.2.1.** $\deg(\alpha(z)-\beta(z)) < \deg\ \alpha(z)$. Let $\alpha(z)-\beta(z) = p_1(z)$. Then $\beta(z) = \alpha(z) - p_1(z)$. And (3.19) can be rewritten as

$$D_4(z)e^{4\alpha(z)} + D_3(z)e^{3\alpha(z)} + D_2(z)e^{2\alpha(z)} \equiv 0,$$

where

$$D_4(z) = C_1(z)e^{-p_1(z)} + C_3(z)e^{-2p_1(z)} + C_2(z)e^{-3p_1(z)},$$
$$D_3(z) = C_4(z) + C_6(z)e^{-p_1(z)} + C_5(z)e^{-2p_1(z)} + C_7(z)e^{-3p_1(z)},$$
$$D_2(z) = C_8(z) + C_9(z)e^{-p_1(z)} + C_{10}(z)e^{-2p_1(z)}.$$

Combining this with (3.20), we obtain that for any $i = 2, 3, 4$,

$$\rho(D_i(z)) < \deg\ \alpha(z).$$

It then follows from Lemma 2.3 that

$$D_4(z) = D_3(z) = D_2(z) \equiv 0.$$

It is easy to deduce that $\Delta_c\beta(z) = \Delta_c\alpha(z) - \Delta_c p_1(z)$ since $\beta(z) = \alpha(z) - p_1(z)$. Hence, by (3.20) and $D_2(z) \equiv 0$, we have

$$e^{\Delta_c\alpha(z)}[-Q_7 + (Q_7 e^{-\Delta_c p_1(z)} + Q_8)e^{-p_1(z)} - Q_8 e^{-\Delta_c p_1(z)} e^{-2p_1(z)}] \equiv 0.$$

Equally,

$$-Q_7 + (Q_7 e^{-\Delta_c p_1(z)} + Q_8)e^{-p_1(z)} - Q_8 e^{-\Delta_c p_1(z)} e^{-2p_1(z)} \equiv 0. \tag{3.21}$$

If deg $p_1(z) \geq 1$, then by (3.8) and Lemma 2.3, we can get $Q_7(z) \equiv 0$, and thus

$$\alpha'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 2.2, we can get a contradiction.

Thus deg $p_1(z) = 0$ and so $p_1(z) = \alpha(z) - \beta(z)$ is a constant and

$$\Delta_c p_1(z) \equiv 0, \quad \alpha'(z) \equiv \beta'(z). \tag{3.22}$$

From (3.22) and (3.8), we can get $Q_7(z) \equiv Q_8(z)$. Combining this, (3.21) and (3.22), we have

$$e^{2p_1(z)} - 2e^{p_1(z)} + 1 \equiv 0.$$

Then $e^{p_1(z)} \equiv 1$, which implies that $e^{\alpha(z)} \equiv e^{\beta(z)}$. This contradicts the assumption $e^{\alpha(z)} \not\equiv e^{\beta(z)}$.

**Subcase 3.2.2.** $\deg(2\alpha(z) - \beta(z)) < \deg \alpha(z)$. Let $2\alpha(z) - \beta(z) = p_2(z)$. Then $\beta(z) = 2\alpha(z) - p_2(z)$. And (3.19) can be rewritten as

$$E_7(z)e^{7\alpha(z)} + E_6(z)e^{6\alpha(z)} + E_5(z)e^{5\alpha(z)} + E_4(z)e^{4\alpha(z)} + E_3(z)e^{3\alpha(z)} + E_2(z)e^{2\alpha(z)} \equiv 0,$$

where

$$E_7(z) = C_2(z)e^{-3p_2(z)}, \quad E_6(z) = C_3(z)e^{-2p_2(z)} + C_7(z)e^{-3p_2(z)},$$
$$E_5(z) = C_1(z)e^{-p_2(z)} + C_5(z)e^{-2p_2(z)}, \quad E_4(z) = C_6(z)e^{-p_2(z)} + C_{10}(z)e^{-2p_2(z)},$$
$$E_3(z) = C_4(z) + C_9(z)e^{-p_2(z)}, \quad E_2(z) = C_8(z).$$

Combining this with (3.20), we obtain that for any $i = 2, ..7$, $\rho(E_i(z)) < \deg \alpha(z)$. It then follows from Lemma 2.3 and (3.20) that $E_2(z) = C_8(z) = -Q_7(z)e^{\Delta_c\alpha(z)} \equiv 0$. Thus $Q_7(z) \equiv 0$. Combining this with (3.8) yields

$$\alpha'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 2.2, we can get a contradiction.

**Subcase 3.2.3.** $\deg(\alpha(z) + \beta(z)) < \deg \alpha(z)$. Let $\alpha(z) + \beta(z) = p_3(z)$. Then $\beta(z) = -\alpha(z) + p_3(z)$. And (3.19) can be rewritten as

$$F_3(z)e^{3\alpha(z)} + F_2(z)e^{2\alpha(z)} + F_1(z)e^{\alpha(z)} + F_0 + F_{-1}(z)e^{-\alpha(z)} + F_{-2}(z)e^{-2\alpha(z)} + F_{-3}(z)e^{-3\alpha(z)} \equiv 0,$$

where

$$F_3(z) = C_4(z), \quad F_2(z) = C_1(z)e^{p_3(z)} + C_8(z), \quad F_1(z) = C_6(z)e^{p_3(z)},$$
$$F_0(z) = C_3(z)e^{2p_3(z)} + C_9(z)e^{p_3(z)}, \quad F_{-1}(z) = C_5(z)e^{2p_3(z)},$$
$$F_{-2}(z) = C_2(z)e^{3p_3(z)} + C_{10}(z)e^{2p_3(z)}, \quad F_{-3}(z) = C_7(z)e^{3p_3(z)}.$$

Combining this with (3.20), we obtain that for any $i = -3, -2, \cdots, 2, 3$,

$$\rho(F_i(z)) < \deg \alpha(z).$$

It then follows from Lemma 2.3 that

$$F_3(z) = F_2(z) = F_1(z) = F_0(z) = F_{-1}(z) = F_{-2}(z) = F_{-3}(z) \equiv 0.$$

Thus from (3.20) and $F_2(z) \equiv F_{-2}(z) \equiv 0$ we have

$$\left\{[a(z+c) - b(z+c)]e^{\Delta_c\beta(z)} - [a(z) - b(z)]\right\} e^{p_3(z)} - Q_7(z) \equiv 0, \tag{3.23}$$

$$\left\{[a(z+c) - b(z+c)]e^{\Delta_c\alpha(z)} - [a(z) - b(z)]\right\} e^{p_3(z)} - Q_8(z) \equiv 0. \tag{3.24}$$

Obviously, by $\deg(\alpha(z) + \beta(z)) < \deg \alpha(z) = \deg \beta(z)$ and $p_3(z) = \alpha(z) + \beta(z)$, we can get $\deg p_3(z) \le \deg \Delta_c\beta(z) = \deg \beta(z) - 1$.

If $\deg p_3(z) < \deg \Delta_c\beta(z) = \deg \beta(z) - 1$, then by (3.23) we have

$$T(r, e^{\Delta_c\beta(z)}) = T\left(r, \frac{Q_7(z) + [a(z) - b(z)]e^{p_3(z)}}{e^{p_3(z)}[a(z+c) - b(z+c)]}\right) \le S(r, e^{\Delta_c\beta(z)}).$$

Thus $e^{\Delta_c\beta(z)}$ is a constant, which contradicts $0 \le \deg p_3(z) < \deg \Delta_c\beta(z)$.

Hence $\deg p_3(z) = \deg \Delta_c\beta(z)$.

If $\deg p_3(z) = \deg \Delta_c\beta(z) \ge 1$, then (3.23) can be rewritten as

$$[a(z+c) - b(z+c)]e^{\Delta_c\beta(z)+p_3(z)} \equiv [a(z) - b(z)]e^{p_3(z)} + Q_7(z), \tag{3.25}$$

where $Q_7(z) \not\equiv 0$. By the second fundamental theroem and (3.25), we have

$$\begin{aligned} T(r, e^{p_3(z)}) &\le N(r, e^{p_3(z)}) + N\left(r, \frac{1}{e^{p_3(z)}}\right) + N\left(r, \frac{1}{e^{p_3(z)} + \frac{Q_7(z)}{a(z)-b(z)}}\right) + S(r, e^{p_3(z)}) \\ &\le N\left(r, \frac{1}{\frac{a(z+c)-b(z+c)}{a(z)-b(z)} e^{\Delta_c\beta(z)+p_3(z)}}\right) + S(r, e^{p_3(z)}) \le S(r, e^{p_3(z)}). \end{aligned}$$

Thus $e^{p_3(z)}$ is a constant, which contradicts $\deg p_3(z) \ge 1$.

If $\deg p_3(z) = \deg(\alpha(z) + \beta(z)) = \deg \Delta_c\beta(z) = 0$, then $\alpha(z)$ and $\beta(z)$ are polynomials of degree one. We let

$$\alpha(z) = a_1 z + a_0, \quad \beta(z) = -a_1 z + b_0, \tag{3.26}$$

where $a_1 \neq 0$, $a_0$ and $b_0$ are all constants. Then it follows from (3.8), (3.23), (3.24) and (3.26) that

(3.27)
$$\left\{[a(z+c)-b(z+c)]e^{-a_1c}-a(z)+b(z)\right\}e^{p_3(z)}-[a(z)-b(z)]a_1+a'(z)-b'(z)\equiv 0,$$

(3.28)
$$\{[a(z+c)-b(z+c)]e^{a_1c}-a(z)+b(z)\}\,e^{p_3(z)}+[a(z)-b(z)]a_1+a'(z)-b'(z)\equiv 0.$$

Combining (3.27) with (3.28) yields

(3.29)
$$\left\{[a(z+c)-b(z+c)](e^{a_1c}+e^{-a_1c})-2[a(z)-b(z)]\right\}e^{p_3(z)}+2[a'(z)-b'(z)]\equiv 0.$$

If $a(z)-b(z)$ is a nonzero constant, then $a(z+c)-b(z+c)$ is a nonzero constant, too. In addition, we can get $a(z+c)-b(z+c)=a(z)-b(z)$ and $a'(z)-b'(z)\equiv 0$. From this and (3.29), we have $e^{a_1c}+e^{-a_1c}-2=0$. Hence $e^{a_1c}=1$. Substituting $e^{a_1c}=1$ into (3.27), we can deduce that $a_1=0$, a contradiction.

If $a(z)-b(z)$ is a nonconstant, then $\deg(a'(z)-b'(z))<\deg(a(z)-b(z))$. Next we let $h(z)=a(z)-b(z)=h_nz^n+\cdots+h_1z+h_0$, where $h_n(\neq 0)$, $h_{n-1}$, $h_{n-2}$, $\cdots$, $h_1$, $h_0$ are all constants and $n\geq 1$. Substituting this into (3.29), we have $e^{a_1c}+e^{-a_1c}-2=0$. Then $e^{a_1c}=1$. Substituting $e^{a_1c}=1$ into (3.27), we can deduce that $a_1=0$, a contradiction.

**Subcase 3.2.4.** $\deg(3\alpha(z)-\beta(z))<\deg\,\alpha(z)$. Let $3\alpha(z)-\beta(z)=p_4(z)$. Then $\beta(z)=3\alpha(z)-p_4(z)$. And (3.19) can be rewritten as

$$\begin{aligned}&G_{10}(z)e^{10\alpha(z)}+G_9(z)e^{9\alpha(z)}+G_8(z)e^{8\alpha(z)}+G_7(z)e^{7\alpha(z)}+G_6(z)e^{6\alpha(z)}\\&+G_5(z)e^{5\alpha(z)}+G_4(z)e^{4\alpha(z)}+G_3(z)e^{3\alpha(z)}+G_2(z)e^{2\alpha(z)}\equiv 0,\end{aligned}$$

where

$$\begin{aligned}&G_{10}(z)=C_2(z)e^{-3p_4(z)},\quad G_9(z)=C_7(z)e^{-3p_4(z)},\\&G_8(z)=C_3(z)e^{-2p_4(z)},\quad G_7(z)=C_5(z)e^{-2p_4(z)},\\&G_6(z)=C_1(z)e^{-p_4(z)}+C_{10}(z)e^{-2p_4(z)},\quad G_5(z)=C_6(z)e^{-p_4(z)},\\&G_4(z)=C_9(z)e^{-p_4(z)},\quad G_3(z)=C_4(z),\quad G_2(z)=C_8(z).\end{aligned}$$

Combining this with (3.20), we obtain that for any $i=2,3,\cdots,10$, $\rho(G_i(z))<\deg\,\alpha(z)$. It then follows from Lemma 2.3 and (3.20) that $G_2(z)=C_8(z)=-Q_7(z)e^{\Delta_c\alpha(z)}\equiv 0$. Thus $Q_7(z)\equiv 0$. Combining this with (3.8) yields

$$\alpha'(z)\equiv\frac{a'(z)-b'(z)}{a(z)-b(z)}.$$

Therefore, using the same method as in the proof of Subcase 2.2, we can get a contradiction.

**Subcase 3.2.5.** $\deg(3\beta(z) - \alpha(z)) < \deg \alpha(z)$. Let $3\beta(z) - \alpha(z) = p_5(z)$. Then $\alpha(z) = 3\beta(z) - p_5(z)$. And (3.19) can be rewritten as

$$\begin{aligned} &J_{10}(z)e^{10\beta(z)} + J_9(z)e^{9\beta(z)} + J_8(z)e^{8\beta(z)} + J_7(z)e^{7\beta(z)} + J_6(z)e^{6\beta(z)} \\ &+ J_5(z)e^{5\beta(z)} + J_4(z)e^{4\beta(z)} + J_3(z)e^{3\beta(z)} + J_2(z)e^{2\beta(z)} \equiv 0, \end{aligned}$$

where

$$\begin{aligned} &J_{10}(z) = C_1(z)e^{-3p_5(z)}, \quad J_9(z) = C_4(z)e^{-3p_5(z)}, \\ &J_8(z) = C_3(z)e^{-2p_5(z)}, \quad J_7(z) = C_6(z)e^{-2p_5(z)}, \\ &J_6(z) = C_2(z)e^{-p_5(z)} + C_8(z)e^{-2p_5(z)}, \quad J_5(z) = C_5(z)e^{-p_5(z)}, \\ &J_4(z) = C_9(z)e^{-p_5(z)}, \quad J_3(z) = C_7(z), \quad J_2(z) = C_{10}(z). \end{aligned}$$

Combining this with (3.20), we obtain that for any $i = 2, 3, \cdots, 10$, $\rho(J_i(z)) < \deg \beta(z)$. It then follows from Lemma 2.3 and (3.20) that $J_2(z) = C_{10}(z) = -Q_8(z)e^{\Delta_c \beta(z)} \equiv 0$. Thus $Q_8(z) \equiv 0$. Combining this with (3.8) yields

$$\beta'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 1.2, we can get a contradiction.

**Subcase 3.2.6.** $\deg(\alpha(z) - 2\beta(z)) < \deg \alpha(z)$. Let $\alpha(z) - 2\beta(z) = p_6(z)$. Then $\alpha(z) = 2\beta(z) + p_6(z)$. And (3.19) can be rewritten as

$$K_7(z)e^{7\beta(z)} + K_6(z)e^{6\beta(z)} + K_5(z)e^{5\beta(z)} + K_4(z)e^{4\beta(z)} + K_3(z)e^{3\beta(z)} + K_2(z)e^{2\beta(z)} \equiv 0,$$

where

$$\begin{aligned} &K_7(z) = C_1(z)e^{3p_6(z)}, \quad K_6(z) = C_3(z)e^{2p_6(z)} + C_4(z)e^{3p_6(z)}, \\ &K_5(z) = C_2(z)e^{p_6(z)} + C_6(z)e^{2p_6(z)}, \quad K_4(z) = C_5(z)e^{p_6(z)} + C_8(z)e^{2p_6(z)}, \\ &K_3(z) = C_7(z) + C_9(z)e^{p_6(z)}, \quad K_2(z) = C_{10}(z). \end{aligned}$$

Combining this with (3.20), we obtain that for any $i = 2, 3, \cdots, 7$, $\rho(K_i(z)) < \deg \beta(z)$. It then follows from Lemma 2.3 and (3.20) that $K_2(z) = C_{10}(z) = -Q_8(z)e^{\Delta_c \beta(z)} \equiv 0$. Thus $Q_8(z) \equiv 0$. Combining this with (3.8) yields

$$\beta'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 1.2, we can get a contradiction.

**Subcase 3.2.7.** $\deg(3\alpha(z) - 2\beta(z)) < \deg \alpha(z)$. Let $3\alpha(z) - 2\beta(z) = p_7(z)$. Then $\beta(z) = \frac{3}{2}\alpha(z) - \frac{1}{2}p_7(z)$. And (3.19) can be rewritten as

$$L_{\frac{11}{2}}(z)e^{\frac{11}{2}\alpha(z)} + L_5(z)e^{5\alpha(z)} + L_{\frac{9}{2}}(z)e^{\frac{9}{2}\alpha(z)} + L_4(z)e^{4\alpha(z)}$$

$$+ L_{\frac{7}{2}}(z)e^{\frac{7}{2}\alpha(z)} + L_3(z)e^{3\alpha(z)} + L_{\frac{5}{2}}(z)e^{\frac{5}{2}\alpha(z)} + L_2(z)e^{2\alpha(z)} \equiv 0,$$

where

$$\begin{aligned} &L_{\frac{11}{2}}(z) = C_2(z)e^{-\frac{3}{2}p_7(z)}, \quad L_5(z) = C_3(z)e^{-p_7(z)}, \\ &L_{\frac{9}{2}}(z) = C_1(z)e^{-\frac{1}{2}p_7(z)} + C_7(z)e^{-\frac{3}{2}p_7(z)}, \quad L_4(z) = C_5(z)e^{-p_7(z)}, \\ &L_{\frac{7}{2}}(z) = C_6(z)e^{-\frac{1}{2}p_7(z)}, \quad L_3(z) = C_4(z) + C_{10}(z)e^{-p_7(z)}, \\ &L_{\frac{5}{2}}(z) = C_9(z)e^{-\frac{1}{2}p_7(z)}, \quad L_2(z) = C_8(z). \end{aligned}$$

Combining this with (3.20), we obtain that for any $i = 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}$,

$$\rho(L_i(z)) < \deg\ \alpha(z).$$

It then follows from Lemma 2.3 and (3.20) that $L_2(z) = C_8(z) = -Q_7(z)e^{\Delta_c\alpha(z)} \equiv 0$. Thus $Q_7(z) \equiv 0$. Combining this with (3.8) yields

$$\alpha'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 2.2, we can get a contradiction.

**Subcase 3.2.8.** $\deg(3\beta(z) - 2\alpha(z)) < \deg\ \alpha(z)$. Let $3\beta(z) - 2\alpha(z) = p_8(z)$. Then $\alpha(z) = \frac{3}{2}\beta(z) - \frac{1}{2}p_8(z)$. And (3.19) can be rewritten as

$$\begin{aligned} &M_{\frac{11}{2}}(z)e^{\frac{11}{2}\beta(z)} + M_5(z)e^{5\beta(z)} + M_{\frac{9}{2}}(z)e^{\frac{9}{2}\beta(z)} + M_4(z)e^{4\beta(z)} \\ &+ M_{\frac{7}{2}}(z)e^{\frac{7}{2}\beta(z)} + M_3(z)e^{3\beta(z)} + M_{\frac{5}{2}}(z)e^{\frac{5}{2}\beta(z)} + M_2(z)e^{2\beta(z)} \equiv 0, \end{aligned}$$

where

$$\begin{aligned} &M_{\frac{11}{2}}(z) = C_1(z)e^{-\frac{3}{2}p_8(z)}, \quad M_5(z) = C_3(z)e^{-p_8(z)}, \\ &M_{\frac{9}{2}}(z) = C_4(z)e^{-\frac{3}{2}p_8(z)} + C_2(z)e^{-\frac{1}{2}p_8(z)}, \quad M_4(z) = C_6(z)e^{-p_8(z)}, \\ &M_{\frac{7}{2}}(z) = C_5(z)e^{-\frac{1}{2}p_8(z)}, \quad M_3(z) = C_7(z) + C_8(z)e^{-p_8(z)}, \\ &M_{\frac{5}{2}}(z) = C_9(z)e^{-\frac{1}{2}p_8(z)}, \quad M_2(z) = C_{10}(z). \end{aligned}$$

Combining this with (3.20), we obtain that for any $i = 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}$,

$$\rho(M_i(z)) < \deg\ \beta(z).$$

It then follows from Lemma 2.3 and (3.20) that $M_2(z) = C_{10}(z) = -Q_8(z)e^{\Delta_c\beta(z)} \equiv 0$. Thus $Q_8(z) \equiv 0$. Combining this with (3.8) yields

$$\beta'(z) \equiv \frac{a'(z) - b'(z)}{a(z) - b(z)}.$$

Therefore, using the same method as in the proof of Subcase 1.2, we can get a contradiction. This completes the proof of Theorem 1.7.

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