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ENTIRE FUNCTIONS AND THEIR HIGH ORDER DIFFERENCE OPERATORS

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Abstract. In this paper, we prove that for a transcendental entire function f of finite order such that $\lambda(f-a)<\rho(f)$, where a is an entire function and satisfies $\rho(a)<\rho(f)$, $n\in\mathbb{N}$, if $\Delta_c^n f$ and f share the entire function b satisfying $\rho(b)<\rho(f)$ CM, where $c\in\mathbb{C}$ satisfies $\Delta_c^n f\not\equiv 0$, then $f(z)=a(z)+de^{cz}$, where d, c are two non-zero constants. In particular, if $a=b$, then a reduces to a constant. This result improves and generalizes the recent results of Chen and Chen [3], Liao and Zhang [10] and Lü et al. [11] in a large scale. Also we exhibit some relevant examples to fortify our results.

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1. Introduction and results

In this paper, a meromorphic function f always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with standard notation and main results of Nevanlinna Theory (see, e.g., [7, 12]). By $S(r,f)$ we denote any quantity that satisfies the condition $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside of an exceptional set of finite logarithmic measure. A meromorphic function a is said to be a small function of f if $T(r,a)=S(r,f)$. Moreover, we use notations $\rho(f)$, $\mu(f)$ and $\lambda(f)$ for the order, the lower order and the exponent of convergence of zeros of a meromorphic function f respectively. As usual, the abbreviation CM means "counting multiplicities", while IM means "ignoring multiplicities".

We now introduce some notations. Let $c\in\mathbb{C}\setminus\{0\}$. Then the forward difference $\Delta_c^n f$ for each integer $n\in\mathbb{N}$ is defined in the standard way by

$$\Delta_c^1 f(z)=\Delta_c f(z)=f(z+c)-f(z)$$

$$\Delta_c^n f(z)=\Delta_c\left(\Delta_c^{n-1}f(z)\right)=\Delta_c^{n-1}f(z+c)-\Delta_c^{n-1}f(z),\ \ n\geq 2.$$

Moreover

$$\Delta_c^n f(z)=\sum_{j=0}^{n}(-1)^{n-j}C_n^j f(z+jc),$$

where C_n^j is a combinatorial number.

In 1996, Brück [2] discussed the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [2].

Conjecture A. *Let f be a non-constant entire function such that*

$$\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}\notin\mathbb{N}\cup\{\infty\}.$$

If f and f' share one finite value a CM, then $f'-a=c(f-a)$, where $c\in\mathbb{C}\setminus\{0\}$.

The conjecture for the special cases (1) $a=0$ and (2) $N\left(r,\frac{1}{f'}\right)=S(r,f)$ had been confirmed by Brück [2]. Though the conjecture is not settled in its full generality, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with its derivatives.

Meromorphic solutions of complex difference equations, and the value distribution and uniqueness of complex differences have become an area of current interest and the study is based on the Nevanlinna value distribution of difference operators established by Halburd and Korhonen [6] and by Chiang and Feng [5] respectively. Recently, many authors (see [3, 4, 10, 11]) have started to consider the sharing values problems of meromorphic functions with their difference operators or shifts. Also it is well known that $\Delta_c f$ can be regarded as the difference counterpart of f'. Now, we recall the following result due to Chen [4], which is difference analogue of the Brück conjecture.

Theorem A. [4] *Let f be a transcendental entire function of finite order which has a finite Borel exceptional value a and let $c(\in\mathbb{C})$ such that $\Delta_c f\not\equiv 0$. If $\Delta_c f(z)$ and $f(z)$ share $b(b\neq a)$ CM then,*

$$\frac{\Delta_c f(z)-b}{f(z)-b}=A,$$

where $A=\frac{b}{b-a}$ is a non-zero constant.

In 2014, Cheng and Cheng [3] further improved Theorem A with the idea of sharing small function and obtained the following result.

Theorem B. [3] *Let f be a transcendental entire function of finite order and a be an entire function such that $\rho(a)<1$ and $\lambda(f-a)<\rho(f)$. Let $n\in\mathbb{N}$ and $c\in\mathbb{C}$ such that $\Delta_c^n f\not\equiv 0$ and b be an entire function such that $b\not\equiv a$ and $\rho(b)<1$. If $\Delta_c^n f$ and f share b CM, then*

$$f(z)=a(z)+de^{cz},$$

where d,c are two non-zero constants.

In 2016, Liao and Zhang [10] improved Theorem B from the case of $\rho(b) < 1$ to the general case of small function such that $\rho(b) < \rho(f)$ and obtained the following result.

Theorem C. [10] *Let f be a transcendental entire function of finite order and a be a small function of f such that $\rho(a) < 1$. Let $n \in \mathbb{N}$ such that $\Delta^n f \not\equiv 0$ and b be an entire function such that $b \not\equiv a$ and $\rho(b) < \rho(f)$. If $\Delta^n f$ and f share b CM, then*

$$\frac{\Delta^n f - b}{f - b} = \frac{b - \Delta^n a}{b - a}.$$

Furthermore f is of the form $f(z) = a(z) + ce^{\beta z}$, where c and β are two non-zero constants such that $\frac{b-\Delta^n a}{b-a} = (e^{\beta} - 1)^n$.

In 2019, Lü et al. [11] asked the following questions.
Question A: Can the condition $\rho(b) < 1$ be weakened in Theorem C.
Question B: Does there exist a joint theorem involve of both cases $a \equiv b$ and $a \not\equiv b$?

In the same paper, Lü et al. [11] gave affirmative answers of Questions A and B by proving the following result.

Theorem D. [11] *Let f be a transcendental entire function of finite order and a be an entire function such that $\lambda(f - a) < \rho(f)$, $\rho(a) < 1$ and $\rho(a) \neq \rho(f)$. Let $n \in \mathbb{N}$ such that $\Delta^n f \not\equiv 0$ and b be an entire function such that $\rho(b) < \max\{1, \rho(f)\}$. If $\Delta^n f$ and f share b CM, then*

$$f(z) = a(z) + ce^{\beta z},$$

where c and β are two non-zero constants. In particular, if $a \equiv b$, then a reduces to a constant.

In the same paper, Lü et al. [11] exhibited the following example to show that the condition $\rho(a) \neq \rho(f)$ is necessary in Theorem D.

Example 1.1. *Let f be a transcendental entire function with $0 < \rho(f) < 1$, $a(z) = f(z) - z$ and $b(z) = 3f(z) - f(z + 1)$. Clearly $\lambda(f - a) = 0 < \rho(f)$, $\rho(b) < 1$ and*

$$\frac{\Delta f - b}{f - b} = 2.$$

Therefore f and Δf share b CM, but f does not satisfies the conclusion of Theorem D.

In the paper, we prove the following main theorem, which extends Theorem D from the case of $\lambda(f - a) < \rho(f)$, $\rho(a) < 1$ and $\rho(a) \neq \rho(f)$ to the general case of entire function such that $\lambda(f - a) < \rho(f)$, $\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$.

Theorem 1.1. *Let f be a transcendental entire function of finite order and a be an entire function such that $\lambda(f-a) < \rho(f)$, $\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $\Delta_c^n f \not\equiv 0$ and b be an entire function such that $\rho(b) < \max\{1, \rho(f)\}$. If $\Delta_c^n f$ and f share b CM, then one of the following cases holds*

(1) *$a = b \in \mathbb{C}$ and $f(z) = a + de^{cz}$, where c and d are two non-zero constants,*
(2) *$a \not\equiv b$ and $f(z) = a(z) + de^{cz}$, where c and d are two non-zero constants.*

Immediately we have the following corollaries.

Corollary 1.1. *Let f be a transcendental entire function such $\rho(f) \geq 1$ and a be an entire function such that $\lambda(f-a) < \rho(f)$ and $\rho(a) < \rho(f)$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $\Delta_c^n f \not\equiv 0$ and b be an entire function such that $\rho(b) < \rho(f)$. If $\Delta_c^n f$ and f share b CM, then one of the following cases holds*

(1) *$a = b \in \mathbb{C}$ and $f(z) = a + de^{cz}$, where c and d are two non-zero constants,*
(2) *$a \not\equiv b$ and $f(z) = a(z) + de^{cz}$, where c and d are two non-zero constants.*

Corollary 1.2. *Let f be a transcendental entire function of finite order and a be an entire function such that $\lambda(f-a) < \rho(f)$, $\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $\Delta_c^n f \not\equiv 0$. If $\Delta_c^n f$ and f share a CM, then a reduces to a constant and $f(z) = a + de^{cz}$, where c and d are two non-zero constants.*

The Corollary 1.2 shows that if a nonzero polynomial a satisfies $\lambda(f-a) < \rho(f)$, then a is not shared CM by $\Delta_c^n f$ and f. For example if we take $f(z) = e^z + z$ and $a(z) = z$, then for any $c \neq 2k\pi i$, $k \in \mathbb{Z}$, we have $\Delta_c f(z) = (e^c - 1)e^z + c$. Hence a is not shared CM by $\Delta_c f$ and f.
This example shows existence of functions which satisfy the conditions of Theorem 1.1.

Example 1.2. *Let $f(z) = e^z$ and $c = \log 2$. Let $a = 0$ and $b \in \mathbb{C} \setminus \{0\}$. Clearly $\lambda(f-a) = 0 < \rho(f)$. Note that*

$$\begin{aligned} \Delta_c^n f(z) &= \sum_{j=0}^{n} (-1)^j C_n^j f(z + (n-j)c) = e^z \sum_{j=0}^{n} (-1)^j C_n^j e^{(n-j)c} \\ &= \left(e^{nc} - C_n^1 e^{(n-1)c} + \ldots + (-1)^n\right) e^z = (e^c - 1)^n e^z = e^z. \end{aligned}$$

Therefore $\Delta_c^n f \equiv f$ and so f and $\Delta_c^n f$ share $b \in \mathbb{C}$ CM.

Following examples show that the condition "$\lambda(f-a) < \rho(f)$" in Theorem 1.1 is sharp.

Example 1.3. *Let* $f(z) = Ae^{z\log(c+1)} - \frac{1-c}{c}$*, where* $c \in \mathbb{R} \setminus \{0\}, c > -1$ *and* A *is an arbitrary constant. Let* $a \in \mathbb{C} \setminus \{0\}$ *such that* $a \neq -\frac{1-c}{c}$ *and* $\frac{1-c}{c} + a = A$*. It is easy to verify that* $\lambda(f-a) = \rho(f)$ *and* $(\Delta_1 f(z) - 1) = c(f(z)-1)$*. Therefore* $\Delta_1 f$ *and* f *share* 1 *CM, but* f *does not satisfy any case of Theorem 1.1.*

Example 1.4. *Let* $f(z) = e^z + 3$*,* $a = 4$ *and* $c = \pi i$*. Clearly* $\lambda(f-4) = \rho(f) = 1$*. Note that* $\Delta_c f(z) = -2e^z$ *and* $\Delta_c f(z) - 2 = -2(f(z)-2)$*. Therefore* $\Delta_c f$ *and* f *share* 2 *CM, but* f *does not satisfy any case of Theorem 1.1.*

It is easy to see that the conditions "$\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$" in Theorem 1.1 is sharp.

Example 1.5. *Let* $f(z) = e^z$*,* $a(z) = e^z - 1$ *and* $c = \log 2$*. Note that* $\rho(a) = \rho(f)$ *and* $\Delta_c f(z) = e^z$*. Clearly* $\lambda(f-a) = 0 < \rho(f)$ *and* f *and* $\Delta_c f$ *share* $b(\in \mathbb{C})$ *CM, but but* f *does not satisfy any case of Theorem 1.1.*

It is easy to see that the condition "$\rho(b) < \max\{1, \rho(f)\}$" in Theorem 1.1 is sharp.

Example 1.6. *Let* $f(z) = ze^z$*,* $a = 0$*,* $b(z) = (z+c)e^z$ *and* $c = \log 2$*. Note that* $\rho(b) = \rho(f)$ *and* $\Delta_c f(z) = ze^z + 2ce^z$*. Clearly* $\lambda(f) = 0 < \rho(f)$ *and* f *and* $\Delta_c f$ *share* b *CM, but* f *does not satisfy any case of Theorem 1.1.*

Following example shows that the condition "$\lambda(f-a) < \rho(f)$" in Corollary 1.2 is sharp.

Example 1.7. *Let* $f(z) = (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c} z\right)$*, where* $\log$ *denotes the principal branch of the logarithm and* $c = 2\pi i$ *such that* $\log(1+\tau) \neq c$*. Let* $a = 0$*. Note that*

$$\begin{aligned}
\Delta_c f(z) &= (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c}(z+c)\right) - (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c} z\right) \\
&= (\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c} z\right)(\exp(\log(1+\tau)) - 1) \\
&= \tau(\exp z - 1)\exp\left(\frac{\log(1+\tau)}{c} z\right) = \tau f(z).
\end{aligned}$$

Clearly f *and* $\Delta_c f$ *share* 0 *CM. On the other hand, we see that* $\rho(f) \leq 1$ *and* $\lambda(f) = \lambda(\exp z - 1) = 1$*. Since* $\lambda(f) \leq \rho(f)$*, it follows that* $\lambda(f) = \rho(f)$*. Also it is clear that* f *does not satisfy any case of Corollary 1.2.*

Following examples show that the condition "$\rho(f) < +\infty$" in Theorem 1.1 and Corollary 1.2 is necessary.

Example 1.8. *Let* $f(z) = e^z\left(e^{s(z)} - 1\right)$, *where* $s(z)$ *is a periodic function with period* $c = \log 2$ *and* $a(z) = -e^z$. *Clearly* $\rho(f) = +\infty$. *Note that* $\Delta_c f = f$ *and so* f *and* $\Delta_c f$ *share* $b(\in \mathbb{C})$ *CM. On the other hand, we see that* $\lambda(f-a) = 0 < \rho(f)$, *but* f *does not satisfy any case of Theorem 1.1.*

Example 1.9. *Let* $f(z) = e^z e^{s(z)}$, *where* $s(z)$ *is a periodic function with period* $c = \log 2$. *Clearly* $\rho(f) = +\infty$. *Note that* $\Delta_c f = f$ *and so* f *and* $\Delta_c f$ *share* 0 *CM. On the other hand, we see that* $\lambda(f) = 0 < \rho(f)$, *but* f *does not satisfy any case of Corollary 1.2.*

Following example assert that Theorem 1.1 does not valid when f is a transcendental meromorphic function.

Example 1.10. *Let* g *be a periodic entire function with period* 1 *such that* $\lambda(g) < \rho(g) = 1$ *and* $g(z)$ *and* $\sin 2\pi z$ *have no common zeros. Let* $a = 0$ *and*

$$f(z) = \frac{g(z)}{\sin 2\pi z} e^{z\log 2}.$$

Clearly $\Delta_1 f$ *and* f *share* 1 *CM, but* f *does not satisfy any case of Theorem 1.1.*

2. Auxiliary lemmas

Lemma 2.1. *[*[12]*, Theorem 1.18] Let* f *and* g *be two non-constant meromorphic functions in the complex plane such that* $\rho(f) < \mu(g)$. *Then* $T(r,f) = o(T(r,g)$ $(r \to \infty)$.

Lemma 2.2. *[*[12]*, Theorem 1.44] Let* g *be a non-constant polynomial and* $f = e^g$. *Then* $\rho(f) = \mu(f) = \deg(g)$.

Lemma 2.3. *(*[8]*, Lemma 1.3.1.) Let* $P(z) = \sum_{i=1}^{n} a_i z^i$ *where* $a_n \neq 0$. *Then* $\forall$ $\varepsilon > 0$, *there exists* $r_0 > 0$ *such that* $\forall$ $r = |z| > r_0$ *the inequalities* $(1-\varepsilon)|a_n|r^n \leq |P(z)| \leq (1+\varepsilon)|a_n|r^n$ *hold.*

Lemma 2.4. [12] *Suppose that* $f_1, f_2, \ldots, f_n$ $(n \geq 2)$ *are meromorphic functions and* $g_1, g_2, \ldots, g_n$ *are entire functions satisfying the following conditions*

(i) $\sum\limits_{j=1}^{n} f_j e^{g_j} = 0$

(ii) $g_i - g_j$ *are non-constants for* $1 \leq i < j \leq n$;

(iii) $T(r, f_j) = o\left(T(r, e^{g_h - g_k})\right)$ $(r \to \infty, r \notin E)$ *for* $1 \leq j \leq n$, $1 \leq h < k \leq n$.

Then $f_j \equiv 0$ *for* $j = 1, 2, \ldots, n$.

Lemma 2.5. [5] *Let f be a meromorphic function of finite order ρ and let $c_1,\ c_2 \in \mathbb{C}$ such that $c_1 \neq c_2$. Then for any $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

Lemma 2.6. [9] *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f), P(z,f), Q(z,f)$ are difference polynomials such that the total degree $\deg\left(U(z,f)\right) = n$ *in $f(z)$ and its shifts and* $\deg\left(Q(z,f)\right) \leq n$. *Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then for each $\varepsilon > 0$,*

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f)$$

possible outside of an exceptional set of finite logarithmic measure.

Remark 2.1. *From the proof of Lemma 2.6 in* [9]*, we can see that if the coefficients of $U(z,f), P(z,f), Q(z,f)$, namely $a_\lambda(z)$ satisfy $m(r,a_\lambda) = S(r,f)$, then the same conclusion still holds.*

Lemma 2.7. [5] *Let f be a meromorphic function with a finite order ρ, $\eta \in \mathbb{C}\setminus\{0\}$. Let $\varepsilon > 0$ be given. Then there exists a sub set $E \subset (1,\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0,1]$, we have*

$$\exp\left(-r^{\rho-1+\varepsilon}\right) \leq \left|\frac{f(z+c)}{f(z)}\right| \leq \exp\left(r^{\rho-1+\varepsilon}\right).$$

Lemma 2.8. [1] *Let g be a transcendental function of order less than 1 and h be a positive constant. Then there exists an ε set E such that*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \quad \frac{g(z+\eta)}{g(z)} \to 1 \ \ \textit{as } z \to \infty \textit{ in } \mathbb{C}\setminus E$$

uniformly in η for $|\eta| \leq h$ Further, the set E may be chosen so that for large $|z| \notin E$, the function g has no zeroes or poles in $|z-\zeta| \leq h$.

Lemma 2.9. *Let f be a transcendental entire function of finite order such that $\rho(f) > 1$ and a be an entire function such that $\lambda(f-a) < \rho(f)$ and $\rho(a) < \rho(f)$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $\Delta_c^n f \not\equiv 0$ and b be an entire function such that $\rho(b) < \rho(f)$. Suppose that f is a solution of the difference equation*

$$\Delta_c^n f - b = (f-b)e^Q,$$

where Q is a polynomial. Then $\deg(Q) = \rho(f) - 1$.

Proof. Proof of the lemma follows directly from the proof of Corollary 2.2. [11].

3. Proof of the theorem

Proof of Theorem 1.1. By the given conditions, we have $\lambda(f-a) < \rho(f)$. Then there exist an entire function $H(\not\equiv 0)$ and a polynomial P such that

$$f = a + He^P, \tag{3.1}$$

where $\lambda(H) = \rho(H) < \rho(f-a)$ and $\deg(P) = \rho(f-a)$.

First we suppose $\rho(f) < 1$. Since $\rho(a) < \max\{1, \rho(f)\}$ and $\rho(a) \neq \rho(f)$, it follows that $\rho(a) < 1$ and so $\rho(f-a) = \max\{\rho(a), \rho(f)\} < 1$. Consequently

$$\lambda(f-a) < \rho(f) \leq \max\{\rho(a), \rho(f)\} = \rho(f-a).$$

Note that 0 and ∞ are the Borel exceptional values of $f-a$. Then $f-a$ is a function of regular growth and so $\rho(f-a) \in \mathbb{N}$. Therefore we arrive at a contradiction.

Next we suppose $\rho(f) \geq 1$. In this case, the given conditions $\rho(a) < \max\{1, \rho(f)\}$, $\rho(a) \neq \rho(f)$ and $\rho(b) < \max\{1, \rho(f)\}$ reduce to $\rho(a) < \rho(f)$ and $\rho(b) < \rho(f)$.

Since $\rho(a) < \rho(f)$, it follows that $\rho(H) < \rho(f)$ and $\deg(P) = \rho(f)$. Let

$$P(z) = a_s z^s + a_{s-1} z^{s-1} + \cdots + a_0, \tag{3.2}$$

where $a_s(\neq 0), a_{s-1}, a_{s-2}, \ldots, a_0 \in \mathbb{C}$ and $s \in \mathbb{N}$. Therefore $\rho(f) = \deg(P) = s$. Also from (3.1), we deduce that

$$\Delta_c^n f = \Delta_c^n a + H_n e^P, \tag{3.3}$$

where

$$H_n(z) = \sum_{j=0}^{n} c_j H(z+jc) e^{P(z+jc)-P(z)}, \quad \text{where} \quad c_j = (-1)^{n-j} C_n^j. \tag{3.4}$$

Since $\rho(H) < \rho(f)$, we have $\rho(H(z+ic)) < \rho(f)$ for $i = 0, 1, \ldots, n$. Note that $\deg(P(z+ic) - P(z)) \leq s-1 = \rho(f)-1$. Then from (3.4), we deduce that $\rho(H_n) < \rho(f)$. Also we see that $\rho(\Delta_c^n a) \leq \rho(a)$.

Since f and $\Delta_c^n f$ share b CM, then there exists a polynomial function Q such that

$$\Delta_c^n f - b = (f-b)e^Q. \tag{3.5}$$

Then from (3.3) and (3.5) we have

$$(\Delta_c^n a - b) - (a-b)e^Q = (He^Q - H_n)e^P. \tag{3.6}$$

Again from (3.5), we deduce that $\deg(Q) = \rho(e^Q) \leq \rho(f)$.

Now we divide the following two cases.

Case 1. Suppose $\rho(f) < 2$. Since $\deg(P) = \rho(f)$, it follows that $\deg(P) < 2$ and so $\deg(P) = 1$. Consequently $\rho(f) = 1$. Therefore by the given conditions, we see that $\lambda(f-a) < 1$, $\rho(a) < 1$, $\rho(a) \neq 1$ and $\rho(b) < 1$.

Now we divide the following two sub-cases.

Sub-case 1.1. Suppose $\deg(Q) = 0$. Let $e^Q = d$. Then from (3.6), we have

$$(\Delta_c^n a - b) - d(a - b) = (dH - H_n)e^P. \tag{3.7}$$

Now from Lemma 2.2, we deduce that $\rho\left((\Delta_c^n a - b) - d(a - b)\right) < \rho(f) = \deg(P) = \rho\left(e^P\right) = \mu\left(e^P\right)$ and $\rho\left(dH - H_n\right) < \rho(f) = \rho\left(e^P\right) = \mu\left(e^P\right)$. Then from Lemma 2.1, we conclude that $T\left(r, (\Delta_c^n a - b) - d(a - b)\right) = S\left(r, e^P\right)$ and $T\left(r, dH - H_n\right) = S\left(r, e^P\right)$. Now from Lemma 2.4 and (3.7), we deduce that

$$\Delta_c^n a - b \equiv d(a - b) \;\text{ and }\; dH \equiv H_n. \tag{3.8}$$

If $a \equiv b$, then from (3.8), we deduce that $\Delta_c^n a \equiv a$.

Now if a is a transcendental entire function with order less than 1, then by Lemma 2.8, we get

$$1 = \frac{\Delta_c^n a(z)}{a(z)} = \sum_{j=0}^{n} (-1)^{n-j} C_n^j \frac{a(z + jc)}{a(z)} \to$$
$$\sum_{j=0}^{n} (-1)^{n-j} C_n^j = (1 - 1)^n = 0$$

as $z \to \infty$ possibly outside a ε set E, which is impossible.

If a is a non-constant polynomial, then $\deg(\Delta_c^n a) < \deg(a)$ and so

$$\deg(a) = \deg(\Delta_c^n a - a) = 0,$$

which is also impossible. Hence a is a constant and then $a = \Delta_c^n a = 0$. Therefore if $a \equiv b$, then $a = b = 0$. Now following Sub-case 1 in the proof of Theorem 4.1 in [11], one can easily conclude that

$$f(z) = a(z) + ce^{\beta z},$$

where c and β are two non-zero constants. In particular, if $a \equiv b$, then $a = b = 0$.

Sub-case 1.2. Suppose $\deg(Q) = 1$. In this case, from Sub-case 2 in the proof of Theorem 4.1 in [11], one can easily conclude that $a = b \in \mathbb{C} \setminus \{0\}$ and

$$f(z) = a + ce^{\beta z},$$

where c and β are two non-zero constants.

Case 2. Suppose $\rho(f) \geq 2$.

Then from Lemma 2.9, we deduce that $\deg(Q) = \rho(f) - 1$. Since $\rho(f) \geq 2$, it follows that $\deg(Q) \geq 1$. Now from Lemma 2.5, we have

$$m\left(r, \frac{H(z + jc)}{H(z)}\right) = O\left(r^{\rho(H) - 1 + \varepsilon}\right), \tag{3.9}$$

where $\varepsilon > 0$ is arbitrary. Since $\rho(H) < \rho(e^P)$, we choose $\varepsilon > 0$ such that $\rho(H) - 1 + 2\varepsilon < \rho(e^P) - 1$. Let

$$b_{n-j}(z) = c_j \frac{H(z + jc)}{H(z)} e^{P_j(z)}, \tag{3.10}$$

for $j = 0, 1, 2, \ldots, n$ and

$$F_n(h) = \sum_{j=0}^{n} b_{n-j} h^j. \tag{3.11}$$

We claim that $H_n - He^Q \equiv 0$. If not, suppose $H_n - He^Q \not\equiv 0$. Then we see that the order of the left side of (3.6) is less than $\rho(f)$, but the order of the right side of (3.6) is equal to $\rho(f)$. This is a contradiction. Hence $H_n - He^Q \equiv 0$. Then from (3.4), (3.10) and (3.11), we have

$$F_n(h) = \sum_{j=0}^{n} b_{n-j} h^j = e^Q. \tag{3.12}$$

Let

$$Q(z) = d_{s-1} z^{s-1} + d_{s-2} z^{s-2} + \cdots + d_0. \tag{3.13}$$

Now from (3.4) and (3.12) , we have

$$\sum_{j=1}^{n} c_j \frac{H(z + jc)}{H(z)} e^{R_j(z)} + (-1)^n - e^{Q(z)} = 0, \tag{3.14}$$

where $R_j(z) = P(z + jc) - P(z)$ $(j = 1, \ldots, n)$. Then from (3.2), we may assume that

$$R_j(z) = jsa_s cz^{s-1} + P_{s-2,j}(z), \tag{3.15}$$

where $\deg(P_{s-2,j}) \leq s - 2$. Clearly $\deg(R_j) = s - 1$ for $j = 1, 2, \ldots, n$.

Now we divide the following two sub-cases.

Sub-case 2.1. Suppose $n = 1$. Then from (3.14), we have

$$c_1 \frac{H(z + c)}{H(z)} e^{R_1(z)} - 1 \equiv e^{Q(z)}. \tag{3.16}$$

Clearly (3.16) shows that $\frac{H(z+c)}{H(z)}$ is entire. Then from (3.9), we deduce that

$$T\left(r, \frac{H(z + c)}{H(z)}\right) = m\left(r, \frac{H(z + c)}{H(z)}\right) = O\left(r^{\rho(H)-1+\varepsilon}\right)$$

and so

$$\rho\left(\frac{H(z + c)}{H(z)}\right) = \rho(H) - 1 < \rho(f) - 1 = s - 1 = \rho(e^{R_1}).$$

Therefore it is easy to conclude that 0 is a Borel exceptional value of the entire function $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)}$. Consequently 1 is not a Borel exceptional of $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)}$ and so $c_1 \frac{H(z+c)}{H(z)} e^{R_1(z)} - 1$ must have infinitely many zeros. Therefore we arrive at a contradiction from (3.16).

Sub-case 2.2. Suppose $n \geq 2$. Then from (3.13) and (3.15), we see that

$$R_j(z) - Q(z) = (jsa_sc - d_{s-1})z^{s-1} + \ldots,$$

where $j = 1, 2, \ldots, n$.

Now we divide following two sub-cases:

Sub-case 2.2.1. Suppose there exists $j_0 (1 \leq j_0 \leq n)$ such that $j_0 sa_sc = d_{s-1}$. Therefore $\deg(R_{j_0} - Q) \leq s - 2$. In this case from (3.14), we have

$$\left(\sum_{\substack{1\leq j\leq n\\ j\neq j_0}} c_j \frac{H(z+cj)}{H(z)} e^{P(z+jc)-P(z+c)} + B_{j_0} e^{P(z+j_0c)-P(z+c)}\right) e^{R_1(z)} = \tag{3.17}$$
$$= (-1)^{n+1},$$

where

$$B_{j_0}(z) = c_{j_0} \frac{H(z+j_0c)}{H(z)} - e^{Q(z)-R_{j_0}(z)}. \tag{3.18}$$

Let $Q_1(z) = e^{R_1(z)}$. Note that

$$Q_1(z+(j-1)c)\ldots Q_1(z+c) = e^{\left(\sum\limits_{i=2}^{j} P(z+ic)-P(z+(i-1)c)\right)} = e^{P(z+jc)-P(z+c)}$$

for $j = 2, 3, \ldots, n$.

Then (3.17) can be written as

$$U(z, Q_1(z))Q_1(z) = (-1)^{n+1}, \tag{3.19}$$

where

$$\begin{aligned} U(z, Q_1(z)) \quad = \quad & \sum_{\substack{1\leq j\leq n\\ j\neq j_0}} c_j \frac{H(z+jc)}{H(z)} Q_1(z+(j-1)c)Q_1(z+(j-2)c)\cdots Q_1(z+c) \\ & + B_{j_0}(z)Q_1(z+(j_0-1)c)Q_1(z+(j_0-2)c)\cdots Q_1(z+c) \end{aligned}$$

if $j_0 \geq 2$ and

$$\begin{aligned} U(z, Q_1(z)) \quad = \quad & \sum_{2\leq j\leq n} c_j \frac{H(z+jc)}{H(z)} Q_1(z+(j-1)c)Q_1(z+(j-2)c)\cdots Q_1(z+c) \\ & + B_{j_0}(z) \end{aligned}$$

if $j_0 = 1$.

From (3.19), it is clear that $U(z, Q_1) \not\equiv 0$ and $\deg(U(z, Q_1)) = n - 1 \geq 1$. Now we want to prove that if a_λ is a coefficient of $U(z, Q_1)$, then $m(r, a_\lambda) = S(r, Q_1)$. Note that from Lemma 2.2, we have

$$\mu(e^{R_1}) = \rho(e^{R_1}) = \deg(R_1) = s - 1$$

and

$$\rho(e^{Q-R_{j_0}}) = \deg(Q - R_{j_0}) \leq s - 2 < s - 1 = \mu(e^{R_1}).$$

Then by Lemma 2.1, we deduce that

$$T(r, e^{Q-R_{j_0}}) = S(r, e^{R_1}) = S(r, Q_1). \tag{3.20}$$

Also it is easy to prove from (3.9) that

$$m\left(r, \frac{H(z+jc)}{H(z)}\right) = S(r, e^{R_1}) = S(r, Q_1) \ (j = 1, 2, \ldots, n). \tag{3.21}$$

Now from (3.18), (3.20) and (3.21), we see that

$$m(r, B_{j_0}(z)) \leq m\left(r, \frac{H(z+j_0c)}{H(z)}\right) + m\left(r, e^{Q(z)-R_{j_0}(z)}\right) \leq S(r, Q_1).$$

Then in view of Remark 2.1 and using Lemma 2.6, we conclude that

$$m(r, Q_1) = S(r, Q_1).$$

Therefore $T(r, Q_1) = m(r, Q_1) = S(r, Q_1)$, which is a contradiction.

Sub-case 2.2.2. Suppose $jsa_sc \neq d_{s-1}$ for $1 \leq j \leq n$. In this case (3.14) can be rewrite as

$$e^{Q(z)} = e^{d_{s-1}z^{s-1}} e^{\tilde{P}_{s-2}(z)} = \sum_{j=0}^{n} c_j \frac{H(z+jc)}{H(z)} e^{R_j(z)}, \tag{3.22}$$

where

$$\tilde{P}_{s-2}(z) = Q(z) - d_{s-1}z^{s-1} = d_{s-2}z^{s-2} + d_{s-3}z^{s-3} + \cdots + d_0. \tag{3.23}$$

Again from (3.15) and (3.22), we have

$$e^{Q(z)} = e^{d_{s-1}z^{s-1}} e^{\tilde{P}_{s-2}(z)} = \sum_{j=1}^{n} c_j \frac{H(z+jc)}{H(z)} e^{jsa_scz^{s-1}} e^{P_{s-2,j}(z)} + (-1)^n. \tag{3.24}$$

Note that

$$ns|a_sc| > (n-1)s|ca_s| > \cdots > s|a_sc|$$

and either $|d_{s-1}| \in \{js|a_sc| : j = 1, 2, \ldots, n\}$ or $|d_{s-1}| \notin \{js|a_sc| : j = 1, 2, \ldots, n\}$. Therefore if we compare $|d_{s-1}|$ with $ns|a_sc|$, $(n-1)s|a_sc|$, $\cdots$, $s|a_sc|$, then it is enough to compare $|d_{s-1}|$ with $ns|a_sc|$. Without loss of generality, we suppose that $ns|a_sc| \leq |d_{s-1}|$.

Let $\arg d_{s-1} = \theta_1$ and $\arg(a_sc) = \theta_2$. Take θ_0 such that $\cos((s-1)\theta_0 + \theta_1) = 1$. Then using Lemma 2.7, we see that for any given ε $(0 < \varepsilon < s - \rho(H))$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure such that for all $z = re^{i\theta_0}$ satisfying $|z| = r \notin [0, 1] \cup E$ we have

$$\exp\left(-r^{\rho(H)-1+\varepsilon}\right) \leq \left|\frac{H(z+jc)}{H(z)}\right| \leq \exp\left(r^{\rho(H)-1+\varepsilon}\right) \ (j = 1, 2, \ldots, n). \tag{3.25}$$

Note that

$$\begin{aligned}(3.26)\qquad & \left|\exp\left(d_{s-1}z^{s-1}\right)\right| \\ = & \left|\exp\left(|d_{s-1}|r^{s-1}\left(\cos((s-1)\theta_0+\theta_1)\right)+i\sin((s-1)\theta_0+\theta_1)\right)\right| \\ = & \exp\left(|d_{s-1}|r^{s-1}\right).\end{aligned}$$

Similarly we can show that

$$(3.27)\quad \left|\exp\left(jsa_scz^{s-1}\right)\right| = \exp\left(js|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2)\right),\ j=1,2,\ldots,n.$$

Using Lemma 2.3 (taking $\varepsilon=\frac{1}{2}$), we deduce from (3.23) that $\left|\tilde{P}_{s-2}(z)\right| \geq \frac{|d_{s-2}|}{2}r^{s-2}$ and so

$$(3.28)\qquad \left|\exp\left(\tilde{P}_{s-2}(z)\right)\right| \geq \exp\left(\frac{|d_{s-2}|}{2}r^{s-2}\right).$$

Again using Lemma 2.3 (taking $\varepsilon=\frac{1}{2}$), we deduce that $|P_{s-2,j}(z)| = O(r^{s-2})$ and so

$$(3.29)\qquad |\exp\left(P_{s-2,j}(z)\right)| = \exp\left(O\left(r^{s-2}\right)\right)\ j=1,2,\ldots,n.$$

Now from (3.25), (3.27) and (3.29), we get

$$\begin{aligned}(3.30)\qquad & \left|\frac{H(z+jc)}{H(z)}e^{jsa_scz^{s-1}}e^{P_{s-2,j}(z)}\right| \\ \leq & \exp\left(js|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2))+r^{\rho(H)-1+\varepsilon}+O\left(r^{s-2}\right)\right) \\ \leq & \exp\left(ns|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2))+r^{\rho(H)-1+\varepsilon}+O\left(r^{s-2}\right)\right)\end{aligned}$$

for $j=1,2,\ldots,n$.

Then from (3.24), (3.26), (3.28) and (3.30), we conclude that

$$\begin{aligned}& \exp\left(|d_{s-1}|r^{s-1}\right) = \left|\exp(d_{s-1}z^{s-1})\right| = \left|\frac{\exp(Q(z))}{\exp(\tilde{P}_{s-2}(z))}\right| \\ (3.31) \leq \quad & \frac{\left|\sum\limits_{j=1}^{n} c_j\frac{H(z+jc)}{H(z)}e^{jsa_scz^{s-1}}e^{P_{s-2,j}(z)}+(-1)^n\right|}{|\exp(\tilde{P}_{s-2}(z))|} \\ \leq \quad & \frac{(n+1)n!\exp\left(ns|a_sc|r^{s-1}\cos((s-1)\theta_0+\theta_2))+r^{\rho(H)-1+\varepsilon}+O\left(r^{s-2}\right)\right)}{\exp\left(\frac{|d_{s-2}|}{2}r^{s-2}\right)}.\end{aligned}$$

Since $\rho(H)-1+\varepsilon<s-1$ and $(n+1)n! = \exp(\log(n+1)n!) = o(r^{s-1})$, from (3.31), we deduce that

$$(3.32)\quad \exp\left(|d_{s-1}|r^{s-1}\right) \leq \exp\left(ns|a_sc|\cos((s-1)\theta_0+\theta_2)r^{s-1}+o(r^{s-1})\right).$$

By assumption, we have $d_{s-1}\neq nsa_sc$ and $ns|a_sc|\leq|d_{s-1}|$.

First we suppose $ns|a_s c| = |d_{s-1}|$. In that case $\cos((s-1)\theta_0 + \theta_2) \neq 1$ and so $\cos((s-1)\theta_0 + \theta_2) < 1$. Therefore

$$ns|a_s c| \cos((s-1)\theta_0 + \theta_2) < ns|a_s c| = |d_{s-1}|.$$

Next we suppose $ns|a_s c| < |d_{s-1}|$. Then obviously

$$ns|a_s c| \cos((s-1)\theta_0 + \theta_2) \leq ns|a_s c| < |d_{s-1}|.$$

Then in either case we have

$$ns|a_s c| \cos((s-1)\theta_0 + \theta_2) < |d_{s-1}|.$$

Therefore there exists $\varepsilon_1 > 0$ such that

$$ns|a_s c| \cos((s-1)\theta_0 + \theta_2) + 2\varepsilon_1 < |d_{s-1}|$$

and so from (3.32), we have

$$\exp\left(|d_{s-1}| r^{s-1}\right) \leq \exp\left(\left(|d_{s-1}| - \varepsilon_1\right) r^{s-1}\right),$$

which is a contradiction. This completes the proof. $\square$

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