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CONVERGENCE OF GENERAL FOURIER SERIES OF DIFFERENTIABLE FUNCTIONS

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Abstract. Convergence of classical Fourier series (trigonometric, Haar, Walsh, ... systems) of differentiable functions are trivial problems and they are well known. But general Fourier series, as it is known, even for the function $f(x) = 1$ does not converge. In such a case, if we want differentiable functions with respect to the general orthonormal system (ONS) (φ_n) to have convergent Fourier series, we must find the special conditions on the functions φ_n of system (φ_n). This problem is studied in the present paper. It is established that the resulting conditions are best possible. Subsystems of general orthonormal systems are considered.

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1. Auxiliary notations and results

By V we denote the class of functions with bounded variation on $[0, 1]$ and $V(f)$ is the finite variation of function f. C_V is the set of functions with $f' \in V$. A is the class of absolute continuous functions. A is a Banach space with the norm

$$\|f\|_A = \|f\|_C + \int_0^1 |f'(x)|\,dx,$$

where C is a class of continuous functions.

Let (φ_n) be an ONS on $[0, 1]$, where φ_n are real-valued functions and $f \in \ell_2$, then the numbers

$$C_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx, \quad n = 1, 2, \dots,$$

are the general Fourier coefficients of function f.

General Fourier series is

$$\sum_{k=1}^{\infty} C_k(f)\varphi_k(x)$$

and its partial sum is

$$S_n(x, f) = \sum_{k=1}^{n} C_k(f)\varphi_k(x).$$

Let $(p = 1, 2, \dots)$

$$B_{np}(x, f) = \sum_{k=n}^{n+p} C_k(f)\varphi_k(x).$$

Lemma 1.1. *Let* (φ_n) *be an ONS on* $[0,1]$ *and* $f \in C_V$*, then*

$$B_{np}(x, f) = f(1) \int_0^1 \sum_{k=n}^{n+p} \varphi_k(u)\varphi_k(x)\, du$$

$$- \int_0^1 f'(u) \sum_{k=n}^{n+p} \int_0^u \varphi_k(v)\, dv\, du\ \varphi_k(x). \tag{1.1}$$

Proof. Integrating by parts, we get

$$C_k(f) = \int_0^1 f(u)\varphi_k(u)\, du = f(1)\int_0^1 \varphi_k(u)\, du - \int_0^1 f'(u)\int_0^u \varphi_k(v)\, dv\, du.$$

From here we can easily obtain (1.1).

Suppose that

$$H_{np}(u, x) = \sum_{k=n}^{n+p} \varphi_k(u)\varphi_k(x)$$

and

$$A_{np}(u, x) = \int_0^u \sum_{k=n}^{n+p} \varphi_k(v)\, dv \varphi_k(x),$$

then by (1.1) we get

$$B_{np}(x, f) = f(1)\int_0^1 H_{np}(u, x)\, du - \int_0^1 f'(u) A_{np}(u, x)\, du. \tag{1.2}$$

The lemma is proved. □

Lemma 1.2. *Let* (φ_n) *be an ONS on* $[0,1]$*. Then if* $N = n + p$*,*

$$\lim_{n\to\infty} n^{-\frac{1}{2}} N^{-\frac{3}{2}} \sum_{k=n}^{N} \varphi_k^2(x) = 0 \quad a.e.\ on\ [0,1].$$

Proof. It is obvious that

$$N^{-\frac{3}{2}} \sum_{k=n}^{N} \varphi_k^2(x) \le \sum_{k=n}^{\infty} k^{-\frac{3}{2}} \varphi_k^2(x).$$

Since

$$\sum_{k=n}^{\infty} k^{-\frac{3}{2}} \int_0^1 \varphi_k^2(x)\, dx = \sum_{k=n}^{\infty} k^{-\frac{3}{2}} < +\infty,$$

according to Levy theorem the series

$$\sum_{k=n}^{\infty} k^{-\frac{3}{2}} \varphi_k^2(x)$$

converges a.e. on $[0,1]$.

So a.e. on $[0,1]$,

$$\lim_{n\to\infty} n^{-\frac{1}{2}} N^{-\frac{3}{2}} \sum_{k=n}^{N} \varphi_k^2(x) = 0.$$

We denote

$$D_N(x) = \max_{1\le i<N} \left| \int_0^{\frac{i}{N}} A_{np}(u,x)\,du \right| = \left| \int_0^{\frac{i_N}{N}} A_{np}(u,x)\,du \right| \quad (1 \le i_N < N). \tag{1.3}$$

Lemma 1.3. *Let (φ_n) be an ONS on $[0,1]$ and $i = 1,2,\dots,N$, then if $n+p=N$,*

$$\int_{\frac{i-1}{N}}^{\frac{i}{N}} |A_{np}(u,x)|\,du \le \frac{1}{N} \left(\sum_{k=n}^{n+p} \varphi_k^2(x) \right)^{\frac{1}{2}}. \tag{1.4}$$

Proof. By Bessel inequality

$$\sum_{k=n}^{n+p} \left(\int_0^u \varphi_k(v)\,dv \right)^2 \le 1.$$

Using Cauchy and Hölder inequalities we get

$$\left| \int_{\frac{i-1}{N}}^{\frac{i}{N}} A_{np}(u,x)\,du \right| \le \frac{1}{\sqrt{N}} \left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\sum_{k=n}^{N} \int_0^u \varphi_k(v)\,dv \varphi_k(x) \right)^2 du \right)^{\frac{1}{2}}$$

$$\le \frac{1}{\sqrt{N}} \left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \sum_{k=n}^{N} \left(\int_0^u \varphi_k(v)\,dv \right)^2 du \sum_{k=n}^{N} \varphi_k^2(x) \right)^{\frac{1}{2}} \le \frac{1}{N} \left(\sum_{k=n}^{n+p} \varphi_k^2(x) \right)^{\frac{1}{2}}.$$

Definition 1.1. *By (W,C,x), $x \in G$, we denote the class of any ONS (φ_n) such that for each of them there exists a sequence $(\varepsilon_n(x))$, where $\lim_{n\to\infty} \varepsilon_n(x) = 0$, and*

$$\left| \sum_{k=n}^{n+p} C_k(f) \varphi_k(x) \right| < m_f \varepsilon_n(x)$$

for any $f' \in V$ and p.

Lemma 1.4. *If $\varphi_n(u) = \cos 2\pi n u$, then $(\varphi_n) \in (W,C,x)$ for any $x \in [0,1]$.*

Proof. Let $f' \in V$, then we have

$$\begin{aligned} C_k(f) &= \int_0^1 f(u)\varphi_k(u)\,du = \int_0^1 f(u)\cos 2\pi k u\,du \\ &= f(1)\int_0^1 \cos 2\pi k u\,du - \frac{1}{2\pi k}\int_0^1 f'(u)\sin 2\pi k u\,du \\ &= -\frac{1}{2\pi k}\int_0^1 f'(u)\sin 2\pi k u\,du. \end{aligned}$$

Since

$$\sum_{k=n}^{n+p} \left(\int_0^1 f'(u)\cos 2\pi k u\,du \right)^2 \le \int_0^1 (f'(u))^2\,du,$$

using the Cauchy inequality we get

$$\Big|\sum_{k=n}^{n+p} C_k(f)\varphi_k(x)\Big| = \frac{1}{2\pi}\Big|\sum_{k=n}^{n+p}\int_0^1 f'(u)\sin 2\pi ku\,du\frac{\sin 2\pi kx}{k}\Big|$$
$$\le \left(\sum_{k=n}^{n+p}\left(\int_0^1 f'(u)\cos 2\pi ku\,du\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=n}^{n+p}\frac{\sin^2 2\pi kx}{k^2}\right)^{\frac{1}{2}} \le m_f\,\frac{1}{\sqrt{n}}\,.$$

Lemma 1.5. *If (X_n) is a Haar system, then $(X_n)\in(W,C,x)$.*

Proof. Let $n=2^m$ and $p\le 2^m$ is any natural number. If $f'\in V$, according to the definition of Haar system (see [20]),

$$|C_{2^m+k}(f)| = O(1)2^{-\frac{3m}{2}} \quad (1\le k\le 2^m)$$

and $(x\in[0,1])$

$$\Big|\sum_{k=2^m}^{2^m+p} X_k(x)\Big| \le 2^{\frac{m}{2}}.$$

Then

$$\Big|\sum_{k=2^m}^{2^m+p} C_k(f)X_k(x)\Big| = O(1)2^{-\frac{3m}{2}}2^{\frac{m}{2}} = O(1)2^{-m}.$$

If $n+p=2^{m+s}$, we get

$$\Big|\sum_{k=2^m}^{2^{m+s}} C_k(f)X_k(x)\Big| = \Big|\sum_{r=m}^{m+s-1}\sum_{k=2^r}^{2^{r+1}} C_k(f)X_k(x)\Big|$$
$$= O(1)\sum_{r=m}^{m+s}2^{-r} = O(1)2^{-m}.$$

Analogously we can proof that

$$\Big|\sum_{k=m}^{m+p} C_k(f)X_k(x)\Big| = O(1)m^{-1}.$$

Theorem 1.1 (Banach [1]). *Let $f\in L_2$ be an arbitrary function ($f\not\simeq 0$). Then there exists an ONS (φ_n) such that*

$$\limsup_{n\to\infty}|S_n(x,f)| = +\infty \ \ \textit{a.e. on } [0,1],$$

where

$$S_n(x,f) = \sum_{k=1}^{n} C_k(f)\varphi_k(x).$$

Theorem 1.2 (see [7]). *Let $F, f \in L_2$, then*

$$\begin{aligned}\int_0^1 f(u)F(u)\,du &= N\sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(f(u)-f\left(u+\frac{1}{N}\right)\right)du\int_0^{\frac{i}{N}}F(u)\,du \\ &+ N\sum_{i=1}^{N}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\int_{\frac{i-1}{N}}^{\frac{i}{N}}(f(u)-f(v))\,dvF(u)\,du \\ &+ N\int_{1-\frac{1}{N}}^{1}f(u)\,du\int_0^1 F(u)\,du. \end{aligned} \tag{1.5}$$

2. The main proposition

Problems of convergence of orthogonal series are well studied. There should be noted the results: E. Men'shov [11], H. Rademacher [12], W. Orlich [14], S. Kachmarcz [8], K. Tandori [15], A. Olevsky [13], etc. On the other hand, the convergence problems of Fourier series of functions from some differentiable class are less studied: J. R. McLaughlin [10], S. V. Bochkarev [2], B. S. Kashin [9], L. Gogoladze, V. Tsagareishvili [7], G. Cagareishvili [3]. In the case when the convergence of the Fourier series of differentiable functions is necessary, certain conditions must be imposed on the functions of ONS. This is necessary because, according to Banach Theorem, in the general case the Fourier series does not converge even for the function $f(x) = 1$, $x \in [0, 1]$ (see Theorem 1.1).

In the present paper, we give special conditions which are imposed on functions of ONS (φ_n) under which the Fourier series of the functions of class C_V will be convergent a.e. on $[0, 1]$.

The similar Problems are studied in the papers [2, 9, 7, 3, 5, 6, 16, 17, 18].

3. The main results

We denote $N = n + p$.

Theorem 3.1. *Suppose that (φ_n) is an ONS on $[0, 1]$ and at the point $x \in G$ the series*

$$\sum_{k=1}^{\infty} C_k(l)\varphi_k(x)$$

converges, where $l(u) = 1$, $u \in [0, 1]$. If at the point $x \in G$ (see (1.3))

$$\lim_{n\to\infty} D_N(x) = 0, \tag{3.1}$$

then the series

$$\sum_{k=1}^{\infty} C_k(f)\varphi_k(x)$$

converges at the point $x \in G$ for any $f \in C_V$.

Proof. Substituting $F(x) = A_{n,p}(u,x)$ and $f = f'$ $(x \in G)$ in (1.5) we get

(3.2)
$$\int_0^1 f'(u)A_{np}(u,x)\,du = N\sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(f'(u) - f'\left(u+\frac{1}{N}\right)\right)du\int_0^{\frac{i}{N}} A_{np}(u,x)\,du$$
$$+ N\sum_{i=1}^{N}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\int_{\frac{i-1}{N}}^{\frac{i}{N}}(f'(u) - f'(v))\,dvA_{n,p}(u,x)\,du$$
$$+ N\int_{1-\frac{1}{N}}^{1} f'(u)\,du\int_0^1 A_{np}(u,x)\,du = a + b + c.$$

Since $f' \in V$ (see (3.1)), we have

$$|a| \le N\frac{1}{N}\sum_{i=1}^{N-1}\sup_{u\in\Delta_{in}}\left|f'(u) - f'\left(u+\frac{1}{N}\right)\right|\max_{1\le i<N}\left|\int_0^{\frac{i}{N}} A_{np}(u,x)\,du\right|$$
$$\le V(f')D_N(x). \tag{3.3}$$

Applying (1.4) and $f' \in V$, we write (see Lemma 1.2 and Lemma 1.3)

$$|b| \le N\,\frac{1}{N}\sum_{i=1}^{N}\max_{u,v\in\Delta_{in}}|f'(u) - f'(v)|\int_{\frac{i-1}{N}}^{\frac{i}{N}}|A_{np}(u,x)|\,du$$
$$\le V(f')\,\frac{1}{N}\left(\sum_{k=n}^{n+p}\varphi_k^2(x)\right)^{\frac{1}{2}}$$
$$= O(1)n^{-\frac{1}{4}}\left(N^{-\frac{3}{2}}\sum_{k=n}^{n+p}\varphi_k^2(x)\right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt[4]{n}}\right). \tag{3.4}$$

Taking into account $f' \in V$ and (1.3), we obtain (see Lemmas 1.2 and 1.3)

$$|c| = O(1)\left|\int_0^1 A_{np}(u,x)\,du\right|$$
$$= O(1)\left(\left|\int_0^{1-\frac{1}{N}} A_{np}(u,x)\,du\right| + \left|\int_{1-\frac{1}{N}}^1 A_{np}(u,x)\,du\right|\right)$$
$$= O\left(D_N(x) + \frac{1}{\sqrt[4]{n}}\right). \tag{3.5}$$

Thus from (3.2), taking into consideration (3.3), (3.4) and (3.5), we have

$$\left|\int_0^1 f'(u)A_n(u,x)\,dx\right| = O\left(D_N(x) + \frac{1}{\sqrt[4]{n}}\right). \tag{3.6}$$

We consider the function $l(u) = 1$, $u \in [0,1]$. Using the formula (1.1) and bear in mind

$$\int_0^1 \varphi_k(u)\,du = \int_0^1 l(u)\varphi_k(u)\,du = C_k(l),$$

we receive

$$\sum_{k=n}^{n+p} C_k(f)\varphi_k(x) = \sum_{k=n}^{n+p} C_k(l)\varphi_k(x) - \int_0^1 f'(u)A_n(u,x)\,du. \tag{3.7}$$

Finally, by condition of Theorem 1,

$$\lim_{n\to\infty}\left|\sum_{k=n}^{n+p} C_k(l)\varphi_k(x)\right| = 0.$$

So, from (3.6) and (3.7) there holds (see (3.1))

$$\lim_{n\to\infty}\left|\sum_{k=n}^{n+p} C_k(f)\varphi_k(x)\right| = 0$$

for any function $f \in C_V$ at the point $x \in G$. □

Theorem 3.2. *Let* (φ_n) *be an ONS on* $[0,1]$ *and* $x \in G$. *If*

$$\limsup_{n\to\infty} D_N(x) > M > 0, \tag{3.8}$$

then $(\varphi_n) \notin (W, C, x)$.

Proof. Suppose on the contrary that $(\varphi_n) \in (W, C, x)$. This means that for any $f \in C_V$,

$$\left|\sum_{k=n}^{n+p} C_k(f)\varphi_k(x)\right| \le m_f \varepsilon_n(x) \quad \Big(\lim_{n\to\infty} \varepsilon_n(x) = 0\Big).$$

For this propose, if $l(u) = 1$,

$$\left|\sum_{k=n}^{n+p} C_k(l)\varphi_k(x)\right| \le m_l \varepsilon_n(x).$$

Also, if in (1.2) we put $f(u) = q(u) = u$, we obtain

$$B_{n,p}(x, q) = \int_0^1 \sum_{k=n}^{n+p} \varphi_k(u)\, du \varphi_k(x) - \int_0^1 A_n(u, x)\, du.$$

Since

$$\int_0^1 \varphi_k(u)\, du = \int_0^1 l(u)\varphi_k(u)\, du = C_n(l),$$

from the last equality we get

$$B_{n,p}(x, q) = B_{n,p}(x, l) - \int_0^1 A_{np}(u, x)\, du. \tag{3.9}$$

Because of $q, l \in C_V$ we have that

$$|B_{n,p}(x, q)| \le m_q \varepsilon_n(x) \quad \text{and} \quad |B_{n,p}(x, l)| \le m_l \varepsilon_n(x).$$

From here and from (3.9) it obviously follows that

$$\left|\int_0^1 A_{np}(u, x)\, du\right| \le (m_q + m_l)\varepsilon_n(x). \tag{3.10}$$

We consider the increasing sequence (Z_n) such that

$$\lim_{n\to\infty} Zn = +\infty, \quad \lim_{n\to\infty} Z_n \varepsilon_n(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} Z_n \frac{1}{\sqrt[4]{n}} = 0. \tag{3.11}$$

Now we define the sequence of functions (h_N) in such a way:

$$h_N(u) = \begin{cases} 0, & u \in [0, \frac{i_N-1}{N}], \\ 1, & u \in [\frac{i_N}{N}, 1], \\ Nx - i_N + 1, & u \in [\frac{i_N-1}{N}, \frac{i_N}{N}]. \end{cases} \tag{3.12}$$

Substituting $f' = h_N$ we can rewrite (3.2) as

$$\begin{aligned} \int_0^1 h_N(u) A_{np}(u,x)\, dx = N \sum_{i=1}^{N-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(h_N(u) - h_N\left(u + \frac{1}{N}\right)\right) du \int_0^{\frac{i}{N}} A_{np}(u,x)\, du \\ + N \sum_{i=1}^{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{i-1}{N}}^{\frac{i}{N}} (h_N(u) - h_N(v))\, dv A_{np}(u,x)\, du \\ + N \int_{1-\frac{1}{N}}^{1} h_N(u)\, du \int_0^1 A_{np}(u,x)\, du = e + f + g. \end{aligned} \tag{3.13}$$

Applying (3.12) we write $|h_N(u) - h_N(v)| \le 1$, $u, v \in [0,1]$. Also, $h_N(u) - h_N(v) = 0$ when $u, v \in [\frac{i-1}{N}, \frac{i}{N}]$, $i = 1, \dots, i_N - 1; i_N + 1, \dots, N$.

For this reason, using Lemma 1.2 and Lemma 1.3, we receive

$$|f| \le N \frac{1}{N} \int_{\frac{N-1}{N}}^{\frac{i_N}{N}} |A_{np}(u,x)|\, du \le \frac{1}{N} \left(\sum_{k=n}^{n+p} \varphi_k^2(x) \right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt[4]{n}}\right). \tag{3.14}$$

We estimate the following integrals:

1) $\displaystyle \int_{\frac{i_N-2}{N}}^{\frac{i_N-1}{N}} \left(h_N(u) - h_N\left(u + \frac{1}{N}\right)\right) du = -\int_{\frac{i_N-1}{N}}^{\frac{i_N}{N}} (Nu - i_N + 1)\, du = -\frac{1}{2N};$

2) $\displaystyle \int_{\frac{i_N-1}{N}}^{\frac{i_N}{N}} \left(h_N(u) - h_N\left(u + \frac{1}{N}\right)\right) du = \int_{\frac{i_N-1}{N}}^{\frac{i_N}{N}} (Nu - i_N + 1)\, du - \frac{1}{N} = -\frac{1}{2N}.$

Taking into consideration these equalities we will show that

$$\begin{aligned} |e| &= N \left| \frac{1}{2N} \left(\int_0^{\frac{i_N}{N}} A_{np}(u,x)\, du + \int_0^{\frac{i_N-1}{N}} A_{np}(u,x)\, du \right) \right| \\ &= \frac{1}{2} \left| 2 \int_0^{\frac{i_N}{N}} A_{np}(u,x)\, du - \int_{\frac{i_N-1}{N}}^{\frac{i_N}{N}} A_{np}(u,x)\, du \right|. \end{aligned}$$

Moreover, according to Lemma 1.2 and Lemma 1.3, we conclude that

$$|e| \ge D_N(x) - O(1) \frac{1}{\sqrt[4]{n}}. \tag{3.15}$$

Next, by (3.10) and $h_N \in C_V$, we get

$$|g| = O(1) \varepsilon_n(x). \tag{3.16}$$

Finally using (3.13) with (3.14), (3.15) and (3.16) we can write

$$\left| \int_0^1 h_N(u) A_{np}(u,x)\, du \right| \ge D_N(x) - O(1)\varepsilon_n(x) - O(1) \frac{1}{\sqrt[4]{n}}. \tag{3.17}$$

We consider the sequence of linear and bounded on A functionals

$$R_n(f) = Z_n \int_0^1 f(u) A_{np}(u,x)\,du.$$

In our case,

$$R_n(h_N) = Z_n \int_0^1 h_N(u) A_{np}(u,x)\,du.$$

According to (3.8), (3.11) and (3.17) ($N = n + p$), we have

$$\lim_{n\to\infty} |R_n(h_N)| \geq \limsup_{n\to\infty} Z_n D_N(x) - O(1) \lim_{n\to\infty} Z_n \varepsilon_n(x) - O(1) \lim_{n\to\infty} Z_n \frac{1}{\sqrt[4]{n}} = +\infty. \tag{3.18}$$

By (3.12),

$$\|h_N\|_A = \|h_N\|_C + \int_0^1 |h'_N(u)|\,du \leq 2.$$

So, according to the Banach–Steinhaus Theorem (see (3.18)), there exists a function $s \in A$ such that

$$\lim_{n\to\infty} |R_n(s)| = \limsup_{n\to\infty} \left| Z_n \int_0^1 s(u) A_{np}(u,x)\,du \right| = +\infty. \tag{3.19}$$

Suppose

$$h(u) = \int_0^u s(v)\,dv.$$

As (see (1.2))

$$\int_0^1 H_{np}(u,x)\,du = \sum_{k=1}^n C_k(l)\varphi_k(x),$$

where $l(u) = 1$, $u \in [0,1]$ and (see p. 7)

$$\left| \sum_{k=1}^n C_k(l)\varphi_k(x) \right| \leq m_l \varepsilon_n(x),$$

then (see (3.11))

$$\lim_n \left| Z_n \int_0^1 H_{np}(u,x)\,du \right| = 0.$$

Using (1.2) when $f = h$, we get

$$B_{np}(x,h) = h(1) \int_0^1 H_{np}(u,x)\,du - \int_0^1 s(u) A_{np}(u,x)\,du.$$

From here

$$Z_n |B_{np}(x,h)| \geq \left| Z_n \int_0^1 s(u) A_{np}(u,x)\,du \right| - \left| h(1) Z_n \int_0^1 H_{np}(u,x)\,du \right|.$$

So, by (3.19), we obtain

$$\limsup_{n\to\infty} Z_n |B_{np}(x,h)| = +\infty. \tag{3.20}$$

On the other hand, as it was assumed $(\varphi_n) \in (W, C, x)$, in view of $h \in C_V$ we have $|B_{np}(x,h)| \le m_h \varepsilon_n(x)$. From here we get $Z_n |B_{np}(x,h)| \le Z_n m_h \varepsilon_n(x)$. Thus we have shown (see (3.11)) that

$$\lim_{n\to\infty} Z_n |B_{np}(x,h)| = m_h \lim_{n\to\infty} Z_n \varepsilon_n(x) = 0 \tag{3.21}$$

holds. Thus we obtain that (3.20) contradicts to (3.21), which means that $(\varphi_n) \notin (W, C, x)$. Theorem 3.2 is completely proved. □

Theorem 3.3. *Let (d_n) be a given increasing sequence. Any ONS (φ_n) contains the subsystem (φ_{n_k}) such that the series*

$$\sum_{k=1}^{\infty} d_k |C_{n_k}(f) \varphi_{n_k}(x)|$$

converges a.e. on $[0,1]$ for any $f \in C_V$.

Proof. We suppose that (φ_n) is the complete ONS. Then according to Parseval equality we have

$$\sum_{n=1}^{\infty} \Big(\int_0^u \varphi_n(v)\, dv \Big)^2 = u.$$

Hence there exists a sequence of natural numbers (n_k) such that uniformly with respect to $u \in [0,1]$,

$$\Big| \int_0^u \varphi_{n_k}(v)\, dv \Big| < \frac{k^{-2}}{d_k}, \quad k = 1, 2, \dots . \tag{3.22}$$

Integrating by parts when $f \in C_V$, we obtain

(3.23)

$$C_{n_k}(f) = \int_0^1 f(u) \varphi_{n_k}(u)\, du = f(1) \int_0^1 \varphi_{n_k}(u)\, du - \int_0^1 f'(u) \int_0^u \varphi_{n_k}(v)\, dv\, du.$$

According to (3.22) we conclude that

$$1) \quad \Big| \int_0^1 \varphi_{n_k}(u)\, du \Big| < \frac{k^{-2}}{d_k}, \quad k = 1, 2, \dots ,$$

$$2) \quad \Big| \int_0^1 f'(u) \int_0^u \varphi_{n_k}(v)\, dv\, du \Big| \le \sup_{u \in [0,1]} |f'(u)| \frac{k^{-2}}{d_k}, \quad k = 1, 2, \dots .$$

From here and (3.23), for any $f \in C_V$ we get

$$|C_{n_k}(f)| = O(1) \frac{k^{-2}}{d_k}, \quad k = 1, 2, \dots .$$

Thus

$$\sum_{k=1}^{\infty} d_k |C_{n_k}(f)| \int_0^1 |\varphi_{n_k}(x)|\, dx = O(1) \sum_{k=1}^{\infty} d_k \frac{k^{-2}}{d_k} \Big(\int_0^1 \varphi_{n_k}^2(x)\, dx \Big)^{\frac{1}{2}} < +\infty.$$

As it is known by Levy theorem, a.e. on $[0,1]$,

$$\sum_{k=1}^{\infty} d_k |C_{n_k}(f) \varphi_{n_k}(x)| < +\infty \quad \text{for any} \quad f \in C_V. \qquad \square$$

4. Problems of efficiency

Theorem 4.1. *The system* $\varphi_n(u) = \cos 2\pi nu$ *on* $[0,1]$ *satisfies the condition (for any* $x \in [0,1]$*)* $\lim\limits_{n\to\infty} D_N(x) = 0$.

Proof. We have $(N = n+p)$

$$A_{np}(u,x) = \sum_{k=n}^{n+p} \int_0^u \cos 2\pi kv\ dv\ \cos 2\pi kx = \frac{1}{2\pi} \sum_{k=n}^{n+p} \frac{1}{k}\ \sin 2\pi ku\ \cos 2\pi kx.$$

By the Hölder inequality we get $(i = 1, 2, \dots, N)$

$$\Big| \int_0^{\frac{i}{N}} A_{np}(u,x)\, du \Big| = \frac{1}{2\pi} \Big| \int_0^{\frac{i}{N}} \sum_{k=n}^{n+p} \frac{1}{k}\ \sin 2\pi k u du\ \cos 2\pi kx \Big|$$

$$= O(1) \Big(\sum_{k=n}^{n+p} \frac{1}{k^2}\ \cos^2 2\pi kx \Big)^{\frac{1}{2}} = O\Big(\frac{1}{\sqrt{n}}\Big). \qquad \square$$

Theorem 4.2. *Haar system* (X_n) *on* $[0,1]$ *satisfies the condition (see* [20]*)*

$$\lim_{n\to\infty} D_N(x) = 0.$$

Proof. Definition of the Haar system imply that

1) $$\Big| \int_0^u X_{2^s+k}(v)\, dv \Big| \le 2^{-\frac{s}{2}}, \ \text{ when } \ u \in \Big[\frac{k-1}{2^s}, \frac{k}{2^s}\Big]$$

and $$\int_0^u X_{2^s+k}(v)\, dv = 0, \ \text{ when } \ u \notin \Big[\frac{k-1}{2^s}, \frac{k}{2^s}\Big];$$

2) $$\Big| \int_0^t \int_0^u X_{2^s+k}(v)\, dv\, du \Big| \le 2^{-s}, \ \text{ when } \ t \in \Big[\frac{k-1}{2^s}, \frac{k}{2^s}\Big]$$

and $$\int_0^t \int_0^u X_{2^s+k}(v)\, dv\, du = 0, \ \text{ when } \ t \notin \Big[\frac{k-1}{2^s}, \frac{k}{2^s}\Big].$$

From here for any $t \in [0,1]$ we get

$$\Big| \int_0^t \sum_{k=1}^{2^s} \int_0^u X_{2^s+k}(v)\, dv\, du X_{2^s+k}(x) \Big| \le 2^{-s} 2^{\frac{s}{2}} = 2^{-\frac{s}{2}}.$$

Hence

$$\Big| \int_0^t \sum_{m=2^r+1}^{2^n} \int_0^u X_m(v)\, dv\, du X_m(x) \Big|$$

$$= \Big| \sum_{s=r}^{n} \int_0^t \sum_{k=1}^{2^s} \int_0^u X_{2^s+k}(v)\, dv\, du\ X_{2^s+k}(x) \Big| \le \sum_{s=r}^{n} 2^{-\frac{s}{2}} = O(1) 2^{-\frac{r}{2}}.$$

Consequently, when $t = \frac{i}{N}$, putting n instead of $2^r + 1$ and $n + p = N$ instead of 2^n, we obtain

$$D_N(x) = \Big| \int_0^{\frac{i_N}{N}} \sum_{m=n}^{n+p} \int_0^u X_m(v)\, dv\, du\ X_m(x) \Big| = O(1)\, \frac{1}{\sqrt{n}}\,.$$

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